Gauge theory and complex manifolds

Vicente Muñoz
St. Hugh's College, Oxford
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To Ana and Carmen

Basic classes

for sums along surfaces

Contents

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Introduction

Since the apparition of Donaldson's invariants [7] nearly ten years ago, these have been very useful in classifying a range of differentiable 4-manifolds. Early results about the vanishing of the invariants for connected sums where both summands have $b^+ > 0$ produce corollaries about the indecomposability of certain four-manifolds (see [13, chapter 10]). Computing the invariants of a given four-manifold has been a major and very challenging problem in most cases. Nonetheless, Kronheimer and Mrowka [38] have discovered that, for a wide range of the manifolds (the so called of simple type), they are encoded in a finite collection of cohomology classes (called the basic classes of the manifold) and some rational numbers attached to them, satisfying very restrictive constraints. The computation of the basic classes for a manifold which appears as a the union of two open manifolds with the same boundary is the key for understanding the behaviour of the invariants under surgery. Here one can proceed in the style of a TQFT, i.e. one attaches to every closed oriented three-manifold Y a vector space V_Y, in some natural way, and for every open manifold X with boundary Y a vector $\phi_X \in V_Y$. When we glue two open manifolds X_1 and X_2 with $\partial X_1 = Y$, $\partial X_2 = \overline{Y}$, the invariant for $X = X_1 \cup_Y X_2$ appears as the product $\phi_{X_1} \cdot \phi_{X_2}$, where $\phi_{X_1} \in V_Y$, $\phi_{X_2} \in V_{\overline{Y}} = V_Y^*$. This program has been carried out to some extent [3][9][10]. The vector spaces are the instanton Floer homology groups of Y. But these are in general difficult to compute.

In the case of $Y = \Sigma \times \mathbb{S}^1$, with Σ a Riemann surface of genus $g \geq 1$, and bundles $E \to Y$ which have odd first Chern class in $H^2(Y;\mathbb{Z})$, there is a description of the Floer groups as the homology of the moduli space of stable bundles of odd degree over Σ. One expects to be able to use this to get some information on the basic classes of a manifold (in the shape of constraints that they satisfy) which is a connected sum along a Riemann surface. In this thesis, we succeed in carrying on this program for the cases of $g = 1$ and 2.

In the late 1994 there was a revolution in gauge theory with the introduction of Seiberg-Witten invariants. They are much easier to compute, but keep the flavour of the former invariants. Conjecturally [63] they give the same information on the manifold whenever $b^+ > 1$, $b_1 = 0$ and the manifold is of simple type. An analogous gluing theory for this case is under development, constructing Floer-Seiberg-Witten homology groups [40][62]. One can use similar sort of arguments to get constraints on the Seiberg-Witten basic classes for connected sums along Riemann surfaces.

The thesis is mainly divided in two parts, the first one corresponding to computations with the Donaldson invariants and the second part dedicated to the Seiberg-Witten invariants. In the first instance our intention was to carry on looking to connected sums along Σ of genus bigger than 2, but the introduction of Seiberg-Witten invariants made more sensible to turn around and look into the direction of using a sort of Floer-Seiberg-Witten theory. This was quite useful, although not at all a trivial problem. Many things remain to be said about the computation of Seiberg-Witten invariants for connected sums along Riemann surfaces or more general three-manifolds.

The first chapter introduces the rudiments and notations of gauge theory and Donaldson invariants and is complemented with an exposition of Floer homology and Fukaya-Floer homology, which we think has not been exploited as much as one could.

The second chapter is very basic and contains essentially topology about connected sums of two four-manifolds along Riemann surfaces. The more interesting part is the introduction of the extended homology groups $H_2^R(X, \partial X)$ for a manifold X with boundary ∂X.

Then we have added a description of algebraic manifolds which are fibred with fibres being elliptic curves and complex curves of genus 2. This has been inserted for the sake of completeness, as these manifolds are examples to which we could apply all the theory developed in this thesis.

In chapter four we use the gluing techniques to compute basic classes for manifolds coming from open manifolds glued along a three-torus. The seed of this work is the first year dissertation of the student, and we thought worthwhile to add our own proof of Friedman's theorem [22], as it uses a very elementary gluing theory. Essentially, we reduce the computation of the low dimensional invariants of an arbitrary simply-connected elliptic surface to those of $\mathbb{T}^2 \times \mathbb{CP}^1$ and then we compute these ones. This is the reason for having a section with a description of the wall-crossing formulae for algebraic manifolds in chapter three (we must say that we have another proof that avoid the wall-crossing formulae, finding out the moduli space of stable bundles explicitly, but it is not as beautiful as this one). We complete the chapter with general results about basic classes for glued manifolds along a $\mathbb{T}^3$ (which are a bit more general than some of the results there are in the literature so far) and for a manifold in which we have performed a log-transform.

The last chapter of the first part is devoted to the case of connected sum along a Riemann surface of genus 2 and it forms the main bulk of the thesis. We have to look at the invariants on three different types of cycles in the glued manifold, according to the way they intersect $Y = \Sigma \times \mathbb{S}^1$, since the Fukaya-Floer groups differ. The main task is to determine the invariants of an open manifold in terms of the invariants of closings (or cappings) of it, and this can be done for $g = 2$ since the dimension of the Floer homology of Y is not too big. Our main argumental line goes through assuming Conjecture 1.22 about the action of the homology of Y on its Floer homology. We also have to assume Conjectures 5.11 and 5.20 which are variations of the former about the action of the homology of Y on its Fukaya-Floer homology.

The principal results, theorems 5.6, 5.17 and 5.23, yield that different identifications for $Y = \Sigma \times \mathbb{S}^1$ inducing the same action on homology give the same invariants, although in principle the resulting manifolds might not be diffeomorphic. We also deduce a finite type condition for manifolds with $b^+ > 1$, $b_1 = 0$ and an embedded Σ of genus 2 representing an odd homology class, theorem 5.16. The main task left for future research is try to remove the use of the conjectures and to generalise these results to the case of connected sums along surfaces of higher genus. One would expect that the finite type condition holds in all cases.

The second part of the thesis deals with the Seiberg-Witten theory. In chapter six we introduce the Seiberg-Witten equations and prove that the basic classes L for a connected sum $X = \bar{X}_1 \# \bar{X}_2$ satisfy the condition $L|_Y = n[\mathbb{S}^1]$, with n an even integer between $-(2g-2)$ and $(2g-2)$ (corollary 6.13).

In chapter seven we make some use of a gluing theory for the Seiberg-Witten invariants which is being developed by many people at the moment (so in this sense the chapter is rather speculative). Now there are no restrictions on the genus of Σ, but there are restrictions on the degree of the line bundle along the curve. We finally compare the results now obtained with the ones obtained previously. For future research it is left the rigorous definition of the Seiberg-Witten-Floer homology and the study of the gluing for line bundles L with $|c_1(L)\cdot\Sigma| < 2g-2$.

We have tried to be short and clear at the same time in our exposition, but we have decided to include some results which overlap whenever we have considered enlightening the different ways provided to prove them.

Chapter 1

Basic notions

1.1 Polynomial invariants

We begin with a review of the definition of Donaldson's polynomial invariants. Throughout X will be any compact oriented connected smooth manifold of dimension four, but we will not suppose in principle that it is simply connected.

1.1.1 Spaces of connections

Here we follow [38] mainly. Very good references for the basics on gauge theory are [13] and [20]. The set-up for gauge theory on X is the following. Fix a line bundle w and a $U(2)$-bundle $E \to X$ (whose fibres are $\mathbb{C}^2$ with the standard representation of $U(2)$) with an isomorphism

$$\psi : \det(E) \to w.$$

The topological type of E is given by its Chern numbers $c_1(E)$ and $c_2(E)$. Let $\mathfrak{g}_E$ denote the bundle of traceless skew-hermitian endomorphisms of E. This is the associated $SO(3)$-bundle with second Stiefel-Whitney class $w_2 \in H^2(X;\mathbb{Z}_2)$ the reduction mod 2 of $c_1(E)$ and with Pontrjagin class $p_1 \in H^4(X;\mathbb{Z}) \cong \mathbb{Z}$ given by

$$\kappa = -\frac{1}{4} < p_1(\mathfrak{g}_E), [X] >=< c_2(E) - \frac{1}{4}c_1^2(E), [X] > .$$

We recall that an $SO(3)$-bundle is uniquely determined by w_2 and p_1, subject to the constraint $p_1 \equiv w_2^2 \pmod 4$.

The gauge group $\mathcal{G} = \mathcal{G}_E$ is the group of determinant one unitary automorphisms of E (i.e. those that respect ψ) and acts in a natural way on $\mathfrak{g}_E$. We denote by $\mathcal{A} = \mathcal{A}_E$ the space of connections in $\mathfrak{g}_E$. Then $\mathcal{G}$ acts on $\mathcal{A}$ and the quotient is denoted by $\mathcal{B} = \mathcal{B}^w_\kappa$. We can think of this space as the moduli space of connections in E all inducing the same connection in $\det(E)$. For all the spaces of connections and gauge groups we have to consider coefficients in suitable Sobolev spaces to make the theory work, but we will not be explicit about this point.

For a connection A in $\mathfrak{g}_E$, we denote its curvature by F_A. By Chern-Weil theory, the action of A is

$$\kappa = -\frac{1}{4} < p_1(\mathfrak{g}_E), [X] >= \frac{1}{32\pi^2} \int \mathrm{tr}(F_A^2).$$

The possible stabilisers of connections are as follows. The typical connection has stabiliser $\pm 1 \subset Z(SU(2)) \subset \mathcal{G}$, in which case it is called irreducible. The space of irreducible connections is $\mathcal{A}^* \subset \mathcal{A}$ and the quotient $\mathcal{A}^*/\mathcal{G}$, denoted by $\mathcal{B}^* = \mathcal{B}^{w,*}_\kappa$, is a Banach manifold. If the connection is non-trivial but preserves a splitting $\mathfrak{g}_E = \mathbb{R} \oplus L$, where L is a complex line bundle, then the stabiliser of the connection is a circle subgroup $\mathbb{S}^1 \subset \mathcal{G}$. These connections are called reducible. If $H^1(X;\mathbb{Z}_2) \neq 0$ then we might have connections preserving a splitting $\mathfrak{g}_E = \lambda \oplus L$, where λ is a non-orientable real line bundle and L is a non-orientable real two plane bundle with orientation bundle isomorphic to λ. These are called twisted reducible connections. Note that the stabiliser of a twisted reducible connection is ± 1. Finally, the trivial connection has stabiliser $SU(2)$.

Now we choose a riemannian metric g on X. We have a Hodge operator $*_g$ and put

$$\mathcal{M}^w_\kappa = \mathcal{M}^w_\kappa(X)_g = \{[A] \in \mathcal{B} \,/\, *_g F_A = -F_A\}/\mathcal{G}$$

for the space of g-**antiselfdual connections** (abbreviated as ASD). We have the following result for generic metrics for a closed manifold

Proposition 1.1 ([20][38, corollary 2.5]) *For a generic metric on X, the moduli spaces $\mathcal{M}^w_\kappa$ are smooth manifolds, except at flat or reducible connections. For a generic path of metrics $\gamma = \{g_t\}_{t\in[0,1]}$, the same is true of the parametrised moduli space $\mathcal{M}^w_\kappa(X)_\gamma = \cup_t \mathcal{M}^w_\kappa(X)_{g_t}$.*

The (formal) dimension of the moduli space $\mathcal{M}^w_\kappa$ is $8\kappa - 3(1 - b_1 + b^+)$. This is its actual dimension for a generic metric at points which correspond to non-flat irreducible connections.

Remark 1.2 *The point of proposition 1.1 is that twisted reducible connections do not appear for closed manifolds, as they are ruled out on dimensional grounds.*

So in that sense, it is rather special for closed manifolds. For open manifolds (manifolds with cylindrical ends) we only have the general result that for a generic metric the moduli spaces of ASD connections are smooth except at locally reducible connections. These also include twisted reducible connections, as well as flat and reducible connections.

To orient the moduli spaces $\mathcal{M}_\kappa^w$ we need to choose a homology orientation Ω. This is an orientation of $H^0(X;\mathbb{R}) \oplus H^1(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R})$. The orientation of the moduli space reverses when we change Ω to $-\Omega$.

Remark 1.3 *Instead of using this $U(2)$ gauge set up, we might have started up with an $SO(3)$-bundle ζ with fibre $\mathbb{R}^3$ and associated principle bundle P, defined by its classes w_2 and p_1 (with the constraint $w_2^2 \equiv p_1 \pmod 4$). The space of connections in this bundle is acted on by the gauge group $\mathcal{G}_\zeta$ consisting of the sections of the adjoint bundle AdP. Now every lift $w \in H^2(X;\mathbb{Z})$ of w_2 to integer coefficients (this might not exist) defines a lift of P to a $U(2)$-bundle E with $c_1(E) = w \equiv w_2 \pmod 2$ and $p_1 = c_1^2 - 4c_2$. Clearly ζ and $\mathfrak{g}_E$ are isomorphic and the spaces of connections coincide, but the gauge groups are not quite the same. One has the following exact sequence*

$$0 \to \mathcal{G}_E/\pm 1 \to Aut(\mathfrak{g}_E) \to Hom(\pi_1(X), \pi_1(SO(3))) = H^1(X;\mathbb{Z}_2) \to 0.$$

Then if $\mathcal{M}_\zeta$ denotes the moduli space of ASD connections in $\zeta = \mathfrak{g}_E$ modulo $\mathcal{G}_\zeta = Aut(\mathfrak{g}_E)$, one has

$$\mathcal{M}_\zeta = \mathcal{M}_\kappa^w / H^1(X;\mathbb{Z}_2).$$

The fixed points of this action of $H^1(X;\mathbb{Z}_2)$ on $\mathcal{M}_\kappa^w$ are the $O(2)$ reductions which are not $SO(2)$ reductions (i.e. the twisted reducibles). The advantage of working with $U(2)$ instead of $SO(3)$ is that the only reductions are to a copy of $U(1)$, no matter whether X is simply connected or not.
The moduli space of $SO(3)$-connections is oriented by choosing a homology orientation Ω of X and an integral lift w of w_2. If we pick another integral lift w', the orientation of the moduli space reverses or not depending on whether $(\frac{w-w'}{2})^2$ is odd or even (see [34]).

Recall that there is a universal $SO(3)$-bundle $\mathcal{P} \to \mathcal{B}^* \times X$ defined as the quotient $(\mathcal{A}^* \times Fr_E)/\mathcal{G}$, where Fr_E is the orthogonal frame bundle of $\mathfrak{g}_E$, and there is a map

$$\mu : H_i(X;\mathbb{R}) \to H^{4-i}(\mathcal{B}^*;\mathbb{R})$$

given by the slant product $\mu(\alpha) = -\frac{1}{4}p_1(\mathcal{P})/\alpha$. Also there is a natural compactification of $\mathcal{M}^w_\kappa$, the Uhlenbeck compactification (see [13, section 4.4]).

$$\overline{\mathcal{M}^w_\kappa} \subset \mathcal{M}^w_\kappa \sqcup (\mathcal{M}^w_{\kappa-1} \times X) \sqcup (\mathcal{M}^w_{\kappa-2} \times \mathrm{Sym}^2 X) \sqcup \ldots \sqcup (\mathcal{M}^w_{\kappa-[\kappa]} \times \mathrm{Sym}^{[\kappa]} X),$$

where $\mathrm{Sym}^i X$ is the i-th symmetric product of X and $[x]$ is the integer part of $x \in \mathbb{R}$. Now Donaldson and Kronheimer [13] and Kronheimer and Mrowka [38] construct geometric representatives for $\mu(\alpha)$ which have good properties with respect to this compactification, i.e. they define cycles $\overline{V}_\alpha \subset \overline{\mathcal{M}^w_\kappa}$, for any homology class $\alpha \in H_*(X)$.

1.1.2 Definition of the invariants

Consider a manifold X with $b^+ > 0$ and a fixed homology orientation Ω. Donaldson's polynomial invariants are diffeomorphism invariants of X (more exactly, X with homology orientation Ω, and only when $b^+ > 1$) and are defined as multilinear functions on the (rational) homology of X. First fix $w \in H^2(X;\mathbb{Z})$. We consider the algebra

$$\mathbb{A}(X) = \mathrm{Sym}^*(H_{\mathrm{even}}(X)) \otimes \bigwedge\nolimits^*(H_{\mathrm{odd}}(X)),$$

where $\deg(\alpha) = 4 - i$ if $\alpha \in H_i(X)$. Throughout this thesis, $H_*(X)$ will denote homology with rational coefficients, unless otherwise stated (and similarly for $H^*(X)$). The Donaldson invariant will be a linear function on $\mathbb{A}(X)$. For a monomial $z = \beta_1\beta_2\ldots\beta_r$ of degree δ, we define $D^{w,\delta}_X(z)$ in the following way. Set $D^{w,\delta}_X(z) = 0$ if

$$\delta \not\equiv -2w^2 - 3(1 - b_1 + b^+) \pmod 8.$$

Otherwise choose $p_1 \equiv w^2 \pmod 4$ such that $\delta = -2p_1 - 3(1 - b_1 + b^+)$ and consider the moduli space $\mathcal{M}^w_\kappa$ with $\kappa = -\frac{1}{4}p_1$. Then proposition 1.1 says that for generic metric, this moduli space has dimension δ and is smooth away from flat or reducible connections. Since $b^+ \geq 1$, reducible non-flat connections will not appear for a generic metric. For the moment let us suppose that flat connections do not appear. Then we can compactify $\mathcal{M}^w_\kappa$ as explained at the end of subsection 1.1.1 and choose generic representatives V_{β_i} of $\mu(\beta_i)$ (compatible with the compactification). They intersect in a finite (transverse) number of points in $\mathcal{M}^w_\kappa$. We define the invariant to be the algebraic count of these points using the orientation of $\mathcal{M}^w_\kappa$, i.e.

$$D^{w,\delta}_X(z) = \# V_{\beta_1} \cap \cdots \cap V_{\beta_r}.$$

This invariant turns out to be linear. The usual cobordism argument along with proposition 1.1 for paths of metrics proves that it does not depend on the (generic) metric in the case $b^+ > 1$. For $b^+ = 1$ the picture is more complicated and will be discussed in subsection 1.1.3. In this case we will have to specify the metric g and denote $D^{w,\delta}_{X,g}$ for the invariants with respect to the generic metric g.

The usual trick to get rid of flat solutions (see [42]) is to blow-up X at one point (the definition of blow-up is well-known but it is recalled in definition 2.1). Call $\tilde{X}$ the blow-up of X and E the exceptional divisor, put $\tilde{\kappa} = \kappa + \frac{1}{4}$, $\tilde{w} = w + E$ and $\tilde{z} = zE \in \mathbb{A}(\tilde{X})$. As $\tilde{w}$ does not vanish when restricted to the sphere representing E, there can not be any flat connections in $\mathcal{M}^{\tilde{w}}_{\tilde{\kappa}}(\tilde{X})$. So now we set

$$D^{w,\delta}_X(z) = D^{w+E,\delta+2}_{\tilde{X}}(zE).$$

The right hand side is always defined. When both sides are defined, they are equal as proved in [34] (they have a factor of -2 because of different gauge group conventions). Otherwise the left hand side is defined to be equal to the right hand side. (This is also valid for $b^+ = 1$ choosing a metric on $\tilde{X}$ which is close to the metric g on the X part, close to the Fubini-Study metric on the $\overline{\mathbb{CP}}^2$ part and gives very large length to the neck joining both parts).

When we change w by $w' = w + 2\alpha$, we obtain the same invariants multiplied by a factor of $(-1)^{\alpha^2}$. This is due to the fact that the moduli spaces $\mathcal{M}^w_\kappa$ and $\mathcal{M}^{w'}_\kappa$ are naturally isomorphic but the orientations differ by $(-1)^{(\frac{w'-w}{2})^2}$ (see remark 1.3).

In general we will be dealing with manifolds with $b_1 = 0$ and so we shall restrict attention to the case $\alpha \in H_2(X)$. There are several ways of wrapping up all the information about the different degrees δ in a single series. We put first $D^w_X = \oplus_\delta D^{w,\delta}_X$. Now for instance, calling x the class of the point, we define for $\alpha \in H_2(X)$

$$D^w_X(e^{t\alpha}) = \sum \frac{D^{w,d}_X(\alpha^d)}{d!} t^d.$$

This is a formal series on t. In the same vein as Witten [63] does, consider

$$\mathcal{D}^w_X(\alpha) = D^w_X(e^{t\alpha+\lambda x}) = \sum \frac{D^{w,d+2a}_X(\alpha^d x^a)}{d!\, a!} t^d \lambda^a,$$

which is a formal series on t and λ, although we do not add these variables in the left hand side.

Definition 1.4 *Let X be a four-manifold and $w \in H^2(X;\mathbb{Z})$. Then we define $d_0 = d_0(X,w) = -w^2 - \frac{3}{2}(1 - b_1 + b^+)$.*

With this notation the only coefficients of the series $D_X^w(e^{t\alpha})$ which are non-vanishing are on degrees $d \equiv d_0 \pmod 4$. Analogously, the non-zero coefficients of $\mathcal{D}_X^w(\alpha)$ have $d + 2a \equiv d_0 \pmod 4$.

Definition 1.5 *Let X have $b^+ > 1$. We say that X is of w-**simple type** when $x^2 - 4$ annihilates the Donaldson invariant D_X^w, that is, when $D_X^w((x^2-4)z) = 0$ for all $z \in \mathbb{A}(X)$.*
*When $b_1 = 0$ and $b^+ > 1$, it is an important fact proved in [38] that whenever X is of w-simple type for some w, it is so for every w', and it is called of **simple type** for brevity.*
*When $b^+ = 1$ the invariant does depend on the (generic) metric g and so we have that X is of w-simple type **with respect to** g when $D_{X,g}^w((x^2-4)z) = 0$ for all $z \in \mathbb{A}(X)$.*

Many manifolds, like elliptic surfaces, algebraic complete intersections in $\mathbb{CP}^n$ and many others are known to be of simple type (see [38]). At the moment there are no examples of simply connected non-simple type manifolds with $b^+ > 1$. When X is of simple type, we have

$$\mathcal{D}_X^w(\alpha) = D_X^w(e^{t\alpha})\cosh 2\lambda + D_X^w(\frac{x}{2}e^{t\alpha})\sinh 2\lambda.$$

Kronheimer and Mrowka defined in [38] another series for simple type manifolds containing the same information

$$\mathbb{D}_X^w(t\alpha) = D_X^w(e^{t\alpha}) + D_X^w(\frac{x}{2}e^{t\alpha}).$$

We warn the reader that this notation differs slightly from that of $\mathcal{D}_X^w$ in that we keep track of the variable t in the left hand side. This series has non-zero coefficients only for $d \equiv d_0 \pmod 2$ and therefore it is even or odd depending on whether d_0 is even or odd. The most fundamental result of [38] [18] is

Proposition 1.6 *Let X be a manifold of simple type with $b_1 = 0$ and $b^+ > 1$ and odd. Then we have*

$$\mathbb{D}_X^w(\alpha) = e^{Q(\alpha)/2}\sum(-1)^{\frac{K_i\cdot w + w^2}{2}} a_i e^{K_i\cdot\alpha}$$

*for finitely many cohomology classes K_i (called **basic classes**) and rational numbers a_i (the collection is empty when the invariants all vanish). These classes are lifts to integral cohomology of $w_2(X)$. Moreover, for any embedded surface $S \hookrightarrow X$ of genus g and with $S^2 > 0$, one has $2g - 2 \geq S^2 + |K_i \cdot S|$.*

Proposition 1.7 ([38]) *Let X be a 4-manifold with $b_1 = 0$ and $b^+ > 1$ and odd. Suppose that it contains a **tight** surface Σ (i.e. an embedded surface $\Sigma \hookrightarrow X$ with $\Sigma^2 > 0$ and genus g satisfying $2g - 2 = \Sigma^2$). Then X is of simple type.*

1.1.3 The case $b^+ = 1$

When $b^+ = 1$ the invariants depend on a fundamental way on the metric g of X, since for a generic path of metrics reducibles may appear. Thanks to proposition 1.1 we only have to deal with reductions of the sort $\mathfrak{g}_E = \mathbb{R} \oplus L$, with L a complex line bundle. If we put $e = c_1(L)$, we have $e \equiv w \pmod 2$, $p_1(\mathfrak{g}_E) = e^2$ and the connection corresponding to the reduction is ASD if and only if $e \in \mathcal{H}^g_- \subset H^2(X;\mathbb{R})$, the subspace consisting of antiselfdual harmonic 2-forms for (the conformal class of) the metric g. This space is of codimension 1 so one expects that for generic 1-families of metrics g_t, $t \in [0,1]$, there will be some t with $e \in \mathcal{H}^{g_t}_-$.

Let $\mathbb{H}$ be the image of $\{x \in H^2(X;\mathbb{R})/x^2 > 0\}$ in $\mathbb{P}(H^2(X;\mathbb{R}))$. Note that this is the model of the hyperbolic space of dimension b^-. Its boundary is the projectivisation of the set $\{x \in H^2(X;\mathbb{R})/x \neq 0, x^2 = 0\}$. The positive harmonic space of g (being of dimension $b^+ = 1$) defines a point ω_g in $\mathbb{H}$ called the period point. The reducibles in $\mathcal{M}^w_{\kappa,g}$ correspond to pairs $\pm e \in H^2(X;\mathbb{Z})$ with $e \equiv w \pmod 2$, $e^2 = p_1(\mathfrak{g}_E)$ and $\omega_g \cdot e = 0$. Finally, for the compactified moduli space $\overline{\mathcal{M}^w_{\kappa,g}}$, the reductions which can appear for different strata and varying metric correspond to pairs $\pm e \in H^2(X;\mathbb{Z})$ such that $e \equiv w \pmod 2$ and $p_1 \leq e^2 \leq 0$.

Remark 1.8 *The second inequality is strict whenever there are no reducible flat connections. The usual trick for ruling out reducible flat connections is to blow-up as explained in subsection 1.1.2. Other case in which these flat reductions are not present is when there is an embedded Riemann surface Σ with $w \cdot \Sigma \equiv 1 \pmod 2$. This case is obviously equivalent to w being odd in $H^2(X;\mathbb{Z})/torsion$.*

We will suppose that w is odd in $H^2(X;\mathbb{Z})/\text{torsion}$. In that case if $e^2 = 0$ and $e \cdot \omega_g = 0$, it should be $e = 0$ and therefore $e \not\equiv w \pmod 2$ (in the $SU(2)$ case one must deal with flat connections). So we define

Definition 1.9 *Let $e \in H^2(X;\mathbb{Z})$ be such that $e \equiv w \pmod 2$ and $p_1 \leq e^2 < 0$. We call the image of the hyperplane $e^- = \{x \in H^2(X;\mathbb{R})/x^2 > 0,\ x \cdot e = 0\}$ in $\mathbb{H}$*

the ***wall*** *defined by e, and denote it by W_e. We say that W_e is a wall associated to (w, p_1).*

The wall W_e divides $\mathbb{H}$ in two connected components. All the walls associated to (w, p_1) partition $\mathbb{H}$ into infinitely many components (but the union of the walls form a locally finite subset) called chambers. When the metric move with its period point in a fixed chamber $\mathcal{C}$ the invariants remain constant, but when it crosses a wall they change. The main issue is whether the fibres of the period map are connected or not. This is not known, so in principle the invariants depend not only on the chambers. Also it might happen that the period map is not surjective (and therefore there are no metrics with period points in some particular chambers at all).

When X has $b_1 = 0$, Kotschick and Morgan [35] have proved that the invariants only depend on the chamber by proving that the change of the invariants under the crossing of a wall is independent of the particular path of metrics (their argument also allows them to define the invariants in chambers in which there are not period points of any metric).

For an algebraic manifold, if we restrict attention to the **ample cone** (the subcone of $\{x \in H^2(X;\mathbb{R})/x^2 > 0\}$ spanned by ample line bundles, see section 3.1 for definition of ampleness), and it happens that the invariants can be computed with moduli spaces of stable bundles (because these were generic, see section 3.1), then these invariants are the same for Hodge metrics with period points in a fixed chamber. We will only use this for $X = \Sigma \times \mathbb{CP}^1$, in which case any point in $\mathbb{H}$ is the period point of some Hodge metric.

Remark 1.10 *Let $[x] \in \mathbb{H}$ be any point not contained in any wall associated to (w, p_1) for any p_1 (this is the case if for instance $x \in H^2(X;\mathbb{Z})$ and $x \cdot w \equiv 1 \pmod 2$). Then for every p_1, $[x]$ is in the interior of a chamber and suppose that we can find a generic metric g in it (this is going to happen in all the cases we will be dealing with). Define $D^{w,\delta}_{X,[x]} = D^{w,\delta}_{X,g}$. In this way we have defined $D^w_{X,[x]}$ (at least when X has $b_1 = 0$ or for the case $X = \Sigma \times \mathbb{CP}^1$).*

We also can allow $[x]$ to lie in the closure of $\mathbb{H}$, i.e. $x^2 = 0$. These invariants appear for metrics g giving a very small volume (with respect to the two-form ω_g) to surfaces representing homology classes orthogonal to x. For instance, if we consider $X = X_1 \cup (Y \times [0,1]) \cup X_2$ and metrics giving a very long neck, we will have $x = P.D.[T]$ with $[T] \in H_2(Y) \subset H_2(X)$. Obviously there are different *ways of stretching the neck, corresponding to different metrics on Y and different $[T] \in H_2(Y)$.*

1.1.4 Definition of $D_X^{(w,\Sigma)}$

Let us suppose now that we have $w, \Sigma \in H^2(X;\mathbb{Z})$ satisfying $w \cdot \Sigma \equiv 1 \pmod 2$ and $\Sigma^2 = 0$. Then we define, for $b^+ > 1$,

$$D_X = D_X^{(w,\Sigma)} = D_X^w + D_X^{w+\Sigma}$$

and similarly $\mathcal{D}_X = \mathcal{D}_X^{(w,\Sigma)} = \mathcal{D}_X^w + \mathcal{D}_X^{w+\Sigma}$. When $b^+ = 1$ we consider the invariants referring to the chambers defined by $[\Sigma]$, i.e. $D_X^{(w,\Sigma)} = D_{X,[\Sigma]}^w + D_{X,[\Sigma]}^{w+\Sigma}$. Obviously, D_X depends only on Σ and $w \pmod \Sigma$, since $D_X^{w+2\Sigma} = D_X^w$ by subsection 1.1.2. Now we note that since $(w+\Sigma)^2 \equiv w^2 + 2 \pmod 4$, we can recover D_X^w and $D_X^{w+\Sigma}$ from the series D_X. This series is even or odd according to whether $d_0 = -w^2 - \frac{3}{2}(1 - b_1 + b^+)$ is even or odd.

Proposition 1.11 *Suppose X is a manifold of simple type with $b_1 = 0$ and $b^+ > 1$ and odd. Write the Donaldson series as $\mathbb{D}_X^w = e^{Q/2} \sum (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j \, e^{K_j}$. Then setting $d_0 = d_0(X,w) = -w^2 - \frac{3}{2}(1+b^+)$ we have*

$$D_X^{(w,\Sigma)}(e^\alpha) = e^{Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod 4} (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j e^{K_j \cdot \alpha} + e^{-Q(\alpha)/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod 4} i^{-d_0} (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j e^{i K_j \cdot \alpha}$$

So giving $\mathbb{D}_X^w$ is equivalent to giving $D_X^{(w,\Sigma)}$.

Proof. Since $((w+\Sigma)^2 + K_j \cdot (w+\Sigma)) = (w^2 + K_j \cdot w) + 2(w \cdot \Sigma + K_j \cdot \Sigma/2)$ we have

$$\mathbb{D}_X^{w+\Sigma} = e^{Q/2} \sum_{K_j \cdot \Sigma \equiv 2 \pmod 4} (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j \, e^{K_j} - e^{Q/2} \sum_{K_j \cdot \Sigma \equiv 0 \pmod 4} (-1)^{\frac{K_j \cdot w + w^2}{2}} a_j \, e^{K_j}$$

Now since the only powers in $D_X^w(e^{t\alpha})$ are those t^d with $d \equiv d_0 \pmod 4$ one has

$$D_X^w(e^{t\alpha}) = \frac{1}{2}(\mathbb{D}_X^w(t\alpha) + i^{-d_0} \mathbb{D}_X^w(it\alpha))$$

and analogously

$$D_X^{w+\Sigma}(e^{t\alpha}) = \frac{1}{2}(\mathbb{D}_X^{w+\Sigma}(t\alpha) - i^{-d_0} \mathbb{D}_X^{w+\Sigma}(it\alpha))$$

since $d_0(X, w+\Sigma) = d_0(X,w)+2$. So we finally get

$$D_X^{(w,\Sigma)}(e^\alpha) = e^{Q(\alpha)/2} \sum_{K_j\cdot\Sigma\equiv 2 \pmod 4} (-1)^{\frac{K_j\cdot w + w^2}{2}} a_j e^{K_j\cdot\alpha} + i^{-d_0} e^{-Q(\alpha)/2} \sum_{K_j\cdot\Sigma\equiv 0 \pmod 4} (-1)^{\frac{K_j\cdot w+w^2}{2}} a_j e^{iK_j\cdot\alpha}$$

□

Remark 1.12 *For the class of the point $x \in H_0(X)$, $a_{j,w} = (-1)^{\frac{K_j\cdot w+w^2}{2}}$,*

$$D_X^{(w,\Sigma)}(x\, e^\alpha) = 2e^{Q(\alpha)/2} \sum_{K_j\cdot\Sigma\equiv 2 \pmod 4} a_{j,w} e^{K_j\cdot\alpha} - 2i^{-d_0} e^{-Q(\alpha)/2} \sum_{K_j\cdot\Sigma\equiv 0 \pmod 4} a_{j,w} e^{iK_j\cdot\alpha}.$$

1.2 Gluing theory

A natural way of computing the invariants for a closed manifold X is to split it into elementary pieces for which the invariants are easily computable. For this we need to understand, for every splitting $X = X_1 \cup_Y X_2$ along a three-manifold Y, how to the record differential-topological information about the open pieces[1] X_i from which we can recover the invariants of X. This was answered in the first place with the (instanton) Floer homology of Y, which allows us to calculate the invariants of X on classes $\alpha \in H_2(X_1) \oplus H_2(X_2)$ (i.e. classes not split in two by Y). The general case is treated with the so called Fukaya-Floer homology.

1.2.1 Instanton Floer homology

Let Y be an oriented three-manifold and let $P_Y \to Y$ be a $U(2)$ bundle such that $c_1(P_Y)$ is odd in $H^2(Y;\mathbb{Z})/\text{torsion}$ (what will be called **odd**). In this situation P_Y only carries irreducible flat connections (with fixed determinant). We say that P_Y is **free of flat reductions.** Possibly after a small perturbation of the flat equations, there will be finitely many flat connections ρ_j, and they will all be non-degenerate. The Floer complex $CF_*(Y)$ is the free abelian group on the generators ρ_j, where the grading is given by the index (see [9] [8]) and lies in an affine $\mathbb{Z}/8$-space (this complex depends on $c_1(P_Y)$ but we do not express this in the notation). When Y is a homology sphere (and P_Y is obviously trivial) the trivial connections were used to fix the index (see [2]). In the general case, the index is only defined up to a constant.

[1] An open manifold will refer to a manifold with boundary or with a cylindrical end. A closed manifold is a compact manifold (without boundary).

Now for every two flat connections ρ_k and ρ_l such that $\mathrm{ind}(\rho_k) = \mathrm{ind}(\rho_l) + 1$, there is a compact zero dimensional moduli space $\mathcal{M}_0(\rho_k, \rho_l)$ of (perturbed) ASD connections on the tube $Y \times \mathbb{R}$ with limits ρ_k and ρ_l modulo translations. This space can be oriented[2] and so we have well defined the algebraic number of its points, $\#\mathcal{M}_0(\rho_k, \rho_l)$. We define the boundary map of the Floer complex to be

$$\begin{aligned} \partial : CF_i(Y) &\rightarrow CF_{i-1}(Y) \\ \rho_k &\mapsto \sum_{\substack{\rho_l \\ \mathrm{ind}(\rho_l)=\mathrm{ind}(\rho_k)-1}} \#\mathcal{M}_0(\rho_k, \rho_l)\rho_l \end{aligned}$$

Lemma 1.13 ([15][2][8]) $\partial^2 = 0$.

Proof. Consider ρ_k and ρ_l flat connections such that $\mathrm{ind}(\rho_l) = \mathrm{ind}(\rho_k) - 2$. Then the moduli space $\mathcal{M}_0(\rho_k, \rho_l)$ is a smooth one dimensional manifold which can be compactified adding the broken instantons in

$$\bigcup_{\substack{\rho_m \\ \mathrm{ind}(\rho_m)=\mathrm{ind}(\rho_k)-1}} \mathcal{M}_0(\rho_k, \rho_m) \times \mathcal{M}_0(\rho_m, \rho_l). \tag{1.1}$$

So this compactification, $\overline{\mathcal{M}}_0(\rho_k, \rho_l)$, is a manifold with boundary given by (1.1). Therefore

$$\sum_{\substack{\rho_m \\ \mathrm{ind}(\rho_m)=\mathrm{ind}(\rho_k)-1}} \#\mathcal{M}_0(\rho_k, \rho_m) \cdot \#\mathcal{M}_0(\rho_m, \rho_l) = 0,$$

from where we get $\partial^2 = 0$. $\square$

We define the Floer homology $HF_*(Y)$ as the homology of this complex (this is what Floer did originally [15]). It can be proved that these groups do not depend on the metric of Y or on the chosen perturbation of the ASD equations. The groups $HF_*(Y)$ are natural under diffeomorphisms of the pair (Y, P_Y). The Floer cohomology $HF^*(Y)$ is defined analogously out of the dual complex and it is naturally isomorphic to $HF_{c-*}(\overline{Y})$, for some constant c (where $\overline{Y}$ is Y with reversed orientation). Therefore we have a natural pairing

$$\sigma : HF_*(Y) \otimes HF_{c-*}(\overline{Y}) \rightarrow \mathbb{Z}.$$

[2]The orientation involves choosing a manifold Z with boundary Y and for every ρ_k a connection A_k on Z with limit ρ_k. Also we need to choose a homology orientation of $X = Z \cup_Y Z$. Then $\mathcal{M}_0(\rho_k, \rho_l)$ is oriented in such a way that the orientations for the moduli spaces $\mathcal{M}(Z, \rho_k)$, $\mathcal{M}_0(\rho_k, \rho_l)$, $\mathcal{M}(Z, \rho_l)$ and $\mathcal{M}(X)$ match up correctly.

It is worth noticing that when Y has an orientation reversing diffeomorphism, i.e. $Y \cong \overline{Y}$, we have a pairing

$$\sigma : HF_*(Y) \otimes HF_{c-*}(Y) \to \mathbb{Z}.$$

In [8] it is explained that there is an extra symmetry which gives an involutive isomorphism $h : HF_i(Y) \to HF_{i+4}(Y)$ lifting degrees by 4, allowing us to consider the Floer homology graded mod 4. This will be done systematically in this thesis. It is equivalent to consider the Floer homology starting with an $SO(3)$-bundle.

Let $\alpha \in H_{3-i}(Y)$. We have cycles V_α, in the moduli spaces $\mathcal{M}(\rho_k, \rho_l)$, of codimension $i+1$, representing $\mu(\alpha \times \mathrm{pt})$, for $\alpha \times \mathrm{pt} \subset Y \times \mathbb{R}$, much in the same way as in the case of a closed manifold. Using them, we construct a map

$$\begin{array}{rcl} \mu(\alpha) : CF_j(Y) & \to & CF_{j-i-1}(Y) \\ \rho_k & \mapsto & \displaystyle\sum_{\substack{\rho_l \\ \mathrm{ind}(\rho_l)=\mathrm{ind}(\rho_k)-i-1}} (\#\mathcal{M}(\rho_k,\rho_l) \cap V_\alpha)\, \rho_l \end{array}$$

It is easily seen that $\partial \circ \mu(\alpha) + \mu(\alpha) \circ \partial = 0$ by considering the 1-dimensional moduli space $\mathcal{M}(\rho_k, \rho_l) \cap V_\alpha$ for $\mathrm{ind}(\rho_l) = \mathrm{ind}(\rho_k) - i - 2$ and counting the number of its ends, which yields

$$\begin{array}{l} \displaystyle\sum_{\substack{\rho_s \\ \mathrm{ind}(\rho_s)=\mathrm{ind}(\rho_l)+1}} (\#\mathcal{M}(\rho_k,\rho_s) \cap V_\alpha) \cdot \#\mathcal{M}(\rho_s,\rho_l) + \\ + \displaystyle\sum_{\substack{\rho_s \\ \mathrm{ind}(\rho_s)=\mathrm{ind}(\rho_k)-1}} \#\mathcal{M}(\rho_k,\rho_s) \cdot (\#\mathcal{M}(\rho_s,\rho_l) \cap V_\alpha) \;=\; 0. \end{array}$$

So $\mu(\alpha)$ descends to a map

$$\mu(\alpha) : HF_*(Y) \to HF_{*-i-1}(Y).$$

Consider now an open (oriented) manifold X with $\partial X = Y$ and[3] let $z = \alpha_1 \alpha_2 \ldots \alpha_r \in \mathbb{A}(X)$ of degree $2d$. We are going to define relative invariants for X. Choose $w \in H^2(X; \mathbb{Z})$ such that $w|_Y = c_1(P_Y)$. We give X a cylindrical end modelled on $Y \times [0, \infty)$ and denote it by X again (we hope no confusion arises out of this). Then we have moduli spaces $\mathcal{M}(X, \rho_l)$ of (perturbed) ASD connections with finite action and asymptotic to ρ_l. The dimension of $\mathcal{M}(X, \rho_l)$ is $\dim \mathcal{M}(X, \rho_l) = \mathrm{ind}(\rho_l) + C$, for some constant C. For orienting the spaces

[3]The orientation of Y followed by the inward normal gives the orientation of X.

$\mathcal{M}(X,\rho_l)$ we have to choose a manifold Z as in the footnote of page 11 and a homology orientation for $X \cup_Y Z$. Then orient $\mathcal{M}(X,\rho_l)$ coherently with the orientations of $\mathcal{M}(Z,\rho_l)$ and $\mathcal{M}(X \cup_Y Z)$. Now we can choose (generic) cycles $V_{\alpha_i} \subset \mathcal{M}(X,\rho_l)$, so we have defined an element

$$\phi_d^w(X,z) = \sum_{\substack{\rho_l \\ \mathrm{ind}(\rho_l)+C=2d}} (\#\mathcal{M}(X,\rho_l) \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_r})\, \rho_l \in CF_*(Y).$$

This element has boundary zero and hence it defines a homology class in $HF_*(Y)$ (see [12]). In the same vein, one defines $\phi_d^w(X,D) = \phi_d^w(X,D^d)$ for $D \in H_2(X)$. Now the important gluing theorem reads as follows

Theorem 1.14 *Let $X = X_1 \cup_Y X_2$ with $b^+(X) > 0$ and $w \in H^2(X;\mathbb{Z})$ with $w|_Y$ odd. Let $T \in H^2(X;\mathbb{Z})$ whose Poincaré dual lies in the image of $H_2(Y;\mathbb{Z}) \to H_2(X;\mathbb{Z})$ and satisfying $w \cdot T \equiv 1 \pmod 2$. Put $w = w_1 + w_2$ (this merely means that $w_i = w|_{X_i} \in H^2(X_i;\mathbb{Z})$, and different w's can be written in this way). Choose $z_i \in \mathbb{A}(X_i)$ of degree $2d_i$. Then*

$$D_X^{(w,T)}(z_1\, z_2) = \sigma(\phi_{d_1}^{w_1}(X_1,z_1), \phi_{d_2}^{w_2}(X_2,z_2)).$$

Also for $\alpha_i \in H_2(X_i)$ and $d_i \geq 0$

$$D_X^{(w,T)}(\alpha_1^{d_1}\alpha_2^{d_2}) = \sigma(\phi_{d_1}^{w_1}(X_1,\alpha_1), \phi_{d_2}^{w_2}(X_2,\alpha_2)).$$

Proof. The essential feature is to consider a family of metrics stretching out the neck joining X_1 and X_2 into a long tube. Then the instantons A on X split into pairs (A_1, A_2) of instantons on X_1 and X_2, whose limits are the same flat connection on Y. Then we also need the result that we can glue two instantons A_1 and A_2 with the same flat limit ρ in a unique way (depending on a parameter $t \in (0,\epsilon)$) and this happens as long as ρ is irreducible (see [12] for details). When $b^+ = 1$ this forces the metric to be in chambers corresponding to metrics giving a long neck to X. This is the case for the chambers defined by $[T]$. The explanation of the appearance of (w,T) is in the remark below. □

Definition 1.15 *We define an **allowable pair** to be a pair (w,T) with $w \in H^2(X;\mathbb{Z})$, $w|_Y$ odd, $T \in H^2(X;\mathbb{Z})$ whose Poincaré dual lies in the image of $H_2(Y;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, and satisfying $w \cdot T \equiv 1 \pmod 2$.*

Remark 1.16 *When we glue two line bundles $L_i \to X_i$, $c_1(L_i) = w_i$, with $L_1|_Y \cong L_2|_{\overline{Y}}$ (under the orientation reversing diffeomorphism $Y \stackrel{\sim}{\to} \overline{Y}$), we have an element of choice lying in $H^1(Y;\mathbb{Z}) \cong H_2(Y;\mathbb{Z})$. An isomorphism $L \stackrel{\sim}{\to} L$, $L = c_1(P_Y)$, covering the identity is given thus by an element $T \in H_2(Y;\mathbb{Z})$. This lifts to an isomorphism $P_Y \stackrel{\sim}{\to} P_Y$ (which is not a gauge transformation, as it does not preserve the determinant). We have*

$$\pi_0(Aut(L)) = Hom(\pi_1(Y), \pi_1(\mathbb{S}^1)) = H^1(Y;\mathbb{Z}) \to$$

$$\to \pi_0(Aut(P_Y)) = Hom(\pi_1(Y), \pi_1(SO(3))) = H^1(Y;\mathbb{Z}_2) \stackrel{\cdot w}{\to} \mathbb{Z}_2.$$

This isomorphism is an element of the $SO(3)$ gauge group, so the induced morphism on $HF_*(Y)$ is the identity when we consider the $\mathbb{Z}/4\mathbb{Z}$ grading. For the $\mathbb{Z}/8\mathbb{Z}$ grading, when $w \cdot T \equiv 0 \pmod 2$, the induced map $HF_*(Y) \to HF_*(Y)$ is the identity. When $w \cdot T \equiv 1 \pmod 2$, it is the involution shifting degrees by 4. (Considering the mapping torus of T as an automorphism of P_Y, we have an $U(2)$-bundle over $Y \times \mathbb{S}^1$ with first Chern class $w + [T] \otimes [\mathbb{S}^1]$. The index is $4 \pmod 8$ precisely when $w \cdot T \equiv 1 \pmod 2$). So if (w,T) is allowable, then for grading mod 8 we have

$$D_X^w(z_1\, z_2) = \sigma(\phi_{d_1}^{w_1}(X_1, z_1), \phi_{d_2}^{w_2}(X_2, z_2))$$

for $\phi_{d_1}^{w_1}(X_1, z_1) \in HF_i(Y)$, $\phi_{d_2}^{w_2}(X_2, z_2) \in HF_{c-i}(Y)$ and

$$D_X^{w+T}(z_1\, z_2) = \sigma(\phi_{d_1}^{w_1}(X_1, z_1), \phi_{d_2}^{w_2}(X_2, z_2))$$

for $\phi_{d_1}^{w_1}(X_1, z_1) \in HF_i(Y)$, $\phi_{d_2}^{w_2}(X_2, z_2) \in HF_{c+4-i}(Y)$. This gives the result for our groups graded mod 4.

If we write

$$\phi^w(X, e^{t\alpha}) = \sum_d \frac{\phi_d^w(X, \alpha^d)}{d!} t^d,$$

we have the following version of theorem 1.14

Theorem 1.17 *Let $X = X_1 \cup_Y X_2$ with $b^+(X) > 0$ and let (w,T) be any allowable pair. Then for $\alpha_i \in H_2(X_i)$*

$$D_X^{(w,T)}(e^{t(\alpha_1+\alpha_2)}) = \sigma(\phi^{w_1}(X_1, e^{t\alpha_1}), \phi^{w_2}(X_2, e^{t\alpha_2})).$$

Computing effectively the Floer homology is a very difficult task. In chapter 5 we will be using the Floer homology of $Y = \Sigma \times \mathbb{S}^1$ where Σ is a Riemann surface of genus 2. The Atiyah-Floer conjecture (which has been already proved in the case of $\Sigma \times \mathbb{S}^1$ by Dostoglou and Salamon [14]) relates these (instanton) Floer homology groups with the (symplectic) Floer homology of the moduli space of flat connections over Σ with odd second Stiefel-Whitney class. In turn these symplectic Floer homology groups are identified with the quantum cohomology through work of many authors [53]. Moreover, quantum multiplication is intertwined with the pair-of-pants product in Floer homology.

Remark 1.18 ([53]) *Let M be a positive symplectic manifold (i.e. $c_1(M_\Sigma^{odd}) = \lambda\omega$, for some positive number λ) of dimension $2n$ with minimal Chern number N. The quantum cohomology of M, $QH^*(M)$ is equal to the usual cohomology of M, as abelian groups, but with a different ring structure given by the quantum multiplication. Quantum multiplication is a deformation of the usual cup product on the cohomology of M. If $\alpha \in QH^i(M)$, $\beta \in QH^j(M)$, the quantum product $\alpha * \beta$ is a sum of terms $(\alpha * \beta)_k \in QH^{i+j-2Nk}(M)$ for $k \geq 0$. The leading term is $(\alpha * \beta)_0 = \alpha \cup \beta$. The other terms, the quantum corrections, are defined by counting pseudoholomorphic curves (for some generic compatible almost complex structure J on M). More precisely $< (\alpha * \beta)_k, \gamma >$, $\gamma \in QH^{2n-(i+j-2Nk)}(M)$, is the number of pseudoholomorphic spheres $f : \mathbb{S}^2 \to M$ with $f(0) \in A$, $f(1) \in B$, $f(\infty) \in C$ for generic cycles A, B, C in M representing Poincaré duals of α, β and γ respectively and $c_1(f_*[\mathbb{S}^2]) = Nk$.*

Proposition 1.19 ([14]) *Let $Y = \Sigma \times \mathbb{S}^1$ and $w_2(P_Y) = P.D.[\mathbb{S}^1] \in H^2(\Sigma \times \mathbb{S}^1; \mathbb{Z}_2)$. Then we have the following isomorphism*

$$HF_*(\Sigma \times \mathbb{S}^1) \cong QH^{(6g-6)-*}(M_\Sigma^{odd}) \tag{1.2}$$

of the Floer cohomology of Y with the homology of the moduli space M_Σ^{odd} of odd degree rank two stable vector bundles on Σ, with the grading considered mod 4. The natural pairings on both sides correspond under this isomorphism.

Proof. In [14], Dostoglou and Salamon prove that

$$HF_*(\Sigma \times \mathbb{S}^1, P_Y) \xrightarrow{\simeq} HF_*^{\text{symp}}(M_\Sigma^{\text{odd}}; \mathbb{Z}).$$

Now M_Σ^{odd} is a positive symplectic manifold (see [9, page 133]). Then there is an isomorphism [53]

$$HF_*^{\text{symp}}(M_\Sigma^{\text{odd}}; \mathbb{Z}) \xrightarrow{\simeq} QH^{(6g-6)-*}(M_\Sigma^{\text{odd}}; \mathbb{Z}).$$

which gives the desired result. $\square$

Remark 1.20 *The number $6g-6$ is rather arbitrary, since the grading of the Floer homology is only defined up to a constant. We use this convention because it produces an isomorphism $HF^*(\Sigma \times \mathbb{S}^1) \cong QH^*(M_\Sigma^{odd})$ preserving the grading.*

Remark 1.21 *We are only going to use the above proposition in the case of genus $g=2$. In this case, M_Σ^{odd} is isomorphic to the intersection of two quadrics in $\mathbb{CP}^5$ (see [45]). Therefore the symplectic form corresponds to the hyperplane section H and the canonical divisor is $K_{M_\Sigma^{odd}} = (2+2-6)H = -4H$. Hence $c_1(M_\Sigma^{odd}) = 4\omega$ and M_Σ^{odd} is positive.*

D. Salamon gave a program for determining the equivalence of the different products for the different Floer theories in [55] (which has not been completed so far). This is believed to be true, and it has been used in several places (see [9]). We state it as a conjecture, in the form which we are going to use later (only for the manifolds $Y = \Sigma \times \mathbb{S}^1$, but including the expected result about the action of $\mu(\mathrm{pt})$).

Conjecture 1.22 *For every homology class $\alpha \in H_*(\Sigma)$ there is an element $\tilde{\mu}(\alpha) \in QH^{(6g-6)-*}(M_\Sigma^{odd})$ given by slanting with $-\frac{1}{4}$ times the first Pontrjagin class of the universal $SO(3)$-bundle. Then the action of $\mu(\alpha)$ in $HF_*(Y)$ is intertwined with quantum multiplication by $\tilde{\mu}(\alpha) \in QH^{(6g-6)-*}(M_\Sigma^{odd})$, for any $\alpha \in H_*(\Sigma)$ $(*=1,2)$. For $pt \in H_0(\Sigma)$, $\mu(pt)$ is quantum multiplication by $\tilde{\mu}(pt)$ plus (possibly) a correction term of lower degree.*

1.2.2 Fukaya-Floer homology

Now we pass on to the definition of the Fukaya-Floer homology for a triple (Y, P_Y, γ), where γ is a loop in Y, i.e. an (oriented) embedding $\mathbb{S}^1 \hookrightarrow Y$ and $c_1(P_Y)$ is odd (see [3]). The complex $CFF_*(Y,\gamma)$ will be the total complex of the double complex $CF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty)$ ($\hat{H}_*(\mathbb{CP}^\infty)$ is the completion of $H_*(\mathbb{CP}^\infty)$). Recall that $H_i(\mathbb{CP}^\infty) = 0$ for i odd and $\mathbb{Z}$ for i even. Therefore

$$CFF_i(Y,\gamma) = CF_i(Y) \times CF_{i-2}(Y) \times \cdots$$

So elements are infinite sequences of (possibly non-zero) Floer chains. This complex is also graded modulo $\mathbb{Z}/8\mathbb{Z}$ in principle, but we will reduce the grading mod

4 again. There is a moduli space $\mathcal{M}_0(\rho_k, \rho_l)$ for every pair of flat connections ρ_k and ρ_l and we can construct generic cycles representing $\mu(\gamma \times \mathbb{R})$ and intersecting transversely in the top stratum of the compactification of $\mathcal{M}_0(\rho_k, \rho_l)$. The boundary will be defined as

$$\begin{array}{rcl} \partial : CFF_i(Y) & \to & CFF_{i-1}(Y) \\ \rho_k & \mapsto & \sum_{\rho_l} \binom{\beta}{\alpha} < \mu(\gamma \times \mathbb{R})^{\beta-\alpha}, \overline{\mathcal{M}}_0(\rho_k, \rho_l) > \rho_l \end{array}$$

for $\rho_k \in CF_{i-2\alpha}$, $\rho_l \in CF_{i-1-2\beta}$ and $\beta \geq \alpha$. $\mathcal{M}_0(\rho_k, \rho_l)$ denotes again the moduli space of instantons on the cylinder with limits ρ_k and ρ_l, quotiented out by the translations, which is of dimension $2(\beta - \alpha) = \text{ind}(\rho_k) - \text{ind}(\rho_l) - 1$. Again

Lemma 1.23 ([3]) $\partial^2 = 0$.

Proof. Consider two flat connections ρ_k and ρ_l, such that $\text{ind}(\rho_l) = \text{ind}(\rho_k) - 2 - 2e$. Then the moduli space $\mathcal{M}_0(\rho_k, \rho_l) \cap V^e_{\gamma\times\mathbb{R}}$ is a one dimensional manifold. Then we compactify it and count the boundary points in the same way as in lemma 1.13 to get

$$\sum_{\substack{\rho_m \\ \text{ind}(\rho_m)=\text{ind}(\rho_k)-1-2f}} \binom{e}{f} \#\mathcal{M}_0(\rho_k, \rho_m) \cap V^f_{\gamma\times\mathbb{R}} \cdot \#\mathcal{M}_0(\rho_m, \rho_l) \cap V^{e-f}_{\gamma\times\mathbb{R}} = 0$$

from where $\partial\partial\rho_k = 0$. $\square$

So we have defined thus the Fukaya-Floer homology $HFF_*(Y, \gamma)$. These groups are independent of metrics and of perturbations of equations (see [3]). There is a filtration $(K^{(i)})_* = CF_*(Y) \otimes (\prod_{*\geq i} \hat{H}_*(\mathbb{CP}^\infty))$ of $CFF_*(Y, \gamma)$ inducing a spectral sequence whose E_3 term is $HF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty)$ and converging to the Fukaya-Floer groups. The boundary d_3 turns out to be

$$\mu(\gamma) : HF_i(Y) \otimes H_{2j}(\mathbb{CP}^\infty) \to HF_{i-3}(Y) \otimes H_{2j+2}(\mathbb{CP}^\infty).$$

The pairing in Floer homology descends to give a pairing for the Fukaya-Floer homology groups

$$\sigma : HFF_*(Y, \gamma) \otimes HFF_*(\overline{Y}, -\gamma) \to \mathbb{Z},$$

where $-\gamma$ is γ with reversed orientation.

To define relative invariants, let X be an open manifold with $\partial X = Y$. We give it a cylindrical end. Choose w such that $w|_Y$ is odd. Let $D \subset X$ be an embedded Riemann surface such that $\partial D = D \cap Y = \gamma$ (more accurately, $D \cap (Y \times [0,\infty)) = \gamma \times [0,\infty)$). One has the moduli spaces $\mathcal{M}(X, \rho_k)$ and we can choose generic cycles $V_D^{(i)}$ representing $\mu(D)$ and intersecting transversely. Then we have an element

$$\phi_d^w(X, D) = \sum_{\substack{\rho_k \\ \mathrm{ind}(\rho_k)+C=2d}} (\#\mathcal{M}(X, \rho_k) \cap V_D^{(1)} \cap \cdots \cap V_D^{(d)})\rho_k$$

in $CF_*(Y) \otimes H_{2d}(\mathbb{CP}^\infty) \subset CFF_*(Y, \gamma)$. We remark that this is **not** a cycle. Then we set $\phi^w(X, D) = \prod_d \phi_d^w(X, D)$, which is a cycle (we also denote by $\phi^w(X, D)$ the Fukaya-Floer homology class it represents). The definition of $\phi_d^w(X, D)$ depends on some choices, but the homology class $\phi^w(X, D)$ only depends on (X, D). Moreover if we have a homology of D which is the identity in the cylindrical end of X, $\phi^w(X, D)$ remains fixed (otherwise stated, $\phi^w(X, D)$ only depends on the class $D \in H_2^R(X, \partial X)$ in the terminology of subsection 2.3.2). Analogously we have $\phi_d^w(X, D^d\, z)$, for any $z \in \mathbb{A}(X)$ of degree δ, and the cycle $\phi^w(X, z, D) = \prod_d \phi_d^w(X, D^d\, z)$. The relevant gluing theorem is:

Theorem 1.24 ([3]) *Let $X = X_1 \cup_Y X_2$ with $b^+(X) > 0$ and let (w, T) be an allowable pair in the sense of definition 1.15. Choose $D \in H_2(X)$ decomposed as $D = D_1 + D_2$ with $D_i \subset X_i$ embedded as above, $w = w_1 + w_2$. Then*

$$D_X^{(w,T)}(D^m) = \sum_i \binom{m}{i} \sigma(\phi_i^{w_1}(X_1, D_1), \phi_{m-i}^{w_2}(X_2, D_2)).$$

Let us remark that the numbers $\sigma(\phi_i^{w_1}(X_1, D_1), \phi_{m-i}^{w_2}(X_2, D_2))$ are dependent of the particular cycle representing the homology class, but their sum is only dependent on the homology class [3].

We write formally

$$\phi^w(X, e^{tD}) = \sum_d \frac{\phi_d^w(X, D)}{d!} t^d.$$

This element lies in $\prod_d V_d\, t^d$, where $\prod_d V_d$ is the graded module associated to the spectral sequence of $CF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty)$ converging to $HFF_*(Y, \gamma)$.

Theorem 1.25 *Let $X = X_1 \cup_Y X_2$ with $b^+(X) > 0$ and let (w, T) be an allowable pair. Choose $D \in H_2(X)$ decomposed as $D = D_1 + D_2$ with $D_i \subset X_i$ embedded as above, $w = w_1 + w_2$. Then*

$$D_X^{(w,T)}(e^{tD}) = \sigma(\phi^{w_1}(X_1, e^{tD_1}), \phi^{w_2}(X_2, e^{tD_2})).$$

Chapter 2

Connected sums

2.1 General definitions

2.1.1 Blow-ups

Suppose given two closed oriented four-manifolds $\bar{X}_1$ and $\bar{X}_2$. The connected sum $X = \bar{X}_1 \# \bar{X}_2$ is formed by removing two small balls from each of the $\bar{X}_i$ and gluing the boundaries by an (orientation reversing) diffeomorphism. The (oriented) diffeomorphism type of the result does not depend on any choices. It is known since the introduction of the invariants that whenever $b^+(\bar{X}_i) > 0$ for both manifolds, the invariants of X vanish. When $b^+(\bar{X}_2) = 0$ the general formula relating the invariants of X and $\bar{X}_i$, $i = 1, 2$ is not yet completely known.

Definition 2.1 *If X is a closed oriented four-manifold, we call $\tilde{X} = X \# \overline{\mathbb{CP}}^2$ the (differentiable)* ***blow-up*** *of X. We call $E = [\overline{\mathbb{CP}}^1] \in H_2(\overline{\mathbb{CP}}^2) \subset H_2(\tilde{X})$ the* ***exceptional divisor*** *of the blow-up. This is represented by an embedded (-1)-sphere.*

When X is a complex manifold, we can perform the blow-up of X in the category of complex manifolds and we get a complex manifold whose underlying differentiable manifold is $\tilde{X}$.

In general, if we have a manifold X with an embedded $\Sigma \subset X$ and we consider $\tilde{X}$, with exceptional divisor E, we have the **proper transform** of Σ defined to be $\tilde{\Sigma} = \Sigma \# (-E)$, the connected sum of Σ and the embedded sphere E (with reversed orientation) joined with a thin tube in $\tilde{X}$ not intersecting either Σ or E. Therefore $\tilde{\Sigma}$ is an embedded Riemann surface of the same genus as Σ but

$\tilde{\Sigma}^2 = \Sigma^2 - 1$. Of course, $\tilde{\Sigma}$ is strictly speaking well-defined, after a choice of a path connecting Σ and E (and with no further intersections to either Σ, E or itself), only up to isotopy. Also note that, in the complex context, when X is a complex surface and Σ a complex curve, if we blow-up at a smooth point in Σ, then $\tilde{\Sigma}$ is the proper transform of Σ in the complex sense.

Fintushel and Stern [16] have found the precise relationship between the invariants of X and those of $\tilde{X}$. Their result settles the case $b^+(\bar{X}_2) = 0$ whenever $\bar{X}_2 = \# n\overline{\mathbb{CP}}^2$. In general, when $\bar{X}_2$ is simply connected, it is homeomorphic to some $\# n\overline{\mathbb{CP}}^2$ (because of a result of Donaldson [6] which establishes that its intersection form is standard) and it is conjectured that the same relationship between the invariants would hold. When X is of simple type the result comes down to the following

Proposition 2.2 *Let X be of simple type with $b_1 = 0$ and $b^+ > 1$ and odd. Let $\mathbb{D}_X = e^{Q/2}\sum a_i e^{K_i}$ be its Donaldson series. Then $\tilde{X}$ is of simple type and has series*

$$\mathbb{D}_{\tilde{X}} = e^{-E^2/2}\cosh E \cdot \mathbb{D}_X = e^{\tilde{Q}/2}\sum\left(\frac{a_i}{2}e^{K_i+E} + \frac{a_i}{2}e^{K_i-E}\right),$$

where $\tilde{Q}$ stands for the quadratic form of $\tilde{X}$ and $E \in H^2(\overline{\mathbb{CP}}^2) \subset H^2(\tilde{X})$ is the cohomology class dual to the exceptional divisor.

2.1.2 Connected sums along Riemann surfaces

In general, we may have a splitting of X along an embedded (oriented) 3-manifold $Y \subset X$. This Y divides X into two manifolds[1] with boundary X_1 and X_2 such that $X = X_1 \cup_Y X_2$. The orientations of all pieces can be arranged to have $\partial X_1 = Y$ and $\partial X_2 = \overline{Y}$ (that is, Y with reversed orientation). Conversely, we could have started with two manifolds with boundary X_1 and X_2 such that $\partial X_1 = Y$ and $\partial X_2 = \overline{Y}$, and choosing a (orientation reversing) diffeomorphism $\phi : \partial X_1 \to \partial X_2$, form the glued manifold $X = X_1 \cup_\phi X_2$. The diffeomorphism type of X depends only on the isotopy class of ϕ.

Consider the case of two manifolds $\bar{X}_1$ and $\bar{X}_2$ together with embeddings $\Sigma \hookrightarrow \bar{X}_i$ of the same Riemann surface. Identify Σ with its image Σ_i and let n_i be the self-intersection of Σ_i. Suppose $n_1 + n_2 = 0$. Then we can choose open tubular neighbourhoods N_i of Σ_i, $X_i = \bar{X}_i - N_i$. The boundary $Y = \partial X_1$

[1] A general 3-manifold Y does not necessarily split X. It does whenever $[Y] = 0 \in H_3(X)$.

is the total space of a circle bundle over the Riemann surface Σ_1 with Chern class equal to n_1. Obviously, $\overline{Y} = \partial X_2$ is the total space of a circle bundle over Σ_2 with Chern class $n_2 = -n_1$. So there is (at least) one orientation reversing diffeomorphism ϕ between the boundaries of X_1 and X_2, namely any bundle isomorphism covering the identity on Σ from ∂X_1 to $\overline{\partial X_2}$. We want to consider the manifold $X = X_1 \cup_\phi X_2$. Obviously the diffeomorphism type of X depends on the isotopy class of ϕ.

It would be important for us to understand the group $\pi_0(\mathrm{Diff}^\pm(Y))$ of isotopy classes of diffeomorphisms where Y is the total space of a circle bundle over the Riemann surface Σ. Let $Y = Y_{g,n}$ be the total space of a circle bundle of degree n over a Riemann surface of genus g. When $g = 1$ and $n = 0$, Y is a three-torus and each isotopy class of diffeomorphisms is characterised by its action on $H_1(Y;\mathbb{Z})$, hence giving that $\pi_0(\mathrm{Diff}^+(Y_{1,0})) = SL(3;\mathbb{Z})$ (see for instance [24, page 145]). In general, the group $\pi_0(\mathrm{Diff}^+(Y_{g,n}))$ is difficult to compute, so we will restrict our attention to the subgroup $\mathrm{Aut}(Y_{g,n}) \subset \mathrm{Diff}^+(Y_{g,n})$ of automorphisms of $Y_{g,n}$ as a circle bundle. Then we have

Proposition 2.3 *Let $Y = Y_{g,n}$ and $(g,n) \neq (1,0)$. Then we have the exact sequence*

$$0 \to H^1(\Sigma;\mathbb{Z}) \to \pi_0(Aut(Y_{g,n})) \to \pi_0(Diff^\pm(\Sigma)) \to 0.$$

Proof. By definition any element $\phi \in \mathrm{Aut}(Y)$ covers a diffeomorphism f of Σ, and isotopic elements in $\mathrm{Aut}(Y)$ give isotopic diffeomorphisms. When f is orientation preserving, ϕ will preserve the orientation of the fibres. If f is orientation reversing, ϕ reverses the orientation of the fibres so $\phi \in \mathrm{Diff}^+(Y_{g,n})$. Any diffeomorphism f of Σ can be lifted to a bundle isomorphism $Y_{g,n} \to Y_{g,n}$, so the map above is surjective. The kernel is the set of isotopy classes of bundle isomorphisms covering the identity (the gauge group of $Y \to \Sigma$). This is $\pi_0(\mathcal{C}^\infty(\Sigma, \mathbb{S}^1)) = H^1(\Sigma;\mathbb{Z})$. $\square$

Remark 2.4 *The group $\pi_0(Diff^\pm(\Sigma))$ is quite big. In general, $\pi_0(Diff^\pm(\Sigma)) = Out(\pi_1(\Sigma))$, the group of outer automorphisms of $\pi_1(\Sigma)$. We have an obvious morphism*

$$\pi_0(Diff^\pm(\Sigma)) \to Aut(H_1(\Sigma;\mathbb{Z}))$$

given by the action on homology. The kernel of this homomorphism is called the Torelli group of Σ. It is an infinite group for $g \geq 2$ and contains very interesting *elements like Dehn twists along separating curves.*

Definition 2.5 *Let $Y = Y_{g,n}$. If $(g,n) \neq (1,0)$, we define an **identification** for Y to be any orientation reversing diffeomorphism $\phi : Y \xrightarrow{\sim} \overline{Y}$ which lies in $Aut(Y) \subset Diff^+(Y) = Diff^-(Y, \overline{Y})$. If $(g,n) = (1,0)$, an identification for Y is any orientation reversing diffeomorphism $\phi : Y \xrightarrow{\sim} \overline{Y}$.*

Definition 2.6 *Let $\bar{X}_i$ be as in the beginning of this subsection. We call $X = X(\phi) = X_1 \cup_\phi X_2$ **the connected sum of** $\bar{X}_1$ **and** $\bar{X}_2$ **along** Σ with identification ϕ, and denote it by $X = \bar{X}_1 \#_\Sigma \bar{X}_2$.*

Remark 2.7 *The fact that $g = 1$, $n = 0$ is a special case will allow us to perform logarithmic transforms which have no analogue when the genus is bigger.*

Suppose now that we have $n_1 + n_2 \geq 0$. By blowing-up $\bar{X}_1$ or $\bar{X}_2$ sufficiently often and replacing Σ_i by its proper transform at each stage, we can reduce $n_1 + n_2$ to zero, and then consider the connected sum along Σ of those manifolds. A simple extension of the arguments in [29] gives the following

Proposition 2.8 *The diffeomorphism type does not depend on the points at which we blow-up. More concretely, blow-up $\bar{X}_i$ at n_i points and fix an (isotopy class of) identification ϕ between the boundaries of the tubular neighbourhoods of the proper transforms of Σ_i. Let $X = X(\phi)$ be the connected sum along Σ. For other choice of integers s_i with $s_1 + s_2 = n_1 + n_2$, blow-up $\bar{X}_i$ at s_i points. Then there is a isotopy class of diffeomorphisms ψ between the boundaries of the tubular neighbourhoods of the proper transforms of Σ_i such that the resulting connected sum along Σ is diffeomorphic to X. Moreover, the correspondence $\phi \mapsto \psi$ is bijective.*

Therefore we have a process consisting of blowing-up the manifolds, taking proper transforms and then doing the connected sum along Σ. What proposition 2.8 tells us is that the only choice (up to diffeomorphism) involved in all the process is an element of $\pi_0(\mathrm{Diff}^+(Y))$ with Y a circle bundle of degree n over Σ (for any n in the range $0 \leq n \leq \max(n_1, n_2)$). Actually we will see in corollary 2.11 that whenever $n_1 + n_2 > 0$ (i.e. we have to perform at least one blow-up) the choice reduces to an element in $\pi_0(\mathrm{Diff}^\pm(\Sigma))$.

When $n_1 = n_2 = 0$, $Y = \Sigma \times \mathbb{S}^1$ is a trivial circle bundle. Note that when $n_1 \geq 0$ and $n_2 \geq 0$ we can lower both quantities to zero by blowing-up to have $Y = \Sigma \times \mathbb{S}^1$.

2.1.3 Characteristic numbers of the connected sum

Let $\bar{X}_1$ and $\bar{X}_2$ be two manifolds with embedded Riemann surfaces $\Sigma_i \hookrightarrow \bar{X}_i$ of the same genus and self-intersection zero, and let $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ be their connected sum along the Riemann surface (for some identification). Let g stand for the genus of Σ. Since $\chi_\Sigma = 2 - 2g$ it is easy to prove that the characteristic numbers are related as follows:

$$\begin{aligned} \chi_X &= \chi_{\bar{X}_1} + \chi_{\bar{X}_2} + 4g - 4 \\ \sigma_X &= \sigma_{\bar{X}_1} + \sigma_{\bar{X}_2} \\ b^\pm(X) - b_1(X) &= (b^\pm(\bar{X}_1) - b_1(\bar{X}_1)) + (b^\pm(\bar{X}_2) - b_1(\bar{X}_1)) + 2g - 1 \end{aligned}$$

As a consequence, when $b^+(\bar{X}_i) > 0$ and $b_1(\bar{X}_i) = 0$ for both sides and $g \geq 1$, one has that $b^+(X) > 1$. Also in general $b^+(X) \geq b^+(\bar{X}_1) + b^+(\bar{X}_2) - 1$. Suppose that $\bar{X}_1$, $\bar{X}_2$ and X are algebraic manifolds (we will see some cases in proposition 2.9 when this actually happens), or even only almost complex manifolds, with canonical classes $K_{\bar{X}_1}$, $K_{\bar{X}_2}$ and K_X respectively. We recall that for an almost complex manifold $K^2 = 2\chi + 3\sigma$. This gives

$$K_X^2 = K_{\bar{X}_1}^2 + K_{\bar{X}_2}^2 + 8(g-1).$$

In the algebraic case, $\chi(\mathcal{O}_X) = \frac{K_X^2 + c_2}{12}$ and so if p_g stands for the geometric genus of X and q for the irregularity

$$p_g(X) - q(X) = (p_g(\bar{X}_1) - q(\bar{X}_1)) + (p_g(\bar{X}_2) - q(\bar{X}_2)) + g.$$

In general, $q(X) \leq q(\bar{X}_1) + q(\bar{X}_2)$, so when $q(\bar{X}_i) = 0$, $i = 1, 2$ we conclude $q(X) = 0$ and $p_g(X) = p_g(\bar{X}_1) + p_g(\bar{X}_2) + g$.

2.2 General results on connected sums

In this section we will state some general results about connected sums along Riemann surfaces. The first one, proposition 2.9, relates the smoothing out of an algebraic manifold with normal crossings with the connected sum of the two irreducible components. The second one, corollary 2.11, gives a condition for the connected sum to be independent of the choice of identification of the normal bundle. The third and last, proposition 2.12, is a result of Gompf about connected sums of symplectic manifolds.

To start with, consider the case when the manifolds $\bar{X}_i$ are complex surfaces and Σ_i are embedded complex curves and let ν_i denote the holomorphic normal bundle to Σ_i. Suppose that the embeddings $\Sigma \stackrel{\sim}{\to} \Sigma_i \subset \bar{X}_i$ are holomorphic (which amounts to say that Σ_1 and Σ_2 are isomorphic complex curves and that we fix an isomorphism). If ν_1 and ν_2^* are isomorphic (obviously this implies that $n_1 + n_2 = 0$), we have an isomorphism $\phi : \nu_1 \stackrel{\sim}{\to} \nu_2^*$ (unique up to a constant factor) and then there is a preferred diffeomorphism $\phi : N_1 \to \overline{N_2}$ between the tubular neighbourhoods of Σ_i (the bar denotes orientation reversed). Then, restricting to the boundaries of the tubular neighbourhoods, there is a preferred identification $\phi : \partial X_1 \to \overline{\partial X_2}$ between the boundaries of $X_i = \bar{X}_i - N_i$. We can use this identification to perform a connected sum of $\bar{X}_i$ along Σ. Note that when $n_1+n_2 > 0$ we need to blow-up. This only can be done if $\nu_1 \otimes \nu_2 \cong \mathcal{O}_\Sigma(p_1+\cdots+p_r)$ for some points $p_i \in \Sigma$. In that case we must blow-up at the r points p_i, but we can choose at which of either $\bar{X}_i$ to blow-up.

Proposition 2.9 *Let $\mathcal{Z} \stackrel{\pi}{\to} \Delta = D(0,1) \subset \mathbb{C}$ be a (flat) family of complex surfaces. Suppose that $Z_t = \pi^{-1}(t)$ are smooth for $t \neq 0$ and that $X = Z_0$ is the union of two surfaces $\bar{X}_1$ and $\bar{X}_2$ intersecting in a normal crossing along Σ (i.e. $\bar{X}_1$ and $\bar{X}_2$ are smooth and intersecting transversely). If $n_1 + n_2 = 0$ (where n_i is the self-intersection of Σ in $\bar{X}_i$), then the diffeomorphism type of a generic fibre is obtained by a connected sum along Σ of $\bar{X}_1$ and $\bar{X}_2$ with the preferred identification alluded above.*

Proof. We note that when $n_1 + n_2 = 0$, $\nu_1 \otimes \nu_2 \cong \mathcal{O}$. So the deformation is d-semistable in the terminology of Friedman [21]. Equivalently, the total deformation space is smooth. In this case the general fibre Z_t is diffeomorphic to the connected sum of $\bar{X}_1$ and $\bar{X}_2$ along Σ (see [24, page 162, lemma 2.13]). □

We cannot hope for a converse of proposition 2.9 (even in the case of d-semistability), as it is shown in [51], where it is constructed an algebraic surface with a normal crossing which can not be deformed into a smooth algebraic surface.

Theorem 2.10 *Let X be a smooth four-manifold with an embedded Riemann surface $\Sigma \subset X$. Suppose that there exists an embedded sphere E of self-intersection -1 intersecting Σ transversely at one point. Then for any element $\gamma \in H^1(\Sigma;\mathbb{Z})$ there exists a diffeomorphism $\phi : X \stackrel{\sim}{\to} X$ fixing Σ and inducing the isomorphism given by γ in the normal bundles to Σ (see proposition 2.3).*

Proof. We shall suppose that $\Sigma^2 = 0$, but the argument is the same for $\Sigma^2 \neq 0$. We can also suppose $\Sigma \cdot E = 1$. We contract E to a point to get a smooth manifold $\hat{X}$ with a smooth embedded Riemann surface $\hat{\Sigma} \subset \hat{X}$ of the same genus as Σ, such that X is the blow-up of $\hat{X}$ with exceptional divisor E and Σ is the proper transform of $\hat{\Sigma}$. Obviously, $H^*(\Sigma) \xrightarrow{\sim} H^*(\hat{\Sigma})$ in a natural way. Now we are going to move the point at which we blow-up around a curve $g : \mathbb{S}^1 \to \hat{\Sigma} \subset \hat{X}$ whose homology class is Poincaré dual of γ in $\hat{\Sigma}$. To move the point at which we blow-up we argue as follows. Consider the family $\hat{\mathcal{X}} = \hat{X} \times \mathbb{S}^1$ with the embedded three-manifold $\hat{\mathcal{S}} = \hat{\Sigma} \times \mathbb{S}^1 \subset \hat{\mathcal{X}}$. Consider the curve $\Delta : \mathbb{S}^1 \to \hat{X} \times \mathbb{S}^1$ given by

$$\Delta(t) = (g(t), t).$$

Then a tubular neighbourhood $N = B \times \mathbb{S}^1$ (B is a four-ball) of the image of Δ has boundary $\mathbb{S}^3 \times \mathbb{S}^1$. We remove it and glue in $(\overline{\mathbb{CP}}^2 - B') \times \mathbb{S}^1$ where B' is a small ball in $\overline{\mathbb{CP}}^2$. The resulting manifold is $\mathcal{X} \to \mathbb{S}^1$ with fibres $X_o \cong X$, for $o \in \mathbb{S}^1$ ($\mathcal{X} = X \times \mathbb{S}^1$ when $g = 0 \in \pi_1(X)$). We also glue together $\hat{\mathcal{S}} = \hat{\Sigma} \times \mathbb{S}^1$ minus the part removed with $(E - B' \cap E) \times \mathbb{S}^1$ with reversed orientation (we choose the ball B' to have centre on the exceptional sphere E), to get $\mathcal{S} = \Sigma \times \mathbb{S}^1 \hookrightarrow \mathcal{X}$. In this way we have constructed a parametrised blow-up along g with a parametrised proper transform of $\hat{X}$. Now fix a trivialisation of the normal bundle of $\Sigma \times \{o\}$ in X_o, $o \in \mathbb{S}^1$ (equivalently fix the homotopy class of $\partial N_{\Sigma \times \{o\}} \xrightarrow{\sim} (\Sigma \times \{o\}) \times \mathbb{S}^1$, for $N_{\Sigma \times \{o\}}$ a tubular neighbourhood of $\Sigma \times \{o\}$ in X_o) and transport the trivialisation along g. To look at the holonomy as we go around the loop, we need to compute the global normal bundle of $\mathcal{S}$ in $\mathcal{X}$. Its first Chern class is

$$[\mathcal{S}]|_{\mathcal{S}} \in H^2(\mathcal{S}) = H^2(\Sigma \times \mathbb{S}^1).$$

Now $[\mathcal{S}] = \text{P.D.}[\hat{\Sigma} \times \mathbb{S}^1] - \text{P.D.}[E \times \mathbb{S}^1] \in H^2(\mathcal{X})$, $[\hat{\Sigma} \times \mathbb{S}^1]|_{\mathcal{S}} = [\mathbb{S}^1]$ (since $\hat{\Sigma}^2 = 1$) and $[E \times \mathbb{S}^1]|_{\mathcal{S}} = [\Delta]$. So

$$[\mathcal{S}]|_{\mathcal{S}} = \text{P.D.}[\mathbb{S}^1] - \text{P.D.}[\Delta] = -\gamma \otimes [\mathbb{S}^1] \in H^2(\Sigma \times \mathbb{S}^1) = H^2(\Sigma) \oplus (H^1(\Sigma) \otimes H^1(\mathbb{S}^1)).$$

This is enough to infer the theorem. $\square$

Actually, the proof is based on the fact that the isotopy class of the proper transform of $\hat{\Sigma}$ does depend on the chosen path from $\hat{\Sigma}$ to the exceptional sphere. Morally, we have been comparing two different proper transforms whose paths differ by juxtaposition of a path which is given by moving g slightly off $\hat{\Sigma}$.

Corollary 2.11 *Let $\bar{X}_1$ and $\bar{X}_2$ be manifolds furnished with embeddings $\Sigma \hookrightarrow \bar{X}_i$ with image Σ_i, with self-intersection number n_i. Suppose $n_1+n_2>0$. Perform the process of blowing-up until the intersection numbers of the proper transforms are zero and do the connected sum along Σ. If we choose two different identifications of the boundaries $Y=\Sigma\times\mathbb{S}^1$, say ϕ and ψ, whose homotopy classes in $Aut(Y)$ differ by an element $\gamma\in H^1(\Sigma;\mathbb{Z})$ then $X_1\cup_\phi X_2$ and $X_1\cup_\psi X_2$ are diffeomorphic.*

We recall here remark 2.4 about the richness of the group $\pi_0(\mathrm{Diff}^{\pm}(\Sigma))$. Although this last result says that the diffeomorphism type of the resulting glued manifold depends on the identification only through the homotopy class it induces in $\pi_0(\mathrm{Diff}^{\pm}(\Sigma))$, this is still quite a big group. Actually, the homology groups of X only depend on the induced element in $\mathrm{Aut}(H_1(\Sigma;\mathbb{Z}))$. Now we pass on to state the last result, which is proved by Gompf [29] and also by McCarthy and Wolfson [41].

Proposition 2.12 *Suppose $\bar{X}_1$ and $\bar{X}_2$ are two symplectic four-manifolds with symplectic submanifolds $\Sigma_i\subset\bar{X}_i$ being Riemann surfaces of the same genus g. Suppose that the self-intersections n_i of Σ_i satisfy $n_1+n_2=0$. Fix an identification ϕ. Then there is a natural isotopy class of symplectic structures for $X=\bar{X}_1\#_\Sigma\bar{X}_2$ associated to the symplectic structures on $\bar{X}_1$, $\bar{X}_2$ and the identification ϕ.*

2.3 Gluing cycles in the connected sum

From now on we are going to consider the case of two four-manifolds $\bar{X}_1$ and $\bar{X}_2$ with $b_1=0$ and with embeddings $\Sigma\hookrightarrow\bar{X}_i$ whose images Σ_i represent cohomology classes $[\Sigma_i]\in H^2(\bar{X}_i;\mathbb{Z})$ which are non-torsion elements and have self-intersection zero. Choose an identification ϕ between the boundaries of tubular neighbourhoods of Σ_i and form the connected sum along Σ, say $X=\bar{X}_1\#_\Sigma\bar{X}_2$.

2.3.1 Computing the (co)homology

The cohomology exact sequence for the pair $(\bar{X}_i, X_i)$ gives the exact sequence[2]

$$0\to H^2(\bar{X}_i;\mathbb{Z})/\mathbb{Z}[\Sigma_i]\to H^2(X_i;\mathbb{Z})\to L_i\to 0, \tag{2.1}$$

[2]The identifications between homology and cohomology groups are through Poincaré duality.

where $L_i \subset H_1(\Sigma;\mathbb{Z})$ is the image of the composition

$$H^2(X_i;\mathbb{Z}) \to H_2(X_i,\partial X_i;\mathbb{Z}) \xrightarrow{\partial} H_1(\Sigma\times\mathbb{S}^1;\mathbb{Z}) \xrightarrow{i_*} H_1(\Sigma\times D^2;\mathbb{Z}) = H_1(\Sigma;\mathbb{Z}),$$

and it is a free abelian group of rank $2g$. We also have isomorphisms

$$H^1(X_i;\mathbb{Z}) \cong H^1(\bar{X}_i;\mathbb{Z}) = 0$$

$$H^3(X_i;\mathbb{Z}) \cong H^3(\bar{X}_i;\mathbb{Z})/(H_1(\Sigma;\mathbb{Z})/L_i)$$

It is worth noticing that $b_1(\bar{X}_i)=0$, $b_1(X_i)=0$, $b^1(\bar{X}_i)=0$ and $b^1(X_i)=0$ are all equivalent (here we use that $[\Sigma_i]$ is non-torsion).

From the exact sequence (2.1) one has that there is a (non-canonical) splitting $H^2(X_i;\mathbb{Z}) = H^2(\bar{X}_i;\mathbb{Z})/\mathbb{Z}[\Sigma_i]\oplus L_i$. Giving one such splitting of the exact sequence is equivalent to giving a subspace $V_i \subset H^2(X_i;\mathbb{Z})$ projecting isomorphically to $L_i \subset H_1(\Sigma;\mathbb{Z})$. Suppose now that two such splittings $V_i \subset H^2(X_i;\mathbb{Z})$ being given. The Mayer-Vietoris sequence for $X = X_1\cup X_2$ gives

$$H^1(X;\mathbb{Z}) \cong H^1(X_1;\mathbb{Z}) \oplus H^1(X_2;\mathbb{Z})$$

and the exact sequence

$$0 \to H^1(Y;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \to H^2(X_1;\mathbb{Z}) \oplus H^2(X_2;\mathbb{Z}) \to$$

$$\to H^2(Y;\mathbb{Z}) = (H^1(\Sigma)\otimes H^1(\mathbb{S}^1)) \oplus H^2(\Sigma) \cong H^1(\Sigma;\mathbb{Z}) \oplus H^2(\Sigma;\mathbb{Z}),$$

where the last map is surjective when we tensor with rational coefficients. The first conclusion is $b_1(X)=0$. Also, under the splittings V_i, we can describe the last map as (we drop $\mathbb{Z}$ in the notation)

$$\begin{array}{rcl} (H^2(\bar{X}_1)/\mathbb{Z}[\Sigma_1]\oplus V_1)\oplus(H^2(\bar{X}_2)/\mathbb{Z}[\Sigma_2]\oplus V_2) & \to & H^1(\Sigma;\mathbb{Z})\oplus H^2(\Sigma;\mathbb{Z}) \\ (\alpha_1,\beta_1,\alpha_2,\beta_2) & \mapsto & (\beta_1-\beta_2, \alpha_1\cdot\Sigma_1-\alpha_2\cdot\Sigma_2). \end{array}$$

Now call G the subgroup of $H^2(\bar{X}_1;\mathbb{Z})/\mathbb{Z}[\Sigma_1] \oplus H^2(\bar{X}_2;\mathbb{Z})/\mathbb{Z}[\Sigma_2]$ consisting of elements (α_1,α_2) such that $\alpha_1\cdot\Sigma_1 = \alpha_2\cdot\Sigma_2$ (note that these pairings make sense). Then we have the exact sequence

$$0 \to H^1(Y;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \xrightarrow{\pi} G\oplus H^1(\Sigma;\mathbb{Z}) \tag{2.2}$$

with the cokernel of last map being a torsion group.

Remark 2.13 *Call m_i the divisibility of Σ_i (that is, the minimum of the positive numbers appearing as $\Sigma_i \cdot D_i$, for $D_i \in H^2(\bar{X}_i;\mathbb{Z})$. Note that $m_i > 0$ as Σ_i is non-torsion. Put m for the least common multiple of m_1 and m_2 and $d = m_1 m_2/m$ for the greatest common divisor. Then m is the divisibility of $\Sigma \subset X$. The cokernel of π is isomorphic to $H_1(\Sigma;\mathbb{Z})/(L_1 \cap L_2)$. So π is surjective if and only if $H^3(X_i;\mathbb{Z}) \cong H^3(\bar{X}_i;\mathbb{Z})$, $i = 1, 2$.*

Now let us rewrite the exact sequences we have for rational coefficients (we recall that $H^*(X)$ stands for rational coefficients)

$$0 \to H^1(Y) \to H^2(X) \to G \oplus H^1(\Sigma) \to 0.$$

Working with homology, we have an analogue exact sequence

$$0 \to H_2(Y) \to H_2(X) \to H_2(X_1, \partial X_1) \oplus H_2(X_2, \partial X_2) \to H_1(Y) \to 0.$$

Also we have the following

$$0 \to H_2(Y) \to H_2(X_1) \oplus H_2(X_2) \to H_2(X) \to H_1(Y) \to 0,$$

or working with cohomology

$$0 \to H^1(Y) \to H^2(X_1, \partial X_1) \oplus H^2(X_2, \partial X_2) \to H^2(X) \to H^2(Y) \to 0. \qquad (2.3)$$

Calling K the cokernel of the first map of (2.3), one has $0 \to K \to H^2(X) \to H^2(Y) \to 0$, which tells us that the homology of X is always an extension of $H^2(Y)$ and K. If we use different identifications ϕ, the first map in (2.3) and hence K are only dependent on the action of ϕ on the 2-homology.

Definition 2.14 *Let $\gamma \in H_1(\Sigma;\mathbb{Z})$ be a primitive class. Then we call T_γ the homology class in $H_2(Y;\mathbb{Z})$ represented by $g \times \mathbb{S}^1 \subset \Sigma \times \mathbb{S}^1$, where g is any loop in Σ with $\gamma = [g]$. So $T_\gamma = \gamma \otimes [\mathbb{S}^1]$ is represented by a torus of self-intersection zero.*

Now, the exact sequence (2.2) admits the following interpretation. The first term corresponds to the 2-homology of Y, i.e. it is generated by Σ and the tori T_γ. The subgroup $\pi^{-1}(G)$ consists of the classes obtained by gluing cycles coming from $\bar{X}_1$ with cycles from $\bar{X}_2$ intersecting Σ_i on the same number of points, equivalently homology classes cutting Y in a multiple of $[\mathbb{S}^1]$. The preimage of $H^1(\Sigma;\mathbb{Z})$ corresponds to 2-homology classes whose intersection with Y is a homology class lying in $H_1(\Sigma) \subset H_1(Y)$. All of these classes are necessarily represented by cycles with a part in X_1 and a part in X_2 going through the neck. The process of gluing two homology classes is not well-defined. If we have homology classes $D_i \in H_2(X_i, \partial X_i)$ with $\partial D_1 = -\partial D_2$, we can glue them together to give $D \in H_2(X)$. For this, we choose cycles representing the homology classes with the same cycle as boundary. The point is that different representatives will lead to homology classes differing by an element in $H_2(Y)$. This is due to the fact that a homology in $(X_1, \partial X_1)$ does not extend to the whole of $\bar{X}_1$.

Now if we think of the exact sequence (2.2) in terms of line bundles and their first Chern classes, then for a line bundle L on X with $c = c_1(L)$, $\pi(c)$ is the (Chern classes of the) restrictions of L to the two open manifolds X_i. Now $H^1(Y;\mathbb{Z})$ expresses the indeterminacy present when we glue two different line bundles $L_i \to X_i$ such that $L_1|_Y \cong L_2|_Y$, that is, the possible choice of identification of both L_i along Y.

2.3.2 The extended homology groups

The reason for refining the second homology groups for the pairs $(X_i, \partial X_i)$ is two-fold. On the one hand, we want to keep track of the indeterminacy of gluing two homology classes or two line bundles on the pieces X_i. On the other hand, we have to use the Fukaya-Floer theory, in which the groups $HFF_*(Y,\gamma)$ depend in principle on the isotopy class of the loop γ and not only on its homology class. For this we define the following concepts.

Definition 2.15 *Let Y be a three-manifold. A* ***loop*** *γ in Y is an embedded oriented $\mathbb{S}^1 \subset Y$. We define $\Omega(Y)$ to be the vector space (over $\mathbb{Q}$) generated by all loops in Y (with the identification of $-\gamma$ with γ with opposite orientation). A* ***framing*** *for a loop γ is a homotopy class of trivialisations of its normal bundle in Y. A* ***framed loop*** *is a loop endowed with a framing. We denote by $F\Omega(Y)$ the vector space generated by all framed loops in Y. Obviously $\mathbb{Z}$ acts freely and transitively on the framings of a loop, and hence it acts freely on $F\Omega(Y)$ with quotient $\Omega(Y)$.*

Now we are ready to define our extended relative homology groups. Let X be an open manifold with boundary $\partial X = Y$. We will give X a cylindrical end, i.e. we will consider $X' = X \cup_Y (Y \times [0,\infty))$. Let $X^o = X' - (Y \times [1,\infty))$ and let $C_2'(X')$ be the subgroup of 2-chains $\Sigma \subset X'$ such that $\Sigma \cap (Y \times [0,\infty)) = \gamma \times [0,\infty)$, for some $\gamma \in \Omega(Y)$. Let $Z_2'(X') \subset C_2'(X')$ be those which are cycles and let $B_2(X^o)$ be the boundaries supported in X^o.

Definition 2.16 *We define the* ***extended non-framed homology groups*** *of the pair $(X, \partial X)$ as*

$$\tilde{H}_2^R(X, \partial X) = \frac{Z_2'(X')}{B_2(X^o) \cap Z_2'(X')}.$$

We have an exact sequence

$$0 \to H_2(X^o) \to \tilde{H}_2^R(X, \partial X) \to \Omega(Y).$$

We define the ***extended homology groups*** *$H_2^R(X, \partial X)$ as the pull-back of $\tilde{H}_2^R(X, \partial X)$ under the projection $F\Omega(Y) \to \Omega(Y)$. So we have*

$$0 \to H_2(X^o) \to H_2^R(X, \partial X) \xrightarrow{q} F\Omega(Y).$$

The image of q is given by those loops $\gamma \in F\Omega(Y)$ whose homology classes lie in the image of $H_2(X; \partial X) \to H_1(Y)$ (so the map is surjective when $H_1(X) = 0$). The group $H_2^R(X, \partial X)$ is natural under diffeomorphisms of X inducing the identity in Y. We have obvious surjective maps $H_2^R(X, \partial X) \to H_2(X, \partial X)$ and $F\Omega(Y) \to H_1(Y)$. If K is the kernel of the first map, one has an exact sequence $H_2(\partial X) \to K \to F\Omega(Y) \to H_1(Y) \to 0$. The kernel of the first map in this exact sequence is the kernel of $H_2(\partial X) \to H_2(X)$.

Let now X_1 and X_2 be two manifolds with $\phi : \partial X_1 \xrightarrow{\sim} \overline{\partial X_2}$ and $X = X_1 \cup_\phi X_2$. We have a natural map

$$\pi : H_2^R(X_1, \partial X_1) \oplus H_2^R(X_2, \partial X_2) \to F\Omega(Y)$$

given by $\pi(D_1, D_2) = \phi_*(q_1(D_1)) - q_2(D_2)$. Its kernel is the diagonal

$$\Delta = \{(D_1, D_2) \in H_2^R(X_1, \partial X_1) \oplus H_2^R(X_2, \partial X_2) / \phi_*(q_1(D_1)) = q_2(D_2)\}.$$

The gluing of cycles is encoded in the following exact sequence

$$0 \to H_2(Y) \to \Delta \to H_2(X) \oplus F\Omega(Y),$$

where the first map is $\alpha \mapsto (\alpha, \phi_*(\alpha))$. This expresses that every pair $(D_1, D_2) \in H_2^R(X_1, \partial X_1) \oplus H_2^R(X_2, \partial X_2)$ with $\phi_*(q_1(D_1)) = q_2(D_2)$ can be glued in a unique way to give a homology class for X. Also every class $D \in H_2(X)$ can be decomposed in this way, i.e. $\Delta \to H_2(X)$ is surjective.

Now we also have to extend the intersection pairing to $H_2^R(X_i, \partial X_i)$ (and here it enters the fact that we have used $F\Omega(Y)$ rather than $\Omega(Y)$). For an open manifold X with boundary $Y = \partial X$ we define

$$Q : H_2^R(X, \partial X) \to \mathbb{Q}$$

in the following way. Let γ be a loop in Y and $D \in H_2^R(X, \partial X)$ with $q(D) = \gamma$. Perturb slightly D by moving its boundary along the framing, so that we get another cycle homologous to D which does not intersect D in $Y \times [0, \infty)$. Then count the intersection points with signs to get $Q(D)$. Now if we represent by $D_n = n \circ D \in H_2^R(X, \partial X)$ the cycle D with the framing of its boundary twisted n times, we have $Q(D_n) = Q(D) + n$.

Now let X_1, X_2 be two open manifolds with boundary $\partial X_1 = \overline{\partial X_2} = Y$, and let Q_i be the intersection pairings in $H_2^R(X_i, \partial X_i)$. If Q denotes the intersection pairing of $X = X_1 \cup_Y X_2$, then for any cycle $D \in H_2(X)$ decomposed as $D = D_1 + D_2$, $D_i \in H_2^R(X_i, \partial X_i)$, one has $Q(D) = Q_1(D_1) + Q_2(D_2)$. Obviously this is independent of the framing since when twisting $D = (D_1)_n + (D_2)_{-n}$, for any $n \in \mathbb{Z}$.

Remark 2.17 *Note that for $\gamma = pt \times \mathbb{S}^1 \subset \Sigma \times \mathbb{S}^1$ there is always a preferred framing since $Y \to \Sigma$ is a circle bundle over Σ. Also if there is a canonical diffeomorphism $Y \cong \Sigma \times \mathbb{S}^1$ and $\gamma \subset \Sigma \subset \Sigma \times \mathbb{S}^1$, there is a preferred framing of γ, namely the framing inside Σ.*

Remark 2.18 *For the manifold $A = \Sigma \times D^2$, the extended group contains a subgroup $\mathbb{Q}[\Delta] \oplus \mathbb{Q}[\Sigma]$ with $\Delta = pt \times D^2$ being a horizontal section, for a fixed point $pt \in \Sigma$. Also $\Delta^2 = 0$, with the framing as in remark 2.17.*

Remark 2.19 *Sometimes we can find a collection $D_{\gamma_i}^{(j)}$, $j = 1, 2$, formed by (-1)-discs, i.e. embedded discs of self-intersection -1 (any self-intersection can be achieved by modifying the framing), for $2g$ framed loops $\gamma_i \subset \Sigma$ whose homology classes form a basis of $H_1(Y; \mathbb{Z})$ together with $[\mathbb{S}^1]$. When this is possible and ϕ matches up the cycles and the framings, the resulting D_γ are embedded (-2)-spheres and $T_\alpha \cdot D_\gamma = \alpha \cdot p(\gamma)$ (for $p : H_1(Y) \to H_1(\Sigma)$). So for every D_γ there is some torus T_α of self-intersection zero intersecting D_γ in one point. Thus $D_\gamma + T_\alpha$ can be represented by a torus of self-intersection zero as well. All the T_α and the D_γ generate a primitive sublattice $V \subset H^2(X; \mathbb{Z})$ hence $V \oplus V^- = H^2(X; \mathbb{Z})$ and* *the exact sequence (2.2) reduces to*

$$0 \to \mathbb{Z}[\Sigma] \to V^- \xrightarrow{\pi} G \to 0. \tag{2.4}$$

Remark 2.20 *In the situation of the previous remark, we can split the (rational) homology of X in a nice way. Choose cohomology classes $D \in V_{\mathbb{Q}}^- \subset H_2(X)$, $\bar{D}_1 \in H_2(\bar{X}_1)$, $\bar{D}_2 \in H_2(\bar{X}_2)$ with $D \cdot \Sigma = 1$, and such that $D = D_1 + D_2$, $\bar{D}_1 = D_1 + \Delta$, $\bar{D}_2 = D_2 + \Delta$. Let $W = \mathbb{Q}[\Sigma, D]$ be the vector subspace of $V_{\mathbb{Q}}^-$ generated by Σ and D. Let $W_i = \mathbb{Q}[\Sigma_i, \bar{D}_i] \subset H_2(\bar{X}_i)$. Then π induces an isomorphism $W^- \xrightarrow{\sim} W_1^- \oplus W_2^-$ preserving the intersection forms. So finally*

$$H_2(X) \cong V_{\mathbb{Q}} \oplus \mathbb{Q}[\Sigma_1, \bar{D}_1]^- \oplus \mathbb{Q}[\Sigma_2, \bar{D}_2]^- \oplus \mathbb{Q}[\Sigma, D]$$

This decomposition is orthogonal. Also note that $\mathbb{Q}[\Sigma_i, \bar{D}_i] \cong H_2(X_i)/\mathbb{Q}[\Sigma_i]$.

We can define the Fukaya-Floer theory of Y as

$$HFF_*(Y) = \bigoplus_{\gamma \text{ framed loop}} (HFF_*(Y, \gamma) \otimes_{\mathbb{Z}} \mathbb{Q}) \to F\Omega(Y),$$

which collects all Fukaya-Floer groups at once. Then every $D \in H_2^R(X, \partial X)$ defines a cycle $\phi^w(X, D) \in HFF_*(Y)$. Now the pairing is defined as $\sigma : HFF_*(Y) \otimes HFF_*(\overline{Y}) \to \mathbb{Q}$ and for a closed manifold $X = X_1 \cup_Y X_2$, $D = D_1 + D_2 \in H_2(X)$, $w = w_1 + w_2 \in H^2(X; \mathbb{Z})$, one has

$$D_X^{(w,T)}(e^{tD}) = \sigma(\phi^{w_1}(X_1, e^{tD_1}), \phi^{w_2}(X_2, e^{tD_2})).$$

Chapter 3

Complex manifolds

This chapter tries to gather together some results concerning the computation of Donaldson invariants for 4-manifolds which are furnished with the extra structure of algebraic surfaces. In this case the invariants can be obtained via algebraic geometry as will be explained in section 3.1. The next two sections contain remarks on (and examples of) surfaces which are fibred as complex manifolds over complex curves, with fibres being generically curves of genus 1 and 2, respectively. These are the basic blocks to which we apply the theory developed in chapters 4 and 5. The last section is devoted to the computation of wall-crossing formulae for some algebraic surfaces with $p_g = 0$.

3.1 Invariants for algebraic surfaces

The first computations of Donaldson invariants for four-manifolds (except in some cases of vanishing) were carried out for complex surfaces. This is due to the fact that in this case one can study the moduli space of stable bundles with fixed Chern classes instead of the moduli space of ASD connections.

Let S be a smooth projective surface and g a Hodge metric on S with corresponding ample line bundle H. Such an H is also called a polarisation of S. The two-form $\omega \in H^2(S;\mathbb{Z})$ given by H is a positive form. By definition of ampleness, there is an embedding of S in some projective space[1] such that the hyperplane section is a positive integer multiple of H, or equivalently, the Kähler form corresponding to the Fubini-Study metric restricts to a positive integer multiple of

[1] For smooth algebraic surfaces, it is equivalent the existence of an ample line bundle H, to be algebraic and to be projective.

ω. Then ω is a Hodge form corresponding to a hermitian metric h, whose real part is the Hodge metric g.

Fix the Chern classes c_1 and c_2 and also a holomorphic line bundle $\mathcal{O}(L)$ with $c_1(L) = c_1$. We consider the set $\mathfrak{M}_H(c_1, c_2)$ of stable bundles $E \to S$ with respect to the polarisation H (for definition see for instance [24, page 322]), with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$ and with a fixed isomorphism $\det(E) \stackrel{\sim}{\to} \mathcal{O}(L)$ of holomorphic line bundles. Different choices of L would give isomorphic moduli spaces.

Proposition 3.1 ([5]) *Under the conditions above, there is an isomorphism of real analytic varieties $\mathcal{M}^w_\kappa \stackrel{\sim}{\to} \mathfrak{M}_H(c_1, c_2)$ from the moduli space of ASD connections on S with respect to the metric g and with $w = c_1$, $\kappa = c_2 - \frac{1}{4}c_1^2$, to the space of stable bundles with respect to H with given Chern classes.*

The metric g is generic in the sense of section 1.1 if the following condition is satisfied:

$$H^0(\mathrm{End}_0(E)) = H^2(\mathrm{End}_0(E)) = 0, \text{ for all } E \in \mathfrak{M}_H(c_1, c_2),$$

where $\mathrm{End}_0(E)$ is the sheaf of trace-free endomorphisms of E. This means that the moduli space of stable bundles is reduced and of the virtual dimension. $\mathfrak{M}_H(c_1, c_2)$ is naturally a complex variety and therefore it has a natural (complex) orientation. The Hodge form ω selects a homology orientation since $H^2_+(S; \mathbb{R}) \cong H^{0,2} \oplus H^{2,0} \oplus \mathbb{R} \cdot \omega$ and $H^1(S; \mathbb{R}) = H^{0,1} \oplus H^{1,0}$ (and $H^{0,i} \oplus H^{i,0}$ are naturally complex vector spaces). We will always consider this homology orientation. The orientations under the isomorphism of proposition 3.1 compare as $(-1)^{\frac{w^2 + K\cdot w}{2}}$, where $K = K_S$ is the canonical class for S.

Now we have to construct the μ-map. In some cases there exists a universal bundle $\mathcal{U} \to \mathfrak{M}_H(c_1, c_2) \times S$, and then we have defined a class

$$-\frac{1}{4}p_1(\mathfrak{g}_{\mathcal{U}}) = \frac{1}{4}c_2(\mathrm{End}_0(\mathcal{U})) = c_2(\mathcal{U}) - \frac{1}{4}c_1(\mathcal{U})^2 \in H^4(\mathfrak{M}_H(c_1, c_2) \times S),$$

as $(\mathfrak{g}_{\mathcal{U}})_{\mathbb{C}} = \mathrm{End}_0(\mathcal{U})$. The good news is that such a class exists even when $\mathcal{U}$ does not [24, section 5.1]. So for an algebraic curve $C \subset S$, we can define

$$\mu([C]) = -\frac{1}{4}p_1(\mathfrak{g}_{\mathcal{U}})/[C] \in H^2(\mathfrak{M}_H(c_1, c_2)).$$

Proposition 3.2 ([24, section 5.1]) *Suppose that $c_1 \cdot C \equiv 0 \pmod 2$. Let θ be a holomorphic line bundle such that $\theta^{\otimes 2} = K_C \otimes \mathcal{O}_C(L^{-1})$ (which exists by the* *assumption on c_1). Put*

$$\mathcal{Z}_{c_1,c_2}(C,\theta) = \{V \in \mathfrak{M}_H(c_1,c_2)/h^0(C;(V|_C)\otimes\theta) \neq 0\}.$$

If $\mathcal{Z}_{c_1,c_2}(C,\theta)$ is a proper hypersurface in $\mathfrak{M}_H(c_1,c_2)$ (i.e. of codimension one and not containing any irreducible component of $\mathfrak{M}_H(c_1,c_2)$) then it is naturally a divisor representing $\mu([C])$, such that the multiplicity of an irreducible component $\mathcal{Z}$ of $\mathcal{Z}_{c_1,c_2}(C,\theta)$ is given by the length of $R^1p_{1*}(\mathcal{U}\otimes p_2^*\theta)$ at a generic point of $\mathcal{Z}$. Here $\mathcal{U} \to \mathfrak{M}_H(c_1,c_2)\times S$ is a universal bundle (at least locally) and p_1 and p_2 denote the projections of $\mathfrak{M}_H(c_1,c_2)\times S$ onto its first and second factors respectively.

We have a word to say about the case $b^+ = 1$. Since for an algebraic surface we have $b^+ = 1 + 2p_g$, $b^+ = 1$ is equivalent to $p_g = 0$. The space of selfdual harmonic forms for S with the Hodge metric g is $H^{0,2}\oplus H^{2,0}\oplus\mathbb{R}\cdot\omega$, so when $p_g = \dim H^{0,2} = 0$ the positive harmonic space is just generated by ω. The space of antiselfdual harmonic forms is its orthogonal complement, i.e. $\mathcal{H}_- = \{D \in H^2(S;\mathbb{R})/D\cdot H = 0\}$. The period point corresponding to the metric g is $[\omega]\in\mathbb{H}$. When h is a hermitian metric coming from a polarisation H, such a point is the line spanned by H.

The moduli spaces $\mathfrak{M}_H(c_1,c_2)$ do not change when we vary H inside a chamber. Therefore if we restrict to the ample cone and the moduli spaces $\mathfrak{M}_H(c_1,c_2)$ are generic (which is going to be the case for the case we need $S = \Sigma\times\mathbb{CP}^1$), then remark 1.10 and the discussion preceding it remain valid.

A very detailed study of the behaviour of Donaldson invariants when crossing a wall will be undertaken in section 3.4.

3.2 Simply connected elliptic surfaces

We feel it necessary to gather some of the results on the classification of smooth projective surfaces since they have been very important in the story of Donaldson invariants. In chapter 4 we will compute the basic classes (and hence the invariants) of elliptic surfaces. Furthermore, all the theory of chapter 5 can be used to compute the basic classes of algebraic surfaces which admit fibrations over complex curves with fibres being generically of genus two.

A rough classification of algebraic surfaces with $b_1 = 0$ can be made using the Enriques-Kodaira classification of surfaces (see [1, page 188]), so we get (κ

denotes the Kodaira dimension):

1. For $\kappa = -\infty$ we have the rational surfaces. This group consists of the projective plane $\mathbb{CP}^2$, its blow-ups $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$, the quadric $\mathbb{S}^2 \times \mathbb{S}^2$ and the ruled surfaces over $\mathbb{CP}^1$ (i.e. the Hirzebruch surfaces) and their blow-ups. So any of these surfaces is diffeomorphic to either $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$.
2. For $\kappa = 0$ we have the Enriques and the $K3$ surfaces. The non-minimal examples are the blow-ups of these.
3. For $\kappa = 1$ we have the rest of the irrational, elliptic surfaces with $b_1 = 0$. All these surfaces admit an elliptic fibration over $\mathbb{CP}^1$. Again the non-minimal examples are their blow-ups.
4. For $\kappa = 2$ we have the surfaces of general type with $b_1 = 0$. This group includes all hypersurfaces in $\mathbb{CP}^3$ of degree $d \geq 5$, and is poorly understood.

All of these surfaces are projective (equivalently, algebraic) except for some examples of $K3$ surfaces and some elliptic surfaces with $\kappa = 1$.

In this section we will discuss briefly the surfaces in the second and third groups. A very good reference for this is the book [24]. An elliptic surface S is a complex manifold with an algebraic projection $\pi : S \to C$ onto a complex curve with fibres being generically smooth elliptic curves. We denote by F the class of the fibre. When $b_1 = 0$, $C = \mathbb{CP}^1$ necessarily and the irregularity is $q = 0$. S is called relatively minimal when there are no exceptional divisors contained in any fibre. S is algebraic if and only if it admits a multisection, i.e. a connected curve $C' \subset S$ not contained in a fibre (see [24, page 34]).

A multiple fibre of multiplicity m is a non-reduced fibre which is m times a reduced divisor. By [1, page 162], the canonical divisor is

$$K_S = (n-2)F + \sum (m_i - 1)F_i,$$

where $n = \chi(\mathcal{O}_S)$ and F_i are the reductions of the multiple fibres (with multiplicities m_i). So always $K_S^2 = 0$ and $c_2 = K_S^2 + c_2 = 12\chi(\mathcal{O}_S) = 12(1 + p_g) = 12n$. Since $n > 0$, there are always singular fibres.

If S has a section, then there are no multiple fibres. In case S has no multiple fibres, S will be deformation equivalent to an elliptic surface with a section [24, page 81]. Such a surface S is characterised, up to deformation equivalence, by its geometric genus $p_g(S) = n - 1$. We denote by S_n the unique, up to deformation, relatively minimal elliptic surface with $b_1 = 0$, no multiple fibres and $p_g(S_n) = n-1$.

Doing the connected sum along any fibre, as explained in section 2.1 (see also [24, page 162]), we have that $S_n \#_F S_m = S_{n+m}$, as differentiable manifolds (where the identification is taken to be the preferred identification as explained in section 2.2). This is true by proposition 2.9 since S_{n+m} can be deformed to a variety with a normal crossing $S_n \cup_F S_m$. The well known $K3$ surface is S_2.

If we allow multiple fibres, still $b_1 = 0$. The fundamental group is finite whenever there are two multiple fibres, of multiplicities m_1 and m_2. When there is only one multiple fibre, we set $m_1 = 1$ since a fibre of multiplicity 1 is a non-multiple fibre. Analogously, if there are no multiple fibres, we put $m_1 = m_2 = 1$. So we can suppose $1 \leq m_1 \leq m_2$. The fundamental group of S is $\mathbb{Z}/d\mathbb{Z}$ where d is the greatest common divisor of m_1 and m_2. So for S to be simply connected, m_1 and m_2 have to be coprime. S is always deformation equivalent to an elliptic surface whose multiple fibres are smooth elliptic curves [24, page 113]. Furthermore, there is a process for creating multiple fibres, called logarithmic transform. This process can be carried out in the analytical setting (see [32, page 564] [1, page 164]), and so the resulting manifold is in a natural way a complex variety (in the general case it is not a projective variety anymore). The differentiable analogue of the logarithmic transform is to remove a tubular neighbourhood of a non-multiple fibre in S, say $A = \mathbb{T}^2 \times D^2$, a 2-torus times a disc, and to replace it by another $A_m = \mathbb{T}^2 \times D^2$ with a twisting. This twisting is given by an orientation preserving diffeomorphism

$$\phi : \partial A_m \xrightarrow{\sim} \overline{\partial(S-A)}$$

which depends, up to isotopy, only on its effect on the first homology groups (see subsection 2.1.2). There are different possibilities for ϕ giving the same 4-manifold. The different ϕ producing the same result (see [24, pages 143-145]) have $\phi_* : H_1(\partial A_m; \mathbb{Z}) \to H_1(\overline{\partial(S-A)}; \mathbb{Z})$ with matrix

$$\begin{pmatrix} * & * & -a \\ * & * & -b \\ * & * & m \end{pmatrix}$$

where g.c.d.$(m, a, b) = 1$. In fact, Gompf [28] has shown that whenever there is a cusp fibre (and we always can suppose this since S is deformation equivalent to an elliptic surface with a cusp fibre, as long as $n > 0$), one can choose any diffeomorphism $\phi : \partial A_m \xrightarrow{\sim} \partial(S-A)$ such that $\pi \circ \phi : \mathrm{pt} \times \mathbb{S}^1 \to \pi(\partial(S-A)) = \mathbb{S}^1$ has winding number $\pm m$, and the resulting manifold is the same. This means that the bottom right entry of the matrix above has to have absolute value m.

We denote by $S_n(m_1, m_2)$ the result of performing two logarithmic transforms of multiplicities m_1 and m_2 in two different fibres of S_n. There are different choices for the logarithmic transform but all of them produce elliptic surfaces which are deformation equivalent since $c_2 > 0$, and deformation equivalent to an algebraic elliptic surface (see [24, section 1.7]). So $S_n(m_1, m_2)$ is unique up to diffeomorphism. We can drop the m_i that are equal to 1, as a log-transform of multiplicity 1 does not change the elliptic surface. The surface S is irrational unless $p_g(S) = 0$ and $m_1 = 1$ and in this latter case S is diffeomorphic to $S_1 = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$. The canonical divisor of $S_n(m_1, m_2)$ is

$$K_S = (n-2)F + (m_1 - 1)F_1 + (m_2 - 1)F_2,$$

where F_i denotes the reduction of the multiple fibre of multiplicity m_i. So it is $F_i = \frac{1}{m_i} F$ in $H^2(S)$.

In S_n we can always find a section $\sigma = \sigma_{S_n}$ such that $\sigma^2 = -(1 + p_g) = -n$. Indeed, thinking of S_1 as the blow-up of $\mathbb{CP}^2$ at the nine points of intersection of two generic cubic curves, we have that any of the nine exceptional divisors is a section of the fibration with self-intersection -1. For

$$S_n = \underbrace{S_1 \#_f \ldots \#_f S_1}_{n}$$

we can obtain σ by smoothly gluing sections of the above form in each of the S_1's. In particular, there is a section σ in $S_2 = K3$ with $\sigma^2 = -2$.

For elliptic surfaces we are going to define suitable metrics following Friedman [22].

Definition 3.3 *Let S be an elliptic surface and let F be the cohomology class corresponding to the fibre. Let $w \in H^2(S; \mathbb{Z})$ and $p_1 \equiv w^2 \pmod 4$ a negative integer, and suppose $w \cdot F \equiv 1 \pmod 2$. Then we say that a Hodge metric is **suitable** (or that the corresponding polarisation is suitable) for (w, p_1) when it belongs to the chamber (associated to (w, p_1)) containing F in its closure (see remark 1.10).*

Remark 3.4 *Roughly stated, a suitable metric is one which assigns small volume to the fibre F in comparison to a section. This is so since a suitable polarisation* *is obtained by taking an ample line bundle H_0 and considering $H = H_0 + nF$, for big n (see [22]). Then for a section σ, Vol(σ) $= \sigma \cdot H$ is much bigger than Vol(F) $= F \cdot H$.*

When we split an elliptic surface as $S = \bar{X}_1 \#_F \bar{X}_2 = X_1 \cup_{F\times\mathbb{S}^1} X_2$, suitable metrics for S correspond to metrics giving a long neck, in the sense that they define the same chambers (at least for appropriate metrics on $Y = F \times \mathbb{S}^1$, see remark 1.10) and hence give the same invariants.

3.3 Algebraic genus 2 fibrations

An algebraic genus 2 fibration is an algebraic surface S together with a holomorphic map $S \to C$ onto a complex curve with fibres being generically connected genus 2 curves. If $b_1 = 0$ then the base curve is $C = \mathbb{CP}^1$. They are called relatively minimal whenever there are no exceptional curves lying in fibres. The best treatise in this subject is [64]. There is not a complete classification but many examples have been constructed, mostly by Xiao [64] and Persson [49]. We will say very little about them. We will just remark that very often they are minimal general type surfaces and that there are two ways of constructing them, for whose exposition we refer to [50].

1. The canonical system (if necessary twisted by fibres) gives a rational map of degree two which image is a ruled surface. So we can construct algebraic genus 2 fibrations by taking the minimal model of the resolution of the singularities of the double cover of a ruled surface branched along a six section.

2. A genus two fibration $X \to C$ can be considered as a rational map $C \to M_2$, the space parametrising genus two curves. There is a surface in M_2 consisting of the double elliptic curves, i.e. those genus 2 curves which are double covers of elliptic curves. When the image of C lies in this surface, X is called doubly elliptic. In practice, such X appears as the minimal model of the resolution of the singularities of the double cover of an elliptic surface branched along a bisection.

3.4 Wall-crossing formulae

In this section we want to compute some wall-crossing formulae for algebraic surfaces. We will follow mainly [26]. In that paper, Friedman and Qin obtain some wall-crossing formulae for algebraic surfaces S with $-K_S$ being effective and the irregularity $q = 0$ (equivalently, $b_1 = 0$). We want to adapt their results to the case $q > 0$ modifying their arguments where necessary. We will suppose that $p_g = 0$ and $-K_S$ is effective throughout.

We introduce some notation. Fix $w \in H^2(S;\mathbb{Z})$ and $p_1 \in \mathbb{Z}$ with $w^2 \equiv p_1 \pmod 4$. Put $d = -p_1 - \frac{3}{2}(1 - b_1 + b^+) = -p_1 - 3(1-q)$. Let ζ define a wall of type (w, p_1). We recall that by definition 1.9 this means $\zeta \equiv w \pmod 2$ and $p_1 \leq \zeta^2 < 0$. We defined the wall corresponding to ζ as $W_\zeta = \{x \in \mathbb{H} / x \cdot \zeta = 0\}$. The walls of type (w, p_1) divide $\mathbb{H}$ into a locally finite collection of chambers (see subsection 1.1.3). In every chamber $\mathcal{C}$ of the ample cone we have defined the Donaldson invariant $D_S^{w,d}(\mathcal{C})$ associated to metrics with period point in that chamber (see remark 1.10). If two chambers $\mathcal{C}_+$ and $\mathcal{C}_-$ are separated by a single wall W_ζ (although there may be more than one class ζ of type (w, p_1) defining W_ζ), there is a **wall-crossing difference term**

$$\delta_S^{w,d}(\mathcal{C}_+, \mathcal{C}_-) = D_S^{w,d}(\mathcal{C}_+) - D_S^{w,d}(\mathcal{C}_-).$$

For an algebraic surface S with a Hodge metric g defined by the polarisation H, we fix the Chern classes c_1, c_2 and the determinant bundle $\mathcal{O}(L)$. Let $w = c_1$ and $p_1 = c_1^2 - 4c_2$. For any chamber $\mathcal{C}$ (of the ample cone) of type (w, p_1), we shall use $\mathfrak{M}_H(c_1, c_2)$ with $H \in \mathcal{C}$, to compute the invariant $D_S^{w,d}(\mathcal{C})$. It turns out that when $\mathcal{C}_+$ and $\mathcal{C}_-$ are adjacent chambers and $H_\pm \in \mathcal{C}_\pm$, one can obtain $\mathfrak{M}_{H_+}(c_1, c_2)$ from $\mathfrak{M}_{H_-}(c_1, c_2)$ by a sequence of blow-ups and blow-downs (what is usually called a flip [61]).

Suppose from now on that $\mathcal{C}_-$ and $\mathcal{C}_+$ are two adjacent chambers separated by a single wall W_ζ of type (w, p_1). For simplicity, we will assume that the wall W_ζ is only represented by the pair $\pm\zeta$ since in the general case we only need to add up the contributions for every pair representing the wall. Set

$$l_\zeta = (\zeta^2 - p_1)/4 \in \mathbb{Z}.$$

Let ζ define the wall separating $\mathcal{C}_-$ from $\mathcal{C}_+$ and put, as in [26, section 2], $E_\zeta^{n_1,n_2}$ to be the set of all isomorphism classes of non-split extensions of the form

$$0 \to \mathcal{O}(F) \otimes I_{Z_1} \to V \to \mathcal{O}(L - F) \otimes I_{Z_2} \to 0,$$

where F is a divisor such that $2F - L$ is homologically equivalent to ζ, and Z_1 and Z_2 are two zero-dimensional subschemes of S with $l(Z_i) = n_i$ and such that $n_1 + n_2 = l_\zeta$. Let us construct $E_\zeta^{n_1,n_2}$ explicitly. Consider $H_i = \mathrm{Hilb}_{n_i}(S)$, $J = \mathrm{Jac}^F(S)$ the Jacobian parametrising divisors homologically equivalent to F, $\mathcal{Z}_i \subset S \times H_i$ the universal codimension 2 scheme, and $\mathcal{F} \subset S \times J$ the universal divisor. Then we define $\mathcal{E}_\zeta^{n_1,n_2} \to J \times H_1 \times H_2$ to be

$$\mathcal{E} = \mathcal{E}_\zeta^{n_1,n_2} = \mathcal{E}\mathrm{xt}^1_{\pi_2}(\mathcal{O}_{S\times(J\times H_1\times H_2)}(\pi_1^* L - \mathcal{F}) \otimes I_{\mathcal{Z}_2}, \mathcal{O}_{S\times(J\times H_1\times H_2)}(\mathcal{F}) \otimes I_{\mathcal{Z}_1}),$$

for $\pi_1 : S \times (J \times H_1 \times H_2) \to S$, $\pi_2 : S \times (J \times H_1 \times H_2) \to J \times H_1 \times H_2$, the projections (we do not denote all pull-backs of sheaves explicitly). This is a vector bundle of rank

$$\mathrm{rk}(\mathcal{E}) = l_\zeta + h^1(\mathcal{O}_S(2F - L)) = l_\zeta + h(\zeta) + q,$$

where $h(\zeta) = \frac{\zeta\cdot K_S}{2} - \frac{\zeta^2}{2} - 1$, by Riemann-Roch [26, lemma 2.6]. Put $N_\zeta = \mathrm{rk}(\mathcal{E}) - 1$. Then $E_\zeta^{n_1,n_2} = \mathbb{P}((\mathcal{E}_\zeta^{n_1,n_2})^\vee)$ which[2] is of dimension $q + 2l_\zeta + (l_\zeta + h(\zeta) + q)$. Also $N_\zeta + N_{-\zeta} + q + 2l_\zeta = d - 1$. We will have to treat the case $\mathrm{rk}(\mathcal{E}) = 0$ (i.e. $l_\zeta = 0$ and $h(\zeta) + q = 0$) separately.

We can modify the arguments in sections 3 and 4 of [26] to get intermediate moduli spaces $\mathfrak{M}_0^{(k)}$ together with embeddings $E_\zeta^{l_\zeta - k, k} \hookrightarrow \mathfrak{M}_0^{(k)}$ and $E_{-\zeta}^{k, l_\zeta - k} \hookrightarrow \mathfrak{M}_0^{(k-1)}$, fitting in the following diagram

$$\begin{array}{ccccccccc}
 & \widetilde{\mathfrak{M}}_0^{(l_\zeta)} & & & \cdots & & & \widetilde{\mathfrak{M}}_0^{(0)} & \\
\swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \\
\mathfrak{M}_0^{(l_\zeta)} & & & \mathfrak{M}_0^{(l_\zeta - 1)} & & \mathfrak{M}_0^{(0)} & & & \mathfrak{M}_0^{(-1)} \\
\| & & & & & & & & \| \\
\mathfrak{M}_- & & & & & & & & \mathfrak{M}_+
\end{array}$$

where $\widetilde{\mathfrak{M}}_0^{(k)} \to \mathfrak{M}_0^{(k)}$ is the blow-up of $\mathfrak{M}_0^{(k)}$ at $E_\zeta^{l_\zeta - k, k}$ and $\widetilde{\mathfrak{M}}_0^{(k)} \to \mathfrak{M}_0^{(k-1)}$ is the blow-up of $\mathfrak{M}_0^{(k-1)}$ at $E_{-\zeta}^{k, l_\zeta - k}$. This is what is called a flip. Basically, the space $E_\zeta = \sqcup E_\zeta^{l_\zeta - k, k}$ parametrises H_--stable sheaves which are H_+-unstable. Analogously, $E_{-\zeta} = \sqcup E_{-\zeta}^{k, l_\zeta - k}$ parametrises H_+-stable sheaves which are H_--unstable. Hence one could say that $\mathfrak{M}_+$ is obtained from $\mathfrak{M}_-$ by removing E_ζ and then attaching $E_{-\zeta}$. The picture above is a nice description of this fact and allows us the find the universal sheaf for $\mathfrak{M}_+$ out of the universal sheaf for $\mathfrak{M}_-$ by a sequence of elementary transforms.

[2] We follow the convention $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\oplus_i S^i(\mathcal{E}))$.

The point is that whenever $-K_S$ is effective (which is a condition quite restrictive for S), we have an embedding $E_\zeta^{0,l_\zeta} \to \mathfrak{M}_-$ (the part of E_ζ consisting of bundles) and rational maps $E_\zeta^{k,l_\zeta-k} \dashrightarrow \mathfrak{M}_-$, $k>0$, but if we blow-up $\mathfrak{M}_-$ at E_ζ^{0,l_ζ}, we have already an embedding from $E_\zeta^{1,l_\zeta-1}$ to this latter space. Now we can proceed inductively for $k = 0, \ldots, l_\zeta$. Analogously, we can have started from $\mathfrak{M}_+$ blowing-up $E_{-\zeta}^{k,l_\zeta-k}$ one by one. The diagram above says that we can perform these blow-ups and blow-downs alternatively, instead of first blowing-up $l_\zeta+1$ times and then blowing-down $l_\zeta+1$ times. We see that the exceptional divisor in $\widetilde{\mathfrak{M}}_0^{(k)}$ is a $\mathbb{P}^{N_\zeta} \times \mathbb{P}^{N_{-\zeta}}$-bundle over $J \times H_{l_\zeta-k} \times H_k$.

When addapting the arguments of [26, sections 3 and 4], the only place requiring serious changes is proposition 3.7 in order to prove proposition 3.6.

Proposition 3.5 ([26, proposition 3.6]) *The map $E_\zeta^{l_\zeta-k,k} \to \mathfrak{M}_0^{l_\zeta-k,k}$ is an immersion. The normal bundle $\mathcal{N}_\zeta^{l_\zeta-k,k}$ to $E_\zeta^{l_\zeta-k,k}$ in $\mathfrak{M}_0^{l_\zeta-k,k}$ is exactly $\rho^*\mathcal{E}_\zeta^{l_\zeta-k,k} \otimes \mathcal{O}_{E_\zeta^{l_\zeta-k,k}}(-1)$, where $\rho : E_\zeta^{l_\zeta-k,k} \to J \times H_{l_\zeta-k} \times H_k$ is the projection.*

The analogue of [26, proposition 3.7] that we need to prove is

Proposition 3.6 *For all nonzero $\xi \in Ext^1 = Ext^1(\mathcal{O}(L-F) \otimes I_{Z_2}, \mathcal{O}(F) \otimes I_{Z_1})$, the natural map from a neighbourhood of ξ in $E_\zeta^{l_\zeta-k,k}$ to $\mathfrak{M}_0^{(\zeta,k)}$ is an immersion at ξ. The image of $T_\xi E_\zeta^{l_\zeta-k,k}$ in $Ext_0(V,V)$ (the tanget space to $\mathfrak{M}_0^{(\zeta,k)}$ at ξ, where V is the sheaf corresponding to ξ) is exactly the kernel of the natural map $Ext_0(V,V) \to Ext^1(\mathcal{O}(F) \otimes I_{Z_1}, \mathcal{O}(L-F) \otimes I_{Z_2})$, and the normal space to $E_\zeta^{l_\zeta-k,k}$ at ξ in $\mathfrak{M}_0^{(\zeta,k)}$ may be canonically identified with $Ext^1(\mathcal{O}(F) \otimes I_{Z_1}, \mathcal{O}(L-F) \otimes I_{Z_2})$.*

Proof. We have that $\mathrm{Ext}^1(I_Z, I_Z)$ parametrises infinitesimal deformations of I_Z as a sheaf. These are of the form $I_{Z'} \otimes \mathcal{O}(D)$ for $D \equiv 0$. The universal space parametrising these sheaves is $\mathrm{Hilb}_r(S) \times \mathrm{Jac}^0(S)$, where r is the length of Z. There is an exact sequence

$$0 \to H^0(\mathcal{E}\mathrm{xt}^1(I_Z, I_Z)) \to \mathrm{Ext}^1(I_Z, I_Z) \to H^1(\mathcal{H}\mathrm{om}(I_Z, I_Z)) \to 0,$$

where $H^0(\mathcal{E}\mathrm{xt}^1(I_Z, I_Z)) = H^0(\mathcal{H}\mathrm{om}(I_Z, \mathcal{O}_Z)) = \mathrm{Hom}(I_Z, \mathcal{O}_Z)$ is the tangent space to $\mathrm{Hilb}_r(S)$ and $H^1(\mathcal{H}\mathrm{om}(I_Z, I_Z)) = H^1(\mathcal{O})$ is the tangent space to the Jacobian. Analogously, $\mathrm{Ext}^1(V,V)$ is the space of infinitesimal deformations of V (but the determinant is not preserved). The infinitesimal deformations preserving the determinant are given by the kernel $\mathrm{Ext}^1_0(V,V)$ of a map $\mathrm{Ext}^1(V,V) \to$

$H^1(\mathcal{H}\mathrm{om}(V,V)) \to H^1(\mathcal{O})$. Now $E = E_\zeta^{l_\zeta - k,k}$ sits inside the bigger space $\tilde{E} = \tilde{E}_\zeta^{l_\zeta-k,k}$ given as

$$\mathbb{P}(\mathcal{E}\mathrm{xt}^1_{\pi_2}(\mathcal{O}_{S\times(J_1\times H_1\times J_2\times H_2)}(\pi_1^*L - \mathcal{F}_2)\otimes I_{\mathcal{Z}_2}, \mathcal{O}_{S\times(J_1\times H_1\times J_2\times H_2)}(\mathcal{F}_1)\otimes I_{\mathcal{Z}_1})^\vee),$$

for $J_1 = J_2 = J$, $\mathcal{F}_i \subset S\times J_i$ the universal divisor, and H_i the Hilbert scheme parametrising Z_i. The arguments in [26, proposition 3.7] go through to prove that for every non-zero $\xi \in \mathrm{Ext}^1 = \mathrm{Ext}^1(\mathcal{O}(L-F)\otimes I_{Z_2}, \mathcal{O}(F)\otimes I_{Z_1})$ we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccc}
T_\xi E & \longrightarrow & \mathrm{Ext}^1_0(V,V) & \longrightarrow & \mathrm{Ext}^1(\mathcal{O}(F)\otimes I_{Z_1}, \mathcal{O}(L-F)\otimes I_{Z_2}) \\
\downarrow & & \downarrow & & \| \\
T_\xi \tilde{E} & \longrightarrow & \mathrm{Ext}^1(V,V) & \longrightarrow & \mathrm{Ext}^1(\mathcal{O}(F)\otimes I_{Z_1}, \mathcal{O}(L-F)\otimes I_{Z_2}) \\
\downarrow & & \downarrow & & \\
H^1(\mathcal{O}) & = & H^1(\mathcal{O}) & &
\end{array}$$

So the natural map from a neighbourhood of ξ in E to $\mathfrak{M}_0^{(\zeta,k)}$ is an immersion at ξ and the normal space may be canonically identified with $\mathrm{Ext}^1(\mathcal{O}(F)\otimes I_{Z_1}, \mathcal{O}(L-F)\otimes I_{Z_2})$. $\square$

Therefore proposition 3.5 is true for $q > 0$. The set up is now in all ways analogous to that of [26]. We fix some notations [26, section 5]:

Notation 3.7 *Let ζ define a wall of type (w, p_1).*

- *λ_k is the tautological line bundle over $E_\zeta^{n_1,n_2} = \mathbb{P}((\mathcal{E}_\zeta^{n_1,n_2})^\vee)$. λ_k will also be used to denote its first Chern class.*
- *$\rho_k : S\times E_\zeta^{l_\zeta-k,k} \to S\times(J\times H_{l_\zeta-k}\times H_k)$ is the natural projection.*
- *$p_k : \widetilde{\mathfrak{M}}_0^{(k)} \to \mathfrak{M}_0^{(k)}$ is the blow-up of $\mathfrak{M}_0^{(k)}$ at $E_\zeta^{l_\zeta-k,k}$.*
- *$q_{k-1} : \widetilde{\mathfrak{M}}_0^{(k)} \to \mathfrak{M}_0^{(k-1)}$ is the contraction of $\widetilde{\mathfrak{M}}_0^{(k)}$ to $\mathfrak{M}_0^{(k-1)}$.*
- *The normal bundle of $E_\zeta^{l_\zeta-k,k}$ in $\mathfrak{M}_0^{(k)}$ is $\mathcal{N}_k = \rho_k^*\mathcal{E}_{-\zeta}^{k,l_\zeta-k}\otimes\lambda_k^{-1}$.*
- *$D_k = \mathbb{P}(\mathcal{N}_k^\vee)$ is the exceptional divisor in $\widetilde{\mathfrak{M}}_0^{(k)}$.*
- *$\xi_k = \mathcal{O}_{\widetilde{\mathfrak{M}}_0^{(k)}}(-D_k)|_{D_k}$ is the tautological line bundle on D_k.*
- *$\mu^{(k)}(\alpha) = -\frac{1}{4}p_1(\mathfrak{g}_{\mathcal{U}^{(k)}})/\alpha$, for $\alpha \in H_2(S;\mathbb{Z})$ and $\mathcal{U}^{(k)}$ a universal sheaf over $S\times\mathfrak{M}_0^{(k)}$. Let $\mu^{(l_\zeta)}(\alpha) = \mu_-(\alpha)$ and $\mu^{(-1)}(\alpha) = \mu_+(\alpha)$.*

We have already mentioned that although $\mathcal{U}^{(k)}$ might not exist, there is always a well-defined element $p_1(\mathfrak{g}_{\mathcal{U}^{(k)}})$. The analogues of lemma 5.2, lemma 5.3 and theorem 5.4 of [26] are

Lemma 3.8 *Let $\alpha \in H_2(S;\mathbb{Z})$ and put $a = (\zeta \cdot \alpha)/2$. Let τ_0, τ_1 and τ_2 be the projections of $E_\zeta^{l_\zeta - k, k}$ to J, $H_{l_\zeta - k}$ and H_k respectively. Then*

$$(Id \times p_k)^* c_1(\mathcal{U}^{(k)})|_{(S\times D_k)} = \pi_1^* L + (p_k|_{D_k})^* \lambda_k$$

$$p_k^* \mu^{(k)}(\alpha)|_{D_k} = (p_k|_{D_k})^* \left[\tau_1^*([\mathcal{Z}_{l_\zeta - k}]/\alpha) + \tau_2^*([\mathcal{Z}_k]/\alpha) - a\lambda_k - \tau_0^*(c_1(\mathcal{F})^2/\alpha)\right]$$

Lemma 3.9 *For $\alpha \in H_2(S;\mathbb{Z})$ we have $q_{k-1}^* \mu^{(k-1)}(\alpha) = p_k^* \mu^{(k)}(\alpha) - aD_k$.*

Theorem 3.10 *Let ζ define a wall of type (w, p_1) and $d = -p_1 - 3(1-q)$. Suppose $l_\zeta + h(\zeta) + q > 0$. For $\alpha \in H_2(S;\mathbb{Z})$, put $a = (\zeta \cdot \alpha)/2$. Then $[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d$ is equal to*

$$\sum_{\substack{0 \le j \le 2l_\zeta \\ 0 \le b \le q \\ 0 \le k \le l_\zeta}} (-1)^{h(\zeta)+l_\zeta+j} \frac{d!}{j!b!(d-j-b)!} a^{d-j-b}([\mathcal{Z}_{l_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot$$
$$\cdot (c_1(\mathcal{F})^2/\alpha)^b \cdot s_{2l_\zeta - j + q - b}(\mathcal{E}_\zeta^{l_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, l_\zeta - k})^\vee),$$

where $s_i(\cdot)$ stands for the Segre class.

We warn the reader to be very careful with signs when checking the formula in theorem 3.10.

Remark 3.11 *If $l_\zeta + h(\zeta) + q = 0$ i.e. $l_\zeta = 0$ and $h(\zeta) + q = 0$, then $\mathfrak{M}_+$ is $\mathfrak{M}_-$ with an additional connected component $E_{-\zeta}^{0,0}$ which is a $\mathbb{P}^{d-q}$-bundle over J, since $E_\zeta^{0,0} = \emptyset$. The universal bundle over $E_{-\zeta}^{0,0}$ is given by an extension*

$$0 \to \pi^* \mathcal{O}_{S\times J}(\pi_1^* L - \mathcal{F}) \otimes p^* \lambda \to \mathcal{U} \to \pi^* \mathcal{O}_{S \times J}(\mathcal{F}) \to 0,$$

where $\pi : S \times E_{-\zeta}^{0,0} \to S \times J$ and $p : S \times E_{-\zeta}^{0,0} \to E_{-\zeta}^{0,0}$ are projections and λ is the tautological line bundle. From this $p_1(\mathfrak{g}_{\mathcal{U}})/\alpha = -4a\lambda + 4c_1(\mathcal{F})^2/\alpha$ (with notations as in theorem 3.10), so $\mu_+(\alpha) = \mu_-(\alpha) - \frac{1}{4}p_1(\mathfrak{g}_{\mathcal{U}})/\alpha = \mu_-(\alpha) + a\lambda - c_1(\mathcal{F})^2/\alpha$. Therefore

$$\mu_+(\alpha)^d - \mu_-(\alpha)^d = \sum_{0 \le b \le q} (-1)^b \binom{d}{b} a^{d-b} (c_1(\mathcal{F})^2/\alpha)^b \cdot s_{q-b}(\mathcal{E}_{-\zeta}).$$

We see that it is important to understand $e_\alpha = c_1(\mathcal{F})^2/\alpha$, for $\alpha \in H_2(S;\mathbb{Z})$, and $e_S = c_1(\mathcal{F})^4/[S]$. Write $c_1(\mathcal{F}) = c_1(F)+\sum \alpha_i \otimes \alpha_i^\#$, $\alpha_i \in H^1(S)$, $\alpha_i^\# \in H^1(J)$, the Künneth decomposition of $c_1(\mathcal{F}) \in H^2(S \times J)$. So

$$e_\alpha = -2\sum_{i<j} < \alpha_i \wedge \alpha_j, \alpha > \otimes \alpha_i^\# \wedge \alpha_j^\# \in H^2(J)$$

$$e_S = \sum_{i,j,k,l} < \alpha_i \wedge \alpha_j \wedge \alpha_k \wedge \alpha_l, [S] > \otimes \alpha_i^\# \wedge \alpha_j^\# \wedge \alpha_k^\# \wedge \alpha_l^\# \in H^4(J).$$

Lemma 3.12 *Let S be a manifold with $b^+ = 1$. Then there is a (rational) cohomology class $\Sigma \in H^2(S)$ such that the image of $\wedge : H^1(S) \otimes H^1(S) \to H^2(S)$ is $\mathbb{Q}[\Sigma]$. Also $e_S = 0$.*

Proof. Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in H^1(S)$. If $\gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4 \neq 0$ then the image of $\wedge : H^1(S) \otimes H^1(S) \to H^2(S)$ contains the subspace V generated by $\gamma_i \wedge \gamma_j$, which has dimension 6, with $b^+ = 3$ and $b^- = 3$. This is absurd, so $\gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4 = 0$. Then $e_S = 0$.

Now let $\Sigma_1 = \gamma_1 \wedge \gamma_2$, $\Sigma_2 = \gamma_3 \wedge \gamma_4 \in H^2(S)$. Then $\Sigma_1^2 = \Sigma_2^2 = 0$ together with the fact that $b^+ = 1$ imply that $\Sigma_1 \cdot \Sigma_2 \neq 0$ unless Σ_1 and Σ_2 are proportional. Since $\Sigma_1 \cdot \Sigma_2 = 0$ by the above, this has to be the case. $\square$

Now write $\gamma_1, \ldots, \gamma_n$ for a basis of $H^1(S)$ and fix a generator Σ of the image of $\wedge : H^1(S) \otimes H^1(S) \to H^2(S)$. Put $\gamma_i \wedge \gamma_j = a_{ij}\Sigma$. The Jacobian of S is $J = H^1(S;\mathbb{R})/H^1(S;\mathbb{Z})$, so naturally $H^1(J) \xrightarrow{\sim} H^1(S)^*$. For the universal bundle $\mathcal{L} \to S \times J$ parametrising divisors homologically equivalent to zero, $c_1(\mathcal{L}) = \sum \gamma_i \otimes \gamma_i^\#$, so

$$e_\alpha = -2\sum_{i<j} a_{ij}(\Sigma \cdot \alpha) \otimes \gamma_i^\# \wedge \gamma_j^\# = -2(\Sigma \cdot \alpha)\omega,$$

where we write $\omega = \sum_{i<j} a_{ij}(\gamma_i^\# \wedge \gamma_j^\#) \in H^2(J)$, which is an element independent of the chosen basis.

The case $l_\zeta = 0$

Now $\mathcal{E}_\zeta = \mathcal{E}_\zeta^{0,0} = R^1\pi_*(\mathcal{O}_{S\times J}(2\mathcal{F} - \pi_1^* L))$, for $\pi : S \times J \to J$ the projection. We note that $H^0(\mathcal{O}_S(2F - L)) = 0$ and $H^0(\mathcal{O}_S(-2F + L) \otimes K) = 0$, as $-K$ is effective, so $R^0\pi_*$ and $R^2\pi_*$ vanish.

$$\mathrm{ch}\, \mathcal{E}_\zeta = -\mathrm{ch}\, \pi_!(\mathcal{O}_{S\times J}(2\mathcal{F} - \pi_1^* L)) = -\pi_*(\mathrm{ch}\, \mathcal{O}_{S\times J}(2\mathcal{F} - \pi_1^* L) \cdot \mathrm{Todd}\ T_S) =$$

$$= -(\frac{\zeta^2}{2} - \frac{\zeta \cdot K}{2} + 1 - q) + e_{K-2\zeta} - \frac{2}{3} e_S = \mathrm{rk}(\mathcal{E}_\zeta) + e_{K-2\zeta},$$

since $e_S = 0$. A fortiori ch $\mathcal{E}^\vee_{-\zeta} = -(\frac{\zeta^2}{2} + \frac{\zeta \cdot K}{2} + 1 - q) - e_{K+2\zeta}$ and

$$\mathrm{ch}\,(\mathcal{E}_\zeta \oplus \mathcal{E}^\vee_{-\zeta}) = (-\zeta^2 + 2q - 2) - 4e_\zeta.$$

Remark 3.13 *The Segre classes of $\mathcal{F}$ are given by $s_t(\mathcal{F}) = c_t(\mathcal{F})^{-1}$. For the relationship between the Chern classes of $\mathcal{F}$ and its Chern character, write a_i for $i!$ times the i-th component of ch $\mathcal{F}$. Then*

$$c_n(\mathcal{F}) = \frac{1}{n!} \begin{vmatrix} a_1 & n-1 & 0 & \cdots & 0 \\ a_2 & a_1 & n-2 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}$$

From this remark, $s_i(\mathcal{E}_\zeta \oplus \mathcal{E}^\vee_{-\zeta}) = \frac{4^i}{i!} e^i_\zeta$. This together with theorem 3.10, and recalling that $[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d$ differs from $\delta^{w,d}_S(\alpha)$ by a factor $\epsilon_S(w) = (-1)^{\frac{w^2 + K_S \cdot w}{2}}$, gives

$$\begin{aligned} \delta^{w,d}_S(\alpha) &= \epsilon_S(w) \sum_{0 \le b \le q} (-1)^{h(\zeta)} \binom{d}{b} a^{d-b} e^b_\alpha \cdot s_{q-b}(\mathcal{E}_\zeta \oplus \mathcal{E}^\vee_{-\zeta}) = \\ &= \epsilon_S(w) \sum_{0 \le b \le q} (-1)^{h(\zeta)} \binom{d}{b} a^{d-b} e^b_\alpha \cdot \frac{4^{q-b}}{(q-b)!} e^{q-b}_\zeta = \\ &= \epsilon_S(w) \sum_{0 \le b \le q} (-1)^{h(\zeta)+q} \frac{2^{3q-b-d}}{(q-b)!} \binom{d}{b} (\zeta \cdot \alpha)^{d-b} (\Sigma \cdot \alpha)^b (\Sigma \cdot \zeta)^{q-b} \omega^q. \end{aligned}$$

Corollary 3.14 *Let $S = \mathbb{CP}^1 \times \mathbb{T}^2$, $\Sigma = \mathbb{CP}^1 \subset S$, ζ defining a wall, $d = -\zeta^2$, $\alpha \in H_2(S)$. Suppose that $l_\zeta + h(\zeta) + q > 0$. Then*

$$\delta^{w,d}_S(\alpha) = \epsilon_S(w)(-1)^{h(\zeta)+1} \left(2^{3-d}(\zeta \cdot \alpha)^d (\Sigma \cdot \zeta) + 2^{2-d}\, d(\zeta \cdot \alpha)^{d-1}(\Sigma \cdot \alpha)\right).$$

Corollary 3.15 *Let $S = \mathbb{CP}^1 \times \mathbb{T}^2$, $\Sigma = \mathbb{CP}^1 \subset S$, ζ defining a wall, $d = -\zeta^2$, $\alpha \in H_2(S)$. Suppose that $l_\zeta + h(\zeta) + q = 0$. Then from remark 3.11*

$$\delta^{w,d}_S(\alpha) = \epsilon_S(w) \left(2^{1-d}(\zeta \cdot \alpha)^d ((K + 2\zeta) \cdot \Sigma) + 2^{2-d} d(\zeta \cdot \alpha)^{d-1}(\Sigma \cdot \alpha)\right).$$

Case $l_\zeta = 1$

We do not want to enter into the detailed computations of the wall-crossing formulae, but just to remark that the pattern laid in [26] can be used here to obtain many of them. For instance, if we write

$$S_{j,b} = \sum_k ([\mathcal{Z}_{l_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot e_\alpha^b \cdot s_{2l_\zeta - j + q - b}(\mathcal{E}_\zeta^{l_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, l_\zeta - k})^\vee),$$

we can give some of the wall-crossing formulae (without proof). For obtaining all of them we would need a better understanding of the Hilbert scheme, not available at the moment.

$$\begin{aligned} S_{2l_\zeta, q} &= \frac{(2l_\zeta)!}{l_\zeta!}(\alpha^2)^{l_\zeta} e_\alpha^q \\ S_{2l_\zeta - 1, q} &= (-4)\frac{(2l_\zeta)!}{l_\zeta!}(\alpha^2)^{l_\zeta - 1} \cdot a \cdot e_\alpha^q \\ S_{2l_\zeta, q-1} &= 4\frac{(2l_\zeta)!}{l_\zeta!}(\alpha^2)^{l_\zeta} e_\alpha^{q-1} e_\zeta \end{aligned}$$

where $a = (\zeta \cdot \alpha)/2$. Also in the case $l_\zeta = 1$,

$$\begin{aligned} S_{0,q} &= (6\zeta^2 + 2K_X^2) e_\alpha^q \\ S_{1,q-1} &= -32\, a\, e_\alpha^{q-1} e_\zeta + 8\, e_\alpha^q \\ S_{2,q-2} &= 16\, \alpha^2\, e_\alpha^{q-2} e_\zeta^2 \end{aligned}$$

Conjectures and speculations

From all the cases we have studied it is natural to give the following conjecture, which is a generalisation of the conjecture about the wall-crossing terms in the case $b_1 = 0$ (see [35]).

Conjecture 3.16 *Let X be an oriented compact four-manifold with $b^+ = 1$ and $b_1 = 2q$ even. Let $w \in H^2(X;\mathbb{Z})$. Choose $\Sigma \in H^2(X)$ generating the image of $\wedge : H^1(X) \otimes H^1(X) \to H^2(X)$. Define $\omega \in H^2(J)$ such that $e_\alpha = -2(\Sigma \cdot \alpha)\omega$ and put $a = \int_J \frac{\omega^n}{n!}$. If ζ defines a wall, then the difference term $\delta_X^{w,d}(\alpha)$ only depends on w, d, b_1, b_2, ζ^2, Q_X, $(\zeta \cdot \alpha)$ and $a(\Sigma \cdot \alpha)^i (\Sigma \cdot \zeta)^{q-i}$, $0 \leq i \leq q$. The coefficients are universal on X.*

This is quite a strong conjecture and one can obviously write down weaker versions. It would allow one to carry out similar arguments to those in [31] and therefore to find out the general shape of the wall-crossing formulae for arbitrary X. This would involve modular forms.

Chapter 4

Connected sums along a torus

In this chapter we are interested in applying all the techniques of gluing theory to the case of connected sum of two manifolds along an embedded two-torus. As we have seen in subsection 2.1.2, we need to apply the gluing theory for the case of $Y = \Sigma \times \mathbb{S}^1$ where Σ is a Riemann surface of genus 1, i.e. a two-torus $\mathbb{T}^2$. We want to point out that results of this kind can be obtained by gluing along circle bundles of degree 1 over a torus [56].

Our first main goal is to recast the results of Friedman in [22] about invariants of elliptic surfaces into a much simpler and tractable form. Instead of using moduli spaces of stable bundles to compute the invariants as in [22], we will use topological methods and the gluing theory from section 1.2. A very similar approach have been carried out in [43]. There, Morgan and Mrowka compute some $SU(2)$ and $SO(3)$ invariants of the elliptic surfaces $X = S_n(m_1, m_2)$ (see section 3.2 for notation) by using the same procedure of splitting $X = X_1 \cup_Y X_2$ along a three-torus $Y = \mathbb{T}^3$. They decompose X into elementary pieces which are elliptic surfaces of geometric genus $p_g = 1$ or $p_g = 2$ and compute some of the invariants inductively using known information about these elementary pieces. In our approach, we use instead only information about the $K3$ surface and some invariants of $S = \mathbb{T}^2 \times \mathbb{CP}^1$, computed in section 4.3. They can not use the Fukaya-Floer groups as in their case P_Y is trivial, and so they have to deal with the presence of reducibles. There are some technicalities, as they need to use the character variety $\chi(\mathbb{T}^3)$ of the three-torus for the possible flat limits of connections along the cylindrical end as X is pulled apart (in our case, the possible limits are only one point, proposition 4.1). This allows them to consider the case of $w \cdot T \equiv 0 \pmod 2$, and therefore elliptic surfaces with multiple fibres of even multiplicities. On the other hand, their argument is restricted as to which classes they can evaluate the polynomial invariant on (they again use classes going across Y, but they are constrained on the number of them across Y they can use). So if we write

$$D_X(\Sigma^d) = \sum_{i=0}^{m} \alpha(n,i)\,(\Sigma\cdot\kappa_X)^{d-2i}(\Sigma^2)^i$$

for the $SU(2)$ Donaldson invariant of $X = S_n(m_1, m_2)$, $\kappa_X = F/m_1m_2$, F denoting the fibre, $m = (d-n+2)/2$, then they are able to compute the two leading terms $\alpha(n,m)$ and $\alpha(n,m-1)$ and conclude from that a differentiable classification of simply connected elliptic surfaces with $b^+ > 1$. Instead, we compute all the terms, but only for the low dimensional invariants (and with the condition $w\cdot\Sigma \equiv 1 \pmod 2$).

Later, in section 4.4, we will have a full computation of the invariants for elliptic surfaces, superseding the proof that we include of theorem 4.2. Nonetheless, we chose to keep this earlier computation because it does not use at all any of the deep results of [38] about basic classes and their properties.

The second goal is to relate the basic classes of the manifold obtained as connected sum along a torus of two manifolds with $b^+ \geq 1$. Applying this to the case of elliptic surfaces and with a little bit more input, we get all their invariants of elliptic surfaces with multiple fibres. This was conjectured by Kronheimer and Mrowka in [37] and proved by Fintushel and Stern [17]. The point of carrying out again these results is to generalise the computations for basic classes when the connected sum along a torus of two elliptic surfaces is not performed in the algebraic setting (see section 2.2). We will see that in this fashion we can also understand the earlier results of Gompf and Mrowka [30] in a very straightforward way.

4.1 Splittings along a three-torus

We will use the following proposition which is an application of the gluing theory from section 1.2. We recall that (w,T) is an allowable pair for a glued manifold $X = X_1\cup_Y X_2$ when $w\in H^2(X;\mathbb{Z})$ with $w|_Y$ odd in $H^2(Y;\mathbb{Z})$ and $T\in H^2(X;\mathbb{Z})$ is any class whose Poincaré dual lies in $H_2(Y;\mathbb{Z})$ and with $w\cdot T\equiv 1 \pmod 2$.

Proposition 4.1 *There exists a vector space V of dimension 1 with the following properties*

1. *Let X be an open manifold with boundary $Y = \partial X = \mathbb{T}^2 \times \mathbb{S}^1$ and with $b^+(X) > 0$. Let $D \in H_2^R(X, \partial X)$ and $w \in H^2(X;\mathbb{Z})$ with $w|_Y$ odd in $H^2(Y;\mathbb{Z})$. Then we have defined a sequence $\phi_i = \phi_i^w(X,D) \in V$, $i = 0, 1, \dots$*

2. *Let X be a closed manifold with $b^+ \geq 1$ which can be written as $X = X_1 \cup_Y X_2$, and let $D \in H_2(X)$ and (w,T) an allowable pair. Then writing $D = D_1 + D_2$, $D_i \in H_2^R(X_i, \partial X_i)$, and $w = w_1 + w_2$, $w_i \in H^2(X_i;\mathbb{Z})$, we have*
$$D_X^{(w,T)}(D^m) = -\sum_i \binom{m}{i} \phi_i^{w_1}(X_1, D_1) \cdot \phi_i^{w_2}(X_2, D_2).$$

Proof. The space V is the Floer homology of the three-torus Y. This is one dimensional because there is only one flat $SO(3)$-connection on Y with $w_2 \equiv w \pmod 2$, and furthermore this connection is generic. Indeed, as explained in [9], there are two flat $U(2)$-connections with the appropriate w (one with holonomy 1 around the $\mathbb{S}^1$-factor, and the other with holonomy -1), but they are interchanged by the involution shifting degrees by 4. So when we reduce the Floer groups mod 4 there is only one generator. We recall that the grading is not well-defined and that we have a pairing $\sigma : HF_*(Y) \otimes HF_{-*}(Y) \to \mathbb{Z}$. The important point is to note that for the generator a, $\sigma : a \otimes a \mapsto -1$. For checking this take $Z = D^2 \times \mathbb{T}^2$ and A the (only) flat connection on Z (with limit the only flat connection on Y). Glue them together to get the flat connection on $X = Z \cup_Y Z = \mathbb{CP}^1 \times \mathbb{T}^2$. This connection contributes with a -1, as the moduli space of flat connections for X is generic, zero-dimensional and consisting of one point. This point is a $+1$ for the complex orientation of the moduli space and hence it contributes as $(-1)^{\frac{w^2 + w\cdot K}{2}} = -1$, since w is an odd multiple of $[\mathbb{CP}^1]$ (see theorem 4.3). □

We remark that we do not need Atiyah-Floer conjecture for the result above. We recall that for defining the invariants $\phi_i^w(X,D)$ we had to orient the moduli spaces (see section 1.2). We can choose Z and A as above and then for every open manifold X with boundary Y, we have to choose a homology orientation of $\bar{X} = X \cup_Y Z$. When $\bar{X}$ is an algebraic surface, the homology orientation is always the one given by the Hodge form ω as in section 3.1. It is important to notice that whenever $\bar{X}_1$, $\bar{X}_2$ are algebraic manifolds with F a complex torus and $F \hookrightarrow \bar{X}_i$ embeddings (with image F_i of self-intersection zero) such that $X = X_1 \#_F X_2$ is also an algebraic manifold, the homology orientations agree (by an excision argument).

Friedman's result [22] (at least the part of it corresponding to odd multiple fibres) is:

Theorem 4.2 *Let X be a simply connected elliptic surface. Let p_g denote its geometric genus and $1 \leq m_1 \leq m_2$ be the multiplicities of the (possibly) multiple fibres. Suppose they are both odd. Let F be the cohomology class corresponding to the fibre of the fibration and let $w \in H^2(X;\mathbb{Z})$ such that $d_0 = -w^2 - \frac{3}{2}(1+b^+) \equiv 0 \pmod 4$ and $w \cdot F \equiv 1 \pmod 2$. So $w^2 \equiv n = 1 + p_g \pmod 4$. Write $\epsilon(w) = (-1)^{\frac{w^2 + K_X \cdot w}{2}}$, so $\epsilon(w+F) = -\epsilon(w)$. Then for $\alpha \in H_2(X)$ we have*

1. $D_X^w(\alpha^0) = \epsilon(w)$
2. $D_X^{w+F}(\alpha^2) = \epsilon(w+F)(\alpha^2 + C_1(F\cdot\alpha)^2)$
3. $D_X^w(\alpha^4) = \epsilon(w)(3(\alpha^2)^2 + 6C_1\alpha^2(F\cdot\alpha)^2 + (3C_1^2 - 2C_2)(F\cdot\alpha)^4)$, *where*

$$C_1 = p_g + 1 - \frac{1}{m_1^2} - \frac{1}{m_2^2}$$
$$C_2 = p_g + 1 - \frac{1}{m_1^4} - \frac{1}{m_2^4}$$

In the case $p_g = 0$, $D_X^{(w,F)}(\alpha^m)$ is computed in the chamber defined by the fibre F (i.e. with respect to a suitable metric).

Recall (section 3.1) that the complex orientation of the moduli space and its natural orientation differ by $\epsilon(w) = (-1)^{\frac{w^2+K\cdot w}{2}}$, and so the difference in our signs. The proof of this theorem proceeds in two stages. The first main goal is to reduce it to a very particular non-simply connected case, which is:

Theorem 4.3 *For the non-simply connected elliptic surface $S = \mathbb{T}^2 \times \mathbb{CP}^1$, $w = [\mathbb{CP}^1] \in H^2(S;\mathbb{Z})$ and $D = [\mathbb{CP}^1] \in H_2(S)$ a horizontal section, we have, with respect to a suitable metric:*

$$D_S^w(D^0) = -1$$
$$D_S^{w+[\mathbb{T}^2]}(D^2) = -2$$
$$D_S^w(D^4) = -16$$

The second stage is to prove theorem 4.3 through explicit calculations in the moduli spaces of stable vector bundles over S with the appropriate Chern classes. Here we must point out that the proof of $D_S^{w+[\mathbb{T}^2]}(D^2) = -2$ has already been carried out by Donaldson in [9].

4.2 Proof of theorem 4.2

The proof of theorem 4.2 is very simple, but it gets rather messy because we have to be very careful in keeping track of the signs for different w's.

Recall that $S_n(m_1, m_2)$ denotes the elliptic surface with $p_g = n-1$ and multiple fibres of (coprime) multiplicities m_1 and m_2. We introduce the following notation:

- A_m will be a tubular neighbourhood of a multiple fibre of multiplicity m. This is always diffeomorphic to $A = \mathbb{T}^2 \times D^2$.
- $B_n = S_n - \mathrm{int}(A_1)$ is the closure of the complement of a tubular neighbourhood of a non-singular fibre in S_n.
- $B_n(m) = S_n(m) - \mathrm{int}(A_1)$.

We have $S_n(m) = B_n \cup_Y A_m$ and $S_n(m_1, m_2) = B_n(m_1) \cup_Y A_{m_2}$. Each of the pieces A_m, B_n, etc. has a natural elliptic fibration and all the unions preserve the elliptic fibration. We have the following facts:

1. Any 2-homology class $D \in H_2(X)$ can be decomposed as $D = D_1 + D_2$ with $D_i \in H_2^R(X_i, \partial X_i)$ (recall subsection 2.3.2). We also have $D^2 = D_1^2 + D_2^2$.
2. From remark 2.18, $H_2^R(A, \partial A)$ contains a subgroup $\mathbb{Q}[F] \oplus \mathbb{Q}[\Delta]$. When $X_2 = A_m$, adding a multiple of $[F]$ to D_1 and arranging things conveniently we can suppose that D_2 is a multiple of $[\Delta]$. Then $\phi_i^w(A_m, D_2)$ only depends on $w \in H^2(A_m;\mathbb{Z}) \cong \mathbb{Z}[\Delta]$ and $F \cdot D_2$. Also note that $D_2^2 = 0$, $D_1^2 = D^2$ and $F \cdot D_1 = F \cdot D_2 = F \cdot D$.
3. Let $X = S_2$ be the $K3$-surface and $w \in H^2(X;\mathbb{Z})$. One has $d_0 = -w^2 - 6 \equiv 0 \pmod 2$. For $D \in H_2(X;\mathbb{Z})$ and x the class of the point, we have
$$D_X^w(D^{2i}x^j) = (-1)^{w^2/2}\, 2^j \frac{(2i)!}{2^i i!}(D^2)^i$$
when $2i + 2j \equiv -w^2 - 6 \pmod 4$ and zero otherwise. This was first proved by O'Grady [46] in the case of $w = 0$ and $j = 0$. There is a complete calculation in [38], but it was well-known before that. If $w \cdot F \equiv 1 \pmod 2$ (see subsection 1.1.4) then
$$D_X^{(w,F)}(D^{2i}) = (-1)^{i+1} \frac{(2i)!}{2^i i!}(D^2)^i.$$

4. For the rational surface $X = S_1$ we have the following easy fact. Let V be a stable bundle. Then V is simple [24, page 323], so $H^0(\mathrm{End}_0(V)) = 0$. Since $-K_X$ is effective, $H^2(\mathrm{End}_0(V)) = H^0(\mathrm{End}_0(V) \otimes K_X) = 0$. So the moduli spaces of stable bundles for X are always generic. In particular, the zero-dimensional moduli space is a collection of points counting positively for the natural complex orientation. So the invariant $D^w_X(D^0)$ is of sign $\epsilon(w)$.

Stage 1 For the $K3$-surface, put $X = S_2 = B_1 \cup_Y B_1$ and fix $w \in H^2(X_2;\mathbb{Z})$ being symmetrical (the restriction to both pieces B_1 are the same) and $w \cdot F \equiv 1 \pmod 2$. Write $\phi_i^j = \phi_i^{w_j}(B_1, D_j)$ with $D = D_1 + D_2$. Then with $T = F$, proposition 4.1 gives

$$\begin{aligned} 1 &= -D^{(w,T)}_X(D^0) = \phi_0^1 \cdot \phi_0^2 \\ -D^2 &= -D^{(w,T)}_X(D^2) = \phi_0^1 \cdot \phi_2^2 + \phi_2^1 \cdot \phi_0^2 \\ 3\,(D^2)^2 &= -D^{(w,T)}_X(D^4) = \phi_0^1 \cdot \phi_4^2 + 6\,\phi_2^1 \cdot \phi_2^2 + \phi_4^1 \cdot \phi_0^2 \end{aligned}$$

From where either $\phi_0^j = 1$, $\phi_2^j = -D_j^2$ and $\phi_4^j = 3\,(D_j^2)^2$ or the opposite signs. Fixing a choice of signs is equivalent to fixing an orientation of the generator of Floer homology. We choose the signs as above.

For A, write $\phi_i = \phi_i^{[\Delta]}(A, [\Delta])$. Then $S = \mathbb{T}^2 \times \mathbb{CP}^1 = A \cup_Y A$ and theorem 4.3 implies

$$\begin{aligned} 1 &= -D_S^{[\mathbb{CP}^1]}([\mathbb{CP}^1]^0) = \phi_0 \cdot \phi_0 \\ 2 &= -D_S^{[\mathbb{CP}^1]+[\mathbb{T}^2]}([\mathbb{CP}^1]^2) = \phi_0 \cdot \phi_2 + \phi_2 \cdot \phi_0 \\ 16 &= -D_S^{[\mathbb{CP}^1]}([\mathbb{CP}^1]^4) = \phi_0 \cdot \phi_4 + 6\,\phi_2 \cdot \phi_2 + \phi_4 \cdot \phi_0 \end{aligned}$$

from where either $\phi_0 = 1$, $\phi_2 = 1$ and $\phi_4 = 5$ or $\phi_0 = -1$, $\phi_2 = -1$ and $\phi_4 = -5$. We have the same invariants for $\phi_i^{(2a+1)[\Delta]}(A, [\Delta])$ (but the sign actually might depend on a).

Now for the rational elliptic surface $X = S_1 = B_1 \cup_Y A_1$, with $w = w_1 + (2a+1)\Delta$ and $D = D_1 + (F \cdot D)\Delta$, we have for $w^2 \equiv 1 \pmod 4$ the invariant $D_X^{(w,T)}(D^0) = \epsilon(w) = (-1)^a$. This forces the invariants $\phi_i^{(2a+1)[\Delta]}(A, [\Delta])$ to be of sign $(-1)^{a-1}$. So finally we get

$$\begin{aligned} D^{(w,T)}_X(D^0) &= \epsilon(w) \\ D^{(w,T)}_X(D^2) &= \epsilon(w+F)(D^2 - (F \cdot D)^2) \\ D^{(w,T)}_X(D^4) &= \epsilon(w)(5(F \cdot D)^4 - 6(F \cdot D)^2 D^2 + 3(D^2)^2) \end{aligned}$$

Stage 2 Now we pass on to prove, by induction on n, that for general B_n and w with $w|_Y = \mathbb{S}^1$ we have, for $D \in H_2^R(B_n, \partial B_n)$ with ∂D a multiple of $\mathbb{S}^1$,

$$\begin{aligned}
\phi_0(n) = \phi_0^w(B_n, D) &= \epsilon_{S_n}(w) \\
\phi_2(n) = \phi_2^w(B_n, D) &= \epsilon_{S_n}(w+F)(D^2 + (n-1)(F\cdot D)^2) \\
\phi_4(n) = \phi_4^w(B_n, D) &= \epsilon_{S_n}(w)(3(D^2)^2 + 6(n-1)D^2(F\cdot D)^2 + \\
&\qquad + (3n^2 - 8n + 5)(F\cdot D)^4)
\end{aligned}$$

where $w \in H^2(S_n;\mathbb{Z})$ is any element restricting to the given $w \in H^2(B_n;\mathbb{Z})$ for $S_n = B_n \cup A_1$, and $w^2 \equiv n \pmod 4$. From this we get that the invariants for S_n are of the desired form.

First, the assertion is true for B_1 as noted in stage 1. For the induction step, note that $B_n \cup B_{n'} = B_{n+n'} \cup A_1$, so obviously $\phi_0(n) = (-1)^{n-1} = \epsilon_{S_n}(w)$ for all n. Now

$$\begin{aligned}
&(-1)^{n'-1}\phi_2(n) + (-1)^{n-1}\phi_2(n') = -\phi_2(n+n') - (-1)^{n+n'-1}(F\cdot D)^2 \\
&(-1)^{n'-1}\phi_4(n) + 6\,\phi_2(n)\cdot\phi_2(n') + (-1)^{n-1}\phi_4(n') = \\
&\qquad = -\phi_4(n+n') - 6\,\phi_2(n+n')\cdot(F\cdot D)^2 - (-1)^{n+n'-1}5\,(F\cdot D)^4
\end{aligned}$$

So suppose the result true for some n. Then

$$\begin{aligned}
\phi_2(n+1) &= -\phi_2(n) - (-1)^n(D_2^2 + (F\cdot D)^2) \\
&= -(-1)^n(D^2 + n(F\cdot D)^2) \\
\phi_4(n+1) &= (-1)^n(3(D_1^2)^2 + 6(n-1)D_1^2(F\cdot D)^2 + (3n^2-8n+5)(F\cdot D)^4 + \\
&\qquad + 6\,(D_1^2 + (n-1)(F\cdot D)^2)D_2^2 + 3(D_2^2)^2 + \\
&\qquad + 6\,(D^2 + n(F\cdot D)^2)(F\cdot D)^2 - 5(F\cdot D)^4) = \\
&= (-1)^n(3(D^2)^2 + 6nD^2(F\cdot D)^2 + \\
&\qquad + (3(n+1)^2 - 8(n+1) + 5)(F\cdot D)^4)
\end{aligned}$$

as required. The invariants of S_n will be

$$\begin{aligned}
D_{S_n}^{(w,T)}(D^0) &= \epsilon(w) \\
D_{S_n}^{(w,T)}(D^2) &= \epsilon(w+F)(D^2 + (n-2)(F\cdot D)^2) \\
D_{S_n}^{(w,T)}(D^4) &= \epsilon(w)(3(D^2)^2 + 6(n-2)D^2(F\cdot D)^2 + (3n^2 - 14n + 16)(F\cdot D)^4)
\end{aligned}$$

Now for general $w \in H^2(B_n;\mathbb{Z})$ with $w\cdot F = 1 \pmod 2$ (but maybe not $w|_Y = \mathbb{S}^1$) and general $D \in H_2^R(B_n, \partial B_n)$, we use $B_n \cup B_1 = S_{n+1}$. From the invariants of S_{n+1}, which we have already computed, and those of B_1, we get that

$$\begin{aligned}
\phi_0^w(B_n, D) &= -\epsilon_{S_{n+1}}(w) \\
\phi_2^w(B_n, D) &= -\epsilon_{S_{n+1}}(w+F)(D^2 + (n-1)(F\cdot D)^2) \\
\phi_4^w(B_n, D) &= -\epsilon_{S_{n+1}}(w)(3(D^2)^2 + 6(n-1)D^2(F\cdot D)^2 + \\
&\qquad +(3n^2 - 8n + 5)(F\cdot D)^4)
\end{aligned}$$

where $w \in H^2(S_{n+1};\mathbb{Z})$ is any element restricting to the given $w \in H^2(B_n;\mathbb{Z})$ with $w^2 \equiv n+1 \pmod 4$.

The case of $F \cdot w = 2a+1$ is in everything analogous to this one and we get an extra factor $(-1)^a$ for $\phi_i(n)$ (the invariants for S_n and hence the last set of formulae remain unchanged). Also note that $(-1)^a \epsilon_{S_n}(w) = -\epsilon_{S_{n+1}}(w) = (-1)^{(n-1)(a+1)}$.

Stage 3 Now for introducing multiple fibres, one has that $S_n(m) = B_n \cup_Y A_m$, with A_m in fact diffeomorphic to $\mathbb{T}^2 \times D^2$. As seen in section 3.2, the gluing between the boundaries is different for different m, but it always preserves the elliptic fibration. Therefore, the class $F \in H_2(B_n)$ is equivalent, as an element of $H_2(S_n(m))$ to the class $m[\mathbb{T}^2] \in H_2(A_m) = H_2(\mathbb{T}^2)$. Call $f \in H_2(S_n(m))$ the homology class corresponding to $[\mathbb{T}^2] \in H_2(A_m)$. Now choose $w \in H^2(S_n(m))$ with $f \cdot w = 1$ and $D \in H_2(S_n(m))$ decomposed as $D = D_1 + \Delta$ for the horizontal class $\Delta \in H_2^R(A_m, \partial A_m)$. Then m odd implies $w_1 \cdot F = m = 2a+1$.

$$\begin{aligned}
D_{S_n(m)}^{(w,T)}(D^0) &= -\phi_0^{w_1}(B_n, D_1)\cdot\phi_0^{[\Delta]}(A_m, \Delta) = (-1)^{(n-1)(a+1)} = \epsilon(w) \\
D_{S_n(m)}^{(w,T)}(D^2) &= \phi_2^{w_1}(B_n, D_1) - (-1)^{(n-1)(a+1)}\phi_2^{[\Delta]}(A_m, \Delta) = \\
&= -(-1)^{(n-1)(a+1)}(D^2 + (n-1)(F\cdot D)^2 - \frac{1}{m^2}(F\cdot D)^2) \\
&= -\epsilon(w)(D^2 + (p_g - \frac{1}{m^2})(F\cdot D)^2) \\
D_{S_n(m)}^{(w,T)}(D^4) &= \epsilon(w)(\,3\,(D^2)^2 + 6\,(p_g - \frac{1}{m^2})D^2\,(F\cdot D)^2 + \\
&\qquad + (3\,p_g^2 - 2\,p_g - 6\,p_g\,\frac{1}{m^2} + \frac{5}{m^4})(F\cdot D)^4)
\end{aligned}$$

If $f \cdot w \neq 1$ but it is still odd, we can work out the same expression. Now putting $S_n(m) = A_1 \cup B_n(m)$, we can calculate

$$\phi_0^w(B_n(m), D) = -\epsilon_{S_{n+1}(m)}(w)$$

$$
\begin{aligned}
\phi_2^w(B_n(m),D) &= -\epsilon_{S_{n+1}(m)}(w+F)(D^2+(p_g-\frac{1}{m^2}+1)(F\cdot D)^2) \\
\phi_4^w(B_n(m),D) &= -\epsilon_{S_{n+1}(m)}(w)(3(D_1^2)^2+6\,(p_g+1-\frac{1}{m^2})D_1^2\,(F\cdot D)^2+ \\
&\quad +(3\,p_g^2+4\,p_g+1-6\,(p_g+1)\frac{1}{m^2}+\frac{5}{m^4})(F\cdot D)^2)
\end{aligned}
$$

where $f\cdot w=1$, $w\in H^2(S_{n+1}(m);\mathbb{Z})$ extends $w\in H^2(B_n(m);\mathbb{Z})$ and has $w^2\equiv n+1 \pmod 4$. If $f\cdot w=1+2a$ we have to multiply the above expressions by a factor $(-1)^a$. In the general case, $S_n(m_1,m_2)=A_{m_2}\cup B_n(m_1)$, m_1 and m_2 both odd and $w\cdot F=2a+1$. We obtain

$$
\begin{aligned}
D^{(w,T)}_{S_n(m_1,m_2)}(D^0) &= \epsilon(w) \\
D^{(w,T)}_{S_n(m_1,m_2)}(D^2) &= \epsilon(w+F)(D^2+(p_g+1-\frac{1}{m_1^2}-\frac{1}{m_2^2})(F\cdot D)^2)= \\
&= \epsilon(w+F)(D^2+C_1(F\cdot D)^2) \\
D^{(w,T)}_{S_n(m_1,m_2)}(D^4) &= \epsilon(w)(3(D^2)^2+6\,(p_g+1-\frac{1}{m_1^2}-\frac{1}{m_2^2})D^2(F\cdot D)^2+ \\
&\quad +(3\,p_g^2+4\,p_g+1-6\,(p_g+1)(\frac{1}{m_1^2}+\frac{1}{m_2^2})+ \\
&\quad +\frac{5}{m_1^4}+\frac{5}{m_2^4}+\frac{6}{m_1^4\,m_2^4})(F\cdot D)^4)= \\
&= \epsilon(w)(3(D^2)^2+6\,C_1D^2(F\cdot D)^2+(3\,C_1^2-2\,C_2)(F\cdot D)^4)
\end{aligned}
$$

as required. □

We tried to push these arguments to handle the case of multiple fibres of even multiplicity, but at some stage we needed the invariants for S_n when $w\cdot F\equiv 0 \pmod 2$. One would need to use the deep results of [38] relating the invariants for different $w\in H^2(X;\mathbb{Z})$.

4.3 Proof of theorem 4.3

For proving theorem 4.3 we need to study the moduli spaces of dimensions 0, 4 and 8 of ASD connections for $S=\mathbb{CP}^1\times\mathbb{T}^2$. This manifold has Betti numbers $b_1=2$, $b_2=2$, $b^+=1$ and therefore we have a chamber structure. We fix some notation. $F=[\mathbb{T}^2]\in H_2(S)$ will stand for the homology class representing the fibre and $D=[\mathbb{CP}^1]\in H_2(S)$ for the horizontal section. The invariants that we want to calculate are with respect to a suitable metric, i.e. metrics having its period point in the chamber containing $\Sigma=\mathbb{T}^2$ in its closure (see definition 3.3). We take h_1,

h_2 fixed Kähler metrics in $\mathbb{CP}^1$ and $\mathbb{T}^2$ respectively such that the volumes with respect to these metrics are normalised, that is, have value 1. Then we consider Kähler metrics of the kind $h = \beta h_1 + \alpha h_2$, whose real part is a Hodge metric g, representative of a conformal class with period point $[\omega_g] = [\alpha D + \beta F] \in \mathbb{H}$. Then h corresponds to the polarisation $H = \alpha D + \beta F$ (strictly speaking, we should only allow α/β be rational, since only in this case can a multiple of H be a true divisor on S). Recall from definition 1.9 that once (w, p_1) are fixed, the chambers associated to (w, p_1) are labelled by pairs $\pm e \in H^2(S;\mathbb{Z})$ with $p_1 \leq e^2 < 0$ and $e \equiv w \pmod 2$.

Since S is an algebraic manifold we use proposition 3.1 and study $\mathfrak{M}_H(c_1, c_2)$ for a suitable polarisation H. One can study directly the moduli space $\mathfrak{M}_H(c_1, c_2)$ to carry out the computations of the invariants. Although more natural, it is not very enlightening and very prone to error. Alternatively, and this is the way we have chosen to develop here, we can use the wall-crossing formulae of section 3.4. This offers the perfect excuse to include that section and produces much cleaner arguments. Fix $w = [\mathbb{CP}^1] \in H^2(S;\mathbb{Z})$. The formal dimension of the moduli space is $2d = 8\kappa - 3(1 - b_1 + b^+) = 8\kappa = -2p_1$, with $\kappa = c_2 - \frac{1}{4}c_1^2$. Note that the moduli spaces $\mathfrak{M}_H(c_1, c_2)$ we will be encountering have natural complex orientations differing from the orientations of $\mathcal{M}_\kappa^{c_1}$ by a factor $\epsilon(c_1) = (-1)^{\frac{K_S \cdot c_1 + c_1^2}{2}}$. We have the following cases:

1. $\kappa = 0$, $c_1 = w$, $c_2 = 0$, $d = 0$ and $p_1 = 0$. The moduli space is of dimension zero. There are no walls, since $p_1 = 0$. The moduli space $\mathcal{M}_\kappa^w$ consists of one point, as there is just one flat connection, which is irreducible, on S for $c_1 = w$. Algebro-geometrically we consider the moduli space $\mathfrak{M}_H(c_1, c_2)$ of stable bundles with $c_1 = w$ and $c_2 = 0$. It's easily seen that this moduli space consists of one point V

   $$0 \to \mathcal{O} \to V \to \mathcal{O}(L) \to 0$$

   (recall that L is the determinant).
   Furthermore $H^1(\mathrm{End}_0(V)) = H^2(\mathrm{End}_0(V)) = 0$, so this point is generic and isolated and counts as $+1$ for the invariant of the 0-dimensional manifold with the complex orientation. Keeping in mind that this orientation differs from the usual one by a factor of $\epsilon(w) = -1$, we have the Donaldson invariant

   $$D_S^w(D^0) = -1.$$

2. $\kappa = \frac{1}{2}$, $c_1 = w + F$, $c_2 = 1$, $d = 2$ and $p_1 = -2$. The moduli space is of dimension four. There is one wall of type (w, p_1) given by $\zeta = D - F$. We denote by $\mathcal{C}_0$ the chamber associated to polarisations $H = \alpha D + \beta F$ with $H \cdot \zeta < 0$ i.e. with β small (say $\beta/\alpha < 1$), and by $\mathcal{C}_F$ the other chamber, the one consisting of (period points of) suitable metrics.

   Since $S = \mathbb{CP}^1 \times \mathbb{T}^2$ is a ruled surface we can apply proposition 2.3 of [54] (with obvious modifications) to conclude that for polarisations $H \in \mathcal{C}_0$ the moduli space $\mathfrak{M}_H(c_1, c_2)$ of H-stable bundles on S is empty. Therefore for $H \in \mathcal{C}_F$ we have
$$D_S^{w+F}(D^2) = \delta_S^{w+F,2}(\mathcal{C}_F, \mathcal{C}_0).$$
   By corollary 3.15 with $\Sigma = D$, $\alpha = D$, $K_S = -2F$, $\zeta \cdot K_S = -2$, $\zeta^2 = -2$, $\zeta \cdot \alpha = -1$, $\epsilon(w + F) = 1$, this term is -2.

3. $\kappa = 1$, $c_1 = w$, $c_2 = 1$, $d = 4$ and $p_1 = -4$. The moduli space is of dimension eight. There is one wall of type (w, p_1) given by $\zeta = D - 2F$. We denote again by $\mathcal{C}_0$ the chamber associated to polarisations $H = \alpha D + \beta F$ with $H \cdot \zeta < 0$ i.e. with $\beta/\alpha < 2$, and by $\mathcal{C}_F$ the chamber consisting of suitable metrics. In this case, unlike the others, the moduli space $\mathcal{M}_1^w$ is not compact and has a natural Uhlenbeck compactification $\overline{\mathcal{M}_1^w} \subset \mathcal{M}_1^w \sqcup (\mathcal{M}_0^w \times S)$ as explained in subsection 1.1.1. Let us study the moduli space $\mathfrak{M}_H(c_1, c_2)$ in detail.

   Theorem 4.4 *Fix $H \in \mathcal{C}_0$ and the determinant $\Lambda = \mathcal{O}(L)$ (for a divisor $L = p_0 \times \mathbb{CP}^1 \hookrightarrow S$. Then the Gieseker-Maruyama compactification of the moduli space $\mathfrak{M}_H(c_1, c_2)$ is given by the set of non-split extensions*
$$0 \to M^{-1} \to V \to \Lambda \otimes M \otimes I_p \to 0 \tag{4.1}$$
   with p a point in S and M a line bundle of degree zero (and hence of the form $\mathcal{O}(D - L)$ for some $D = pt \times \mathbb{CP}^1 \hookrightarrow S$). All such extensions are stable. There is a space $\mathfrak{M}$ parametrising all extensions like (4.1) which is a compactification of a (generically) double cover of $\mathfrak{M}_H(c_1, c_2)$.

   Proof. First, let V be an H-stable bundle. By Riemann-Roch [1, page 21], we have
$$\chi(V \otimes \mathcal{O}(F)) = \frac{1}{2}((L + 2F)^2 - 2 \cdot 2) + \frac{1}{2}(L + 2F) \cdot 2F + 0 = 1$$

But, because of the stability, there is no nontrivial morphism $\mathcal{O}(L+3F) \to V$, and so $h^2(V \otimes \mathcal{O}(F)) = h^0(V^\vee \otimes \mathcal{O}(-F-2F)) = h^0(V \otimes \mathcal{O}(-L-3F)) = 0$. So $h^0(V \otimes \mathcal{O}(F)) > 0$ and thus there is a monomorphism $\mathcal{O}(-F) \to V$. Then there is a divisor $C \geq 0$ such that the morphism above factors through $\mathcal{O}(C-F) \to V$ with a torsion free cokernel. The H-stability of V leaves us only with three cases:

$$0 \to \mathcal{O}(-F) \to V \to \Lambda \otimes \mathcal{O}(F) \otimes I_Z \to 0$$

$$0 \to \mathcal{O} \to V \to \Lambda \otimes I_p \to 0$$

$$0 \to \mathcal{O}(D-F) \to V \to \Lambda \otimes \mathcal{O}(F-D) \to 0$$

where I_Z is the ideal defining a 0-dimensional subscheme Z of length 2, I_p is the maximal ideal of some point $p \in S$ and $D = \mathrm{pt} \times \mathbb{CP}^1 \hookrightarrow S$. In the first case there is a divisor $M = \mathcal{O}(D-L)$ of degree zero such that Z is contained in the divisor that $M \otimes \Lambda \otimes \mathcal{O}(F)$ defines[1]. Therefore $h^0(V \otimes M) > 0$ and hence we can suppose that we are always in one of the cases

$$0 \to M^{-1} \to V \to \Lambda \otimes M \otimes I_p \to 0$$

$$0 \to M^{-1} \otimes \Lambda \otimes \mathcal{O}(-F) \to V \to M \otimes \mathcal{O}(F) \to 0$$

In the second case $h^0(V \otimes M^{-1}) = 2$ so it gets reduced to the first.

For dealing with the issue of uniqueness, we check that whenever $h^0(V \otimes M') > 0$ then either $(M')^{-1}$ factors through $M^{-1} \to V$ (and therefore p and the extension class are uniquely determined) or p belongs to the divisor defined by $\Lambda \otimes M \otimes M'$. This second case happens for a unique M'. So the space parametrising these extensions $\mathfrak{M}$ looks like a (ramified) double cover of $\mathfrak{M}_H(c_1, c_2)$. Off the ramification locus (i.e. when p is not in the divisor defined by $\Lambda \otimes M^2$), $h^0(V \otimes M) = 1$ and hence $\mathfrak{M}$ is properly a double cover of $\mathfrak{M}_H(c_1, c_2)$. In the ramification locus it might happen (and actually does) that $h^0(V \otimes M) = 2$, so the map $\mathfrak{M} \to \mathfrak{M}_H(c_1, c_2)$ sends some lines to points.

We also want to check that all the extensions are stable. Suppose that $h^0(V \otimes N^{-1}) > 0$ for some line bundle N. Then either $N \leq M^{-1}$ of p is

[1] If Z consists of two points, consider one in F and the other in D. If Z is supported in one point, consider $D + F$ with D and F intersecting in the supporting point of Z. In any case $\mathcal{O}(-D-F) \subset I_Z$.

contained in an effective divisor defined by $\Lambda \otimes M \otimes N^{-1}$. If the first case N is of negative degree. In the second, either $\Lambda \otimes M \otimes N^{-1} = \mathcal{O}(D)$ or $\Lambda \otimes M \otimes N^{-1} = \mathcal{O}(F)$, i.e. N is homologically trivial or homologically $L - F$. This proves the stability of V.

Now we want to construct a space $\mathfrak{M}$ parametrising the extensions. First we have

$$0 \to H^1(\Lambda^{-1} \otimes M^{-2}) \to \mathrm{Ext}^1(M \otimes \Lambda \otimes I_p, M^{-1}) \to \mathcal{O}_p = \mathbb{C} \to 0$$

So $\mathrm{Ext}^1 \cong \mathbb{C}^2$ and we get a fibration $\mathbb{CP}^1 \to \mathfrak{M} \xrightarrow{\pi} J \times S$, where J is the Jacobian of S (of divisors homologically equivalent to zero) and $\mathfrak{M} = \mathbb{P}(\mathcal{E})$ for $\mathcal{E} = \mathcal{E}\mathrm{xt}^1_{\pi_{J\times S}}(\mathcal{M} \otimes \pi_S^*\Lambda \otimes I_\Delta, \mathcal{M}^{-1})$, $\pi_S : (J \times S) \times S \to S$, $\pi_{J\times S} : (J \times S) \times S \to J \times S$ the projections, $\mathcal{M} \to J \times S$ the universal sheaf, $\Delta \subset S \times S$ the diagonal. $\mathfrak{M}$ is obviously of dimension 4 and in every fibre there is exactly one extension which is not a bundle (the one corresponding to the extension in $H^1(\Lambda^{-1} \otimes M^{-2})$, not giving a unit in $\mathcal{O}_p$). Such V lies in a diagram with $V^{\vee\vee} \in \mathfrak{M}_H(c_1, 0) = \mathrm{pt}$,

$$\begin{array}{ccccc}
M^{-1} & \longrightarrow & V & \longrightarrow & M \otimes \Lambda \otimes I_p \\
\| & & \downarrow & & \downarrow \\
M^{-1} & \longrightarrow & V^{\vee\vee} & \longrightarrow & M \otimes \Lambda \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{O}_p & = & \mathcal{O}_p
\end{array}$$

So $\mathfrak{M}$ is a compactification of (a double cover of) $\mathfrak{M}_H(c_1, c_2)$. The complement is a divisor $U \cong J \times S$. Projection onto the second factor gives a map $U \to S = \mathcal{M}_0^w \times S$ relating the two compactifications of $\mathfrak{M}_H(c_1, c_2)$.

The moduli space $\mathfrak{M}$ is generic since for all $V \in \mathfrak{M}$, V is simple as it is stable. So $H^0(\mathrm{End}_0(V)) = 0$ and hence $H^2(\mathrm{End}_0(V)) = H^0(\mathrm{End}_0(V) \otimes K_S) = 0$ as $-K_S$ is effective. $\square$

Let V be a extension like (4.1) and let $D = \mathrm{pt} \times \mathbb{CP}^1 \subset S$. Then restricting the exact sequence to D we have that

$$0 \to \mathcal{O}_D \to V|_D \to \mathcal{O}_D \to 0 \qquad \text{if } p \notin D$$

$$0 \to \mathcal{O}_D(1) \to V|_D \to \mathcal{O}_D(-1) \to 0 \qquad \text{if } p \in D$$

So $\mathcal{Z}_{c_1,c_2}(D, \mathcal{O}_D(-1)) = \{V/h^0(V \otimes \mathcal{O}_D(-1)) \neq 0\} = \{V/p \in D\}$. Therefore $\mu([D])^2 = 0$ and $\mu([D])^4 = 0$. So finally for $H \in \mathcal{C}_F$ we have

$$D_S^w(D^4) = \delta_S^{w,4}(\mathcal{C}_F, \mathcal{C}_0).$$

By corollary 3.14 with $\Sigma = D$, $\alpha = D$, $K_S = -2F$, $\zeta \cdot K_S = -2$, $\zeta^2 = -4$, $\zeta \cdot \alpha = -2$, $\epsilon(w) = -1$, we have that this term is -16.

4.4 Basic classes for elliptic surfaces

Here we will use the gluing formulae for a quicker computation of the invariants of elliptic surfaces. This is by now widely known, but it will be added because of the simplicity of our argument. Furthermore, in section 4.5 we will extend the results to cases in which the resulting manifold is not an elliptic surface (not even a complex manifold), we will allow log-transforms of even multiplicity and extend the results of [17], since we do not suppose the existence of cusp fibres. The tool we use is the following proposition from section 1.2.

Proposition 4.5 *There exists a vector space V of dimension 1 with the following properties*

1. *Let X be an open manifold with boundary $Y = \partial X = \mathbb{T}^2 \times \mathbb{S}^1$ and with $b^+(X) > 0$. Let $D \in H_2^R(X, Y)$ and $w \in H^2(X; \mathbb{Z})$ with $w|_Y$ odd in $H^2(Y; \mathbb{Z})$. Then we have defined an element $\phi^w(X, D) = \phi^w(X, e^{tD}) \in V[[t]]$.*

2. *Let X be a closed manifold with $b^+ \geq 1$ which can be written as $X = X_1 \cup_Y X_2$, and let $D \in H_2(X)$ and (w, T) an allowable pair. Write $D = D_1 + D_2$, $D_i \in H_2^R(X_i, Y)$, and $w = w_1 + w_2$, $w_i \in H^2(X_i; \mathbb{Z})$, then we have*

$$D_X^{(w,T)}(e^{tD}) = -\phi^{w_1}(X_1, D_1) \cdot \phi^{w_2}(X_2, D_2).$$

Proof. Recall proposition 4.1. Then we put

$$\phi^w(X, D) = \sum_{i \geq 0} \frac{\phi_i^w(X, D)}{i!} t^i$$

for an open manifold X. In the notation of subsection 1.2.2, we have that $V_i = V = HF_*(Y)$, for all i. When X is closed we have

$$D_X^{(w,T)}(e^{tD}) = \sum_{i \geq 0} \frac{D_X^{(w,T)}(D^i)}{i!} t^i.$$

So the result is equivalent to proposition 4.1. $\square$

The $K3$-surface has series $\mathbb{D}_X^w = (-1)^{w^2/2} e^{Q/2}$. Therefore proposition 1.11 yields

$$D_X^{(w,T)}(e^{tD}) = -e^{-Q(tD)/2},$$

for any allowable pair (w, T). We want to prove

Theorem 4.6 ([37][18][17]) *Let $X = S_n(m_1, \dots, m_r)$ be an elliptic surface with $b_1 = 0$, $p_g = n - 1 > 0$ and multiple fibres of multiplicities $1 \le m_1 \le \cdots \le m_r$ (we do not suppose them odd or coprime). Let F be the class of the fibre and $\alpha \in H_2(X)$. Then*

$$\mathbb{D}_X(\alpha) = e^{Q(\alpha)/2} \frac{\sinh^n(F \cdot \alpha)}{\sinh(\frac{1}{m_1} F \cdot \alpha) \sinh(\frac{1}{m_2} F \cdot \alpha)}. \tag{4.2}$$

For the sake of simplicity, we will prove theorem 4.6 when $r = 2$ (i.e. there are at most 2 multiple fibres). An application of proposition 1.11 produces the following

Proposition 4.7 *Let $X = S_n(m_1, m_2)$ be an elliptic surface with $b_1 = 0$, $p_g = n - 1 > 0$ and multiple fibres of multiplicities $1 \le m_1 \le m_2$. Let F be the class of the fibre and $\alpha \in H_2(X)$. Then the statement of theorem 4.6 is equivalent to either of the following:*

- *If $w \cdot F \equiv 1 \pmod 2$ (in particular m_1, m_2 are both odd) then*

$$D_X^{(w,T)}(e^\alpha) = i^{3(K \cdot w + n)} \, e^{-Q(\alpha)/2} \frac{\cos^n(F \cdot \alpha)}{\cos(\frac{1}{m_1} F \cdot \alpha) \cos(\frac{1}{m_2} F \cdot \alpha)}.$$

- *If both $w \cdot \frac{1}{m_i} F \equiv 1 \pmod 2$ and m_i are even, then*

$$D_X^{(w,T)}(e^\alpha) = i^{3(K \cdot w + n) + n} \, e^{-Q(\alpha)/2} \frac{\sin^n(F \cdot \alpha)}{\cos(\frac{1}{m_1} F \cdot \alpha) \cos(\frac{1}{m_2} F \cdot \alpha)}.$$

- *If $w \cdot \frac{1}{m_1} F \equiv 1 \pmod 2$, $w \cdot \frac{1}{m_2} F \equiv 0 \pmod 2$ (note that m_1 is even), then*

$$D_X^{(w,T)}(e^\alpha) = i^{3(K \cdot w + n) + n - 1} \, e^{-Q(\alpha)/2} \frac{\sin^n(F \cdot \alpha)}{\cos(\frac{1}{m_1} F \cdot \alpha) \sin(\frac{1}{m_2} F \cdot \alpha)}.$$

- *If both* $w \cdot \frac{1}{m_i} F \equiv 0 \pmod 2$, *then*

$$D_X^{(w,T)}(e^{\alpha}) = i^{3(K\cdot w+n)+n-2}\, e^{-Q(\alpha)/2} \frac{\sin^n(F\cdot\alpha)}{\sin(\frac{1}{m_1}F\cdot\alpha)\sin(\frac{1}{m_2}F\cdot\alpha)},$$

where K stands for the canonical class.

Proof of theorem 4.6. First we have $S_2 = B_1 \cup_Y B_1$. Choose any allowable pair (w, T) with $w_1 = w_2 \in H^2(B_1;\mathbb{Z})$ (i.e. w is symmetrical), and $D \in H_2(S_2)$ also symmetrical (i.e. $D = D_1 + D_2$, with $D_1 = D_2 \in H_2^R(B_1, \partial B_1)$). Then

$$D_{S_2}^{(w,T)}(e^{tD}) = -\phi^{w_1}(B_1, D_1)\cdot\phi^{w_2}(B_1, D_2) = -e^{-Q(tD)/2} = -e^{-Q(tD_1)/2}e^{-Q(tD_2)/2}$$

implies $\phi^{w_1}(B_1, D_1) = e^{-Q(tD_1)/2}$ (as there is an indeterminacy on signs, we fix them in such a way that we get a plus here). Now we have to compute $\mathbb{D}_{S_3}$. We know by section 3.2 that S_3 has an embedded sphere σ of self-intersection -3 which intersects the torus T representing the fibre $F = [T]$, in one positive point. Therefore the class $[\sigma] + 2F$ can be represented by a genus 2 embedded surface Σ of self-intersection 1. The canonical class is $K_{S_3} = F$ and so $w_2 \cdot \Sigma \equiv 1 \pmod 2$. By [23], S_3 has big monodromy with respect to K_{S_3} (being a minimal simply connected elliptic surface). So the invariants can be written as polynomials on Q and K and therefore the basic classes have to be multiples of K. Now by proposition 1.6, every basic class K_i is a lift to integral coefficients of w_2 and so $K_i = rF$ with r odd. Also $2 = 2g - 2 \geq \Sigma^2 + |K_i \cdot \Sigma| = 1 + |r|$ so $r = \pm 1$ and the basic classes are $\pm F$. We conclude $\mathbb{D}_{S_3} = c\, e^{Q/2} \sinh F$ (sinh since d_0 is odd) for some constant c, i.e. $D_{S_3}^{(w,T)} = c\, e^{-Q/2}\cos F$ (for $w \cdot F \equiv 1 \pmod 4$). From the computations in section 4.2 we have $c = 1$. So

$$\mathbb{D}_{S_3} = e^{Q/2}\sinh F.$$

We use $S_3 = B_1 \cup_Y B_2$, with (w, T) an allowable pair and $D \in H_2(S_3)$. Then using proposition 4.7 we get

- If $w \cdot F \equiv 1 \pmod 2$

$$D_{S_3}^{(w,T)}(e^{tD}) = -\phi^{w_1}(B_1, D_1)\cdot\phi^{w_2}(B_2, D_2) = i^{3(F\cdot w+3)}\, e^{-Q(tD)/2}\cos(F\cdot tD)$$

and $\phi^{w_2}(B_2, D_2) = i^{3F\cdot w_2 - 1}\, e^{-Q(tD_2)/2}\cos(F\cdot tD_2)$.

- If $w \cdot F \equiv 0 \pmod 2$

  $$D_{S_3}^{(w,T)}(e^{tD}) = -\phi^{w_1}(B_1, D_1) \cdot \phi^{w_2}(B_2, D_2) = i^{3(F\cdot w+3)+1}\, e^{-Q(tD)/2} \sin(F \cdot tD)$$

  and $\phi^{w_2}(B_2, D_2) = i^{3F\cdot w_2}\, e^{-Q(tD_2)/2} \sin(F \cdot tD_2)$.

But now $S_2 = B_2 \cup_Y A_1$, from where we get, using $w = \text{P.D.}[\Delta] \in H^2(A_1; \mathbb{Z})$ and $D \in H_2^R(A_1, \partial A_1)$,

$$\phi^w(A_1, D) = -e^{-Q(tD)/2} \frac{1}{\cos(F \cdot tD)}. \tag{4.3}$$

Analogously

$$\phi^{(2a+1)[\Delta]}(A_1, D) = -(-1)^a\, e^{-Q(tD)/2} \frac{1}{\cos(F \cdot tD)}.$$

Theorem 4.8 *The manifold $S = \mathbb{CP}^1 \times \mathbb{T}^2 = A_1 \cup_Y A_1$ has series*

$$D_S^{(\Delta,\mathbb{T}^2)}(e^{tD}) = -e^{-Q(tD)/2} \frac{1}{\cos^2(F \cdot tD)}.$$

Recall that the invariants are computed for suitable metrics (those giving big volume to $\mathbb{CP}^1$).

For the case $S_n = B_n \cup A_1 = B_{n-1} \cup B_1$, one proves by induction that, for $w \in H^2(B_n; \mathbb{Z})$ with $w|_Y = \mathbb{S}^1$,

$$\phi^w(B_n, e^{tD}) = (-1)^{n-1}\, e^{-Q(tD)/2} \cos^{n-1}(F \cdot tD),$$

so for $w \in H^2(S_n; \mathbb{Z})$ with $w \cdot F = 1$,

$$D_{S_n}^{(w,T)}(e^{tD}) = (-1)^{n-1}\, e^{-Q(tD)/2} \cos^{n-2}(F \cdot tD).$$

Also for the rational surface $S_1 = A_1 \cup B_1$,

$$D_{S_1}^{(w,T)}(e^{tD}) = e^{-Q(tD)/2} \frac{1}{\cos(F \cdot tD)}.$$

Using proposition 4.7, we have proved so far that the statement of theorem 4.6 is true for all elliptic surfaces S_n, $n \geq 2$. Using proposition 4.7 again, we know the shape of all invariants for S_n, $n \geq 2$ and all $w \in H^2(S_n; \mathbb{Z})$. Now for introducing multiple fibres, one has that $X = S_n(m) = B_n \cup_Y A_m$, with A_m in fact diffeomorphic to $D^2 \times \mathbb{T}^2$. We have a class $F \in H_2(B_n)$ which is equivalent, as an element of $H_2(S_n(m))$, to the class $m[\mathbb{T}^2] \in H_2(A_m) = H_2(\mathbb{T}^2)$. Call $f \in H_2(S_n(m))$ the class corresponding to $[\mathbb{T}^2] \in H_2(A_m)$. Now choose $w \in H^2(S_n(m))$ with $f \cdot w = 1$ and $D \in H_2(S_n(m))$ decomposed as $D = D_1 + \Delta$ for the horizontal class $\Delta \in H_2^R(A_m)$. Then there are two cases:

- If m is odd, we have $\phi^{w_1}(B_n, D_1) = -i^{3(K_{S_{n+1}}\cdot w+n+1)}\, e^{-Q(tD_1)/2} \cos^{n-1}(F\cdot tD_1)$ (use $S_{n+1} = B_n \cup B_1$ and knowledge about S_{n+1}). Now $K_{S_{n+1}} \cdot w = (n-1)m$ and $K_X \cdot w = (n-2)m + m - 1$,

$$D_X^{(w,T)}(e^\alpha) = i^{3(K_X\cdot w+n)}\, e^{-Q(\alpha)/2} \frac{\cos^{n-1}(F\cdot\alpha)}{\cos(\frac{1}{m}F\cdot\alpha)}.$$

- If m is even, then $\phi^{w_1}(B_n, D_1) = -i^{3(K_{S_{n+1}}\cdot w+n+1)+n-1}\, e^{-Q(tD_1)/2} \sin^{n-1}(F\cdot tD_1)$. Now $K_{S_{n+1}} \cdot w = (n-1)m$ and $K_X \cdot w = (n-2)m + m - 1$,

$$D_X^{(w,T)}(e^\alpha) = i^{3(K_X\cdot w+n)+n-1}\, e^{-Q(\alpha)/2} \frac{\sin^{n-1}(F\cdot\alpha)}{\cos(\frac{1}{m}F\cdot\alpha)},$$

and theorem 4.6 is true for all $S_n(m)$, $n \geq 2$. Analogously we get (for instance for m_1 and m_2 both odd and $w \cdot F \equiv 1 \pmod 2$)

$$D^{(w,T)}_{S_n(m_1,m_2)}(e^{tD}) = (-1)^{3(K\cdot w+n)}\, e^{-Q(tD)/2} \frac{\cos^n(F\cdot tD)}{\cos(\frac{1}{m_1}\cdot tD)\cos(\frac{1}{m_2}\cdot tD)}.$$

□

Remark 4.9 *The proof above also works for $n = 1$ and $w \cdot F \equiv 1 \pmod 2$. So the expression in the first item of proposition 4.7 is true when $n = 1$ (the metrics are always suitable in this case).*

Remark 4.10 *The first three terms of the power series of the first item in proposition 4.7 are the equalities of theorem 4.2. So this proof is a generalisation of that former proof.*

4.5 Fundamental results

In this section we try to gather together the most fundamental results on invariants of connected sums along a torus $\mathbb{T}^2$. They are indeed proved in the same fashion as the proofs of last section. The first one establishes that the manifolds we are dealing with are always of simple type. The second allows us to compute the basic classes of the connected sum along a torus. Finally, the third gives the invariants of a manifold in which we have performed a logarithmic transform (this was carried out by Fintushel and Stern [17] but only under the assumption of the existence of a cusp fibre).

Theorem 4.11 *Let X be a manifold with $b^+ > 1$ and $b_1 = 0$ containing an embedded torus $\mathbb{T}^2 \subset X$ representing an odd homology class and of self-intersection zero. Then X is of simple type. When $b^+ = 1$, X is of w-simple type with respect to the invariants defined by $[\mathbb{T}^2]$, i.e. $D^w_{X,[\mathbb{T}^2]}$, for any $w \in H^2(X;\mathbb{Z})$ with $w \cdot \mathbb{T}^2 \equiv 1 \pmod 2$.*

Proof. Put A for a tubular neighbourhood of $F = \mathbb{T}^2$. Decompose $X = X^o \cup_Y A$. Then for any $D \in H_2(X)$ we put $D = D_1 + a\Delta$, with $a = F \cdot D$. Now proposition 4.5 and equation (4.3) yield

$$D^{(w,F)}_{X,[F]}(e^{tD}) = \phi^w(X_1, D_1) \cdot \frac{1}{\cos(tF \cdot D)},$$

$$D^{(w,F)}_{X,[F]}((x^2-4)e^{tD}) = -\phi^w(X_1, D_1)\phi^w(A, (x^2-4)e^{t\,a\Delta}).$$

We have examples of simple type fulfilling the conditions of the statement (for instance the K3 surface) with non-trivial invariants. This forces $\phi^w(A, (x^2 - 4)e^{t\,a\Delta}) = 0$, whence the result. $\square$

Theorem 4.12 *Let $\bar{X}_1$ and $\bar{X}_2$ be two four-manifolds with $b^+ \geq 1$ and containing embedded tori $F_i \subset \bar{X}_i$ representing odd homology classes and of self-intersection zero . Let $X = \bar{X}_1 \#_F \bar{X}_2$ be their connected sum along F and call $F_i \in H_2(X)$ the class induced by $F_i \in H_2(\bar{X}_i)$. Consider $w \in H^2(X;\mathbb{Z})$ with $F \cdot w \equiv 1 \pmod 2$. Then for $D \in H_2(X)$ with $D|_Y = a\mathbb{S}^1 \in H_1(Y)$, we put $D = D_1 + D_2$ and $\bar{D}_i = D_i + a\Delta$, so we have*

$$D^{(w,F)}_{X,[F]}(e^{tD}) = -D^{(w,F)}_{\bar{X}_1,[F_1]}(e^{t\bar{D}_1}) \cdot D^{(w,F)}_{\bar{X}_2,[F_2]}(e^{t\bar{D}_2}) \cdot \cos(tF_1 \cdot \bar{D}_1)\cos(tF_2 \cdot \bar{D}_2).$$

Proof. From proposition 4.5 and equation (4.3) we have

$$D^{(w,F)}_{\bar{X}_1,[F_1]}(e^{t\bar{D}_1}) = \phi^w(X_1, D_1) \cdot \frac{1}{\cos(tF_1 \cdot \bar{D}_1)}$$

and analogously for $\bar{X}_2$. The result is immediate from proposition 4.5 (recall from subsection 2.1.3 that $b^+(X) \geq 1$). $\square$

Theorem 4.13 *Let $\bar{X}_1$ and $\bar{X}_2$ be two four-manifolds with $b^+ > 1$ and $b_1 = 0$ and containing embedded tori $F_i \subset \bar{X}_i$ representing odd homology classes and of self-intersection zero. By theorem 4.11, they are of simple type. Let $\mathbb{D}_{\bar{X}_1}$ and*

$\mathbb{D}_{\bar{X}_2}$ *be their invariants, respectively. Then* $X = \bar{X}_1 \#_F \bar{X}_2$ *is of simple type and calling* $F_i \in H_2(X)$ *the class induced by* $F_i \in H_2(\bar{X}_i)$*, one has for* $D \in H_2(X)$ *with* $D|_Y = a\mathbb{S}^1 \in H_1(Y)$ *(put* $D = D_1 + D_2$ *and* $\bar{D}_i = D_i + a\Delta$*),*

$$\mathbb{D}_X(tD) = \mathbb{D}_{\bar{X}_1}(t\bar{D}_1) \cdot \mathbb{D}_{\bar{X}_2}(t\bar{D}_2) \cdot \sinh(tF_1 \cdot \bar{D}_1) \sinh(tF_2 \cdot \bar{D}_2).$$

Proof. Note first that $b_1(X) = 0$, $b^+(X) > 1$ from section 2.3. This is a formal consequence of the previous theorem. We write $\mathbb{D}_{\bar{X}_1} = e^{Q/2} \sum a_i e^{K_i}$, so $D^{(w,F)}_{\bar{X}_1} = i^{-d_0(\bar{X}_1)} e^{-Q/2} \sum (-1)^{\frac{K_i \cdot w + w^2}{2}} a_i e^{iK_i}$, and analogously for $\bar{X}_2$ with $\mathbb{D}_{\bar{X}_2} = e^{Q/2} \sum b_j e^{L_j}$. From the formula in the previous theorem

$$\mathbb{D}_X(tD) = e^{Q(tD)/2} \sum \epsilon_{ij} \frac{1}{4} a_i b_j e^{t(K_i + L_j \pm F_1 \pm F_2) \cdot D}$$

with

$$\epsilon_{ij} = -i^{d_0(X) - d_0(\bar{X}_1) - d_0(\bar{X}_2)} (-1)^{\frac{K_i \cdot w_1 + w_1^2}{2} + \frac{L_j \cdot w_2 + w_2^2}{2} - \frac{(K_i + L_j \pm F_1 \pm F_2) \cdot w + w^2}{2}}$$

which is $-(-1)^{\frac{(\pm F_1 \pm F_2) \cdot w}{2}}$, since $w_1^2 + w_2^2 \equiv w^2 \pmod 2$ and $-\frac{3}{2}(1 + b^+)$ behaves additively for connected sums along tori. (In the expression of $\mathbb{D}_X$ there might be in principle different basic classes corresponding to the same exponential $e^{t(K_i + L_j \pm F_1 \pm F_2) \cdot D}$, but all of them contribute with the same sign to ϵ_{ij}). $\square$

If we want to deal with other homology classes $D \in H_2(X)$, we should consider closings of X_i with B_1 instead of A.

Theorem 4.14 *Let* $\bar{X}_1$ *and* $\bar{X}_2$ *be two four-manifolds with* $b^+ \geq 1$ *and* $b_1 = 0$ *and containing embedded tori* $F_i \subset \bar{X}_i$ *representing odd homology classes and of self-intersection zero. Let* $\bar{X}_i^{(1)} = \bar{X}_i \#_{F_i} S_1$ *and let* $\mathbb{D}_{\bar{X}_1^{(1)}}$ *and* $\mathbb{D}_{\bar{X}_2^{(1)}}$ *be their invariants, respectively. Then* $X = \bar{X}_1 \#_F \bar{X}_2$ *is of simple type and, calling* $F_i \in H_2(X)$ *the class induced by* $F_i \in H_2(\bar{X}_i)$*, one has for* $D \in H_2(X)$ *(put* $D = D_1 + D_2$ *and* $\bar{D}_i \in H_2(\bar{X}_i^{(1)})$ *restricting to* D_i *in* X_i *with* $\bar{D}_i^2 = D_i^2$*),*

$$\mathbb{D}_X(tD) = \mathbb{D}_{\bar{X}_1^{(1)}}(t\bar{D}_1) \cdot \mathbb{D}_{\bar{X}_2^{(1)}}(t\bar{D}_2).$$

Proof. Note that X and both $\bar{X}_i^{(1)}$ have $b_1 = 0$, $b^+ > 1$ and are of simple type. We want to get

$$D^{(w,F)}_X(e^{tD}) = -D^{(w,F)}_{\bar{X}_1^{(1)}}(e^{t\bar{D}_1}) \cdot D^{(w,F)}_{\bar{X}_2^{(1)}}(e^{t\bar{D}_2})$$

and work then as in theorem 4.13. This is analogous to theorem 4.12 but now

$$D^{(w,F)}_{\bar{X}_1^{(1)}}(e^{t\bar{D}_1}) = -\phi^w(X_1, D_1).$$

□

Theorem 4.15 *Let X be a manifold with $b^+ > 1$ and $b_1 = 0$ containing an embedded torus $\mathbb{T}^2 \hookrightarrow X$ representing an odd homology class F of self-intersection zero. Perform a logarithmic transform of degree p on F and call $X(p)$ the resulting manifold. Suppose $b_1(X(p)) = 0$. Then for $D \in H_2(X(p))$ and $\bar{D} \in H_2(X^{(1)})$ agreeing on X^o, and with $D^2 = \bar{D}^2$,*

$$\mathbb{D}_{X(p)}(tD) = \mathbb{D}_{X^{(1)}}(t\bar{D}) \cdot \frac{1}{\sinh(tD \cdot \frac{1}{p}F)},$$

where $F \in H_2(X(p);\mathbb{Z})$ denotes the class induced by $F \in H_2(X;\mathbb{Z})$ on $X(p)$.

Proof. Straightforward. □

Application to prove a result of [30]

Consider a Dolgachev surface $\bar{X}_1$ with one multiple fibre of (odd) multiplicity p_1, i.e. $\bar{X}_1 = S_1(p_1)$, which is diffeomorphic to the rational surface S_1. Let F be the homology class of the fibre, which is an odd class. Consider $\bar{X}_2$ another Dolgachev surface with one multiple fibre of (odd) multiplicity p_2, and let F be the homology class of the fibre. Then we can consider the connected sum $\bar{X}_1 \#_F \bar{X}_2$, under different identifications. We have, for $(\frac{1}{p_i}F) \cdot w = 2a + 1$,

$$D^{(w,T)}_{\bar{X}_i}(e^\alpha) = (-1)^a \, e^{-Q(\alpha)/2} \frac{1}{\cos(\frac{1}{p_i}F \cdot \alpha)}.$$

1. We glue with the preferred holomorphic identification of proposition 2.9 and the discussion preceding it. Then the classes F from both pieces correspond canonically and give the class of the fibre of the natural fibration for $X = \bar{X}_1 \#_F \bar{X}_2 = S_2(p_1, p_2)$, which is the elliptic surface with $p_g = 1$ and two multiple fibres of multiplicities p_1 and p_2. The invariants for X are given by (for $F \cdot w \equiv 1 \pmod 2$)

$$D^{(w,T)}_X(e^\alpha) = (-1)^{\frac{p_1+p_2}{2}} \, e^{-Q(\alpha)/2} \frac{\cos^2(F \cdot \alpha)}{\cos(\frac{1}{p_1}F \cdot \alpha)\cos(\frac{1}{p_2}F \cdot \alpha)}.$$

2. We glue with the identification which is the preferred identification composed with a twist in the $\mathbb{S}^1$ factor. More explicitly, if we identify $F \times D^2$ with the tubular neighbourhood of the fibre F which we are using for gluing, in such a way that the holomorphic projection is the projection in the second factor, then the gluing that we will use is $\phi : \partial(F \times D^2) \to \overline{\partial(\bar{X}_2 - F \times D^2)}$, inducing ϕ_* in $H_1(F \times \mathbb{S}^1)$ with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

This gluing is considered in [29] and [30]. Consider $\alpha, \beta \in H_1(F; \mathbb{Z})$ some basis such that $\alpha \cdot \beta = 1$, $\phi_*(\alpha) = \alpha$ and $\phi_*(\beta) = \beta + [\mathbb{S}^1]$. Then the class F in $\bar{X}_1$ corresponds to the class $F - T_\alpha$ in $\bar{X}_2$, with $T_\alpha = \alpha \otimes [\mathbb{S}^1] \in H_2(F \times \mathbb{S}^1)$. Call $X' = X_1 \cup_\phi X_2$, and let $F_i \in H_2(X')$ be the class induced by the class F in $\bar{X}_i$, for $i = 1, 2$. Then (for $F \cdot w \equiv 1 \pmod 2$)

$$D_X^{(w,T)}(e^\alpha) = (-1)^{\frac{p_1+p_2}{2}} e^{-Q(\alpha)/2} \frac{\cos(F_1 \cdot \alpha)\cos(F_2 \cdot \alpha)}{\cos(\frac{1}{p_1} F_1 \cdot \alpha)\cos(\frac{1}{p_2} F_2 \cdot \alpha)}.$$

In the first case all the basic classes span a one-dimensional subspace of $H^2(X)$ and in the second they span a two-dimensional subspace. Therefore the manifolds X and X' cannot be diffeomorphic (although they are homeomorphic) which is the main result of [30]. X' is a homotopy $K3$-surface and by proposition 2.12, it is a symplectic manifold.

Chapter 5

Connected sums along Riemann surfaces of genus 2

5.1 Introduction

In this chapter we carry on studying the behaviour of the Donaldson invariants under connected sums along Riemann surfaces in the next natural case, i.e. when the genus of the surface is 2. For this case, the computations can then be carried out quite explicitly.

We recall first the Floer homology of $Y = \Sigma \times \mathbb{S}^1$ when $g = 2$ from section 1.2 (see proposition 1.19). Let P_Y be an $U(2)$-bundle whose associated $SO(3)$-bundle has $w_2 \in H^2(Y;\mathbb{Z}_2)$ dual to the class $[\mathbb{S}^1]$. Then there is the isomorphism $HF^*(\Sigma \times \mathbb{S}^1) \stackrel{\sim}{\to} QH^*(M_\Sigma^{\text{odd}})$ with the quantum cohomology of M_Σ^{odd}, the moduli space of rank-2 stable bundles over Σ with odd determinant. As usual, the universal bundle yields a map $\tilde{\mu} : H_*(\Sigma) \to H^{4-*}(M_\Sigma^{\text{odd}})$ given by slanting with $-\frac{1}{4}$ times the first Pontrjagin class. In this chapter it will be necessary to make use of Conjecture 1.22 which states that this corresponds with the μ map in Floer homology for $[\Sigma] \in H_2(\Sigma)$ and elements $\alpha \in H_1(\Sigma)$, but there will be a correction term for $x \in H_0(\Sigma)$ (see remark 5.3). Nonetheless some of the results can be proved independently of the aforementioned conjecture, and we will give an alternative proof in such cases.

The quantum cohomology of M_Σ^{odd} in the case of Σ of genus 2 was first computed by Donaldson [9]. He uses the fact that for Σ hyperelliptic there is an explicit description of M_Σ^{odd} as an algebraic manifold [4], which, in the case of genus 2, gives that M_Σ^{odd} is the intersection of two quadrics in $\mathbb{CP}^5$ [45]. From this description one can compute the space of lines (rational curves of degree 1) in M_Σ^{odd}. This turns out to be the only necessary data for finding the quantum product. M_Σ^{odd} has (real) dimension 6 and has (integral) cohomology equal to $\mathbb{Z}$ in degrees 0, 2, 4 and 6. The generators are 1, h, l and p which correspond, in the description of M_Σ^{odd} as the intersection of two quadrics in $\mathbb{CP}^5$, to the fundamental class, a plane, a line and a point. The map μ gives an isomorphism $\mu : H_1(\Sigma) \to H^3(M_\Sigma^{\mathrm{odd}})$, describing the other non-zero bit of the cohomology of M_Σ^{odd}. So far the description of $QH^*(M_\Sigma^{\mathrm{odd}})$ as abelian group. Regarding the multiplicative structure, we have the following (see [9])

$$\begin{aligned} h * h &= 4l + 4 \\ h * h * h &= 4p + 12h \\ h * h * h * h &= 16h * h \\ \mu(\alpha) * \mu(\beta) &= (\alpha \cdot \beta)(\frac{1}{4}h * h * h - 4h) \end{aligned} \tag{5.1}$$

In general, we will drop the $*$ symbol for denoting the quantum product. $QH^*(M_\Sigma^{\mathrm{odd}})$ has a natural $\mathbb{Z}/4$-grading given by reducing the $\mathbb{Z}$-grading above. The standard basis is given by $e_i = h^i$, $0 \leq i \leq 3$, and elements $\mu(\alpha_j)$, where $\{\alpha_j\}$ is a basis of $H_1(\Sigma; \mathbb{Z})$. Note that the matrix $< e_i, e_j >$ is

$$4 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 1 & 0 & 16 & 0 \end{pmatrix}$$

The pairings $< \mu(\alpha), e_i >$ are all zero, so $QH^3(M_\Sigma^{\mathrm{odd}})$ is orthogonal to the "even" part of $QH^*(M_\Sigma^{\mathrm{odd}})$. This is an important remark since it will allow us to ignore the "odd" part in later computations.

We recall our set up. Let $\bar{X}_1$ and $\bar{X}_2$ be two manifolds with $\Sigma_i \subset \bar{X}_i$ embedded surfaces of the same genus $g = 2$ and self-intersection zero. Remove a tubular neighbourhood of Σ_i and call the resulting manifold X_i (it has boundary $Y = \Sigma \times \mathbb{S}^1$). The (closure of the) tubular neighbourhood removed is always diffeomorphic to $A = \Sigma \times D^2$. Consider some identification ϕ for Y (see definition 2.5) and let $X = X(\phi) = X_1 \cup_\phi X_2$ be the connected sum of $\bar{X}_1$ and $\bar{X}_2$ along Σ.

Our starting hypotheses are that $b_1 = 0$ for both $\bar{X}_i$ and that Σ_i are odd in $H_2(\bar{X}_i; \mathbb{Z})/\text{torsion}$ (equivalently, the homology class $[\Sigma_i]$ is an odd multiple of a non-torsion primitive class). Therefore there exist $w_i \in H^2(\bar{X}_i; \mathbb{Z})$ with

$w_i \cdot \Sigma_i \equiv 1 \pmod 2$. We fix w's in all the manifolds involved ($\bar{X}_1$, $\bar{X}_2$ and X) once and for all in a compatible way (i.e. the restriction of w to $X_i \subset X$ coincides with the restriction of w_i to $X_i \subset \bar{X}_i$) and such that $w_i \cdot \Sigma_i \equiv 1 \pmod 2$. We drop the subindices, so we will not differentiate these w's, since the context makes always clear to which manifold they refer.

The series $\mathcal{D}^w_X(\alpha)$ is determined by its action on:

1. $\alpha = \alpha_1 + \alpha_2 \in H_2(X_1) \oplus H_2(X_2) \subset H_2(X)$. These elements will be called classes of the **first type.**

2. elements $D \in H_2(X)$ such that $D = D_1 + D_2$ with $\partial D_1 = -\partial D_2$ being a multiple of $[\mathbb{S}^1] \in H_1(Y)$. These elements will be called classes of the **second type.**

3. elements D_γ such that $D_\gamma = D_1 + D_2$ with $\partial D_1 = -\partial D_2$ defining a class $0 \neq \gamma \in H_1(Y)$ which is not a multiple of $[\mathbb{S}^1]$. These elements will be called classes of the **third type.**

For studying the behaviour on classes of the first type we will use the Floer homology groups $HF_*(Y)$ explained in subsection 1.2.1. When the classes are of the second and third type we must use the extension of the Floer theory provided by the Fukaya-Floer homology groups $HFF_*(Y, \gamma)$, for loops γ, as explained in subsection 1.2.2.

This chapter is a bit long, but it is naturally divided into sections corresponding to the study of the invariants on classes of first, second and third type. The fundamental results of each of these sections are theorems 5.6, 5.17 and 5.23, respectively. All throughout sections 5.2 and 5.3 we will be using Conjecture 1.22 (in section 5.3 we use also an extension of it, Conjecture 5.11), and so the results are in principle dependent on it (except obviously proposition 5.5). In subsection 5.3.2 we give an alternative way of proving theorem 5.17, completely independent of Conjectures 1.22 and 5.11. So theorems 5.17 and 5.6 (which is a consequence of the former) and corollary 5.9 are proved and not conjectural. Also corollary 5.15 is independent of the conjectures. Finally, our proof of theorem 5.16, about the finite type condition of the manifolds we are dealing with, does depend on Conjecture 5.11.

On the other hand, section 5.4 depends on Conjecture 1.22, since the computation of the Fukaya-Floer homology groups $HFF_*(Y, \gamma)$ is based on our description of $\mu(\gamma)$, and on Conjecture 5.20, which is another extension of the former (nonetheless, Conjecture 5.20 could have been avoided with arguments similar to those of subsection 5.3.2). So the main theorem in section 5.4, theorem 5.23, remains as conjectural. We use this theorem to produce some nice explicit results about basic classes, corollaries 5.9, 5.25 and 5.26.

Let us remark that the three conjectures we are using in this chapter, Conjectures 1.22, 5.11 and 5.20, refer to the action of $\mu(\Sigma)$ on $HF_*(Y)$, $HFF_*(Y, \mathbb{S}^1)$ and $HFF_*(Y, \gamma)$ (with γ representing a homology class not a multiple of $[\mathbb{S}^1]$), respectively.

We finalise with a section in which we try to give some ideas for generalising these results to the case $g > 2$.

5.2 Classes of the first type

Here we suppose $g \geq 2$, since the remarks we are going to use now are not specific for genus 2. We deal with the invariants for classes $\alpha \in H_2(X_1) \subset H_2(X)$ and $\beta \in H_2(X_2) \subset H_2(X)$. We are going to use Conjecture 1.22 to make explicit computations.

Fix a set $\{z_l\}$ of elements of the shape $z_l = \Sigma^a x^b \gamma_1 \cdots \gamma_r \in \mathbb{A}(\Sigma)$, where x corresponds to the class of a point and $\gamma_i \in H_1(\Sigma)$, with the property that the corresponding $e_l = \mu(\Sigma)^a \mu(x)^b \mu(\gamma_1) \cdots \mu(\gamma_r)$ form a basis for $HF_*(M_\Sigma^{\text{odd}})$ (quantum multiplication is understood throughout). Then we have the following

Lemma 5.1 *$\phi^w(A, z_l) = e_l$, after possible renormalisation (see proof of theorem for explanation of this word).*

Proof. From Conjecture 1.22, $\phi^w(A, z_l) = e_l \phi^w(A, 1)$. It only remains to check $\phi^w(A, 1) = 1$. Put $e_a = \phi^w(A, 1)$. Now

$$D^{(w,\Sigma)}_{\mathbb{CP}^1 \times \Sigma}(z_l) = < e_a e_l e_a, 1 > = < e_a^2 e_l, 1 > .$$

Making z_l run through the basis, the left hand side is zero (by dimensionality) except for the element of degree $6g - 6$, in which case it is equal to $< e_l, 1 >$ (since the corresponding moduli space is isomorphic to M_Σ^{odd}). The conclusion is that $e_a^2 = 1$. If $e_a = 1$ we have finished, if not we have to renormalise defining

$$\hat{\phi}^w(X, D) = \phi^w(X, z_a D)$$

(obvious meaning for z_a). This does not modify any of the arguments of the chapter and makes $\hat{\phi}^w(A, z_l) = e_l$. $\square$

For the open manifold X_1 we write $\phi^w(X_1,z)=\sum<\phi^w(X_1,z),e_l>e_l^*$, where $\{e_l^*\}$ is the dual basis of $\{e_l\}$. Transferring the cycles contained in z_l to X_1 we get, for $z\in\mathbb{A}(X_1)$,

$$\phi^w(X_1,z)=\sum_l<\phi^w(X_1,z),\phi^w(A,z_l)>e_l^*=$$

$$=\sum_l<\phi^w(X_1,z_lz),\phi^w(A,1)>e_l^*=\sum_l D^{(w,\Sigma)}_{\bar{X}_1}(z_lz)e_l^*.$$

Then

$$D^{(w,\Sigma)}_X(e^{t(\alpha+\beta)})=<\phi^w(X_1,e^{t\alpha}),\phi^w(X_2,e^{t\beta})>=$$
$$=\sum_{l,m}D^{(w,\Sigma)}_{\bar{X}_1}(e^{t\alpha}z_l)D^{(w,\Sigma)}_{\bar{X}_2}(e^{t\beta}z_m)<e_l^*,e_m^*>. \tag{5.2}$$

Therefore the series for X is determined by the series for both sides. We can work out $\mathcal{D}_X(\alpha+\beta)$ adding the class of the point to either side, and we get

$$\mathcal{D}^{(w,\Sigma)}_X(\alpha+\beta)=\sum_{l,m}D^{(w,\Sigma)}_{\bar{X}_1}(e^{t\alpha+\lambda x}z_l)D^{(w,\Sigma)}_{\bar{X}_2}(e^{t\beta}z_m)<e_l^*,e_m^*>.$$

Note that on cycles of the type $T_\gamma=\gamma\times\mathbb{S}^1\subset\Sigma\times\mathbb{S}^1$ the series $\phi^w(X_1,e^{tT_\gamma})$ is constant (since T_γ represents the zero homology class in $\bar{X}_i$). This agrees with the case when X is of simple type, where tori of self-intersection zero have intersection zero with all basic classes (and so the series is constant on such tori). Before going further, let us prove a simple lemma.

Lemma 5.2 *For $g=2$ one has $\mu(\Sigma)=\frac{1}{2}h$ and $\mu(x)=\frac{1}{4}h^2-2$. In particular $\mu(x)^2-4=0$.*

Proof. Take X to be a $K3$ surface blown-up in two points. Consider a tight surface S of self-intersection 2 (and therefore of genus 2) in the $K3$ surface (whose existence is guaranteed by [38]). Let E_1 and E_2 be the exceptional divisors in X and let Σ be the proper transform of the tight surface, i.e. $\Sigma=S-E_1-E_2$. Put $w=E_1$, so $w\cdot\Sigma=1$ and Σ has genus 2 and self-intersection zero. X is of simple type and $\mathbb{D}^w_X=-e^{Q/2}\cosh E_2\sinh E_1$. So $\mathbb{D}^w_X(t\Sigma)=-\cosh t\sinh t$. The moduli spaces of connections on X are of dimensions $2d\equiv 2d_0\equiv 6 \pmod 8$. From these remarks we have

$$D^w_X(\Sigma^{3+4n})=-2^{2+4n}$$

Write $X=X_1\cup_Y A$. Then $\mu(\Sigma)$ is a multiple of h, say ah. Now, for $n\equiv 1 \pmod 4$,

$$D^w_X(\Sigma^{n+6})=<\phi^w_n(X_1,\Sigma^n),(ah)^6>=<\phi^w_n(X_1,\Sigma^n),a^6\,16^2h^2>=a^4\,16^2D^w_X(\Sigma^{n+2})$$

from where $a = \pm\frac{1}{2}$. There is an indeterminacy in the sign coming from choosing an orientation of generators of $CF_*(\Sigma \times \mathbb{S}^1)$. We make choices such that $\mu(\Sigma) = \frac{1}{2}h$. For computing $\mu(x)$, we put $\mu(x) = ah^2 + b$. From proposition 1.11 and remark 1.12 we have

$$D_X^{(w,\Sigma)}(e^\alpha) = -e^{Q(\alpha)/2}\frac{1}{2}\sinh((E_1+E_2)\cdot\alpha) + e^{-Q(\alpha)/2}\frac{1}{2}\sin((E_1-E_2)\cdot\alpha)$$

$$D_X^{(w,\Sigma)}(xe^\alpha) = -2e^{Q(\alpha)/2}\frac{1}{2}\sinh((E_1+E_2)\cdot\alpha) - 2e^{-Q(\alpha)/2}\frac{1}{2}\sin((E_1-E_2)\cdot\alpha)$$

Now $D_X^{(w,\Sigma)}(xe^{t\Sigma}) =< ah^2 + b, \phi^w(X_1, e^{t\Sigma}) >= aD_X^{(w,\Sigma)}(4\Sigma^2 e^{t\Sigma}) + bD_X^{(w,\Sigma)}(e^{t\Sigma})$ yields the equation

$$-\sinh(2t) = 4a\frac{d^2}{dt^2}(-\frac{1}{2}\sinh(2t)) + b(-\frac{1}{2}\sinh(2t)),$$

from where $16a + b = 2$. Now put $\beta = -E_1 + E_2$, so $\beta\cdot\Sigma = 0$ and use the same argument with $\alpha = t\beta + s\Sigma$. This yields $b = -2$, so $a = \frac{1}{4}$. Note that $\mu(x)^2 - 4 = (16a^2 - 4a)h^2 = 0$. $\square$

Remark 5.3 *Note that $\mu(\Sigma) = \tilde{\mu}(\Sigma) \in QH^2(M_\Sigma^{odd})$. Instead $\mu(x) = l - 1$, so $\tilde{\mu}(x) = l \in QH^4(M_\Sigma^{odd})$ and there is a correction term $-1 \in QH^0(M_\Sigma^{odd})$ (we recall that $l = \frac{1}{4}h \cup h$ is the generator of $QH^4(M_\Sigma^{odd})$).*

Corollary 5.4 *Let $\bar{X}_1$ have $b_1 = 0$, $b^+ > 1$ and an embedded surface Σ of genus 2 and self-intersection zero which represents an odd homology class (as supposed so far). Let $w \in H^2(\bar{X}_1;\mathbb{Z})$ with $w\cdot\Sigma \equiv 1 \pmod 2$. Then $D^w_{\bar{X}_1}(x^2 z) = 4D^w_{\bar{X}_1}(z)$ for every $z \in \mathbb{A}(X_1) \subset \mathbb{A}(\bar{X}_1)$. The same statement is true for $b^+ = 1$ if we use the invariants with respect to $[\Sigma]$.*

Proof. Let $2d$ be the degree of z. Suppose for instance that $w\cdot\Sigma = 1$. If $2d \not\equiv 2d_0 \pmod 8$, then both sides vanish. Otherwise

$$D^w_{\bar{X}_1}(x^2 z) =< \phi_d^{w_1}(X_1, z), \phi^w(A, x^2) >=< \phi_d^{w_1}(X_1, z), \mu(x)^2 >= 4D^w_{\bar{X}_1}(z).$$

$\square$

We have a simple result about the simple type condition of some of the manifolds we are dealing with from chapter 4.

Proposition 5.5 *Let $\bar{X}_1$, $\bar{X}_2$ have $H_1(\bar{X}_i;\mathbb{Z})=0$, $b^+>0$ and embedded surfaces Σ_i of genus 2 and self-intersection zero which represent odd homology classes. Then $X=\bar{X}_1\#_\Sigma \bar{X}_2$ is of simple type.*

Proof. The manifold X has $b_1=0$ and $b^+>1$. It also contains tori $T_\alpha \in H_2(Y;\mathbb{Z})\subset H_2(X;\mathbb{Z})$ of self-intersection zero and representing an odd homology class (use remark 2.13 with $L_i=H_1(\Sigma;\mathbb{Z})$). By theorem 4.11, X is of simple type. □

In theorem 5.16 we finalise the proof that a manifold $\bar{X}_1$ with $b_1=0$ and an embedded surface of genus 2 and self-intersection zero representing an odd homology class is always of finite type. For the moment, let us suppose that both $\bar{X}_i$ have $b_1=0$, $b^+>0$ and are of simple type. Then we write

$$\mathbb{D}^w_{\bar{X}_1}(\alpha)=e^{Q(\alpha)/2}\sum a_{i,w}e^{K_i\cdot\alpha}, \qquad \mathbb{D}^w_{\bar{X}_2}(\beta)=e^{Q(\beta)/2}\sum b_{j,w}e^{L_j\cdot\beta}.$$

Recall that $a_{i,w}=(-1)^{\frac{w^2+K_i\cdot w}{2}}a_i$. First note $D^w_{\bar{X}_1}((1+\frac{x}{2}e^\alpha\Sigma^a)=e^{Q(\alpha)/2}\sum a_i(K_i\cdot\Sigma)^a\cdot e^{K_i\cdot\alpha}$. Also

$$D^{(w,\Sigma)}_{\bar{X}_1}(e^\alpha)=e^{\pm Q(\alpha)/2}\sum \tilde{a}_{i,w}e^{\tilde{K}_i\cdot\alpha}, \qquad D^{(w,\Sigma)}_{\bar{X}_2}(e^\beta)=e^{\pm Q(\beta)/2}\sum \tilde{b}_{j,w}e^{\tilde{L}_j\cdot\beta}.$$

where

$$(\tilde{a}_{i,w},\tilde{K}_i)=\begin{cases}(a_{i,w},K_i) & \text{if } K_i\cdot\Sigma\equiv 2 \pmod 4\\ (i^{-d_0}a_{i,w}, i\,K_i) & \text{if } K_i\cdot\Sigma\equiv 0 \pmod 4\end{cases}$$

and analogously for $(\tilde{b}_{j,w},\tilde{L}_j)$. The sign of the exponent is $+$ when $K_i\cdot\Sigma\equiv 2 \pmod 4$ and $-$ otherwise. Now from formula (5.2), $D^{(w,\Sigma)}_X(e^{\alpha+\beta})$ is equal to

$$e^{\pm Q(\alpha)/2\pm Q(\beta)/2}\sum_{\substack{i,j,l,m\\ r=r'=0}} \tilde{a}_{i,w}\tilde{b}_{j,w}\, e^{\tilde{K}_i\cdot\alpha+\tilde{L}_j\cdot\beta}((\tilde{K}_i\cdot\Sigma)^a(\pm 2)^b(\tilde{L}_j\cdot\Sigma)^{a'}(\pm 2)^{b'} < e^*_l, e^*_m >) \tag{5.3}$$

(recall $e_l=\mu(\Sigma)^a\mu(x)^b\mu(\alpha_1)\cdots\mu(\alpha_r)$). Here a, b and r correspond to l and a', b' and r' correspond to m. It must be $r=r'=0$ since $b_1=0$. The ± 2 are due to remark 1.12. Using $\alpha=t\Sigma$, $\beta=(1-t)\Sigma$ we see that $K_i\cdot\Sigma=L_j\cdot\Sigma$ for non-zero summands. Moreover, from proposition 1.6, this number is even and less or equal than $2g-2$ in absolute value. Suppose now that X is of simple type. Then $\mathbb{D}^w_X(e^{\alpha+\beta})$ equals

$$e^{Q(\alpha+\beta)/2}\sum_{\substack{l,m\\ r=r'=0\\ K_i\cdot\Sigma=L_j\cdot\Sigma}} c_{ij,w}\, e^{K_i\cdot\alpha+L_j\cdot\beta}((\tilde{K}_i\cdot\Sigma)^a(\pm 2)^b(\tilde{L}_j\cdot\Sigma)^{a'}(\pm 2)^{b'} < e^*_l, e^*_m >),$$

where $c_{ij,w} = a_{i,w} b_{j,w}$ if $K_i \cdot \Sigma \equiv 2 \pmod 4$. If $K_i \cdot \Sigma \equiv 0 \pmod 4$ then $c_{ij,w} = i^{-d_0(\bar{X}_1) - d_0(\bar{X}_2) + d_0(X)} a_{i,w} b_{j,w}$.

The genus 2 case

When the genus is $g = 2$, $K_i \cdot \Sigma$ has to be -2, 0 or 2. The standard basis has only elements with $b = b' = 0$. Also for the case $K_i \cdot \Sigma = 0$ the coefficient in (5.3) vanishes, so we have:

Theorem 5.6 *Let $\bar{X}_i$ be as in section 5.1 with $b_1 = 0$, $b^+ > 1$ and of simple type. Write $\mathbb{D}^w_{\bar{X}_1} = e^{Q/2} \sum a_{i,w} e^{K_i}$ and $\mathbb{D}^w_{\bar{X}_2} = e^{Q/2} \sum b_{j,w} e^{L_j}$. Let $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ (for some identification) and suppose X is of simple type. Then for $\alpha \in H_2(X_1)$ and $\beta \in H_2(X_2)$, it is*

$$\mathbb{D}^w_X(\alpha + \beta) = e^{Q(\alpha+\beta)/2} \Big(\sum_{K_i \cdot \Sigma = L_j \cdot \Sigma = 2} 32\, a_{i,w}\, b_{j,w}\, e^{K_i \cdot \alpha + L_j \cdot \beta} + \sum_{K_i \cdot \Sigma = L_j \cdot \Sigma = -2} -32\, a_{i,w}\, b_{j,w}\, e^{K_i \cdot \alpha + L_j \cdot \beta} \Big).$$

Proof. We use the standard basis $\{e_i\}$ for $QH^*(M^{\mathrm{odd}}_\Sigma)$ and note that h corresponds to 2Σ from lemma 5.2. So in the expression (5.3) one has $b = b' = 0$ and $0 \le a, a' \le 3$. The matrix $< e^*_l, e^*_m >$ is

$$\frac{1}{4} \begin{pmatrix} 0 & -16 & 0 & 1 \\ -16 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

hence for $K_i \cdot \Sigma = 2$, the coefficient is computed to be 32 and for $K_i \cdot \Sigma = -2$ it will be -32. $\square$

Remark 5.7 *Actually we should say "for an appropriate homology orientation for X".*

Remark 5.8 *The reason for the different signs is easy to work out. First, w^2 for X is always congruent $\pmod 2$ with the sum of both of w^2 for $\bar{X}_i$. Also $b^+(X) = b^+(\bar{X}_1) + b^+(\bar{X}_2) + (2g-1)$, so $-\frac{3}{2}(1 + b^+(X)) = -\frac{3}{2}(1 + b^+(\bar{X}_1)) - \frac{3}{2}(1 + b^+(\bar{X}_2)) - 3(g-1)$. Recalling that $d_0 = -w^2 - \frac{3}{2}(1 + b^+)$ and $g = 2$, we have $d_0(X, w) \equiv d_0(\bar{X}_1, w) + d_0(\bar{X}_2, w) + 1 \pmod 2$. Now the sign comes from the fact that the coefficient for the basic class $-\kappa$ is $(-1)^{d_0} c_\kappa$, being c_κ the coefficient for the basic class κ.*

Corollary 5.9 *Let $\bar{X}_i$ have $b_1 = 0$, $b^+ > 1$ and be of simple type. Suppose that there are embedded Riemann surfaces $\Sigma_i \subset \bar{X}_i$ of genus 2, self-intersection zero and odd. Put $\mathbb{D}^w_{\bar{X}_1} = e^{Q/2} \sum a_{i,w} e^{K_i}$ and $\mathbb{D}^w_{\bar{X}_2} = e^{Q/2} \sum b_{j,w} e^{L_j}$. Perform a connected sum $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ (with some identification) and suppose it is of simple type. Let $\mathbb{D}^w_X = e^{Q/2} \sum c_{\kappa,w} e^{\kappa}$ be its Donaldson series. Then for any pair $(K, L) \in H^2(X_1; \mathbb{Z}) \oplus H^2(X_2; \mathbb{Z})$, we have*

$$\sum_{\{\kappa / \kappa|_{X_1} = K, \, \kappa|_{X_2} = L\}} c_{\kappa,w} = \pm 32 \, (\sum_{K_i|_{X_1} = K} a_{i,w}) \cdot (\sum_{L_j|_{X_2} = L} b_{j,w})$$

whenever $K|_Y = L|_Y = \pm 2 P.D.[\mathbb{S}^1]$. Otherwise the sum in the left hand side is zero.

5.3 Classes of the second type

Let us now have a homology class D of the second type. Using the extended homology groups (see subsection 2.3.2), we can write $D = D_1 + D_2$ with $D_i \in H_2^R(X_i, \partial X_i)$ and $\partial D_1 = -\partial D_2 = \mathbb{S}^1 \subset \Sigma \times \mathbb{S}^1$ (substitute D by a rational multiple if necessary).

In subsection 5.3.1 we use again Conjecture 1.22 to carry all explicit computations of the invariants of the connected sum along Σ, for D of second type. This conjecture tells us the action of $\mu(\Sigma)$ on $HF_*(Y)$. We will have to make an analogous conjecture on the action of $\mu(\Sigma)$ on $HFF_*(Y, \mathbb{S}^1)$, Conjecture 5.11. In subsection 5.3.2 we will give a short argument to get basically theorem 5.17 (and hence theorem 5.6 and corollary 5.9) and corollary 5.15 avoiding the conjectures. This is very nice from the point of view of having our main results proved, but we lose all the understanding of the way in which the Fukaya-Floer groups HFF_* enter in the arguments.

5.3.1 Explicit computations

In this section we need to work with the Fukaya-Floer homology version of the gluing theory. We recall from subsection 1.2.2 that $HFF_*(Y, \mathbb{S}^1)$ appears as the limit of a spectral sequence whose E_3 term is $HF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty)$ and whose differential d_3 is multiplication by $\mu(\mathbb{S}^1)$. Thus all the differentials in the E_3 term of the spectral sequence are of the form $H_{\text{odd}}(M_\Sigma^{\text{odd}}) \to H_{\text{even}}(M_\Sigma^{\text{odd}})$ and $H_{\text{even}}(M_\Sigma^{\text{odd}}) \to H_{\text{odd}}(M_\Sigma^{\text{odd}})$. Now the boundary cycle is $\mathbb{S}^1$ and thus invariant under the action of the group $\mathrm{Diff}^+(\Sigma)$ on $Y = \Sigma \times \mathbb{S}^1$, so the differentials commute with the action of $\mathrm{Diff}^+(\Sigma)$. As there are elements $\rho \in \mathrm{Diff}^+(\Sigma)$ acting as -1 on $H^1(\Sigma)$, we have that ρ acts as -1 on $H_{\mathrm{odd}}(M_\Sigma^{\mathrm{odd}})$ and as 1 on $H_{\mathrm{even}}(M_\Sigma^{\mathrm{odd}})$. Therefore the differentials are zero and the spectral sequence degenerates in the third term. Hence

$$\phi^w(X_i, D) = (\phi_0, \phi_1, \phi_2, \ldots) \in HF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty),$$

where we can interpret $\phi_k = \phi^w(X_i, D^k) \in QH_*(M_\Sigma^{\mathrm{odd}})$. So we have from theorem 1.24

$$D_X^{(w,\Sigma)}(D^m) = \sum \begin{pmatrix} m \\ i \end{pmatrix} \sigma(\phi^w(X_1, D_1^i), \phi^w(X_2, D_2^{m-i})),$$

or with the invariants in power series form, theorem 1.25 says

$$D_X^{(w,T)}(e^{tD}) = \sigma(\phi^w(X_1, e^{tD_1}), \phi^w(X_2, e^{tD_2})).$$

Now we can decompose $\phi^w(X_1, e^{tD_1}) = \phi^w(X_1, e^{tD_1})_{\mathrm{even}} + \phi^w(X_1, e^{tD_1})_{\mathrm{odd}}$ in components lying in $HF_{\mathrm{even}} \otimes \hat{H}_*(\mathbb{CP}^\infty)$ and $HF_{\mathrm{odd}} \otimes \hat{H}_*(\mathbb{CP}^\infty)$.

Lemma 5.10 *Suppose that $b_1(\bar{X}_1) = 0$ (equivalently, $b_1(X_1) = 0$). Then*

$$\phi^w(X_1, e^{tD_1})_{odd} = 0.$$

Proof. Since $b_1(\bar{X}_1) = 0$, for any $\gamma \in H_1(\Sigma)$ we have

$$0 = D_{\bar{X}_1}^{(w,T)}(e^{tD}\gamma) =< \phi^w(X_1, e^{tD_1}), \phi^w(A, e^{t\Delta}\gamma) > .$$

Now $\gamma \mapsto \phi^w(A, e^{t\Delta}\gamma)$ is $\mathrm{Diff}(\Sigma)$-equivariant. So the even part of $\phi^w(A, e^{t\Delta}\gamma)$ is zero and the odd part is a combination of the shape $\lambda_1\gamma + \lambda_2 i_\omega(\gamma)$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ and i_ω denoting contraction with the natural symplectic form ω in $H_1(\Sigma)$ (everything under the identification $\mu : H_1(\Sigma) \xrightarrow{\sim} QH^3(M_\Sigma^{\mathrm{odd}}) = HF^3(\Sigma \times \mathbb{S}^1)$). If $(\lambda_1, \lambda_2) \neq (0, 0)$ then the image of $\lambda_1\gamma + \lambda_2 i_\omega(\gamma)$, $\gamma \in H_1(\Sigma)$, is the total space $H_1(\Sigma)$. From this we conclude $\phi^w(X_1, e^{tD_1})_{\mathrm{odd}} = 0$. We still have to rule out the possibility $(\lambda_1, \lambda_2) = (0, 0)$. Suppose it happens, then it would be $\phi^w(A, e^{t\Delta}\gamma) = 0$ and so

$$D_{\mathbb{CP}^1 \times \Sigma}^{(w,T)}(e^{tD}\gamma_1\gamma_2) =< \phi^w(A, e^{tD_1}\gamma_1), \phi^w(A, e^{t\Delta}\gamma_2) >= 0.$$

But $D_{\mathbb{CP}^1 \times \Sigma}^{(w,T)}(\gamma_1\gamma_2) =< \mu(\gamma_1), \mu(\gamma_2) >= \omega(\gamma_1, \gamma_2) \neq 0$ (see equation (5.1)) in general. This finishes the proof. $\square$

Now we use the trick of transferring Σ from X_1 to X_2.

$$D_X^{(w,\Sigma)}(D^m\Sigma) = \sum \binom{m}{i} < \phi^w(X_1, D_1^i\Sigma), \phi^w(X_2, D_2^{m-i}) >=$$

$$= \sum \binom{m}{i} < D_{X_1}^w(D^i), P_\Sigma D_{X_2}^w(D^{m-i}) > .$$

The map $P_\Sigma : HFF_*(Y,\gamma) \to HFF_*(Y,\gamma)$ can be defined at the level of chains as

$$\begin{aligned} P_\Sigma : CFF_i(Y) &\to CFF_{i-2}(Y) \\ \rho_k &\mapsto \sum_{\rho_l} \binom{\beta}{\alpha} < \mu(\gamma\times\mathbb{R})^{\beta-\alpha}\mu(\Sigma\times\{t_0\}), \overline{\mathcal{M}}(\rho_k,\rho_l) > \rho_l \end{aligned}$$

for $\rho_k \in CF_{i-2\alpha}$, $\rho_l \in CF_{i-2-2\beta}$ and $\beta \geq \alpha$. Then $\partial\circ P_\Sigma + P_\Sigma\circ\partial = 0$ and the map descends to homology, so we get that, for every open manifold X with boundary Y and $D \in H_2^R(X,\partial X)$ with $\partial D = \gamma$,

$$\prod_m \phi_m^w(X, \Sigma D^m) = P_\Sigma(\prod_m \phi^w(X, D^m)).$$

We recall that $\phi^w(X, D^m)$ is not a cycle, so the expression $\phi_m^w(X, \Sigma D^m) = P_\Sigma(\phi^w(X, D^m))$ has no meaning. Actually P_Σ can be decomposed regarding at its action in the E_3 term of the spectral sequence as an infinite sum of maps

$$\binom{m+a}{a} P_{\Sigma,a} : HF_i(Y)\otimes H_{2m}(\mathbb{CP}^\infty) \to HF_{i-2-2a}(Y)\otimes H_{2m+2a}(\mathbb{CP}^\infty).$$

For $a=0$, the map is obviously multiplication by $\mu(\Sigma)$ in Floer homology. For the invariants in exponential form, we have

$$P_\Sigma(\sum_m \phi_m^w(X, D^m)\frac{t^m}{m!}) = \sum_{m,a}\binom{m+a}{a} P_{\Sigma,a}(\phi_m^w(X,D^m))\frac{t^{m+a}}{(m+a)!} =$$

$$= \sum_a \frac{t^a P_{\Sigma,a}}{a!}(\sum_m \phi_m^w(X, D^m)\frac{t^m}{m!}). \tag{5.4}$$

Here we interpret $P_{\Sigma,a} : HF_i(Y) \to HF_{i-2-2a}(Y)$. There is an exponential operator

$$Q_\Sigma = e^{sP_\Sigma} : HFF_*(Y,\gamma) \to HFF_*(Y,\gamma).$$

We have the following conjecture about the structure of P_Σ which we do not prove in the thesis (actually it can be regarded as an extension of Conjecture 1.22). In the next subsection we will give an alternative way avoiding it to prove our final result, theorem 5.17 and some evidence towards this conjecture.

Conjecture 5.11 *$P_{\Sigma,0}$ is quantum multiplication by $\mu(\Sigma)$. $P_{\Sigma,1}$ is the identity on the ± 2-eigenspaces of $\mu(\Sigma)$ and minus the identity on the 0-eigenspace of $\mu(\Sigma)$. $P_{\Sigma,a}$ is zero for $a \geq 2$.*

From lemma 5.2 we obtain the following expression

$$e^{s\mu(\Sigma)} = 1 + \frac{s}{2}h + \frac{\cosh 2s - 1}{16}h^2 + \frac{\sinh 2s - 2s}{64}h^3. \tag{5.5}$$

Corollary 5.12 *As a consequence of the above conjecture we get (recall $\Sigma \cdot D = 1$)*

$$D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) =< \phi^w(X_1, e^{tD_1}e^{s\Sigma}), \phi^w(X_2, e^{tD_2}) >=$$

$$= e^{(s\Sigma)(tD_1)} < \phi^w(X_1, e^{tD_1}), e^{s\mu(\Sigma)} * \phi^w(X_2, e^{tD_2})_{\pm 2} > +$$

$$+ e^{-(s\Sigma)(tD_1)} < \phi^w(X_1, e^{tD_1}), e^{s\mu(\Sigma)} * \phi^w(X_2, e^{tD_2})_0 >,$$

where $\phi^w(X_2, e^{tD_2})_{\pm 2}$ and $\phi^w(X_2, e^{tD_2})_0$ are the components of $\phi^w(X_2, e^{tD_2})$ in the ± 2-eigenspaces and 0-eigenspace respectively.

So the operator above is (for $\Sigma \cdot D = 1$)

$$e^{sP_\Sigma} = e^{-ts}(1 - \frac{1}{16}h^2 + \frac{s}{2}h - \frac{2s}{64}h^3) + e^{ts}(\frac{\cosh 2s}{16}h^2 + \frac{\sinh 2s}{64}h^3). \tag{5.6}$$

Let us write $\phi^w(X_1, e^{tD_1}) = (\phi_0, \phi_1, \phi_2, \phi_3)$, $\phi^w(X_2, e^{tD_2}) = (\psi_0, \psi_1, \psi_2, \psi_3)$ and $\phi^w(A, e^{t\Delta}) = (a_0, a_1, a_2, a_3)$ with respect to the standard basis $\{e_i\}$ (recall that $\Delta = \mathrm{pt} \times D^2 \subset \Sigma \times D^2 = A$). We do not consider the odd part of the Floer homology thanks to lemma 5.10. We have

$$D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = (\phi_0, \phi_1, \phi_2, \phi_3) B \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$B(t,s) = \frac{1}{4}\begin{pmatrix} 0 & -16 & 0 & 1 \\ -16 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \times$$

$$\times \begin{pmatrix} e^{-ts} & \frac{1}{2}se^{-ts} & \frac{1}{16}(e^{ts}\cosh 2s - e^{-ts}) & \frac{1}{64}(e^{ts}\sinh 2s - 2se^{-ts}) \\ 0 & e^{-ts} & \frac{1}{4}e^{ts}\sinh 2s & \frac{1}{16}(e^{ts}\cosh 2s - e^{-ts}) \\ 0 & 0 & e^{ts}\cosh 2s & \frac{1}{4}e^{ts}\sinh 2s \\ 0 & 0 & 4e^{ts}\sinh 2s & e^{ts}\cosh 2s \end{pmatrix} =$$

$$
= \frac{1}{4}\begin{pmatrix} 0 & -16e^{-ts} & 0 & e^{-ts} \\ -16e^{-ts} & -8se^{-ts} & e^{-ts} & \frac{1}{2}se^{-ts} \\ 0 & e^{-ts} & \frac{1}{4}e^{ts}\sinh 2s & \frac{1}{16}(e^{ts}\cosh 2s - e^{-ts}) \\ e^{-ts} & \frac{1}{2}se^{-ts} & \frac{1}{16}(e^{ts}\cosh 2s - e^{-ts}) & \frac{1}{64}(e^{ts}\sinh 2s - 2se^{-ts}) \end{pmatrix}
$$

Now we separate according to coefficients corresponding to functions on s in the expression of $D_X^{(w,\Sigma)}(e^{s\Sigma+tD})$.

$$
\begin{pmatrix} \text{coef. of } e^{2s}e^{ts} \\ \text{coef. of } e^{-2s}e^{ts} \\ \text{coef. of } e^{-ts} \\ \text{coef. of } se^{-ts} \end{pmatrix} = A_\psi \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{5.7}
$$

where

$$
A_\psi = \frac{1}{4}\begin{pmatrix} 0 & 0 & \frac{1}{128}(4\psi_3+16\psi_2) & \frac{1}{128}(4\psi_2+\psi_3) \\ 0 & 0 & \frac{1}{128}(4\psi_3-16\psi_2) & \frac{1}{128}(4\psi_2-\psi_3) \\ \psi_3-16\psi_1 & \psi_2-16\psi_0 & \psi_1-\frac{1}{16}\psi_3 & \psi_0-\frac{1}{16}\psi_2 \\ 0 & \frac{1}{2}(\psi_3-16\psi_1) & 0 & \frac{1}{2}(\psi_1-\frac{1}{16}\psi_3) \end{pmatrix}
$$

Therefore $(e^{2s}e^{ts}, e^{-2s}e^{ts}, e^{-ts}, se^{-ts})A_\psi = (\psi_0,\psi_1,\psi_2,\psi_3)B$. The matrix A_a will correspond to $A = \Sigma \times D^2$.

$$
A_a = \frac{1}{4}\begin{pmatrix} 0 & 0 & \frac{1}{128}(4a_3+16a_2) & \frac{1}{128}(4a_2+a_3) \\ 0 & 0 & \frac{1}{128}(4a_3-16a_2) & \frac{1}{128}(4a_2-a_3) \\ a_3-16a_1 & a_2-16a_0 & a_1-\frac{1}{16}a_3 & a_0-\frac{1}{16}a_2 \\ 0 & \frac{1}{2}(a_3-16a_1) & 0 & \frac{1}{2}(a_1-\frac{1}{16}a_3) \end{pmatrix}
$$

Lemma 5.13 *The matrix A_a is invertible.*

Proof. That the determinant vanishes would imply that either $a_3 = 4\,a_2$, $a_3 = -4\,a_2$ or $a_3 = 16\,a_1$. The first two cases give that the first or second row of A_a is zero respectively, which is contradictory as there are examples where the left hand side of (5.7) has non-zero first two entries (see subsection 5.3.2). The case $a_3 = 16\,a_1$ implies that the series for any such X is always of the form $f_1(t)e^{2s}e^{ts} + f_2(t)e^{-2s}e^{ts} + f_3(t)e^{-ts}$. This would also be valid for $X = \mathbb{CP}^1 \times \Sigma = A \cup_Y A$ (using the invariance under $\mathrm{Diff}(\Sigma)$, as in the proof of lemma 5.10, we get that $\phi^w(A, e^{tD_2})_{\mathrm{odd}} = 0$ and so the odd part of the Floer homology does not intervene in the series). Particularising for $t = 0$, $D_X^{(w,\Sigma)}(e^{s\Sigma})$ would be a linear combination of e^{2s}, e^{-2s} and 1. But from lemma 5.1 and equation (5.5) we get that

$$D_X^{(w,\Sigma)}(e^{s\Sigma}) = \frac{1}{16}(\sinh 2s - 2s).$$

□

Corollary 5.14 *The ϕ_i are determined by the series $D_{\bar{X}_1}^{(w,\Sigma)}(e^{t\bar{D}_1+s\Sigma})$ where $\bar{D}_1 = D_1 + \Delta$.*

Corollary 5.15 *Let $\bar{X}_1$ be of simple type with $b_1 = 0$, $b^+ > 1$ and an embedded surface Σ_1 of genus 2, self-intersection zero and representing an odd homology class. Let $\bar{X}_2$ be an arbitrary four-manifold with an embedded surface Σ_2 satisfying the same conditions as Σ_1. Then for $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ and $D \in H_2(X)$ of second type, $D_X^w((x^2-4)e^{tD}) = 0$.*

Proof. By corollary 5.14, $D_{\bar{X}_1}^{(w,\Sigma)}((x^2-4)e^{t\bar{D}_1+s\Sigma}) = 0$ implies $\phi^w(X_1,(x^2-4)e^{tD_1}) = 0$, and so $D_X^{(w,\Sigma)}((x^2-4)e^{tD}) = 0$ for any $D \in H_2(X)$ of second type. □

Theorem 5.16 *Let $\bar{X}_1$ have $b_1 = 0$, $b^+ > 1$ and an embedded surface Σ_1 of genus 2, self-intersection zero and representing an odd homology class. Then $\bar{X}_1$ is of w-**finite type** (i.e. $D_{\bar{X}_1}^w((x^2-4)^n e^{tD})$ vanishes identically for some $n > 0$), for any $w \in H^2(\bar{X}_1;\mathbb{Z})$ with $w \cdot \Sigma_1 \equiv 1 \pmod 2$. If we suppose $b^+ = 1$, the result remains true for the invariants with respect to $[\Sigma_1]$.*

Proof. We are going to check that $D_{\bar{X}_1}^{(w,\Sigma)}((x^2-4)^2 e^{tD}) = 0$, for all $D \in H_2(\bar{X}_1)$. When $D \in H_2(X_1)$ this is a consequence of corollary 5.4. Suppose now that $D \in H_2(\bar{X}_1)$ has $D \cdot \Sigma_1 = 1$. Put $D = D_1 + \Delta$. If $\bar{X}_1$ is of simple type, we have

$$0 = D_{\bar{X}_1}^{(w,\Sigma)}((x^2-4)e^{t\bar{D}_1+s\Sigma}) = < \phi^w(X_1, e^{tD_1}), \phi^w(A,(x^2-4)e^{t\Delta+s\Sigma}) > .$$

The vectors $\phi^w(X_1, e^{tD_1})$ (with $\bar{X}_1$ being of simple type) generate a 3-dimensional subspace (see subsection 5.3.2). Moreover this subspace is given by the equation $\phi_3 = 16\phi_1$ (see remark 5.18). So $\phi^w(A,(x^2-4)e^{t\Delta+s\Sigma})$ is a multiple of $(1,0,0,0)$ and therefore

$$< \phi^w(A,(x^2-4)e^{t\Delta+s\Sigma}), \phi^w(A,(x^2-4)e^{t\Delta}) >= D_{\Sigma\times\mathbb{CP}^1}^{(w,\Sigma)}((x^2-4)^2 e^{t\Delta+s\Sigma}) = 0,$$

from where $\phi^w(A,(x^2-4)^2 e^{t\Delta}) = 0$ and hence the result. □

Now we invert A_a (lemma 5.13) and do the matrix product

$$D_X^{(w,\Sigma)}(e^{tD}) =< (\phi_0,\phi_1,\phi_2,\phi_3),(\psi_0,\psi_1,\psi_2,\psi_3) >=$$

$$(v_1)^T \begin{pmatrix} \frac{32}{(a_3+4a_2)^2} & 0 & 0 & 0 \\ 0 & -\frac{32}{(a_3-4a_2)^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{-8}{(a_3-16a_1)^2} \\ 0 & 0 & \frac{-8}{(a_3-16a_1)^2} & 32\frac{a_2-16a_0}{(a_3-16a_1)^3} \end{pmatrix} (v_2) \qquad (5.8)$$

where

$$v_i = \begin{pmatrix} \text{coef. of } e^{2s}e^{ts} \\ \text{coef. of } e^{-2s}e^{ts} \\ \text{coef. of } e^{-ts} \\ \text{coef. of } se^{-ts} \end{pmatrix}$$

in $D_{\bar{X}_i}^{(w,\Sigma)}(e^{s\Sigma+t\bar{D}_i})$. When $\bar{X}_1$ is of simple type with $b_1=0$, $b^+>1$, we can use $\mathbb{D}^w_{\bar{X}_1}$ instead of $D_{\bar{X}_1}^{(w,\Sigma)}$.

Let us suppose $D^2=0$ (by adding a suitable multiple of Σ to D we can always arrange to have this). This does no harm to the argument. We put $\bar{D}_i = D_i + \Delta$. Also suppose $\bar{D}_i^2 = 0$. Then we can write

$$v_1 = \begin{pmatrix} \sum_{K_j\cdot\Sigma=2} a_{j,w}e^{tK_j\cdot\bar{D}_1} \\ \sum_{K_j\cdot\Sigma=-2} a_{j,w}e^{tK_j\cdot\bar{D}_1} \\ \sum_{K_j\cdot\Sigma=0} i^{-d_0}a_{j,w}e^{t\,i\,K_j\cdot\bar{D}_1} \\ 0 \end{pmatrix}$$

When both $\bar{X}_i$ are of simple type with $b_1=0$ and $b^+>1$, and also X is of simple type (from subsection 2.3.1, $b_1(X)=0$, $b^+(X)>1$), we have

$$\mathbb{D}^w_X(tD) = \qquad (5.9)$$

$$= \Big(\sum_{K_i\cdot\Sigma=2} a_{i,w}e^{tK_i\cdot\bar{D}_1}, \sum_{K_i\cdot\Sigma=-2} a_{i,w}e^{tK_i\cdot\bar{D}_1}\Big) \begin{pmatrix} \frac{32}{(a_3+4a_2)^2} & 0 \\ 0 & -\frac{32}{(a_3-4a_2)^2} \end{pmatrix} \begin{pmatrix} \sum_{L_j\cdot\Sigma=2} b_{j,w}e^{tL_j\cdot\bar{D}_2} \\ \sum_{L_j\cdot\Sigma=-2} b_{j,w}e^{tL_j\cdot\bar{D}_2} \end{pmatrix}$$

Theorem 5.17 *Let $\bar{X}_i$ be as in section 5.1 with $b_1=0$, $b^+>1$ and of simple type. Write $\mathbb{D}^w_{\bar{X}_1} = e^{Q/2}\sum a_{i,w}e^{K_i}$ and $\mathbb{D}^w_{\bar{X}_2} = e^{Q/2}\sum b_{j,w}e^{L_j}$. Let $X=\bar{X}_1\#_\Sigma\bar{X}_2$ (for some identification) and suppose X is of simple type. Let $D\in H_2(X)$ be* *of second type with $D \cdot \Sigma = 1$. Write $D = D_1 + D_2$, $D_i \in H_2^R(X_i, \partial X_i)$. Put $\bar{D}_i = D_i + \Delta$, so $D^2 = \bar{D}_1^2 + \bar{D}_2^2$. Then (for appropriate homology orientations)*

$$\mathbb{D}_X^w(tD) =$$

$$= e^{Q(tD)/2}\Big(\sum_{K_i\cdot\Sigma = L_j\cdot\Sigma = 2} 32 a_{i,w} b_{j,w}\, e^{(K_i\cdot\bar{D}_1 + L_j\cdot\bar{D}_2 + 2)t} + \sum_{K_i\cdot\Sigma = L_j\cdot\Sigma = -2} -32 a_{i,w} b_{j,w}\, e^{(K_i\cdot\bar{D}_1 + L_j\cdot\bar{D}_2 - 2)t}\Big).$$

Proof. The statement is equivalent to proving that the square matrix in equation (5.9) is

$$\begin{pmatrix} 32e^{2t} & 0 \\ 0 & -32e^{-2t} \end{pmatrix}.$$

We can again suppose $D^2 = \bar{D}_1^2 = \bar{D}_2^2 = 0$ (in the general case we only have to add some extra terms $e^{Q(tD)/2}$, $e^{Q(t\bar{D}_i)/2}$). For proving this, it would be enough to find examples of manifolds $\bar{X}_1$, $\bar{X}_2$ and X whose basic classes are known. Instead we use an indirect argument. Since all the manifolds involved can be chosen of simple type with $b_1 = 0$, $b^+ > 1$, the non-zero entries of the matrix are finite sums of exponentials, i.e.

$$\begin{pmatrix} \sum c_n\, e^{nt} & 0 & 0 \\ 0 & \sum d_n\, e^{nt} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we evaluate the series on $tD + r_1\alpha_1 + r_2\alpha_2$, for $\alpha_i \in H^2(X_i;\mathbb{Z})$, put $t = 0$ and use theorem 5.6 to get $\sum c_n = 32$ and $\sum d_n = -32$. Let $S = \mathbb{CP}^2\#10\overline{\mathbb{CP}}^2$ the rational elliptic surface blown-up once. Denote by $E_1, \ldots, E_{10}$ the exceptional divisors and let $T_1 = C - E_1 - \cdots - E_9$, $T_2 = C - E_1 - \cdots - E_8 - E_{10}$, where C is the cubic curve in $\mathbb{CP}^2$. So T_1 and T_2 can be represented by smooth tori of self-intersection zero and with $T_1 \cdot T_2 = 1$. We can glue two copies of S along T_1. The result is a K3 surface $S\#_{T_1}S$ blown-up twice. The T_2 pieces glue together to give a genus 2 Riemann surface Σ_2 of self-intersection zero which intersects T_1 in one point. Now set $X = (S\#_{T_1}S)\#_{\Sigma_2}(S\#_{T_1}S)$, call $\Sigma = \Sigma_2$ and get D piecing together both T_1 in $S\#_{T_1}S$. So (choose $w = T_1$ on $S\#_{T_1}S$)

$$D_X^{(D,\Sigma)}(e^{tD+s\Sigma}) = e^{Q(tD+s\Sigma)/2}\Big(\sum_{K_i\cdot\Sigma = L_j\cdot\Sigma = 2} c_n a_i b_j\, e^{2s+nt} + \sum_{K_i\cdot\Sigma = L_j\cdot\Sigma = -2} d_n a_i b_j\, e^{-2s+nt}\Big) =$$

$$= e^{ts}\Big(\sum \frac{c_n}{16} e^{2s+nt} + \sum \frac{d_n}{16} e^{-2s+nt}\Big)$$

since T_1 evaluates 0 on basic classes being a torus of self-intersection zero (the coefficient $\frac{1}{16}$ appears from the explicit computation of the basic classes of the K3 surface blown-up in two points). The trick is now to use the symmetry fact that $X=(S\#_{T_2}S)\#_{\Sigma_1}(S\#_{T_2}S)$ where Σ_1 comes from gluing together both T_1. Under this diffeomorphism $D=\Sigma_1$ and Σ comes from piecing together both T_2 in $S\#_{T_2}S$. Hence

$$D_X^{(\Sigma,D)}(e^{tD+s\Sigma})=e^{ts}(\sum\frac{c_n}{16}e^{2t+ns}+\sum\frac{d_n}{16}e^{-2t+ns}).$$

Both expressions are equal, $D_X^{(D,\Sigma)}(e^{tD+s\Sigma})=\mathbb{D}_X^{D+\Sigma}(tD+s\Sigma)=D_X^{(\Sigma,D)}(e^{tD+s\Sigma})$ (X is of simple type from proposition 5.5). From here we deduce that $c_n=0$ unless $n=\pm 2$ and $d_n=0$ unless $n=\pm 2$. Also $c_{-2}=d_2$, $c_2+c_{-2}=32$, $d_2+d_{-2}=-32$, so $c_2-d_{-2}=64$. But $c_2=\pm d_{-2}$, so it has to be $c_{-2}=d_2=0$, whence the result. $\square$

Remark 5.18 *Note that when $\bar{X}_1$ is of simple type there is no summand in $D_{\bar{X}_1}^{(w,\Sigma)}$ corresponding to se^{-ts}. Therefore from equation (5.7) with $A_\psi=A_a$, $0=\phi_1(a_3-16a_1)+\phi_3(a_1-\frac{1}{16}a_3)$ and then $(\phi_3-16\phi_1)(a_3-16a_1)=0$, so $\phi_3=16\phi_1$. In particular, for any X_2, the manifold $X=\bar{X}_1\#_\Sigma\bar{X}_2$ has a series without the coefficient corresponding to se^{-ts}. In this case we have*

$$D_X^{(w,\Sigma)}(e^{tD+s\Sigma})=$$

$$=\frac{1}{4}(0,(\phi_2-16\phi_0)e^{-ts},(\frac{\sinh 2s}{4}\phi_2+\frac{\cosh 2s}{16}\phi_3)e^{ts},$$

$$(\phi_0-\frac{\phi_2}{16})e^{-ts}+(\frac{\cosh 2s}{16}\phi_2+\frac{\sinh 2s}{64}\phi_3)e^{ts})\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix}=$$

$$=\frac{1}{64}(\psi_3-16\psi_1)(16\phi_0-\phi_2)e^{-ts}+\frac{e^{ts}}{256}(\psi_2,\psi_3)\begin{pmatrix}16\sinh 2s & 4\cosh 2s\\ 4\cosh 2s & \sinh 2s\end{pmatrix}\begin{pmatrix}\phi_2\\ \phi_3\end{pmatrix}.$$

So if both $\bar{X}_1$ and $\bar{X}_2$ are of simple type, we get

$$D_X^{(w,\Sigma)}(e^{tD+s\Sigma})=\frac{e^{ts}}{16}(\psi_1,\psi_2)\begin{pmatrix}16\sinh 2s & 4\cosh 2s\\ 4\cosh 2s & \sinh 2s\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}.$$

5.3.2 Getting around Conjecture 5.11 and Conjecture 1.22

Now we want to give an argument to reach theorem 5.17 without the use of Conjecture 5.11 about the explicit description of P_Σ and also without the use of Conjecture 1.22. We argue as follows. We have, instead of corollary 5.12,

$$D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) =< \phi^w(X_1, e^{tD_1}), Q_\Sigma \phi^w(X_2, e^{tD_2}) >,$$

for some symmetric map $Q_\Sigma = Q_\Sigma(s,t)$. Write $\phi^w(X_1, e^{tD_1}) = (\phi_0, \phi_1, \phi_2, \phi_3)$, $\phi^w(X_2, e^{tD_2}) = (\psi_0, \psi_1, \psi_2, \psi_3)$ and $\phi^w(A, e^{t\Delta}) = (a_0, a_1, a_2, a_3)$ as before, so

$$D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = (\phi_0, \phi_1, \phi_2, \phi_3) B(t,s) \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

for some matrix $B(t,s)$. Separating according to coefficients of s in the expression $D_{\bar{X}_1}^{(w,\Sigma)}(e^{s\Sigma+t\bar{D}_1})$,

$$\begin{pmatrix} \text{coef. of } e^{2s}e^{ts} \\ \text{coef. of } e^{-2s}e^{ts} \\ \text{coef. of } e^{-ts} \\ \text{coef. of } g(t,s) \end{pmatrix} = A_a(t) \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{5.10}$$

where $g(t,s)$ is a function (to be found later) linearly independent with $e^{2s}e^{ts}$, $e^{-2s}e^{ts}$ and e^{-ts} over the field $\mathcal{F}(t)$ of (Laurent) formal power series on t. For proving the invertibility of A_a it is enough to find four linearly independent vectors in $\mathcal{F}(t) \otimes \mathbb{R}^4$ for the left hand side of formula (5.10). For this we use the following set of examples:

- X a $K3$ surface blown-up twice with E_1 and E_2 the two exceptional divisors, $\Sigma = S - E_1 - E_2$ for S a tight surface of genus 2 in $K3$, $w = E_1$, D a cohomology class coming form the $K3$ such that $D \cdot S = 1$, $D^2 = 0$. We get $D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = -e^{ts}\frac{e^{2s}-e^{-2s}}{4}$ and therefore the vector $(1,-1,0,0)$.

- X, Σ, D as before, but now $w \in H^2(K3)$, with $w \cdot S = 1$. We will get $D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = (-1)^{\frac{w^2}{2}} e^{ts}\frac{e^{2s}+e^{-2s}}{4} - \frac{1}{2}e^{-ts}$ and the vector $(1,1,\pm 2,0)$.

- X a $K3$ surface, Σ a tight torus with an added trivial handle to make it of genus 2, $w \in H^2(X;\mathbb{Z})$ such that $w \cdot \Sigma = 1$ and D with $D \cdot \Sigma = 1$, $D^2 = 0$. Then $D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = -e^{-ts}$ and the vector we get is $(0,0,1,0)$.

- $S = \mathbb{CP}^1 \times \Sigma$, $w = \text{P.D.}[\mathbb{CP}^1]$, $D = \mathbb{CP}^1$. Then $D_S^{(w,\Sigma)}(e^{s\Sigma+tD})$ has a summand of the form $s + t\,f(t,s)$, since when we set $t = 0$ there is a summand which is a multiple of s. So we get a vector with non-vanishing last component.

So this means that we have the expression of equation (5.10) with $g(t,s) = s + t\,f(t,s)$ and with an invertible matrix A_a (lemma 5.13). Having reached this point we know of the existence of a universal matrix as in (5.8). For the simple type case we get the expression

$$D_X^{(w,\Sigma)}(e^{tD}) = (v_1)^T\, M(t)\, v_2$$

with

$$v_1 = \left(\sum_{K_j\cdot\Sigma=2} a_{j,w} e^{tK_j\cdot\bar{D}_1}, \sum_{K_j\cdot\Sigma=-2} a_{j,w} e^{tK_j\cdot\bar{D}_1}, \sum_{K_j\cdot\Sigma=0} i^{-d_0(\bar{X}_1)} a_{j,w} e^{t\,i\,K_j\cdot\bar{D}_1} \right)^T,$$

and analogously for v_2. The 3 by 3 matrix $M(t)$ is universal. This matrix is diagonal since obviously it is always the case $K_i \cdot \Sigma = L_j \cdot \Sigma$, for nonzero summands. Now consider the case in which both $\bar{X}_i$ and Σ_i are as in the third example above. Then $X = \bar{X}_1 \# \bar{X}_2$ splits off a $\mathbb{S}^2 \times \mathbb{S}^2$, so its invariants are zero. Therefore the third diagonal entry in the matrix is zero. The other two coefficients are computed in the proof of theorem 5.17.

Also, the fact that A_a is invertible implies corollary 5.15, so this corollary does not depend on any conjecture. On the other hand, our proof of theorem 5.16 does depend on the conjectures.

As promised, we are going to support Conjecture 5.11 with some evidence. By equation (5.4) one can write

$$Q_\Sigma = e^{sP_\Sigma} = e^{sP_0 + stP_1 + \frac{1}{2}st^2P_2 + \cdots}$$

Suppose that all maps $P_i = P_{\Sigma,i}$ leave invariant the 2-eigenspace of P_0, then P_i acts as a complex number λ_i in that subspace and Q_Σ as $e^{2s+st\lambda_1+\frac{1}{2}st^2\lambda_2+\cdots}$. This produces a function of that shape in the Donaldson series $D_X^{(w,\Sigma)}(e^{tD+s\Sigma})$. Since we have seen that the term e^{2s+ts} is the only one appearing with a $2s$ summand in the exponent, it must be $\lambda_1 = 1$ and $\lambda_i = 0$ for $i \geq 2$. The same argument works for the 0-eigenvector, so it is very plausible that Conjecture 5.11 be true.

5.4 Classes of the third type

In this section we need to use Conjecture 1.22 to compute the Fukaya-Floer groups $HFF_*(Y,\gamma)$ (with $[\gamma]$ not a multiple of $[\mathbb{S}^1]$). Also we shall need another conjectural result about the action of $\mu(\Sigma)$ on $HFF_*(Y,\gamma)$, Conjecture 5.20. The results of this section are dependent on them. At the end of the section, we give some nice applications of our main result, theorem 5.23, which should also be treated as conjectural.

Consider a homology class $D \in H_2(X)$ of third type. Substituting it by a rational multiple if necessary, one can always write D as $D = D_1 + D_2$ with $D_i \in H_2^R(X_i, \partial X_i)$, $\partial D_1 = -\partial D_2 = \gamma \subset \Sigma \times \mathbb{S}^1$ and γ a loop such that $\gamma \hookrightarrow \Sigma \times \mathbb{S}^1 \to \Sigma$ is an embedding (so the class $[\gamma]$ is primitive and not a multiple of $[\mathbb{S}^1]$). Now we need to work out the groups $HFF_*(Y,\gamma)$. There is an identification $Y \stackrel{\sim}{\to} \Sigma \times \mathbb{S}^1$ carrying γ to a loop α_1 inside Σ. The E_3 term of the usual spectral sequence is $HF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty)$ with differential d_3 given by

$$\mu(\alpha_1) : QH_i(M_\Sigma^{\mathrm{odd}}) \otimes H_j(\mathbb{CP}^\infty) \to QH_{i-3}(M_\Sigma^{\mathrm{odd}}) \otimes H_{j+2}(\mathbb{CP}^\infty).$$

From this we write the E_5 term of the spectral sequence and see that the differential d_5, being invariant under the subgroup of diffeomorphisms of Σ fixing α_1, has to be zero. Hence $HFF_*(Y,\gamma)$ is equal to this E_5 term. Let us write it down. Set $HF_2^{\mathrm{red}} = < h^2 >$, $HF_0^{\mathrm{red}} = HF^2 / < h^3 - 16h >$, $W_0 = \mu(\alpha_1)^-$ and $W = \mu(\alpha_1)^- / < \mu(\alpha_1) >$. Then

$$\begin{array}{rcccccccccc}
HFF_0(Y,\gamma) & = & HF_0 & \oplus\ 0\ \oplus & HF_2^{\mathrm{red}} & \oplus\ 0\ \oplus & HF_0^{\mathrm{red}} & \oplus & \cdots \\
HFF_1(Y,\gamma) & = & 0 & \oplus\ 0\ \oplus & W & \oplus\ 0\ \oplus & 0 & \oplus & \cdots \\
HFF_2(Y,\gamma) & = & HF_2^{\mathrm{red}} & \oplus\ 0\ \oplus & HF_0^{\mathrm{red}} & \oplus\ 0\ \oplus & HF_2^{\mathrm{red}} & \oplus & \cdots \\
HFF_3(Y,\gamma) & = & W_0 & \oplus\ 0\ \oplus & 0 & \oplus\ 0\ \oplus & W & \oplus & \cdots
\end{array}$$

So finally we put $e_0 = h^2 \in HF_2^{\mathrm{red}}$, $e_1 = h \in HF_0^{\mathrm{red}}$ and $e_3 = h^3 - 16h \in HF_0$. We have

$$HFF_{\mathrm{even}} = (HF_{\mathrm{even}}^{\mathrm{red}} \otimes \hat{H}_*(\mathbb{CP}^\infty)) \oplus (< e_3 > \otimes H_0(\mathbb{CP}^\infty)).$$

There is an intersection pairing for $HFF_*(Y,\gamma)$ induced by the pairing on Floer homology. For that, we have $e_0 \cdot e_0 = 0$, $e_0 \cdot e_1 = 4$, $e_1 \cdot e_1 = 0$ and $e_0 \cdot e_3 = 0$, $e_1 \cdot e_3 = 0$.

For every open manifold X_1, $D_1 \in H_2^R(X_1, \partial X_1)$ and $w \in H^2(X_1; \mathbb{Z})$,

$$\phi^w(X_1, D_1) = (\phi_0, \phi_1, \phi_2, \ldots) \in HFF_*(Y,\gamma),$$

which we write as $\phi^w(X_1, e^{tD_1})$. We decompose $\phi^w(X_1, e^{tD_1}) = \phi^w(X_1, e^{tD_1})_{\text{even}} + \phi^w(X_1, e^{tD_1})_{\text{odd}}$ in components lying in HFF_{even} and HFF_{odd}. Clearly,

$$\phi^w(X_1, e^{tD_1})_{\text{even}} = \sum_{i \geq 0} \frac{\phi_i}{i!} t^i = f_0(t)\, e_0 + f_1(t)\, e_1 + f_3\, e_3$$

with $f_3 \in \mathbb{Q}$. We expect that $\phi^w(X_1, e^{tD_1})_{\text{odd}} = 0$, for any $D_1 \in H_2^R(X_1, \partial X_1)$, whenever $b_1(\bar{X}_1) = 0$, as in lemma 5.10. This is due to the fact that the pairing in $HFF_{\text{odd}}(Y, \gamma)$ is antisymmetric though $< \phi^w(X_1, e^{tD_1}), \phi^w(X_2, e^{tD_2}) > = < \phi^w(X_2, e^{tD_2}), \phi^w(X_1, e^{tD_1}) >$ (this argument replaces the proof of lemma 5.10).

Consider another pair X_2, D_2, with corresponding $(g_0(t), g_1(t), g_3)$. The pairing formula reads

$$D_X^{(w,\Sigma)}(e^{tD}) = (f_0, f_1) \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \frac{1}{4}\, (f_0 g_1 + f_1 g_0).$$

Remark 5.19 *If we take an identification $\phi : Y \stackrel{\sim}{\to} Y$, then the induced map on the even part of the Fukaya-Floer homology is the identity.*

We use again the trick of transferring Σ from X_1 to X_2 and the map $P_\Sigma : HFF_*(Y, \gamma) \to HFF_*(Y, \gamma)$, as in the previous section. We have the following conjecture about the structure of P_Σ analogous to Conjecture 5.11. We also could avoid using it as in subsection 5.3.2.

Conjecture 5.20 *$P_{\Sigma,0}$ is quantum multiplication by $\mu(\Sigma)$. $P_{\Sigma,1}$ is the identity on the ± 2-eigenspaces for the action of $\mu(\Sigma)$ (and preserves the 0-eigenspace generated by e_3). $P_{\Sigma,a} = 0$, for $a \geq 2$.*

Corollary 5.21 *As a consequence of the above conjecture we get*

$$D_X^{(w,\Sigma)}(e^{s\Sigma + tD}) = < \phi^w(X_1, e^{tD_1}), e^{(s\Sigma)(tD_1)} e^{s\mu(\Sigma)} * \phi^w(X_2, e^{tD_2}) > .$$

Clearly $e^{s\mu(\Sigma)} * e_3 = e_3$ since $\mu(\Sigma) * e_3 = 0$ and

$$e^{s\mu(\Sigma)} : \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \mapsto \begin{pmatrix} \cosh 2s & \frac{1}{4} \sinh 2s \\ 4 \sinh 2s & \cosh 2s \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

Easily we obtain

$$e^{s\mu(\Sigma)} = 1 + \frac{\cosh 2s - 1}{16} e_0 + \frac{\sinh 2s}{4} e_1 \tag{5.11}$$

Let us write $\phi^w(X_1, e^{tD_1}) = (f_0, f_1)$ and $\phi^w(X_2, e^{tD_2}) = (g_0, g_1)$. Then

$$D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = e^{ts(D\cdot\Sigma)}(f_0, f_1)\begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}\begin{pmatrix} \cosh 2s & \frac{1}{4}\sinh 2s \\ 4\sinh 2s & \cosh 2s \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} =$$

$$= \frac{1}{16}\, e^{ts(D\cdot\Sigma)}(f_0, f_1)\begin{pmatrix} 16\sinh 2s & 4\cosh 2s \\ 4\cosh 2s & \sinh 2s \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}. \qquad (5.12)$$

We consider now the following very important example. Let $\bar{A}$ be the $K3$ surface blown-up in two points and let E_1 and E_2 stand for the exceptional divisors. Let $S \subset K3$ be a tight embedded surface of genus 2 and put $\Sigma = S - E_1 - E_2$ for its proper transform, which is represented by an embedded surface of genus 2 and self-intersection zero. A will be the complement of a tubular neighbourhood of Σ in $\bar{A}$. Call $X = \bar{A}\#_\Sigma\bar{A}$ the double of $\bar{A}$, i.e. the connected sum of $\bar{A}$ with itself with the identification which is given by the natural orientation reversing diffeomorphism of $Y = \partial A$ to itself. As in the proof of theorem 5.17, we choose D to be the embedded surface obtained by piecing together two fibres of the natural elliptic fibration of $\bar{A}$. Then D is a genus 2 Riemann surface of self-intersection zero. Also take $w = \text{P.D.}[D] \in H^2(X;\mathbb{Z})$. Then

$$D_X^{(w,\Sigma)}(e^{tD+s\Sigma}) = e^{ts}(2\,e^{2s+2t} - 2\,e^{-2s-2t}).$$

We can take a collection α_i, $1 \le i \le 4$, of framed loops in a fibre $\Sigma \subset \partial A$, which together with $\mathbb{S}^1$ form a basis for $H_1(Y)$, such that they can be capped off with embedded (-1)-discs D_i (writing $\bar{A} = S\#_{T_1}S$, as in the proof of theorem 5.17, we consider the vanishing discs of the elliptic fibration of S with fibre T_2, see [24, page 167], since they do not intersect T_1). Now these discs can be glued together pairwise when forming $X = A \cup_Y A$, since the framings are respected (see remark 2.19), to give a collection of (-2)-embedded spheres $S_i = D_i \cup_{\alpha_i} D_i$. Everyone of these discs has a dual torus T_i, by considering another loop in $\Sigma \subset \partial A$, say β_i, with $\alpha_i \cdot \beta_i = 1$, and putting $T_i = \beta_i \times \mathbb{S}^1 \subset \Sigma \times \mathbb{S}^1$. Then the elements $S_i + T_i$ are represented by embedded tori of self-intersection zero. Since the manifold X is of simple type (proposition 5.5), the basic classes evaluate zero on T_i and on $S_i + T_i$. Our conclusion is

$$D_X^{(w,\Sigma)}(e^\alpha) = 4\,e^{Q(\alpha)/2}\sinh(K\cdot\alpha),$$

with $K \in H^2(X;\mathbb{Z})$ being the only cohomology class with

- $K \cdot \alpha = (E_1 + E_2) \cdot \alpha$ for $\alpha \in H_2(A)$.
- $K \cdot \Sigma = K \cdot D = 2$.
- $K \cdot S_i = K \cdot T_i = 0$, for all i.

i.e. K plays the role of the canonical class of X (if it were an algebraic surface). Now we split K into two symmetric pieces $K_i \subset A$. The boundary of K_i is $\partial K_i = 2\mathbb{S}^1$ and $K_i^2 = 2$ since $K^2 = 4$.

Lemma 5.22 *For every framed loop γ in Y (with $\gamma \hookrightarrow Y \to \Sigma$ embedding), there is a $D_\gamma \in H_2^R(A, \partial A)$, whose boundary is γ and such that $(f_0, f_1) = e^{tD_\gamma^2/2}(2,0)$ or $e^{tD_\gamma^2/2}(0,8)$ (up to sign). Moreover any such D_γ satisfy the condition as long as $D_\gamma \cdot K_1 = 0$.*

Proof. Choose any $D_\gamma \in H_2^R(A, \partial A)$ with boundary γ. If $D_\gamma \cdot K_1 \neq 0$, then add a rational multiple of Σ to D_γ to get $D_\gamma \cdot K_1 = 0$ (possible since $K_1 \cdot \Sigma = 2$). Suppose without loss of generality that $D_\gamma^2 = 0$. Then consider the embedded surface $D = D_\gamma + D_\gamma \in H_2(X)$, which has $D \cdot K = 0$ and so

$$D_X^{(w,\Sigma)}(e^{s\Sigma + tD}) = 4\, e^{ts(\Sigma \cdot D)} \sinh(2s) =$$

$$= \frac{1}{16}\, e^{ts(D\cdot\Sigma)}(f_0, f_1) \begin{pmatrix} 16 \sinh 2s & 4\cosh 2s \\ 4\cosh 2s & \sinh 2s \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix},$$

where (f_0, f_1) corresponds to (A, D_γ). Therefore we have the equations

$$\begin{aligned} 16 f_0^2 + f_1^2 &= 64 \\ 4 f_0 f_1 + 4 f_1 f_0 &= 0 \end{aligned}$$

from where either $(f_0, f_1) = (2, 0)$ or $(f_0, f_1) = (0, 8)$. $\square$

Return to the manifold $\bar{X}_1$, and consider any $D_\gamma \in H_2^R(X_1, \partial X_1)$ with boundary $\gamma \subset Y$. By the result above, we can cap off D_γ in $\tilde{X}_1 = \bar{X}_1 \#_\Sigma \bar{A} = X_1 \cup_Y A$, to get $D = \tilde{D}_\gamma = D_\gamma + D_2$, where $D_2 \cdot K_1 = 0$ (this intersection only makes sense when ∂D_2 and ∂K_1 are disjoint, but we always can suppose that). Suppose for the calculations that $D_\gamma^2 = D_2^2 = 0$. Then

$$D_{\tilde{X}_1}^{(w,\Sigma)}(e^{s\Sigma + tD}) = \frac{1}{16}\, e^{ts(D\cdot\Sigma)}(f_0, f_1) \begin{pmatrix} 16 \sinh 2s & 4\cosh 2s \\ 4\cosh 2s & \sinh 2s \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix},$$

where $\phi^w(X_1, e^{tD_\gamma}) = (g_0, g_1)$. The possibility $(f_0, f_1) = (2, 0)$ yields

$$\begin{pmatrix} \text{coef. of } e^{2s+ts} \\ \text{coef. of } e^{-2s+ts} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}.$$

The possibility $(f_0, f_1) = (0, 8)$ yields

$$\begin{pmatrix} \text{coef. of } e^{2s+ts} \\ \text{coef. of } e^{-2s+ts} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}.$$

Now $\tilde{X}_1$ has $b^+ > 1$ and $b_1 = 0$ (since $b_1(\bar{X}_1) = 0$). Let us see that it is of simple type. For this it is enough to prove that $\phi^w(A, (x^2-4)e^{tD_2}) = 0$ for D_2 of third type. $X = \bar{A}\#_\Sigma\bar{A}$ has $D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = 4e^{ts(\Sigma\cdot D)}\sinh(2s)$ if $D \cdot K = 0$. In the same fashion, we find that for $Z = \bar{A}\#_\Sigma\bar{A}\#_\Sigma\bar{A}$, Z is of simple type and $D_X^{(w,\Sigma)}(e^\alpha) = 16e^{Q(\alpha)/2}\cosh(K_Z \cdot \alpha)$, where K_Z satisfy conditions analogous to those of K. So $D_Z^{(w,\Sigma)}(e^{s\Sigma+tD}) = 16e^{ts(\Sigma\cdot D)}\cosh(2s)$, whenever $D \cdot K_Z = 0$. This implies that the vectors (f_0, f_1) given by A and by $X^o = X - N_\Sigma$ are linearly independent. The vector $\phi^w(A, (x^2-4)e^{tD_2})$ is orthogonal to both $\phi^w(A, e^{tD_1})$ and $\phi^w(X^o, e^{tD_1})$, as X and Z are of simple type, so it is zero. Then we can write $\mathbb{D}^w_{\tilde{X}_1} = e^{Q/2}\sum \tilde{a}_{i,w}e^{\tilde{K}_i}$ for its Donaldson series. So either

$$\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} \sum_{\tilde{K}_i\cdot\Sigma=2} \tilde{a}_{i,w}e^{t\tilde{K}_i\cdot D} \\ \sum_{\tilde{K}_i\cdot\Sigma=-2} \tilde{a}_{i,w}e^{t\tilde{K}_i\cdot D} \end{pmatrix}$$

or

$$\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \sum_{\tilde{K}_i\cdot\Sigma=2} \tilde{a}_{i,w}e^{t\tilde{K}_i\cdot D} \\ \sum_{\tilde{K}_i\cdot\Sigma=-2} \tilde{a}_{i,w}e^{t\tilde{K}_i\cdot D} \end{pmatrix}.$$

If we have two manifolds $\bar{X}_1$ and $\bar{X}_2$ (with $\mathbb{D}^w_{\tilde{X}_1} = e^{Q/2}\sum \tilde{a}_{i,w}e^{\tilde{K}_i}$ and $\mathbb{D}^w_{\tilde{X}_2} = e^{Q/2}\sum \tilde{b}_{j,w}e^{\tilde{L}_j}$) and consider the connected sum $X = \bar{X}_1\#_\Sigma\bar{X}_2$ along Σ, let $D \in H_2(X)$ be decomposed as $D_1 + D_2$ with $D_i \in H_2^R(X_i, \partial X_i)$ and consider two cappings $\tilde{D}_i$ in $\tilde{X}_i$ as before. Then we use equation (5.12) and the different possible combinations of the cases above to get (in all cases)

$$D_X^{(w,\Sigma)}(e^{s\Sigma+tD}) = \tag{5.13}$$

$$= e^{Q(s\Sigma+tD)/2}\Big(\sum_{\tilde{K}_i\cdot\Sigma=2} \tilde{a}_{i,w}e^{t\tilde{K}_i\cdot\tilde{D}_1}, \sum_{\tilde{K}_i\cdot\Sigma=-2} \tilde{a}_{i,w}e^{t\tilde{K}_i\cdot\tilde{D}_1}\Big)\begin{pmatrix} \frac{e^{2s}}{2} & 0 \\ 0 & -\frac{e^{-2s}}{2} \end{pmatrix}\begin{pmatrix} \sum_{\tilde{L}_j\cdot\Sigma=2} \tilde{b}_{j,w}e^{t\tilde{L}_j\cdot\tilde{D}_2} \\ \sum_{\tilde{L}_j\cdot\Sigma=-2} \tilde{b}_{j,w}e^{t\tilde{L}_j\cdot\tilde{D}_2} \end{pmatrix}$$

$$= e^{Q(s\Sigma+tD)/2}\Big(\sum_{\tilde{K}_i\cdot\Sigma=\tilde{L}_j\cdot\Sigma=2}\frac{1}{2}\tilde{a}_{i,w}\tilde{b}_{j,w}e^{t(\tilde{K}_i\cdot\tilde{D}_1+\tilde{L}_j\cdot\tilde{D}_2)+2s}-\sum_{\tilde{K}_i\cdot\Sigma=\tilde{L}_j\cdot\Sigma=-2}\frac{1}{2}\tilde{a}_{i,w}\tilde{b}_{j,w}e^{t(\tilde{K}_i\cdot\tilde{D}_1+\tilde{L}_j\cdot\tilde{D}_2)-2s}\Big).$$

The choice of signs in lemma 5.22 does not matter for this final result. Also if we do not suppose $D^2=0$ we get the same expression.

We also want to remark that the formula remains valid for any $\alpha\in H_2(X)$. For this it is enough to consider $D+r\alpha$ with α of the first or second type and make $t\to 0$ while keeping $rt=1$.

Theorem 5.23 *Let $\bar{X}_i$ be two manifolds with $b_1=0$ and Riemann surfaces $\Sigma_i\subset\bar{X}_i$ of genus 2, self-intersection zero and representing odd homology classes. Consider $\tilde{X}_i=\bar{X}_i\#_\Sigma\bar{A}$, which are of simple type. Put $\mathbb{D}^w_{\tilde{X}_1}=e^{Q/2}\sum\tilde{a}_{i,w}e^{\tilde{K}_i}$ and $\mathbb{D}^w_{\tilde{X}_2}=e^{Q/2}\sum\tilde{b}_{j,w}e^{\tilde{L}_j}$. Let $X=\bar{X}_1\#_\Sigma\bar{X}_2$ (for some identification). Then X is of simple type. For every $D\in H_2(X)$, consider any cappings $\tilde{D}_i\in H_2(\tilde{X}_i)$ with the condition above. Then*

$$\mathbb{D}^w_X(tD)=e^{Q(tD)/2}\Big(\sum_{\tilde{K}_i\cdot\Sigma=\tilde{L}_j\cdot\Sigma=2}\frac{1}{2}\tilde{a}_{i,w}\tilde{b}_{j,w}e^{t(\tilde{K}_i\cdot\tilde{D}_1+\tilde{L}_j\cdot\tilde{D}_2)}-\sum_{\tilde{K}_i\cdot\Sigma=\tilde{L}_j\cdot\Sigma=-2}\frac{1}{2}\tilde{a}_{i,w}\tilde{b}_{j,w}e^{t(\tilde{K}_i\cdot\tilde{D}_1+\tilde{L}_j\cdot\tilde{D}_2)}\Big).$$

Proof. The only remaining point is to prove that X is of simple type. This is proved as in corollary 5.15, using that (f_0,f_1) is determined by $D^{(w,\Sigma)}_{\tilde{X}_1}(e^{t\tilde{D}_1+s\Sigma})$ (analogue of corollary 5.14). $\square$

Corollary 5.24 *Under the conditions of theorem 5.23, X has no basic classes κ with $\kappa\cdot\Sigma=0$.*

Now we pass on to give some nice and simple applications of theorem 5.23. Probably, many results like the following can be obtained in the same fashion. We only want to give some examples to show its usefulness.

Corollary 5.25 *Let $\bar{X}_1$ and $\bar{X}_2$ be two manifolds with $b_1=0$ and embedded $\Sigma_i\subset\bar{X}_i$ of genus 2 and odd. Let ϕ and ψ be two different identifications for $Y=\Sigma\times\mathbb{S}^1$ and consider the two different connected sums along Σ, $X(\phi)$ and $X(\psi)$. Suppose that $\phi_*=\psi_*:H_1(Y)\to H_1(Y)$. Then there is an (non-canonical) isomorphism of vector spaces $H^2(X(\phi))\xrightarrow{\sim}H^2(X(\psi))$ sending the basic classes of $X(\phi)$ to those of $X(\psi)$ such that the rational numbers attached to them coincide.*

Proof. First we observe that we have a natural identification of the images I_ϕ of $H_2(X_1)\oplus H_2(X_2)\to H_2(X(\phi))$ and I_ψ of $H_2(X_1)\oplus H_2(X_2)\to H_2(X(\psi))$

since the kernels coincide. Now consider a splitting $H_2(X(\phi)) \cong \mathrm{Im}(I_\phi) \oplus V$ with $V \xrightarrow{\sim} H_1(Y)$. Choose an integral basis $\{\alpha\}$ for $H_1(Y;\mathbb{Z})$. For every α we have an element $D_\alpha \in H_2(X(\phi))$ which can be split as $D_\alpha = D_1 + D_2$, for $D_i \in H_2^R(X_i, \partial X_i)$ with $\partial D_1 = \gamma$, $-\partial D_2 = \phi(\gamma)$ and $\alpha = [\gamma]$. Now we leave D_1 (and $\tilde{D}_1 \in H_2(\tilde{X}_1))$ fixed and modify D_2 to glue it to D_1 in $H_2(X(\psi))$. Write $\tilde{D}_2 = D_2 + D_3 \in H_2(\tilde{X}_2)$. The loops $\phi(\gamma)$ and $\psi(\gamma)$ are homologous and hence there is homology $C = \mathbb{S}^1 \times [0,1] \hookrightarrow \Sigma \subset \Sigma \times \mathbb{S}^1$ between them. Consider

$$D_3' = \Big[D_3 \cup_{\phi(\gamma)} C \cup_{\psi(\gamma)} (\psi(\gamma) \times [0,\infty))\Big] + n\Sigma \in H_2^R(A, \partial A)$$

$$D_2' = \Big[D_2 \cup_{\phi(\gamma)} (-C) \cup_{\psi(\gamma)} (-\psi(\gamma) \times [0,\infty))\Big] - n\Sigma \in H_2^R(X_2, \partial X_2)$$

where n is chosen so that $D_3' \cdot K_1 = 0$. So $\tilde{D}_2' = D_2' + D_3'$. Consider $D_\alpha' = D_1 + D_2' \in H_2(X(\psi))$. The map $D_\alpha \mapsto D_\alpha'$ gives the sought isomorphism $H^2(X(\phi)) \xrightarrow{\sim} H^2(X(\psi))$, since $\tilde{D}_2 = \tilde{D}_2'$. $\square$

This corollary says that although in principle $X(\phi)$ and $X(\psi)$ might not be diffeomorphic (and probably in many cases this happens), they can not be distinguished by the number and coefficients of their basic classes. Still the polynomial invariants can differentiate both manifolds (maybe the intersection matrix of the basic classes could help). It would be desirable to find examples when this happens. The identifications to try out could be Dehn twists, as mentioned in remark 2.4.

Corollary 5.26 *Let $\bar{X}_1$ and $\bar{X}_2$ be two manifolds with $b_1 = 0$ and embedded $\Sigma_i \subset \bar{X}_i$ of genus 2 and odd. Let ϕ and ψ be two different identifications for $Y = \Sigma \times \mathbb{S}^1$ and consider the two different connected sums along Σ, $X(\phi)$ and $X(\psi)$. Suppose that $X(\phi)$ has only two basic classes $\pm\kappa$. Then the same is true for $X(\psi)$ and the coefficients coincide (up to sign). Also if the invariants of $X(\phi)$ vanish (no basic classes), so do the invariants of $X(\psi)$.*

Proof. We do the case of two basic classes. The other one is analogous. Suppose $\phi = \mathrm{Id}$, put $X = X(\phi)$ and let $\pm\kappa$ be the two basic classes, with $\kappa \cdot \Sigma = 2$. Let $c_{\kappa,w}$ be its coefficient. We now want to prove that this implies that there is only one basic class $\tilde{K}_i$ with $\tilde{K}_i \cdot \Sigma = 2$ and only one basic class $\tilde{L}_j$ with $\tilde{L}_j \cdot \Sigma = 2$. The result is obvious from that applying theorem 5.23.

Suppose that we can find $S_i \in H_2(\tilde{X}_i)$ with $\alpha = S_1 \cap [Y] = -S_2 \cap [Y] \in H_1(Y;\mathbb{Z})$ such that all the values $\tilde{K}_i \cdot S_1$ are different among them, and all the values $\tilde{L}_j \cdot S_2$ are also different among them (where $\tilde{K}_i$ and $\tilde{L}_j$ run through all the basic classes in $\tilde{X}_1$ and $\tilde{X}_2$ evaluating 2 on Σ). Then reorder the subindices in such a way that

$$\tilde{K}_1 \cdot S_1 < \tilde{K}_2 \cdot S_1 < \cdots < \tilde{K}_{n_1} \cdot S_1$$
$$\tilde{L}_1 \cdot S_2 < \tilde{L}_2 \cdot S_2 < \cdots < \tilde{L}_{n_2} \cdot S_2$$

We can easily arrange $D_i \in H_2^R(X_i, \partial X_i)$ with $\partial D_1 = -\partial D_2 = \gamma \in F\Omega(Y)$ with $[\gamma] = \alpha$ such that $\tilde{D}_i = S_i$. Set $D = D_1 + D_2 \in H_2(X)$ and apply equation (5.13). We have

$$c_{\kappa,w} e^{t\kappa \cdot D} = \sum_{\tilde{K}_i \cdot \Sigma = \tilde{L}_j \cdot \Sigma = 2} \frac{1}{2} \tilde{a}_{i,w} \tilde{b}_{j,w} e^{t(\tilde{K}_i \cdot S_1 + \tilde{L}_j \cdot S_2)}$$

Considering the exponentials with the smallest and with the largest exponents, we get that it has to be $\tilde{K}_1 \cdot S_1 + \tilde{L}_1 \cdot S_2 = \tilde{K}_{n_1} \cdot S_1 + \tilde{L}_{n_2} \cdot S_2$, from where the result.

To find the required collection of S_i, we consider all the differences $\alpha_{ij} = \tilde{K}_i - \tilde{K}_j$, $\beta_{ij} = \tilde{L}_i - \tilde{L}_j$, $i \neq j$. Consider $\alpha \in H_1(Y; \mathbb{Z})$ such that $\alpha \cdot \alpha_{ij} \neq 0$ for any α_{ij} which happens to be in the image of the homomorphism $H^1(Y) \cong H_2(Y) \hookrightarrow H_2(\tilde{X}_1) \cong H^2(\tilde{X}_1)$, and $\alpha \cdot \beta_{ij} \neq 0$ when β_{ij} is in the same condition with $\tilde{X}_2$ replacing $\tilde{X}_1$. Now we can choose $S_1 \in H_2(\tilde{X}_1)$ with $S_1 \cap [Y] = \alpha$ such that $\alpha_{ij} \cdot S_1 \neq 0$ (indeed the bad set is a finite union of hyperplanes). Analogously we choose S_2. $\Box$

5.5 The case of higher genus

We propose the following (see corollary 5.9)

Conjecture 5.27 *Let $\bar{X}_i$ have $b_1 = 0$, $b^+ > 1$ and be of simple type. Suppose that there are embedded surfaces $\Sigma_i \subset \bar{X}_i$ of genus $g \geq 3$, representing odd homology classes of self-intersection zero. Form $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ with some identification. Then X is of simple type and every basic class κ of X intersects Y in $n[\mathbb{S}^1]$ where n is an even integer with $-(2g-2) \leq n \leq (2g-2)$. Moreover the sum of the coefficients $c_{\kappa,w}$ of the different basic classes κ agreeing with $(K, L) \in H^2(X_1; \mathbb{Z}) \oplus H^2(X_2; \mathbb{Z})$ (i.e. $\kappa|_{X_1} = K$ and $\kappa|_{X_2} = L$) is zero unless $K|_Y = L|_Y = \pm(2g-2) P.D.[\mathbb{S}^1]$. In this latter case, it is $\pm 2^{7g-9}$ times the product*

$$\Big(\sum_{K_i|_{X_1} = K} a_{i,w}\Big) \cdot \Big(\sum_{L_j|_{X_2} = L} b_{j,w}\Big),$$

where $a_{i,w}$ are the coefficients of the basic classes K_i of X_1 (and similarly for $b_{j,w}$, L_j and X_2).

The factor $\pm 2^{7g-9}$ is the one agreeing with section 7.3. Here we propose a way of tackling this Conjecture. Call

$$HF_* = HF_*(\Sigma \times \mathbb{S}^1) = QH^{(6g-6)-*}(M_\Sigma^{\text{odd}}),$$

and let $u = \mu(\text{pt})$, $h = \mu(\Sigma)$ and $\Gamma = \sum \mu(\alpha_{2i})\mu(\alpha_{2i+1})$ be the generators of the invariant part of HF_* ($\{\alpha_i\}$ is a basis of $H_1(\Sigma)$ with $\alpha_{2i} \cdot \alpha_{2i+1} = 1$ and the other pairings zero). Actually this invariant part is generated as a vector space by $u^i h^j \Gamma^p$ with $i + 2p < g$ and $j + 2p < g$ [33]. Now we define I to be the ideal in HF generated by the image of $H^1(\Sigma)$ under μ. The space HF/I is generated by elements of the form $u^i h^j$ with $i < g$, $j < g$ (in principle they might not be linearly independent). Consider V any subspace of HF containing the orthogonal complement I^- of I such that it has generators $e_{ij} = u^i h^j \pmod{\text{I}}$, $i < g$, $j < g$. The dimension of V is $N = g^2$. We decompose $HF_* = V \oplus W$ with $W = V^- \subset I$. Obviously, we intend to get rid of the part W corresponding to the 1-homology.

Now we write

$$E = e^{sh + \lambda u + \alpha\Gamma} = \sum f_{ijp}(s, \lambda, \alpha) u^i h^j \Gamma^p.$$

We have that for every relation $R(h, u, \Gamma) = 0$ it is $R(\frac{\partial}{\partial s}, \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \alpha})E = 0$ and so $R(\frac{\partial}{\partial s}, \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \alpha}) f_{ijp} = 0$. Note also that $\frac{\partial}{\partial s^i}\frac{\partial}{\partial \lambda^j}\frac{\partial}{\partial \alpha^p} f_{i'j'p'}(0,0,0) = \delta^{ijp}_{i'j'p'}$ for $i + 2p < g$ and $j + 2p < g$. So the f_{ijp} are linearly independent functions. E defines a map from V to V (which we keep on calling E) by multiplication followed by orthogonal projection. This map is of the form $N_c g_c(s, \lambda, \alpha)$, where N_c are constant endomorphisms of V, $c = (i, j)$, $0 \leq i, j < g$ and $g_c(s, \lambda, \alpha)$ are linearly independent functions.

Let now X be an open manifold with boundary $Y = \Sigma \times \mathbb{S}^1$ and $D \in H_2^R(X, \partial X)$ with $\partial D = \mathbb{S}^1$. Then $\phi^w(X, D) \in HFF_*(Y, \mathbb{S}^1) = HF_* \otimes \hat{H}_*(\mathbb{CP}^\infty)$ has components $\phi^w(X, D)_V$ in V and $\phi^w(X, D)_W$ in W. When $b_1(X) = 0$ one expects to have $\phi^w(X, D)_W = 0$, as in lemma 5.10. Hence it would be

$$D_X^{(w,\Sigma)}(e^{tD}) = <\phi^w(X_1, e^{tD_1})_V, \phi^w(X_2, e^{tD_2})_V> .$$

So if either of $\bar{X}_i$ has $b_1 = 0$, then $D_X^{(w,\Sigma)}(e^{tD + s\Sigma + \lambda x}) = D_X^{(w,\Sigma)}(e^{tD + s\Sigma + \lambda x + \alpha\Gamma})$ is equal to

$$e^{ts} < \phi^w(X_1, e^{tD_1})_V, e_{ij} > (< e_{ij}, e_{i'j'} >)^{-1} < e_{i'j'}, E * \phi^w(X_2, e^{tD_2})_V > .$$

Write $\phi_a = \phi_a(t)$ for the components of $\phi^w(X_1, e^{tD_1})$ in V and $\psi_a = \psi_a(t)$ for the components of $\phi^w(X_2, e^{tD_2})$. Then

$$D_X^{(w,\Sigma)}(e^{tD+s\Sigma+\lambda x}) = e^{ts}\,\phi_a(t)\,M_{abc}\,\psi_b(t)\,g_c(s,\lambda,\alpha),$$

for some universal matrices M_{abc} depending only on Y. Now we can decompose $D_X^{(w,\Sigma)}(e^{tD+s\Sigma+\lambda x}) = e^{ts}\,D_{X,c}\,g_c(s,\lambda,\alpha)$ so $D_{X,c} = \phi_a\,(M_{abc}\,\psi_b)$ (note that we can choose the g_c corresponding to non-vanishing $D_{X,c}$ to be independent of α).

When $X_2 = A = D^2 \times \Sigma$, we put $a_b = \psi_b$. So we have constructed a map

$$\begin{array}{rcl} V \otimes \mathcal{F}(t) & \to & \mathbb{R}^N \otimes \mathcal{F}(t) \\ (\phi_a)_a & \mapsto & (\phi_a\,M_{abc}\,a_b)_c \end{array}$$

where $\mathcal{F}(t)$ is (for instance) the field of (Laurent) formal power series. Essentially, the map sends the "relative invariants" of X_1 to the "closed invariants" of $\bar{X}_1$, that is $\phi^w(X_1, e^{tD_1})$ is mapped to $D_{\bar{X}_1}^{(w,\Sigma)}(e^{t\bar{D}_1+s\Sigma+\lambda x})$ (more accurately to $D_{\bar{X}_1,c}$) To see that $D_{\bar{X}_1}$ determines ϕ_a we need to prove the injectivity of this linear map between vector spaces of the same dimension (or equivalently the surjectivity). Obviously,

Lemma 5.28 *Suppose we find a collection of $N = g^2$ quadruples (X, Σ, w, D) consisting of closed manifolds with embedded surfaces $\Sigma \subset X$ of genus g, self-intersection zero and representing odd homology classes with $D \cdot \Sigma = 1$, $D^2 = 0$, such that the functions $e^{-ts}D_X(e^{tD+s\Sigma+\lambda x})$ are linearly independent over $\mathcal{F}(t)$. Then the map above is an isomorphism.*

If this were proved, we could mimic the argument of section 5.3 to obtain the existence of some N by N universal matrix P whose coefficients depend on t and λ such that

$$D_X^{(w,\Sigma)}(e^{tD+\lambda x}) = (D_{\bar{X}_1,a}^{(w,\Sigma)}(e^{t\bar{D}_1}))(P_{ab}(t,\lambda))(D_{\bar{X}_2,b}^{(w,\Sigma)}(e^{t\bar{D}_2})).$$

When X is of simple type, $D_X^{(w,\Sigma)}(e^{tD+s\Sigma+\lambda x})$ is a linear combination of the functions $e^{2\lambda}e^{(2+4n)s}$ $(-[\frac{g+1}{2}] \leq n \leq [\frac{g-2}{2}])$ and $e^{-2\lambda}e^{4ns}$ ($-[\frac{g}{2}] \leq n \leq [\frac{g-1}{2}]$), see remark 1.12. So these functions are among the g_c (or they are combinations of them) and without loss of generality we can suppose they are the first $2g-1$ of the lot. With the same sort of arguments and one non-trivial example of the gluing where the basic classes were known, we would get the corresponding $(2g-1)$ by

$(2g-1)$ minor to be (conjecturally)

$$\begin{pmatrix} \text{coef} \cdot e^{2t \pm 2\lambda} & 0 & \cdots & 0 \\ 0 & \text{coef} \cdot e^{-2t \pm 2\lambda} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

We can further obtain more information from $X = \Sigma \times \mathbb{CP}^1$, but this gives us a total of $(2g)$ times $(2g)$ coefficients (very far from the g^2 times g^2 we seek for).

Chapter 6

Seiberg-Witten invariants

Since their introduction in late 1994, the Seiberg-Witten invariants have proved to be at least as useful as their close relatives the Donaldson invariants. When $b^+ > 1$, these provide differentiable invariants of a smooth oriented 4-manifold, whose construction is very similar in nature to the Donaldson invariants. Conjecturally, they give the same information about the 4-manifold, but they are much easier to compute in many cases, e.g. algebraic surfaces (see [63][25]). They have been used very successfully to obtain information about the differentiable structure of 4-manifolds [47][25] and about submanifolds of 4-manifolds [39][19]. Also Taubes has used them to prove strong theorems about symplectic manifolds [57][58].

The classification problem of simply connected 4-dimensional manifolds can not be solved with these invariants, but they may be the key for understanding the subcategory of symplectic 4-manifolds (see [57]). In any case it is intriguing to compute them for a general 4-manifold. The first step towards it is obviously to relate the invariants of a manifold with those of the manifold which results after some particular surgery on it. Some cases have been dealt with [63][17]. We are interested in the behaviour of this Seiberg-Witten invariants under a connected sum along a Riemann surface, as we have studied the behaviour of Donaldson invariants under the same operation and this can be a testing ground for the conjecture that both set of invariants are equivalent.

Also we would like to mention that Morgan, Szabó and Taubes [44] have carried out very similar work independently. This was pointed out to me by Szabó, who provided me with a copy of their work [44].

6.1 Seiberg-Witten invariants

We start off by recalling the definition of the Seiberg-Witten invariants. Let X be a smooth compact oriented four-manifold with $b^+ > 0$ (we will suppose later that $b_1 = 0$). We furnish it with a Riemannian metric g.

The $\mathrm{Spin}^{\mathbb{C}}(n)$ group is

$$\mathrm{Spin}^{\mathbb{C}}(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1).$$

A $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{c}$ on a Riemannian n-manifold is a lifting of the principle $SO(n)$ tangent bundle to a principle $\mathrm{Spin}^{\mathbb{C}}(n)$-bundle. There is a morphism $\mathrm{Spin}^{\mathbb{C}} \to U(1)$ given by $[B, \zeta] \mapsto \zeta^2$. Accordingly, every $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{c}$ has an associated complex line bundle L called its **determinant line bundle**. In general, the first Chern class of the determinant line bundle $c = c_1(L)$ is a lift of $w_2(X)$ to integer coefficients (we call that a **characteristic cohomology class**). Conversely, for any such a lift c, the possible $\mathrm{Spin}^{\mathbb{C}}$ structures with $c_1(L) = c$ are parametrised by the 2-torsion part of $H^2(X;\mathbb{Z})$. Therefore, if X is simply connected the $\mathrm{Spin}^{\mathbb{C}}$ structures are determined by c, which may be any characteristic class. Said otherwise, the set of $\mathrm{Spin}^{\mathbb{C}}$ structures is an affine space modelled on $H^2(X;\mathbb{Z})$, and fixing $\mathfrak{c}_0$ with determinant line bundle L_0, the other $\mathrm{Spin}^{\mathbb{C}}$ structures are $\mathfrak{c} = \mathfrak{c}_0 \otimes \mu$, $\mu \in H^2(X;\mathbb{Z})$, with determinant line bundle $L = L_0 \otimes \mu^2$.

In dimension four, there is an exact sequence

$$\mathrm{Spin}^{\mathbb{C}}(4) \hookrightarrow U(2)_+ \times U(2)_- \to U(1),$$

where the last map is $(A, B) \mapsto \det(A)\det(B)^{-1}$. So a $\mathrm{Spin}^{\mathbb{C}}$ structure has associated $U(2)$-bundles $W^+ = W^+_{\mathfrak{c}}$ and $W^- = W^-_{\mathfrak{c}}$. These are the two inequivalent irreducible $\mathrm{Spin}^{\mathbb{C}}$ bundles and $L = \Lambda^2 W^+ = \Lambda^2 W^-$ is the determinant line bundle of $\mathfrak{c}$. Clifford multiplication consists of a couple of maps

$$\rho : \Lambda^1_{\mathbb{C}} \to \mathrm{Hom}_{\mathbb{C}}(W^{\pm}, W^{\mp}).$$

They induce a map on two-forms

$$\rho : \Lambda^2_{\mathbb{C}} \to \mathrm{Hom}_{\mathbb{C}}(W^{\pm}, W^{\pm}).$$

If we consider an orthonormal basis $e_1, \ldots, e_4$ of $T^*_x X$, we have $\rho(e_i \wedge e_j) = \rho(e_i)\rho(e_j)$, $i \neq j$. We have that the map ρ splits as two homomorphisms

$$\rho : (\Lambda^2_{\pm})_{\mathbb{C}} \to \mathrm{Hom}_{\mathbb{C}}(W^{\pm}, W^{\pm}).$$

We can choose the map ρ to act as

$$\begin{aligned}
\frac{1}{2}(e_1 \wedge e_2 + e_3 \wedge e_4) &\mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
\frac{1}{2}(e_1 \wedge e_3 - e_2 \wedge e_4) &\mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
\frac{1}{2}(e_1 \wedge e_4 + e_2 \wedge e_3) &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

so $\rho : (\Lambda^2_+)_{\mathbb{C}} \to \mathfrak{sl}(W^+)$, taking the real forms into $\mathfrak{su}(W^+)$ and the imaginary form to self-adjoint operators.

Every connection A on L induces a Dirac operator $\not{D}_A : \Gamma(W^+) \to \Gamma(W^-)$ when coupled with the Levi-Civita connection on the tangent bundle of X. The monopole equations introduced by Seiberg and Witten [63] for a pair (A, Φ) of connection A on the line bundle L and section $\Phi \in \Gamma(W^+)$ are

$$\begin{cases} \not{D}_A \Phi = 0 \\ \rho(F_A^+) = (\Phi \otimes \Phi^*)_0 \end{cases} \tag{6.1}$$

where $\Phi^* \in (W^+)^* \cong \bar{W}^+$ is the conjugate of Φ obtained using the hermitian metric and $(\Phi \otimes \Phi^*)_0 \in \mathfrak{sl}(W^+)$ is the trace-free part of $(\Phi \otimes \Phi^*) \in \mathrm{End}(W^+)$.

The gauge group $\mathcal{G} = \mathcal{C}^\infty(X, \mathbb{S}^1)$ acts on the configuration space $\mathcal{A}(L) \times \Gamma(W^+)$ by $\sigma(A, \Phi) = (A - 2\sigma^{-1} d\sigma, \sigma\Phi)$ with quotient $\mathcal{B}$ (here we need to use suitable Sobolev completions). A solution (A, Φ) is reducible (i.e. has non-trivial stabiliser) if and only if $\Phi = 0$. Let $\mathcal{B}^*$ denote the subset of irreducible pairs. The cohomology ring of $\mathcal{B}^*$ is $H^*(\mathbb{CP}^\infty; \mathbb{Z}) \otimes \Lambda^* H^1(X; \mathbb{Z})$. When $b_1 = 0$, it will be generated by an element μ of degree two. If we consider evaluation in one point $\mathcal{G} \stackrel{\mathrm{ev}}{\to} \mathbb{S}^1$, denote the kernel by $\mathcal{G}^o$. Then $\mathcal{B}^o = \mathcal{A}^*/\mathcal{G}^o$ and $\mathcal{B}^o \to \mathcal{B}^*$ is a $U(1)$-bundle whose first Chern class is μ.

The moduli space of solutions of the equations (6.1) sits in $\mathcal{B}$ and will be denoted by $\mathcal{W}_{X,g}(\mathfrak{c})$. It has expected dimension

$$d = \frac{c^2 - (2\chi + 3\sigma)}{4},$$

where χ is the Euler characteristic of X and σ its signature. The moduli space is always compact. Whenever $b^+ > 0$ and $c_1(L)$ is not torsion, reducibles can be avoided for a generic metric. For obtaining a smooth moduli space we have to perturb the equations by adding a self-dual 2-form to F_A^+ [39]. As a self-dual 2-form is always the self-dual projection of a closed two form, we consider the equations

$$\begin{cases} \not{D}_A \Phi = 0 \\ \rho((F_A + i\,\eta)^+) = (\Phi \otimes \Phi^*)_0 \end{cases} \tag{6.2}$$

for η a closed real two-form. The moduli space of solutions is denoted by $\mathcal{W}_{X,g,\eta}(\mathfrak{c})$. For generic perturbation η there are no reducible solutions. Also this moduli space is smooth for generic perturbation (moreover we can suppose it is supported in a small ball). Both statements are also valid when $c_1(L)$ is torsion.

The moduli space is orientable and an orientation is determined by a choice of homology orientation for X (see [63]).

Let X be a compact, oriented 4-manifold endowed with a Riemannian metric g. Fix a homology orientation of X. Then for generic closed two forms η, the moduli space $\mathcal{W}_{X,g,\eta}(\mathfrak{c})$ is smooth, compact, oriented, free of reducibles and of dimension the expected dimension d. So if $d < 0$, it will be empty.

Definition 6.1 *We define the* ***Seiberg-Witten invariant*** $SW_{X,g,\eta}(\mathfrak{c})$ *for the* $Spin^{\mathbb{C}}$ *structure* $\mathfrak{c}$ *to be zero if* $d < 0$ *or if* d *is odd and to be*

$$SW_{X,g,\eta}(\mathfrak{c}) = < \mu^{d/2}, [\mathcal{W}_{X,g,\eta}(\mathfrak{c})] >$$

for d *even. Note that* d *is even when* b^+ *is odd (since* $b_1 = 0$*). When* $d = 0$, $SW_{X,g,\eta}(\mathfrak{c})$ *is the number of points of* $\mathcal{W}_{X,g,\eta}(\mathfrak{c})$ *counted with signs. When* $b^+ > 1$ *this number is independent of metrics and perturbations and is denoted* $SW_X(\mathfrak{c})$.

When $b^+ = 1$ we have to deal with reducibles that appear for generic paths of metrics and perturbations η. Let $b = [\eta] \in H^2(X;\mathbb{R})$. For fixed $\mathfrak{c}$ with $c_1(L) = c$ there are reducibles when $F_A + i\eta$ is ASD, i.e. when $[c + \frac{1}{2\pi} b] \cdot \omega_g = 0$, for $[\omega_g] \in \mathbb{H}$ the period point of g. So for every $b \in H^2(X;\mathbb{R})$ there is at most one wall. There are two possible invariants[1] $SW_X^{\pm}(\mathfrak{c})$ depending on whether $\pm[c + \frac{1}{2\pi} b] \cdot \omega_g > 0$.

Definition 6.2 *Let* c *be a characteristic cohomology class. We define* $SW_X(c)$ *to be the sum of* $SW_X(\mathfrak{c})$ *over all* $Spin^{\mathbb{C}}$ *structures* $\mathfrak{c}$ *with determinant line bundle with first Chern class* c *(note that there are a finite number of them).*

[1] We have to choose a component of the cone $\{x \in H^2(X;\mathbb{R}) / x^2 > 0\}$ and then ask the period point of every metric to lie in it. We suppose that this has always been done.

Definition 6.3 *Let X be a compact, oriented 4-manifold with $b_1 = 0$, $b^+ > 1$ and odd. Then we say that a characteristic cohomology class $c \in H^2(X;\mathbb{Z})$ is a* ***Seiberg-Witten basic class*** *(or a basic class for brevity) for X if $c^2 = 2\chi + 3\sigma$ and $SW_X(c) \neq 0$.*

One important remark in place is the fact that the set of classes with non-vanishing Seiberg-Witten invariant is always finite [63].

Definition 6.4 *For X compact, oriented 4-manifold with $b_1 = 0$ and $b^+ > 1$ and odd, we say that it is of* ***(Seiberg-Witten) simple type*** *if $SW_X(c) = 0$ whenever $d = (c^2 - (2\chi + 3\sigma))/4 > 0$.*

In this chapter and the next, basic class will always refer to Seiberg-Witten basic class and simple type to Seiberg-Witten simple type.

Witten has proved [63] that every Kähler surface is of simple type and ideas of Kronheimer and Mrowka give the basic classes explicitly (see [36]). Moreover Taubes [57] [58] has proved that for a symplectic four-manifold (X, ω) with $b^+ > 1$ the canonical class $K = -c_1(TX)$ is a basic class and that for any other basic class $\kappa \neq \pm K$, one has $|\kappa \cdot [\omega]| < K \cdot [\omega]$.

6.2 Seiberg-Witten equations for a three-manifold

Our main interest is the study of the behaviour of the Seiberg-Witten invariants under elementary surgeries. This amounts to splitting X along an embedded 3-manifold $Y \subset X$. So we have $X = X_1 \cup_Y X_2$, where X_1 and X_2 are manifolds with boundary. We orient Y so that $\partial X_1 = Y$ and $\partial X_2 = \overline{Y}$, Y with reversed orientation. We will have to consider families of metrics giving longer and longer necks. So we need to study the equations (6.1) for cylinders $Y \times \mathbb{R}$.

The simplest cases are those for which Y admits a metric of positive scalar curvature. For instance, for $Y = \mathbb{S}^3$ (i.e. X is a connected sum) we have that the hypothesis $b^+(X_i) > 0$ for both X_i leads in a straightforward way to the vanishing of all the invariants for X [63]. The case $b^+(X_1) = 0$ and $b^+(X_2) > 0$ is also of interest and we have, for instance, the following theorem about the behaviour of the Seiberg-Witten invariants under blowing-ups [19]

Proposition 6.5 *If X is of simple type and $\{K_i\}$ is the set of basic classes of X, then the blow-up $\tilde{X} = X \# \overline{\mathbb{CP}}^2$ is of simple type and (denoting by E the exceptional divisor) the set of basic classes are $\{K_i \pm E\}$.*

If we are interested on the study of equations (6.1) for a manifold $X = X_1 \cup_Y X_2$, the standard technique in Donaldson theory is to pull apart X_1 and X_2 so that we are led to consider metrics giving X_i a cylindrical end and L^2-solutions of the equations in these open manifolds. So we will have to use an analogue of this process in the Seiberg-Witten setting, first introduced in [39].

First we need to study the equations on the cylinder $Z = Y \times \mathbb{R}$. Let $\pi : Z \to Y$ be the projection. The coordinate on $\mathbb{R}$ will be t. Choose a product metric, i.e. $g = \pi^* h + dt \otimes dt$, for a metric h on Y.

The $\mathrm{Spin}^{\mathbb{C}}$ structures on Z correspond to $\mathrm{Spin}^{\mathbb{C}}$ structures on Y by pull-back (this corresponds to the natural morphism $\mathrm{Spin}^{\mathbb{C}}(3) \hookrightarrow \mathrm{Spin}^{\mathbb{C}}(4)$ induced by $SO(3) \hookrightarrow SO(4)$). Given a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{c}_Y$ on Y, there is only one spin (irreducible) bundle W_Y up to isomorphism. This is a rank two hermitian complex vector bundle with determinant line bundle L_Y. If $\mathfrak{c}$ is the corresponding $\mathrm{Spin}^{\mathbb{C}}$ structure on Z then its determinant line bundle is $L = \pi^*(L_Y)$. Also $W^\pm \cong \pi^* W_Y$ and the restriction of the action of the Clifford algebra $\mathrm{Cl}(Z)$ on W^+ to its even part corresponds with the action of $\mathrm{Cl}(Y)$ on W_Y under the isomorphism

$$\begin{array}{rcl} \pi^*\mathrm{Cl}(Y) & \to & \mathrm{Cl}_0(Z) \\ \alpha_0 + \alpha_1 & \mapsto & \alpha_0 + \alpha_1 dt \end{array} \tag{6.3}$$

There is also an isomorphism

$$\begin{array}{rcl} \pi^*\Lambda^1_Y & \to & \Lambda^2_+ \\ \alpha & \mapsto & \frac{1}{2}(\alpha \wedge dt + *_Y \alpha) \end{array} \tag{6.4}$$

Under this isomorphism, $\rho : \Lambda^1_Y \to \mathfrak{su}(W_Y)$ is given by (we denote $e_4 = dt$)

$$\begin{array}{rcl} e_1 & \mapsto & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ e_2 & \mapsto & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ e_3 & \mapsto & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \end{array}$$

Now for every pair (A, Φ) of connection on L and section of W^+, there is a gauge transformation putting A in temporal gauge. So we can suppose that A does not have component with dt. We can interpret this pair as a path $(A(t), \phi(t))$, $t \in \mathbb{R}$, in the configuration space $\mathcal{A}(L_Y) \times \Gamma(W_Y)$. The gauge group for Y will be $\mathcal{G}_Y = \mathcal{C}^\infty(Y, \mathbb{S}^1)$ and the quotient $\mathcal{B}_Y$. It is worth noticing that the gauge transformations preserving the temporal gauge conditions are pull-backs of elements in $\mathcal{G}_Y$, so there is an equivalence between points in $\mathcal{B}$ and paths in $\mathcal{B}_Y$. Again, the irreducible set will be denoted $\mathcal{B}_Y^*$.

Now let us rewrite equations (6.1) in this set up.

$$\not{D}_A \Phi = \sum e_i \cdot \nabla_{e_i} \Phi + e_4 \cdot \nabla_{e_4} \Phi \in \Gamma(W^-).$$

Multiplying by e_4 we get $-\sum e_i e_4 \cdot \nabla_{e_i} \Phi - \nabla_{e_4} \Phi \in \Gamma(W^+)$, which corresponds to

$$-\sum e_i \cdot \nabla_{e_i} \phi(t) - \frac{\partial \phi}{\partial t} \in \Gamma(W_Y).$$

Also $F_A = F_{A(t)} - \frac{\partial}{\partial t} A(t)\, dt$, so $F^+ = \frac{1}{2}(F + *F)$ corresponds to the one-form $*F_{A(t)} - \frac{\partial A}{\partial t}$. Equations (6.1) read

$$\begin{cases} d\phi/dt = -\not{\partial}_{A(t)} \phi(t) \\ dA/dt = *F_{A(t)} - i\tau(\phi) \end{cases} \tag{6.5}$$

where we have written $\tau : W_Y \to \Lambda_Y^1$ for the quadratic map $\tau(\phi) = -i\rho^{-1}((\phi \otimes \phi^*)_0)$. The solutions of equation (6.5) are the downward gradient lines of the functional

$$CSW(A, \phi) = \frac{1}{8\pi^2} \left(\int_Y F_{A_0} \wedge a + \frac{1}{2} \int_Y a \wedge da + \int_Y < \phi, \not{\partial}_A \phi > d\mathrm{vol} \right),$$

where A_0 is a fixed connection on L and $A = A_0 + a$ (changing the base connection changes the functional by addition of a constant). To prove this we have to compute

$$\nabla CSW_{(A,\phi)}(a, \delta) =$$

$$= \frac{1}{8\pi^2} \left(\int_Y F_A \wedge a + \int_Y < \phi, \frac{1}{2} a \cdot \phi > + \int_Y (< \delta, \not{\partial}_A \phi > + < \not{\partial}_A \phi, \delta >) d\mathrm{vol} \right),$$

where we have used that $\not{\partial}_A$ is self-adjoint. The $1/2$ appears because A is the induced connection on the determinant line bundle by the connection on W_Y and $L = \Lambda^2 W_Y$.

Lemma 6.6 *For $\phi \in W_Y$ and $a \in i\Lambda^1_Y$ purely imaginary form, one has $<\phi, a\cdot\phi> = -2 < a, i\tau(\phi)>$ (the first is a hermitian product, the second is a bilinear one).*

Proof. This is a calculation in a local basis at every point $x \in Y$. Choose an orthonormal basis e_i. We only have to use that the map $\tau : W_Y \to \mathfrak{su}(W_Y) \to \Lambda^1_Y$ is given by

$$\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(|\phi_0|^2 - |\phi_1|^2) & \phi_0\overline{\phi}_1 \\ \overline{\phi}_0\phi_1 & \frac{1}{2}(|\phi_1|^2 - |\phi_0|^2) \end{pmatrix} \mapsto$$

$$\mapsto -\mathrm{Im}(\phi_0\overline{\phi}_1)e_1 - \mathrm{Re}(\phi_0\overline{\phi}_1)e_2 + \frac{|\phi_0|^2 - |\phi_1|^2}{2}e_3.$$

□

So finally,

$$\nabla CSW_{(A,\phi)}(a,\delta) = \frac{1}{8\pi^2}(- < -(*F_A - i\tau(\phi)), a >_{L^2} + 2\,\mathrm{Re} < \partial\!\!\!/_A\phi, \delta >_{L^2}),$$

and the gradient of CSW (with respect to a metric in $\mathcal{A}(L_Y) \times \Gamma(W_Y)$ which is the L^2 metric on imaginary forms and twice the real part of the hermitian metric on spinors) gives equations (6.5).

Now we have an exact sequence $0 \to \mathcal{G}_0 \to \mathcal{G}_Y \to H^1(Y;\mathbb{Z}) \to 0$, where $\mathcal{G}_0$ is the component of the identity of $\mathcal{G}_Y$. For an element $\sigma \in \mathcal{G}_Y$

$$CSW(\sigma(A,\phi)) = CSW(A,\phi) + < [\sigma], c_1(L) > .$$

So CSW takes values in $\mathbb{R}/\mathbb{Z}$. The universal cover of $\mathcal{B}_Y$ is $\tilde{\mathcal{B}}_Y = \mathcal{A}(L_Y) \times \Gamma(W_Y)/\mathcal{G}_0$. Then $\mathcal{B}_Y = \tilde{\mathcal{B}}_Y/H^1(Y;\mathbb{Z})$. Therefore CSW can be lifted to a well defined functional on $\tilde{\mathcal{B}}_Y$ (up to a constant).

The critical points of CSW correspond to translation invariant solutions on the tube $Z = Y \times \mathbb{R}$. They are solutions to

$$\begin{cases} \partial\!\!\!/_A\phi = 0 \\ *F_A = i\tau(\phi) \end{cases} \tag{6.6}$$

The reducible solutions are those for which $\phi = 0$ and therefore $F_A = 0$. So they only appear when $c_1(L_Y) = 0$.

We consider a perturbation as in (6.2) given by a closed real two-form η. We suppose it to be translation invariant and with no dt-term, i.e. the pull-back of a two-form on Y. Then $\eta^+ = \frac{1}{2}(\eta + *_Y\eta\wedge dt) = \frac{1}{2}(*_Y\rho + \rho\wedge dt)$ and so it corresponds to a real coclosed one-form $\rho = *_Y\eta$ on Y. The equations we are led to are

$$\begin{cases} \partial\!\!\!/_A\phi = 0 \\ *F_A = i\tau(\phi) - i\rho \end{cases} \tag{6.7}$$

For any perturbation with $[*\rho] \neq -2\pi c_1(L)$ there are no reducibles. For generic perturbation, the moduli space is zero-dimensional [40, lemma 2.4] and therefore is a finite collection of points. So reducibles can be ruled out when $b_1(Y) \neq 0$. The space of perturbations giving a finite collection of irreducible points is path connected whenever $b_1(Y) \geq 2$. The perturbed functional is

$$CSW_\rho(A,\phi) = CSW(A,\phi) + \frac{1}{8\pi^2}\int_Y i*\rho\wedge a.$$

Remark 6.7 *The functional above is not full gauge invariant. Actually*

$$CSW_\rho(\sigma(A,\phi)) = CSW_\rho(A,\phi) + <[\sigma], c_1(L)> + < \left[\frac{*\rho}{2\pi}\right], [\sigma] > .$$

Since we are interested in small ρ, and we have to let it vary, the functional CSW will have a gauge behaviour dependent on the perturbation. To get around this problem we need to consider more general perturbations. For instance, we might try the following: take a neighbourhood of the singular set of CSW small enough such that the preimage under $\tilde{\mathcal{B}}_Y \to \mathcal{B}_Y$ is a collection of disjoint open sets. We fix one of them and consider a perturbation which is of the form (6.7) in a smaller neighbourhood and zero in the complement of the original neighbourhood. We define the perturbation in the other open sets by requiring the same gauge behaviour as CSW. Now we should prove that for a generic small perturbation of equations (6.6) like this, the moduli space of solutions on Y is finite and generic, and the deformed CSW functional does have a good gauge behaviour (this is not meant to be a proof).

We also have to define an index $\mathrm{ind}(A,\phi)$ associated to every (deformed) solution (A,ϕ). This is done fixing one of them and considering the spectral flow of the relevant operator as in [40]. The index is defined modulo N, where $< c_1(L), H^1(Y;\mathbb{Z}) >= N\mathbb{Z}$, since

$$\mathrm{ind}(\sigma(A,\phi)) = \mathrm{ind}(A,\phi) + <[\sigma], c_1(L)> .$$

For proving this, we should consider the mapping torus of σ, i.e. a line bundle $\mathcal{L} \to Y\times\mathbb{S}^1$ with $c_1(\mathcal{L}) = c_1(L) + 2[\sigma]\otimes[\mathbb{S}^1]$ (the factor 2 is due to the factor 2 in the action of σ on A). So the index will be $d = c_1(\mathcal{L})^2/4 =< [\sigma], c_1(L) >$.

6.3 Seiberg-Witten equations for $Y = \Sigma \times \mathbb{S}^1$

We are interested in the computation of basic classes of connected sums along Riemann surfaces. First we are going to recall briefly the set up from chapter 2. $\bar{X}_i$ are two smooth oriented four-manifolds and Σ is a Riemann surface of genus $g \geq 1$ such that we have embeddings $\Sigma \hookrightarrow \bar{X}_i$ with image Σ_i, representing a non-torsion element in homology, whose self-intersection is zero. We form the connected sum $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ by removing tubular neighbourhoods N_i of Σ_i in both $\bar{X}_i$ and gluing the boundaries Y and $\overline{Y}$ with some identification ϕ. We put $X_i = \bar{X}_i - N_i$, so $X = X_1 \cup_\phi X_2$. The boundaries are diffeomorphic to $\Sigma \times \mathbb{S}^1$. The diffeomorphism type of the resulting manifold depends on the isotopy class of ϕ (see subsection 2.1.2). There is an exact sequence

$$0 \to H^1(Y;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \xrightarrow{\pi} G \oplus H^1(\Sigma;\mathbb{Z}),$$

with G the subgroup of $H^2(\bar{X}_1;\mathbb{Z})/\mathbb{Z}[\Sigma_1] \oplus H^2(\bar{X}_2;\mathbb{Z})/\mathbb{Z}[\Sigma_2]$ consisting of elements (α_1, α_2) such that $\alpha_1 \cdot \Sigma_1 = \alpha_2 \cdot \Sigma_2$. The cokernel is a finite group (so it is torsion). There are two interpretations for this. The first one (reading the exact sequence in homology through Poincaré duality) says that the 2-homology of X is composed out of the 2-homology of Y plus those cycles which restrict to X_1 and X_2 having the same boundary 1-cycle in Y (here to be in $\pi^{-1}(G)$ means to have intersection with $Y = \Sigma \times \mathbb{S}^1$ a multiple of $[\mathbb{S}^1]$). The second interpretation says that a line bundle in X comes from gluing two line bundles in X_1 and X_2 and that the possible gluings are parametrised by $H^1(Y;\mathbb{Z})$.

Now we pass on to study equations (6.6) for $Y = \Sigma \times \mathbb{S}^1$. Call $p : Y \to \Sigma$ the projection and put θ for the $\mathbb{S}^1$ coordinate. We choose a metric for Y which is rotation invariant, i.e. $h = p^* g_\Sigma + d\theta \otimes d\theta$, with g_Σ a metric on Σ. Let $L \to \Sigma \times \mathbb{S}^1$ be a line bundle. We pull it back to $\Sigma \times [0,1]$ under the map identifying $\Sigma \times \{0\}$ with $\Sigma \times \{1\}$, and denote it by $\tilde{L}$. Then this line bundle is topologically the pull-back of some line bundle L_Σ on Σ. Clearly $c_1(L_\Sigma)$ is the restriction of $c_1(L)$ to Σ and L is obtained by pulling back L_Σ to $\Sigma \times [0,1]$ and gluing with some gauge transformation $g \in \mathcal{G}_\Sigma = \mathcal{C}^\infty(\Sigma, \mathbb{S}^1)$. The homotopy class of g is the component of $c_1(L)$ in $H^1(\Sigma;\mathbb{Z}) \otimes H^1(\mathbb{S}^1;\mathbb{Z})$ under the isomorphisms

$$[\Sigma; \mathbb{S}^1] \cong H^1(\Sigma;\mathbb{Z}) \cong H^1(\Sigma;\mathbb{Z}) \otimes H^1(\mathbb{S}^1;\mathbb{Z}),$$

where the last one is product with the fundamental class of the circle.

Any connection A on L gives a connection $\tilde{A}$ on $\tilde{L}$. This $\tilde{A}$ has a representative in its gauge equivalence class with no $d\theta$ component. This is unique up to constant gauge transformation (i.e. a gauge pulled-back from Σ). So giving A (up to gauge) is equivalent to giving a family A_θ, $\theta \in [0,1]$, of connections on L_Σ (up to constant gauge) with the condition $A_1 = g^*(A_0)$, with $g \in \mathcal{G}_\Sigma$ in the homotopy class determined by L.

Suppose now that we have a Spin$^{\mathbb{C}}$ structure $\mathfrak{c}$ on Y, which is determined uniquely by its determinant line bundle L (since $H^2(Y;\mathbb{Z})$ has no two-torsion). The Spin$^{\mathbb{C}}$ structure $\mathfrak{c}$ induces a Spin$^{\mathbb{C}}$ structure on Σ with determinant line bundle L_Σ. This latter one has two (irreducible) spin bundles, S^+ and S^-, which are $U(1)$-bundles. If K_Σ denotes the canonical bundle,

$$S^+ = \Lambda^0 \otimes \mu, \qquad\qquad S^- = \Lambda^{0,1} \otimes \mu,$$

where $\mu = (K_\Sigma \otimes L_\Sigma)^{1/2}$. So $L_\Sigma = K_\Sigma^{-1} \otimes \mu^2$. For a connection A on L_Σ the induced Dirac operators are

$$(\sqrt{2})\bar{\partial}_A : \Omega^0(\Sigma;\mu) \to \Omega^{0,1}(\Sigma;\mu)$$

and its adjoint $(\sqrt{2})\bar{\partial}_A^* : \Omega^{0,1}(\Sigma;\mu) \to \Omega^0(\Sigma;\mu)$. When restricting to every fibre $\Sigma \times \mathrm{pt} \subset \Sigma \times \mathbb{S}^1$, W_Y splits as $S^+ \oplus S^-$ ((ϕ_0,ϕ_1) corresponds to $\phi_0 + \phi_1 \cdot \frac{d\bar{z}}{\sqrt{2}}$, as $|d\bar{z}| = \sqrt{2}$) and the action of the Clifford algebra $\mathrm{Cl}(Y)$ corresponds with the action of $\mathrm{Cl}(\Sigma)$ under the inclusion.

We interpret every pair (A,ϕ) of connection on L and section of W_Y as a path (A_θ,ϕ_θ), $\theta \in [0,1]$, in the configuration space $\mathcal{A}(L_\Sigma) \times \Gamma(S^+ \oplus S^-)$. We again denote $\mathcal{B}_\Sigma$ for the quotient of this space with $\mathcal{G}_\Sigma$ and $\mathcal{B}_\Sigma^*$ for the subset of irreducible pairs. Writing $\phi_\theta = \alpha_\theta + \beta_\theta \in \Gamma(S^+ \oplus S^-)$, we have that the Dirac operator $\partial\!\!\!/_A : \Gamma(W_Y) \to \Gamma(W_Y)$ is identified with (we denote $e_3 = d\theta$)

$$\partial\!\!\!/_A \phi = \sum e_i \cdot \nabla_{e_i}\phi_\theta + e_3 \cdot \nabla_{e_3}\phi_\theta = \left[(\sqrt{2})\bar{\partial}_A\alpha + (\sqrt{2})\bar{\partial}_A^*\beta\right] + \frac{\partial}{\partial\theta}(i\beta - i\alpha).$$

Recall that the connection A_θ is on the bundle L_Σ and induces uniquely a connection B_θ on the bundle μ. Actually $F_{A_\theta} = 2F_{B_\theta} - F_{K_\Sigma}$, where F_{K_Σ} is the curvature of the Levi-Civita connection on K_Σ.

By the proof of lemma 6.6, $\phi = \alpha + \beta = \phi_0 + \phi_1 \cdot \frac{d\bar{z}}{\sqrt{2}}$ corresponds to

$$\tau(\phi) = -\mathrm{Im}(\phi_0\overline{\phi}_1)e_1 - \mathrm{Re}(\phi_0\overline{\phi}_1)e_2 + \frac{|\phi_0|^2 - |\phi_1|^2}{2}e_3 =$$

$$= -i\frac{\overline{\phi_0}\phi_1}{2}d\bar{z} + i\frac{\phi_0\overline{\phi_1}}{2}dz + \frac{|\phi_0|^2 - |\phi_1|^2}{2}e_3 =$$

$$= -\frac{\overline{\phi_0}\phi_1}{2} * d\bar{z} - \frac{\phi_0\overline{\phi_1}}{2} * dz + \frac{|\phi_0|^2 - |\phi_1|^2}{2}e_3 = - * \left(\frac{\alpha\overline{\beta}}{\sqrt{2}} + \frac{\overline{\alpha}\beta}{\sqrt{2}}\right) + \frac{|\alpha|^2 - |\beta|^2}{2}e_3,$$

since $*$ is complex linear and $dz = e_1 + ie_2$, $d\bar{z} = e_1 - ie_2$. Now $*F_A = \Lambda F_{A_\Sigma} e_3 + *_\Sigma(\frac{\partial A}{\partial \theta})$.

Lemma 6.8 *Let $(A, \phi) \in \mathcal{A}(L_Y) \times \Gamma(W_Y)$. Consider the family (A_θ, ϕ_θ), $\theta \in [0,1]$, determined by the pair and write $\phi_\theta = (\alpha_\theta, \beta_\theta) \in \Omega^0(\mu) \oplus \Omega^{0,1}(\mu)$. Then the solutions of (6.6) correspond to solutions of:*

$$\begin{cases} \frac{\partial \alpha}{\partial \theta} = -i\sqrt{2}\bar{\partial}^*_{A_\theta}\beta \\ \frac{\partial \beta}{\partial \theta} = i\sqrt{2}\bar{\partial}_{A_\theta}\alpha \\ \sqrt{2}\frac{\partial A_\theta}{\partial \theta} = -i(\alpha\overline{\beta} + \beta\overline{\alpha}) \\ 2\, i\Lambda F_{A_\theta} = -|\alpha|^2 + |\beta|^2 \end{cases} \tag{6.8}$$

In the third equation $\alpha\overline{\beta} + \beta\overline{\alpha} \in \Omega^1$ is a real two-form. Recall that the connection A_θ is equivalent to the holomorphic structure $\bar{\partial}_{A_\theta}$, so we can rewrite third line as either

$$\frac{\partial}{\partial \theta}(\bar{\partial}_{A_\theta}) = -\frac{i}{\sqrt{2}}\overline{\alpha}\beta \qquad \text{or} \qquad \frac{\partial}{\partial \theta}(\partial_{A_\theta}) = -\frac{i}{\sqrt{2}}\alpha\overline{\beta}.$$

Proposition 6.9 *Let $(\alpha, \beta) \in \Omega^0(\mu) \oplus \Omega^{0,1}(\mu)$ and A_θ, $\theta \in [0,1]$, such that*

$$\begin{cases} \frac{\partial \alpha}{\partial \theta} = -\sqrt{2}i\, \bar{\partial}^*_{A_\theta}\beta \\ \frac{\partial \beta}{\partial \theta} = \sqrt{2}i\, \bar{\partial}_{A_\theta}\alpha \\ \frac{\partial}{\partial \theta}(\partial_{A_\theta}) = -\frac{i}{\sqrt{2}}\alpha\overline{\beta} \end{cases} \tag{6.9}$$

Then A_θ, α and β are constant and either $\alpha = 0$ and $\bar{\partial}^*_{A_0}\beta = 0$ or $\beta = 0$ and $\bar{\partial}_{A_0}\alpha = 0$.

Proof. We work out the following expression (using $i\bar{\partial}^* = \Lambda\partial$ in $(0,1)$-forms and $|\beta|^2 = -i\Lambda\beta \wedge \overline{\beta}$)

$$\frac{\partial}{\partial \theta}(\bar{\partial}^*\beta) = -\frac{\partial}{\partial \theta}(i\Lambda\partial)\beta + \bar{\partial}^*(\frac{\partial \beta}{\partial \theta})$$

with the given equalities to get

$$-\frac{1}{\sqrt{2}i}\frac{\partial^2 \alpha}{\partial \theta^2} = i\Lambda\frac{i}{\sqrt{2}}\alpha\overline{\beta}\beta - \bar{\partial}^*(-i\sqrt{2}\bar{\partial}\alpha),$$

$$-\frac{\partial^2 \alpha}{\partial \theta^2} + \alpha|\beta|^2 + 2\bar{\partial}^*\bar{\partial}\alpha = 0,$$

where we drop subindices for convenience of notation. Take scalar product with α and integrate along Σ by parts to get, for every $\theta \in [0,1]$,

$$-\int_\Sigma < \frac{\partial^2 \alpha}{\partial \theta^2}, \alpha > + \int_\Sigma |\alpha|^2 |\beta|^2 + 2 \int_\Sigma |\bar{\partial} \alpha|^2 = 0.$$

This equation makes sense in $\mathbb{S}^1$, since the values for $\theta = 0$ and $\theta = 1$ coincide. Then we can integrate again by parts to get

$$|| \frac{\partial}{\partial \theta} \alpha ||^2 + ||\alpha \beta||^2 + 2 ||\bar{\partial} \alpha||^2 = 0.$$

The result is immediate from this. □

Now the fourth equation in (6.8) is constant. From lemma 6.8, $c_1(L|_\Sigma) = \frac{1}{4\pi} \int_\Sigma (|\alpha|^2 - |\beta|^2)$. So $\alpha = 0$ when $c_1(L|_\Sigma) < 0$ and $\beta = 0$ when $c_1(L|_\Sigma) > 0$. When $c_1(L|_\Sigma) = 0$ it must be $\alpha = \beta = 0$ and the solution will be reducible.

Corollary 6.10 *If the line bundle L admits any solution to (6.6) then L is pulled-back from Σ. Any solution is invariant under rotations in the $\mathbb{S}^1$ factor.*

Now let L be a characteristic line bundle on Y which is the pull-back of a line bundle in Σ. Since $\Sigma \cdot \Sigma = 0$ we have that $c_1(L) \cdot \Sigma$ is even. Consider the Spin$^{\mathbb{C}}$ structure $\mathfrak{c}_Y$ with determinant line bundle L. Then we have

Corollary 6.11 ([27]) *Suppose that $c_1(L_\Sigma) > 0$. Then the solutions of equations (6.8) are equivalent to the solutions of*

$$\begin{cases} \bar{\partial}_A \alpha = 0 \\ 2i\Lambda F_A = -|\alpha|^2 \end{cases} \tag{6.10}$$

on Σ. These are the typical vortex equations. The solutions are parametrised by the smooth algebraic variety $M_\Sigma = s^k \Sigma$ (the k-th symmetric product of Σ), where $c_1(L_\Sigma) = 2g - 2 - 2k$. If $c_1(L_\Sigma) > 2g - 2$ this space is empty.

Theorem 6.12 *Let X have $b_1 = 0$ and $b^+ > 0$ and odd. Suppose we have $\Sigma \subset X$ of genus $g \geq 1$ with self-intersection zero and representing a non-torsion class in homology. Let Y be the boundary of a tubular neighbourhood of Σ. If $SW_X(L) \neq 0$ then $|c_1(L) \cdot \Sigma| \leq 2g - 2$ and $L|_Y$ is a line bundle pulled back from Σ.*

Proof. This is a direct consequence of corollary 6.10 and the fact that the functional CSW is bounded when we stretch the neck into an infinite tube $Y \times \mathbb{R}$, as proved in [44, section 7.1] (they use the tube $\mathbb{R} \times Y$, which has opposite orientation, so in their case the functional is increasing for solutions on the tube). We can give a more direct proof, as suggested by Paul Seidel. For a closed manifold X and a pair (A, Φ) of connection and spinor as in section 6.1, we have, as in [11], the functional

$$E_X(A,\Phi) = \int_X \left(|\nabla_A \Phi|^2 + \frac{1}{4}|F_A|^2 + \frac{1}{8}(|\Phi|^2 + s)^2 \right) d\mathrm{vol},$$

where s is the scalar curvature. This can be rewritten as

$$E_X(A,\Phi) = \int_X \left(|\not{D}_A \Phi|^2 + \frac{1}{2}|F_A^+ - \rho^{-1}(\Phi \otimes \Phi^*)_0|^2 + \frac{1}{8}s^2 \right) d\mathrm{vol} - \pi^2 c_1(L)^2.$$

The Seiberg-Witten solutions minimise this functional to $-\pi^2 c_1(L)^2 + \int_X \frac{s^2}{8}$. For $Z = Y \times [0,T]$, we have the same functional, but when rewriting it there is an extra boundary term $\frac{1}{2}(CSW(A(0),\phi(0)) - CSW(A(T),\phi(T)))$.

Now consider $X = X_1 \cup (Y \times [0,T]) \cup X_2$. Then

$$E_X = E_{X_1} + E_{Y \times [0,T]} + E_{X_2}.$$

For solutions to the Seiberg-Witten equations on X, $E_X \leq \int_X \frac{s^2}{8} + K$ (with K a constant), $E_{Y\times[0,T]} = \int_{Y\times[0,T]} \frac{s^2}{8} + \frac{1}{2}(CSW(A(0),\phi(0)) - CSW(A(T),\phi(T)))$ and always $E_{X_i} \geq 0$. From here we deduce the boundness of the functional CSW. $\square$

Corollary 6.13 *Let $\bar{X}_i$ be smooth oriented manifolds with $b_1 = 0$ and $b^+ > 0$ and odd. Suppose we have $\Sigma_i \subset \bar{X}_i$ of the same genus $g \geq 1$ with self-intersection zero and representing a non-torsion class in homology. Construct $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ (choosing an identification). Then the intersection of the basic class with Y is $n[\mathbb{S}^1]$. Moreover, n is an even integer between $-(2g-2)$ and $(2g-2)$. In other words, every basic class of X lies in $\pi^{-1}(G)$ (see exact sequence (2.2) for definition of π).*

Remark 6.14 *We can paraphrase corollary 6.13 by saying that any basic class is orthogonal to $H_1(\Sigma;\mathbb{Z}) \otimes H_1(\mathbb{S}^1;\mathbb{Z}) \hookrightarrow H_2(Y;\mathbb{Z}) \hookrightarrow H_2(X;\mathbb{Z})$. We can prove that using the adjunction inequalities. For every torus $T_\gamma \subset Y$ (see definition 2.14), T_γ has self-intersection zero and hence $K \cdot T_\gamma = 0$ for any basic class K.*

To deal with the case of $c_1(L|_\Sigma) = 0$ we have to introduce perturbations. In this case the moduli space of solutions of (6.8) consists uniquely of reducibles and is isomorphic to the Jacobian of line bundles of degree $g-1$ over Σ. The perturbation we use is a real closed two-form η on Y which is $\mathbb{S}^1$-invariant (i.e. of the form $2\pi t\omega$, for ω the symplectic form of Σ), such that $< [\eta], \mathrm{P.D.}[\mathbb{S}^1] >= 2\pi t > 0$ and equation (6.10) becomes

$$\left\{ \begin{array}{l} \bar{\partial}_A \alpha = 0 \\ 2i\Lambda F_A = -|\alpha|^2 + 4\pi t \end{array} \right. \tag{6.11}$$

The solution space for these equations is $s^{g-1}\Sigma$. Now CSW_η takes values in $\mathbb{R}/t\mathbb{Z}$. We can further choose another extra (small) perturbation with a two-form not $\mathbb{S}^1$-invariant to make the moduli space zero-dimensional, but we should do this without destroying the behaviour of CSW_η under gauge transformations, as in remark 6.7.

When $b^+ = 1$, $c_1(L)\cdot\Sigma \neq 0$, one has to consider the Seiberg-Witten invariants corresponding to a metric with a long tube, i.e. to a metric with a period point $\omega_g \in \mathbb{H}$ close to $[\Sigma]$. So we consider $SW_X^\pm(L)$ for $\pm c_1(L)\cdot\Sigma > 0$. When $b^+ = 1$, $c_1(L)\cdot\Sigma = 0$, and we have chosen a perturbation of the form $\eta = 2\pi t\omega$ (plus a second small perturbation), we always refer to the invariant $SW_X^+(L)$, since $[\eta]\cdot\Sigma > 0$.

Chapter 7

Seiberg-Witten gluing theory

7.1 Seiberg-Witten-Floer homology

In the Seiberg-Witten context there is a parallel of the usual Floer theory for the Donaldson invariants developed in section 1.2. At the moment this topic is under development. Some nice few remarks about the case relevant to us appear in [11] and details have been carried out in [40] [62]. Wang [62] has studied the case of a homology 3-sphere and Marcolli [40] has analysed the case of a general three-manifold with a line bundle of non-zero first Chern class. Nonetheless, Seiberg-Witten-Floer theory must be considered still under construction while all the checking of details has to be completed. In this sense, this chapter is an application of this theory and relies upon the results of [11] [40] [62] which, although expected to be true, might require eventually minor modifications. Therefore this chapter is rather speculative and some of the results conjectural.

In general, for a three-manifold Y and a line bundle L_Y on Y, we perturb the Seiberg-Witten equations in the three-manifold as in equation (6.7). We will only have a finite number of solutions which are non-degenerate and irreducible. The problem with this perturbation is that the functional CSW_ρ is not well-defined, so we have to consider a more general perturbation as in remark 6.7. The solutions of the perturbed equation will be the generators of $CFSW_*(Y; L_Y)$. There is also an index $\mathrm{ind}(a)$ attached to every translation invariant solution $a = (A, \phi)$ (which is defined up to addition of a constant). We recall that this index is defined only in $\mathbb{Z}/N\mathbb{Z}$ with N such that $< c_1(L_Y), H^1(Y;\mathbb{Z}) >= N\mathbb{Z}$ (so when $c_1(L) = 0$, the index is an integer). For generic perturbations (they might need to be more generic than in equation (6.7) or remark 6.7), the moduli spaces $\mathcal{M}(a, b)$ are of dimension $\mathrm{ind}(b)-\mathrm{ind}(a)$, and they admit a free $\mathbb{R}$-action, with quotient $\mathcal{M}_0(a,b)$. These moduli spaces are also orientable[1], so we can define the boundary map as

$$\begin{array}{rcl} \partial : CFSW_i(Y) & \to & CFSW_{i-1}(Y) \\ a & \mapsto & \sum\limits_{\substack{b \\ \mathrm{ind}(a)=\mathrm{ind}(b)-1}} \#\mathcal{M}_0(a,b)b \end{array}$$

Then $\partial^2 = 0$ and one defines the **Seiberg-Witten-Floer homology groups** $HFSW_*(Y;L|_Y)$ as the homology of this complex. Whenever $c_1(L)$ is not torsion, these groups are independent of metrics and of (small) perturbations. When $c_1(L)$ is torsion, this is true as long as $b_1 \geq 2$, since the space of perturbations giving rise to reducibles is of codimension at least two.

Actually we could have defined these groups a bit more generally for every $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{c}_Y$ on Y.

The cohomology groups $HFSW^*(Y;L_Y)$ are defined similarly out of the dual complex and are naturally identified with the homology groups of Y with reversed orientation. We have a natural intersection pairing

$$\sigma : HFSW_*(Y) \otimes HFSW_{c-*}(Y) \to \mathbb{Z},$$

for some constant c.

Let X_1 be an open manifold with cylindrical end Y. For a line bundle L over X_1 whose restriction to Y is L_Y, the limit values of solutions to the (deformed) Seiberg-Witten equations give an element

$$\phi(X_1, L) = \sum_a \#\mathcal{M}(X_1, a)a \in CFSW_*(Y;L_Y).$$

This element is actually a cycle and defines a Floer-Seiberg-Witten homology class which is independent of metrics and deformations under the conditions above. When we have two open manifolds X_1 and X_2, which we want to glue along the common boundary Y (with a fixed diffeomorphism of the boundaries), and line bundles $L_i \to X_i$ with $L_i|_Y \cong L_Y$, there is an indeterminacy for choosing the identification of the line bundles over Y resulting in different line bundles for $X = X_1 \cup_Y X_2$, as it was explained in subsection 2.3.1.

Theorem 7.1 *For every compact oriented three-manifold Y and every characteristic line bundle L_Y, with either $c_1(L_Y) \neq 0$ or $b_1 \geq 2$, there are well-defined*

[1]The problem of giving orientations is analogous to the case of instanton Floer homology.

Seiberg-Witten-Floer homology groups $HFSW_*(Y; L_Y)$ (graded modulo N) independent of metrics and perturbations, with the following properties:

1. *Let X be an open manifold with boundary Y and $b^+ > 0$. Let L be a characteristic line bundle on X restricting to L_Y on Y. Then there is a well-defined a homology class $\phi(X, L) \in HFSW_*(Y; L_Y)$.*

2. *Let X be a closed manifold with $b^+ \geq 1$ which can be written as $X = X_1 \cup_Y X_2$ and let L_i be characteristic line bundles on X_i with $L_i|_Y = L_Y$. Then we have the following pairing formula*

$$\sigma(\phi(X_1, L_1), \phi(X_2, L_2)) = \sum_{\{L/L|_{X_i} = L_i,\ i=1,2\}} SW_X(L)$$

   If $b^+ = 1$ then the invariants refer to the chamber given as in the end of chapter 6.

In the case $b_1(X_i) = 0$ for both i, the possible L are parametrised by $H^1(Y; \mathbb{Z})$.

For $Y = \Sigma \times \mathbb{S}^1$ we only need to consider characteristic line bundles L whose restrictions to Y have $c_1(L|_Y) = 2m[\mathbb{S}^1]$, for $|m| \leq g-1$, as already established in theorem 6.13. Fix m, i.e. the topological type of $L|_Y$, and put $k = (g-1) - |m|$. Then (after a perturbation in the case $m = 0$), the moduli space of translation invariant solutions is $M_\Sigma = s^k\Sigma$. We have the following

Conjecture 7.2 ([11]) *For $c_1(L_Y) = 2m[\mathbb{S}^1]$, $k = (g-1) - |m|$, we have the following isomorphism*

$$HFSW_*(Y; L_Y) \xrightarrow{\sim} H_*(s^k\Sigma),$$

where the grading is reduced modulo $N = 2|m| = 2(g-1-k)$.

Remark 7.3 *Actually, Morgan, Szabó and Taubes [44] have developed the analytical details for the case $k = 0$, $g \geq 2$ of the above conjecture, proving it in that case, and using it to get a proof of the Thom conjecture for symplectic manifolds.*

Corollary 7.4 *Let $\bar{X}_1$ be a compact oriented four-manifold with $b^+ > 0$ such that there is an embedded Riemann surface of genus $g \geq 2$ and self-intersection zero representing a non-torsion homology class. Let L be a line bundle with*

$c_1(L)\cdot\Sigma = 2m \neq 0$, $|m| \leq g-1$. Let $A = \Sigma \times D^2$. Then there exists an element $\alpha = \phi(A, L|_A) \in H_*(s^k\Sigma)$ such that

$$\sum_{\{L'/c_1(L')=c_1(L)+n\Sigma\}} SW_{\bar{X}_1}(L') =< \phi(X_1, L|_{X_1}), \alpha >,$$

where $\phi(X_1, L|_{X_1}) \in H_*(s^k\Sigma)$ is the relative Seiberg-Witten invariant for X_1. When $\bar{X}_1$ is of simple type at most one of the L' can appear in the sum above, since at most one of them has $c_1(L')^2 = 2\chi + 3\sigma$.

When $b^+ = 1$, $c_1(L)\cdot\Sigma \neq 0$, one has to consider the Seiberg-Witten invariants corresponding to a metric with a long tube, i.e. to a metric with a period point $\omega_g \in \mathbb{H}$ close to $[\Sigma]$. So we consider $SW_X^{\pm}(L)$ for $\pm c_1(L)\cdot\Sigma > 0$.

7.2 Computations of basic classes

Now we state the gluing theorem about basic classes for a connected sum. Let us suppose $g \geq 2$. The different cases to be treated correspond to the possible restrictions of L to Σ. The easiest case is when $c_1(L)\cdot\Sigma = \pm(2g-2)$ and the situation gets more and more complicated as $k = (g-1) - \frac{1}{2}|c_1(L)\cdot\Sigma|$ gets bigger. Morgan, Szabó and Taubes have worked out the case $c_1(L)\cdot\Sigma = \pm(2g-2)$, carrying out the analysis explicitly and using perturbations as in equation (6.7). We deem that the analysis would be probably easier for perturbations as in remark 6.7, but details are yet to be carried out.

Theorem 7.5 ([44]) *Suppose that $\bar{X}_i$ have $b_1 = 0$, $b^+ > 1$ and are of simple type, and $g \geq 2$. Fix $\kappa_i \in H^2(\bar{X}_i;\mathbb{Z})$ characteristic with $\kappa_i^2 = 2\chi_{\bar{X}_i} + 3\sigma_{\bar{X}_i}$, such that $\kappa_i\cdot\Sigma_i = \pm(2g-2)$. Then*

$$\sum_{L|_{X_i}=\kappa_i|_{X_i},\, i=1,2} SW_X(L) = (\pm 1)^{g-1} SW_{\bar{X}_1}(\kappa_1)\cdot SW_{\bar{X}_2}(\kappa_2),$$

for appropriate homology orientations.

Proof. In the case $c_1(L_Y) = (2g-2)[\mathbb{S}^1]$, one has $k=0$ so $M_\Sigma = s^0\Sigma$ is a point and $H_*(M_\Sigma) \cong \mathbb{Z}$. Fix line bundles $L_i \to X_i$ and $L_A \to A$ that are isomorphic to L_Y when restricted to the boundary. Put $\alpha = \phi(A, L_A)$. Then

$$SW_{\bar{X}_i}(\kappa_i) = \sigma(\phi(X_i, L_i), \alpha) = x\,\phi(X_i, L_i)\alpha,$$

where $x = \sigma(1,1) \in \mathbb{Z}$ and $L_i = \kappa_i|_{X_i}$. Now for $\mathbb{CP}^1 \times \Sigma$ with a metric giving $\mathbb{CP}^1$ long volume, one has only the basic class $(2(g-1)\mathbb{CP}^1 - 2\Sigma)$ with Seiberg-Witten invariant 1. So $x\,\alpha^2 = 1$ and it must be $x = 1$ and $\alpha = \pm 1$. Therefore the sum in the left hand side above is

$$\sum_{\substack{L|_{X_1}=L_1 \\ L|_{X_2}=L_2}} SW_X(L) = \sigma(\phi(X_1, L_1), \phi(X_2, L_2)) = SW_{\bar{X}_1}(\kappa_1) \cdot SW_{\bar{X}_2}(\kappa_2).$$

For the case $c_1(L_Y) = -(2g-2)[\mathbb{S}^1]$, we have that $\mathbb{CP}^1$ has only the basic class $-(2(g-1)\mathbb{CP}^1 - 2\Sigma)$ with Seiberg-Witten invariant $(-1)^{g-1}$, responsible for the sign. $\square$

The sign above can be checked as in remark 5.8, since $d_0(X) \equiv d_0(\bar{X}_1) + d_0(\bar{X}_2) - 3(g-1) \pmod 2$. Now we analyse some examples in which the information already gathered in theorem 7.5 is enough to find the basic classes for the glued manifold.

Proposition 7.6 *Suppose that we are in the situation of remark 2.19 and suppose that both $\bar{X}_i$ are of simple type and $g \geq 2$. Then $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ is of simple type and the basic classes κ of X such that $\kappa \cdot \Sigma = \pm(2g-2)$ are in one-to-one correspondence with pairs of basic classes (κ_1, κ_2) for $\bar{X}_1$ and $\bar{X}_2$ respectively, such that $\kappa_1 \cdot \Sigma_1 = \kappa_2 \cdot \Sigma_2 = \pm(2g-2)$. Moreover, κ is determined in an explicit way.*

Proof. By remark 2.19 we have a primitive lattice $V \subset H_2(X;\mathbb{Z})$ generated by homology classes represented by tori of self-intersection zero. The basic classes vanish on all of these homology classes (see for instance theorem 6.12). So if κ is a basic class for X, we have argued that P.D.$[\kappa] \in V^-$.

Let S be a (-2)-sphere provided by remark 2.19 such that $[S] \in V$ and so $L \cdot S = 0$ whenever $SW_X(L) \neq 0$. From [19, theorem 1.3], we know that if L is a line bundle with $SW_X(L) \neq 0$ and $\dim \mathcal{W}_X(L) > 0$ then $SW_X(L \pm 2S) \neq 0$. But this is a contradiction as $(L \pm 2S) \cdot S \neq 0$. So X is of simple type.

Now let κ be a basic class for X with $\kappa \cdot \Sigma = 2g-2$. Corollary 6.13 tells us how the image of κ under π is. From the previous theorem, there are basic classes κ_1 and κ_2 in $\bar{X}_1$ and $\bar{X}_2$ such that $\kappa \cdot \Sigma = \kappa_1 \cdot \Sigma_1 = \kappa_2 \cdot \Sigma_2 = 2g-2$. As $g \geq 2$, we have that $\kappa^2 \neq (\kappa + n\Sigma)^2$ for $n \neq 0$, so at most one of the $\kappa + n\Sigma$ can be basic class. Also κ is determined as the only class in V^- agreeing with both $\kappa_i|_{X_i}$ and with square $\kappa^2 = 2\chi_X + 3\sigma_X = \kappa_1^2 + \kappa_2^2 + (8g-8)$. $\square$

Remark 7.7 *In the situation of the last proposition, and under the splittings of remark 2.20, we have*

$$\kappa_i = \alpha_i + (2g-2)\bar{D}_i + r_i\Sigma_i \in W_i^- \oplus \mathbb{Q}[\Sigma_i, \bar{D}_i],$$

for the basic classes κ_i of X_i. The corresponding basic class for X is

$$\kappa = 0 + \alpha_1 + \alpha_2 + (2g-2)D + (r_1 + r_2 + 2)\Sigma \in V \oplus W_1^- \oplus W_2^- \oplus \mathbb{Q}[\Sigma, D],$$

where the coefficient of Σ is found out using the requirement on κ^2. So formally

$$\kappa = \kappa_1 + \kappa_2 + 2\Sigma. \tag{7.1}$$

The condition in proposition 7.6 is indeed a very natural condition. For example if we have a Kähler manifold which is a fibration $X \to C$ over a complex curve C with fibres being generically genus g Riemann surfaces and if we take a smooth fibre $\Sigma \subset X$ and a vanishing cycle $\gamma \subset \Sigma$ (see [24, page 167]), then the vanishing disc is a (-1)-disc (the framings are the natural framings of γ **inside** Σ). If for instance there is a rational fibre then all the 1-cycles in Σ are vanishing cycles and the hypotheses in the theorem above are satisfied. As a consequence, when we glue two of these fibrations along a fibre we get the same Seiberg-Witten invariants for basic classes κ with $\kappa \cdot \Sigma = \pm(2g-2)$, regardless of the chosen gluing, although in general one expects that only for one particular gluing the resulting manifold is a Kähler surface.

Remark 7.8 *Suppose both $\bar{X}_i$ are symplectic manifolds and Σ_i are symplectic submanifolds. Then from recent work of Taubes [59], $\bar{X}_i$ are of simple type. Now $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ can be given a symplectic structure by proposition 2.12 (regardless of the homotopy class of the chosen gluing ϕ). Taubes [57] [58] has proved that the canonical class $K = -c_1(TX)$ is a basic class and that for any other basic class $\kappa \neq \pm K$, one has $|\kappa \cdot [\omega]| < K \cdot [\omega]$, with ω the symplectic form. Since $T_\gamma \cdot [\omega] = 0$, none of the $K + \sum n_\gamma T_\gamma$ can be basic classes unless all $n_\gamma = 0$. Hence in the formula of theorem 7.5 only one term appears in the sum. Notice that Taubes also proves that this number is ± 1.*

The result of the last remark falls very short since it does not even tell us about the other basic classes that might appear when we glue two basic classes K_i for $\bar{X}_i$ with $K_i \cdot \Sigma = 2g-2$ but K_i are not the canonical classes. In some situations we get more: suppose that (both) $\bar{X}_i$ have $b^+ > 1$ and are the blow-up of some symplectic manifolds M_i of simple type at points on $\Sigma_i' \hookrightarrow M_i$ (and Σ_i is the proper transform of Σ_i') and that the cohomology class defined by the symplectic forms in both M_i are Poincaré dual to $[\Sigma_i']$ (i.e. $[\Sigma_i']$ are ample classes). Then one has $|\kappa_i \cdot [\Sigma_i]| < K \cdot [\Sigma_i] = 2g - 2$ for every basic class κ_i in $\bar{X}_i$. So we conclude that the only basic classes with $\kappa \cdot \Sigma = \pm(2g-2)$ for X are $\kappa = \pm K$.

Suppose now the case when the manifolds involved are complex surfaces and Σ_i are embedded complex curves. If the holomorphic normal bundles to Σ_i are (orientation reversing) isomorphic then there is a preferred identification as explained in section 2.2.

Proposition 7.9 *Suppose that both $\bar{X}_i$ are Kähler manifolds with embedded complex curves Σ_i of self-intersection zero. Suppose that X is an algebraic surface which is deformation equivalent to the variety $\bar{X}_1 \cup_\Sigma \bar{X}_2$ with a normal crossing along Σ. Then the basic classes κ of X such that $\kappa \cdot \Sigma = \pm(2g-2)$ are in one-to-one correspondence with pairs of basic classes (κ_1, κ_2) for X_1 and X_2 respectively, such that $\kappa_1 \cdot \Sigma_1 = \kappa_2 \cdot \Sigma_2 = \pm(2g-2)$.*

Proof. Recall from proposition 2.9 that X has the diffeomorphism type of the (preferred) connected sum of $\bar{X}_1$ and $\bar{X}_2$ along Σ. First, it is known after Witten [63] that all Kähler manifolds with $b^+ > 1$ are of simple type. Also by proposition 2.9, $X = \bar{X}_1 \#_\Sigma \bar{X}_2$. As in the proof of proposition 7.6, we just need to prove that if κ is a basic class for X and $T = \sum n_\beta T_\beta \neq 0$ in $H_2(Y;\mathbb{Z}) \subset H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$ then $\kappa + T$ is not basic class. In the Kähler case we know that the basic classes are in $H^{1,1}$, so it is enough to show that $T \notin H^{1,1}$. But $T^2 = 0$ and $T \cdot [\omega] = 0$, for the symplectic form ω. If T were in $H^{1,1} \cap H^2(X;\mathbb{Z})$, it would represent a divisor with $T^2 = 0$ and orthogonal to an ample class, but this is impossible. $\Box$

The case of $k = 1$

The following natural case to pursue is $c_1(L_\Sigma) = \pm(2g-4)$, with $g \geq 3$. Obviously this corresponds to $k = (g-1) - \frac{1}{2}|c_1(L_\Sigma)| = 1$. So $HFSW_*(\Sigma \times \mathbb{S}^1, L_Y) = H_*(\Sigma)$, with the grading being modulo $N = g-2$. Every open manifold X with $b^+ > 0$ and boundary $\partial X = Y$, and line bundle $L \to X$ with $c_1(L|_Y) = (2g-4)[\mathbb{S}^1]$, defines a class

$$\phi(X, L) = (f_0, f_1, f_2) \in H_*(\Sigma).$$

For a closed manifold X with $b^+ \geq 1$ which can be written as $X = X_1 \cup_Y X_2$ and L_i characteristic line bundles on X_i with $L_i|_Y = L_Y$, we put $\phi(X_1, L_1) = (f_0, f_1, f_2)$ and $\phi(X_2, L_2) = (g_0, g_1, g_2)$. Then

$$\sum_{\{L/L|_{X_i}=L_i,\, i=1,2\}} SW_X(L) = f_0 \cdot g_2 + f_2 \cdot g_0 + f_1 \cdot g_1.$$

If we have that for different identifications the sum in the left hand side remains unchanged, then $f_1 \cdot \phi_*(g_1)$ is constant, for all $\phi \in \mathrm{Diff}^+(\Sigma)$, which forces either $f_1 = 0$ or $g_1 = 0$. For example, for $A = \Sigma \times D^2$, one has $X = A \cup_\phi A = \Sigma \times \mathbb{CP}^1$ and the invariants $SW_X(L) = 0$ for any line bundle L with $c_1(L) \cdot \Sigma = 2g - 4$ (in the chamber given by $[\Sigma]$), as there are no zero-dimensional non-empty moduli spaces with that condition. So writing $\phi(A, L_A) = (a_0, a_1, a_2)$, it must be $a_1 = 0$ and either $a_0 = 0$ or $a_2 = 0$. Let us suppose $(a_0, a_1, a_2) = (0, 0, 1)$. Then for every closed manifold of simple type $X = X^o \cup_Y A$ and characteristic line bundle L with $c_1(L) = \pm(2g - 4)$, one has $SW_X(L) = f_0$, where $\phi(X^o, L|_{X^o}) = (f_0, f_1, f_2)$.

Now one should look to different cappings X_2 to extract f_1 and f_2 from the Seiberg-Witten invariants of $\tilde{X} = X^o \cup X_2$. We will not say anymore about this, but it seems very promising and we hope to come back to it for future research.

7.3 Final remarks on the comparison of Donaldson and Seiberg-Witten theories

Here we would like to point out the close relationship between the results in both parts of the thesis. Witten [63] has conjectured that for a simply connected manifold the condition of being simple type and Seiberg-Witten simple type are equivalent, and that in that case the basic classes are the same as the Seiberg-Witten basic classes[2], the shape of the Donaldson series being

$$\mathbb{D}_X^w = e^{Q/2} \sum a_{i,w} e^{K_i},$$

where

$$a_{i,w} = (-1)^{\frac{K_i \cdot w + w^2}{2}} 2^{2 + \frac{1}{4}(7\chi + 11\sigma)} SW_X(K_i).$$

[2]Let us remark here that Pidstrigatch and Tyurin [52] have a program to prove rigorously this relationship in general.

We have from theorem 7.5 that for $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ and basic classes K_i for $\bar{X}_i$ such that $K_1 \cdot \Sigma = K_2 \cdot \Sigma = 2g - 2$ one has

$$\sum_{L|_{X_i} = K_i|_{X_i},\, i=1,2} SW_X(L) = SW_{\bar{X}_1}(K_1) SW_{\bar{X}_2}(K_2).$$

Now we recall that the topological numbers are computed in subsection 2.1.3 and are $\chi_X = \chi_{\bar{X}_1} + \chi_{\bar{X}_2} + 4g - 4$ and $\sigma_X = \sigma_{\bar{X}_1} + \sigma_{\bar{X}_2}$. So

$$2 + \frac{1}{4}(7\chi_X + 11\sigma_X) = 2 + \frac{1}{4}(7\chi_{\bar{X}_1} + 11\sigma_{\bar{X}_1}) + 2 + \frac{1}{4}(7\chi_{\bar{X}_2} + 11\sigma_{\bar{X}_2}) + (7g - 9).$$

When $g = 2$, this tells us that the sum of the coefficients of the (Seiberg-Witten) basic classes L such that $L|_{X_i} \cong K_i|_{X_i}$, $i = 1, 2$, is 32 times the product of the coefficients of K_i and L_j. This agrees with corollary 5.9 about the (instanton) basic classes. We want to remark here that the result in the Seiberg-Witten context is more general in the sense that we do not impose restriction in the genus $g \geq 2$, but the results in chapter 5 are far more general in the sense that they also give information about basic classes K with $K \cdot \Sigma = 0$ and more explicit information about the structure of the Donaldson invariants. Nonetheless, it is highly likely that Seiberg-Witten-Floer theory can provide results of this kind.

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