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Pedro Daniel GONZÁLEZ PÉREZ

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Étude des singularités quasi-ordinaires d'hypersurfaces au moyen de la géométrie torique

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M. Alain CHENCINER	Examinateur
M. Vincent COSSART	Rapporteur
Mme. Monique LEJEUNE-JALABERT	Examinatrice
M. Joseph LIPMAN	Rapporteur
M. Ignacio LUENGO VELASCO	Examinateur
M. Bernard TEISSIER	Directeur

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Introduction

Dans ce mémoire nous utilisons des méthodes de la géométrie des variétés toriques pour étudier les singularités quasi-ordinaires d'hypersurfaces et en particulier leurs résolutions plongées des singularités.

Un germe de variété analytique complexe est quasi-ordinaire s'il existe un morphisme fini sur l'espace affine complexe tel que son lieu discriminant est contenu dans un diviseur à croisements normaux. Ces singularités, qui généralisent les singularités des courbes, apparaissent classiquement dans la *méthode de Jung* pour analyser les singularités d'une hypersurface.

Un germe réduit d'hypersurface quasi-ordinaire $(S,0) \subset (\mathbf{C}^{d+1},0)$ analytiquement irréductible à l'origine peut être défini par une équation $f=0$, où $f \in \mathbb{C}\{X\}[Y]$ est le polynôme minimal d'une série de puissances fractionnaires ζ dans les variables $X=(X_1,\dots,X_d)$ munie d'un nombre fini de *monômes caractéristiques* qui généralisent les exposants caractéristiques de Newton-Puiseux obtenus lorsque $d=1$. On dispose, grâce aux travaux de Gau et Lipman, d'une caractérisation du type topologique plongé de cette hypersurface en fonction des monômes caractéristiques. Ce résultat est une généralisation de la caractérisation du type topologique plongé d'un germe de courbe plane complexe par les exposants caractéristiques des composantes irréductibles de la courbe et les multiplicités d'intersection des paires des ces composantes ; cette donnée est équivalente à la connaissance de certains invariants de la résolution minimale de la courbe.

Le résultat principal de ce mémoire est la construction d'une résolution plongée d'une hypersurface quasi-ordinaire *réduite* $(S,0) \subset (\mathbf{C}^{d+1},0)$ comme composition de morphismes toriques qui sont déterminés par les monômes caractéristiques. Ceci donne une réponse positive à une question posée par Lipman dans [L5]. Dans le cas analytiquement irréductible nous construisons, suivant le travail de Goldin et Teissier pour les courbes planes, une deuxième résolution plongée du germe $(S,0)$, replongé dans l'espace affine $(\mathbf{C}^{d+g},0)$ en utilisant certaines racines approchées de f, avec un seul morphisme torique. Ce morphisme est aussi une résolution plongée d'une variété torique affine $Z^{\Gamma} \subset \mathbf{C}^{d+g}$ obtenue de $S \subset \mathbf{C}^{d+g}$ par spécialisation et définie par un semi-groupe Γ de rang d. Nous faisons une comparaison entre ce deux résolutions et nous montrons que la première coïncide avec la restriction de la deuxième à certaine variété lisse de dimension $d+1$ contenant la transformée stricte de S.

Le semi-groupe Γ généralise le semi-groupe classique d'une branche plane obtenu lorsque $d=1$.

Ses éléments sont les sommets des polyèdres de Newton des résultants des polynômes f et h, lorsque h parcourt les polynômes de $\mathbb{C}\{X\}[Y]$ qui ne sont pas divisibles par f. Nous montrons que ce semi-groupe est un invariant complet du type topologique plongé de $(S,0)$ tel que caractérisé par les travaux de Gau et Lipman.

Puisque les singularités quasi-ordinaires sont des revêtements ramifiés de $\mathbb{C}^d$ étales sur $(\mathbb{C}^*)^d$, il est naturel d'étudier les revêtements ramifiés d'une variété torique affine normale Z_ρ qui sont non ramifiés au dessus de tore $(\mathbb{C}^*)^d$, que nous appelons *singularités quasi-ordinaires toriques.* Le cas hypersurface relative à la base Z_ρ apparaît naturellement dans les étapes intermédiaires de la résolution plongée de $(S,0) \subset (\mathbb{C}^{d+1},0)$. Pour cette raison nous construisons les résolutions toriques plongées pour ces singularités, en généralisant à ce cas la notion de monômes caractéristiques et certaines de ses propriétés dans le cas classique.

Les singularités quasi-ordinaires d'hypersurface sont étudiés classiquement dans le cas algebroïde, sur un corps algebrébriquement clos de caractéristique zéro, en utilisant les paramétrisations en séries de puissances fractionnaires dont l'existence est garantie par le théorème de Jung-Abhyankar. Les techniques et résultats du premier chapitre permettent d'obtenir le lien entre ce deux points de vue, en particulier grâce à l'algorithme à la Newton Puiseux qui représente les racines d'un polynôme de Weierstrass $F \in \mathbb{C}\{X\}[Y]$ comme des paramétrisations des singularités quasi-ordinaires toriques définies par $F=0$ après des changements de base déterminés par le polyèdre de Newton du discriminant $\Delta_Y F$ de F par rapport à Y.

Motivations

Résolution des singularités

Nous rappelons quelque définitions et motivations sur la notion de résolution des singularités.

Une *résolution des singularités* d'une variété $\mathcal{X}$ est un morphisme propre $\mathcal{Y} \to \mathcal{X}$ d'une variété lisse $\mathcal{Y}$ qui est un isomorphisme hors du lieu singulier de $\mathcal{X}$. Si $\mathcal{X}$ est une sous-variété de $\mathcal{W}$ de complémentaire dense, *une résolution plongée* de $\mathcal{X} \subset \mathcal{W}$ est un morphisme propre $\mathcal{W}' \to \mathcal{W}$, où $\mathcal{W}'$ est une variété lisse, tel que la restriction à la transformée stricte $\mathcal{X}'$ de $\mathcal{X}$ du morphisme $\mathcal{X}' \to \mathcal{X}$ est une résolution des singularités de $\mathcal{X}$ et que $\mathcal{X}'$ est transverse au diviseur exceptionnel.

On peut interpréter ici *variété* comme variété algébrique ou comme variété analytique complexe ou réelle. Un diviseur à croisements normaux est le type le plus simple de singularité : en tout point de l'espace ambiant l'idéal de définition du diviseur est engendré par un monôme dans un système local de coordonnées. Hironaka a montré l'existence d'une résolution plongée de singularités des variétés définies sur un corps de caractéristique zéro et d'espaces analytiques complexes ou réels (pour des références voir [L2]). Bierstone et Milman, et Villamayor ont donné des algorithmes canoniques de résolution basés sur le travail d'Hironaka.

La résolution (non plongée) d'un germe de courbe plane complexe coïncide avec sa normalisation et est obtenue grâce aux paramétrisations des ses composantes irréductibles (fournies par le Théorème de Newton-Puiseux). La résolution plongée (minimale) d'un germe de courbe plane, qui remonte à M. Noether, est obtenue en éclatant de forme successive les points singuliers de la courbe et de ses transformées jusqu'à obtenir que la transformée totale de la courbe est un diviseur à croisements normaux. Le rapport entre les paramétrisations à la Newton-Puiseux et la résolution plongée est bien connu depuis les travaux d'Enriques et Chisini : par exemple si le germe est une *branche plane*, c'est à dire analytiquement irréductible à l'origine, la suite des multiplicités des branches planes obtenues en éclatant les points singuliers des transformées totales de la courbe jusqu'à obtenir la résolution plongée minimale est déterminée et détermine les *exposants caractéristiques* de la paramétrisation de Newton Puiseux lorsque les axes de coordonnées ne sont pas tangents au germe, voir [Z7] et [Z3].

Jung a prouvé qu'il est possible de recouvrir un voisinage du point singulier d'un représentant d'un germe de surface par un nombre fini de paramétrisations en séries de puissances à deux variables. Les singularités quasi-ordinaires apparaissent naturellement comme celles qui sont recouvertes par une seule paramétrisation qui est obtenue par des arguments topologiques (voir [J]). L'existence de cette paramétrisation est prouvé par Abhyankar dans un cadre plus général en utilisant des arguments algébriques (voir [A1]). La preuve donnée par Jung fournit une méthode pour analyser les singularités d'une hypersurface à partir d'une résolution plongée du lieu discriminant réduit de sa projection sur une variété lisse par un morphisme fini (voir [L2], Lecture 2). Par exemple, si $\mathcal{X} \subset \mathbb{C}^{d+1}$ est une hypersurface il existe une projection $\mathbb{C}^{d+1} \to \mathbb{C}^d$ dont sa restriction à $\mathcal{X}$ est un morphisme fini $\pi : \mathcal{X} \to \mathbb{C}^d$. Le lieu discriminant réduit du morphisme π, c'est-à-dire, l'image des points de $\mathcal{X}$ où le morphisme π n'est pas un isomorphisme local, est une hypersurface $\mathcal{D} \subset \mathbb{C}^d$. Si le morphisme $\mathcal{Y} \to \mathbb{C}^d$ est une résolution plongée de $\mathcal{D} \subset \mathbb{C}^d$ nous obtenons alors un diagramme commutatif canonique :

$$\begin{tikzcd}
\mathcal{W} \arrow[d] & \supset & \tilde{\mathcal{X}} \arrow[r, "\pi'"] \arrow[d] & \mathcal{Y} \arrow[d] & \supset & \mathcal{D}' \arrow[d] \\
\mathbb{C}^{d+1} & \supset & \mathcal{X} \arrow[r, "\pi"] & \mathbb{C}^d & \supset & \mathcal{D}
\end{tikzcd}$$

où la variété $\mathcal{W}$ est non singulière. La projection finie π' est un isomorphisme local hors du diviseur à croisements normaux D' (qui est l'image inverse de l'hypersurface D) donc les singularités de l'hypersurface $\tilde{\mathcal{X}}$ sont quasi-ordinaires. Des travaux fondamentaux sur les singularités et leurs résolutions dans le cas de surfaces et de variétés de dimension trois ont été faits par Walker, Zariski, Abhyankar et Hirzebruch entre autres, suivant la méthode de Jung.

Quelques résultats sur les hypersurfaces quasi-ordinaires

Un germe d'hypersurface quasi-ordinaire peut être défini par une équation $f = 0$, où $f \in \mathbb{C}\{X\}[Y]$ est un polynôme de Weierstrass tel que le discriminant de f par rapport à Y a un monôme dominant dans les variables $X = (X_1, \ldots, X_d)$, c'est-à-dire qu'il est de la forme $\Delta_Y f = X^\delta \epsilon$ pour ϵ une unité dans l'anneau des séries convergentes (ou formelles). Le théorème de Jung-Abhyankar garantit l'existence de paramétrisations de $(S, 0)$ de la forme $Y = \zeta$ où ζ est une série de puissances fractionnaires en X, qui est en fait une racine de f dans une extension d'anneaux convenable. Les séries fractionnaires qui sont racines d'un tel polynôme s'appellent des *branches quasi-ordinaires*[1]. Puisque le discriminant de f est le produit des différences de racines différentes de f, chacune de ces différences a un monôme dominant. Nous appelons les monômes ainsi obtenus les *monômes caractéristiques* de f. Les résultats principaux sur les singularités quasi-ordinaires d'hypersurface concernent principalement le cas où le germe $(S, 0)$ est analytiquement irréductible à l'origine.

L'approche classique pour obtenir une résolution d'une surface quasi-ordinaire S ramène le problème à sa normalisation $\bar{S}$, qui est encore une surface quasi-ordinaire et de plus est une singularité quotient (voir [L4], §2). Une résolution de la surface $\bar{S}$ peut être obtenue en éclatant un idéal de dimension zéro (voir [L2]). La normalisation $\bar{S}$ est munie d'une structure de surface torique (voir [B-P-V], Chapter III, Theorem 5.2 et plus généralement Proposition 3.14) et nous pouvons obtenir sa résolution minimale par un morphisme torique (voir [Od]).

Lipman donne une autre méthode pour obtenir une résolution non plongée d'une surface quasi-ordinaire en fonction des monômes caractéristiques. D'abord, il associe à une branche quasi-ordinaire arbitraire une branche quasi-ordinaire *normalisée*[2] par un *lemme d'inversion*. Ensuite, il construit cette résolution comme composition des transformations quadratiques et monoïdales qui déterminent et sont déterminées par les monômes caractéristiques de la branche quasi-ordinaire normalisée (voir [L1] et [L3]). En particulière, il obtient que ces monômes caractéristiques sont un invariant analytique du germe. Luengo donne une autre preuve de ce résultat (voir [Lu]).

De manière plus générale Lipman a remarqué que les monômes caractéristiques d'une branche quasi-ordinaire déterminent le *type topologique plongé*[3] du germe grâce aux résultats de Zariski sur la saturation des anneaux locaux (voir [Z5], [L4] et [Oh]). Gau a montré qu'en fait ils définissent un invariant complet du type topologique. Sa preuve passe par la description donnée par Lipman du groupe local de classes de diviseurs d'un germe d'hypersurface quasi-ordinaire en fonction des monômes caractéristiques (voir [Gau] et [L4]). Ce résultat est une généralisation de la caractérisation

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[1] Il n'y a pas lieu de confusion entre cette notion algébrique et la notion géométrique de branche plane

[2] Il s'agit d'une condition technique sur la forme des monômes caractéristiques qui dans le cas des courbes planes veut dire que la courbe $X = 0$ n'est pas contenue dans le cône tangent au germe.

[3] Deux germes d'hypersurface analytique complexe $(S, 0)$ et $(S', 0)$ dans $(\mathbb{C}^{d+1}, 0)$ ont le même *type topologique plongé* s'il existe un homéomorphisme de deux voisinages ouverts de l'origine qui transforme un représentant du germe $(S, 0)$ en un représentant du germe $(S', 0)$.

du type topologique d'un germe de courbe plane complexe par les exposants caractéristiques des composants irréductibles de la courbe et les multiplicités d'intersection des paires des composantes (voir [Re], [Z5] et [Z6]).

Singularités quasi-ordinaires et généralisations de la notion d'équisingularité

Zariski a développé la notion d'équisingularité en codimension un et caractéristique zéro, obtenant plusieurs conditions équivalentes au fait que dans une hypersurface V qui définit une famille de courbes planes à fibre speciale réduite le type topologique des fibres est constant, voir [Z3],[Z6], [A2] et [T3].

L'équisingularité en codimension un du germe $(V, 0)$ implique que son lieu singulier est de codimension un et est équivalent, grâce au critère discriminant de Zariski (voir [Z6]), à ce que le germe soit quasi-ordinaire et que le lieu discriminant réduit de la projection quasi-ordinaire associée ait une seule composante irréductible (qui est donc non singulière). Dans ce cas, une résolution de $(V, 0)$ est obtenue par éclatements successifs du lieu singulier de V et ses transformés stricts qui correspondent aux éclatements de points infiniment voisins nécessaires pour obtenir une résolution minimale de l'une des fibres. Réciproquement l'existence d'une telle résolution implique la constance du type topologique des fibres.

Dans [L5], Lipman étudie des généralisations possibles de la notion d'équisingularité en particulier le rapport entre la classification par équirésolution, par le type topologique et le *type dimensionnel* de Zariski ; (ce dernièr est défini par récurrence à partir du type dimensionnel du lieu discriminant d'une projection finie générique sur un espace lisse, le cas d'équisingularité en codimension un correspondant à type dimensionnel égal à un).

Un pas important dans cette direction est le problème ouvert 5.1 de [L5] : déterminer si les monômes caractéristiques associés à un germe de singularité quasi-ordinaire permettent de définir une procédé canonique de résolution plongée. Un tel procédé a été trouvé par Ban and McEwan (voir [B-M]) dans le cas des germes de surfaces analytiquement irréductibles et par Villamayor en toute généralité (voir [Vi]). Les deux procédés, qui suivent des points de vues très différents, utilisent les algorithmes de résolution qui ont étés développés suivant le travail de Hironaka. Dans le troisième chapitre nous donnons une autre solution au problème de Lipman.

Contributions de ce mémoire

Les contributions principales de ce mémoire sont :

- Une généralisation de la construction des paramétrisations à la Newton-Puiseux pour des germes d'hypersurface analytique complexe qui fait apparaître les singularités quasi-ordinaires en rapport avec les variétés toriques.
- L'association d'un semi-groupe à un germe analytiquement irréductible d'hypersurface quasi-ordinaire. Ce semi-groupe, qui généralise le semi-groupe classique d'une branche plane, est un invariant complet du type topologique plongé du germe tel que caracterisé par Gau.
- La construction de procédés de résolution plongée des singularités d'un germe $(S,0)$ d'hypersurface quasi-ordinaire qui ne dependent que des monômes caractéristiques. Si le germe $(S,0)$ est analytiquement irréductible nous présentons deux résolutions : la première est une résolution plongée comme hypersurface qui est composée de g de morphismes toroïdaux (où g est le nombre de monômes caractéristiques) ; la deuxième est un morphisme torique qui est une résolution plongée simultanée du germe $(S,0)$, replongé dans l'espace affine $(\mathbb{C}^{d+g},0)$, et d'une variété monomiale obtenue du plongement de $(S,0)$ par spécialisation et définie par le semi-groupe associé au germe. Nous comparons ces deux résolutions et nous montrons que la première est la restriction de la deuxième à une section transverse à la fibre exceptionnelle. Nous obtenons ainsi que le type topologique plongé du germe $(S,0)$ détermine une résolution plongée.

Le **premier chapitre**, qui a été aussi le premier chronologiquement, introduit des méthodes de la géométrie des variétés toriques qui sont le lien entre la constructions des paramétrisations et l'étude postérieure des singularités quasi-ordinaires et de leurs résolutions plongées. Le sujet de ce chapitre est la représentation des racines $Y(X)$ d'un polynôme de Weierstrass $F \in \mathbb{C}\{X\}[Y]$ à coefficients dans l'anneau $\mathbb{C}\{X\}$ des séries de puissances en les variables $X = (X_1, \dots, X_d)$ absolument convergentes dans un voisinage de l'origine. Il s'agit de généraliser le Théorème de Newton-Puiseux. Nous poursuivons une direction inaugurée par McDonald dans [McD], et précisons ses résultats.

Notre approche est d'étudier d'abord le problème dans le cas d'un polynôme F en l'indéterminée Y à coefficients dans l'anneau des germes des fonctions holomorphes à l'origine d'une variété affine normale Z_τ de dimension d. Nous résolvons le probleme lorsque $F=0$ défini une *hypersurface quasi-ordinaire torique*, c'est-à-dire, lorsque le *discriminant* $\Delta_Y F$ de F par rapport à Y ne s'annule pas sur le tore $(\mathbb{C}^*)^d \subset Z_\tau$ dans un voisinage de l'origine. Ceci est en fait une généralisation de l'étude classique des singularités quasi-ordinaires obtenues lorsque $Z_\tau = \mathbb{C}^d$. La réduction du cas général à ce cas particulier fait appel à des constructions combinatoires sur le polyèdre de Newton $\mathcal{N}(F) \subset \mathbb{R}^{d+1}$ de F. La plus importante, déjà utilisée dans [McD] est celle du *polyèdre-fibre* $\mathcal{Q}(F) \subset \mathbb{R}^d$ de la projection du polyèdre de Newton de F sur l'espace des exposants des monômes de Z_τ. Les sommets du polyèdre $\mathcal{Q}(F)$ correspondent à certains chemins dans les arêtes de $\mathcal{N}(F)$. Le polyèdre-fibre est également relié au polyèdre de Newton du discriminant de F par rapport à Y. Si $F = a_0 + a_1 Y + \cdots + a_r Y^r$, on a l'inclusion de polyèdres de Newton

$$\mathcal{N}(\Delta_Y F) + \mathcal{N}(a_0) + \mathcal{N}(a_r) \subseteq \mathcal{Q}(F)$$

(où la somme est la somme de Minkowski), avec l'égalité sous des hypothèses de non-dégénérescence des coefficients de F par rapport à $\mathcal{N}(F)$, (Théorème 1.4).

Le résultat principal de ce chapitre, le théorème 1.3, garantit que les représentations de racines du polynôme $F \in \mathbb{C}\{X\}[Y]$, obtenues à la Newton-Puiseux grâce au théorème 1.2, correspondent aux paramétrisations des singularités quasi-ordinaires toriques définies par $F = 0$ après des changements de base qui sont déterminés par le polyèdre de Newton du discriminant de F. Nous utilisons ces résultats pour donner une description des sommets du polyèdre de Newton du *résultant* $\mathrm{Res}_Y(F, G)$ des polynômes $F, G \in \mathbb{C}\{X\}[Y]$ à partir des polyèdres-fibres $\mathcal{Q}(F)$, $\mathcal{Q}(G)$ et $\mathcal{Q}(FG)$; nous calculons enfin les coefficients des monômes correspondant aux sommets du polyèdre de Newton de $a_0 a_r \Delta_Y F$ et de $\mathrm{Res}_Y(F, G)$ sous des hypothèses de non dégénérescence. Ces coefficients dépendent entre autres des résultants des paires de polynômes à une variable obtenus en ne regardant de F et de G que les termes dont les exposants appartiennent à des paires parallèles d'arêtes de $\mathcal{N}(F)$ et $\mathcal{N}(G)$. On montre un résultat analogue pour le discriminant. Nous trouvons aussi, avec une méthode très différente, des résultats de même nature que ceux de Gel'fand, Kapranov et Zelevinski dans [G-K-Z].

Dans le **deuxième chapitre** nous étudions un germe analytiquement irréductible d'hypersurface quasi-ordinaire d'équation $f = 0$. Nous pouvons suposser que f est un polynôme unitaire dans $\mathbb{C}\{X\}[Y]$, pour $X = (X_1, \ldots, X_d)$, tel que le discriminant $\Delta_Y f$ est de la forme $\Delta_Y f = X^{\delta}\epsilon$, où $\epsilon(0) \neq 0$.

Nous généralisons à ce cas certaines propriétés de la multiplicité d'intersection d'une branche plane avec d'autres germes de courbe plane en étudiant la structure combinatoire de l'ensemble des faces compactes des polyèdres de Newton du résultant de f et des polynômes dans l'anneau $\mathbb{C}\{X\}[Y]$. En particulier nous montrons que l'ensemble des sommets des polyèdres de Newton des résultants $\mathrm{Res}_Y(f, h)$, pour les polynômes h qui ne sont pas divisibles par f, est un semi-groupe de rang d (voir Theorem 2.1 et Corollaire 2.2). Lorsque $d = 1$ nous obtenons le semi-groupe classique d'une branche plane[4]. Les polyèdres de Newton des résultants de f et des polynômes minimaux de certaines sommes partielles de la série ζ ainsi que les coordonnées $X_1, \ldots, X_d$ ont un seul sommet. L'ensemble des sommets ainsi obtenus est un système minimal de générateurs du semi-groupe. Le même résultat reste valable avec les mêmes générateurs, lorsque on remplace chacun des polynômes minimaux par la *racine approchée*[5] de f du même degré. De plus ces racines approchées sont des polynômes quasi-ordinaires irréductibles et leurs racines ont des ordres de coïncidence prescrits, qui sont certains monômes caractéristiques, avec la série ζ (voir Proposition 2.14). L'ensemble des générateurs du semi-groupe obtenu est relié aux monômes caractéristiques par des formules analogues

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[4] Le semi-groupe d'une branche plane est l'ensemble de ses multiplicités d'intersection avec les germes de courbe plane qui ne la contiennent pas comme composante.

[5] Voir [A-M], [G-P] et [PP1] pour la notion de racine approchée et ses propriétés dans le cas des germes de courbes planes

à celles qui apparaissent dans le cas des branches planes, le lien étant fait grâce à un ensemble fini de réseaux et à des entiers associés aux monômes caractéristiques. Nous montrons que les semi-groupes associés à une branche quasi-ordinaire ζ et à la branche normalisée associée par le lemme d'inversion de Lipman sont isomorphes. Le système minimal de générateurs du semi-groupe, qui est unique, correspond aux monômes caractéristiques d'une branche normalisée quelconque définissant la même singularité. Nous obtenons ainsi que le semi-groupe est un invariant complet du type topologique de $(S, 0)$ tel que caractérisé par les travaux de Gau et Lipman. Nous utilisons le semigroupe pour donner une autre preuve des formules d'inversion entre les exposants caractéristiques de ζ et de la branche quasi-ordinaire normalisée associée.

Dans le **troisième chapitre** nous construisons une résolution plongée d'un germe quasi-ordinaire réduit $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$ qui est une composition de morphismes toriques déterminés par les monômes caractéristiques. Comme dans le cas des germes de courbes planes, nous distinguons ceux qui sont associés aux composantes irréductibles du germe $(S, 0)$ de ceux qui donnent les ordres de coïncidence entre les paramétrisations des composantes différentes du germe $(S, 0)$ (voir Theorem 3.2).

Nous commençons par un résultat en collaboration avec Teissier : la construction d'une résolution torique plongée d'une variété torique affine non nécessairement normale plongée de façon équivariante (voir Proposition 3.6, [GP-T], et Proposition 6.4 [T5]). Nous généralisons cette méthode au cas d'une hypersurface quasi-ordinaire avec un seul monôme caractéristique en utilisant un argument de *non dégénérescence* (voir l'article de Khovanskii [Kho]). De manière plus générale, le premier morphisme torique de la résolution est défini par le diagramme de Newton dual d'un polynôme adéquat $f \in \mathbb{C}\{X\}[Y]$ définissant le plongement $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$. Adéquat veut dire que Y est une *bonne coordonnée*, une notion définie en fonction des ordres de coïncidence des paramétrisations des facteurs irréductibles de f. Nous montrons que les singularités de la transformée stricte de S sont plus simples et sont munies d'une projection quasi-ordinaire canonique. En particulier si le germe $(S, 0)$ est analytiquement irréductible, à chaque pas de la résolution la transformée stricte perd un monôme caractéristique et l'une des racines approchées du polynôme f a le contact maximal : elle fournit une bonne coordonnée pour le morphisme torique suivant. Par récurrence nous obtenons une *résolution plongée torique partielle* : une variété normale de dimension $d + 1$ munie d'un structure de plongement toroïdale, déterminée par les monômes caractéristiques, et d'une modification de $(\mathbb{C}^{d+1}, 0)$ telle que la transformée stricte du germe $(S, 0)$ est une section transversale à la fibre exceptionnelle. Ainsi, toute résolution torique de l'espace ambiant fournit une résolution plongée de la transformée stricte et à posteriori de la singularité S. De plus, cette résolution partielle est une *normalisation plongée* car sa restriction à la transformé stricte de S est justement le morphisme de normalisation. L'existence de cette résolution partielle est un pas important pour aboutir à la construction des résolutions plongées des hypersurfaces suivant la méthode de Jung (voir [Z1] et [L2]). Dans le cas des germes de courbes planes notre point de vue se rapproche de celui de A'Campo et Oka pour une branche plane (voir [A'C-Ok]) : notre méthode fournit une construction torique de la résolution minimale de la courbe à partir des exposants caractéristiques et des ordres de coïncidence lorsque la courbe $X = 0$ n'est pas contenue dans le cône tangent au germe. Voir aussi les travaux de Lê et Oka [Le-Ok] et [Ok].

Dans le cas analytiquement irréductible nous construisons une autre résolution plongée du germe $(S, 0)$ en généralisant la méthode de Goldin et Teissier pour les branches planes (voir [G-T]). Nous définissons un morphisme torique π qui est une résolution plongée partielle du germe S, convenablement replongé dans une variété torique affine Z_Δ en utilisant certaines racines approchées de f, et d'une variété torique affine $Z^\Gamma \subset Z_\Delta$ qui est obtenu de $S \subset Z_\Delta$ par spécialisation (voir Theorem 3.4). La variété Z^Γ est la variété torique définie par le semi-groupe Γ associé au germe $(S, 0)$. Finalement, nous faisons une comparaison entre ces deux résolutions partielles et nous montrons que la première coïncide avec la restriction de la deuxième à une certaine variété lisse de dimension $d + 1$ contenant la transformée stricte de S lorsque le morphisme torique π est convenablement choisi. Des résultats de nature analogue sont obtenus par Lejeune et Reguera pour des singularités sandwich des surfaces et pour des singularités des branches planes (voir [LJ-R] et [LJ-R2]).

Quelques perspectives et problèmes ouverts

Nous mentionnons ici quelques résultats et problèmes ouverts reliés a ce travail.

Dans le cadre de l'étude topologique des singularités, il reste le problème de déterminer si les monômes caractéristiques d'un polynôme quasi-ordinaire *réduit* déterminent un invariant complet du type topologique de la singularité qu'il définit, en généralisant ainsi la situation pour les germes de courbes planes et le cas quasi-ordinaire analytiquement irréductible. La résolution plongée que nous définissons est un outil important pour décrire la fonction zêta de la monodromie d'une singularité quasi-ordinaire d'hypersurface. En collaboration avec McEwan et Némethi nous obtenons que si les variables $X_1, \dots, X_d$ sont bien ordonnées la fonction zêta associé à $f = 0$ coïncide avec celle de la courbe plane $f = X_2 = \dots X_d = 0$.

Une autre application possible des procédés de résolution plongée des hypersurfaces quasi-ordinaires est la construction d'une méthode de résolution plongée des hypersurfaces suivant l'approche de Jung.

Il reste à déterminer une preuve algébrique de l'invariance du semi-groupe associé à une hypersurface quasi-ordinaire. Dans le cas de dimension deux Popescu-Pampu a fourni une telle preuve en utilisant la structure du lieu singulier de la singularité et les propriétés de la résolution minimale de la surface (voir [PP2]). Cet objectif est important pour pouvoir déterminer si l'on peut associer un semi-groupe à une singularité quasi-ordinaire non hypersurface. Une idée qui pourrait être utile pour cela est la généralisation des développements d'Hamburger Noether (voir [Ca] pour le cas des courbes).

Une autre problème se pose en rapport avec la théorie de saturation : est-ce que tout germe d'hypersurface quasi-ordinaire de semi-groupe Γ est projection générique d'une variété torique affine Z^{Λ} déterminée par le semi-groupe Γ ? (voir [Z3] et [T1] pour le cas des courbes planes).

L'analyse des hypersurfaces polaires des germes d'hypersurface quasi-ordinaire, commencé dans la thèse de Popescu-Pampu (voir [PP2]) et completé ensuite dans un travail en collaboration avec García Barroso (voir [GB-GP]), pourrait être employé pour étudier la courbure des fibres de Milnor (voir [GB-T] pour le cas des courbes planes).

Chapitre 1

Singularités quasi-ordinaires toriques et polyèdre de Newton du discriminant

Paru dans Canadian Journal of Mathematics (voir [GP1])

Résumé. [1] Nous étudions les polynômes $F \in \mathbb{C}\{S_\tau\}[Y]$ à coefficients dans l'anneau de germes de fonctions holomorphes au point spécial d'une variété torique affine. Nous généralisons à ce cas la paramétrisation classique des singularités quasi-ordinaires. Cela fait intervenir d'une part une généralization de l'algorithme de Newton-Puiseux, et d'autre part une relation entre le polyèdre de Newton du discriminant de F par rapport à Y et celui de F au moyen du polytope-fibre de Billera et Sturmfels ([Bi-St]). Cela nous permet enfin de calculer, sous des hypothèses de non dégénérescence, les sommets du polyèdre de Newton du discriminant a partir de celui de F, et les coefficients correspondants à partir des coefficients des exposants de F qui sont dans les arêtes de son polyèdre de Newton.

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[1] Numéros de classification : principal 14M25, secondaire 32S25.

1.1 Introduction

Le sujet de la première partie de ce travail est la représentation des racines $Y(X)$ d'une équation polynôme $F(X_1, \ldots, X_d; Y) = 0$ par des séries à exposants fractionnaires en les variables $X = (X_1, \ldots, X_d)$. Il s'agit de généraliser le théorème de Newton-Puiseux. Nous poursuivrons dans une direction inaugurée par McDonald dans [McD], et précisons ses résultats.

Notre approche est d'étudier d'abord le problème dans le cas d'un polynôme $F \in \mathbb{C}\{S_\tau\}[Y]$, où $\mathbb{C}\{S_\tau\}$ est l'anneau des germes des fonctions holomorphes au point spécial d'une variété torique affine correspondant à un cône rationnel strictement convexe, $\tau \subset (\mathbb{R}^d)^*$, de dimension d. Nous résolvons le probleme, lorsque le *discriminant* $\Delta_Y F$ de F par rapport à Y est de la forme $X^u \epsilon$ où ϵ est une unité dans $\mathbb{C}\{S_\tau\}$ et u appartient au semigroupe $S_\tau := \tau^\vee \cap \mathbb{Z}^d$ des élements de $\mathbb{Z}^d$ qui appartiennent au cône dual $\tau^\vee := \{w \in \mathbb{R}^d / \langle w, u \rangle \geq 0, \ \forall u \in \tau\}$. Ceci est en fait une généralisation de l'étude classique des singularités quasi-ordinaires, qui correspondent au cas où τ est le quadrant positif.

La réduction du cas général à ce cas fait appel à des constructions combinatoires sur le polyèdre de Newton $\mathcal{N}(F) \subset \mathbb{R}^{d+1}$ de F. La plus importante, déjà utilisée dans [McD] est celle du *polyèdre-fibre* $\mathcal{Q}(F) \subset \mathbb{R}^d$ de $\mathcal{N}(F)$ par rapport à sa projection $\mathcal{N}(F) \rightarrow \mathbb{R}^d$ sur l'espace des exposants des monômes en X. Les points extrêmes de $\mathcal{Q}(F)$ correspondent à certains chemins dans les arêtes de $\mathcal{N}(F)$.

Le polyèdre-fibre est également relié au polyèdre de Newton du discriminant de F par rapport à Y. Si $F = a_0(X) + \cdots + a_r(X) Y^r$, on a l'inclusion de polyèdres de Newton

$$\mathcal{N}(\Delta_Y F) + \mathcal{N}(a_0) + \mathcal{N}(a_r) \subseteq \mathcal{Q}(F)$$

(où la somme est la somme de Minkowski), avec l'égalité sous des hypothèses de non-dégénérescence des coefficients de F par rapport à $\mathcal{N}(F)$, (théorème 1.4).

Un cône $\tau \subset (\mathbb{R}^d)^*$ est compatible avec des polyèdres $\mathcal{P}_1, \ldots, \mathcal{P}_s \subset \mathbb{R}^d$ s'il est constitué des fonctions linéaires qui prennent toutes leur valeur minimale sur $\mathcal{P}_1, \ldots, \mathcal{P}_s$ en des point fixés $p_1 \in \mathcal{P}_1, \ldots, p_s \in \mathcal{P}_s$. On décrit le résultat principal, le théorème 1.3, dans le cas où F est un polynôme réduit dans l'anneau $\mathbb{C}\{X\}[Y]$. Si $\tau \subset (\mathbb{R}^d_+)^\vee$ est un cône rationnel strictement convexe de dimension d compatible avec les polyèdres $\mathcal{N}(\Delta_Y F)$, $\mathcal{N}(a_r)$ et $\mathcal{Q}(F)$, alors l'homomorphisme $\mathbb{C}\{X\} \rightarrow \mathbb{C}\{S_\tau\}$ étendant l'inclusion des algèbres $\mathbb{C}[\mathbb{Z}^d_+] \rightarrow \mathbb{C}[S_\tau]$ transforme F en un polynôme $F_\tau \in \mathbb{C}\{S_\tau\}[Y]$ dont toutes les racines sont de la forme $X^u \varepsilon(X)$ où $u \in \frac{1}{k}\mathbb{Z}^d$, $\varepsilon(X)$ est une unité dans l'anneau $\mathbb{C}\{\frac{1}{k} S_\tau\}$ et k est un entier positif. La construction des racines est donnée par un algorithme qui généralise celui du théorème de Newton-Puiseux, (théorème 1.2), et qui pourrait se développer à l'aide d'un logiciel de calcul formel. On peut comparer l'algorithme obtenu avec celui de [A-L-R], developpé pour le cas quasi-ordinaire.

Nous utilisons ces résultats pour donner une description des sommets du polyèdre de Newton du *résultant* $\mathrm{Res}_Y(F, G)$ des polynômes $F, G \in \mathbb{C}\{S_\tau\}[Y]$ à partir des polyèdres-fibres $\mathcal{Q}(F)$, $\mathcal{Q}(G)$ et $\mathcal{Q}(FG)$; nous calculons enfin les coefficients des monômes correspondants aux sommets du polyédre de Newton de $a_0 a_r \Delta_Y F$ et de $\mathrm{Res}_Y(F, G)$ sous des hypothèses de non dégénérescence. Ces coefficients dépendent entre autres des résultants des paires de polynômes à une variable obtenus en-regardant de F et de G que les termes donts les exposants appartiennent à des paires parallèles d'arêtes de $\mathcal{N}(F)$ et $\mathcal{N}(G)$. On montre un résultat analogue pour le discriminant. Nous trouvons aussi, avec une méthode très différente, des résultats de même nature que ceux de Gel'fand, Kapranov et Zelevinski dans [G-K-Z].

1.2 Paramétrisation de singularités quasi-ordinaires toriques

1.2.1 L'algèbre des germes de fonctions holomorphes au point distingué d'une variété torique affine

Soit τ un cône convexe rationnel de dimension d dans $(\mathbb{R}^d)^*$. Cette condition garantit que le cône dual $\tau^\vee := \{w \in \mathbb{R}^d / \langle w, u \rangle \geq 0,\ \forall u \in \tau\}$ est un cône rationnel strictement convexe. Pour cette raison, chaque élément de S_τ peut s'exprimer commme somme d'éléments du semigroupe S_τ d'un nombre fini de manières. L'ensemble des séries formelles à exposants dans S_τ est un anneau, que nous notons par $\mathbb{C}[[S_\tau]]$. La propriété de finitude précédente permet de garantir que les coefficients de la série produit sont des fonctions polynomiales des coefficients des facteurs. Ces anneaux sont définis dans [McD], pour construire des racines d'un polynôme $F \in \mathbb{C}[X_1, \ldots, X_d][Y]$, à la Newton-Puiseux.

On va donner une interprétation géométrique de ces anneaux au moyen de la variété torique associée au cône τ dans le cas où τ est un cône rationnel strictement convexe de dimension d dans $\mathbb{R}^d$.

Soit $N \subset (\mathbb{R}^d)^*$ un réseau de dimension d, de réseau dual $M \subset \mathbb{R}^d$. Le cône τ définit le semi-groupe de type fini $S_\tau := \tau^\vee \cap M$. Soit $\mathbb{C}[S_\tau]$ l'algèbre du semi-groupe S_τ à coefficients dans $\mathbb{C}$. Associons à (τ, N) la variété torique affine $Z_\tau :=$ Spec $\mathbb{C}[S_\tau]$. Chaque point fermé de Z_τ est définit par un homomorphisme de semi-groupes $S_\tau \to \mathbb{C}$. La valeur de la fonction $X^u \in \mathbb{C}[S_\tau]$ au point x est $x(u)$. L'orbite de dimension 0 de la variété Z_τ est le point spécial z_τ défini par l'homomorphisme de semi-groupes $S_\tau \to \mathbb{C}$ qui applique $0 \mapsto 1$ et $u \mapsto 0$ si $u \neq 0$. (Pour tout ceci voir [F1] ou [Od]).

L'anneau des séries convergentes à exposants dans S_τ , que nous notons par $\mathbb{C}\{S_\tau\}$, est l'ensemble des séries de $\mathbb{C}[[S_\tau]]$ qui sont absolument convergentes dans un voisinage du point spécial z_τ de la variété torique Z_τ. Si τ est un cône convexe rationnel de dimension d, on défini $\mathbb{C}\{S_\tau\} = \cap\, \mathbb{C}\{S_\sigma\}$ où σ parcourt les cônes rationnels strictement convexes de dimension d contenus dans τ.

Lemme 1.1. *L'algèbre locale des germes de fonctions holomorphes au point* z_τ *de* Z_τ *est isomorphe à* $\mathbb{C}\{S_\tau\}$.

Preuve. Soient $u_1, \dots, u_s$ des générateurs du semi-groupe S_τ. L'homomorphisme $\mathbb{C}[U_1, \dots, U_s] \to \mathbb{C}[S_\tau]$ défini par $U_i \mapsto X^{u_i} \in \mathbb{C}[S_\tau]$ est surjectif. Son noyau est un idéal premier I. Ce morphisme définit un plongement $Z_\tau \subset \mathbb{C}^s$ de la variété torique affine $Z_\tau :=$ Spec $\mathbb{C}[S_\tau]$ définie par le cône τ, et l'image du point distingué z_τ est l'origine de $\mathbb{C}^s$. Soit R l'algèbre des germes fonctions holomorphes dans un voisinage de z_τ dans Z_τ.

Remarquons que l'homomorphisme composé $\mathbb{C}[U_1, \dots, U_s] \to \mathbb{C}[S_\tau] \hookrightarrow \mathbb{C}[[S_\tau]]$ s'étend en un homomorphisme $\chi : \mathbb{C}\{U_1, \dots, U_s\} \to \mathbb{C}[[S_\tau]]$ dont l'image est $\mathbb{C}\{S_\tau\}$. En effet, l'image du monôme U^λ, où $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbb{N}^s$, est le monôme $X^{u(\lambda)}$, où $u(\lambda) = \sum \lambda_i u_i \in S_\tau$. Donc, l'image de ϕ est la série $\chi(\phi) = \sum_{u \in S_\tau} (\sum_{u(\lambda)=u} c_\lambda) X^u$, qui est bien définie parce que τ est strictement convexe. Supposons que ϕ est absolument convergente au point $x' = (x'_1, \dots, x'_s)$ correspondant au point $x \in Z_\tau$ par le plongement torique $Z_\tau \subset \mathbb{C}^s$. La valeur de la fonction $X^u \in \mathbb{C}[S_\tau]$ au point du Z_τ, ne dépend pas de l'immersion, donc si $u = u(\lambda)$, on a $x(u) = x'^\lambda$ et la série $\chi(\phi)$ est absolument convergente au point x. Ceci implique que la série $\chi(\phi) \in \mathbb{C}[[S_\tau]]$ est convergente. Avec un raisonnement analogue, on peut montrer que l'homomorphisme d'algèbres $\chi : \mathbb{C}\{U_1, \dots, U_s\} \to \mathbb{C}\{S_\tau\}$ est surjectif.

Par ailleurs, on a montré aussi que $\chi(\phi)$ est une fonction holomorphe dans un voisinage de z_τ dans Z_τ, définissant un unique élément de R. Clairement, tous les éléments de R sont obtenus de cette forme. Si la fonction $\chi(\phi)$ est nulle dans un voisinage de z_τ dans Z_τ, la série ϕ est dans l'ideal engendré par I dans $\mathbb{C}\{U_1, \dots, U_s\}$ donc $\chi(\phi) = 0$. ◇

Comme conséquence de ce lemme, on déduit que l'anneau $\mathbb{C}\{S_\tau\}$ est noethérien et intégralement clos parce que Z_τ est une variété normale (voir [Ka], §71).

1.2.2 Extensions galoisiennes

Soit $k \in \mathbb{Z}$ un entier positif fixé. Considérons les réseaux $N' = kN \subset N$. Leurs réseaux duaux respectifs sont $M' = \frac{1}{k} M \supset M$. Un cône τ strictement convexe dans $(\mathbb{R}^d)^*$ est rationnel pour les deux réseaux en même temps. Nous notons Z_τ (resp. Z'_τ) la variété torique associée à (τ, N) (resp. à (τ, N')). Le semi-groupe associé à (τ, N') est $S'_\tau := \frac{1}{k} S_\tau \subset M'$. L'homomorphisme de semi-groupes $M \supset S_\tau \hookrightarrow S'_\tau \subset M'$ définit un morphisme torique $f_k : Z'_\tau \to Z_\tau$. L'image du point distingué de Z'_τ est le point distingué du Z_τ, donc on obtient un morphisme de germes irréductibles $(Z'_\tau, z'_\tau) \to (Z_\tau, z_\tau)$. En utilisant le lemme 1.1 on vérifie que l'homomorphisme des algèbres intègres associées est $\mathbb{C}\{S_\tau\} \hookrightarrow \mathbb{C}\{S'_\tau\}$.

L'homomorphisme des semi-groupes $M \hookrightarrow M'$ définit le morphisme $f_k : T' \to T$ obtenu en restreignant f_k aux tores respectifs de Z'_τ et Z_τ On peut vérifier directement que le noyau de ce morphisme $f_{k\,|T'}$, comme morphisme de groupes algébriques, est le sous-groupe fini H de T', formé des éléments $(w_1, \dots, w_d)$ tels que $w_i^k = 1$, pour $i = 1, \dots, d$. Ce morphisme est un revêtement galoisien à k^d feuilles de la variété T, parce que le groupe H agit transitivement sur les fibres. Donc on a une extension galoisienne des corps des fonctions rationnelles $\mathbb{C}(T) \hookrightarrow \mathbb{C}(T')$. On va montrer qu'on a une situation analogue pour les corps des fonctions méromorphes aux points distingués des variétés toriques correspondantes.

Soit L (resp. L') le corps des fractions de $\mathbb{C}\{S_\tau\}$, (resp. de $\mathbb{C}\{S'_\tau\}$). L'homomorphisme $\mathbb{C}\{S_\tau\} \hookrightarrow \mathbb{C}\{S'_\tau\}$ définit une extension de corps $L \hookrightarrow L'$.

Lemme 1.2. *L'extension de corps $L \hookrightarrow L'$ est galoisienne. Soit G son groupe de Galois. L'action de H sur les monômes définit un épimorphisme de groupes $H \to G$ et l'ensemble des éléments G-invariants de l'anneau $\mathbb{C}\{S'_\tau\}$ est $\mathbb{C}\{S_\tau\}$.*

Preuve. Clairement, $L \hookrightarrow L'$ est une extension normale finie. A chaque $w \in H$ est associé l'homomorphisme d'algèbres $\mathbb{C}\{S'_\tau\} \to \mathbb{C}\{S'_\tau\}$ qui applique $X^{\frac{u}{k}} \mapsto w(u)X^{\frac{u}{k}}$. Cela définit un homomorphisme de groupes $H \to G$.

Remarquons que $X^{\frac{u}{k}} \mapsto w(u)X^{\frac{u}{k}}$ définit l'action de l'élément $w \in H$ sur un monôme de $\mathbb{C}[S'_\tau]$. Le corollaire 1.16 de [Od] garantit que le morphisme $Z'_\tau \to Z_\tau$ coïncide avec la projection du quotient de Z_τ par rapport à l'action du groupe H. C'est-à-dire que $\mathbb{C}[S_\tau]$ est l'ensemble des éléments invariants de l'algèbre $\mathbb{C}[S'_\tau]$ par l'action de H.

Si H' est l'image de H dans G on a montré que le sous-corps fixe de L' par H' coïncide avec L, donc $(L')^G \subset L$, c'est-à-dire que $L \subset L'$ est une extension galoisienne et donc $H' = G$. ◇

1.2.3 Paramétrisation de singularités quasi-ordinaires toriques

Supposons que $F \in \mathbb{C}\{X_1, \dots, X_d\}[Y]$ est un polynôme réduit tel que $0 \in \mathbb{C}$ est une racine de multiplicité $r \geq 1$ du polynôme $F(0, Y)$ et que le discriminant de F soit de la forme $X^q \varepsilon$ où ε est une unité de $\mathbb{C}\{X_1, \dots, X_d\}$. D'après le théorème de préparation de Weierstrass, il existe un pseudo-polynôme à la Weierstrass H de degré r en Y, et une unité ϵ dans $\mathbb{C}\{X_1, \dots, X_d, Y\}$ tels que $F = \epsilon H$. Par définition, la projection du germe $(\mathcal{X}, 0) \subset \mathbb{C}^d \times \mathbb{C}$ défini par le polynôme $H \in \mathbb{C}\{X_1, \dots, X_d\}[Y]$ sur $(\mathbb{C}^d, 0)$ est quasi-ordinaire. D'après [A1], Theorem 3, il existe $k \in \mathbb{N}$ tel que H ait ses r racines dans l'anneau $\mathbb{C}\{X_1^{1/k}, \dots, X_d^{1/k}\}$.

On va généraliser la construction de racines associées à une projection quasi-ordinaire, au cas où le germe $(\mathbb{C}^d, 0)$ est remplacé par un germe de variété torique affine (Z_τ, z_τ) au point distingué.

Théorème 1.1. *Pour tout polynôme $F \in \mathbb{C}\{S_\tau\}[Y]$ réduit tel que le discriminant de F soit de la forme $X^{u_0}\varepsilon$, où ε est une unité dans l'anneau $\mathbb{C}\{S_\tau\}$ et que $0 \in \mathbb{C}$ soit une racine de multiplicité $r \geq 1$ du polynôme $F(z_\tau, Y)$ il existe $k \in \mathbb{N}$ tel que F ait r racines sans terme constant dans l'anneau $\mathbb{C}\{\frac{1}{k}S_\tau\}$.*

Preuve. Nous fixons un nombre fini de générateurs du semi-groupe S_τ. Cela permet de définir un plongement de la variété torique affine $Z_\tau \subset \mathbb{C}^s$. Il lui est associé un épimorphisme d'algèbres $\chi : \mathbb{C}\{U_1, \ldots, U_s\} \to \mathbb{C}\{S_\tau\}$ (voir le lemme 1.1). Considérons un polynôme $G \in \mathbb{C}\{U_1, \ldots, U_s\}[Y]$ tel que $G^\chi = F$. On a $G(0, Y) = F(z_\tau, Y)$. D'après le théorème de préparation de Weierstrass il existe un pseudo-polynôme à la Weierstrass H de degré r en Y, et une unité ε dans $\mathbb{C}\{U_1, \ldots, U_s, Y\}$ tels que $G = \varepsilon H$. Clairement, les germes définis au point $(z_\tau, 0)$ par F et par H^χ coïncident. Donc, on peut supposer que F est un polynôme réduit de degré r tel que $F(z_\tau, Y) = Y^r$.

Soit L le corps de fractions de l'anneau intégre $\mathbb{C}\{S_\tau\}$. Les facteurs F_i de la factorisation de F en polynômes irréductibles dans $L[Y]$ sont dans $\mathbb{C}\{S_\tau\}[Y]$ parce que, d'après le lemme 1.1, l'anneau $\mathbb{C}\{S_\tau\}$ est intégralement clos et le coefficient de Y^r est une unité (voir le théorème 5, §3, chap V, [Z-S]). De plus, le discriminant de F_i divise le discriminant de F donc $\Delta_Y F_i$ est de la forme $X^u \epsilon$ où $\epsilon \in \mathbb{C}\{S_\tau\}$ est une unité. On peut donc supposer que F est irréductible, engendrant un ideal premier (F) dans $\mathbb{C}\{S_\tau\}[Y]$. Considérons le germe de variété analytique irréductible $(\mathcal{X}, x) \subset (Z_\tau \times \mathbb{C}, x)$ au point x correspondant à l'algèbre intègre $R = \mathbb{C}\{S_\tau\}[Y]/(F)$.

Soit $(\mathcal{X}, x) \to (Z_\tau, z_\tau)$ la projection des germes, et choisissons un représentant fini $\pi : \mathcal{X} \to Z_\tau$ tel que $\pi^{-1}(z_\tau) = \{x\}$. L'hypothèse sur le discriminant implique qu'il existe un voisinage W du point z_τ dans Z_τ tel que π est non ramifié sur $W^* := W \cap T$. Par continuité, comme $\pi^{-1}(z_\tau) = \{x\}$, on peut supposer que $\pi^{-1}(W)$ est un sous-ensemble relativement compact de $\mathbb{C}^{s+1}$.

Comme π est un morphisme fini, l'intersection de l'image inverse du lieu discriminant de π avec $\mathcal{X}$ est une sous-variété analytique fermée propre de $\mathcal{X}$. Son complémentaire est un ouvert $\mathcal{X}^* \subset \mathcal{X}$, connexe parce que $\mathcal{X}$ est analytiquement irréductible. Ceci montre que $\pi : \mathcal{X}^* \to W^*$ est un revêtement connexe à r feuillets.

On peut supposer que l'ouvert W^* du tore T est $W^* = (\mathbb{D}^*)^d$ où $\mathbb{D}^* = D(0,1) \backslash \{0\} \subset \mathbb{C}^*$. Soit J le sous-groupe du groupe fondamental $\pi_1(W^*, w) \cong \mathbb{Z}^d$ associé au revêtement $\pi : X^* \to W^*$. Puisque J est d'indice fini, il existe $k \in \mathbb{N}$ tel que $k\mathbb{Z}^d \subset J$. Le revêtement $f_k : W^* \to W^*$, défini par $x \mapsto (x_1^k, \ldots, x_d^k)$, est associé au sous-groupe $k\mathbb{Z}^d$ du $\mathbb{Z}^d$. Donc, il existe un revêtement $p : W^* \to X^*$ tel que $\pi \circ p = f_k$, (voir [F2], chap. 13).

Clairement, p est holomorphe, et borné dans le complémentaire dans W d'un ensemble analytique fermé, c'est-à-dire que p est une fonction faiblement holomorphe dans W. Comme $W \subset Z_\tau$ est une variété normale, toute fonction faiblement holomorphe est holomorphe, (voir [Ka], §71). Donc, p s'étend en un morphisme $W \to \mathcal{X}$. La fonction holomorphe $\pi \circ p$ coïncide sur W^* avec le morphisme torique $f_k : Z'_\tau \to Z_\tau$ (où on considère $W^* \subset Z'_\tau$ et aussi $W^* \subset Z_\tau$). Donc, elle est égale à la restriction du morphisme f_k à W. Nous remarquons que $p(z'_\tau) = x$ parce que $f_k(z'_\tau) = z_\tau$ et $\pi^{-1}(z_\tau) = \{x\}$.

En utilisant le lemme 1.1, on voit que l'homomorphisme d'algèbres intègres associé au morphisme f_k aux points distingués est $\mathbb{C}\{S_\tau\} \to \mathbb{C}\{S'_\tau\}$. Considérons le monomorphisme d'algèbres $R \to$

$\mathbb{C}\{S'_\tau\}$ correspondant au morphisme de germes $p : (W, z'_\tau) \to (X, x)$. L'algèbre R est une sous-$\mathbb{C}\{S_\tau\}$-algèbre de $\mathbb{C}\{S'_\tau\}$ parce que $\pi \circ p = f_k$. Nous avons donc un diagramme :

$$\begin{tikzcd}
R \arrow[r] & \mathbb{C}\{S'_\tau\} \\
\mathbb{C}\{S_\tau\} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Soit L (resp. K, L') le corps des fractions de $\mathbb{C}\{S_\tau\}$, (resp. de $R, \mathbb{C}\{S'_\tau\}$). Par construction, $L \subset K \subset L'$ et $K = L(\zeta)$ où ζ est l'image de $Y \in R$ dans $\mathbb{C}\{S'_\tau\}$. D'après le lemme 1.2, l'extension de corps $L \hookrightarrow L'$ est galoisienne, et la série ζ a ses r conjugués dans l'anneau $\mathbb{C}\{S'_\tau\}$. Ces conjugués sont les racines de F dans $\mathbb{C}\{S'_\tau\}$ qui paramétrisent $(\mathcal{X}, x)$. $\diamond$

Remarque 1.3. *Si F est irréductible, on peut prendre $k = r$ dans le théorème 1.1.*

En effet, puisque le polynôme F est irréductible, l'indice du sous-groupe J (dans la preuve du théorème 1.1), est égal à r. L'ordre du sous-groupe engendré par l'image d'un vecteur de la base canonique de $\mathbb{Z}^d$ dans $\mathbb{Z}^d/J$ est un diviseur de r. Donc, on a $r\mathbb{Z}^d \subset J$.

1.3 Racines à la Newton Puiseux

1.3.1 Valuation induite par un vecteur irrationnel

Soit R un anneau commutatif et Γ un groupe totalement ordonné. Une valuation ω de R dans Γ est une application $\omega : R\backslash\{0\} \to \Gamma$ tel que

$$\begin{array}{ll} (i) & \omega(ab) = \omega(a) + \omega(b) \ \text{ pour } \ 0 \neq a, b \in R, \\ (ii) & \omega(a-b) \geq \inf \ (\omega(a), \omega(b)) \ \text{ pour } \ 0 \neq a, b \in R \ \text{ et } a \neq b, \ \text{ avec égalité si } \omega(a) \neq \omega(b). \end{array}$$

On associe à chaque $\lambda \in \Gamma$ l'ensemble $\mathcal{I}_\lambda = \{a \in R/\omega(a) > \lambda\}$. L'ensemble $\mathcal{I}_\lambda$ est un ideal de R et on a $\lambda > \beta \Rightarrow \mathcal{I}_\lambda \subset \mathcal{I}_\beta$. La topologie ω-adique sur R, est la topologie qui fait de R un groupe topologique dans lequelle l'ensemble des ideaux $\{\mathcal{I}_\lambda\}_{\lambda \in \Gamma}$ est un système fondamental de voisinages de $0 \in R$.

La valuation ω est *archimédienne* si Γ est isomorphe comme groupe totalement ordonné à un sous-groupe de $\mathbb{R}$. Soit $(R, \mathfrak{M})$ est un anneau local de corps de fractions L. La valuation ω de L est *centrée sur* R si $\omega(a) \geq 0$ pour $a \in R$ et $\omega(a) > 0$ pour $a \in \mathfrak{M}$.

Lemme 1.4. *Soient $(R, \mathfrak{M})$ un anneau local noethérien de corps de fractions L, et ω une valuation archimédienne de L centrée sur R Alors, la topologie $\mathfrak{M}$-adique coïncide avec la topologie ω-adique de R.*

Preuve. Puisque R est noethérien, le semi-groupe $\omega(R\backslash\{0\})$ est bien ordonné, et il existe un plus petit élément λ de l'ensemble $\omega(\mathfrak{M}\backslash\{0\})$. Si $\beta \in \omega(\mathfrak{M}\backslash\{0\})$, il existe $n \in \mathbb{N}$ tel que $n\lambda > \beta$ donc $\mathfrak{M}^n \subset \mathcal{I}_\beta \subset \mathfrak{M}$. ◇

On appelle un vecteur $w \in \mathbb{R}^d$ *irrationnel* si ses coordonneés sont linéairement indépendantes sur $\mathbb{Q}$. Associé à un vecteur irrationnel $w \in \mathbb{R}^d$ nous définissons un ordre total sur $\mathbb{Q}^d$ par :

$$u <_w u' \Leftrightarrow \langle u, w\rangle \ < \ \langle u', w\rangle.$$

Remarque 1.5. *Soit τ un cône rationnel strictement convexe de dimension d dans $\mathbb{R}^d$. Un vecteur irrationnel $w \in \tau^\vee$ définit une valuation archimédienne de l'anneau local complet $\mathbb{C}[[S_\tau]]$ par $w(\sum_{u\in S_\tau} c_u X^u) = \ min_{c_u\neq 0}(\langle u, w\rangle)$. Cette valuation vérifie les hypothèses du lemme 1.4.*

Si $(\phi_j) \subset \mathbb{C}[[S_\tau]]$ vérifie que la suite $(w(\phi_j)) \subset \mathbb{R}$ est strictement croissante alors ϕ_j tend vers $0 \in \mathbb{C}[[S_\tau]]$.

On appelle *l'exposant initial* d'une série $\phi \in \mathbb{C}[[S_\tau]]$ par rapport a w, l'exposant u de ϕ tel que $w(X^u) = w(\phi)$. L'exposant initial est le plus petit, pour l'ordre $<_w$, parmi les exposants de ϕ.

1.3.2 Chemins monotones dans le polyèdre de Newton

Un polyèdre $\mathcal{N}$ dans $\mathbb{R}^d$ est l'intersection d'une famille de demi-espaces d'équation $\langle \omega, u\rangle \geq \lambda_\omega$, pour $\omega \in \Xi \subset (\mathbb{R}^d)^*$. On dira que le polyèdre $\mathcal{N}$ est *rationnel* si ses sommets sont dans le réseau $\mathbb{Z}^d$ et si ses faces ont des équations à coefficients dans $\mathbb{Q}$. Le cône τ associé au sommet u d'un polyèdre rationnel $\mathcal{N}$ dans $\mathbb{R}^d$ est l'ensemble des fonctions linéaires qui atteignent leur valeur minimale sur $\mathcal{N}$ au sommet u. Le cône τ est rationnel de dimension d, et τ est strictement convexe si et seulement si le polyèdre $\mathcal{N}$ est de dimension d. Dans ce cas, l'ensemble des cônes associés au polyèdre $\mathcal{N}$ forme un éventail Σ dans $(\mathbb{R}^d)^*$ avec un nombre éventuellement infini de cônes; le support $|\Sigma| = \cup_{\sigma\in\Sigma}\sigma$ de l'éventail associé à $\mathcal{N}$ n'est pas nécessairement fermé. Si $\tau \subset |\Sigma|$ est un cône rationnel strictement convexe de dimension d, l'ensemble des cônes $\tau\cap\sigma$, pour $\sigma \in \Sigma$ est une subdivision de τ. En particulier, c'est l'éventail associé à la somme de Minkowski $\tau^\vee + \mathcal{N}$. Cette subdivision est finie parce que si S^{d-1} est l'sphère unité $\{\tau\cap\sigma\cap S^{d-1}\}$ est un complexe polyèdral de support l'ensemble compact $\tau\cap\sigma\cap S^{d-1}$.

Considérons $\mathbb{R}^d \times \mathbb{R}$ avec des coordonnées fixées (u, v). On dira qu'une arête bornée e d'un polyèdre $\mathcal{N} \subset \mathbb{R}^d \times \mathbb{R}$ est *admissible* si elle n'est pas parallèle à l'hyperplan $v = 0$. Une arête admissible est de la forme $[p_{v_1}, p_{v_2}]$ où $p_{v_i} = (u_{v_i}, v_i), \in \mathbb{R}^d \times \mathbb{R}$ avec $v_1 < v_2$. Nous appellerons le vecteur $q_e := \frac{u_{v_1}-u_{v_2}}{v_2-v_1}$ l'*inclinaison* , et le nombre $l_e = v_2 - v_1 \in \mathbb{N}$ la *longueur* de l'arête. Considérons la projection $\pi_e : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \times \{0\}$ parallèlement à l'arête e, définie par $\pi_e(u, v) = u + vq_e$. Le cône $\sigma(e) \subset (\mathbb{R}^d)^*$, associé au sommet $\pi_e(e)$ du polyèdre $\pi_e(\mathcal{N})$ et de dimension d et on a :

Lemme 1.6. *Pour $w \in (\mathbb{R}^d)^*$, les propriétés suivantes sont équivalentes :*

1. *$w \in \sigma(e)$.*
2. *La fonction lineaire w atteint sa valeur minimale sur chaque section $v = \lambda$ de $\mathcal{N}$ au point $(u(\lambda), \lambda)$ de l'arête e.* ◇

On dit qu'un chemin γ dans les arêtes de $\mathcal{N} \subset \mathbb{R}^d \times \mathbb{R}$ est *monotone* si on peut le paramétriser par $\gamma(\lambda) = (u(\lambda), \lambda)$. Supposons que le chemin γ a pour sommets $\{p_0, p_{i_1}, \ldots, p_{i_t}, p_n\}$ avec $p_j = (u_j, j)$ pour $j \in \{i_0, i_1, \ldots, i_t, i_{t+1}\}$ avec $0 = i_0 < \cdots < i_{t+1} = n$. Nous notons q_r l'inclinaison du segment $e_r = [p_{i_r}, p_{i_{r+1}}]$, pour $r = 0, \ldots, t$. Le chemin monotone γ est *cohérent* si il existe $w \in (\mathbb{R}^d)^*$ tel que $\gamma(\lambda)$ est l'unique point de la section $v = \lambda$ du polyèdre $\mathcal{N}$ en lequel w atteint sa valeur minimale sur cette section, pour $\lambda \in [0, n]$.

Lemme 1.7. *Avec les notations précedentes, si $w \in (\mathbb{R}^d)^*$ est un vecteur irrationnel définissant le chemin monotone cohérent γ dans les arêtes du polyèdre rationnel $\mathcal{N}$, alors les inclinaisons des arêtes de γ vérifient :*

$$q_t <_w q_{t-1} \cdots <_w q_0$$

Preuve. Notons π_s la projection parallèlement au segment e_s. Le vecteur $\pi_{s-1}(e_{s-1}) - \pi_s(e_s)$ a le même sens que le vecteur $\pi_s(p_{i_{s-1}}) - \pi_s(e_s)$. Par le lemme précedent w appartient à $\sigma(e_s)$, donc on a $\langle w, \pi_{s-1}(e_{s-1}) - \pi_s(e_s)\rangle = i_s \langle w, q_{s-1} - q_s\rangle \geq 0$, pour $s = 1, \ldots t$. ◇

Soit $\rho \subset (\mathbb{R}^d)^*$ un cône strictement convexe de dimension d. Nous notons $\mathbb{C}((S_\rho))$ (resp. $\mathbb{C}\{\{S_\rho\}\}$) l'anneau de fractions de l'anneau $\mathbb{C}[[S_\rho]]$ (resp. $\mathbb{C}\{S_\rho\}$) pour l'ensemble multiplicativement fermé correspondant aux monômes X^u pour $u \in S_\rho$.

Définition 1.1. *Le ρ-polyèdre de Newton d'une série $\phi \in \mathbb{C}((S_\rho))$ non nulle est la somme de Minkowski de l'enveloppe convexe de ses exposants et du cône $\rho^\vee$. Le ρ-polyèdre de Newton d'un polynôme $F \in \mathbb{C}((S_\rho))[Y]$, est la somme de Minkowski de l'enveloppe convexe de ses exposants et du cône $\rho^\vee \times \{0\}$.*

Le ρ-polyèdre de Newton de ϕ, que nous notons $\mathcal{N}_\rho(\phi)$, est un polyèdre de dimension d ayant un nombre fini de sommets. L'éventail associé est la subdivision du cône ρ induite par l'éventail associé à l'enveloppe convexe des exposants de ϕ.

Nous notons $\mathcal{N}_\rho(F)$ le polyèdre de Newton d'un polynôme $F \in \mathbb{C}((S_\rho))[Y]$. Remarquons que le ρ-polyèdre de Newton de F ne dépend que des exposants de F, il dépend aussi de l'anneau dans lequel on considère que se trouvent les coefficients de F. Pour tout cône $\tau \subset (\mathbb{R}^d)^*$ rationnel strictement convexe de dimension d, l'inclusion d'algèbres $\mathbb{C}[X_1, \ldots, X_d] \to \mathbb{C}\{\{S_\tau\}\}$, permet de considérer un polynôme $F \in \mathbb{C}[X_1, \ldots, X_d][Y]$ comme élement de $\mathbb{C}\{\{S_\tau\}\}[Y]$. L'enveloppe convexe des exposants de F est un polyèdre compact, $\mathcal{P}(F)$, mais le polyèdre $\mathcal{N}_\rho(F)$ n'est pas compact.

On définit la relation suivante parmi les vecteur irrationnels du cône ρ : $w \sim w'$, si et seulement si, ils définissent le même chemin polygonal dans les arêtes du polyèdre $\mathcal{N}_\rho(F)$. Par le lemme 1.6, cette relation définit un éventail qui subdivise le cône ρ. Cet éventail est défini par un polyédre que nous allons décrire maintenant.

1.3.3 Le polyèdre-fibre de la projection du polyèdre de Newton

Soient $\mathcal{P} \subseteq \mathbb{R}^N$ un polytope et $\pi : \mathbb{R}^N \to \mathbb{R}^M$ une application affine surjective, l'image de $\mathcal{P}$ est un polytope $\mathcal{Q}$. L'*intégrale de Minkowski* de l'application $\pi : \mathcal{P} \to \mathcal{Q}$ est l'ensemble des intégrales $\int_{\mathcal{Q}} \gamma \in \mathbb{R}^N$ lorsque γ parcourt l'ensemble des sections Borel-mesurables $\gamma : \mathcal{Q} \to \mathcal{P}$ de π. D'après [Bi-St], l'intégrale de Minkowski est un polytope convexe de dimension égale à $\dim \mathcal{P} - \dim \mathcal{Q}$.

Si F est un polynôme dans $\mathbb{C}[X_1, \dots, X_d][Y]$, son polytope de Newton $\mathcal{P}(F) \subset \mathbb{R}^{d+1}$ est l'enveloppe convexe de ses exposants. Considérons un polynôme de la forme $F = \sum_{k=0}^n a_k Y^k$, où les a_k sont des polynômes dans $\mathbb{C}[X_1, \dots, X_d]$ avec $a_0 a_n \neq 0$. Soit $\mathcal{P}(F)$ le polytope de Newton de F ; nous allons décrire l'intégrale de Minkowski de la projection $\pi : \mathcal{P}(F) \subset \mathbb{R}^d \times \mathbb{R} \to [0, n] \subset \mathbb{R}$. Ceci est un cas particulier du théorème 7.3 de [Bi-St]. Une section de π est une application monotone de la forme $t \mapsto (\gamma(t), t) \in \mathcal{P}(F)$, pour $t \in [0, n]$, et il lui est associé le point $\int_{[0,n]} \gamma$ dans l'intégrale de Minkowski de π. Il est montré en [Bi-St] que les sommets de l'intégrale de Minkowski correspondent à des intégrales des chemins monotones cohérents dans les arêtes de $\mathcal{P}(F)$.

Ces chemins sont décrits par une collection $\{p_0, p_{i_1}, \dots, p_{i_s}, p_n\}$ de sommets de $\mathcal{P}(F)$, avec $p_j = (u_j, j)$, où u_j est un sommet de $\mathcal{P}(a_j)$ pour $j \in \{0, i_1, \dots, i_s, n\}$, avec $0 < i_1 < \dots < i_s < n$. Si $v < v'$ on peut paramétriser le segment $[(u, v), (u', v')] \subset \mathbb{R}^d \times \mathbb{R}$, par $\gamma(\lambda) = u + \frac{\lambda - v}{v' - v}(u' - u)$, avec $\lambda \in [v, v']$, donc $\int_{[v,v']} \gamma = \frac{v' - v}{2}(u + u')$.

L'intégrale du chemin γ correspondant à la collection de sommets $\{p_0, p_{i_1}, \dots, p_{i_s}, p_n\}$ est :

$$\begin{aligned}
\int_{[0,n]} \gamma &= \int_{[0,i_1]} \gamma + \dots + \int_{[i_s,n]} \gamma \\
&= \frac{1}{2}(i_1(u_0 + u_{i_1}) + \sum_{k=2}^{s} (i_k - i_{k-1})(u_{i_k} + u_{i_{k-1}}) + (n - i_s)(u_{i_s} + u_n)) \\
&= \frac{1}{2}(i_1 u_0 + i_2 u_{i_1} + \sum_{k=2}^{s-1} (i_{k+1} - i_{k-1}) u_{i_k} + (n - i_{s-1}) u_{i_s} + (n - i_s) u_n) \qquad (1.1)
\end{aligned}$$

Ces considérations motivent la définition suivante :

Définition 1.2. *Soient un cône strictement convexe $\rho \subset \mathbb{R}^d$ de dimension d et $F \in \mathbb{C}((S_\rho))[Y]$ un polynôme de degré n de terme constant non nul. Soit $\mathcal{Q}$ l'enveloppe convexe des intégrales $\int \gamma_w$ des chemins monotones γ dans le polyèdre $\mathcal{N}_\rho(F)$ définis par des vecteurs irrationnels $w \in \rho$. Le ρ-polyèdre-fibre de F est la somme de Minkowski $\mathcal{Q}_\rho(F) := 2(\mathcal{Q} + \rho^\vee)$.*

Le ρ-polyèdre-fibre $\mathcal{Q}_\rho(F)$ est un polyèdre rationnel. Il dépend du ρ-polyèdre de Newton de F. L'éventail Σ associé au polyèdre-fibre $\mathcal{Q}_\rho(F)$ est une subdivision rationnelle finie du cône ρ. Si w, w' sont des vecteurs dans l'interieur d'un cône de dimension d de Σ, ils définissent le même chemin polygonal dans les arêtes du polyèdre $\mathcal{N}_\rho(F)$.

Dans le cas où F est un polynôme dans l'anneau $\mathbb{C}[X_1, \dots, X_d][Y]$ on appelle *polytope-fibre* de F le polytope $\mathcal{Q}(F) := 2\mathcal{Q}$ où $\mathcal{Q}$ est l'intégrale de Minkowski de la la projection du polytope de Newton $\pi : \mathcal{P}(F) \subset \mathbb{R}^d \times \mathbb{R} \to [0, n] \subset \mathbb{R}$.

1.3.4 Théorème de Newton-Puiseux

On va généraliser un résultat de [McD].

Théorème 1.2. *Soient ρ un cône rationnel strictement convexe de dimension d et $F \in \mathbb{C}((S_\rho))[Y]$ un polynôme non constant. Pour tout vecteur irrationnel $w \in \rho$ il existe un cône rationnel strictement convexe σ_w de dimension d, et $k \in \mathbb{N}$ tels que $w \in \sigma_w \subset \rho$ et que F se décompose dans l'anneau $\mathbb{C}((\frac{1}{k}S_{\sigma_w}))[Y]$.*

Preuve. Elle est essentiellement la même que celle de [McD]. Un vecteur irrationnel $w \in \rho \subset (\mathbb{R}^d)^*$ définit un chemin monotone cohérent γ dans les arêtes du polyèdre rationnel $\mathcal{N}_\rho(F)$. Fixons une arête $e = [(u, v), (u', v')] \subset \mathbb{R}^d \times \mathbb{R}$ du chemin γ, avec $v < v'$. L'inclinaison de e est un vecteur $q \in \frac{1}{l}\mathbb{Z}^d$ où l est la longueur $v' - v$. La restriction de F à l'arête e est le polynôme $F_{|e} = \sum_{I \in e} \alpha_I X_1^{i_1} \dots X_d^{i_d} Y^{i_{d+1}}$. On associe à l'arête e le polynôme $f_e \in \mathbb{C}[t]$ par $F_{|e}(1, \dots, 1, t) = t^v f_e$ où $f_e(0) \neq 0$. Le polynôme f_e est de degré l et toutes ses racines sont non nulles.

Soit c une racine de f_e, définissons le polynôme $F_2 = F(Y + cX^q) \in \mathbb{C}((\frac{1}{l}S_\rho))[Y]$. Clairement, F_2 est un polynôme de degré r et on a :

$$F_2 = \sum_I \alpha_I \sum_{j=0}^{i_{d+1}} \begin{pmatrix} i_{d+1} \\ j \end{pmatrix} c^j X_1^{i_1 + j q_1} \dots X_d^{i_d + j q_d} Y^{i_{d+1} - j}.$$

On en déduit que :

1. Les exposants de F_2 sont de la forme $I + j(q, -1)$ où $I = (i_1, \dots, i_{d+1})$ est un exposant de F et $j \in \{0, \dots, i_{d+1}\}$, donc $\pi_e(\mathcal{N}_\rho(F_2))$ est contenu dans $\pi_e(\mathcal{N}_\rho(F))$.
2. Le coefficient du terme constant de F_2 d'exposant dans la droite E défini par l'arête e est, $\sum_{I \in e} \alpha_I c^{i_{d+1}} = f_e(c)$, nul par construction.
3. Le coefficient du terme de F_2 d'exposant $p_{v'}$ coïncide avec celui de F.
4. Si Y ne divise pas F_2, le polyèdre de F_2 a toujours des points dans l'hyperplan $v = 0$.

5. L'exposant de F_2 dans la droite E correspondant au terme de plus petit degré en Y est un sommet du ρ-polyèdre de Newton de F_2. Cet exposant est de la forme (u, m) où m est la multiplicité de c comme racine de f_e. En effet, la plus petite ordonnée des exposants de F_2 dans la droite E est le nombre m de fois qu'il faut dériver pour que $\frac{\partial^m}{\partial Y^m}(F_{2\,|E})$, ait un terme constant non nul. Comme $F_{2\,|E} = F_{|e}(Y + cX^q)$ le coefficient du terme constant de Y de $\frac{\partial^k}{\partial Y^k}(F_{2\,|E})$ est égal à $\sum_{I \in e} \alpha_I i_{d+1} \dots (i_{d+1} - k + 1) c^{i_{d+1}-k} = \frac{d^k f_e}{dt^k}(c)$.

Le sommet (u, m) du polyèdre $\mathcal{N}_\rho(F_2)$ défini par 5. est un sommet du chemin monotone défini par w dans les arêtes de $\mathcal{N}_\rho(F_2)$. Si Y ne divise pas F_2 on va considérer la partie finale du chemin entre le sommet (u, m) et l'hyperplan $v = 0$.

Parmi les segments de cette partie finale du chemin polygonal on choisit une arête e_2 d'inclinaison q_2 et de longueur l_2. On choisit une racine c_2 du polynôme associé f_{e_2} de multiplicité m_2 et on définit $F_3 := F_2(Y + c_2 X^{q_2})$. On continue par récurrence. Le polynôme F_n est un élément de l'anneau $\mathbb{C}((\frac{1}{l_1 \dots l_{n-1}} S_\rho))[Y]$.

On obtient une suite décroissante de nombres entiers positifs : $l \geq m \geq l_2 \geq m_2 \geq \cdots > 0$, qui est donc stationnaire ; il existe $n_0 \in \mathbb{N}$ tel que pour tout $n \geq n_0$ on a $l_n = m_n = m_{n_0} = m$. Ceci implique que $f_{e_n} = \theta(t - c_n)^m$, et aussi que la partie finale du chemin défini par w dans les arêtes de $\mathcal{N}_\rho(F_n)$ est le segment e_n. De plus les sommets de e_n et e_{n+1} qui ne sont pas dans l'hyperplan $v = 0$ coïncident, pour $n > n_0$.

Pour $n > n_0$, on a le segment $e_n = [(u_n, 0), (u_0, m)]$ d'inclinaison $q_n = \frac{1}{m}(u_n - u_0)$. L'intersection de la droite définie par e_n avec l'hyperplan $v = 0$ est le point $p_n := u_0 + m q_n$. Par définition du ρ-polyèdre de Newton, le cône $\sigma(e_{n_0})$ associé au sommet p_{n_0} du polyèdre $\pi_{e_{n_0}}(\mathcal{N}_\rho(F_{n_0}))$ est contenu dans ρ. Le lemme 1.6 implique que $w \in \sigma(e_{n_0})$.

On vérifie qu'il existe $k \in \mathbb{N}$ tel que les inclinaisons construites sont dans un réseau $\frac{1}{k}\mathbb{Z}^d$. On sait que $u_0 = \frac{\beta_0}{k}$, $u_{n_0} = \frac{\beta_{n_0}}{k}$ où $\beta_0, \beta_{n_0} \in \mathbb{Z}^d$ et $k = l_1 l_2 \dots l_{n_0 - 1}$ est un entier. L'inclinaison de e_{n_0} est $q_{n_0} = \frac{\beta_{n_0} - \beta_0}{km} = \frac{\beta}{k\lambda}$ où $\beta \in \mathbb{Z}^d$ et $\lambda \in \mathbb{N}$ est premier avec une coordonnée de β. Si $(\frac{\beta'}{k}, h)$ est un exposant de $F_{|e}^{n_0}$ on a $q_{n_0} = \frac{\beta_{n_0} - \beta'}{kh} = \frac{\beta}{k\lambda}$. Comme $\lambda(\beta_{n_0} - \beta') = h\beta$, on déduit que λ divise h donc $f_{e_{n_0}}$ est un polynôme en t^λ. Par ailleurs $f_{e_n} = \theta(t - c_n)^m$ et comme la caractéristique de $\mathbb{C}$ est zéro, on a $\lambda = 1$. Par récurrence on obtient que $q_n \in \frac{1}{k}\mathbb{Z}^d$, pour $n \geq n_0$ et donc pour $n \in \mathbb{N}$.

Montrons que les inclinaisons q_j sont dans un cône affine strictement convexe.

Le ρ-polyèdre de Newton de F_{n_0} est contenu dans le cone affine :

$$W(e_{n_0}) := \{\lambda(u - u') \;/ u \in \mathcal{N}_\rho(F_{n_0}),\ u' \in e_{n_0},\ \lambda \geq 0\},$$

associé à l'arête e_{n_0}. Comme $u_{n_0+1} \in W(e_{n_0})$, par construction on a l'inclusion $\mathcal{N}_\rho(F_{n_0+1}) \subset W(e_{n_0})$. Par récurrence, en utilisant que le sommet (u_0, m) de e_n est sur la droite qui contient le segment e_{n_0}, on montre que $u_n \in W(e_{n_0})$ et que $\mathcal{N}_\rho(F_n) \subset W(e_{n_0})$, pour $n > n_0$. Ceci implique pour tout $w' \in \sigma(e_{n_0})$ que $\langle w', q_n - q_{n_0} \rangle = \frac{1}{m} \langle w', p_n - p_{n_0} \rangle$ est ≥ 0. Donc les exposants construits sont dans le cône rationnel affine $q_{n_0} + \sigma(e_{n_0})^\vee$ pour $n \geq n_0$.

Notons σ pour $\sigma(e_{n_0})$. Il existe $u_0 \in \frac{1}{k}\mathbb{Z}^d$ tel que les inclinaisons q_n appartiennent à $u_0 + \frac{1}{k}S_\sigma$. Définissons les sommes partielles, $\phi_n = \sum_{j=1}^n c_j X^{q_j}$ pour $n \in \mathbb{N}$. On a $\phi_n \in \mathbb{C}((\frac{1}{k}S_\sigma))$ et $X^{-u_0}\phi_n \in \mathbb{C}[[\frac{1}{k}S_\sigma]]$. Par construction, et par le lemme 1.7, on sait que $q_j <_w q_{j+1}$ pour $j \in \mathbb{N}$. Par la remarque 1.5, ceci implique que la série formelle $\phi := \sum c_j X^{q_j}$ est égale à $X^{u_0} \lim_{n \to \infty} X^{-u_0}\phi_n$ où la limite est dans l'anneau complet $\mathbb{C}[[\frac{1}{k}S_\sigma]]$.

Comme $w \in \sigma \subset \rho$, on peut considérer F comme élément de $\mathbb{C}((\frac{1}{k}S_\sigma))[Y]$. La série formelle ϕ est une racine de F. En effet, si $n \geq n_0$ la série $F(\phi_{n-1}) = F_n(0)$ a tous ses exposants dans le cône rationnel affine $p_{n_0} + \sigma(e_{n_0})^\vee$. L'égalité suivante $F(\phi) = X^{p_{n_0}} \lim_{n\to\infty} F(\phi_{n-1})X^{-p_{n_0}}$ est clair. Si $n \geq n_0$ l'exposant initial de $F(\phi_{n-1})X^{-p_{n_0}}$ par rapport à w est $p_{n+1} - p_n = m(q_{n+1} - q_{n_0})$ et on a $\lim_{n\to\infty} F_n(0)X^{-p_{n_0}} = 0$.

On vérifie que la multiplicité de ϕ comme racine de F est $\geq m$. La multiplicité de c_n comme racine de f_{e_n} est $\geq m$, donc c_n est une racine de $\frac{d^s f_{e_n}}{dt^k}$ pour $1 \leq s \leq m-1$. Le polynôme $\frac{d^s f_{e_n}}{dt^s}$ est le polynôme de l'arête e_n^s du polyèdre $\mathcal{N}_\rho(\frac{\partial^s F}{\partial Y^s})$ qui est sur le segment $-(0, \ldots, 0, s) + e_n$. L'arête e_n^s est déterminée par le vecteur irrationnel w. Comme $\frac{\partial^s F_n}{\partial Y^s} = \frac{\partial^s F}{\partial Y^s}(Y + \phi_{n-1})$ on obtient que ϕ est une racine de $\frac{\partial^s F}{\partial Y^s}$ pour $1 \leq s \leq m-1$.

On a montré que, associés à chaque arête e du chemin monotone γ, il existe $k \in \mathbb{N}$ et un cône rationnel σ_e strictement convexe de dimension d tel que $w \in \sigma_e \subset \rho$, tels que F ait au moins l_e racines à la Newton-Puiseux dans $\mathbb{C}[[\frac{1}{k}S_{\sigma_e}]]$. On peut choisir $k \in \mathbb{N}$ valable pour toutes les arêtes de γ. Comme le vecteur w est irrationnel, le cône rationnel $\tau = \cap_{e \in \gamma} \sigma_e$ est de dimension d. L'existence d'un homomorphisme d'algèbres $\mathbb{C}[[\frac{1}{k}S_{\sigma_e}]] \hookrightarrow \mathbb{C}[[\frac{1}{k}S_\tau]]$, pour chaque arête e de γ, garantit que F se décompose dans $\mathbb{C}[[\frac{1}{k'}S_\tau]]$, parce que les exposants initiaux par rapport à $<_w$ des séries correspondantes à segments différents de γ sont différents. ◇

Remarque 1.8. *Soient* $F \in \mathbb{C}[[S_\rho]]$ *un polynôme de degré* ≥ 1 *et* $w \in \rho$ *un vecteur irrationnel, définissant un chemin polygonal* γ *dans les arêtes de* $\mathcal{N}_\rho(F)$. *La démonstration du théorème 1.2 montre que, associées à chaque arête* e *de* γ, *il existe* l_e *racines de* F *telles que leur exposant initial par rapport à* w *est l'inclinaison* q_e.

1.3.5 Rapport avec les paramétrisations des singularités quasi-ordinaires

Théorème 1.3. *Soit* $F = \sum_{j=0}^n a_j Y^j$ *un polynôme réduit de degré* $n \geq 1$ *avec* $a_j \in \mathbb{C}\{\{S_\rho\}\}$.

1. *Pour tout cône* τ *de dimension* d *de l'éventail associé au polyèdre* $\mathcal{N}_\rho(a_n \Delta_Y F)$, *il existe* $k \in \mathbb{N}$ *tel que* F *se décompose dans l'anneau* $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$.

2. *Si* $a_0 \neq 0$, *pour tout cône* τ *de dimension* d *de l'éventail associé au polyèdre* $\mathcal{N}_\rho(a_n \Delta_Y F) + \mathcal{Q}_\rho(F)$ *il existe* $k \in \mathbb{N}$ *tel que* F *se décompose dans l'anneau* $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$, *et de plus les racines de* F *sont des unités.*

Preuve. Soit τ un cône de dimension d de l'éventail associé au polyèdre $\mathcal{N}_\rho(a_n \Delta_Y F)$. Par définition de ρ-polyèdre de Newton, le cône τ est contenu dans ρ, et on a l'homomorphisme d'algèbres $\mathbb{C}\{S_\rho\} \hookrightarrow \mathbb{C}\{S_\tau\}$ qui permet de considérer F comme élément de $\mathbb{C}\{\{S_\tau\}\}[Y]$.

Tout vecteur irrationnel $w \in \tau$ atteint sa valeur minimale sur les exposants de a_n au même point u_n. On montre par récurrence sur n qu'il existe $q_0 \in \mathbb{Z}^d$ tel que le polyèdre $\mathcal{N}_\tau(F)$ est contenu dans le cône affine :

$$W := \{(u_n, n) + \lambda(q_0, -1) + (u', 0) \ / \ \lambda \in [0, n], \ u' \in \tau^\vee\}$$

Si $n = 1$, il suffit de prendre $q \in \mathbb{Z}^d$ tel que $\mathcal{N}_\tau(a_0)$ soit contenu dans le cone affine $u_n + q + \tau^\vee$. Si $n > 1$, par récurrence on a construit q pour le polynôme $(F - a_0)Y^{-1}$. Il suffit de prendre $q_0 \in \mathbb{Z}^d$ tel que le polyèdre $u_n + nq + \tau^\vee + \mathcal{N}_\tau(a_0)$ soit contenu dans le cone affine $u_n + nq_0 + \tau^\vee$.

Nous notons p_0 le point $u_n + nq_0$, e_0 le segment $[(u_n, n), (p_0, 0)]$ et $\pi_{e_0} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \times \{0\}$ la projection parallèlement à l'arête e_0. On a $\pi_{e_0}(u, v) = u + vq_0$.

On définit le changement :

$$G = X^{-p_0} F(X^{q_0} Y).$$

On en déduit :

1. Si $F = \sum_{j=0}^n a_j Y^j$ avec $a_i \in \mathbb{C}\{\{S_\tau\}\}$ on obtient que $G = \sum_{j=0}^n a_j X^{jq_0 - p_0} Y^j$, et donc l'exposant de G qui correspond à l'exposant (u, j) de F est $(\pi_{e_0}(u) - p_0, j)$. Par construction, comme $\mathcal{N}_\tau(F) \subset W$, le vecteur $\pi_{e_0}(u) - p_0$ appartient au cône $\tau^\vee$. Ceci implique que G est un polynôme dans l'anneau $\mathbb{C}\{S_\tau\}$. De plus, l'exposant de G qui correspond à l'exposant (u_n, n) de F est $(0, n)$, donc $G(z_\tau, Y) \in \mathbb{C}[Y]$ est un polynôme de degré n.
2. La quasi-homogénéité et l'homogénéité du discriminant générique impliquent que le discriminant de G par rapport à Y est de la forme $\Delta_Y G = X^{u_0} \varepsilon$ où ε est une unité dans l'anneau $\mathbb{C}\{S_\tau\}$.

En appliquant le théorème 1.1, on voit qu'il existe $k \in \mathbb{N}$ tel que G se décompose dans l'anneau $\mathbb{C}\{\frac{1}{k}S_\tau\}$. Les racines correspondantes de F sont dans $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$.

Soit τ est un cône de dimension d de la subdivision finie de ρ induite par le polyèdre $\mathcal{N}_\rho(a_n \Delta_Y F) + \mathcal{Q}_\rho(F)$; vérifions que les racines construites sont des unités dans $\mathbb{C}\{\{S_\tau\}\}$.

Soit $w \in \tau$ un vecteur irrationnel. D'après le théorème 1.2, il existe un cône rationnel strictement convexe σ qui contient w, et $k \in \mathbb{N}$ tels que le polynôme F se décompose sur l'anneau $\mathbb{C}((\frac{1}{k}S_\sigma))$.

Puisque le vecteur $w \in \tau \cap \sigma$ est irrationnel, le cône rationnel $\tau \cap \sigma$ est nécessairement un cône de dimension d, et son cône dual $\tau^\vee + \sigma^\vee$ est strictement convexe. L'anneau intègre $\mathbb{C}((\frac{1}{k}S_{\tau\cap\sigma}))$ contient $\mathbb{C}((\frac{1}{k}S_\sigma))$ et $\mathbb{C}((\frac{1}{k}S_\tau))$ comme sous-anneaux.

D'abord, les racines de F obtenues par le théorème 1.1 sont dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$, et donc elles doivent coïncider avec les racines obtenues à la Newton-Puiseux. Nous affirmons que ces racines sont des éléments inversibles dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$. Par hypothèse, chaque élément irrationnel $w \in \tau$ définit le même chemin γ dans les arêtes de $\mathcal{N}_\tau(F)$. Donc l'exposant initial u par rapport à w d'une racine ϕ ne dépend pas de $w \in \tau$ (par la remarque 1.8). On obtient que $\phi = X^u\varepsilon$ où ε est une unité de $\mathbb{C}\{\frac{1}{k}S_\tau\}$, c'est-à-dire que ϕ est une unité dans $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$. ◇

On obtient aussi la version polynomiale du théorème précedent :

Corollaire 1.9. *Soit un polynôme* $F = \sum a_j Y^j \in \mathbb{C}[X_1, \dots, X_d][Y]$ *réduit de degré* $n \geq 1$.

1. *Si le polytope* $\mathcal{P} = \mathcal{P}(a_n\Delta_Y F)$ *est de dimension* d, *pour tout cône* τ *de dimension* d *de l'éventail associé à* $\mathcal{P}$ *il existe* $k \in \mathbb{N}$ *tel que* F *se décompose dans l'anneau* $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$.
2. *Si* $a_0 \neq 0$ *et si* τ *est un cône associé à un sommet du polytope* $\mathcal{P}(\Delta_Y F) + \mathcal{Q}(F)$, *toutes les racines de* F *sont des unités dans* $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$.

Preuve. Si le polytope $\mathcal{P}(a_n\Delta_Y F)$ est de dimension d, le cône τ associé à un sommet de $\mathcal{P}(a_n\Delta_Y F)$ est de dimension d. On applique le théorème 1.3 à F vu comme élément de $\mathbb{C}\{\{S_\tau\}\}[Y]$.

Si le polytope $\mathcal{P} := \mathcal{P}(\Delta_Y F) + \mathcal{Q}(F)$ est de dimension $< d$, le cône τ associé à un sommet de $\mathcal{P}$ est rationnel de dimension d mais il n'est pas strictement convexe. Le cône $\tau^\vee$ est strictement convexe et τ définit l'algèbre $\mathbb{C}[[S_\tau]]$. Si $\sigma \subset \tau$ est un cône strictement convexe de dimension d, on considére F comme élément de $\mathbb{C}\{\{S_\sigma\}\}[Y]$ et on obtient que il existe $k \in \mathbb{N}$ tel que F se décompose dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\sigma\}\}$.

On peut recouvrir le cône τ par un nombre fini de cônes rationnels strictement convexes de dimension d, $\{\sigma_i\}_{1 \geq i \geq s}$, tels que $\sigma_i \cap \sigma_{i+1}$ soit d'intérieur non vide, pour $i = 1, \dots, s-1$. Ceci implique que les racines de F obtenues par le théorème 1.1 correspondant à σ_i et à σ_{i+1} vont coïncider, et le terme initial d'une racine par rapport à la valuation induite par un vecteur irrationel w ne dépend pas de $w \in \sigma_i \cup \sigma_{i+1}$. Donc toutes les racines de F sont des séries à exposants dans un translaté du cône $\cap_{i=1}^s \sigma_i^\vee = \tau^\vee$. ◇

Remarque 1.10. *Soit* $F \in \mathbb{C}[X_1, \dots, X_d][Y]$ *un polynôme de degré* n *tel que* 0 *soit une racine simple de* $F(0, Y)$, *le théorème des fonctions implicites garantit qu'il existe une unique série* $\phi \in \mathbb{C}\{X_1, \dots, X_d\}$ *telle que* $F(\phi) = 0$. *Si on a* $a_n\Delta_Y F = X^u\varepsilon$ *où* $\varepsilon(0) \neq 0$ *le théorème 1.3 montre que les exposants de* ϕ *sont dans un translaté entier du cône dual associé au sommet* u *de* $\mathcal{P}(a_n\Delta_Y F)$ *lorsque ce polytope est de dimension* d. *(Voir l'exemple 1).*

1.4 Application aux polyèdres de Newton du discriminant et du résultant

1.4.1 Les conditions discriminantales pour le polyèdre de Newton

Suivant [McD], on dit que un polynôme $F \in \mathbb{C}((S_\rho))[Y]$ vérifie la *condition discriminantale* si pour toute arête admissible e de son ρ-polyèdre de Newton, le polynôme f_e n'a que des racines simples.

Théorème 1.4. *Soit* $F = \sum_{k=0}^{n} a_k Y^k$ *un polynôme à coefficients dans* $\mathbb{C}((S_\rho))$ *tels que* $a_0 a_n \neq 0$. *On a l'inclusion de polyèdres*

$$\mathcal{N}_\rho(a_0) + \mathcal{N}_\rho(a_n) + \mathcal{N}_\rho(\Delta_Y F) \subseteq \mathcal{Q}_\rho(F)$$

où $\Delta_Y F$ *est le discriminant de* F *par rapport à* Y. *On a l'égalité si* F *vérifie la condition discriminantale.*

Preuve. On va trouver les conditions génériques que doivent vérifier les coefficients des termes qui apparaissent dans F, pour garantir l'égalité dans le théorème.

Soit $w \in \rho$ un vecteur irrationnel, et soit γ le chemin monotone défini par w dans les arêtes du ρ-polyèdre de Newton. Le chemin γ a des sommets $\{p_0, p_{i_1}, \dots, p_{i_t}, p_n\}$ dans $\mathcal{N}_\rho(F)$, avec $p_j = (u_j, j)$ pour $j \in \{0 = i_0, i_1, \dots, i_t, i_{t+1} = n\}$ avec $i_0 < \cdots < i_{t+1}$. Nous notons $q_r := \frac{-u_{i_r} + u_{i_{r-1}}}{i_r - i_{r-1}} \in \mathbb{Q}^d$ et $l_r := i_r - i_{r-1}$ l'inclinaison et la longueur du segment $e_r = [p_{i_{r-1}}, p_{i_r}]$ du chemin γ, pour $r = 1, \dots, t+1$.

D'après le théorème 1.2, il existe un cône rationnel strictement convexe σ_w, et $k \in \mathbb{N}$ tels que F se décompose dans l'anneau $\mathbb{C}((\frac{1}{k} S_{\sigma_w}))[Y]$. A chaque segment $e_r = [p_{i_{r-1}}, p_{i_r}]$ du chemin γ sont associées $i_r - i_{r-1}$ racines de F de la forme :

$$\phi_j = c_j X^{q_r} + \dots,$$

où q_r est l'inclinaison du segment e_r, et c_j parcourt les racines de f_{e_r} comptées avec leur multiplicité. De plus q_r est l'exposant initial par rapport à w des termes qui apparaissent dans ϕ_j. On indexe les racines ϕ_j correspondant à e_r, par $j \in \mathcal{A}_r := \{i_{r-1} + 1, \dots, i_r\}$.

En appliquant le lemme 1.7, on voit que parmi les termes qui peuvent apparaître dans $\phi_k - \phi_j$, celui d'exposant de le plus petit par rapport à $<_w$ est égal à :

$$\begin{cases} (c_k - c_j) X^{q_r} & \text{si } \; k, j \in \mathcal{A}_r \\ c_j X^{q_m} & \text{si } \; k \in \mathcal{A}_r, \; j \in \mathcal{A}_m \text{ et } \; r < m. \end{cases}$$

Comme,

$$\Delta_Y F = (-1)^{\frac{1}{2}n(n-1)} a_n^{2(n-1)} \prod_{k<j} (\phi_k - \phi_j)^2,$$

le terme d'exposant le plus petit par rapport à $<_w$ qui peut apparaître dans $a_0 a_n \Delta_Y F$ est égal a ABC où :

$$\begin{aligned} A &= (-1)^{\frac{1}{2}n(n-1)} \alpha_{p_0} \alpha_{p_n}^{2n-1} X^{u_0+(2n-1)u_n} \\ B &= \prod_{r=1}^{t+1} \prod_{i_{r-1}<k_r<j_r\leq i_r} (c_{k_r} - c_{j_r})^2 X^{2q_r} \\ C &= \prod_{r=1}^{t} \prod_{k_r\in\mathcal{A}_r} \prod_{m=r+1}^{t+1} \prod_{j_m\in\mathcal{A}_m} c_{j_m}^2 X^{2q_m}. \end{aligned}$$

L'exposant correspondant à B est :

$$\begin{aligned} & 2\Big(\binom{i_1}{2} q_1 + \binom{i_2 - i_1}{2} q_2 + \cdots + \binom{i_{t+1} - i_t}{2} q_{t+1}\Big) \\ = \; & (i_1 - 1)(u_0 - u_{i_1}) + (i_2 - i_1 - 1)(u_{i_1} - u_{i_2}) + \cdots + (i_{t+1} - i_t - 1)(u_{i_t} - u_{i_{t+1}}) \\ = \; & (i_1 - 1)u_0 + (i_2 - 2i_1)u_{i_1} + (i_3 - 2i_2 + i_1)u_{i_2} + \cdots + (n - 2i_t + i_{t-1})u_{i_t} + (-n + i_{t-1})u_n \end{aligned}$$

L'exposant correspondant à C est :

$$\begin{aligned} 2(\sum_{r=1}^{t}(i_r - i_{r-1}) \sum_{m=r+1}^{t+1} q_m(i_m - i_{m-1})) &= 2(\sum_{r=1}^{t}(i_r - i_{r-1})(-u_n + u_{i_r})) \\ &= 2(i_1 u_{i_1} + (i_2 - i_1)u_{i_2} + \cdots + (i_t - i_{t-1})u_{i_t} - i_t u_n) \end{aligned}$$

L'exposant u correspondant à ABC coïncide avec $2\int\gamma$, (voir la formule (1.1), §1.3.3). Clairement, le coefficient correspondant à ABC est non nul si et seulement si les segments du chemin γ vérifient la condition discriminantale. Ceci termine la preuve, parce que le vecteur irrationnel w est arbitraire. ◇

Corollaire 1.11. *Avec les notations précedentes, le coefficient du terme de la série $a_0 a_n \Delta_Y F$ d'exposant égal à $2\int\gamma$ est :*

$$c(\gamma) := (-1)^k \alpha_{p_{i_0}} \alpha_{p_{i_1}}^2 \ldots \alpha_{p_{i_t}}^2 \alpha_{p_{i_{t+1}}} \Delta f_{e_1} \ldots \Delta f_{e_{t+1}},$$

où $f_{e_r} = \alpha_{p_{i_{r-1}}} + \cdots + \alpha_{p_{i_r}} t^{l_r}$ est le polynôme de l'arête $e_r = [p_{i_{r-1}}, p_{i_r}]$ du chemin γ, son discriminant est Δf_{e_r} et $k = \frac{1}{2}\big(n(n-1) + \sum_{r=1}^{t+1} l_r(l_r - 1)\big)$.

Preuve. En utilisant que $\Delta f_{e_r} = (-1)^{\frac{1}{2}l_r(l_r-1)} \alpha_{p_{i_r}}^{2(l_r-1)} \prod_{i_{r-1}<k_r<j_r\leq i_r} (c_{k_r} - c_{j_r})^2$ on obtient que le coefficient de B est

$$\prod_{r=1}^{t+1} (-1)^{\frac{1}{2}l_r(l_r-1)} \alpha_{p_{i_r}}^{-2(l_r-1)} \Delta f_{e_r}.$$

Comme le produit des racines de f_{e_m} est égal à $(-1)^{l_m}\frac{\alpha_{p_{i_{m-1}}}}{\alpha_{p_{i_m}}}$, on déduit que le le coefficient de C est

$$\prod_{r=1}^{t}\prod_{k_r\in\mathcal{A}_r}\prod_{m=r+1}^{t+1}\Big(\frac{\alpha_{p_{i_{m-1}}}}{\alpha_{p_{i_m}}}\Big)^2=\prod_{r=1}^{t}\prod_{k_r\in\mathcal{A}_r}\Big(\frac{\alpha_{p_{i_r}}}{\alpha_{p_{i_{t+1}}}}\Big)^2=\prod_{r=1}^{t}\Big(\frac{\alpha_{p_{i_r}}}{\alpha_{p_{i_{t+1}}}}\Big)^{2l_r}$$

Donc le coefficient de ABC est :

$$(-1)^k\Delta f_{e_1}\dots\Delta f_{e_{t+1}}\alpha_{p_{i_0}}\alpha_{p_{i_{t+1}}}^{2n-1-2(n-l_{t+1})-2(l_{t+1}-1)}\prod_{r=1}^{t}\alpha_{p_{i_r}}^{2l_r-2(l_r-1)}=c(\gamma)$$

◇

On déduit des théorèmes 1.3 et 1.4 :

Corollaire 1.12. *Soit le polynôme* $F=\sum_{k=0}^{n}a_kY^k$, *où* a_k *sont des séries dans* $\mathbb{C}((S_\rho))$ *telles que* $a_0a_n\neq 0$. *Si* F *vérifie la condition discriminantale, pour tout cône* τ *de dimension* d *de l'éventail associé au polyèdre* $\mathcal{Q}_\rho(F)$, *il existe* $k\in\mathbb{N}$ *tel que* F *se décompose dans l'anneau* $\mathbb{C}((\frac{1}{k}S_\tau))$, *et de plus les racines de* F *sont des unités.* ◇

Remarque 1.13. *Le corollaire 4.1 de [McD] énonce une "version polynomiale" incorrecte du corollaire précedent. Il est dit que des racines de* F *correspondants aux cônes associés aux sommets différents du polytope-fibre sont différentes. Supposons que le polytope* $\mathcal{P}:=\mathcal{P}(\Delta_YF)+\mathcal{P}(a_n)$ *soit de dimension* d, *et que l'éventail* Σ *associé au polytope-fibre soit une sous-division stricte de l'éventail* Σ' *associé à* $\mathcal{P}$. *Par le corollaire 1.9, les racines de* F *correspondants aux cônes de* Σ *qui subdivisent un cône* $\tau\in\Sigma'$ *de dimension* d *vont coïncider dans l'anneau* $\mathbb{C}\{\{\frac{1}{k}S_\tau\}\}$. *Elles ne seront pas toutes des unités dans cet anneau. (Voir l'exemple 1).*

Corollaire 1.14. *Soit le polynôme* $F=\sum_{i=1}^{n}a_iY^i$ *où* a_i *sont des polynômes dans* $\mathbb{C}[X_1,\dots,X_d]$ *tels que* $a_0a_n\neq 0$. *On a l'inclusion de polytopes*

$$\mathcal{P}(a_0)+\mathcal{P}(a_n)+\mathcal{P}(\Delta_YF)\subseteq\mathcal{Q}(F)$$

et on a l'égalité si et seulement si le polynôme F *vérifie la condition discriminantale.*

Preuve. Soit ρ un cône de dimension d strictement convexe. On va considérer le polynôme F comme un élément de l'anneau $\mathbb{C}[[S_\rho]][Y]$. En appliquant le théorème 1.4 pour chaque $w\in\rho$, on voit que $\mathcal{Q}(F)+\rho^\vee\supseteq\mathcal{P}(a_0)+\mathcal{P}(a_n)+\mathcal{P}(\Delta_YF)+\rho^\vee$, et que l'on a l'égalité si et seulement si toutes les arêtes admissibles du polyèdre $\mathcal{P}(F)+\rho^\vee\times\{0\}$ vérifient la condition discriminantale. Ceci termine la preuve parce que ρ est arbitraire. ◇

Remarque 1.15. *En utilisant le corollaire 1.14 et le théorème 7.3 de [Bi-St], on peut déduire de ce qui précède les théorèmes 2.2, et 2.3, Chp. 12, de [G-K-Z]. Ces résultats donnent le polytope de Newton du discriminant générique (c'est-à-dire le discriminant du polynôme* $F=X_nY^n+\dots X_1Y+X_0\in\mathbb{C}[X_0,X_1,\dots,X_n][Y]$ *par rapport à* Y*) et les coefficients des termes correspondant aux sommets du polytope.*

En effet, le polytope $\mathcal{P}(F)$ est un simplexe de dimension n, de sommets $(u_j, j) \in \mathbb{R}^{n+1} \times \mathbb{R}$, où $\{u_j\}_{j=0}^{n}$ sont les vecteurs de la base canonique dans $\mathbb{R}^{n+1}$. Comme le polynôme F vérifie la condition discriminantale on a $\mathcal{Q}(F) = \mathcal{P}(X_0 X_n \Delta_Y F)$. Chaque sous-ensemble $\{i_1, \dots, i_s\}$ de $\{1, \dots, n-1\}$ correspond de manière unique à un chemin monotone $\gamma_{\{i_1,\dots,i_s\}}$ dans les arêtes de $\mathcal{P}(F)$. Comme $\mathcal{P}(F)$ est un simplexe, il existe un vecteur irrationnel $w \in (\mathbb{R}^{n+1})^*$ définissant le chemin $\gamma_{\{i_1,\dots,i_s\}}$. Le sommet de l'intégrale de Minkowski $\int \gamma_{\{i_1,\dots,i_s\}}$ est décrit par la formule (1.1). En appliquant le corollaire 1.11, on obtient aussi le coefficient correspondants aux sommets du polytope de Newton du discriminant générique.

1.4.2 Application au polyèdre de Newton du résultant

On dira que deux polynômes $F, G \in \mathbb{C}((S_\rho))[Y]$ vérifient la *condition résultante* si pour toute paire d'arêtes e de $\mathcal{N}_\rho(F)$ et e' de $\mathcal{N}_\rho(G)$ ayant la même inclinaison, les polynômes des arêtes respectives $f_e, g'_e \in \mathbb{C}[t]$ n'ont pas de racines en commun.

Soit w un vecteur irrationnel dans un cône τ de dimension d de l'éventail associé au polyèdre $\mathcal{Q}_\rho(F) + \mathcal{Q}_\rho(G)$. Le vecteur w détermine des chemins monotones uniques $\gamma_{_F}$ et $\gamma_{_G}$ dans les arêtes des ρ-polyèdres de Newton de F et de G. Le chemin $\gamma_{_F}$ a des arêtes e_i d'inclinaisons q_i, pour $i = 1, \dots, m$ et par le lemme 1.7 on a $q_m <_w \dots <_w q_1$. Le chemin $\gamma_{_G}$ a des arêtes e'_j d'inclinaisons q'_j, pour $j = 1, \dots, m'$ tels que $q'_{m'} <_w \dots <_w q'_1$.

Par contre, l'ordre défini par w dans $q_1, \dots q_m, q'_1, \dots q'_{m'}$ peut varier lorsque w parcourt τ. Nous considérons la subdivision finie rationnelle la moins fine de τ possédant la propriété suivante : des vecteurs irrationnels qui sont dans le même cône de la subdivision définissent le même ordre sur l'ensemble des inclinaisons $q_1, \dots q_m, q'_1, \dots q'_{m'}$. On définit de cette manière une subdivision Σ de l'éventail associé à $\mathcal{Q}_\rho(F) + \mathcal{Q}_\rho(G)$.

Proposition 1.16. *L'éventail associé à l'intégrale de Minkowski $\mathcal{Q}_\rho(FG)$ est égal à Σ.*

Preuve. Soit $\gamma_{_{FG}}$ le chemin monotone dans le polyèdre $\mathcal{N}_\rho(FG) = \mathcal{N}_\rho(F) + \mathcal{N}_\rho(G)$ défini par un vecteur irrationnel $w \in \tau \in \Sigma$. Chaque point $\gamma(t)$ est la somme de deux points situés dans les chemins $\gamma_{_F}$ et $\gamma_{_G}$ définis par w dans les polyèdres respectifs. Clairement on a $\gamma_{_{FG}}(n+n') = \gamma_{_F}(n) + \gamma_{_G}(n')$. Si $q_m \geq_w q'_m$, le segment $l_{m+m'} := \gamma_{_G}(n') + e_m$ est contenu dans $\gamma_{_{FG}}$. Ce segment n'est pas une arête de $\gamma_{_{FG}}$ si et seulement si, on a $q_m = q'_m$. Par récurrence, on subdivise $\gamma_{_{FG}}$ en $m+m'$ segments, $l_1, \dots, l_{m+m'}$, tels qu'il existe une bijection $\{l_1, \dots, l_{m+m'}\} \to \{e_1, \dots, e_m, e'_1, \dots, e'_{m'}\}$ qui préserve l'inclinaison et la longueur. De plus, $\gamma_{_{FG}}$ est complètement déterminé par $\gamma_{_F}$, $\gamma_{_G}$ et l'ordre des inclinaisons. Ceci implique que Σ est un éventail plus fin que l'éventail associé à $\mathcal{Q}_\rho(FG)$.

Réciproquement, si w, w' sont des vecteurs irrationnels dans un cône de l'éventail associé au polyèdre $\mathcal{Q}_\rho(FG)$, ils définissent un unique chemin monotone γ et par le lemme 1.7 les inclinaisons de ses arêtes ont le même ordre par rapport à $<_w$ et $<_{w'}$, donc w, w' sont dans le même cône de Σ. ◇

Proposition 1.17. *Soient $F, G \in \mathbb{C}((S_\rho))[Y]$ des polynômes de degrés $n, n' \geq 1$ ayant des termes constants non nuls. Si F, G vérifient la condition résultante, alors :*

1. *L'éventail Σ associé au polyèdre-fibre $\mathcal{Q}_\rho(FG)$ est une subdivision de l'éventail du ρ-polyèdre de Newton du résultant de F et G.*
2. *Soit $\tau \in \Sigma$ un cône de dimension d, définissant les chemins monotones $\gamma_{_F}$, $\gamma_{_G}$ et $\gamma_{_{FG}}$ dans les polyèdres de Newton $\mathcal{N}_\rho(F)$, $\mathcal{N}_\rho(G)$ et $\mathcal{N}_\rho(FG)$. Le sommet du ρ-polyèdre de Newton du résultant de F et G associé a τ est $\int \gamma_{_{FG}} - \int \gamma_{_F} - \int \gamma_{_G}$.*

Preuve. Soit $w \in \tau \in \Sigma$ un vecteur irrationnel, on montre d'abord que l'exposant le plus petit par rapport à $<_w$ qui peut apparaître dans $\mathrm{Res}\,(F,G)$ est le même pour tout $w \in \tau$.

Le chemin $\gamma_{_F}$ a des arêtes $e_r = [p_{r-1}, p_r]$ de pente q_r est de longueur l_r pour $r = 1, \ldots, m$. Le polynôme associé à l'arête e_r est $f_r = \alpha_{p_{r-1}} + \cdots + \alpha_{p_r} t^{l_r}$. Nous notons $\{c_r^1, \ldots, c_r^{l_r}\}$ ses racines comptées avec multiplicité.

Le chemin $\gamma_{_G}$ a des arêtes $e'_s = [p'_{s-1}, p'_s]$ de pente q'_s est de longueur l'_s pour $s = 1, \ldots, m'$. Le polynôme associé à l'arête e'_s est $g_s = \beta_{p'_{s-1}} + \cdots + \beta_{p'_s} t^{l'_s}$. Nous notons $\{d_s^1, \ldots, d_s^{l'_s}\}$ ses racines comptées avec multiplicité.

Par le théorème 1.2, le terme initial par rapport à $<_w$, d'une racine ϕ_r^i de F correspondant au segment e_r est $c_r^i X^{q_r}$, et celui d'un racine ψ_s^j de G correspondant au segment e'_s est $d_s^j X^{q'_s}$.

Si a_n et $b_{n'}$ sont les coefficients des termes de degré n et n' de F et G respectivement, on a $\mathrm{Res}\,(F,G) = a_n^{n'} b_{n'}^n \prod(\phi_r^i - \psi_s^j)$. Le coefficient du terme d'exposant le plus petit par rapport à $<_w$ qui peut apparaître dans $\mathrm{Res}\,(F,G)$ est le produit $ABCD$ où le facteur A correspond à $a_n^{n'} b_{n'}^n$:

$$
\begin{aligned}
A &= \alpha_{p_m}^{n'} \beta_{p'_{m'}}^n, \\
B &= \prod_{\substack{r,s \\ q_r = q'_s}} \prod_{i=1,\ldots,l_r} \prod_{j=1,\ldots l'_s} (c_r^i - d_s^j), \\
C &= \prod_{\substack{r,s \\ q_r >_w q'_s}} \prod_{j=1,\ldots l'_s} \prod_{i=1,\ldots,l_r} c_r^i, \\
D &= \prod_{\substack{r,s \\ q_r <_w q'_s}} \prod_{j=1,\ldots l'_s} \prod_{i=1,\ldots,l_r} -d_s^j.
\end{aligned}
$$

Comme F et G vérifient la condition résultante, on a $ABCD \neq 0$, et le terme obtenu ne varie pas lorsque w parcourt τ.

Nous notons $u_{\Delta(F)}$, $u_{D(G)}$ et $u_{\Delta(FG)}$ l'exposant le plus petit par rapport à $<_w$ qui peut apparaître parmi les exposants du discriminant de F, G et FG respectivement. Par le théorème 1.4 on a :

$$\begin{array}{rcl} 2\int\gamma_F & = & \gamma_F(0)+\gamma_F(n)+u_{\Delta(F)} \\ 2\int\gamma_G & = & \gamma_G(0)+\gamma_G(n')+u_{\Delta(G)} \\ 2\int\gamma_{FG} & = & \gamma_{FG}(0)+\gamma_{FG}(n+n')+u_{\Delta(FG)} \end{array}$$

On considère l'expression

$$\Delta_Y(F)\,\Delta_Y(G)(\text{Res }(F,G))^2 = \Delta_Y(FG) \tag{1.2}$$

en fonction des racines de F et de G et on déduit que si u_0 est l'exposant initial par rapport à $<_w$ de Res (F,G) :

$$2u_0 = u_{\Delta(FG)} - u_{\Delta(F)} - u_{\Delta(G)} = 2\int\gamma_{FG} - 2\int\gamma_F - 2\int\gamma_G,$$

parce que l'on a $\gamma_{FG}(0)=\gamma_F(0)+\gamma_G(0)$ et $\gamma_{FG}(n+n')=\gamma_F(n)+\gamma_G(n')$. ◇

Corollaire 1.18. *Dans les hypothèses de la proposition 1.17, chaque cône $\tau\in\Sigma$ de dimension d définit un sommet du ρ-polyèdre de Newton du résultant de F et G de coefficient :*

$$\alpha_{p_m}^{n'}\beta_{p'_m}^{n}\Big(\prod_{q_r=q'_s}^{r,s}\alpha_{p_r}^{-l'_s}\beta_{p'_s}^{-l_r}\text{Res}\,(f_r,g_s)\Big)\Big(\prod_{q_r>_w q'_s}^{r,s}(-1)^{l_r l'_s}\Big(\frac{\alpha_{p_{r-1}}}{\alpha_{p_r}}\Big)^{l'_s}\Big)\Big(\prod_{q_r<_w q'_s}^{r,s}\Big(\frac{\beta_{p'_{s-1}}}{\beta_{p'_s}}\Big)^{l_r}\Big)$$

Preuve. Comme $\text{Res}\,(f_r,g_s)=\alpha_{p_r}^{l'_s}\beta_{p'_s}^{l_r}\prod_{i,j}(c_r^i-d_s^j)$ on a :

$$B=\prod_{q_r=q'_s}^{r,s}\alpha_{p_r}^{-l'_s}\beta_{p'_s}^{-l_r}\text{Res}\,(f_r,g_s).$$

En utilisant que $\prod_{i=1,\dots,l_r}c_r^i=(-1)^{l_r}\frac{\alpha_{p_{r-1}}}{\alpha_{p_r}}$, et que $\prod_{j=1,\dots,l'_s}-d_s^j=\frac{\beta_{p'_{s-1}}}{\beta_{p'_s}}$ on déduit :

$$C=\prod_{q_r>_w q'_s}^{r,s}(-1)^{l_r l'_s}\Big(\frac{\alpha_{p_{r-1}}}{\alpha_{p_r}}\Big)^{l'_s},\qquad D=\prod_{q_r<_w q'_s}^{r,s}\Big(\frac{\beta_{p'_{s-1}}}{\beta_{p'_s}}\Big)^{l_r}.$$

◇

Exemple 1.1. *Considérons le polynôme $F=U^4V^2Y^5+U^3V^2Y^2-Y+U^2V+V^2$. Le discriminant du polynôme F par rapport à Y est*

$$\begin{aligned}\Delta_Y F &= 108\,U^{25}V^{15}+3125\,U^{24}V^{12}+108\,U^{23}V^{16}+12500\,U^{22}V^{13}\\ &\quad-2250\,U^{22}V^{12}+18750\,U^{20}V^{14}-4500\,U^{20}V^{13}-27\,U^{20}V^{12}\\ &\quad+12500\,U^{18}V^{15}-2250\,U^{18}V^{14}+1600\,U^{17}V^{9}+3125\,U^{16}V^{16}\\ &\quad+1600\,U^{15}V^{10}-256\,U^{12}V^{6}\\ &= U^{12}V^{6}\,\varepsilon\end{aligned}$$

où ε est une unité de l'anneau $\mathbb{C}\{S_\Delta\}$, où $\Delta=pos\{(-2,5),(2,-1)\}$. Nous allons montrer de deux manières qu'il existe $k\in\mathbb{N}$ tel que F se décompose dans l'anneau $\mathbb{C}\{\{\frac{1}{k}S_\Delta\}\}$.

1. D'abord, on définit $G := U^6\,V^8\,F(U^{-2}\,V^{-2}) = Y^5 + U^5\,V^6\,Y^2 - U^4\,V^6\,Y + U^8\,V^9 + U^6\,V^{10}$ et on vérifie que $G \in \mathbb{C}\{S_\Delta\}[Y]$ et que $G(z_\Delta, Y) = Y^5$. Comme le discriminant de F est une unité dans $\mathbb{C}\{\{S_\Delta\}\}$ par le théorème 1.1, il existe $k \in \mathbb{N}$ tel que G se décompose dans $\mathbb{C}\{\frac{1}{k}S_\Delta\}$ et donc F se décompose dans $\mathbb{C}\{\{\frac{1}{k}S_\Delta\}\}$.

2. Le polynôme F vérifie la condition discriminantale donc on a $\mathcal{N}_\Delta(U^4\,V^2(U^2\,V + V^2)\Delta_Y F) = \mathcal{Q}_\Delta(F)$. Le Δ-polyèdre-fibre $\mathcal{Q}_\Delta(F)$ a deux sommets correspondant aux chemins monotones cohérents $(\gamma_i)_{i=1,2}$. Le premier, γ_1, correspondant aux termes $U^4V^2Y^5$, Y, V^2 et définissant le sommet $2\int\gamma_1 = (8,10)$ du polyèdre $\mathcal{Q}_\Delta(F)$. Le deuxième γ_2, correspondant à $U^4V^2Y^5$, Y, U^2V et définissant le sommet $2\int\gamma_2 = (8,9)$. L'éventail associé est la subdivision de Δ par les cônes $\sigma_1 = \text{pos}\{(2,-1),(1,2)\}$, et $\sigma_2 = \text{pos}\{(-2,5),(1,2)\}$. Fixons un vecteur irrationnel $w \in \Delta$. Si $w \in \sigma_1$, (resp. $w \in \sigma_2$), il détermine le chemin γ_1, (resp. γ_2).

Les 4 racines de F correspondant au segment $e = [(4,2,5),(0,0,1)]$ de γ_1 (resp. de γ_2) par le théorème 1.2 ont un terme d'exposant $(-1,-\frac{1}{2})$. On définit $F_2 = F(Y + \lambda\,U^{-1}V^{-1/2})$ où $\lambda^4 = 1$.

$$\begin{aligned} F_2 := \;& Y^5\,U^4\,V^2 + 5\,\lambda\,Y^4\,U^3\,V^{3/2} + 10\,\lambda^2\,Y^3\,U^2\,V + Y^2\,U^3\,V^2 \\ & +10\,\lambda^3\,Y^2\,U\,V^{1/2} + 2\,\lambda\,Y\,U^2\,V^{3/2} + 5\;Y - Y \\ & +U^2\,V + \lambda^2\,U\,V + \lambda\,U^{-1}\,V^{-1/2} - \lambda\,U^{-1}\,V^{-1/2} + V^2 \end{aligned}$$

Par la preuve du théorème 1.2 les exposants des racines correspondant à e sont dans le cône affine de sommet $\pi_e(e) = (-1,\frac{-1}{2})$ qui contient les exposants de $\pi_e(\mathcal{N}_\Delta(F_2))$ cette-à-dire le cône $(-1,\frac{-1}{2}) + \text{pos}\{(2,3),(2,1)\}$. Comme le cône $\text{pos}\{(2,3),(2,1)\}$ est contenu dans $\Delta^\vee$, ces racines sont des éléments de $\mathbb{C}\{\{\frac{1}{2}S_\Delta\}\}$.

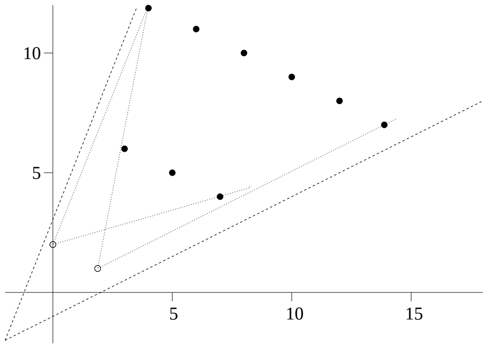

FIG. 1.1 – Les sommets noirs sont les exposants de $F_3^{(1)}(U,V,0)$

Par ailleurs, $F(0,Y) = -Y$, et en appliquant le théorème des fonctions implicites, il existe un unique $\phi \in \mathbb{C}\{U,V\}$ tel que $F(\phi) = 0$. Clairement, ϕ doit coïncider avec la série correspondant au segment $[(0,0,1),(0,2,0)]$ déterminé par $w \in \sigma_1$, (resp. au segment $[(0,0,1),(0,1,2)]$ déterminé par $w \in \sigma_2$).

Pour $w \in \sigma_1$, on définit $F_2^{(1)} := F(Y + V^2)$ et on remarque que le terme initial par la valuation w de $F_2^{(1)}(0) = U^4V^{12} + U^3V^6 + U^2V$ est U^2V. On défini $F_3^{(1)} := F_2^{(1)}(Y + U^2V) = F(Y + V^2 + U^2V)$.

On sait que les exposants de la série $\phi - V^2$ sont dans le cône affine de sommet $(2,1)$ qui contient les exposants de

$$\begin{aligned} F_3^{(1)}(0) = \; & U^3\,V^6 + 2\,U^5\,V^5 + U^7\,V^4 + \\ & + U^4\,V^{12} + 5\,U^6\,V^{11} + 10\,U^8\,V^{10} + 10\,U^{10}\,V^9 + 5\,U^{12}\,V^8 + U^{14}\,V^7 \end{aligned}$$

C'est-à-dire le cône $(2,1) + \mathrm{pos}\{(1,5),(2,1)\}$.

(Pour $w \in \sigma_2$, on définit $F_2^{(2)} = F(Y + U^2 V)$, on vérifie que $F_3^{(2)} = F_2^{(2)}(Y + V^2) = F(Y + V^2 + U^2 V) = F_3^{(1)}$, et que les exposants de la série construit $\phi - U^2 V$ sont dans le cône affine $(0,2) + \mathrm{pos}\{(2,5),(7,2)\}$.)

On obtient que les exposants de ϕ sont dans le cône affine $(-2,-2) + \mathrm{pos}\{(2,5),(1,2)\}$. Comme le cône $\Delta^\vee = \mathrm{pos}\{(2,5),(1,2)\}$ la série ϕ est dans l'anneau $\mathbb{C}\{\{S_\Delta\}\}$ (voir la figure 1.1).

Chapter 2

The semigroup of a quasi-ordinary hypersurface

Abstract. An analytically irreducible hypersurface germ $(S,0) \subset (\mathbb{C}^{d+1},0)$ is quasi-ordinary if it can be defined by the vanishing of the minimal polynomial $f \in \mathbb{C}\{X\}[Y]$ of a fractional power series in the variables $X = (X_1, \ldots, X_d)$ which has *characteristic monomials*, generalizing the classical Newton-Puiseux characteristic exponents of the plane branch case ($d = 1$). We prove that the set of vertices of Newton polyhedra of the resultant of f and h with respect to the indeterminate Y, for those polynomials h which are not divisible by f, is a semigroup of rank d, generalizing the classical semigroup appearing in the plane branch case. We show that some of the *approximate roots* of the polynomial f are irreducible quasi-ordinary polynomials and that together with the coordinates $X_1, \ldots, X_d$ provide a set of generators of the semigroup from which we can recover the characteristic monomials and vice-versa. Finally, we prove that the semigroups corresponding to any two parametrizations of $(S,0)$ are isomorphic and that this semigroup is a complete invariant of the *embedded topological type* of the germ $(S,0)$ as characterized by the work of Gau and Lipman.

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[1] Math. Subject Class.: 14M25, 32S25

Introduction

The set of intersection multiplicities at the origin of a plane branch $(\mathcal{C}, 0)$ with those curves which do not contain this branch as a component is a sub-semigroup of $\mathbb{Z}_{\geq 0}$ which is in fact a complete invariant of the embedded topological type of the germ $(\mathcal{C}, 0) \subset (\mathbb{C}^2, 0)$ (see [Re]). If $f \in \mathbb{C}\{X\}[Y]$ is a Weierstrass polynomial defining the germ $\mathcal{C}$ we can build a system of generators of the semigroup by considering the curves defined by $X = 0$ and by the vanishing of some of the approximate roots of the polynomial f. If X defines a transversal parameter, i.e., the curve $X = 0$ is not contained in the tangent cone of the germ $(\mathcal{C}, 0)$, this set of generators is the minimal one (see [Z8], [A-M], [PP1] and [G-P]).

In this paper we generalize some of the above results to the class of analytically irreducible quasi-ordinary hypersurface germs. A germ of complex analytic variety (S, o) is a *quasi-ordinary* if there exist a finite morphism $(S, o) \rightarrow (\mathbb{C}^d, 0)$ such that the discriminant locus is contained (germ-wise) in a normal crossing divisor. Quasi-ordinary hypersurface singularities arise classically in Jung's approach to analyse a hypersurface by using embedded resolution of the discriminant of a finite projection to the affine space (see [L2], [W]).

A quasi-ordinary hypersurface germ can be defined by an equation $f = 0$ where where $f \in \mathbb{C}\{X\}[Y]$ is a quasi-ordinary polynomial, i.e., monic polynomial such that the discriminant $\Delta_Y f$ is of the form $X^\delta \varepsilon$ for a unit ε (where X denotes $(X_1, \dots, X_d)$). The Jung-Abhyankar theorem implies that the roots of quasi-ordinary polynomials, called quasi-ordinary branches, are fractional power series in the ring $\mathbb{C}\{X^{1/n}\}$ for some positive integer n (see [J] and [A1]). Since the difference $\zeta - \zeta'$ of any two roots of f divides the discriminant it must be of the form $X^\lambda \epsilon$, where $\epsilon \in \mathbb{C}\{X^{1/n}\}$ is a unit and λ is a d-tuple of non negative rationals. The monomials X^λ so obtained are called the *characteristic*. Lipman builds an inversion lemma which associates to any quasi-ordinary branch ζ a *normalized* quasi-ordinary branch parametrizing the same germ whose characteristic monomials are obtained from those of ζ by an inversion formulae similar to that of the plane curve case (see [L1] and [Gau] Appendix). Being normalized is a technical condition which in the plane curve case means that the kernel of the projection is not contained in the tangent cone of the germ. In the two dimensional case Lipman proved that the characteristic monomials of a normalized quasi-ordinary branch are an analytical invariant of the surface (see [L1], [L3]). Luengo gives another proof of this result (see [Lu]). Lipman remarked, using general results of Zariski on saturation of local rings, that the characteristic monomials of a normalized quasi-ordinary branch determine the topological type of the germ it parametrizes (see [L4] and also [Oh] for another proof); Gau proved the converse: these monomials define a complete invariant of the *embedded topological type* of the germ. Gau's proof involves Lipman's results on topological invariants of quasi-ordinary singularities: the description of the local divisor class group in terms of the characteristic monomials (see [Gau] and [L4]).

When the quasi-ordinary polynomial f is irreducible we generalize some of the properties of the intersection multiplicity of plane curve germs by studing the combinatorial structure of the set of compact faces of the Newton polyhedra of the resultants with respect to the indeterminate Y of the polynomial f and polynomials in the ring $\mathbb{C}\{X\}[Y]$. In particular, we prove that the set of vertices of Newton polyhedra of the resultant $\mathrm{Res}_Y(f,h)$, for those polynomials h which are not divisible by f, is a semigroup of rank d (see Theorem 2.1 and Corollary 2.2). When $d=1$ we obtain the classical semigroup associated a plane branch. In general we prove that the Newton polyhedra of the resultants of f and of the minimal polynomials of some suitable truncations of the series ζ together with the coordinates $X_1,\ldots,X_d$, have only one vertex and that these vertices generate the semigroup. The same result holds when we replace the minimal polynomials of the truncations of ζ by approximate roots of f of the same degrees. We prove that these approximate roots are irreducible quasi-ordinary polynomials and that their roots are fractional power series which coincide up to prescribed terms, which are some of the characteristic monomials, with the series ζ (see Proposition 2.14). The set of generators of the semigroup is related with the characteristic monomials by formulae similar to those which appear in the plane branch case. Relevant to these formulae and to the properties of the semigroup are a set of integers and rank d lattices determined by the characteristic monomials, which we use to give properties characterizing those semigroups associated to a quasi-ordinary branch.

In the last section we use Theorem 2.1 and Corollary 2.2 to prove that the semigroups associated to a quasi-ordinary branch and its normalized quasi-ordinary branch by the inversion lemma of Lipman are isomorphic. As a consequence we obtain another proof of the inversion formulae for the characteristic monomials. Our main result, Theorem 2.4, states that the semigroup we build from a quasi-ordinary branch is a complete invariant of the embedded topological type (as characterized by Gau): two quasi-ordinary branches parametrizing the same hypersurface germ have isomorphic semigroups and this semigroup determines the normalized characteristic exponents. Up to Theorem 2.4, which uses the characterization of the topological type, the results of this work are valid also in the algebroid case when we replace $\mathbb{C}\{X\}$, the ring of germs of holomorphic functions at the origin of the affine complex space, by the ring of formal power series with coefficients in an algebraically closed field of characteristic zero.

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2.1 Quasi-ordinary hypersurface singularities

A germ of complex analytic variety (S,o) is a *quasi-ordinary singularity* if there exist a finite morphism $(S,o)\rightarrow(\mathbb{C}^d,0)$ (called a *quasi-ordinary projection*) such that the discriminant locus is contained (germ-wise) in a normal crossing divisor. This means that there exist some analytical coordinates $X = (X_1, \dots, X_d)$ at the origin such that the projection is unramified over the torus $X_1 \cdots X_d = 0$ in a neighborhood of the origin. The finite map $(S, o) \to (\mathbb{C}^d, 0)$ corresponds algebraically to a local homomorphism $\mathbb{C}\{X\} \to R$ of their analytic algebras which gives R the structure of finite $\mathbb{C}\{X\}$-module. The germ (S, o) is a quasi-ordinary *hypersurface* if R is generated by one element. In this case we have a surjection $\mathbb{C}\{X\}[Y] \to R$ corresponding geometrically to an embedding $(S, o) \hookrightarrow (\mathbb{C}^d \times \mathbb{C}, 0)$. The kernel of the homomorphism above is a principal ideal generated by a monic polynomial $f \in \mathbb{C}\{X\}[Y]$ such that $f(0, Y) = Y^n$ (where $n = \deg f$ is also equal to the degree of the quasi-ordinary projection) and the discriminant $\Delta_Y f$ of f with respect to the variable Y is of the form $X^\delta H$ for a unit H in $\mathbb{C}\{X\}$. We say that a polynomial f in $\mathbb{C}\{X\}[Y]$ satisfying these conditions is *quasi-ordinary* since it defines a quasi-ordinary hypersurface for the projection $(X, Y) \mapsto X$. The Jung-Abhyankar theorem (see [A1], Th. 3 for an algebraic proof, see [J]) implies that the roots of the quasi-ordinary polynomial f are fractional power series in the ring $\mathbb{C}\{X^{1/k}\}$ for some positive integer k where $X^{1/k} = (X_1^{1/k}, \dots, X_d^{1/k})$. The roots of quasi-ordinary polynomials are called *quasi-ordinary branches.* The difference $\zeta^{(s)} - \zeta^{(t)}$ of two distinct roots of f divides the discriminant in $\mathbb{C}\{X^{1/k}\}$, thus it is necessarily of the form $X^{\lambda_{st}} H_{st}$ where H_{st} is a unit in $\mathbb{C}\{X^{1/k}\}$. When $d = 1$ the exponents λ_{st} obtained are the classical Newton-Puiseux characteristic exponents of the (analytically) irreducible components of the germ (S, o) in general coordinates and the orders of coincidence of the fractional parametrizations of distinct components.

If the polynomial f is irreducible we can take $k = n := \deg f$ (see [L4] page 52 and remarque 1 of [GP1]). The analytic algebra $R = \mathbb{C}\{X\}[Y]/(f)$ can be viewed as the subring $\mathbb{C}\{X\}[\zeta]$ of $\mathbb{C}\{X^{1/n}\}$ for ζ any root of f. If L (resp. L_n) is the field of fractions of $\mathbb{C}\{X\}$ (resp. of $\mathbb{C}\{X^{1/n}\}$), the field extension $L \subset L_n$ is finite and Galois. Its Galois group is obtained from the action of d-tuples $(\eta_1, \dots, \eta_d)$ of n^{th}-roots of unity given by $X_i^{1/n} \mapsto \eta_i X_i^{1/n}$, for $i = 1, \dots, d$. It follows that the field extension obtained from $R \subset \mathbb{C}\{X^{1/n}\}$ by taking fields of fractions is Galois therefore the roots of f are the conjugates of ζ by the above action. The fractional monomials $X^{\lambda_{st}}$ (resp. the vector exponents λ_{st}) are called *characteristic or distinguished* monomials (resp. exponents) of the quasi-ordinary branch ζ.

If f is a reduced quasi-ordinary polynomial it follows from the geometrical definition that all its irreducible factors are quasi-ordinary polynomials.

Definition 2.1. *(see [GP2]) The polynomials $f^{(i)}$ and $f^{(j)}$ in $\mathbb{C}\{X\}[Y]$ have order of coincidence $\lambda_{(i,j)}$ if $f^{(i)} f^{(j)}$ is a quasi-ordinary polynomial and $\lambda_{(i,j)}$ is the largest exponent of the set $\{\lambda_{st} / f^{(i)}(\zeta^{(s)}) = 0, f^{(j)}(\zeta^{(t)}) = 0\}$.*

It follows in this case that the λ_{st} above are orders of coincidence of pairs of different factors or characteristic exponents of the factors.

The partial order in $\mathbb{Q}^d$ defined by $\lambda \leq \lambda'$ if and only if we have $\leq$ coordinate-wise, induces a total order in the set of characteristic exponents of a quasi-ordinary branch (see [L4], lemma 5.6). We relabel them in a unique form $\lambda_1 < \lambda_2 < \cdots < \lambda_g$ (where $<$ means $\leq$ and $\neq$).

Lemma 2.1. *([L3], prop. 1.5, [Gau], prop 1.3) Let* $\zeta = \sum c_\lambda X^\lambda$ *be a non unit in* $\mathbb{C}\{X^{1/n}\}$*. Then* ζ *is a quasi-ordinary branch if and only if there exists elements* $\lambda_i \in \frac{1}{k}\mathbb{Z}^d_{\geq 0}$ $(1 \leq i \leq g)$ *such that*

1. $\lambda_1 < \lambda_2 < \cdots < \lambda_g$*, and* $c_{\lambda_i} \neq 0$ *for* $1 \leq i \leq g$*.*
2. *If* $c_\lambda \neq 0$ *then* λ *is in the subgroup of* $\mathbb{Q}^d$ *given by* $\mathbb{Z}^d + \sum_{\lambda_i \leq \lambda} \mathbb{Z}\lambda_i$*.*
3. λ_j *is not in the subgroup of* $\mathbb{Q}^d$ *given by* $\mathbb{Z}^d + \sum_{\lambda_i < \lambda_j} \mathbb{Z}\lambda_i$*, for* $j = 1, \ldots, g$*.*

If such elements exist, they are uniquely determined by ζ*, and they are the characteristic exponents of* ζ*.*

Remark 2.2. *Lemma 2.1 give us a canonical way of writing the terms of a quasi-ordinary branch:*

$$\zeta = p_0 + p_1 + \cdots + p_g,$$

where p_0 *is in* $\mathbb{C}\{X\}$ *and* X^λ *appears in* p_i *with non zero coefficient implies that* $\lambda_i \leq \lambda$ *and* $\lambda_{i+1} \not\leq \lambda$*.*

It follows from lemma 2.1 that the series $p_0 + p_1 + \cdots + p_i$ obtained by truncating ζ are quasi-ordinary branches with i-characteristic exponents for $i = 0, \ldots, g$.

We say that the quasi-ordinary branch ζ has *well ordered variables* if the g-tuples $(\lambda_{1,i}, \ldots, \lambda_{g,i})$ of i-coordinates of the characteristic exponents $\lambda_1, \ldots, \lambda_g$ are ordered lexicographically, i.e., we have that

$$(\lambda_{1,i}, \ldots, \lambda_{g,i}) \geq_{\text{ lexicographically }} (\lambda_{1,j}, \ldots, \lambda_{g,j}) \text{ for } 1 \leq i < j \leq d.$$

It is clear that given a quasi-ordinary branch ζ we can relabel the variables $X_1, \ldots, X_d$ in order to satisfy this condition.

Definition 2.2. *The quasi-ordinary branch* ζ*, is normalized, if its given with well ordered variables and if it happens that the first characteristic exponent is of the form* $\lambda_1 = (\lambda_{1,1}, 0, \ldots, 0)$ *then we have* $\lambda_{1,1} > 1$*.*

This condition in the case of plane curve germs means that the kernel of the projection $(X, Y) \mapsto X$ is not contained in the tangent cone of the germ. Lipman proved that any quasi-ordinary hypersurface is parametrized by a normalized quasi-ordinary branch ζ (see [L1] and [Gau] Appendix).

We denote the lattice $\mathbb{Z}^d$ by M and by M_j the lattice $\mathbb{Z}^d + \sum_{\lambda_i < \lambda_{j+1}} \mathbb{Z}\lambda_i$, for $j = 1, \ldots, g$ with the convention $\lambda_{g+1} = +\infty$. Following Lipman (see [L4], page 61) we associate to the characteristic exponents sequences of lattices and integers. In the plane branch case the sequence of integers coincide with the first component of the characteristic pairs in arbitrary coordinates.

Definition 2.3. *The characteristic lattices are the lattices* M_i *defined above. The characteristic integers are the the indexes* n_j *of* M_{j-1} *in* M_j *for* $j = 1, \dots, g$ *.*

We denote by $e_{i-1} = n_i \cdots n_g$, for $i = 1, \dots, g$ and we set $n_0 := 1$.

We denote by N the dual lattice of M and by $N_{\mathbb{R}}$ the real vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by the lattice N. We denote by $\rho \subset N_{\mathbb{R}}$ the cone spanned by the dual basis of the canonical basis of M. The exponents of the quasi-ordinary branch ζ belong to the semigroup $\rho^\vee \cap M_g$ where $\rho^\vee = \{u \in M_{\mathbb{R}} / \langle u, v\rangle \geq 0,\ \forall v \in \rho\}$ is the *dual cone* of the cone ρ. The series ζ can be viewed as an element of the ring $\mathbb{C}\{\rho^\vee \cap M_g\}$ of germs of holomorphic functions at the special point of the *affine toric variety* Spec $\mathbb{C}[\rho^\vee \cap M_g]$ (see [GP4]). We denote the ring $\mathbb{C}\{X\}$ by $\mathbb{C}\{\rho^\vee \cap M\}$. The advantage of these notations is that we can define ring homomorphisms by changing the lattice or the cones, for instance the ring extension $R = \mathbb{C}\{\rho^\vee \cap M\}[\zeta] \to \mathbb{C}\{\rho^\vee \cap M_g\}$ is the inclusion in the integral closure (see Proposition 14 of [GP4]).

The field of fractions of R is $L[\zeta]$ since ζ is algebraic over L.

Lemma 2.3. *([L4], lemma 5.7) We have the following equality:* $L[\zeta] = L[X^{\lambda_1}, \dots, X^{\lambda_g}]$. ◇

Remark 2.4. *The integers* n_j *and* e_i *are the degrees of the Galois extensions:*

$$\begin{aligned} e_i &:= [L[\zeta] : L[X^{\lambda_1}, \dots, X^{\lambda_i}]], \ \textit{for } i = 1, \dots, g. \\ n_j &:= [L[X^{\lambda_1}, \dots, X^{\lambda_j}] : L[X^{\lambda_1}, \dots, X^{\lambda_{j-1}}]], \ \textit{for } j = 1, \dots, g. \end{aligned} \tag{2.1}$$

In particular we have that $\deg f = e_0 = n_1 \cdots n_g = n$ *.*

It follows from lemma 2.3 that the polynomial f has all its roots in the field $L[\zeta]$ (since it is also equal to the field of fractions of $\mathbb{C}\{\rho^\vee \cap M_g\}$) thus the extension $L[\zeta] : L[X^{\lambda_1}, \dots, X^{\lambda_i}]$ is Galois. The assertion on the degrees follows then from definition 2.3 and lemma 2.3 using that the minimal polynomial of X^{λ_j} over $L[X^{\lambda_1}, \dots, X^{\lambda_{j-1}}]$ is $Y^{n_j} - X^{n_j \lambda_j}$.

The *Newton polyhedron* of a series $0 \neq \phi = \sum c_a X^a \in \mathbb{C}\{\rho^\vee \cap M\}$ is the convex hull of the set $\bigcup_{c_a \neq 0} a + \rho^\vee$. We define the Newton polyhedron of $0 \neq \phi \in \mathbb{C}\{\rho^\vee \cap M'\}$ for M' a lattice containing M in the same way. It follows that the Newton polyhedron does not change after a lattice extension. The face determined by $\eta \in \rho$ on the Newton polyhedron $\mathcal{N}(\phi)$ is the set $\{v \in \mathcal{N}(\phi) / \langle \eta, v\rangle = \inf_{v' \in \mathcal{N}(\phi)} \langle \eta, v'\rangle\}$. All faces of the polyhedron $\mathcal{N}(\phi)$ can be recovered in this way, in particular the compact faces are determined by the vectors η in the interior of ρ. The *cone* $\sigma(\mathcal{F}) \subset \rho$ *associated to the face* $\mathcal{F}$ of the polyhedron $\mathcal{N}(\phi)$ is

$$\sigma(\mathcal{F}) := \{\eta \in \rho \,/ \langle \eta, v\rangle = \inf_{v' \in \mathcal{N}(\phi)} \langle \eta, v'\rangle, \ \forall v \in \mathcal{F}\}.$$

The set of cones $\sigma(\mathcal{F})$, for $\mathcal{F}$ running through the set of faces of the polyhedron $\mathcal{N}(\phi)$ is called the *dual Newton diagram* of ϕ. The *Newton principal part* $\phi_{|\mathcal{N}}$ of ϕ is the sum of those terms of ϕ having exponents lying on the compact faces of its Newton polyhedron (see [Kou] for the terminology).

2.2 The semigroup of a quasi-ordinary branch

In the following sections we study the singularity (S, o) defined parametrized by a quasi-ordinary branch ζ with $g \geq 1$ characteristic exponents $\lambda_1, \dots, \lambda_g$.

If $d = 1$ then S is a plane branch and the set of intersection multiplicities $(S, S')_0$ of S which those plane curve germs S' not containing S as a component forms a sub-semigroup of $(\mathbb{Z}_{\geq 0}, +)$ which is an invariant of the germ S. A set of generators of the semigroup is obtained from the characteristic exponents by the following formula (see [Z8]):

$$\bar{\gamma}_1 = n\lambda_1, \quad \bar{\gamma}_{j+1} = n_j \bar{\gamma}_j + n\lambda_{j+1} - n\lambda_j, \quad \text{for } j = 1, \dots, g-1. \tag{2.2}$$

For $j = 0, \dots, g-1$, we expand:

$$\begin{aligned} \bar{\gamma}_{j+1} &= n\left((n_1 - 1)n_2 \cdots n_j \lambda_1 + (n_2 - 1)n_3 \cdots n_j \lambda_2 + \cdots + (n_j - 1)\lambda_j + \lambda_{j+1}\right) \\ &\overset{\text{Def. 2.3}}{=} n_1 \dots n_j \left((e_0 - e_1)\lambda_1 + (e_1 - e_2)\lambda_2 + \cdots + (e_{j-1} - e_j)\lambda_j + e_j \lambda_{j+1}\right) \end{aligned} \tag{2.3}$$

We denote $\frac{1}{n}\bar{\gamma}_i$ by γ_i for $i = 1, \dots, g$. We have:

$$\gamma_1 = \lambda_1, \quad \gamma_{j+1} = n_j \gamma_j + \lambda_{j+1} - \lambda_j, \quad \text{for } j = 1, \dots, g-1. \tag{2.4}$$

We associate to the characteristic monomials of the quasi-ordinary branch ζ the sequence of semigroups

$$\Gamma_j = \rho^\vee \cap M + \gamma_1 \mathbb{Z}_{\geq 0} + \cdots + \gamma_g \mathbb{Z}_{\geq 0} \text{ for } j = 0, \dots, g.$$

We denote $n\Gamma_j$ by $\bar{\Gamma}_j$ for $j = 0, \dots, g$. If $d = 1$ the semigroup $\bar{\Gamma}_g \subset \mathbb{Z}_{\geq 0}$ is the classical semigroup of the plane branch S.

Lemma 2.5. *(See [T4], Chapitre I, Lemma 2.2.1 in the plane branch case and [GP2])*

1. *The sub-lattice of* M_g *generated by* Γ_j *is equal to* M_j, *for* $0 \leq j \leq g$.
2. *The order of the image of* γ_j *in the group* M_j / M_{j-1} *is equal to* n_j *for* $j = 1, \dots, g$.
3. *We have that* $\gamma_j > n_{j-1}\gamma_{j-1}$ *for* $j = 2, \dots, g$.
4. *If a vector* $u_j \in \rho^\vee \cap M_j$ *then we have* $u_j + n_j \gamma_j \in \Gamma_j$.
5. *The vector* $n_j \gamma_j$ *belongs to the semigroup* Γ_{j-1} *for* $j = 1, \dots, g$, *moreover we have a unique relation:*
$$n_j \gamma_j = \alpha^{(j)} + l_1^{(j)} \gamma_1 + \cdots + l_{j-1}^{(j)} \gamma_{j-1} \tag{2.5}$$
such that $0 \leq l_i^{(j)} \leq n_i - 1$ *and* $\alpha^{(j)} \in M_0$, *for* $j = 1, \dots, g$.

Proof. The first assertion follows from (2.4). The second assertion follows from the definition of the integers n_i and the fact that $\gamma_i = \lambda_i \mod M_{i-1}$ for $i = 1, \ldots, g$. We deduce the third from (2.4) and the inequality:

$$\begin{aligned} n_j\gamma_j - n_{j-1}\gamma_{j-1} &= n_{j-1}(n_j - 1)\gamma_{j-1} + n_j(\lambda_j - \lambda_{j-1}) \\ &> (n_j - 1)(n_{j-1}\gamma_{j-1} + \lambda_j - \lambda_{j-1}) \\ &= (n_j - 1)\gamma_j, \end{aligned} \tag{2.6}$$

The assertion *4* is easy for $j = 1$. We suppose it is true for $j - 1 \geq 1$. A vector $u_j \in \rho^\vee \cap M_j$ is of the form $u_j = \alpha_j\gamma_j + u'_j$ for a unique $0 \leq \alpha_j < n_j$ and $u'_j \in M_{j-1}$. By (2.6) the vector $u_{j-1} := u'_j + n_j\gamma_j - n_{j-1}\gamma_{j-1}$ belongs to $\rho^\vee \cap M_{j-1}$. By induction hypothesis the vector $u_{j-1} + n_{j-1}\gamma_{j-1} = u'_j + n_j\gamma_j$ is in the semigroup Γ_{j-1} hence the vector $u_j + n_j\gamma_j = \alpha_j\gamma_j + u'_j + n_j\gamma_j$ belongs to the semigroup Γ_j.

We deduce from formula (2.4) that $n_j\gamma_j = n_j n_{j-1}\gamma_{j-1} + n_j(\lambda_j - \lambda_{j-1})$. By assertions *1* and *2* the vector $n_j(\lambda_j - \lambda_{j-1})$ is in the lattice M_{j-1} and by lemma 2.1 it belongs to $\rho^\vee$. Now we apply *4* to obtain the first assertion of *5*. The existence of relations of the form (2.5) follows by induction on g using Euclidean division: For $g = 1$ is clear. We suppose true for $g - 1$, then by induction we have the required relations for $j = 1, \ldots, g - 1$. We have proved that $n_j\gamma_g \in \Gamma_g$, thus we have a relation:

$$n_j\gamma_g = \alpha_0 + l_1\gamma_1 + \cdots + l_{g-1}\gamma_{g-1}. \tag{2.7}$$

We divide the non negative integer l_{g-1} by n_{g-1} and we obtain $l_{g-1} = kn_{g-1} + l^{(g)}_{g-1}$ with $0 \leq l^{(g)}_{g-1} \leq n_{g-1} - 1$; then we substitute $l_{g-1}\gamma_{g-1}$ by $l^{(g)}_{g-1}\gamma_{g-1} + +k(\alpha_0^{(g-1)} \sum_{i=1}^{g-2} l_i^{(g-1)}\gamma_i)$ in formula (2.7) and we obtain a formula of the same type where $l_{g-1} = l^{(g)}_{g-1}$. The required expansion is obtained by iterating this procedure. The unicity follows from *2*. ◇

We say that a set of generators of a semigroup is *minimal* if none of the generators belongs to the semigroup spanned by the others.

Lemma 2.6. *The semigrup Γ_g has a unique minimal set of generators. If ζ is normalized this set is the canonical basis of M_0 union $\{\gamma_1, \ldots, \gamma_g\}$.*

Proof. Since the semigroup Γ_ζ is contained in the cone with vertex $\rho^\vee$ we can use lemmas 3.6 and 3.5, chapter V of [Ew] to proof that Γ_ζ has a minimal set of generators which is unique. We show that the canonical basis vectors and $\gamma_1, \ldots, \gamma_g$ are a minimal set of generators of Γ_ζ. The condition of being normalized implies that the canonical basis vectors are the first elements of the semigroup Γ_ζ appearing on the edges of the cone $\rho^\vee$, hence we cannot eliminate any of them while preserving the semigroup. They generate the subsemigroup $\Gamma_0 = \rho^\vee \cap M_0$ of Γ_g. If we have a relation of the form $\gamma_k = u + \sum a_j\gamma_j$ with $u \in \Gamma_0$, $a_j \in \mathbb{Z}_{\geq 0}$ and $a_k = 0$, then assertion *3* of lemma 2.5 implies that $a_j = 0$ for $j > k$. The relation obtained contradicts assertion *2* of lemma 2.5. ◇

Remark 2.7. *The properties* 1, 2, 3, 5 *of lemma 2.5 characterize those semigroups* Λ *of the form*

$$\Lambda = \rho^\vee \cap M_0 + \gamma_1 \mathbb{Z}_{\geq 0} + \cdots + \gamma_g \mathbb{Z}_{\geq 0} \text{ for } \gamma_i \in \rho^\vee \cap M_{\mathbb{Q}},\ i = 1, \dots, g$$

which are associated to the characteristic monomials of a quasi-ordinary branch.

Proof. In this proof we denote by M_j the lattice $M_0 + \sum_{i=1}^{j} \gamma_i \mathbb{Z}$, by Γ_j the semigroups $\rho^\vee \cap M_0 + \sum_{i=1}^{j} \gamma_i \mathbb{Z}_{\geq 0}$ for $i = 0, \dots g$ and by n_i the order of γ_i modulo M_{i-1} for $i = 1, \dots, g$. We define vectors in $M_{\mathbb{Q}}$: $\alpha_1 := \gamma_1$ and $\alpha_{j+1} := \gamma_{j+1} - \gamma_j + \alpha_j$ for $j = 1, \dots, g-1$. It follows by property *3* and induction that α_{j+1} is $> \alpha_j$ and belongs to the semigroup $\rho^\vee \cap M_g$ for $j = 1, \dots, g-1$. By lemma 2.1 the series $\zeta := X^{\alpha_1} + \cdots + X^{\alpha_g}$ is a quasi-ordinary branch with minimal polynomial in $\mathbb{C}\{\rho^\vee \cap M_0\}[Y]$, with characteristic exponents α_i for $i = 1, \dots, g$ and with the previously defined characteristic integers $n_1, \dots, n_g$ and lattices $M_1, \dots, M_g$. It follows that $\Lambda = \Gamma_g$. ◇

Recall that if $q \in \mathbb{C}\{X\}[Y]$ defines a germ of curve S' at the origin the intersection multiplicity $(S, S')_0 = \dim_{\mathbb{C}} \mathbb{C}\{X, Y\}/(f, q)$ coincides with the order in X of the resultant $\operatorname{Res}_Y(f, q)$ of the polynomials f and q with respect to Y.

Proposition 2.8. *Let* $q \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ *be any irreducible quasi-ordinary polynomial of degree* $n_0 n_1 \dots n_j$ *for* $0 \leq j \leq g-1$. *The following are equivalent:*

1. *The polynomial* q *has order of coincidence* λ_{j+1} *with* f
2. $\operatorname{Res}_Y(f, q) = X^{\bar{\gamma}_{j+1}} \varepsilon_j$ *for a unit* $\varepsilon_j \in \mathbb{C}\{\rho^\vee \cap M\}$.
3. $q(\zeta) = X^{\gamma_{j+1}} \epsilon_j$ *for a unit* $\epsilon_j \in \mathbb{C}\{\rho^\vee \cap M_g\}$.

Proof. The result is trivial for $j = 0$. Suppose first that q is an irreducible quasi-ordinary polynomial of degree ≥ 1 comparable with f and having order of coincidence α. Since the roots $\{\zeta^{(k)}\}_{k=1}^{n}$ of f are a complete set of conjugates over L and the polynomial q has its coefficients on this ring, the series $\{q(\zeta^{(k)})\}_{k=1}^{n}$ are a complete set of conjugates over L, thus we have:

$$\mathcal{N}(\operatorname{Res}_Y(f, q)) = \mathcal{N}(\prod_{k=1}^{n} q(\zeta^{(k)})) = \deg f \mathcal{N}(q(\zeta)). \tag{2.8}$$

We deduce from (2.8) the equivalence between assertions *2* and *3*. If τ is any root of the irreducible polynomial q we deduce by symmetry from (2.8) that

$$\mathcal{N}(\operatorname{Res}_Y(f, q)) = \deg q \mathcal{N}(f(\tau)).$$

Take a root $\zeta^{(k)}$ of f such that $\zeta^{(k)} - \tau = X^{\alpha} \cdot$ unit. By definition of the order of coincidence, the biggest characteristic exponent of τ that is $< \alpha$, if it exist, is a characteristic exponent λ_i of $\zeta^{(k)}$.

Any other root of f verifying this property is obtained from $\zeta^{(k)}$ by the action an element of the Galois group of the extension $L[X^{\lambda_1}, \dots, X^{\lambda_j}] \subset L[\zeta]$ by remark 2.4 and conversely thus:

$$\#\{\text{Roots } \zeta^{(l)} \text{ of } f \text{ such that } \zeta^{(l)} - \tau = X^{\alpha}. \text{ unit}\} = [L[\zeta] : L[X^{\lambda_1}, \dots, X^{\lambda_i}]] = e_i \tag{2.9}$$

Analogously, we obtain that:

$$\#\{\text{Roots } \zeta^{(l)} \text{ of } f \text{ such that } \zeta^{(l)} - \tau = X^{\lambda_k}. \text{ unit}\} = e_{k-1} - e_k, \text{ for } k = 1, \dots, i. \tag{2.10}$$

By (2.8) and (2.9) this implies that $\operatorname{Res}_Y(f, q) = X^{\gamma} \cdot$ unit where:

$$\gamma = \deg q((e_0 - e_1)\lambda_1 + (e_1 - e_2)\lambda_2 + \cdots + (e_{i-1} - e_i)\lambda_i + e_i\alpha) \tag{2.11}$$

If q is a j-semi-root we have that $\deg q = n_1 \dots n_j$ and $\alpha = \lambda_{j+1}$ thus $i = j$ and $\gamma = \bar{\gamma}_{j+1}$ by (2.3).

Conversely if $1 \leq i < j$ or if α is $\leq$ than any characteristic exponent of τ we deduce from formulae (2.3) and (2.11) that $\alpha \geq \lambda_{j+1}$. This implies that λ_j is $< \alpha$ and it is also a characteristic exponent of τ a contradiction. If $j < i$, we obtain from (2.3) and (2.11) that $\gamma > \bar{\gamma}_{j+1}$. Therefore, $\gamma = \bar{\gamma}_{j+1}$ implies that $j = i$ and $\alpha = \lambda_{j+1}$. ◇

Remark 2.9. *With the previous notations the leading exponent δ of the discriminant of f (which is defined by $\Delta_Y f = X^{\delta}\varepsilon$, for ε a unit) is equal to $\delta = \sum_{k=1}^{g} (e_{k-1} - e_k)\lambda_k$.*

The proof follows using (2.10) for the roots of f and the relation $\Delta_Y f = \prod_{i \neq j} (\zeta^{(i)} - \zeta^{(j)})$.

We denote by q_i the minimal polynomial of the truncation $p_0 + p_1 + \cdots + p_i$ of the parametrization of f for $i = 0, \dots, g$. The polynomials q_i are quasi-ordinary by lemma 2.1 and $\deg q_i = n_0 \cdots n_i$ for $i = 0, \dots, g$.

Any non zero element of the ring $R = \mathbb{C}\{\rho^{\vee} \cap M\}[\zeta]$ is of the form $h(\zeta)$ for a polynomial $h \in \mathbb{C}\{\rho^{\vee} \cap M\}[Y]$ of degree $< n$. The following proposition (see [GP2]) is a generalization of a result of plane branches (see [Z8], Chapitre II, Th. 3.9).

Theorem 2.1. *If $\deg h < n_0 \dots n_j$ then the Newton principal part of $h(\zeta)$ belongs to the ring $\mathbb{C}[\Gamma_j]$, for $j = 0, \dots, g$.*

Proof. We show the result by induction. For $j = 0$, if $\deg h < n_0 = 1$ the result is trivial since $\Gamma_0 = \rho^{\vee} \cap M$. We suppose the result true for degrees $< n_0 \dots n_{j-1}$. If $\deg h < n_1 \dots n_j$, using Euclidean division several times we can expand the polynomial h in a unique way, the q_{j-1}-adic expansion of h:

$$h = a_0 + a_1 q_{j-1} + \cdots + a_s q_{j-1}^{s}, \tag{2.12}$$

where a_k are polynomials in $\mathbb{C}\{\rho^{\vee} \cap M\}[Y]$ of degree $< n_1 \cdots n_{j-1} = \deg q_{j-1}$, for $k = 0, \dots, s$ and $0 \leq s < n_j$. The Newton principal part of $(a_k q_{j-1}^{k})(\zeta)$ is the product of the Newton principal parts of the factors. We have that $(q_{j-1}^{k}(\zeta))_{|\mathcal{N}} = X^{k\gamma_j}$ and by induction hypothesis $a_k(\zeta)_{|\mathcal{N}}$ belongs to the ring $\mathbb{C}[\Gamma_{j-1}]$. The exponents of the Newton principal part of $(a_k q_{j-1}^{k})(\zeta)$ belong to the set $k\gamma_j + M_{j-1}$ for $k = 0, \dots, s$, and by lemma 2.5 these sets are disjoint since the order n_j of γ_j in the group M_j/M_{j-1} is $> s$. It follows that the terms of the Newton principal parts of $(a_k q_{j-1}^{k})(\zeta)$ can not cancel each other thus the polynomial $h(\zeta)_{|\mathcal{N}}$ is a sum of the terms of the Newton principal parts of $(a_k q_{j-1}^{k})(\zeta)$ for $k = 0, \dots, s$. ◇

Corollary 2.2. *The Newton principal part of* $Res_Y(f,h)$*, for those polynomials* h *in* $\mathbb{C}\{X\}[Y]$ *which are not divisible by* (f)*, runs through the elements of the ring* $\mathbb{C}[\bar{\Gamma}_g]$*.*

The set of vertices of Newton polyhedra of $Res_Y(f,h)$ *(resp. of* $h(\zeta)$*) for* $h \in (\mathbb{C}\{X\}[Y] \backslash (f))$*, is a semigroup with respect to the addition which is isomorphic to* Γ_g*.*

Proof. The proof is consequence of formulae (2.8) and of theorem 2.1. ◇

This corollary gives the analogy with the classical definition of the semigroup of a plane branch using the canonical valuation of the integral closure of the ring R. We say that the semigroup Γ_g is associated to the quasi-ordinary branch ζ and we denote it also by Γ_ζ.

Remark 2.10. *An alternate way of defining the semigroup* Γ_ζ *is given by Popescu-Pampu in [PP2]: he introduce first the set* $\mathcal{C}_f$ *of functions* $h \in \mathbb{C}\{X\}[Y] \backslash (f)$ *such that the Newton polyhedron of* $h(\zeta)$ *has only one vertex* γ_h*; then he define the semigroup by* $\Gamma_\zeta = \{\gamma_h / h \in \mathcal{C}_f\}$*.*

Remark 2.11. *Any polynomial* $h \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ *can be written in a unique way as*

$$h = \sum c_{l_1,\dots,l_{g+1}} q_0^{l_1} q_1^{l_2} \cdots q_g^{l_{g+1}} \tag{2.13}$$

with $c_{l_1,\cdots,l_{g+1}} \in \mathbb{C}\{\rho^\vee \cap M\}$*,* $0 \leq l_k \leq n_k - 1$ *for* $k = 1, \dots, g$ *and* $l_{g+1} \in \mathbb{Z}_{\geq 0}$*.*

This follows by computing the q_g expansion of h first and then the q_{g-1} expansion of the coefficients and iterating the procedure (see [A4] in the case of plane branches). By using lemma 2.5 we deduce that the following fact (see [PP2]):

Remark 2.12. *The exponents of the Newton principal part of the terms appearing in the summation (2.13) can not cancel each other.*

2.3 Semi-roots and approximate roots

In the plane branch case several authors have studied the properties of those curves S' such that the intersection multiplicity with S at the origin belongs to the unique minimal set of generators of the semigroup of the branch (see [Z8]). In [LJ1] Lejeune introduced the notion of curves of maximal contact of higher genus with a given plane curve germ for curves defined over a field of arbitrary characteristic in terms of the resolution (see [LJ2]). If the characteristic is zero it turns out that both notions are equivalent (see [Ca]). If the projection (X, Y) is transversal we can study these curves by means of the minimal polynomials of suitable truncations of the roots of f. When we do this with respect to an arbitrary projection the curves we obtain provide a non necessarily minimal set of generators of the semigroup of the branch S. We call them *semi-roots* following the terminology of Abhyankar and Popescu-Pampu (see [A4] and [PP1]). They are also called *pseudoroots* by Gwoździewicz and Ploski (see [G-P]). *Approximate roots* are used by Abhyankar and Moh, (see [A-M]) to obtain several results on the local and global geometry of plane curves. We show that the approximate roots of f of degrees e_j for $j = 0, \dots, g$ are semi-roots of f (see [GP2]). We follow the approach of the plane curve case given by Gwoździewicz, J., Ploski (see [G-P]) and by Popescu-Pampu in the survey [PP1].

Definition 2.4. *A j^{th}-semi-root of f is an irreducible quasi-ordinary polynomial in $\mathbb{C}\{\rho^\vee \cap M\}[Y]$ of degree $n_0 \dots n_j$ which has order of coincidence equal to λ_{j+1} with f, for $j = 0, \dots, g$.*

The minimal polynomials of the quasi-ordinary branches $p_0 + \cdots + p_j$ obtained by truncating ζ in remark 2.2 are j^{th}-semi-roots of f for $j = 0, \dots, g$.

Proposition 2.13. *Let $q \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ a monic polynomial of degree $n_0 \dots n_j$. Then q is a j-semi-root of f if and only if $q(\zeta) = X^{\gamma_{j+1}} \varepsilon_j$ for a unit ε_j in $\mathbb{C}\{\rho^\vee \cap M_g\}$.*

Proof. We have to show that if $q(\zeta)$ is of the form $X^{\gamma_{j+1}} \varepsilon_j$ for a unit ε_j then q is necessarily an irreducible quasi-ordinary polynomial with order of coincidence λ_{j+1} with f.

The polynomial q is irreducible: Suppose that it is equal to the product $q_1 q_2$ of polynomials of degrees $< n_1 \cdots n_j$. Then we have that $q(\zeta) = q_1(\zeta) q_2(\zeta) = X^{\gamma_{j+1}} \varepsilon_{j+1}$ for a unit $\varepsilon_{j+1} \in \mathbb{C}\{\rho^\vee \cap M_g\}$. The Newton polyhedron of $q(\zeta)$ has only one vertex and since this polyhedron is equal to the Minkowski sum $\sum_{i=1,2} \mathcal{N}(q_i(\zeta))$ and it follows that each the polyhedron $\mathcal{N}(q_i(\zeta))$ has only one vertex u_i. We deduce that $q_i(\zeta) = X^{u_i} \epsilon_i$ for ϵ_i a unit in $\mathbb{C}\{\rho^\vee \cap M_g\}$. Then proposition 2.1 implies that $\gamma_{j+1} = u_1 + u_2$ belongs to Γ_j contradicting lemma 2.5.

By proposition 2.8 the result follows by proving that the polynomial q is quasi-ordinary, i.e., that the Newton polyhedron of the discriminant $\Delta_Y q$ has only one vertex. The cone associated to a vertex v of the polyhedron $\mathcal{N}(\Delta_Y q)$,

$$\sigma_v = \{u \in \rho / \min_{v' \in \mathcal{P}} \langle u, v' \rangle = \langle u, v \rangle\}$$

is of dimension d. Let $\sigma \subset \sigma_v$ be any cone generated by a basis of the lattice N (see [Ew] for its existence). The dual cone $\sigma^\vee$ of σ is generated by a basis of the dual lattice M and contains $\rho^\vee$. We have the local ring extension $\mathbb{C}\{\rho^\vee \cap M\} \hookrightarrow \mathbb{C}\{\sigma^\vee \cap M\}$ defined by the semigroup inclusion

$\rho^\vee \cap M \hookrightarrow \sigma^\vee \cap M$. We obtain a similar extension by replacing the lattice M by M_g. The ring $\mathbb{C}\{\sigma^\vee \cap M\}$ is isomorphic to the ring $\mathbb{C}\{T_1, \ldots, T_d\}$, the isomorphism maps T_i to X^{t_i} for $t_1, \ldots, t_d$ the basis of M which spans $\sigma^\vee$ as a cone.

We denote an element ϕ in $\mathbb{C}\{\rho^\vee \cap M\}$ (resp. in $\mathbb{C}\{\rho^\vee \cap M\}[Y]$) viewed as an element of $\mathbb{C}\{\sigma^\vee \cap M\}$ (resp. of $\mathbb{C}\{\sigma^\vee \cap M\}[Y]$) by ϕ_σ. The polynomial f_σ is quasi-ordinary: its roots $\zeta_\sigma^{(i)}$, for $\zeta^{(i)}$ running through the roots of f, are conjugated quasi-ordinary branches since the and the characteristic exponents and lattices coincide for ζ and ζ_σ by lemma 2.1. The polynomial q_σ is quasi-ordinary since its Newton polyhedron is equal to the Minkowski sum $\mathcal{N}(\Delta_Y q) + \sigma^\vee$ and it has a unique vertex v since $\sigma \subset \sigma_v$. By the argument given in the first paragraph of this proof the polynomial q_σ is irreducible and it follows from proposition 2.8 that it is a j-semi-root of f_σ. The vertex v is determined by the first j characteristic exponents of f and the associated characteristic integers (more precisely we have $v = \sum_{k=1}^{j} (e_{k-1} - e_k)\lambda_k$ by remark 2.9). This implies that the polyhedron $\mathcal{N}(\Delta_Y(q))$ has only one vertex v and $q \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ is a quasi-ordinary polynomial. $\diamond$

Let A a ring containing $\mathbb{Q}$ as a subring. If p is any monic polynomial and k divides the degree of p there is a unique monic polynomial r in $A[Y]$ of degree $\frac{\deg p}{k}$ such that $\deg(p - r^k) < \deg p - \frac{\deg p}{k}$. We denote this polynomial $\sqrt[k]{p}$. For instance if $p = Y^n - a_1 Y^{n-1} + \cdots + a_0$, we have that $\sqrt[n]{p} = Y - \frac{a_1}{n}$. If $k = k_1 k_2$ divides $\deg p$ then we have that $\sqrt[k]{p} = \sqrt[k_1]{\sqrt[k_2]{p}}$. (see proposition 3.3 in [PP1]). If $q \in A[Y]$ is any monic polynomial of degree $\frac{\deg p}{k}$ we expand: $p = q^k + a_1 q^{k-1} + \cdots + a_0$, with polynomials $a_i \in A[Y]$ of degree $< \frac{\deg p}{k}$. The map $\mathcal{T}_p$ between the set of monic polynomials of degree $\frac{\deg p}{k}$ defined by

$$\mathcal{T}_p(q) = q + \frac{1}{k} a_1,$$

is called the k*-Tschirnhausen operator*. It is shown that the k-approximate roots can be computed by iterating the k-Tschirnhausen operator, i.e,

$$\sqrt[k]{p} = \overbrace{\mathcal{T}_p \circ \cdots \circ \mathcal{T}_p}^{\deg p/k}(q),$$

where q is any monic polynomial of degree $\frac{\deg p}{k}$ (see Proposition 6.3 in [PP1]).

Proposition 2.14. *The* e_j*-approximate roots of* f *are* j*-semi-roots of* f.

Proof. This is trivial for $j = 0$ and it is also trivial for $j = g$, since we have that $\sqrt[1]{f} = f$ satisfies the condition (for $\lambda_{g+1} = \infty$). Suppose the result is true for $1 < j < g$. We show that it is true for $j - 1$. We have that $e_{j-1} = n_j e_j$ and $\sqrt[e_{j-1}]{f} = \sqrt[n_j]{\sqrt[e_j]{f}}$. If we set $p = \sqrt[e_j]{f}$, we obtain that

$$\sqrt[e_{j-1}]{f} = \overbrace{\mathcal{T}_p \circ \cdots \circ \mathcal{T}_p}^{n_j}(q),$$

where q is any monic polynomial of degree $n_1 \ldots n_{j-1}$.

It is sufficient to prove that if q is a $(j-1)$-semi-root the polynomial $\mathcal{T}_p(q)$ is a $(j-1)$-semi-root. We expand, $p = q^{n_j} + a_1 q^{n_j - 1} + \cdots + a_0$, with $a_i \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ polynomials of degree $< \deg q$. By the induction hypothesis p is a j-semi-root of f thus: $\mathcal{N}(a_1(\zeta) q^{n_j - 1}(\zeta)) \subset \mathcal{N}(p(\zeta)) = \gamma_{j+1} + \rho^\vee$. It means that if u is any vertex of $\mathcal{N}(a_1(\zeta))$ then $u \geq \gamma_{j+1} - (n_j - 1)\gamma_j$ is $> \gamma_j$ by lemma 2.5. It follows that $q + \frac{1}{n_j} a_1$ is a $(j-1)$-semi-root of f by proposition 2.13. ◇

2.4 Invariance of the semigroup and inversion formulae

In this section prove that the semigroups associated to a quasi-ordinary branch ζ and its normalized by Lipman's inversion lemma are isomorphic. We obtain from this another proof of the inversion formulae relating their characteristic monomials. We show then that this semigroup is a complete topological invariant of the embedded topological type of the hypersurface parametrized by ζ, as characterized by the work of Gau and Lipman.

If ζ is of the form: $\zeta = X_1^{k/n} H(X_1^{1/n}, \ldots, X_d^{1/n})$ with $H(0) \neq 0$ with minimal polynomial $f \in \mathbb{C}\{X\}[Y]$, the order of the series $f(X_1, 0, \ldots, 0; 0)$ is equal to k (for $n = \deg f$). The Weierstrass polynomial $f' \in \mathbb{C}\{Y, X_2, \ldots, X_d\}[X_1]$ associated to f defines the same hypersurface germ. The inversion lemma of Lipman (see [L1], lemma 2.3, or [Gau], Appendix) guarantees that f' is quasi-ordinary polynomial and if ζ is not normalized any root τ of f' is. This lemma is a generalization of the classical inversion lemma of plane branches (see [Z5] and [A2]).

A root τ of f' can be obtained as follows: Let F be a unit such that $F^k = H$. The series ζ is equal to $\zeta = \zeta_0^k$ where $\zeta_0 = X_1^{1/n} F(X_1^{1/n}, \ldots, X_d^{1/n})$. The series $Y^{1/k} - \zeta_0$ is of order one in $X_1^{1/n}$ and by the Weierstrass preparation theorem there exists a unit ε such that:

$$\varepsilon(Y^{1/k} - \zeta_0) = X_1^{1/n} - \tau_0 \tag{2.14}$$

where τ_0 is of the form $\tau_0 = Y^{1/k} F'(Y^{1/k}, X_2^{1/n}, \ldots, X_d^{1/n})$ for a unit F'.

We build from (2.14) an isomorphism:

$$\mathbb{C}\{X_1^{1/n}, \ldots, X_d^{1/n}\} \rightarrow \mathbb{C}\{Y^{1/k}, X_2^{1/n}, \ldots, X_d^{1/n}\} \tag{2.15}$$

which maps $X_1^{1/n} \mapsto \tau_0$, $X_i^{1/n} \mapsto X_i^{1/n}$ for $i = 2, \ldots, d$, and such that the inverse image of $Y^{1/k}$ (resp. of Y) is equal to ζ_0 (resp. to $\zeta_0^k = \zeta$). Since ζ is a root of f we have $f'(\zeta, X_2, \ldots, X_d; X_1) = 0$ thus its image by (2.15) is $f'(Y, X_2, \ldots, X_d, \tau_0^n) = 0$. Therefore $\tau := \tau_0^n$ is a root of f'.

Moreover, we have a commutative diagram

$$\begin{tikzcd}
R = \mathbb{C}\{X_1, X_2, \ldots, X_d, Y\}/(f') \arrow[r, hook] & \mathbb{C}\{Y^{1/k}, X_2^{1/n}, \ldots, X_d^{1/n}\} \\
R = \mathbb{C}\{X_1, X_2, \ldots, X_d, Y\}/(f) \arrow[r, hook] \arrow[u] & \mathbb{C}\{X_1^{1/n}, \ldots, X_d^{1/n}\} \arrow[u]
\end{tikzcd} \tag{2.16}$$

where the horizontal arrows are the ring extensions defined by τ and ζ and the left vertical arrow is the identity and the right vertical arrow is the isomorphism (2.15).

If the Newton polyhedron of $0 \neq \phi \in \{X_1^{1/n}, \dots, X_d^{1/n}\}$ has only one vertex the same happens for the Newton polyhedron of the image of ϕ by the isomorphism (2.15). By corollary 2.2 the semigroup Γ_ζ is isomorphic to the semigroup of Newton polyhedra (with the Minkowski sum) whose elements are $\mathcal{N}(\phi)$ of $0 \neq \phi \in R$ such that $\mathcal{N}(\phi)$ has only one vertex. The Newton polyhedron of $\phi \in R$ is defined in this case as the Newton polyhedron of the image of ϕ by the ring extension $R \hookrightarrow \mathbb{C}\{X_1^{1/n}, \dots, X_d^{1/n}\}$.

The same assertion holds, with respect to the ring extension $R \hookrightarrow \mathbb{C}\{Y^{1/k}, X_2^{1/n}, \dots, X_d^{1/n}\}$, for the semigroup Γ_τ. The isomorphism (2.15) induces an isomorphism of semigroups $\Gamma_\zeta \cong \Gamma_\tau$ since the diagram (2.16) is commutative and the Newton polyhedron of a product is the Minkowski sum of the factors. We deduce from this the following:

Proposition 2.15. *The semigroups Γ_ζ and Γ_τ are isomorphic.* ◇

Two hypersurfaces germs $(H, 0)$ and $(H', 0)$ in $\mathbb{C}^{d+1}$ have the same embedded topological type if and only if there is a homeomorphism $U \to U'$, between two open neighborhoods of the origin, which maps representatives $H \cap U$ to $H' \cap U'$. The characteristic exponents of a quasi-ordinary branch ζ determine the *embedded topological type* of the hypersurface it defines. This is deduced from results of Zariski on saturation of local rings (see [Z5], and [L3], §2 and also [Oh] for another proof). Gau, using Lipman's description of the divisor class group of the singularity, proves the equivalence:

Theorem 2.3. *(see [Gau], Theorem 1.6) A pair of analytically irreducible quasi-ordinary hypersurface germs $(S, 0)$ and $(S', 0)$ in $(\mathbb{C}^{d+1}, 0)$ have the same topological type if and only if any two normalized quasi-ordinary branches parametrizing $(S, 0)$ and $(S', 0)$ respectively have the same characteristic exponents.*

We deduce from proposition 2.15 and Gau's characterization the following result:

Theorem 2.4. *If ζ and ζ' are quasi-ordinary branches parametrizing the same quasi-ordinary hypersurface then the semigroups Γ_ζ and $\Gamma_{\zeta'}$ are isomorphic. The isomorphism class of this semigroup determines and is determined by the embedded topological type of the germ $(S, 0)$.*

Proof. We can suppose that ζ and ζ' are given with well ordered variables. The first assertion follows from proposition 2.15 and Gau's characterization. Then we can recover from the unique minimal set of generators of the semigroup Γ_ζ (by lemma 2.6) the characteristic exponents of a normalized quasi-ordinary branch parametrizing the same hypersurface germ (see remark 2.7). ◇

In the case of quasi-ordinary surfaces the analytical invariance of the semigroup is deduced without using the Gau's characterization of the topological type, the analytical invariance of the characteristic monomials being proved by Lipman: he builds a non embedded resolution of the surface as a composition of monoidal and cuadratic transforms which are determined and determines the characteristic monomials of a normalized quasi-ordinary branch parametrizing the surface (see [L1], [L3]). A direct proof of analytical invariance of the semigroup associated to a quasi-ordinary surface, which implies the invariance properties of the characteristic monomials, has recently been given by Popescu-Pampu (see [PP2] and [PP3]). In the general case the semigroup appears in an intrinsic manner by analysing, with the generalized Abhyankar's expansions (see remark 2.12), some algebraic properties of the minimal resolution of the normalization of S at the singular points of the reduced normal crossing divisor defined by the transform of the singular locus of S.

Another application of proposition 2.15 is the following proof of the inversion formulae relating the characteristic exponents of ζ and τ (stated in [L1] page 78 and [L3] page 170).

Lemma 2.16. *Let ζ be a quasi-ordinary branch of the form $\zeta = X^{\lambda_0} H \in \mathbb{C}\{\rho^\vee \cap M_g\}$, with $\lambda_0 \in \rho^\vee \cap M$ and H a unit. For any positive integer k, the series ζ^k is a quasi-ordinary branch and its characteristic exponents are $\lambda'_j = \lambda_j + (k-1)\lambda_0$, for $j = 1, \dots, g$ where $\{\lambda_j\}_{j=1}^g$ are the characteristic exponents of ζ.*

Proof. Firstly, we have that $\lambda'_1 < \cdots < \lambda'_g$. The characteristic lattices M_i associated to ζ coincide with the lattices we can associate to $\{\lambda'_j\}_{j=1}^g$ by the same formulae. We show that λ'_j are exponents of ζ^k. We expand $\zeta = p_0 + p_1 + \cdots + p_g$ using remark 2.2. Each summand

$$\left(\begin{array}{c} k \\ s \end{array} \right) (p_0 + \cdots + p_{j-1})^{k-s} (p_j + \cdots + p_g)^s$$

appearing in the binomial expansion of $((p_0 + \cdots + p_{j-1}) + (p_j + \cdots + p_g))^k$ is of the form $X^{(k-s)\lambda_0 + s\lambda_j}$. unit , for $s = 0, \dots, k$, because $\lambda_0 < \lambda_1$ by the hypothesis.

We have $(k-s)\lambda_0 + s\lambda_j \geq (k-1)\lambda_0 + \lambda_j = \lambda'_j$, for $s = 1, \dots, k$, hence ζ^k is of the form:

$$\zeta^k = (p_0 + \cdots + p_{j-1})^k + X^{\lambda'_j}. \text{ unit }.$$

The exponents of the terms appearing in $(p_0 + \cdots + p_{j-1})^k$ belongs to the group M_{j-1} by lemma 2.1. Therefore the monomial $X^{\lambda'_j}$ appears in ζ^k with non zero coefficient. It also follows from the formula above, that if we have $\lambda'_{j-1} \leq \lambda$ and $\lambda'_j \not\leq \lambda$, for λ the exponent of a term appearing in ζ^k, then this term appears in $(p_0 + \cdots + p_{j-1})^k$, hence λ is in the group M_{j-1}. The conditions in lemma 2.1 are satisfied for ζ^k and $\{\lambda'_j\}_{j=1}^g$. ◇

Proposition 2.17. *Suppose that the rational number $\alpha = k/n$ is not an integer. Let $\{\lambda_i\}_{i=1}^g$ be the characteristic exponents of ζ. Denote by $\lambda'_i \in \mathbb{Q}^d$ the vector with coordinates: $\lambda'_{i,1} = \alpha^{-1}\lambda_{i,1} + \alpha^{-1} - 1$ and $\lambda'_{i,q} = \lambda_{i,q}$ for $q = 2, \dots, d$ and $i = 1, \dots, g$. Then the characteristic exponents of τ are*

$$\left\{ \begin{array}{l} \lambda'_1, \dots, \lambda'_g \text{ if } \alpha, \alpha^{-1} \notin \mathbb{Z} \\ \lambda'_2, \dots, \lambda'_g \text{ if } \alpha^{-1} \in \mathbb{Z} \end{array} \right.$$

Proof. We follow the notations above. Let m be the l.c.d. of k and n. We have $k = u'm$ and $n = um$ for positive integers, u, u' with $(u, u') = 1$. We consider $U = X_1^{1/u}$ as an indeterminate. As a consequence of lemma 2.1, the series ζ defines a quasi-ordinary branch: $\zeta_1 = U^{u'} F^{u'm}(U^{1/m}, X_2^{1/n}, \dots, X_d^{1/n})$, whose characteristic exponents $\{\lambda_i(\zeta_1)\}_{i=2}^g$ are obtained from $\{\lambda_i\}_{i=2}^g$ multiplying the first coordinate by u.

By lemma 2.16 the series, $\zeta_2 = UF^m(U^{1/m}, X_2^{1/n}, \dots, X_d^{1/n})$, is a quasi ordinary branch with characteristic exponents $\{\lambda_i(\zeta_2)\}_{i=2}^g$ with coordinates: $\lambda_{i,1}(\zeta_2) = \lambda_{i,1}(\zeta_1) - u' + 1 = u\lambda_{i,1} - u' + 1$, and, $\lambda_{i,q}(\zeta_2) = \lambda_{i,q}(\zeta_1) = \lambda_{i,q}$ for $q = 2, \dots, d$.

We set $V = Y^{1/u'}$, and define:

$$\tau_1 = V^u G^{um}(V^{1/m}, X_2^{1/n}, \dots, X_d^{1/n}) \text{ and } \tau_2 = VG^m(V^{1/m}, X_2^{1/n}, \dots, X_d^{1/n}).$$

In the same manner, we deduce that the characteristic exponents, $\{\lambda_i(\tau)\}_{i=2}^{g'}$ of τ which are $> \lambda'_1$, are related with the characteristic exponents $\{\lambda_i(\tau_2)\}_{i=2}^{g'}$ of τ_2 by: $\lambda_{i,1}(\tau_2) = u'\lambda_{i,1}(\tau) - u + 1$ and $\lambda_{i,q}(\tau_2) = \lambda_{i,q}(\tau)$, for $q = 2, \dots, d$.

It is easy to see that $\zeta_2 = \zeta_0^m$ and that $\tau_2 = \tau_0^m$ thus by the isomorphism (2.15) restricts to an isomorphism

$$\mathbb{C}\{U, X_2, \dots, X_d, \zeta_2\} \to \mathbb{C}\{V, X_2, \dots, X_d, \tau_2\}$$

therefore ζ_2 and τ_2 parametrize the same hypersurface germ. It follows from the fact that the isomorphism (2.15) preserves Newton polyhedra that the semigroups Γ_{ζ_2} and Γ_{τ_2} are isomorphic. Since ζ_2 and τ_2 are both normalized and they have the same semigroup it follows that they have the same characteristic exponents. This implies that $g = g'$ and $\lambda_i(\tau) = \lambda'_i$ for $i = 2, \dots, g$.

If $\alpha^{-1} \in \mathbb{Z}$, the characteristic exponents of τ are $\{\lambda_i\}_{i=2}^g$ since $u' = 1$ and $\tau = \tau_1$. Since $\alpha \notin \mathbb{Z}$ the first characteristic exponent of ζ is $\lambda_1 = (\alpha, 0, \dots, 0)$ and if $\lambda'_{1,1} = \alpha^{-1} \notin \mathbb{Z}$, then λ'_1 is the first characteristic exponent of τ. ◇

Chapter 3

Toric embedded resolutions of quasi-ordinary singularities

Abstract. We build a toric embedded resolution of a reduced quasi-ordinary hypersurface singularity $(S,0) \subset (\mathbb{C}^{d+1},0)$ which is a composition of toric morphisms determined by the *characteristic monomials* associated to a quasi-ordinary projection $(S,0) \to (\mathbb{C}^{d},0)$. This gives a positive answer to a a question of Lipman (see [L5]). If the germ $(S,0)$ is analytically irreducible the characteristic monomials define a complete invariant of the embedded topological type of the singularity. We show in this case that some of the approximate roots of a suitable Weierstrass polynomial defining the embedding $(S,0) \subset (\mathbb{C}^{d+1},0)$ have *maximal contact* with $(S,0)$: at each step of the resolution there is one of them which is resolved and becomes a *good coordinate* for the following morphism; then we build a toric morphism π which is a simultaneous toric embedded resolution of the irreducible germ S, re-embedded in a toric variety Z_Δ by using these approximate roots, and of an affine toric variety $Z^\Gamma \subset Z_\Delta$ obtained from $S \subset Z_\Delta$ by specialization and defined by a rank d semigroup Γ generalizing the classical semigroup of a plane branch. Finally we compare both resolutions and we prove that the first resolution is the restriction of the toric morphism π to a $(d+1)$-smooth variety containing the strict transform of S provided that the morphism π is suitably chosen.

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Introduction

A germ of complex analytic variety is *quasi-ordinary* if there exists a finite projection, called quasi-ordinary, to the complex affine space $(\mathbf{C}^d, 0)$ with discriminant locus contained in a normal crossing divisor (for instance, the singularities of complex analytic curves are quasi-ordinary). These singularities appear classically on Jung' strategy to obtain resolution of singularities of surfaces from the embedded resolution of plane curves (see [J], [W] and [L2]). Some properties of complex analytic curve singularities generalize to quasi-ordinary hypersurface singularities: for instance, Jung-Abhyankar's theorem guarantees the existence of fractional power series parametrizations generalizing the classical Newton-Puiseux parametrizations of the plane curve case; by comparing these parametrizations we obtain a finite set of *distinguished* or *characteristic monomials* which generalize the notion of characteristic exponents in the plane branch case.

The results on quasi-ordinary hypersurface singularities concern mainly the analytically irreducible case: Lipman builds a non embedded resolution procedure of a quasi-ordinary surface where only quasi-ordinary singularities occurs and uses it to prove the the analytical invariance properties of the characteristic monomials (see [L1], and [L3]); another proof of this result was given by Luengo in [Lu]; more generally Gau proved that the characteristic monomials define a complete invariant of the *embedded topological type* of the quasi-ordinary hypersurface singularity (see [Gau]); Gau's proof involves Lipman's description of the divisor class group of the singularity in terms of the characteristic monomials (see [L4]).

The embedded topological type of a complex plane curve singularity is completely characterized by the characteristic exponents of the branches of the curve and their intersection multiplicities (see [Re]), these data being equivalent to the structure of the minimal embedded resolution of the curve (see [Z6] and [Z3]). An important step to establish the relations between the topological type and the embedded resolutions of a hypersurface singularity is to determine if the characteristic monomials of an hypersurface quasi-ordinary singularity determine a canonical procedure of embedded resolution. This is the content of Lipman's open problem 5.1 (see [L5]) which is stated in the context of the generalizations of *equisingularity*, in particular by using Zariski's work on the *dimensionality type* with respect to the classification by *equiresolution*. Such a procedure has been found in the case of an analytically irreducible quasi-ordinary surface germ by Ban and McEwan (see [B-M]) and in general by Villamayor with a different approach (see [Vi]). Both procedures make use of the algorithms of resolution developed from the work of Hironaka. In this paper we give another solution to Lipman's problem. We build an embedded resolution of a reduced quasi-ordinary hypersurface germ $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$ as a composition of toric morphisms which are determined by the characteristic monomials where, as in the plane curve case, we have to distinguish between those which are characteristic monomials of some irreducible component of the germ $(S, 0)$ and those which arise as *orders of coincidence* between the parametrizations of different irreducible components of $(S, 0)$ (see Theorem

3.2). The embedded resolution we build is explicit enough to obtain some significative results about the zeta function of the monodromy. In collaboration with Némethi and McEwan we show that the zeta function of $(S,0)$ coincides with the zeta function of the plane curve germ obtained from $(S,0)$ by intersection with $d-1$ coordinate hyperplanes which are determined by the characteristic monomials (see [M-N] and [GP-M-N]). The construction of this embedded resolution seems to be an important result in order to to build embedded resolution of higher dimensional hypersurface singularities following Jung's approach (see [Z1] and [L2]).

We summarize below the contents of this paper. We begin by giving a result in collaboration with Teissier: we build embedded resolutions of non necessarily normal affine toric varieties (see Proposition 3.6, [GP-T], and Proposition 6.4 of [T5]). If there is only one characteristic monomial a toric embedded resolution of the hypersurface germ $(S,0) \subset (\mathbb{C}^{d+1},0)$ is obtained by using this result combined with a *non degenerescence* argument (see Khovanskii's paper [Kho]). In general, the first toric morphism of the resolution is defined by the *dual Newton diagram* of a suitable polynomial $f \in \mathbb{C}\{X_1,\ldots,X_d\}[Y]$ defining the embedding $(S,0) \subset (\mathbb{C}^{d+1},0)$. Suitable here means that Y is a *good coordinate*, a notion defined in terms of the coincidence of the parametrizations of the irreducible factors of f. We show that the singularities of the strict transform of S are simpler and quasi-ordinary with respect to a quasi-ordinary projection canonically defined. In particular when the germ $(S,0)$ is analytically irreducible at each step of the resolution the strict transform loose one characteristic monomial and there is a suitable *approximate root* of the polynomial f with *maximal contact* (it becomes a good coordinate for the following morphism). By iterating we obtain a *partial embedded resolution*: a normal variety of dimension $d+1$ with a toroidal embedding structure (prescribed by the characteristic monomials), provided with a modification of $\mathbb{C}^{d+1}$ such that the strict transform of S is a section of the ambient toroidal embedding which is transversal to the exceptional fiber. This implies that any toric resolution of the ambient space is an embedded resolution of the strict transform and provides a fortiori an embedded resolution of S. Moreover, the partial resolution we build is an *embedded normalization* of S, its restriction to the strict transform of S provides the normalization map. In the case of plane curve germs we compare our approach with that of Lê, Oka and A'Campo (see [Le-Ok], [Ok], and [A'C-Ok]). We show that our procedure provides the minimal resolution of the curve germ when the axis $X = 0$ is not contained in its tangent cone.

When the germ $(S,0)$ is analytically irreducible we build an embedded resolution by generalizing the method of Goldin and Teissier for plane branches (see [G-T]). We define a toric morphism π which is a simultaneous partial embedded resolution of the irreducible germ S, re-embedded in a toric variety Z_Δ by using some of the approximate roots of f, and of an affine toric variety $Z^\Gamma \subset Z_\Delta$ obtained from $S \subset Z_\Delta$ by specialization (see Theorem 3.4). The variety Z^Γ is defined by a rank d semigroup Γ, generalizing the classical semigroup of a plane branch. In [GP2] we proved indeed that the semigroup Γ does not depend on the quasi-ordinary projection and it defines a complete invariant of the embedded topological type as characterized by Gau. Finally we compare both partial resolutions and we prove that the first is the restriction of π to a $(d+1)$-dimensional smooth variety containing the strict transform of S provided that the modification π is suitably chosen (see Theorem 3.5). Analogous results were obtained by Lejeune and Reguera for sandwiched surface singularities (see [LJ-R]) and for plane curve germs (see [LJ-R2]).

Since quasi-ordinary singularities are defined as branched coverings of the torus $(\mathbb{C}^*)^d \subset \mathbb{C}^d$ is natural to study the branched coverings of $(\mathbb{C}^*)^d$ embedded in a normal affine toric variety Z_ρ. We called these branched coverings *toric quasi-ordinary singularities.* The *toric quasi-ordinary hypersurfaces* (relative to the basis Z_ρ) appear naturally as the singularities of the intermediate steps between the *classical* quasi-ordinary singularity S and its normalization in the first procedure of partial resolution. For this reason we build both partial resolution procedures directly for toric quasi-ordinary hypersurfaces generalizing to this case the notions of characteristic monomials and many of their properties in the classical case. Classically quasi-ordinary singularities are studied in the algebroid case over an algebraically closed field by means of the fractional power series parametrizations provided by the Jung-Abhyankar theorem. The link with the toric geometry methods employed in this paper can be obtained by the Newton-Puiseux type algorithm which builds the roots of a Weierstrass polynomial F as parametrizations of the toric quasi-ordinary hypersurfaces defined by the polynomial after toric base changes determined by the Newton polyhedron of the discriminant of F (see [GP1]).

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3.1 Toric maps, Newton polyhedra and partial resolution of singularities

We introduce the notations and basic definitions of toric geometry, we define the notion of partial embedded resolution and we use it to build embedded resolutions of non necessarily normal affine toric varieties.

3.1.1 A reminder of toric geometry

We give some definitions and notations (see [F1], [Ew] and [Od] for proofs). If $N \cong \mathbb{Z}^{d+1}$ is a lattice we denote by $N_\mathbb{R}$ the real vector space $N \otimes_\mathbb{Z} \mathbb{R}$ spanned by N and by M the dual lattice.

A *rational convex polyhedral cone* σ in $N_{\mathbb{R}}$ is the set non negative linear combinations of vectors $a^1, \dots, a^s \in N$. In what follows a *cone* will mean a rational convex polyhedral cone. The cone σ is *strictly convex* if σ contains no linear subspace of dimension > 0; the cone σ is *regular* if the *primitive integral vectors* defining the 1-dimensional faces belong to a basis of the lattice N. We denote by $\overset{\circ}{\sigma}$ the *relative interior* of a cone σ. The *dual* cone $\sigma^{\vee}$ (resp. *orthogonal* cone $\sigma^{\perp}$) of σ is the set $\{w \in M_{\mathbb{R}} / \langle w, u \rangle \geq 0,$ (resp. $\langle w, u \rangle = 0$) $\forall u \in \sigma\}$. A *fan* Σ is a family of *strictly convex cones* in $N_{\mathbb{R}}$ such that any face of such a cone is in the family and the intersection of any two of them is a face of each. The *support* of the fan Σ is the set $\bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}$. The i-skeleton $\Sigma^{(i)}$ is the subset of i-dimensional cones of Σ. The fan Σ is *regular* if all its cones are regular.

Any non necessarily normal affine toric variety over the field $\mathbb{C}$ of complex numbers is of the form $Z^{\Lambda} = \operatorname{Spec} \mathbb{C}[\Lambda]$ where Λ is a *monoid*, i.e., a sub-semigroup of finite type of a lattice $-\Lambda + \Lambda$ which generates it as a group. The closed points of Z^{Λ} correspond to homomorphisms of semigroups $\Lambda \to \mathbb{C}$ where $\mathbb{C}$ is considered as a semigroup with respect to multiplication. The torus embedded in Z^{Λ} is the group of homomorphisms of semigroups $\Lambda \to \mathbb{C} - \{0\}$ and acts naturally on the closed points of Z^{Λ}. The *normalization* of Z^{Λ} is obtained from the inclusion $\Lambda \to \mathbb{R}_{\geq 0}\Lambda \cap (-\Lambda + \Lambda)$ where $\mathbb{R}_{\geq 0}\Lambda$ is the cone spanned by the elements of Λ (see [KKMS]). The action of the torus has a fixed point if and only if the cone $\mathbb{R}_{\geq 0}\Lambda$ is strictly convex, then this point is defined by the ideal $(X^u / u \in \Lambda - \{0\})$ of $\mathbb{C}[\Lambda]$ and coincides with the 0-dimensional orbit; the analytic algebra $\mathbb{C}\{\Lambda\}$ of Z^{Λ} at this point can be viewed as a subring of the ring $\mathbb{C}[[\Lambda]]$ of formal complex power series with exponents in the semigroup Λ (see [GP1] lemme 1.1).

In particular, if σ is a cone in the fan Σ the semigroup $\sigma^{\vee} \cap M$ is of finite type, it spans the lattice M and the variety $Z^{\sigma^{\vee} \cap M}$, which we denote also by $Z_{\sigma, N}$ or by Z_{σ} when the lattice is clear from the context, is *normal.*

If $\sigma \subset \sigma'$ are cones in the fan Σ then we have an open immersion $Z_{\sigma} \subset Z_{\sigma'}$; the affine varieties Z_{σ} corresponding to cones in a fan Σ glue up to define the *toric variety* Z_{Σ}. The torus, $(\mathbb{C}^*)^{d+1}$, is embedded as an open dense subset $Z_{\{0\}}$ of Z_{Σ}, which acts on each chart Z_{σ}; these actions paste to an action on Z_{Σ} that extends the product on the torus. General toric varieties are defined by this property, the toric varieties which can be defined using fans are precisely the normal ones (see [KKMS]). The toric variety Z_{Σ} is non singular if and only if the fan Σ is regular.

We describe the orbits of the action of the torus on the variety Z_{Σ}. The orbit $\mathbf{O}_{\sigma, N}$ (which we denote also by $\mathbf{O}_{\sigma}$) is the Zariski closed subset of Z_{σ} defined by the ideal $(X^w / w \in (\sigma^{\vee} - \sigma^{\perp}) \cap M)$ of $\mathbb{C}[\sigma^{\vee} \cap M]$. This orbit is a torus for $0 \leq \dim \sigma < \operatorname{rk} N$, since the associated coordinate ring is the $\mathbb{C}$-algebra of the sub-lattice $M(\sigma) := M \cap \sigma^{\perp}$ of M of codimension equal to $\dim \sigma$. On the closed orbit $\mathbf{O}_{\sigma}$ we consider the *special point* o_{σ} defined by $X^u(o_{\sigma}) = 1$ for all $u \in M(\sigma)$. If $\dim \sigma = \operatorname{rk} N$ the orbit $\mathbf{O}_{\sigma}$ is reduced to the special point. If $\dim \sigma < \operatorname{rk} N$ we have an exact sequence of lattices:

$$0 \to M(\sigma) \to M \xrightarrow{j} M_{\sigma} \to 0.$$

If $0 \to N_\sigma \xrightarrow{j^*} N \to N(\sigma) \to 0$ is the dual exact sequence the lattice N_σ spanned by $\sigma \cap N$ is of dimension equal to $\dim \sigma$ and the semigroup $\sigma^\vee_{N_\sigma}$ associated to the cone σ with respect to the lattice N_σ is isomorphic to $j(\sigma^\vee \cap M)$. If we choose a splitting $M \cong M(\sigma) \oplus M_\sigma$ we obtain a semigroup isomorphism $\sigma^\vee \cap M \cong M(\sigma) \oplus (\sigma^\vee_{N_\sigma} \cap M_\sigma)$ inducing an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}[M(\sigma)] \otimes_{\mathbb{C}} \mathbb{C}[\sigma^\vee_{N_\sigma} \cap M_\sigma]$ which defines (non canonically) the product structure

$$Z_{\sigma,N} \cong \mathbf{O}_{\sigma,N} \times Z_{\sigma,N_\sigma}. \tag{3.1}$$

The map that sends a cone σ in Σ to the orbit $\mathbf{O}_\sigma \subset Z_\Sigma$ is a bijection between the fan Σ and the set of orbits. If σ is a face of τ then Z_σ is an open subset of Z_τ and the orbit $\mathbf{O}_\tau$ is contained in the closure of $\mathbf{O}_\sigma$ in Z_τ since $\tau^\perp \subset \sigma^\perp$, thus the closure of the orbit of σ in Z_Σ is $\overline{\mathbf{O}_\sigma} = \bigcup \mathbf{O}_\tau$ where τ runs through the cones of Σ which have σ as a face.

The orbit closures are normal toric varieties by themselves with respect to the lattice $N(\sigma)$. The cones of the fan associated to $\overline{\mathbf{O}_\sigma}$ are of the form $\tau + (N_\sigma)_\mathbb{R} \subset N_\mathbb{R}/(N_\sigma)_\mathbb{R}$ for $\tau \in \Sigma$ containing σ as a face.

Remark 3.1. *The singular locus of* Z_Σ *is the union of those orbits* $\mathbf{O}_\sigma$ *for* σ *a non regular cone.*

This follows from formula (3.1) by noticing that the orbit $\mathbf{O}_\sigma$ is contained in the singular locus of Z_σ if and only if o_{σ,N_σ} is a singular point of Z_{σ,N_σ} if and only the cone σ is not a regular cone.

Definition 3.1. *A fan* Σ' *is a subdivision of the fan* Σ *if both fans have the same support and if any cone of* Σ' *is contained in a cone of* Σ. *The fan* Σ' *is regular subdivision if* Σ' *is a regular fan. A regular subdivision* Σ' *is a resolution of the fan* Σ *if any regular cone of* Σ *belongs to* Σ'.

Associated to a subdivision of fans there is a *modification* $\pi_\Sigma : Z_{\Sigma'} \to Z_\Sigma$ inducing an isomorphism between their tori.

Exemple 3.1. *Let* Σ *be a regular subdivision of the cone* $\sigma := \mathbb{R}^{d+1}_{\geq 0}$ *with lattice* $N := \mathbb{Z}^{d+1}$. *This subdivision defines a modification* $\pi_\Sigma : Z_\Sigma \to Z_\sigma = \mathbb{C}^{d+1}$ *which we describe in detail:*

The variety Z_Σ is non singular, for each cone σ of maximal dimension the variety Z_σ is isomorphic to $\mathbb{C}^{d+1}$ and the restriction $\pi_\sigma : Z_\sigma \to \mathbb{C}^{d+1}$ of the morphism π_Σ is induced by the semigroup inclusion $\mathbb{R}^{d+1}_{\geq 0} \cap M \to \sigma^\vee \cap M$. The set of primitive vectors in the 1-skeleton σ is a basis of N and its dual basis is a minimal set of generators of the semigroup $\sigma^\vee \cap M$. These generators give us coordinates to describe the map $\pi_\sigma : Z_\sigma \to \mathbb{C}^{d+1}$ in the form:

$$\begin{array}{rcl} X_1 & = & U_1^{a_1^1} U_2^{a_1^2} \cdots U_{d+1}^{a_1^{d+1}} \\ X_2 & = & U_1^{a_2^1} U_2^{a_2^2} \cdots U_{d+1}^{a_2^{d+1}} \\ & \cdots & \\ X_{d+1} & = & U_1^{a_{d+1}^1} U_2^{a_{d+1}^2} \cdots U_{d+1}^{a_{d+1}^{d+1}} \end{array} \tag{3.2}$$

where $(a_1^i, a_2^i, \ldots, a_{d+1}^i)$ is the coordinate of the primitive vector a^i in the 1-skeleton of σ, for $i = 1, \ldots, d+1$. Since the fan Σ is regular, it is easy to see directly from formula (3.2) that the map π_Σ is an isomorphism over the torus $X_1 \cdots X_{d+1} \neq 0$ of $\mathbb{C}^{d+1}$.

A *resolution of singularities* of a variety Z is a smooth variety Z' with a modification $Z' \to Z$ which is an isomorphism outside the singular locus of Z. The resolution of singularities of normal toric varieties is reduced to a combinatorial property of faces (see [KKMS]). More precisely we have that: Given any fan Σ there is a resolution Σ' of Σ (see definition 3.1). The associated toric morphism $Z_{\Sigma'} \to Z_\Sigma$ is a resolution of singularities of the variety Z_Σ (see [Co], Theorem 5.1).

We describe now the exceptional locus associated to a subdivision Σ' of a fan Σ. Taking away the cone σ from the fan of the cone σ means geometrically to take away the orbit $\mathbf{O}_\sigma$ from the variety Z_σ. We deduce that:

$$\pi^{-1}(\mathbf{O}_\sigma) = \bigcup_{\tau \in \Sigma', \overset{\circ}{\tau} \subset \overset{\circ}{\sigma}} \mathbf{O}_\tau \tag{3.3}$$

It follows from (3.3) that the *exceptional fibers*, i.e., the union of subvarieties of dimension > 1 which are mapped to points, are given by:

$$\bigcup_{\dim \sigma = \mathrm{rk} N, \sigma \notin \Sigma'} \pi^{-1}(\mathbf{O}_\sigma)$$

and that the *exceptional locus*, i.e., the subvarieties that are mapped on a variety of smaller dimension, is:

$$\bigcup_{\sigma \notin \Sigma'} \pi^{-1}(\mathbf{O}_\sigma) = \bigcup_{\tau \text{ minimal } \in \Sigma' - \Sigma} \overline{\mathbf{O}_\tau}.$$

The *discriminant locus*, i.e., the image of the exceptional locus, is equal to

$$\bigcup_{\sigma \text{ minimal } \in \Sigma - \Sigma'} \overline{\mathbf{O}_\sigma}. \tag{3.4}$$

3.1.2 Newton polyhedra and partial resolution of singularities

The *Newton polyhedron* $\mathcal{N}(\phi)$ of a non zero series $\phi = \sum c_a X^a \in \mathbb{C}\{X\}$ with $X = (X_1, \ldots, X_{d+1})$ is the convex hull of the set $\bigcup_{c_a \neq 0} a + \mathbb{R}_{\geq 0}^{d+1}$. More generally the Newton polyhedron of any non-zero germ $\phi = \sum c_a X^a$ of holomorphic function at the special point o_ρ of a normal affine toric variety $Z_\rho = \mathrm{Spec}\mathbb{C}[\rho^\vee \cap M]$ (for a strictly convex cone $\rho^\vee$) is the convex hull of the subset $\bigcup_{c_a \neq 0} a + \rho^\vee$ of $M_\mathbb{R}$. We denote it by $\mathcal{N}_\rho(\phi)$ or by $\mathcal{N}(\phi)$ if the cone ρ is clearly determined by the context. Many of the properties associated with classical Newton polyhedra hold in this case; for instance, if $0 \neq \phi = \phi_1 \cdots \phi_s$ we have that $\mathcal{N}(\phi)$ is the Minkowski sum $\mathcal{N}(\phi_1) + \cdots + \mathcal{N}(\phi_1)$ since the series ϕ_i have coefficients in a domain. It follows from this property that:

Remark 3.2. *If* $0 \neq \phi = \phi_1 \cdots \phi_s$ *and* $\mathcal{N}(\phi)$ *has only one vertex the same holds for each of the Minkowski terms* $\mathcal{N}(\phi_i)$ *for* $i = 1, \dots, s$ *.*

The *face* $\mathcal{F}_u$ *of the polyhedron* $\mathcal{N}_\rho(\phi)$ *defined by a vector* in $u \in \rho$ is the set of vectors $v \in \mathcal{N}_\rho(\phi)$ such that $\langle u, v \rangle = \inf_{v' \in \mathcal{N}_\rho(\phi)} \langle u, v' \rangle$. All faces of the polyhedron $\mathcal{N}_\rho(\phi)$ can be recovered in this way. The face of $\mathcal{N}_\rho(\phi)$ defined by u is compact if and only if $u \in \overset{\circ}{\rho}$.

The *cone* $\sigma(\mathcal{F}) \subset \rho$ *associated to the face* $\mathcal{F}$ of the polyhedron $\mathcal{N}_\rho(\phi)$ is

$$\sigma(\mathcal{F}) := \{ u \in \rho \;/\; \forall v \in \mathcal{F}, \text{ we have } \langle u, v \rangle = \inf_{v' \in \mathcal{N}_\rho(\phi)} \langle u, v' \rangle \}.$$

The cones $\sigma(\mathcal{F})$, for $\mathcal{F}$ running through the set of faces of the polyhedron $\mathcal{N}_\rho(\phi)$, define a subdivision $\Sigma(\mathcal{N}_\rho(\phi))$ of the fan of the cone ρ called the *dual Newton diagram.* The relative interiors of the cones in the fan $\Sigma(\mathcal{N}_\rho(\phi))$ are equal to the equivalence classes of vectors in ρ by the relation: $u \sim u' \Leftrightarrow \mathcal{F}_u = \mathcal{F}_{u'}$. We say that a fan Σ supported on the cone ρ is *compatible* with a set of series $\phi_1, \dots, \phi_s \in \mathbb{C}\{\rho^\vee \cap M\}$ if it subdivides the fan $\Sigma(\mathcal{N}_\rho(\phi))$ with $\phi = \phi_1 \cdots \phi_s$. A cone in the fan $\Sigma(\mathcal{N}_\rho(\phi_1 \cdots \phi_s))$ is intersection of cones of the fans $\Sigma(\mathcal{N}_\rho(\phi_i))$ therefore Σ is compatible with all the polyhedra $\mathcal{N}_\rho(\phi_i)$. If Σ is compatible with $\mathcal{N}_\rho(\phi)$ all vectors in $\overset{\circ}{\sigma}$ define the same face $\mathcal{F}_\sigma$ of $\mathcal{N}_\rho(\phi)$, for $\sigma \in \Sigma$.

Definition 3.2. *Let* $0 \neq \phi = \sum c_a X^a \in \mathbb{C}\{\rho^\vee \cap M\}$ *. The symbolic restriction* $\phi_{|\mathcal{F}}$ *of* ϕ *to the compact face* $\mathcal{F}$ *of the polyhedron* $\mathcal{N}_\rho(\phi)$ *is the polynomial* $\phi_{|\mathcal{F}} := \sum_{a \in \mathcal{F}} c_a X^a \in \mathbb{C}[\rho^\vee \cap M]$ *. The Newton principal part* $\phi_{|\mathcal{N}}$ *of* ϕ *is the sum of those terms of* ϕ *having exponents lying on the compact faces of the Newton polyhedron* $\mathcal{N}_\rho(\phi)$

We follow here the terminology of [Kou] and [Ok]. The Newton principal part $\phi_{|\mathcal{N}} \in \mathbb{C}[\rho^\vee \cap M]$ does not change if we change the ring $\mathbb{C}[[\rho^\vee \cap M]]$ by extending the lattice M .

Let Σ be any fan supported on ρ defining the modification $\pi_\Sigma : Z_\Sigma \to Z_\rho$. Let $\mathcal{V}$ be a subvariety of Z_ρ such that the intersection of the discriminant locus of π_Σ with each irreducible component $\mathcal{V}_i$ of $\mathcal{V}$ is nowhere dense on $\mathcal{V}_i$. For instance if $\mathcal{V}$ is irreducible this condition holds if the torus is an open dense subset of $\mathcal{V}$. The *strict transform* $\mathcal{V}_\Sigma \subset Z_\Sigma$ is the subvariety of $\pi_\Sigma^{-1}(\mathcal{V})$ such that the restriction $\mathcal{V}_\Sigma \to \mathcal{V}$ is a modification.

If the fan Σ is regular, the toric map $\pi_\Sigma : Z_\Sigma \to Z_\rho$ is a (toric) *embedded pseudo-resolution* of $\mathcal{V}$ if the restriction $\mathcal{V}_\Sigma \to \mathcal{V}$ is a modification such that the strict transform $\mathcal{V}_\Sigma$ is non singular and transversal to the orbit stratification of the exceptional locus of Z_Σ . The modification π_Σ is a (toric) *embedded resolution* of $\mathcal{V}$ if the restriction to the strict transform $\mathcal{V}_\Sigma \to \mathcal{V}$ is an isomorphism outside the singular locus of $\mathcal{V}$ (see [G-T]). If π_Σ is only a pseudo-resolution we can only guarantee that the map $\mathcal{V}_\Sigma \to \mathcal{V}$ is an isomorphism outside the intersection of $\mathcal{V}$ with the discriminant locus of π_Σ . In this case, this set contains the singular locus of $\mathcal{V}$ but it is not necessarily equal to it.

Definition 3.3. *If* Σ *is a (non necessarily regular) subdivision of* ρ *the toric morphism* $\pi_\Sigma : Z_\Sigma \to Z_\rho$ *is a partial (toric) embedded resolution of* $\mathcal{V}$ *if for any resolution* Σ' *of the fan* Σ *the map* $\pi_{\Sigma'} \circ \pi_\Sigma$ *is an embedded resolution of* $\mathcal{V}$*.*

Let $\mathcal{V} \subset Z_\rho$ an irreducible subvariety such that the intersection with the torus is an open dense subset. Let Σ be a subdivision ρ compatible with a set of generators $\phi_1, \dots, \phi_s$ of the ideal of $\mathcal{V} \subset Z_\rho$. We give a combinatorial condition on the Newton polyhedra of $\phi_1, \dots, \phi_s$ for the intersection of the strict transform with the exceptional fiber being non empty.

Lemma 3.3. *Let* σ *a cone in* Σ *such that* $\mathring{\sigma} \subset \mathring{\rho}$*. If* $\mathbf{O}_\sigma \cap \mathcal{V}_\Sigma \neq \emptyset$ *then the face* $\mathcal{F}_i$ *of the Newton polyhedron* $\mathcal{N}(\phi_i)$ *of* ϕ_i *defined by* σ *is of dimension* ≥ 1 *for* $1 \leq i \leq s$*.*

Proof. We have that $\phi_i - \phi_{i|\mathcal{F}_i}$ belongs to the ideal generated by $\{X^u / u \in (\mathcal{N}(\phi_i) - \mathcal{F}_i) \cap M\}$. Since Σ is compatible with the ϕ_i the cone $\sigma^\vee$ contains the cone spanned by elements in the polyhedron $-u_0 + \mathcal{N}(\phi_i)$ for any $u_0 \in \mathcal{F}_i$. Let $u_i \in \mathcal{F}_i$ be a vertex then we can factor in the ring $\mathbb{C}[\sigma^\vee \cap M]$:

$$\phi_i \circ \pi_\sigma = X^{u_i} \psi_i \text{ and } \phi_{i|\mathcal{F}_i} \circ \pi_\sigma = X^{u_i} \psi_{i|\mathcal{F}_i} \text{ with } \psi_{i|\mathcal{F}_i} \in \mathbb{C}[\sigma^\perp \cap M]$$

in such a way that the exponent of a term appearing in $X^{-u_i}(\phi_i \circ \pi_\sigma - \phi_{i|\mathcal{F}_i} \circ \pi_\sigma)$ belongs to $(\sigma^\vee - \sigma^\perp) \cap M$ and thus this term vanishes on the orbit $\mathbf{O}_\sigma$. By definition the elements $X^{-u_i} \phi \circ \pi_\sigma$ for $1 \leq i \leq s$ belong to the ideal defining the strict transform of $\mathcal{V}$. If the face $\mathcal{F}_i$ is a vertex for some i the ideal of $\mathbf{O}_\sigma \cap \mathcal{V}_\Sigma$ in Z_σ is equal to (1) thus $\mathcal{V}_\Sigma \cap \mathbf{O}_\sigma$ is empty. ◇

The following lemma is an easy consequence of the implicit function theorem.

Let $\rho \subset N_\mathbb{R}$ be a rational strictly convex cone of dimension equal to rk N. We denote by Δ the cone $\rho \oplus \mathbb{R}^g_{\geq 0} \subset (N_\Delta)_\mathbb{R}$ where N_Δ is the lattice $N \oplus \mathbb{Z}^g$ with dual lattice M_Δ. The semigroup $\Delta^\vee \cap M_\Delta$ is of the form $(\rho^\vee \cap M) \oplus \mathbb{Z}^g_{\geq 0}$. The monomial corresponding to $(\alpha, v) \in \Delta^\vee \cap M_\Delta$ is denoted by $X^\alpha U^v$ or by $X^\alpha U_1^{v_1} \dots U_g^{v_g}$.

Lemma 3.4. *If* $\phi_1, \dots, \phi_g \in \mathbb{C}\{\Delta^\vee \cap M_\Delta\}$ *verify that* $\phi_1(o_\sigma, U) = U_i$*, for* $i = 1, \dots, g$ *then there exist series* $\epsilon_i \in \mathbb{C}\{\rho^\vee \cap M\}$ *for* $i = 1, \dots, g$ *such that the ideals of* $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$ *generated by* $\phi_1, \dots, \phi_g$ *and* $U_1 - \epsilon_1, \dots, U_g - \epsilon_g$ *coincide.*

Proof. An homomorphism of semigroups $\mathbb{Z}^s_{\geq 0} \xrightarrow{\psi} \rho^\vee \cap M$ extends to an homomorphism $\mathbb{Z}^{s+k}_{\geq 0} \xrightarrow{\psi \times \mathrm{Id}} \Delta^\vee \cap M_\Delta$. If ψ is surjective it defines an equivariant embedding $Z_\rho \subset \mathbb{C}^s$ which extends (by using the homomorphism $\psi \times \mathrm{Id}$) to an equivariant embedding $Z_\Delta = Z_\rho \times \mathbb{C}^g \subset \mathbb{C}^{s+g}$. If $\varphi_1, \dots, \varphi_g$ are power series defining holomorphic functions at $(\mathbb{C}^{s+g}, 0)$ representing $\phi_1, \dots, \phi_g$ the implicit function theorem guarantees the existence of power series ε_i in s variables such that the ideals $(\varphi_1, \dots, \varphi_g)$ and $(U_1 - \varepsilon_1, \dots, U_g - \varepsilon_g)$ coincide. The result follows by passing to the quotient by the binomial ideal defining the embedding $Z_\Delta \subset \mathbb{C}^{s+g}$. ◇

3.1.3 Embedded resolution of non necessarily normal toric varieties

We build an embedded resolution of non necessarily normal affine toric variety Z^Λ equivariantly embedded in a normal affine toric variety Z_ρ (for $\rho^\vee$ a strictly convex cone). We build first a *partial embedded resolution* which is a toric morphism providing an *embedded normalization* inside a normal toric ambient space. Then any toric resolution of the singularities of the ambient space, which always exists, provides an embedded resolution. The advantage is method is that the partial resolution is completely determined by the embedding $Z^\Gamma \subset Z_\rho$. This result is the fruit of discussions with Professor B. Teissier (see [T5], §6, Proposition 6.4 and [GP-T]).

Let Λ be a monoid. An *equivariant embedding* of Z^Λ in the normal affine toric variety Z_ρ is given by a surjective homomorphism of semigroups $\rho^\vee \cap M \rightarrow \Lambda$ which extends to a lattice homomorphism $\varphi : M \rightarrow -\Lambda + \Lambda$ and a vector space homomorphism $\varphi_{\mathbb{R}} : M_{\mathbb{R}} \rightarrow (-\Lambda + \Lambda)_{\mathbb{R}}$. The torus of Z^Λ is equivariantly embedded in the torus of Z_ρ, the embedding is obtained from the homomorphism φ. The linear subspace $(\mathrm{Ker}(\varphi_{\mathbb{R}}))^\perp \subset N_{\mathbb{R}}$, denoted by ℓ in what follows, is of dimension equal to $\mathrm{rk}\Lambda$ and the same holds for the cone $\sigma_0 := \ell \cap \rho$. The ideal of the embedding $Z^\Lambda \subset Z_\rho$ is generated by the binomials:

$$X^u - X^v \in \mathbb{C}[\rho^\vee \cap M] \text{ such that } \varphi(u) = \varphi(v) \quad \text{(see [St], Chapter 4).} \tag{3.5}$$

Lemma 3.5. *With the above notations suppose that the cone $\rho^\vee$ is strictly convex. Let Σ be any fan compatible with a finite set of binomial equations $X^{u_i} - X^{v_i} = 0$ for $i \in I$ defining the embedding $Z^\Lambda \subset Z_\rho$. Then the fan Σ is compatible with the linear subspace ℓ. If $\sigma \in \Sigma$ and $\overset{\circ}{\sigma} \subset \overset{\circ}{\rho}$ then $\mathbf{O}_\sigma \cap Z^\Lambda_\Sigma \neq \emptyset$ implies that $\sigma \subset \ell$. Moreover, if $\sigma \subset \ell$ and $\dim \sigma = \dim \ell$ the intersection $Z^\Lambda_\Sigma \cap \mathbf{O}_\sigma$ as schemes is the simple point o_σ and $Z^\Lambda_\Sigma \cap Z_\sigma$ is isomorphic to Z_{σ, N_σ}. If Σ is regular the morphism π_Σ is an embedded pseudo-resolution of singularities of $Z^\Lambda \subset Z_\rho$.*

Proof. The cone $\sigma_0 = \rho \cap \ell$ is associated to the Minkowski sum of compact edges of $\mathcal{N}(X^{u_i} - X^{v_i})$ for $i \in I$ since $\langle w, u_i \rangle = \langle w, v_i \rangle, \forall i \in I$ if and only if $w \in \ell$. Since the fan Σ is compatible with the binomial equations of Z^Λ it follows that a subdivision of σ_0 is contained in Σ, i.e., this fan is compatible with the linear subspace ℓ.

We deduce by duality from the equality $\sigma_0 = \rho \cap \ell$ that:

$$\sigma_0^\vee = \rho^\vee + \ell^\vee = \rho^\vee + \ell^\perp = \rho^\vee + \mathrm{Ker}(\varphi_{\mathbb{R}}) \tag{3.6}$$

Since the cone $\rho^\vee$ is strictly convex formula (3.6) implies that

$$\sigma_0^\perp = \mathrm{Ker}(\varphi_{\mathbb{R}}) \tag{3.7}$$

and thus

$$\mathrm{Ker}(\varphi) \subset \sigma_0^\vee \cap M \tag{3.8}$$

Let $\sigma \in \Sigma$ with $\overset{\circ}{\sigma} \subset \overset{\circ}{\rho}$, since Σ is compatible with the binomials $X^{u_i} - X^{v_i}$, the ideal generated by $1 - X^{u_i - v_i}$ (up to relabeling) is contained in the ideal defining the strict transform Z^{Λ}_{Σ} in the chart Z_σ. Thus the variety Z', defined by $X^{u_i - v_i} - 1 = 0$ for $i \in I$, contains $Z^{\Lambda}_{\Sigma} \cap Z_\sigma$. Then we have:

$$Z' \cap \mathbf{O}_\sigma \neq \emptyset \Leftrightarrow \ \exists p \in Z_\sigma : X^{u_i - v_i}(p) = 1 \, \forall i \in I, X^{u}(p) = 0 \, \forall u \in (\sigma^\vee - \sigma^\perp) \cap M \Leftrightarrow$$
$$u_i - v_i \in \sigma^\perp \cap M, \forall i \in I \Leftrightarrow \ \mathrm{Ker}\,(\varphi) \subset \sigma^\vee \Leftrightarrow \sigma \subset \rho \cap \ell$$

The chart Z_σ is isomorphic to $\mathbf{O}_\sigma \times Z_{\sigma, N_\sigma}$ by formula (3.1).

If $\sigma \subset \ell$ and $\dim \sigma = \dim \ell$ we have that $\sigma^\perp = \sigma_0^\perp$ coincides with $\mathrm{Ker}(\varphi_{\mathbb{R}})$ by (3.7). We deduce an isomorphism

$$Z' \cong \{o_\sigma\} \times Z_{\sigma, N_\sigma} \subset Z_\sigma. \tag{3.9}$$

from (3.1) since the lattice $\sigma^\perp \cap M = \mathrm{Ker}(\varphi)$ is generated by $\{u_i - v_i\}_{i \in I}$. Therefore the variety Z' is irreducible and of dimension equal to $\mathrm{rk}\,\Lambda$. We deduce from (3.9) that Z^{Λ}_{Σ} intersects the orbit $\mathbf{O}_\sigma$ transversally since the coordinate ring of $Z^{\Lambda}_{\Sigma} \cap \mathbf{O}_\sigma$ is $\mathbb{C}$. Since $Z^{\Lambda}_{\Sigma} \cap Z_\sigma$ is a subvariety of the irreducible variety Z' and both are of the same dimension they coincide.

If Σ is regular we deduce that Z^{Λ}_{Σ} is smooth and intersects transversally the orbit stratification of the exceptional locus of Z_Σ thus π_Σ is an embedded pseudo-resolution of Z^Λ. ◇

With the notations of lemma 3.5 we have:

Proposition 3.6. *Suppose that the cone $\rho^\vee$ is strictly convex. Let Σ be a subdivision of ρ containing the cone σ_0.*

1. *The strict transform Z^{Λ}_{Σ} of Z^Λ by the morphism π_Σ is isomorphic to $Z_{\sigma_0, N_{\sigma_0}}$ and the restriction $\pi_\Sigma | Z^{\Lambda}_{\Sigma} : Z^{\Lambda}_{\Sigma} \to Z^\Lambda$ is the normalization map.*
2. *The morphism π_Σ is a partial embedded resolution of $Z^\Lambda \subset Z_\rho$.*

Proof. We keep notations of lemma 3.5. If we choose a splitting $M \cong \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$ we obtain using (3.8) a semigroup isomorphism

$$\sigma_0^\vee \cap M \cong \mathrm{Ker}(\varphi) \oplus \varphi(\sigma_0^\vee \cap M),$$

which corresponds geometrically to the isomorphism $Z_{\sigma_0} \cong \mathbf{O}_{\sigma_0} \times Z_{\sigma_0, N_{\sigma_0}}$ of (3.1).

We deduce from (3.6) that $\sigma_0^\vee = \varphi_{\mathbb{R}}^{-1}(\varphi_{\mathbb{R}}(\rho^\vee))$ and it follows that the semigroup

$$\varphi(\sigma_0^\vee \cap M) = \varphi_{\mathbb{R}}(\rho^\vee) \cap \varphi(M) \tag{3.10}$$

is the saturated semigroup $\mathbb{R}_{\geq 0}\Lambda \cap (-\Lambda + \Lambda)$ of Λ in the lattice it spans; therefore the variety $Z_{\sigma_0, N_{\sigma_0}}$ is isomorphic to the normalization of Z^Λ (see [KKMS]).

Let Σ' be a subdivision of Σ compatible with the equations of Z^{Λ}. By lemma 3.5 if $\sigma \in \Sigma'$, $\overset{\circ}{\sigma} \subset \overset{\circ}{\rho}$ and $\mathbf{O}_\sigma \cap Z^{\Lambda}_{\Sigma'} \neq \emptyset$ then we have $\sigma \subset \ell$. A fortiori the same property holds replacing Σ' by Σ as a consequence of (3.3). It follows that the strict transform of the germ (Z^{Λ}, o_ρ) is contained in the chart corresponding to the cone σ_0. This implies that $Z^{\Lambda}_{\Sigma} \subset Z_{\sigma_0}$ since the morphism π_Σ is equivariant and Z^{Λ} is equivariantly embedded. It follows also from the proof of lemma 3.5 that the restriction of π_Σ to $Z^{\Lambda}_{\Sigma} \to Z^{\Lambda}$ corresponds algebraically to the inclusion of $\mathbb{C}[\Lambda]$ in its integral closure thus it is the normalization map.

A resolution Σ' of the fan Σ is subdivided by a regular fan Σ'' which is compatible with the equations of Z^{Λ}. By lemma 3.5 the map $\pi_{\Sigma''} \circ \pi_{\Sigma'} \circ \pi_{\Sigma}$ is a pseudo-resolution of Z^{Λ}. A fortiori the same holds for $p_{\Sigma'} \circ \pi_\Sigma$ by (3.3). By definition if $\sigma' \in \Sigma$ is a regular cone then $\sigma' \in \Sigma'$, thus $Z_{\Sigma'} \to Z_{\Sigma}$ is an isomorphism over the points of the $\mathbf{O}_{\sigma'}$. By remark 3.1 the singular locus of Z^{Λ}_{Σ} is defined by the intersection of those orbits $\mathbf{O}_{\sigma'}$ for those cones σ' running through the set of non regular faces of σ_0. This shows that $Z^{\Lambda}_{\Sigma'} \to Z^{\Lambda}_{\Sigma}$ is a resolution of singularities of the normalization Z^{Λ}_{Σ} of Z^{Λ}. A fortiori the map $Z^{\Lambda}_{\Sigma} \to Z^{\Lambda}$ is a resolution of singularities. ◇

3.1.4 Equivariant branched coverings of normal toric varieties

Some branched coverings of normal toric varieties are equivariant. Typically, if σ is a rational cone for N it is also rational for $N' \subset N$ a sub-lattice of the same rank and we have a homomorphism of semigroups $\sigma^\vee \cap M \to \sigma^\vee \cap M'$ where $M \subset M'$ is the inclusion of lattices corresponding to $N' \subset N$ by duality. This homomorphism defines an equivariant morphism

$$Z_{\sigma,N'} \longrightarrow Z_{\sigma,N} \tag{3.11}$$

extending the homomorphism of tori $T' \to T$ defined by the lattice extension $M \subset M'$, which has kernel a finite subgroup H of T'. Each $w \in H$ corresponds to a morphism $Z_{\sigma,N'} \to Z_{\sigma,N'}$ given by the homomorphism $\mathbb{C}[\sigma^\vee \cap M'] \to \mathbb{C}[\sigma^\vee \cap M']$ mapping $X^u \mapsto w(u)X^u$. The ring $\mathbb{C}[\sigma^\vee \cap M]$ is the set of invariants of $\mathbb{C}[\sigma^\vee \cap M']$ by the action of the group H and the morphism (3.11) coincides with canonical projection of the quotient of Z'_σ with respect to the action of the group H by Corollary 1.16 of [Od]. If σ is of maximal dimension the 0-orbit o'_σ of $Z_{\sigma,N'}$ projects to the 0-orbit o_σ of $Z_{\sigma,N}$ and we have that $(Z_{\sigma,N'}, o'_\sigma) \to (Z_{\sigma,N}, o_\sigma)$ is a morphism of analytically irreducible germs. The corresponding homomorphism of analytic algebras $\mathbb{C}\{\sigma^\vee \cap M\} \to \mathbb{C}\{\sigma^\vee \cap M'\}$ extends to a homomorphism $L \to L'$ of their fields of fractions of degree equal to the cardinality of H, i.e., the index of M as a subgroup of M'. This field extension is Galois and the Galois group is obtained from the automorphisms of $\mathbb{C}\{\sigma^\vee \cap M'\}$ defined by the elements of H (see [GP1]).

Let $\nu_1, \dots, \nu_g \in M'$ and define from them a sequence of lattices and integers:

$$\left\{ \begin{array}{c} M_0 := M, M_i := M_{i-1} + \nu_i \mathbb{Z}, \text{ for } i = 1, \dots, g \\ n_0 := 1, n_i = \# M_i / M_{i-1} \text{ for } i = 1, \dots, g \end{array} \right. \tag{3.12}$$

The lattices M_i are all sub-lattices of finite index of M'. We have the inclusions of lattices $N' \subset N_g \subset \cdots \subset N_1 \subset N_0 = N$ where N_i denotes the dual lattice of M_i.

Lemma 3.7. *The field of fractions of* $\mathbb{C}\{\rho^\vee \cap M_j\}$ *is* $L[X^{\nu_1}, \ldots, X^{\nu_j}]$. *If* $\lambda \in \rho^\vee \cap M'$ *then* $X^\lambda \in \mathrm{Fix}(\mathrm{Gal}(L'/L[X^{\nu_1}, \ldots, X^{\nu_j}]))$ *if and only if* $\lambda \in \rho^\vee \cap M_j$.

Proof. The homomorphism of analytic algebras $\mathbb{C}\{\rho^\vee \cap M\} \to \mathbb{C}\{\rho^\vee \cap M_j\}$ is finite and defines an extension of the corresponding fields of fractions of degree $n_1 \cdots n_j$ equal to the order of the finite group M_j/M. We prove the first assertion by induction on j: for $j = 1$ the roots of the minimal polynomial of X^{ν_1} over L are the different conjugates of X^{ν_1} by the action of the elements of the Galois group of L/L'. We deduce from this that the minimal polynomial of X^{ν_1} is $Y^{n_1} - X^{n_1 \nu_1}$ where $n_1 = \# M_1/M_0$ is also the degree of the extension $L[X^{\nu_1}]/L$. Since $L[X^{\nu_1}]$ is contained in the field of fractions of $\mathbb{C}\{\rho^\vee \cap M_1\}$ and both fields define extensions of L of the same degree they are equal. By induction hypothesis the field of fractions of $\mathbb{C}\{\rho^\vee \cap M_{j-1}\}$ is $L[X^{\nu_1}, \ldots, X^{\nu_{j-1}}]$ and we can replace L, ν_1 and n_1 in the previous argument by $L[X^{\nu_1}, \ldots, X^{\nu_{j-1}}]$, ν_j and n_j respectively to obtain the assertion for j.

If $\nu \in \rho^\vee \cap M_j$ it is clear that ν is fixed by any element of the Galois group of the extension $L'/L[X^{\nu_1}, \ldots, X^{\nu_j}]$. The converse follows by the first assertion and Corollary 1.16 of [Od] applied to the inclusion of semigroups $\rho^\vee \cap M_j \subset \rho^\vee \cap M'$. ◇

3.1.5 A reminder on toroidal embeddings

Let $\mathcal{X}$ be a normal variety of dimension $d+1$, and let E_i be a finite set of normal hypersurfaces with complement $\mathcal{U}$ in $\mathcal{X}$. A *toroidal embedding without self intersection* is defined by requiring the triple $(\mathcal{X}, \mathcal{U}, x)$ at any point $x \in \mathcal{X}$ to be formally isomorphic to $(Z_\sigma, T = (\mathbb{C}^*)^{d+1}, z)$ for z a point in some toric variety Z_σ. This means that there is a formal isomorphism between the completions of the local rings at respective points which sends the ideal of $\mathcal{X} - \mathcal{U}$ into the ideal of $Z_\sigma - T$; (see [KKMS]). The variety $\mathcal{X}$ is naturally stratified, with strata $\bigcap_{i \in K} E_i - \bigcup_{i \notin K} E_i$ and the open stratum $\mathcal{U}$.

The *star of a stratum* $\mathfrak{S}$, $\mathrm{star}\,\mathfrak{S}$, is the union of the strata containing $\mathfrak{S}$ in their closure. We associate to the stratum $\mathfrak{S}$ the set $M^{\mathfrak{S}}$ of Cartier divisors supported on $\mathrm{star}\,\mathfrak{S} - \mathcal{U}$. We denote by $N^{\mathfrak{S}}$ the dual group $\mathrm{Hom}(M^{\mathfrak{S}}, \mathbb{Z})$. The semigroup of effective divisors defines in the real vector space $M^{\mathfrak{S}}_{\mathbb{R}} := M^{\mathfrak{S}} \otimes \mathbb{R}$ a rational convex polyhedral cone and we denote its dual cone in $N^{\mathfrak{S}}_{\mathbb{R}} := N^{\mathfrak{S}} \otimes \mathbb{R}$ by $\rho^{\mathfrak{S}}$. If $\mathfrak{S}'$ is a stratum in $\mathrm{star}\,\mathfrak{S}$, we have a group homomorphism defined by restriction of Cartier divisors $M^{\mathfrak{S}} \to M^{\mathfrak{S}'}$ which is onto; by duality we obtain an inclusion $N^{\mathfrak{S}'} \to N^{\mathfrak{S}}$ and the cone $\rho^{\mathfrak{S}'}$ is mapped onto a face of $\rho^{\mathfrak{S}}$ (see [KKMS]). We can associate in this way to a toroidal embedding without self-intersection a *conic polyhedral complex with integral structure* (*c.p.c.* in what follows) see [KKMS]. This generalizes the way of recovering from a normal toric variety the associated fan.

This complex is *combinatorially isomorphic* to the cone over the dual graph of intersection of the divisors E_i. We have that the strata of the stratification are in one-to-one correspondence with the faces of the conic polyhedral complex. For instance, the conic polyhedral complex associated to the toroidal embedding defined by Z_Σ and the normal hypersurfaces $\{\overline{\mathbf{O}_\sigma}\}_{\sigma \in \Sigma^{(1)}}$ is isomorphic to the conic polyhedral complex (with integral structure) (Σ, N) defined by the fan Σ and the lattice N.

We can define, in an analogous manner to the case of a fan, a regular subdivision of a conic polyhedral complex. Associated to a subdivision we have an induced *toroidal modification* (see [KKMS] Th. 6* and 8*), i.e, a normal variety $\mathcal{X}'$ with a toroidal embedding $\mathcal{U} \subset \mathcal{X}'$ and a modification $\mathcal{X}' \to \mathcal{X}$ provided with a commutative diagram:

$$\begin{tikzcd}
\mathcal{U} \arrow[r] \arrow[d] & \mathcal{X} \\
\mathcal{X}' \arrow[ur] &
\end{tikzcd}$$

The notion of toric partial embedded resolution generalize easily in the toroidal case.

3.2 Toric quasi-ordinary singularities

We introduce toric quasi-ordinary singularities and we extend to this case many notions and properties of quasi-ordinary singularities.

Let (S, o) be a germ of analytically irreducible complex variety of dimension d. We denote by R the associated analytic algebra. A sufficiently small representative $S \to S'$ of a finite map germ $(S, o) \to (S', o')$ has finite fibers, its image is an open neighborhood of o' and the maximal cardinality of the fibers is equal to to the *degree* of the map. The *discriminant locus*, i.e., the set of points of having fibers of cardinality less than the degree, is an analytical subvariety of S'. Outside the discriminant locus, the map is an *unramified* covering. We can think of the discriminant locus as an analytic space or as a germ at o'.

Definition 3.4. *A germ of complex analytic variety (S, o) is a quasi-ordinary singularity if there exist a finite morphism $(S, o) \to (\mathbb{C}^d, 0)$ (called a quasi-ordinary projection) and some analytical coordinates $(X_1, \dots, X_d)$ at 0, such that the morphism is unramified over the torus $X_1 \dots X_d \neq 0$ in a neighborhood of the origin.*

The class of quasi-ordinary singularities contains all curve singularities. The Jung-Abhyankar Theorem guarantees that R can be viewed as a subring of $\mathbb{C}\{X_1^{1/m}, \dots, X_d^{1/m}\}$ for some suitable integer m (see [J] for a topological proof in the surface case and [A1], Th. 3 for an algebraic proof).

The finite map germ $(S, o) \to (S', o')$ corresponds algebraically to a local homomorphism $R' \to R$ of their analytic algebras which gives R the structure of finite module over R'. In particular if R is generated over R' by one element there is a surjection $R'[Y] \to R$ which corresponds geometrically to an embedding $(S,o) \subset (S' \times \mathbb{C}, (o',0))$. We say that (S,o) is an *hypersurface relative to the base* (S',o'). We define toric quasi-ordinary singularities by replacing the base $(\mathbb{C}^d, 0)$ by the germ (Z_ρ, o_ρ) of an affine toric variety at its zero orbit (for a strictly convex cone $\rho^\vee$).

Definition 3.5. *(see [GP1]) The germ (S,o) is a toric quasi-ordinary singularity if there exist a finite morphism $(S,o) \to (Z_\rho, o_\rho)$ unramified over the torus in a neighborhood of the zero-orbit o_ρ of a suitable normal affine toric variety Z_ρ.*

Remark 3.8. *The classical quasi-ordinary singularities are obtained when $(\rho, M) = (\mathbb{R}^d_{\geq 0}, \mathbb{Z}^d)$.*

By definition the analytic algebra R of a toric quasi-ordinary singularity is a $\mathbb{C}\{\rho^\vee \cap M\}$-algebra of finite type. The germ (S,o) is an hypersurface relative to the toric base if there exists $x \in R$ such that $R = \mathbb{C}\{\rho^\vee \cap M\}[x]$. Then the $\mathbb{C}\{\rho^\vee \cap M\}$-algebra homomorphism $\mathbb{C}\{\rho^\vee \cap M\}[Y] \to R$ that maps $Y \mapsto x$ is surjective. Its kernel is a principal ideal generated by a monic polynomial f such that $f(o_\rho, Y) = Y^{\deg f}$ and $\deg f$ is equal to the degree of the map $(S,o) \to (Z_\rho, o_\rho)$. The polynomial f is a *quasi-ordinary polynomial*, i.e., the discriminant $\Delta_Y f$ of f with respect to Y is of the form:

$$\Delta_Y f = X^\eta H \text{ with } H(o_\rho) \neq 0 .$$

Conversely each monic quasi-ordinary polynomial $f \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ such that $f(o_\rho, Y) = Y^{\deg f}$ defines a germ of toric quasi-ordinary hypersurface. The $\mathbb{C}\{\rho^\vee \cap M\}$-algebra homomorphism $\mathbb{C}\{\rho^\vee \cap M\}[Y] \to R$ defines an embedding $S \subset Z_\rho \times \mathbb{C}$ that maps $o \mapsto (o_\rho, 0)$. The quasi-ordinary projection of (S,o) is induced by the first projection of the product $Z_\rho \times \mathbb{C}$.

The product $Z_\rho \times \mathbb{C}$ is the toric variety Z_ϱ defined by the cone $\varrho = \rho \times \mathbb{R}_{\geq 0}$ with respect to the lattice N' dual to the lattice $M' := M \oplus y\mathbb{Z}$. Then we have $\varrho^\vee \cap M' \cong (\rho^\vee \cap M) \oplus y\mathbb{Z}_{\geq 0}$. We denote the monomial corresponding to $u + sy \in (\rho^\vee \cap M) \oplus y\mathbb{Z}_{\geq 0}$ by $X^u Y^s$.

If f is an irreducible quasi-ordinary polynomial the associated analytic algebra R is the domain $R = \mathbb{C}\{\rho^\vee \cap M\}[Y]/(f)$. There exists a fractional power series $\zeta \in \mathbb{C}\{\rho^\vee \cap \frac{1}{n}M\}$ which is a root of f where n is the degree of f (see Théorème 1.1 and Remarque 1 of [GP1]). The inclusion $\mathbb{C}\{\rho^\vee \cap M\} \subset \mathbb{C}\{\rho^\vee \cap \frac{1}{n}M\}$ corresponds to a branched covering of a normal affine toric variety and defines a Galois extension $L \subset L_n$ of their corresponding fields of fractions (see subsection 3.1.4). The minimal polynomial of the root ζ over the field L is equal to f, we have $R \cong \mathbb{C}\{\rho^\vee \cap M\}[\zeta]$ and the field of fractions of R is $L[\zeta]$ since ζ is finite over L. The conjugates $\zeta^{(i)}$ of ζ by the action of the Galois group of $L \subset L_n$ define all the roots of f since the extension $L[\zeta] \subset L_n$ is Galois.

We call *(toric) quasi-ordinary branches* the roots of (toric) quasi-ordinary polynomials.

If f is a reduced quasi-ordinary polynomial of degree n then it splits on $\mathbb{C}\{\rho^\vee \cap \frac{1}{n!}M\}$. The difference $\zeta^{(s)} - \zeta^{(t)}$ of two different roots of f divides the discriminant of f on the ring $\mathbb{C}\{\rho^\vee \cap \frac{1}{n!}M\}$. By remark 3.2, the Newton polyhedron of $\zeta^{(s)} - \zeta^{(t)}$ has only one vertex therefore $\zeta^{(s)} - \zeta^{(t)}$ is of the form $X^{\lambda_{st}} H_{st}$ where H_{st} is a unit in $\mathbb{C}\{\rho^\vee \cap \frac{1}{n!} M\}$. It follows that the irreducible factors of f are quasi ordinary polynomials. The monomials $X^{\lambda_{st}}$ so obtained are called *characteristic monomials* and the exponents $\lambda_{st} \in \rho^\vee \cap \frac{1}{n} M$ are called the *characteristic exponents*. If $\mathrm{rk} M = 1$ and if f is irreducible the characteristic exponents correspond to the classical Puiseux characteristic exponents in arbitrary coordinates. We do not need the classical argument to define the characteristic monomials which uses the factoriality of the ring $\mathbb{C}\{X_1, \dots, X_d\}$ (see [L3]), a property which does not hold for the rings of the form $\mathbb{C}\{\rho^\vee \cap M\}$ in general. The notion of characteristic monomials in the classical quasi-ordinary case is already present in Zariski's work (see [Z4]); in the analytically irreducible hypersurface case many geometrical and topological features of these singularities are determined in terms of the characteristic monomials by Lipman, Luengo, Gau and others (see [L1], [L3], [L4], [Lu] and [Gau]).

We define a partial order $\leq_\rho$ (or $\leq$ for short) on the cone $\rho^\vee$:

$$u \leq_\rho u' \Leftrightarrow u' \in u + \rho^\vee \Leftrightarrow \forall w \in \rho : \langle u' - u, w \rangle \geq 0.$$

We can extend this partial ordering to a total one on the subset $\rho^\vee \cap M$ by taking an *irrational* vector $\eta \in \rho$, i.e., the coordinates of η with respect to any base of the lattice N are linearly independent over $\mathbb{Q}$, and defining then $\leq_\eta$ by $u \leq_\eta u' \Leftrightarrow \langle \eta, u - u' \rangle \leq 0$.

Lemma 3.9. *(see [Z4] and [L4] in the classical case). Let f_1 be an irreducible factor of the reduced toric quasi-ordinary polynomial f. If $f_1(\zeta^{(s_0)}) = 0$ then we have:*

$$\left\{\lambda_{s_0 t} / \zeta^{(s_0)} \neq \zeta^{(t)},\ f(\zeta^{(t)}) = 0 \right\} = \left\{\lambda_{st} / \zeta^{(s)} \neq \zeta^{(t)},\ f(\zeta^{(t)}) = 0 \text{ and } f_1(\zeta^{(s)}) = 0\right\}$$

and this set is totally ordered by $\leq_\rho$.

Proof. The equality above follows since the extension $L[\zeta^{(s_0)}] \subset L_{n!}$ is Galois and the elements of the Galois group act on a series in $\mathbb{C}\{\rho^\vee \cap \frac{1}{n!} M\}$ by changing the coefficients of its terms. Then, if $\zeta^{(t)} \neq \zeta^{(t')}$ are roots of f different to $\zeta^{(s_0)}$ we have that:

$$X^{\lambda_{t't}} H_{t't} = \zeta^{(t')} - \zeta^{(t)} = \zeta^{(t')} - \zeta^{(s_0)} - (\zeta^{(t)} - \zeta^{(s_0)}) = X^{\lambda_{t's_0}} H_{t's_0} - X^{\lambda_{ts_0}} H_{ts_0}.$$

Therefore $\lambda_{t't} = \min_\rho\{\lambda_{t's_0}, \lambda_{ts_0}\}$ and the assertion follows. ◇

Definition 3.6. *(see [GP2]) Two irreducible quasi-ordinary polynomials $f^{(i)}$ and $f^{(j)}$ have order of coincidence $\lambda_{(i,j)}$ if their product $f^{(i)} f^{(j)}$ is a quasi-ordinary polynomial and $\lambda_{(i,j)}$ is the largest exponent of the set $\{\lambda_{st} / f^{(i)}(\zeta^{(s)}) = 0, f^{(j)}(\zeta^{(t)}) = 0\}$.*

We say that the order of coincidence of $f^{(i)}$ with itself is $\lambda_{(i,i)} := +\infty$. We deduce from the proof of lemma 3.9 and definition 3.6 the following property:

Lemma 3.10. *If* $f = f^{(1)} \cdots f^{(r)}$ *is the factorization of a quasi-ordinary polynomial with monic irreducible factors we have that:* $\min\{\lambda_{(i,j)}, \lambda_{(j,l)}\} \geq \lambda_{(i,l)}$ *with equality if* $\lambda_{(i,j)} \neq \lambda_{(j,l)}$ *for* $i, j, l \in \{1, \dots, r\}$. $\diamond$

In particular when f is irreducible it follows that the set of characteristic exponents is totally ordered by $<_\rho$ (see [L3]). In this case we relabel the characteristic exponents by $\lambda_1 <_\rho \lambda_2 <_\rho \cdots <_\rho \lambda_g$ and we denote $\lambda_{g+1} = +\infty$. Following Lipman (see [L4], page 61) we associate to the characteristic exponents sequences of lattices and integers. In the plane branch case the sequence of integers coincide with the first component of the characteristic pairs in arbitrary coordinates.

Definition 3.7. *The lattices* M_i *and the integers* n_i *associated to the sequence of characteristic exponents* $\lambda_1, \dots, \lambda_g$ *for* $i = 0, \dots, g$ *by formulae (3.12) are called characteristic.*

We denote by $e_{i-1} = n_i \cdots n_g$, for $i = 1, \dots, g$ and we set $n_0 := 1$. We denote by $N_g \subset \cdots \subset N_1 \subset N_0 = N$ the sequence of dual lattices of $M = M_0 \subset \cdots \subset M_g$.

If f is reduced the set of characteristic exponents is not totally ordered by $\leq_\rho$, for example the characteristic exponents $(1,0), (\frac{3}{2}, 0), (1, \frac{3}{2})$ of $f = ((Y - X_1)^2 - X_1^3))((Y + X_1)^2 - X_1^2 X_2^3))$ are not totally ordered for $\leq_{\mathbb{R}^2_{\geq 0}}$.

Lemma 3.11. *(see [L3]) If* f *is an irreducible toric quasi-ordinary polynomial and if* ζ *is a root of* f *we have:*

1. *The characteristic integers* n_i *verify that* $n_i > 1$ *for* $i = 1, \dots, g$ *and* $n_1 \cdots n_g = \deg f$.
2. *The field of fractions of* R *is equal to* $L[\zeta] = L[X^{\lambda_1}, \dots, X^{\lambda_g}]$.

Proof. Let ζ' be a conjugate of ζ by an element of the Galois group of the field extension $L_n \supset L[X^{\lambda_1}, \dots, X^{\lambda_j}]$. If $\zeta' \neq \zeta$ we have $\zeta' - \zeta = X^{\lambda_k} H_k$ for a unit H_k and $k > j$ (since $X^{\lambda_1}, \dots, X^{\lambda_j}$ are fixed for this Galois group). In particular for $j = g$ the only possibility is $\zeta' = \zeta$ thus $\zeta \in L[X^{\lambda_1}, \dots, X^{\lambda_g}]$ since the extension $L_n \supset L[X^{\lambda_1}, \dots, X^{\lambda_j}]$ is Galois. Conversely any element of the Galois group of the extension $L_n \supset L[\zeta]$ fix ζ and therefore all the terms appearing in ζ, in particular $X^{\lambda_1}, \dots, X^{\lambda_g}$, belong to $L[\zeta]$ since the extension $L_n \supset L[\zeta]$ is Galois. It follows that $n_i > 1$ for $i = 1, \dots, g$, and that the degree n of the extension $L[\zeta] \supset L$ is equal to $n_1 \cdots n_g$. $\diamond$

We have the following conditions for a power series $\zeta \in \mathbb{C}\{\rho^\vee \cap \frac{1}{n} M\}$ to be a quasi-ordinary branch (see [L3], prop. 1.5 or [Gau], prop 1.3 in the classical case).

Lemma 3.12. *Let* $\zeta = \sum c_\lambda X^\lambda$ *be a non unit in* $\mathbb{C}\{\rho^\vee \cap \frac{1}{m} M\}$. *Then the minimal polynomial of* ζ *over* $\mathbb{C}\{\rho^\vee \cap M\}$ *is quasi-ordinary if and only if there exists elements* $\lambda_i \in \rho^\vee \cap \frac{1}{m} M$ *for* $1 \leq i \leq g$ *such that*

1. $\lambda_1 <_\rho \lambda_2 <_\rho \cdots <_\rho \lambda_g$, *and* $c_{\lambda_i} \neq 0$ *for* $1 \leq i \leq g$.

2. *If* $c_\lambda \neq 0$ *then* λ *is the sub-lattice* $M + \sum_{\lambda_i \leq_\rho \lambda} \mathbb{Z}\lambda_i$ *of* $M_{\mathbb{Q}}$.
3. λ_j *is not in the sub-lattice* $M + \sum_{\lambda_i <_\rho \lambda_j} \mathbb{Z}\lambda_i$, *of* $M_{\mathbb{Q}}$ *for* $j = 1, \dots, g$.

If such elements exist they are uniquely determined by ζ *and they are the characteristic exponents of* ζ.

Proof. If the minimal polynomial of ζ over $\mathbb{C}\{\rho^\vee \cap M\}$ is quasi-ordinary then the result follows from lemmas 3.9, 3.11 and 3.7 applied to sequence of characteristic exponents. Conversely, if ζ' is the conjugate of ζ by an element of the Galois group of $L_n \supset L$ and if $\zeta \neq \zeta'$ let us consider the sequence of lattices M_i and integers n_i associated to $\lambda_1, \dots, \lambda_g$ by (3.12). There is some $j \geq 1$ such that the monomials X^ν are fixed for $\nu \in M_{j-1}$ and X^{λ_j} is not fixed by this element by lemma 3.7 and hypothesis 3. Then hypothesis 1 and 2 imply that the difference $\zeta' - \zeta$ is of the form $\zeta' - \zeta = X^{\lambda_j} H_j$ for a unit H_j. ◇

Remark 3.13. *The characteristic lattices associated to* f *provide a canonical way of writing the terms of its roots:*

$$\zeta = p_0 + p_1 + \dots + p_g,$$

where $p_0 \in \mathbb{C}\{\rho^\vee \cap M\}$ *and the monomial* X^λ *appears in the summand* p_j *implies that* $\lambda_j \leq_\rho \lambda$ *and* $\lambda_{j+1} \not\leq_\rho \lambda$ *for* $j = 1, \dots, g$.

It is shown by Lipman (see [L4], remark 7.3.2) that an analytically irreducible quasi-ordinary hypersurface germ of dimension d is *normal* if and only if it is isomorphic to a germ of the form $Y^n - X_1 \dots X_c = 0$ for some $1 \leq c \leq d$; otherwise it is well known that its normalization is a *quotient singularity* (see [L4]); in the two dimensional case it is the germ of an affine toric surface (see [B-P-V], Chapter III, Theorem 5.2). In [GP2] is proved that the normalization of an irreducible quasi-ordinary hypersurface germ is isomorphic to the germ of an affine normal toric variety at its zero orbit and that this singularity is determined from the set of characteristic exponents. The following proposition generalizes this fact for toric quasi-ordinary hypersurface germs.

Proposition 3.14. *The integral closure of the analytic algebra* R *in its field of fractions is equal to* $\mathbb{C}\{\rho^\vee \cap M_g\}$.

Proof. The analytic algebra of the quasi-ordinary hypersurface is of the form $R = \mathbb{C}\{\rho^\vee \cap M\}[\zeta]$. By lemma 3.12 we have a ring extension $R \subset \mathbb{C}\{\rho^\vee \cap M_g\}$ which is integral since $\mathbb{C}\{\rho^\vee \cap M_g\}$ is integral over $\mathbb{C}\{\rho^\vee \cap M\}$. By lemmas 3.7 and 3.11 the rings R and $\mathbb{C}\{\rho^\vee \cap M_g\}$ have the same field of fractions. These two conditions imply that both rings have the same integral closure over their field of fractions. The ring $\mathbb{C}\{\rho^\vee \cap M_g\}$ is integrally closed since it is the analytic algebra of the normal variety Z_{ρ, N_g} at the point o_ρ. ◇

3.2.1 The Eggers-Wall tree of a reduced quasi-ordinary polynomial

We structure the partially ordered set of characteristic monomials of a reduced toric quasi-ordinary polynomial in a labeled tree. When $\mathrm{rk} M = 1$ the germ S defined by a reduced quasi-ordinary polynomial $f \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ at the origin is just a germ of complex plane curve. It is well known that the intersection multiplicities of the different branches of the curve at the origin and the semigroups associated to each of them are a complete invariant of the *embedded topological type* of the plane curve germ $(S,0)$ (see [Re]). Eggers shows that this information can be encoded by structuring in a labeled tree the characteristic exponents of each irreducible factor and the orders of coincidence between any two of them (see [Eg]). Wall (see [Wa]) gives a different definition of Egger's tree to give a new proof of a theorem of García Barroso on the structure of polar curves (see [GB1] or [GB2]). Wall's definition encodes the same amount of information as Egger's definition does and involves the use of a simplicial 1-chain on the tree which is defined from the sequence of characteristic integers of the irreducible factors (see definition 3.7). In the case of a classical quasi-ordinary hypersurface, Zariski's result stated in lemma 3.9 can be reformulated as follows: If $f = 0$ defines a classical quasi-ordinary hypersurface and if f_1 is an irreducible factor of f the set of characteristic exponents of f_1 union the set of orders of coincidence of f_1 with the factors of f is totally ordered with respect to the partial order defined by the divisibility of the corresponding monomials. Zariski's observation and the sequences of characteristic integers are exactly what is necessary to extend Wall's definition to the quasi-ordinary case in terms of a fixed quasi-ordinary projection $(X,Y) \mapsto X$. This is done more generally by Popescu-Pampu (see [PP2]) for a quasi-ordinary polynomial f, obtaining a generalization of Wall's proof which describes the structure of $\frac{\partial f}{\partial Y}$ in terms of the tree of f when $\frac{\partial f}{\partial Y}$ is quasi-ordinary. This result on the structure of the factorization of $\frac{\partial f}{\partial Y}$ remains true without any hypothesis on the derivative (see [GB-GP]).

The definition of the tree in our case runs as follows: Let $f = f^{(1)} \cdots f^{(r)}$ be the factorization in monic irreducible polynomials of f. Each factor $f^{(i)}$ of f is quasi-ordinary and the subset $\theta(f^{(i)})^{(0)}$ of $\rho^\vee \cap M_g \cup \{+\infty\}$ whose elements are 0, the characteristic exponents $\lambda_1^{(i)} <_\rho \cdots <_\rho \lambda_{g(i)}^{(i)}$ of $f^{(i)}$ (if they exist) and the orders of coincidence of $f^{(i)}$ with the irreducible factors of f is totally ordered by lemma 3.9; we denote by $n_k^{(i)}$ and $e_k^{(i)}$ for $k = 1, \ldots, g(i)$, the sequences of integers associated to $f^{(i)}$ by definition 3.7 for $i = 1, \ldots, r$.

We denote by $\theta(f^{(i)})^{(1)}$ the set of closed segments joining pairs of consecutive elements in $\theta(f^{(i)})^{(0)}$ and by $\theta(f^{(i)})$ the abstract simplicial complex with vertices on $\theta(f^{(i)})^{(0)}$ and edges on $\theta(f^{(i)})^{(1)}$. The underlying topological space is homeomorphic to the segment $[0, +\infty]$. We denote the vertex of $\theta(f^{(i)})$ corresponding to $\lambda \in \theta(f^{(i)})^{(0)}$ by $P_\lambda^{(i)}$. The simplicial complex $\theta(f)$ obtained from the disjoint union $\bigsqcup_{i=1}^{s} \theta(f^{(i)})$ by identifying in $\theta(f^{(i)})$ and $\theta(f^{(j)})$ the sub-simplicial complex corresponding to $\overline{P_0^{(i)} P_{\lambda_{(i,j)}}^{(i)}}$ and $\overline{P_0^{(j)} P_{\lambda_{(i,j)}}^{(j)}}$ for $1 \leq i < j \leq r$ is a tree.

We denote by $C_0(f)$ the 0-chain that attaches the value $\lambda \in \mathbb{Q}^d \cup \{+\infty\}$ to each vertex $P_\lambda^{(i)}$ in the Eggers tree (counting each vertex only once). We define a integral 1-chain $C_1(f^{(i)})$ in $\theta(f^{(i)})$ by subdividing the chain

$$\overline{P_0^{(i)} P_{\lambda_1^{(i)}}} + n_1^{(i)} \overline{P_{\lambda_1^{(i)}}^{(i)} P_{\lambda_2^{(i)}}^{(i)}} + \cdots + n_1^{(i)} \cdots n_{g(i)}^{(i)} \overline{P_{\lambda_g^{(i)}}^{(i)} P_{+\infty}^{(i)}}$$

with the points corresponding to the orders of coincidence of $f^{(i)}$ with other factors of f with respect to the order defined by lemma 3.9. The vertex $P_\lambda^{(i)}$, if $\lambda \neq 0, +\infty$ is not a characteristic exponents of $f^{(i)}$ if and only if the vertex $P_\lambda^{(i)}$ appears in two segments with the same coefficient. It follows from definition 3.7 that the 1-chains $C_1(f^{(i)})$ paste on $\theta(f)$ defining a 1-chain $C_1(f)$. The chains $C_1(f)$ and $C_0(f)$ determine the number of factors of f, the characteristic exponents of each factor and the orders of coincidence.

Definition 3.8. *The Eggers-Wall tree is the simplicial complex $\theta(f)$ with the chains $C_1(f)$ and $C_0(f)$.*

3.3 Embedded resolution procedure

In this section we build an embedded resolution of a reduced quasi-ordinary polynomial which is a composition of toric morphism determined by the characteristic monomials.

3.3.1 Definition of good coordinates

We introduce the notion of Y being a *good coordinate* in terms of the coincidence of the parametrizations of f. In the following section we build the toric morphisms of the resolution using this notion. Different choices of good coordinates provide isomorphic morphisms.

We keep the notations of section 3.2.1. We suppose that f is a quasi-ordinary polynomial with r irreducible factors $f^{(1)}, \ldots, f^{(r)}$. Define $\mathcal{A}(i) := (M \cap \{\lambda_{(i,j)}\})_j \cup \{\lambda_1^{(i)}\}$ for $1 \leq i \leq r$. By lemma 3.9, if the set $\mathcal{A}(i)$ is non empty it is totally ordered by $<_\rho$.

Then we can define:

$$\lambda_{\kappa(i)} := \left\{ \begin{array}{l} \min \mathcal{A}(i) \text{ if } \mathcal{A}(i) \neq \emptyset \\ +\infty \text{ otherwise} \end{array} \right\} \text{ for } i = 1, \ldots, r. \tag{3.13}$$

Lemma 3.15.

1. *If $\lambda_{\kappa(i)} \nleq \lambda \notin M$ the term X^λ does not appear in the expansions of the roots of $f^{(i)}$. In particular if X^λ appears in the expansions of the roots of $f^{(j)}$ then λ is $\geq \lambda_{(i,j)}$ and the equality $\lambda = \lambda_{(i,j)}$ implies that $\lambda_{(i,j)} = \lambda_1^{(j)}$.*

2. *The case* $\lambda_{\kappa(i)} = +\infty$ *happens if and only if* $f^{(i)}$ *is the only factor of* f *without characteristic exponents and* $\lambda_{(i,j)} = \lambda_1^{(j)}$ *for all* $j \neq i$ *.*

3. *If* $\lambda_{\kappa(i_0)} \in M$ *then* $\lambda_{\kappa(i_0)}$ *is* $\geq \lambda_{\kappa(j)}$ *for all* $j \neq i_0$ *.*

4. *The set* $\{\lambda_{\kappa(1)}, \ldots, \lambda_{\kappa(r)}\}$ *is totally ordered by* $<$ *.*

Proof. If $f^{(i)}$ has no characteristic exponent the terms in the expansion of its root have exponents in $\rho^\vee \cap M$. Otherwise, $\lambda_{\kappa(i)} \not\leq \lambda \notin M$ implies that $\lambda_1^{(i)} \not\leq \lambda \notin M$ thus the term X^λ does not appear in the expansion of the roots of $f^{(i)}$ by lemma 3.12. If X^λ appears in the expansion of the roots of $f^{(j)}$ then it appears in any difference of roots of $f^{(i)}$ and $f^{(j)}$ thus $\lambda \geq \lambda_{(i,j)}$. Moreover, if $\lambda = \lambda_{(i,j)}$ then $\lambda \notin M$ implies that $\lambda_{(i,j)} \geq \lambda_1^{(j)}$ by lemma 3.12. Since $\lambda_{\kappa(i)} \not\leq \lambda_{(i,j)}$ we have that $\lambda_{\kappa(i)} \not\leq \lambda_1^{(j)} \notin M$ and therefore $\lambda_1^{(j)} \geq \lambda_{(i,j)}$ and the equality $\lambda_{(i,j)} = \lambda_1^{(j)}$ follows.

For the second assertion notice that if $f^{(i)}$ and $f^{(j)}$ are two different factors without characteristic exponents then $\lambda_{(i,j)}$ belongs to M thus $\lambda_{\kappa(i)}, \lambda_{\kappa(j)} \neq +\infty$. If $\lambda_{\kappa(i)} = +\infty$ then $\lambda_{(i,j)}$ is not in M for all $j \neq i$; thus the exponent $\lambda_{(i,j)}$ appears on a term of the parametrization of $f^{(j)}$ and therefore we have $\lambda_{(i,j)} \geq \lambda_1^{(j)}$ by lemma 3.12. The first assertion for $\lambda = \lambda_1^{(j)}$ implies that $\lambda_{(i,j)} \leq \lambda_1^{(j)}$ and equality follows.

Now suppose that $\lambda_{\kappa(i_0)} \in M$. If $j \neq i$ the exponents $\lambda_{\kappa(i_0)}$ and $\lambda_{(i_0,j)}$ are comparable by lemma 3.9. We distinguish two cases:

(a) $\lambda_{\kappa(i_0)} \leq \lambda_{(i_0,j)}$. Notice that assertion *1* implies that if $f^{(j)}$ has some characteristic exponent then $\lambda_1^{(j)} > \lambda_{\kappa(i_0)}$. If $\lambda_{\kappa(i_0)} < \lambda_{(i_0,j)}$ there is $j \neq l_0 \neq i_0$ such that $\lambda_{\kappa(i_0)} = \lambda_{(i_0,l_0)} = \min\{\lambda_{(i_0,l_0)}, \lambda_{(i_0,j)}\} = \lambda_{(j,l_0)}$ by lemma 3.10; hence the exponents $\lambda_{(j,l)}$ and $\lambda_{\kappa(i_0)}$ are comparable by lemma 3.9. If $\lambda_{\kappa(i_0)} = \lambda_{(i_0,j)}$ set $l_0 = j$.

If $\lambda_{(j,l)} < \lambda_{\kappa(i_0)}$ we deduce from lemma 3.10 that:

$$\lambda_{(j,l)} = \min\{\lambda_{(j,l_0)}, \lambda_{(j,l)}\} = \lambda_{(l,l_0)} = \min\{\lambda_{(i_0,l_0)}, \lambda_{(l,l_0)}\} = \lambda_{(i_0,l)},$$

and $\lambda_{(j,l)}$ does not belong to M by definition of $\lambda_{\kappa(i_0)}$. This shows that $\lambda_{\kappa(j)} = \lambda_{\kappa(i_0)}$.

(b) $\lambda_{(i_0,j)} < \lambda_{\kappa(i_0)}$. By definition of $\lambda_{\kappa(i_0)}$ we have that $\lambda_{(i_0,j)} \notin M$ and then assertion *1* implies that $\lambda_{(i_0,j)} = \lambda_1^{(j)}$. If $\lambda_{(j,l)} < \lambda_1^{(j)}$ we deduce using lemma 3.10 that $\lambda_{(j,l)} = \min\{\lambda_{(j,l)}, \lambda_{(i_0,j)}\}$ is equal to $\lambda_{(i_0,l)}$ and $< \lambda_{\kappa(i_0)}$. It follows that $\lambda_{(i_0,l)} \notin M$, thus $\lambda_{\kappa(j)} = \lambda_1^{(j)} < \lambda_{\kappa(i_0)}$.

For the last assertion we only have to prove that if $\lambda_{\kappa(i)} = \lambda_1^{(i)}$ and $\lambda_{\kappa(j)} = \lambda_1^{(j)}$ they are comparable by $<$. By lemma 3.9, $\lambda_{(i,j)}$ is comparable with $\lambda_1^{(i)}$ and $\lambda_1^{(j)}$. The case $\lambda_{(i,j)} < \lambda_1^{(i)}, \lambda_1^{(j)}$ implies that $\lambda_{(i,j)} \in M$ by lemma 3.12, thus $\lambda_{\kappa(i)} \leq \lambda_{(i,j)}$ a contradiction. Therefore we can assume that $\lambda_1^{(i)} \leq \lambda_{(i,j)}$, replacing i by j if necessary. It follows from the definition of order of coincidence that if $\lambda_1^{(i)} < \lambda_{(i,j)}$ then $\lambda_1^{(i)} = \lambda_1^{(j)}$. If $\lambda_1^{(i)} = \lambda_{(i,j)}$ then the result follows from lemma 3.9. ◇

We relabel the factors $f^{(i)}$ of f in order to have: $\lambda_{\kappa(1)} \leq \lambda_{\kappa(2)} \leq \cdots \leq \lambda_{\kappa(r)}$. If $\lambda \in \rho \cap M$, the monomial X^λ appears in all the roots of $f^{(r)}$ with the same coefficient $c_\lambda^{(r)}$. Then we define:

$$\phi_0 := \sum_{\lambda_{\kappa(r)} \not\leq \lambda \in \rho^\vee \cap M} c_\lambda^{(r)} X^\lambda,$$

$$Y' := \begin{cases} Y + \phi_0 \text{ if } \lambda_{\kappa(r)} \notin M \\ Y + \phi_0 + cX^{\lambda_{\kappa(r)}}, \text{ for } c \in \mathbb{C}^* \text{ generic, if } \lambda_{\kappa(r)} \in M. \end{cases} \tag{3.14}$$

Generic here means that if $\lambda_{\kappa(r)} = \lambda_{\kappa(l)} \in M$ then $c - c^{(l)}_{\lambda_{\kappa(l)}} \neq 0$.

Lemma 3.16. *The order of coincidence of the polynomials Y' and $f^{(i)}$ is equal to $\lambda_{\kappa(i)}$ for $i = 1, \ldots, r$.*

Proof. It follows from lemma 3.15 that if $\lambda_{\kappa(i)} < \lambda_{\kappa(r)}$ then $\lambda_{\kappa(i)}$ is the order of coincidence of $f^{(i)}$ and $f^{(r)}$ (remark that $\lambda_{\kappa(i)} \notin M$ by assertion *3* of lemma 3.15, thus $\lambda_{\kappa(i)} = \lambda_1^{(i)}$ is $\geq \lambda_{(i,r)}$ by assertion *1* of lemma 3.15; it follows from this fact that $\lambda_{(i,r)} \notin M$ thus $\lambda_1^{(i)} \leq \lambda_{(i,r)}$ by lemma 3.12). This implies that the order of coincidence of Y' with $f^{(i)}$ is well defined and equal to $\lambda_{\kappa(i)}$. The generic choice of c guarantees in the case $\lambda_{\kappa(r)} \in M$ that the order of coincidence of Y' with those factors $f^{(i)}$ of f with $\lambda_{\kappa(i)} = \lambda_{\kappa(r)}$ is $\lambda_{\kappa(r)}$. ◇

Definition 3.9. *We say that Y is a good coordinate for the reduced quasi-ordinary polynomial $f \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ if the order of coincidence of Y with $f^{(i)}$ is well defined and equal to $\lambda_{\kappa(i)}$, for $i = 1, \ldots, r$.*

If Y is not a good coordinate for f then the $\mathbb{C}\{\rho^\vee \cap M\}$-automorphism of $\mathbb{C}\{\rho^\vee \cap M\}[Y]$ that maps $Y \mapsto Y'$, for Y' defined in the lemma 3.16, transforms $f \mapsto f' \in \mathbb{C}\{\rho^\vee \cap M\}[Y']$. The polynomial f' is quasi-ordinary, f' and f have the same Eggers-Wall tree and Y' is a good coordinate for f'.

3.3.2 The first toric morphism of the embedded resolution

We build the first toric morphism of the embedded resolution and we prove that it simplifies the singularity preserving at the same time the quasi-ordinary structure.

The case of a Newton polyhedron with only one compact edge

We deal first with the case when all the irreducible factors of f are parametrized by series of the form $X^\lambda \varepsilon$ with $\varepsilon(o_\rho) = c$.

We denote by M_λ the lattice $M + \lambda\mathbb{Z}$ for $\lambda \in \frac{1}{n}M$ (resp. N_λ for the dual lattice), by M'_λ the lattice $M_\lambda \oplus y\mathbb{Z}$ (resp. N'_λ for the dual lattice) and by n_λ the integer $|M_\lambda / M|$. Let Σ be a subdivision of ϱ containing cone $\sigma := \varrho \cap \ell$ where ℓ is the linear subspace of $N'_{\mathbb{R}}$ orthogonal to the compact face $[n\lambda, ny]$ of the polyhedron $\mathcal{N}(f)$ (where $n = \deg f$). The subdivision Σ of ϱ is rational for the lattices N'_g and N_λ. We have the following commutative diagram of equivariant maps:

$$\begin{tikzcd}
Z_{\Sigma, N'_\lambda} \arrow[r, "\Pi_\Sigma"] \arrow[d] & Z_{\varrho, N'_\lambda} \arrow[d] \\
Z_{\Sigma, N'} \arrow[r, "\pi_\Sigma"] & Z_{\varrho, N'}
\end{tikzcd} \tag{3.15}$$

where the vertical arrows are defined by lattice extension and the horizontal arrows are defined by the subdivision Σ. Often we do not precise the lattice if it is corresponds to the below line of the diagram 3.15.

Lemma 3.17. *The lattice homomorphism $\varphi : M' \to M_\lambda$ that maps $y \mapsto \lambda$ and fixes $u \in M$ induces an isomorphism:*

$$M_\sigma \cong M_\lambda \tag{3.16}$$

If we choose an splitting $M' \cong M_\sigma \oplus Ker(\varphi)$ we have a semigroup isomorphism:

$$\sigma^\vee \cap M' \cong n_\lambda(y - \lambda_1)\mathbb{Z} \oplus (\rho^\vee \cap M_\lambda) \tag{3.17}$$

which corresponds to an isomorphism $Z_{\sigma,N'} \cong \mathbf{O}_{\sigma,N'} \times Z_{\rho,N_\lambda}$.

Proof. We use the combinatorial arguments in the proofs of lemma 3.5 and proposition 3.6 to prove (3.16) using that $\sigma^\perp = \mathrm{Ker}(\varphi_{\mathbb{R}})$ by (3.7); then (3.17) holds by (3.10). ◇

We denote by $S^{(i)}_\Sigma$ the strict transform of the germ $S^{(i)}$ defined by the irreducible factor $f^{(i)}$ of f for $i = 1, \ldots, r$.

Lemma 3.18. *The intersection $S^{(i)}_\Sigma \cap \pi_\Sigma^{-1}(o_\rho)$ is the point $o_1 = (c', o_\rho) \in \mathbf{O}_\sigma$ counted $e^{(i)}_\lambda := (\deg f^{(i)})/n_\lambda$ times, where $c' = c^{n_\lambda}$ and c is the coefficient of X^{λ_1} in any root of the polynomial $f^{(i)}$ defining $S^{(i)}$. In particular, the intersection $S^{(i)}_\Sigma \cap \mathbf{O}_\sigma$ is transversal if and only if $e^{(i)}_\lambda = 1$. The strict transform S_Σ of S is a germ at the point o_1.*

Proof. To simplify the proof we drop the super-index (i). If $\tau \in \Sigma$ with $\overset{\circ}{\tau} \subset \overset{\circ}{\varrho}$ then $S_\Sigma \cap \mathbf{O}_\tau \neq \emptyset$ implies that $\tau = \sigma$ since the face of $\mathcal{N}_\varrho(f)$ defined by σ is of dimension ≥ 1 (by lemma 3.3). The strict transform S_Σ is defined on Z_σ by $X^{-n\lambda} f = 0$ and it follows that the ideal of $\mathbf{O}_\sigma \cap S_\Sigma$ is generated by $(X^{n_\lambda(y-\lambda)} - c^{n_\lambda})^{e_\lambda}$ where c is the coefficient of X^λ in any root of f. This implies that the intersection of the strict transform S_Σ with $\pi_\Sigma^{-1}(o_\varrho)$ is reduced to the point $o_1 = (c', o_\rho)$ counted e_λ times. In particular, the intersection is transversal if and only if $e_\lambda = 1$. This shows also that the strict transform S_Σ is a germ at the point o_1 since this is the only point of intersection with the exceptional fiber. ◇

Proposition 3.19. *The restriction of the projection* $\mathbf{O}_\sigma \times Z_{\rho,N_\lambda} \cong Z_{\sigma,N'} \to Z_{\rho,N_\lambda}$ *to* (S_Σ, o_1) *is quasi-ordinary. The germ* (S_Σ, o_1) *is defined by a quasi-ordinary polynomial* $f_\Sigma \in \mathbb{C}\{\rho^\vee \cap M_\lambda\}[W]$ *(where* $W = Y^{n_\lambda} X^{-n_\lambda \lambda} - c^{n_\lambda}$ *) with characteristic exponents* $\lambda' - \lambda$ *for those characteristic exponents* $\lambda' > \lambda$ *of* f. *If* $\lambda_{(i,j)}$ *is the order of coincidence between the irreducible components* $f^{(i)}$ *and* $f^{(j)}$ *of* f *then the order of coincidence of* $f_\Sigma^{(i)}$ *and* $f_\Sigma^{(j)}$ *is* $\lambda_{(i,j)} - \lambda$. *If* (S, o) *is irreducible the same holds for* (S_Σ, o_1).

Proof. We deal first with the case $\lambda \in M$, i.e., $n_\lambda = 1$. By lemma 3.18 the chart Z_σ contains the strict transform S_Σ. By hypothesis the roots $\zeta^{(i)}$ of f are of the form $\zeta^{(i)} = cX^\lambda + \sum_{\lambda' > \lambda} c_{\lambda'}^{(i)} X^{\lambda'}$, i.e., the coefficient of the monomial X^λ is the same for all of them. By lemma 3.18 the strict transform of $Y - \zeta^{(i)} = 0$ by the morphism $Z_{\sigma,N'_g} \to Z_{\varrho,N'_g}$ is defined by:

$$0 = X^{y-\lambda} - c + \sum_{\lambda' > \lambda} c_{\lambda'}^{(i)} X^{\lambda' - \lambda} \tag{3.18}$$

where the terms $X^{\lambda' - \lambda}$ vanish on the orbit $\mathbf{O}_\sigma$ for all $\lambda' > \lambda$. By lemma 3.17 the chart Z_{σ,N'_g} (resp. $Z_{\sigma,N}$) is isomorphic to $\mathbf{O}_{\sigma,N'_g} \times Z_{\rho,N_g}$ (resp. to $\mathbf{O}_{\sigma,N'} \times Z_{\rho,N}$). Since $n_\lambda = 1$ the toric morphism $Z_{\sigma,N'_g} \to Z_{\sigma,N'}$ restricts to an isomorphism of the orbits $\mathbf{O}_{\sigma,N'_g} \cong \mathbf{O}_{\sigma,N'} = \mathbf{O}_\sigma$ by (3.17), the coordinate ring of the orbit $\mathbf{O}_\sigma$ being equal to $\mathbb{C}[YX^{-\lambda}]$. We study the strict transform of $Y - \zeta^{(i)} = 0$ (resp. of S) at the point of intersection with the orbit $\mathbf{O}_\sigma$ by replacing the invertible term $X^{y-\lambda}$ by the unit $c + W$ on (3.18) (resp. on $X^{-n\lambda} f = 0$). We obtain a polynomial $f_\Sigma \in \mathbb{C}\{\rho^\vee \cap M\}[W]$ from $X^{-n\lambda} f$ which splits in $\mathbb{C}\{\rho^\vee \cap M_g\}[W]$: $f_\Sigma = \prod_{i=1}^n (W - \tau^{(i)})$; where $\tau^{(i)} = \sum_{\lambda' > \lambda} c_{\lambda'}^{(i)} X^{\lambda' - \lambda}$. It follows from lemma 3.12 that the series $\tau^{(i)}$ are quasi-ordinary branches and that their characteristic exponents are obtained from those of $\zeta^{(i)}$ by subtracting λ. If f is irreducible the same thing happens for f_Σ. Otherwise, we have $\tau^{(i)} - \tau^{(j)} = X^{-\lambda}(\zeta^{(i)} - \zeta^{(j)})$ and this implies the assertion about the orders of coincidence.

If $n_\lambda > 1$ we reduce to the previous case by passing through the diagram (3.15):

Each irreducible factor of f splits into n_λ irreducible factors in $\mathbb{C}\{\rho^\vee \cap M_\lambda\}[Y]$ having order of coincidence equal to λ. We factor f as a product $F_1 \cdots F_{n_\lambda}$ in $\mathbb{C}\{\rho^\vee \cap M_\lambda\}[Y]$, the F_i being defined by the properties: the order of coincidence of $F_i \neq F_j$ (resp. of any two factors of F_i) is $= \lambda$ (resp. is $> \lambda$). The Eggers-Wall tree of F_i is obtained from the Eggers- Wall tree of f by deleting the vertex P_λ and dividing by n_λ the coefficients of the chain $C_1(f)$ between P_λ and the extreme points $P_{+\infty}^{(j)}$ of the tree (this follows from lemma 3.11 and definition 3.7). Then the strict transforms of $F_i = 0$ by Π_Σ are disjoint germs at the n_λ points of intersection with $\mathbf{O}_{\sigma,N'_\lambda}$ by lemma 3.18.

By lemma 3.17 the toric morphism $Z_{\sigma,N'_\lambda} \to Z_{\sigma,N}$ corresponds to the semigroup inclusion

$$n_\lambda(y - \lambda)\mathbb{Z} \oplus (\rho^\vee \cap M_\lambda) \to (y - \lambda)\mathbb{Z} \oplus (\rho^\vee \cap M_\lambda).$$

This map is an unramified covering of degree n_λ and it commutes with the projections onto the factor Z_{ρ,N_λ} of Z_{σ,N'_λ} and $Z_{\sigma,N}$. This provides an isomorphism between the strict transform of F_i

by Π_Σ and S_Σ which commutes with the projection onto factor $Z_{\rho^{,}N_\lambda}$ for $i = 1, \ldots, n_\lambda$. A fortiori the restriction of the projection $Z_{\sigma, N'} \to Z_{\rho, N_\lambda}$ to S_Σ is quasi-ordinary and the result follows. $\diamond$

With the same hypothesis of proposition 3.19 we have:

Corollary 3.1. *If (S, o) is analytically irreducible and if $\lambda = \lambda_1$ is the only characteristic exponent of ζ the strict transform S_Σ of S is isomorphic to the germ Z_{ρ, N_1} and the restriction of π_Σ to $S_\Sigma \to S$ is the normalization map. The morphism π_Σ is a partial embedded resolution of $S \subset Z_\varrho$. If Σ' is a resolution of the fan Σ the map $\pi_{\Sigma'} \circ \pi_\Sigma$ is an embedded resolution of $S \subset Z_\varrho$.*

Proof. It follows from lemma 3.18 that S_Σ is isomorphic to the germ (Z_{ρ, N_1}, o_ρ) and to the normalization of (S, o) by proposition 3.14. We argue as in proposition 3.6 and lemma 3.5 to extend the result in this case. $\diamond$

The following remark is a consequence of the proof of proposition 3.19.

Remark 3.20. *If f is irreducible, $\lambda = \lambda_1$ and if $f_1 \in \mathbb{C}\{\rho^\vee \cap M_\lambda\}[W]$ defines a good coordinate for f_Σ then the image of $f_1 = 0$ by π_Σ is defined by an irreducible quasi-ordinary polynomial in $\mathbb{C}\{\rho^\vee \cap M\}[Y]$ with only one characteristic exponent λ_1 and with maximal order of coincidence with f.*

The following result has been suggested by Némethi and McEwan see ([M-N] and [GP-M-N]).

Lemma 3.21. *The morphism π_Σ of proposition 3.19 is an isomorphism over $Z_\varrho - S$.*

Proof. The discriminant of the morphism Π_Σ is described by (3.4). It follows from this formula that the functions X^λ and Y vanishes on those orbits of Z_{ϱ, N'_λ} which are contained in the discriminant locus of Π_Σ. The image of these orbits by the map $Z_{\varrho, N'_\lambda} \to Z_\varrho$ is the discriminant of π_Σ and it is contained in S since all the roots of f are of the form $Y = X^\lambda$ (up to multiplication by a unit). $\diamond$

The general case

We build the first toric morphism of the embedded resolution in the general case.

We suppose from now on that Y is a good coordinate for f. The Newton polyhedron of each irreducible factor $f^{(i)}$ with $\lambda_{\kappa(i)} \neq +\infty$ is an edge with vertices $(\deg f^{(i)}, 0)$ and $(0, \deg f^{(i)} \lambda_{\kappa(i)})$ where $X^{\lambda_{\kappa(i)}}$ is the initial monomial of any root of $f^{(i)}$. Since the set of $\{\lambda_{\kappa(i)}\}$ is completely ordered by $<_\rho$ the set of compact faces of $\mathcal{N}_\rho(f)$ defines a *monotone polygonal path* with *inclinations* running through $\{\lambda_{\kappa(1)}, \ldots, \lambda_{\kappa(r)}\} - \{+\infty\}$ independently of the choice of good coordinate (see [GP1] for the terminology). This fact is a special feature of quasi-ordinary singularities and it is a generalization of the plane curve case.

The dual fan Σ_1 of the polyhedron $\mathcal{N}(f)$ is obtained by intersecting ϱ with the linear hyperplanes $\ell_{\kappa(j)} := \langle y - \lambda_{\kappa(j)}, u \rangle = 0$ for those $\lambda_{\kappa(j)} \neq +\infty$. Since we have that $\{\lambda_{\kappa(j)}\}$ is totally ordered by $<_\rho$ we find that the cones $\varrho \cap \ell_{\kappa(j)}$ belong to Σ_1 since they cannot intersect in the interior of ϱ. Geometrically, this implies that the exceptional locus of π_{Σ_1} is a *bamboo* of $\mathbf{P}^1_{\mathbb{C}}$, each one of them being the closure of the orbit $\mathbf{O}_{\varrho \cap \ell_{\kappa(j)}}$ (we say that a curve is a bamboo if the dual graph of intersection of its irreducible components is isomorphic to the subdivision of a segment).

Proposition 3.22. *If* $\lambda_{\kappa(i)} \neq +\infty$ *then we have:*

1. *The strict transform of* $S^{(i)}$ *by* π_{Σ_1} *is a germ* $(S^{(i)}_{\Sigma_1}, o_1^{(i)})$ *at the point of intersection with the exceptional curve* $\pi^{-1}_{\Sigma_1}(o_\varrho)$.
2. *The Eggers-Wall tree of a polynomial defining the strict transform* S_{Σ_1} *at the point* $o_1^{(i)}$ *is obtained from* $\theta(f)$ *by removing the segment* $[P_0^{(j)}, P^{(j)}_{\lambda_{\kappa(i)}}[$ *from the sub-tree of* $\theta(f)$ *given by* $\bigcup \theta(f^{(j)})$, *for* $f^{(j)}$ *with order of coincidence* $> \lambda_{\kappa(i)}$ *with* $f^{(i)}$. *The coefficients of the vertices of the resulting tree are obtained by subtracting* $\lambda_{\kappa(i)}$. *The coefficients of the associated* 1*-chain are obtained by dividing by* $n_{\lambda_{\kappa(i)}}$.

Proof. The first assertion follows from lemma 3.18. It follows from proposition 3.19 that $o_1^{(i)} = o_1^{(j)}$ if and only if the irreducible factors of the symbolic restrictions of $f^{(i)}$ and $f^{(j)}$ to the compact edges of their Newton polyhedra coincide

This is equivalent $f^{(i)}$ and $f^{(j)}$ have order of coincidence $> \lambda_{\kappa(i)} = \lambda_{\kappa(j)}$. The second assertion follows from proposition 3.19 since the characteristic exponents and the order of coincidence of $f^{(i)}_\Sigma$ and $f^{(j)}_\Sigma$ are obtained from those corresponding to $f^{(i)}$ and $f^{(j)}$ by subtracting λ_1. The strict transform $S^{(i)}_\Sigma$ is a toric quasi-ordinary hypersurface relative to the base $Z_{\rho, N_{\lambda_{\kappa(i)}}}$ by proposition 3.19 and the statement about the coefficients of the associated 1-chain follows from this change of lattice by lemma 3.7. ◇

Remark 3.23. *If* $\lambda_{\kappa(i)} = +\infty$ *the strict transform of* $S^{(i)}$ *is the germ of the closure of the orbit associated to the edge* $y\mathbb{R}_{\geq 0}$ *at the point of intersection with the exceptional curve* $\pi^{-1}_{\Sigma_1}(o_\varrho)$.

The assertion follows from the description of the exceptional locus and the discriminant locus of a toric modification given in section 3.1.1 once it is noticed that the point of intersection $o_1^{(i)}$ of $S^{(i)}$ with $\pi^{-1}_{\Sigma_1}(o_\varrho)$ is the orbit associated to the $(d+1)$-dimensional cone of Σ_1 which contains the cone $y\mathbb{R}_{\geq 0}$.

3.3.3 The toric embedded resolution

We show the way to iterate the procedure of the previous section to build an embedded resolution of $S \subset Z_\varrho$ by first eliminating the characteristic exponents and then by resolving the toric singularities of the ambient space.

By proposition 3.22, the germs defined by the strict transform at each of the points $o_1^{(i)}$ of intersection with the exceptional curve are simpler toric quasi-ordinary hypersurface singularities. In a finite number of iterations of this procedure the strict transform becomes a union of r toric quasi-ordinary hypersurface germs with no characteristic exponents at all, i.e., is collection of r germs of affine toric varieties at the special points. It follows from propositions 3.19 and 3.14 that the strict transform of (S, o) is its normalization. Thus this method provides an *embedded normalization* (in a normal environment) of the germ $(S, o) \subset (Z_\varrho, o_\varrho)$. We keep the information of the toric singularities of the ambient space by defining at each stage a toroidal embedding without self-intersection:

First, we associate to the toric quasi-ordinary hypersurface $(S, o) \subset (Z_\varrho, o_\varrho)$ embedded with a good coordinate the toroidal embedding defined by (Z_ϱ, N_0'). Its conic polyhedral complex Θ_0 is equal to (ϱ, N_0). Then, we associate to each point of intersection $o_1^{(i)}$ of the strict transform S_{Σ_1} with the exceptional fiber a normal hypersurface $S_1^{(i)}$ defined by taking a good coordinate for the quasi-ordinary projection of S_{Σ_1} of proposition 3.19. Obviously, if $o_1^{(i)} = o_1^{(j)}$ we have $S_1^{(i)} = S_1^{(j)}$ (see remark 3.20).

Lemma 3.24. *The c.p.c.* Θ_1 *associated to the toroidal embedding defined by the variety* Z_{Σ_1} *and the set normal hypersurfaces* $\{(S_1^{(i)}, o_1^{(i)})\}_{\lambda_{\kappa(i)} \neq +\infty} \cup \{\overline{\mathbf{O}_\sigma}\}_{\sigma \in \Sigma_1^{(1)}}$ *is obtained from the c.p.c.* Σ_1 *by adding for each point in the set* $\{o_1^{(i)}\}_{i=1,\dots,r}^{\lambda_{\kappa(i)} \neq +\infty}$ *the c.p.c.* $(\varrho, N'_{\lambda_{\kappa(i)}})$ *and pasting it to* Σ_1 *by identifying* $(\rho \times \{0\}, N_{\lambda_{\kappa(i)}} \times \{0\})$ *with* $(\varrho \cap \ell_{\kappa(i)}, N_{\varrho \cap \ell_{\kappa(i)}})$ *by the lattice isomorphism corresponding to (3.16) by duality. The c.p.c.* Θ_1 *is independent of the choice of good coordinates.*

Proof. To simplify the proof we drop the index i, we denote $\lambda_{\kappa(i)}$ by λ and we keep notations of proposition 3.19 and lemmas 3.17 and 3.18. The germ (S_1, o_1) is defined by the vanishing of a monic polynomial $f_1 \in \mathbb{C}\{\rho^\vee \cap M_\lambda\}[W]$ of degree one where $W = X^{n_\lambda(y-\lambda)} - c$. We deduce from lemma 3.17 that the analytic algebra of the germ (Z_Σ, o_1) is isomorphic to $\mathbb{C}\{\varrho^\vee \cap M'_\lambda\}$ by the isomorphism that maps $f_1 \mapsto X^{y_1}$ and $X^u \mapsto X^u$ for all $u \in \rho \cap M_\lambda$. Since the c.p.c. associated to the torus embedding of Z_{ϱ, N_1} is (ϱ, N_1) the same holds for the toroidal embedding corresponding to $\Sigma_1^{(1)}$ and the set of normal hypersurfaces $\mathcal{H} = \{\overline{\mathbf{O}_\sigma}\}_{\sigma \in (\varrho \cap \ell)^{(1)}} \cup \{S_1\}$. The sub-c.p.c. associated to the toroidal embedding corresponding to $\mathcal{H} - \{S_1\}$ is $(\varrho \cap \ell, N_{\varrho \cap \ell})$; it is isomorphic to (ρ, N_1), the pasting isomorphism being obtained from (3.16) by duality. ◇

Then we continue as follows:

If the quasi-ordinary polynomial defining the germ of the strict transform $(S_{\Sigma_1}, o_1^{(i)})$ has some characteristic exponent we put it in good coordinates; then its Newton polyhedron defines a subdivision of $(\varrho, N'_{\lambda_{\kappa(i)}})$, for $1 \leq i \leq r$. These subdivisions glue up to define a subdivision Σ_2 of the c.p.c. Θ_1 since the pasting cones $(\rho \times \{0\}, N_{\lambda_{\kappa(i)}})$ are not subdivided, for $1 \leq i \leq r$.

The corresponding toric modifications, defined locally, paste into a toroidal modification $\pi_2 : Z_2 \rightarrow Z_1$; (we denote the variety Z_{Σ_1} by Z_1, the morphism π_{Σ_1} by π_1, and S_{Σ_1} by S_1). By iterating this procedure we obtain: A modification $\pi_k : Z_k \to Z_{k-1}$, where the variety Z_k is given with the structure of toroidal embedding (Σ_k denoting its associated c.p.c.). The strict transform S_k of S by $\pi_k \circ \cdots \circ \pi_1$ at the points of intersection with the exceptional fiber is given with a quasi-ordinary projection and the associated Eggers-Wall tree is obtained from the eventually non connected tree of S_{k-1} as indicated by proposition 3.22. If the quasi-ordinary polynomial defining the germ S_k at any of these points has some characteristic monomial we define a a finer toroidal embedding for Z_k (with c.p.c. Θ_k defined by using lemma 3.24) and a subdivision Σ_{k+1} of Θ_k with associated modification $\pi_{k+1} : Z_{k+1} \to Z_k$. In a finite number k_0 of steps the quasi-ordinary polynomials defining the germ S_{k_0} at the points of intersection with the exceptional fiber have no characteristic monomials. Then it follows from corollary 3.1 that:

Theorem 3.2. *The proper morphism* $\pi = \pi_{k_0} \circ \cdots \circ \pi_1$ *is a partial embedded resolution of the quasi-ordinary hypersurface germ* $(S, o) \subset (Z_\rho, o_\rho)$. *The restriction* $S' \to S$ *of* π *to the strict transform* S' *of* S *is the normalization map.* ◇

An embedded resolution of $S \subset Z_\varrho$ is obtained by composing π with any toric resolution of the toroidal embedding Z_{k_0} with the c.p.c. Σ_{k_0} (or also with the c.p.c. Θ_{k_0}).

Remark 3.25. *The irreducible components of the exceptional fiber* $\pi^{-1}(o_\varrho)$ *of the partial resolution are projective lines* $\mathbb{P}^1_{\mathbb{C}}$. *The dual intersection graph of the components of* $\pi^{-1}(o_\varrho)$ *is obtained from the Eggers-Wall tree* $\theta(f)$ *by deleting the extremal segments.*

One of this segment joins the base vertex P_0 to one defined by the first characteristic exponent of the reduced f and the others corresponds to the segment containing the point $P_{+\infty}^{(i)}$ for $i = 1, \ldots, r$.

3.3.4 The case of plane curve germs

The case of plane curve germs corresponds to $\mathrm{rk}\, N = 1$. We keep the same notations. The partial resolution procedure depends only on the Eggers-Wall tree of $f \in \mathbb{C}\{X\}[Y]$ with respect to the projection $(X, Y) \mapsto X$ or more precisely on the choice of the curve $X = 0$. If f is irreducible then our construction is closely related to the construction of the "*Tschirnhausen good resolution tower*" of A'Campo and Oka (see [A'C-Ok], Theorem 4.5). In particular if the curve $X = 0$ is not contained in the tangent cone of S we show that this procedure leads to a minimal embedded resolution of the curve.

Let $f \in \mathbb{C}\{X\}[Y]$ be a reduced polynomial with Y a good coordinate for f. We keep notations of theorem 3.2 and we give some more definitions and notations. We denote by $\Theta_{k_0}^{reg}$ the minimal regular subdivision of the c.p.c. Θ_{k_0} (for the minimal regular subdivision in the toric case see Proposition 1.19 of [Od]). This provides a resolution $p : Z_{\Theta_{k_0}^{reg}} \to \mathbb{C}^2$ where $p := \pi \circ \pi_{\Theta_{k_0}^{reg}}$ which is canonically determined from the projection $(X, Y) \mapsto X$.

Denote by $\mathcal{G}(p,0)$ (resp. $\mathcal{G}(p,f)$) the subset of $\Theta_{k_0}^{reg}$ whose elements are the cones corresponding to non empty intersections of pairs of components of the exceptional divisor of the resolution p, (resp. of the total transform of S by p). Denote by $\mathcal{G}(\pi,0)$ (resp. by $\mathcal{G}(\pi,f)$) the subset of Θ_{k_0} of those cones corresponding to non empty intersections of pairs of components of the exceptional divisor of the partial resolution π (resp. of the total transform of S by π).

Recall that each edge of $\Theta_{k_0}^{reg}$ corresponds to an irreducible divisor in the toroidal embedding and any pair of these divisors intersect if and only if the corresponding edges belong to the same cone. It follows that $\mathcal{G}(p,0)$ (resp. $\mathcal{G}(p,f)$) is combinatorially isomorphic to the *resolution graph of the resolution* (resp. to the *total resolution graph of the resolution*), we just drop the dimension of the faces by one. We deduce from proposition 3.22, remark 3.23 and an easy induction that the Eggers-Wall tree $\theta(f)$ is combinatorially isomorphic to $\mathcal{G}(\pi, Xf)$.

The valency of a cone e in a conic polyhedral complex is the number of cones of the complex containing e as a facet.

Lemma 3.26. *Let $f \in \mathbb{C}\{X\}[Y]$ be a reduced polynomial of degree > 1, such that Y is a good coordinate for f. For any edge e in $\mathcal{G}(\pi,0)$ we have:*

1. *The valency $\delta(e)$ of e in $\mathcal{G}(p,Xf)$ is ≥ 3.*

2. *If $\lambda_{\kappa(1)}^{-1} \notin \mathbb{Z}_{>1}$ then the valency $\delta(e)$ of e in $\mathcal{G}(p,f)$ is ≥ 3.*

Proof. Recall that we have relabel the factors of f in order to have $\lambda_{\kappa(1)} \leq \cdots \leq \lambda_{\kappa(r)}$. We show first the assertion for the exceptional divisors appearing in the first toric modification π_{Σ_1}. The extremal edges of the fan Σ_1, which are defined by the vectors u_1, u_2 of the canonical basis, correspond to the divisors $X=0$ and $Y=0$ respectively. If $\lambda_{\kappa(j)} \neq +\infty$, there is an exceptional divisor $D_{\lambda_{\kappa(j)}}$ of π_{Σ_1} corresponding to $d_{\lambda_{\kappa(j)}} \in \mathcal{G}(\pi,f)$. We denote by the same letter the edge $d_{\lambda_{\kappa(j)}}$ of Σ_1 and the primitive vector $(n_{\lambda_{\kappa(j)}}, n_{\lambda_{\kappa(j)}}\lambda_{\kappa(j)})$ on this edge for the lattice N_0'. We say that a two dimensional cone σ is on the *left* (resp. on the *right*) of the vector $d_{\lambda_{\kappa(j)}} \in \sigma$ if $\sigma \subset \langle d_{\lambda_{\kappa(j)}}, u_2\rangle$ (resp. $\sigma \subset \langle u_1, d_{\lambda_{\kappa(j)}}\rangle$).

By proposition 3.22, the divisor $D_{\lambda_{\kappa(j)}}$ meets the strict transform of S by π_{Σ_1}.

If $\lambda_{\kappa(r)} > \lambda_{\kappa(j)}$ (resp. if $\lambda_{\kappa(j)} > \lambda_{\kappa(1)}$) then there exist a two dimensional cone on the left (resp. right) of $d_{\lambda_{\kappa(j)}}$ in $\mathcal{G}(p,f)$, obtained from the minimal regular subdivision of the cone $\sigma \in \mathcal{G}(\pi,f)$, on the right (resp. on the left) of $d_{\lambda_{\kappa(j)}}$. Therefore if $\lambda_{\kappa(r)} > \lambda_{\kappa(j)} > \lambda_{\kappa(1)}$ we have $\delta(d_{\lambda_{\kappa(j)}}) \geq 3$.

If $\lambda_{\kappa(r)} = +\infty$ then Y divides f and $Y=0$ is a component of the strict transform of S by π_{Σ_1}. If $\lambda_{\kappa(r)} \neq +\infty$ we two cases may occur: a) if the cone $\sigma = \langle d_{\lambda_{\kappa(r)}}, u_2\rangle$ is not regular we have a two dimensional cone in $\mathcal{G}(\pi,f)$ on the left of $d_{\lambda_{\kappa(r)}}$; b) the cone σ is regular thus $\lambda_{\kappa(r)} \in M$. By the proof of lemma 3.15 there exist $i \neq r$ such that $\lambda_{\kappa(i)} = \lambda_{(i,r)} = \lambda_{\kappa(r)}$. By proposition 3.22 this implies that the strict transforms of $f^{(i)} = 0$ and $f^{(r)} = 0$ meet the divisor $D_{\lambda_{\kappa(r)}}$ in two different points so that we have $\delta(d_{\lambda_{\kappa(r)}}) \geq 3$.

Now we deal with the divisor $D_{\lambda_{\kappa(1)}}$. The cone $\langle u_1, d_{\lambda_{\kappa(1)}} \rangle$ belongs to $\mathcal{G}(\pi, Xf)$ and we deduce from this that $\delta(d_{\lambda_{\kappa(1)}}) \geq 3$ in $\mathcal{G}(\pi, Xf)$. If the cone $\langle u_1, d_{\lambda_{\kappa(1)}} \rangle$ is not regular we can argue as before to show the existence of a two dimensional cone of $\mathcal{G}(\pi, f)$ on the right of $d_{\lambda_{\kappa(1)}}$. Otherwise we have $n_{\lambda_{\kappa(1)}} \lambda_{\kappa(1)} = 1$ and if $\lambda_{\kappa(1)}^{-1} \notin \mathbb{Z}_{>1}$ the only possibility is $d_{\lambda_{\kappa(1)}} = (1,1)$. Then we have $\lambda_{\kappa(1)} \in M$ and thus $\lambda_{\kappa(r)} = \lambda_{\kappa(1)}$ by lemma 3.15. This case has already been solved.

These facts give the assertion for e corresponding to an exceptional divisor of π_{Σ_1}. When we iterate, the curve $X = 0$ corresponds to the equation of the exceptional divisor meeting the strict transform thus after the first step we are always in the case *1* and proposition follows. ◇

An exceptional divisor D of the resolution p is *collapsible* if it has *self-intersection number* equal to -1 and the corresponding edge $d \in \mathcal{G}(p, 0)$ has valency ≤ 2 in $\mathcal{G}(p, f)$. If the divisor D is collapsible, the modification obtained by blowing down D is still a resolution and the corresponding resolution graph is obtained from $\mathcal{G}(p, 0)$ by deleting the point corresponding to D. The self intersection of the divisors which are images of compact divisors meeting D is increased by one. In a finite number of steps we obtain a *minimal resolution*, i.e., a resolution in which no exceptional divisor is collapsible. The minimal resolution is unique up to isomorphism (see [Lau]).

Corollary 3.3. *If $\lambda_{\kappa(1)}^{-1} \notin \mathbb{Z}_{>1}$, in particular if the projection $(X, Y) \mapsto X$ is transversal for all the components of f then the morphism p is the minimal resolution.*

Proof. The self intersection numbers of the exceptional divisors of the minimal resolution of a toric surface singularity are ≤ -2 (see proposition 1.19 of [Od]). This implies that the exceptional divisors corresponding to edges in $\mathcal{G}(p, f) - \mathcal{G}(\pi, f)$ are not collapsible. Then the corollary follows from lemma 3.26. ◇

Remark 3.27. *The number of local toroidal morphisms used in the partial resolution π is not necessarily equal to the complexity of the resolution (as defined by [Le-Ok]).*

For instance $f = ((Y - X)^2 - X^3))((Y + X)^2 - X^5))$ has characteristic exponents $\{1, \frac{3}{2}, \frac{5}{2}\}$. The projection $(X, Y) \mapsto X$ is transversal for the two irreducible components. It follows easily that the number of local toroidal morphisms used to define the partial resolution in this coordinates is three. On the other hand, the resolution graph is a bamboo so that the resolution complexity is equal to one.

3.4 The semigroup associated to a toric quasi-ordinary branch

We associate to the quasi-ordinary branch ζ a semigroup Γ which is determined from the characteristic exponents; the construction of Γ involves also a generalization of the notion of the plane curves with maximal contact with a given branch given by Lejeune [LJ2] and this relation can be described by using the *approximate roots* of the polynomial f. The main part of the results and the proofs of this section is given in [GP3].

In the following sections we study a fixed toric quasi-ordinary singularity S parametrized by a toric quasi-ordinary branch $\zeta \in \mathbb{C}\{\rho^\vee \cap \frac{1}{n}M\}$ with $g \geq 1$ characteristic exponents $\{\lambda_1, \dots, \lambda_g\}$ and with minimal polynomial $f \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$. If $\mathrm{rk}\, M = 1$ then the singularity S is a plane branch and the set of intersection multiplicities $(S, S')_0$ of S which those plane curve germs S' not containing S as a component forms a sub-semigroup of $(\mathbb{Z}_{\geq 0}, +)$ which is an invariant of the germ S and which is generated by the following elements (see [Z8]):

$$\bar{\gamma}_1 = n\lambda_1, \quad \bar{\gamma}_{j+1} = n_j \bar{\gamma}_j + n\lambda_{j+1} - n\lambda_j, \quad \text{for } j = 1, \dots, g-1. \tag{3.19}$$

For $j = 0, \dots, g-1$, we expand:

$$\begin{aligned} \bar{\gamma}_{j+1} &= n\left((n_1 - 1)n_2 \cdots n_j \lambda_1 + (n_2 - 1)n_3 \cdots n_j \lambda_2 + \cdots + (n_j - 1)\lambda_j + \lambda_{j+1}\right) \\ &\overset{\text{Def. 3.7}}{=} n_1 \dots n_j \left((e_0 - e_1)\lambda_1 + (e_1 - e_2)\lambda_2 + \cdots + (e_{j-1} - e_j)\lambda_j + e_j \lambda_{j+1}\right) \end{aligned} \tag{3.20}$$

We denote $\frac{1}{n}\bar{\gamma}_i$ by γ_i for $i = 1, \dots, g$ and we have:

$$\gamma_1 = \lambda_1, \gamma_{j+1} = n_j \gamma_j + \lambda_{j+1} - \lambda_j, \quad \text{for } j = 1, \dots, g-1. \tag{3.21}$$

Definition 3.10. *We associate to the quasi-ordinary branch* ζ *the sequence of semigroups* $\Gamma_j = \rho^\vee \cap M + \gamma_1 \mathbb{Z}_{\geq 0} + \cdots + \gamma_g \mathbb{Z}_{\geq 0}$ *for* $j = 0, \dots, g$.

We denote Γ_g by Γ and $n\Gamma_j$ by $\bar{\Gamma}_j$ for $j = 0, \dots, g$. The classical semigroup of a plane branch is $\bar{\Gamma}_g$.

If ζ is a *classical* quasi-ordinary branch suitably *normalized*[1]. Lipman proved that the sequence of characteristic exponents is an analytical invariant of the germ it parametrizes when $\dim S = 2$, by building a (non embedded) resolution of the germ (see [L1], [L3]) which determines the characteristic exponents. Luengo gives another proof also using resolutions (see [Lu]). If the germ is analytically irreducible the characteristic exponents define a complete invariant of the embedded topological type of the hypersurface $S \subset \mathbb{C}^{d+1}$ it parametrizes (see [Gau] and [L4]). We proved in [GP2] that if τ and ζ are quasi-ordinary branches parametrizing S then the semigroups associated to them are isomorphic and moreover that the minimal set of generators of this semigroup defines the sequence of characteristic exponents of any normalized quasi-ordinary branch parametrizing S. By Gau's characterization it follows that the semigroup Γ defined above is a complete topological invariant of the embedded topological type of germ $(S, 0)$.

The following lemma generalizes the properties of the semigroups of plane branches (see [T4], Chapitre I, Lemma 2.2.1) to the quasi-ordinary hypersurface case (see [GP2]).

---

[1] In the case of a plane branch this condition means that $X = 0$ is not contained in its the tangent cone

Lemma 3.28. *(See [GP3])*

1. *The sub-lattice of* M *generated by* Γ_j *is equal to* M_j, *for* $0 \leq j \leq g$.
2. *The order of the image of* γ_j *in the group* M_j/M_{j-1} *is equal to* n_j *for* $j = 1, \dots, g$.
3. *We have that* $\gamma_j > n_{j-1}\gamma_{j-1}$ *for* $j = 2, \dots, g$.
4. *If a vector* $u_j \in \rho^\vee \cap M_j$ *then we have* $u_j + n_j\gamma_j \in \Gamma_j$.
5. *The vector* $n_j\gamma_j$ *belongs to the semigroup* Γ_{j-1} *for* $j = 1, \dots, g$, *moreover we have a unique relation:*
$$n_j\gamma_j = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \cdots + l_{j-1}^{(j)}\gamma_{j-1} \tag{3.22}$$
such that $0 \leq l_i^{(j)} \leq n_i - 1$ *and* $\alpha^{(j)} \in M_0$, *for* $j = 1, \dots, g$.

In the plane branch case several authors have studied the properties of those curves S' such that the intersection multiplicity with S at the origin belongs to the unique minimal set of generators of the semigroup of the branch (see [Z8]). Lejeune introduced the notion of curves of maximal contact with a given plane curve germ for curves defined over a field of arbitrary characteristic in terms of the resolution (see [LJ2]). If the characteristic is zero it turns out that both notions are equivalent (see [Ca]). If the projection (X, Y) is transversal we can study these curves by means of the minimal polynomials of suitable truncations of the roots of f. When we do this with respect to an arbitrary projection, the curves we obtain provide a non necessarily minimal set of generators of the semigroup of the branch S. These curves can be represented by some of the *approximate roots* of the polynomial f (see [A-M]) and we call them *semi-roots.* following the terminology of [A4]. See Popescu-Pampu's survey [PP1] for more on the notion of semi-root.

Definition 3.11. *A* j^{th}*-semi-root of* f *is an irreducible quasi-ordinary polynomial in* $\mathbb{C}\{\rho^\vee \cap M\}[Y]$ *of degree* $n_0 \dots n_j$ *which has order of coincidence equal to* λ_{j+1} *with* f, *for* $j = 0, \dots, g$.

The minimal polynomials of the quasi-ordinary branches $p_0 + \cdots + p_j$ obtained by truncating ζ in remark 3.13 are j^{th}-semi-roots of f for $j = 0, \dots, g$.

Proposition 3.29. *(see [GP2] and [GP3]) Let* $q \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ *a monic polynomial of degree* $n_0 \dots n_j$. *Then* q *is a* j*-semi-root of* f *if and only if* $q(\zeta) = X^{\gamma_{j+1}}\varepsilon_j$ *for a unit* ε_j.

The notion of semi-root extends the properties of maximal contact with respect to the resolution to the quasi-ordinary case (see the proof of theorem 3.2 and remark 3.20).

Remark 3.30. *The polynomial* q_j *is a* j*-semi-root of* f *is and only if the strict transform of* $q_j = 0$ *by the morphism* $\pi_j \circ \cdots \circ \pi_1$ *is a germ defined by a good coordinate and conversely.*

This follows from the proof of theorem 3.2 and remark 3.20.

Let A a ring containing $\mathbb{Q}$ as a subring. *Approximate roots* are defined by Abhyankar and Moh, (see [A-M], [G-P], and [PP1]). If p is any monic polynomial and k divides the degree of p there is a unique monic polynomial r in $A[Y]$ of degree $\frac{\deg p}{k}$ such that $\deg(p - r^k) < \deg p - \frac{\deg p}{k}$. We say that r is a k-semi-root of p. We can use proposition 3.29 to prove that the e_j-approximate roots of a quasi-ordinary polynomial f are semi-roots, and therefore are irreducible quasi-ordinary polynomials with a prescribed order of coincidence with the polynomial f (see [GP2] and [GP3]).

3.4.1 Expansion in terms of semi-roots

The expansions in terms of semi-roots are introduce by Abhyankar in the plane curve case (see [A4]) and used by Popescu-Pampu in the case of a quasi-ordinary hypersurface singularity (see [PP2]).

We fix from now on a *complete set* $q_0, \dots, q_g$ *of semi-roots of* f ($\deg q_i = n_0 \cdots n_i$ for $i = 0, \dots, g$). We assume that the coefficient of the term $X^{\gamma_{j+1}}$ appearing in $q_j(\zeta)$ by proposition 3.29 is equal to one for $j = 0, \dots, g-1$ in order to simplify some computations.

We recall now the classical q-adic expansion of a polynomial $p_0 \in A[Y]$ with coefficients on a domain A in terms of a polynomial $q \in A[Y]$ having invertible leading term (see [Z8]). The sequence of Euclidean divisions:

$$p_0 = p_1 q + a_0, \quad p_1 = p_2 q + a_1, \quad \dots, \quad p_s = p_{s+1} q + a_s,$$

(where s is the first integer for which $p_{s+1} = 0$) provides a unique decomposition of the form

$$H = a_0 + a_1 q + a_2 q^2 + \cdots + a_s q_s^s, \text{ for } 0 \leq \deg a_i \leq \deg q - 1.$$

Lemma 3.31. *(see [PP2]) Any polynomial $h \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ can be written in a unique way as*

$$h = \sum c_{l_1,\dots,l_{g+1}} q_0^{l_1} q_1^{l_2} \cdots q_g^{l_{g+1}} \tag{3.23}$$

with $c_{l_1,\cdots,l_{g+1}} \in \mathbb{C}\{\rho^\vee \cap M\}$, $0 \leq l_k \leq n_k - 1$ for $k = 1, \dots, g$ and $l_{g+1} \in \mathbb{Z}_{\geq 0}$.

If $c_{l_1,\dots,l_g,0}$ and $c_{l'_1,\dots,l'_g,0}$ are two different coefficients of the expansion the Newton principal parts of $c_{l_1,\cdots,l_g,0} q_0^{l_1}(\zeta) \cdots q_{g-1}^{l_g}(\zeta)$ and $c_{l'_1,\cdots,l'_g,0} q_0^{l'_1}(\zeta) \cdots q_{g-1}^{l'_g}(\zeta)$ (viewed in the ring $\mathbb{C}\{\rho^\vee \cap M_g\}$) have no term in common.

Proof. The q_g-adic expansion of h is of the form: $h = a_0^{(g)} + a_1^{(g)} q_g + \cdots + a_{s_g}^{(g)} q_g^{s_g}$. We build the q_{g-1}-adic expansions of the coefficients:

$$a_j^{(g)} = a_{0,j}^{(g-1)} + a_{1,j}^{(g-1)} q_{g-1} + \cdots + a_{s_{g-1},j}^{(g-1)} q_{g-1}^{s_{g-1}}$$

where $0 \leq \deg a_{l,j}^{(g-1)} \leq n_0 \cdots n_{g-1} - 1$ for $0 \leq l \leq s_{g-1}$ and $0 \leq s_{g-1} \leq n_g - 1$ since $a_j^{(g)}$ is of degree $< n_0 \cdots n_g = n$. An expansion satisfying the required properties is obtained by iterating this procedure. The unicity follows from the unicity of Euclidean division. For the last assertion, remark that by lemma 3.29 the Newton principal part of $q_{k-1}(\zeta)$ (viewed in $\mathbb{C}\{\rho^\vee \cap M_g\}$) is equal to X^{γ_k} for $k = 1, \dots, g$. It follows from *2* in lemma 3.28 that the Newton principal parts of $c_{l_1,\dots,l_g,0} q_0^{l_1}(\zeta) \cdots q_{g-1}^{l_g}(\zeta)$ and of $c_{l'_1,\dots,l'_g,0} q_0^{l'_1}(\zeta) \cdots q_{g-1}^{l'_g}(\zeta)$ do not have any term in common if $(l_1, \dots, l_g) \neq (l'_1, \dots, l'_g)$. ◇

The following proposition (see [GP2]) generalizes [Z8], Chapitre II, Th. 3.9. in the plane branch case.

Proposition 3.32. *If $h \in \mathbb{C}\{\rho^\vee \cap M\}[Y]$ is of degree $< n_0 n_1 \dots n_j$ then the Newton principal part of $h(\zeta)$ belongs to $\mathbb{C}[\Gamma_j]$, for $j = 1, \dots, g$.*

Proof. The result is trivial if $\deg h = 0$. If $\deg h < n_1 \dots n_j$ then the $(q_0, \dots, q_g)$-expansion of h is of the form: $h = \sum c_{l_1,\dots,l_j} q_0^{l_1} q_1^{l_2} \cdots q_{j-1}^{l_j}$. By lemma 3.31 the Newton principal parts of $c_{l_1,\dots,l_j} q_0^{l_1}(\zeta) \cdots q_{j-1}^{l_j}(\zeta)$ and of $c_{l'_1,\dots,l'_j} q_0^{l'_1}(\zeta) \cdots q_{j-1}^{l'_j}(\zeta)$ do not have terms in common, thus the polynomial $h(\zeta)_{|\mathcal{N}}$ is a sum of some of the terms in the Newton principal parts of the summands $c_{l_1,\dots,l_j} q_0^{l_1}(\zeta) \cdots q_{j-1}^{l_j}(\zeta)$ and therefore it belongs to the ring $\mathbb{C}[\Gamma_j]$ by proposition 3.29, for $j = 1, \dots, g$. ◇

We call the expansion (3.23) above the $(q_0, \dots, q_g)$*-expansion of* h.

Lemma 3.33. *The $(q_0, \dots, q_g)$-expansion of $q_{j-1}^{n_j}$ is of the following form, for $1 \leq j \leq g$:*

$$q_{j-1}^{n_j} = c_j^* q_j + \sum c_{l_1,\dots,l_j}^{(j)} q_0^{l_1} q_1^{l_2} \cdots q_{j-1}^{l_j} \tag{3.24}$$

where $c_j^* \in \mathbb{C}^*$, the other coefficients belong to $\mathbb{C}\{\rho^\vee \cap M\}$, we have $0 \leq l_k \leq n_{k+1} - 1$ for $k = 0, \dots, j-1$. The coefficient $c^{(j)}_{l_1^{(j)},\dots,l_{j-2}^{(j)},0}$ appears and it is of the form $X^{\alpha^{(j)}} \cdot$ unit where the integers $l_1^{(j)}, \dots, l_{j-2}^{(j)}$ and the exponent $\alpha^{(j)}$ are given by formula (3.22). Moreover, if $X^{\alpha'}$ appears on the coefficient $c^{(j)}_{l_1,\dots,l_j}$ then

$$n_j \gamma_j \leq_\rho \alpha' + l_1 \gamma_1 + \cdots + l_j \gamma_j \tag{3.25}$$

and equality holds if and only if $(l_1, \dots, l_j) = (l_1^{(j)}, \dots, l_{j-2}^{(j)}, 0)$ and $\alpha' = \alpha^{(j)}$.

Proof. Since $\deg q_{j-1}^{n_j} = n_1 \cdots n_j$ the algorithm to calculate the $(q_0, \dots, q_g)$-expansion begins by dividing $q_{j-1}^{n_j}$ by q_j. This gives $q_{j-1}^{n_j} = c_{j+1}^* q_{j+1} + r_j$, where $c_{j+1}^* \in \mathbb{C}^*$ since both polynomials have the same degree. The q_k that may appear in the expansion of r_j are those of degree $\leq \deg r_j < n_1 \cdots n_j$. We deduce from the second assertion of lemma 3.31 that

$$\mathcal{N}(c^{(j)}_{l_1,\dots,l_j} q_0^{l_1}(\zeta) \cdots q_{j-1}^{l_{j-1}}(\zeta)) \subset n_j \gamma_j + \rho^\vee = \mathcal{N}(q_{j-1}^{n_j}(\zeta)).$$

This implies that if $X^{\alpha'}$ appears on the coefficient $c^{(j)}_{l_1,\dots,l_j}$ then formula (3.25) holds. If equality in (3.25) holds for a term the term $X^{\alpha'}$ appearing on the series $c^{(j)}_{s_1^{(j)},\dots,s_j^{(j)}}(\zeta)$ it follows that the series is the form $X^{\alpha'} \cdot \text{unit}$. Assertion *2* of lemma 3.28 implies that $s_j^{(j)} = 0$ in the relation $n_j \gamma_j = \alpha_j + l_1\gamma_1 + \cdots + l_j\gamma_j$. Then it follows that $(l_1, \dots, l_{j-2}) = (l_1^{(j)}, \dots, l_{j-2}^{(j)})$ and that $\alpha' = \alpha^{(j)}$ by unicity in (3.22). ◇

3.5 Partial embedded resolution with one toric morphism

In this section we build a partial embedded resolution of the toric quasi-ordinary germ embedded in an affine toric variety by using the semi-roots. We follow the approach of [G-T] for irreducible germs of plane curves.

We denote by Δ the cone $\rho \oplus \mathbb{R}^g_{\geq 0} \subset (N_\Delta)_\mathbb{R}$ where N_Δ is the lattice $N \oplus \mathbb{Z}^g$ with dual lattice M_Δ. We denote by $u_1, \dots, u_g$ the canonical basis of $\{0\} \oplus \mathbb{Z}^g$. An element of $\Delta^\vee \cap M_\Delta$ is of the form (α, v) where $\alpha \in \rho^\vee \cap M$ and $v = v_1 u_1^* + \cdots v_g u_g^*$ where $u_1^*, \dots, u_g^*$ is the dual basis of $u_1, \dots, u_g$ and $v_i \in \mathbb{Z}_{\geq 0}$. We denote the monomial corresponding to (α, v) by $X^\alpha U^v$, $X^\alpha U_1^{v_1} \dots U_g^{v_g}$ or $X^{\alpha + \sum v_i u_i^*}$ depending on the context.

The embedding $S \subset Z_\Delta$ which is studied in this section corresponds algebraically to the homomorphism of $\mathbb{C}\{\rho^\vee \cap M\}$-algebras:

$$\Psi_0 : \mathbb{C}\{\rho^\vee \cap M\}[U_1, \dots, U_g] \to R \text{ given by } U_j \mapsto q_{j-1}(\zeta), \text{ for } j = 1, \dots, g \tag{3.26}$$

(which is surjective since in particular $R = \mathbb{C}\{\rho^\vee \cap M\}[q_0(\zeta)]$).

In the plane branch case Teissier shows that this embedding specializes to the *monomial curve*, an affine curve monomially embedded with the same semigroup (see [T4]). In the general case the generalization of monomial curve is given by an equivariant embedding $Z^\Gamma \subset Z_\Delta$ which is defined from the restriction of the lattice homomorphism

$$\varphi : M_\Delta \to M_g \text{ that maps } \alpha + v \mapsto \alpha + v_1\gamma_1 + \cdots + v_g\gamma_g \tag{3.27}$$

to the semigroup $\Delta^\vee \cap M_\Delta$ and its image Γ.

3.5.1 Specialization through graded rings

In the plane branch case the embedding of the monomial curve is determined by a system of generators the *graded ring*[2] associated to the *filtration* of R induced by the powers of the maximal ideal of its integral closure (see [T4]). In our case we show that the homomorphism Ψ_0 can be filtered in such a way that the homomorphism of the associated graded rings, forgetting the graded structure, defines the embedding $Z^\Gamma \subset Z_\Delta$ above.

---

[2] See [Bbk] for the definitions and properties of commutative algebra used in the following sections.

The filtration of the ring $\mathbb{C}\{\rho^\vee \cap M\}$ (resp. of $\mathbb{C}[[\rho^\vee \cap M]]$) defined by a vector $\eta \in \rho$ is given by the ideals:

$$\mathcal{I}_j = \left\{ \sum_{u \in \rho^\vee \cap M} c_u X^u \, / \, \min_{c_u \neq 0} \langle \eta, u \rangle \geq j \right\} \text{ for } j \in \eta(\rho^\vee \cap M).$$

Since the ring $\mathbb{C}\{\rho^\vee \cap M\}$ is Noetherian the ordered sub-semigroup $\eta(\rho^\vee \cap M)$ of $\mathbb{R}_{\geq 0}$ is isomorphic to $\mathbb{Z}_{\geq 0}$ (see the proof of Lemme 1.4 of [GP1]). The vector η defines a *weighted filtration* of $\mathbb{C}\{\rho^\vee \cap M\}[U_1, \dots, U_g]$ (resp. of $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$ or $\mathbb{C}[[\Delta^\vee \cap M_\Delta]]$) given by the ideals $\mathcal{J}_j$ generated by those series having only terms $X^\alpha U^v$ of *weights* $w := \varphi(\alpha, v)$ such that $\langle \eta, w \rangle \geq j$, for j running through the semigroup $\eta(\rho^\vee \cap M_g)$. The homomorphism Ψ_0 is *filtered* since $\Psi_0(\mathcal{J}_k) \subset \mathcal{I}_k$ for all $k \in \eta(\rho^\vee \cap M_g)$, and then it defines an homomorphism of the associated graded rings.

Proposition 3.34. *The sequence of graded ring homomorphisms associated to the filtered sequence of homomorphisms (with the filtrations defined by* $\eta \in \overset{\circ}{\rho}$*)*

$$\mathbb{C}\{\rho^\vee \cap M\}[U_1, \dots, U_g] \xrightarrow{\Psi_0} R \hookrightarrow \mathbb{C}\{\rho^\vee \cap M_g\} \tag{3.28}$$

is isomorphic to

$$\mathbb{C}[\rho^\vee \cap M][U_1, \dots, U_g] \longrightarrow \mathbb{C}[\Gamma] \hookrightarrow \mathbb{C}[\rho^\vee \cap M_g]$$

where the first homomorphism is defined by $X^\alpha U^v \mapsto X^{\varphi(\alpha, v)}$*, and the graduations are defined by* η*. If the vector* η *is irrational the semigroup* Γ *is determined by the graduation.*

Proof. If $\eta \in \overset{\circ}{\rho}$ the symbolic restriction $\phi_{|\eta}$ of $\phi \in \mathbb{C}\{\rho^\vee \cap M_g\}$ to the face defined by η on the polyhedron $\mathcal{N}_\rho(\phi)$ belongs to $\mathbb{C}[\rho^\vee \cap M_g]$ since this face is compact. If $\phi \in \mathbb{C}\{\rho^\vee \cap M_g\}$ there exists a unique integer k such that $\phi \in \mathcal{I}_k - \mathcal{I}_{k+1}$ and then we have $\phi = \phi_{|\eta} \mod \mathcal{I}_{k+1}$. It follows from the property: $\phi_{|\eta} \phi'_{|\eta} = (\phi\phi')_{|\eta}$ for $0 \neq \phi, \phi' \in \mathbb{C}\{\rho^\vee \cap M_g\}$, that the graded ring associated to this filtration is isomorphic to the graded ring $\mathbb{C}[\rho^\vee \cap M_g]$ where the j-homogeneous term of the graduation is $\bigoplus_{\langle \eta, u \rangle = j} \mathbb{C} X^u$ for $j \in \eta(\rho^\vee \cap M_g)$. We deduce analogously that the graded ring associated to the weighted filtration is isomorphic to $\mathbb{C}[\Delta^\vee \cap M_\Delta]$ where the non zero elements in the j-homogeneous term are those polynomials such that $\langle \eta, w \rangle = j$ for w running through the weights of the monomials appearing on them.

Under these identifications we have that:

- The graded ring associated to R with the induced filtration is isomorphic to the graded subring of $\mathbb{C}[\rho^\vee \cap M_g]$ generated as a $\mathbb{C}$-algebra by the symbolic restrictions $\phi_{|\eta}$ of $0 \neq \phi \in R$ to the face defined by η on the polyhedron $\mathcal{N}(\phi)$. We deduce from proposition 3.32 and proposition 3.29 that this graded subring is equal to $\mathbb{C}[\Gamma]$.

- The initial term of $\Psi_0(U_i) = q_{i-1}(\zeta)$ is equal to X^{γ_i} (the coefficient has been normalized to be one) thus the homomorphism $\operatorname{gr}(\Psi_0)$ corresponds to the $\mathbb{C}[\rho^\vee \cap M]$-homomorphism $\mathbb{C}[\rho^\vee \cap M][U_1, \dots, U_g] \to \mathbb{C}[\Gamma]$ that maps $U_i \mapsto X^{\gamma_i}$ for $i = 1, \dots, g$.

If the vector η is *irrational* we can recover the semigroup $\rho^\vee \cap M_g$ (resp. Γ) from the graduation of $\mathbb{C}[\rho^\vee \cap M]$ (resp. of $\mathbb{C}[\Gamma]$) since each term of the graduation is of dimension one (resp. zero or one) over $\mathbb{C}$, the vector η defining a total ordering on $\rho^\vee \cap M_g$. ◇

Remark 3.35. *The sequence of homomorphisms (3.28) extends to the sequences:*

$$\begin{tikzcd}
\mathbb{C}[[\Delta^\vee \cap M_\Delta]] \arrow[r, "\hat{\Psi}"] & \hat{R} \arrow[r, hook] & \mathbb{C}[[\rho^\vee \cap M_g]] \\
\mathbb{C}\{\Delta^\vee \cap M_\Delta\} \arrow[r, "\Psi"] \arrow[u] & R \arrow[r, hook] \arrow[u] & \mathbb{C}\{\rho^\vee \cap M_g\} \arrow[u]
\end{tikzcd} \tag{3.29}$$

where $\hat{R}$ denotes the completion of the ring R with respect to the maximal ideal $\mathfrak{M}_R$. The assertion of proposition 3.34 remains true for each line of the above diagram.

We notice that $\hat{R}$ coincides with the completion with respect to the filtration defined by η: we have that $\mathfrak{M}_R^{s_j} \subset \mathcal{I}_j$ where s_j is the minimal power of $\mathfrak{M}_R$ containing the set of monomials in $\mathcal{I}_j - \mathcal{I}_{j+1}$ which is finite since $\eta \in \overset{\circ}{\rho}$.

3.5.2 Equations for the embeddings

We build equations of the embeddings of $Z^\Gamma \subset Z_\Delta$ and $S \subset Z_\Delta$.

Proposition 3.36. *The ideal of the embedding $Z^\Gamma \subset Z_\Delta$ is generated by the binomials*

$$h_1 = -U_1^{n_1} + X^{\alpha^{(1)}}, \quad h_2 = -U_2^{n_2} + X^{\alpha^{(2)}} U_1^{l_1^{(2)}}, \quad \ldots, \quad h_g = -U_g^{n_g} + X^{\alpha^{(g)}} U_1^{l_1^{(g)}} \ldots U_{g-1}^{l_{g-1}^{(g)}}, \tag{3.30}$$

which correspond to relations (3.22).

Proof. The ideal I of the embedding $Z^\Gamma \subset Z_\Delta$ is generated by the binomials $X^\alpha U^\omega - X^{\alpha'} U^{\omega'}$ of $\mathbb{C}[\Delta^\vee \cap M_\Delta]$ verifying (see (3.5)):

$$\varphi(\alpha, \omega) = \varphi(\alpha', \omega') \tag{3.31}$$

The binomials $h_1, \ldots, h_g$ above verify this condition by lemma 3.28. If B is a binomial in I, we can factor the common term in U_g to obtain a binomial in I of the form $X^\alpha U^\omega - X^{\alpha'} U^{\omega'}$ with $w'_g = 0$. Then the integer ω_g is a multiple of n_g (since $n_g \gamma_g \in M_{g-1}$ by lemma 3.28 we obtain from the equality (3.31) a relation $r\gamma_g \in M_{g-1}$ where r is the reminder of the Euclidean division of ω_g by n_g and then lemma 3.28 implies that $r = 0$). We can show by induction on ω_g / n_g that the reminder of the Euclidean division of $X^\alpha U^\omega - X^{\alpha'} U^{\omega'}$ by h_g as polynomials in U_g is a binomial B_1 in $\mathbb{C}[\rho^\vee \cap M][U_1, \ldots, U_{g-1}]$. The binomial B_g obtained by iterating this procedure belongs to $\mathbb{C}[\rho^\vee \cap M]$ and to the ideal I. The relation (3.31) corresponding to B_g is trivial since the homomorphism φ is injective on M thus $B_g = 0$. This implies that the ideal I is generated by $h_1, \ldots, h_g$. ◇

Proposition 3.37. *The ideal of the embedding* $S \subset Z_\Delta$ *defined by (3.26) is generated by the following elements of the ring* $\mathbb{C}\{\rho^\vee \cap M\}[U_1, \dots, U_g]$

$$H_j := -U_j^{n_j} + c_j^* U_{j+1} + c^{(j)}_{l_1^{(j)},\dots,l_{j-1}^{(j)},0} U_1^{l_1^{(j)}} \cdots U_{j-1}^{l_{j-1}^{(j)}} + \sum_{(l_1,\dots,l_j)\neq(l_1^{(j)},\dots,l_{j-1}^{(j)},0)} c^{(j)}_{l_1,\dots,l_j} U_1^{l_1} U_2^{l_2} \cdots U_j^{l_j}, \qquad (3.32)$$

for $j = 1, \dots, g$ *where the coefficients and the exponents are those of formula (3.24) and* $U_{g+1} = 0$. *The weight of a term* $X^\alpha U_1^{l_1} U_2^{l_2} \cdots U_j^{l_j}$ *appearing in the expansion of* H_j *is* $\geq n_j \gamma_j$ *and equality holds only for* $U_j^{n_j}$ *and* $X^{\alpha^{(j)}} U_1^{l_1^{(j)}} \cdots U_{j-1}^{l_{j-1}^{(j)}}$ *for* $j = 1, \dots, g$.

Proof. It follows from the definition of the homomorphism Ψ_0 and formula (3.24) that the polynomials H_i above belong to the kernel of Ψ_0 (and then to the kernels of Ψ and $\hat{\Psi}$). By proposition 3.36 and lemma 3.33 their initial forms with respect to the filtration defined by $\eta \in \overset{\circ}{\rho}$ generate $\mathrm{Ker}(\mathrm{gr}(\hat{\Psi}))$. Then we have that $\mathrm{gr}(\mathrm{Ker}(\hat{\Psi})) = \mathrm{Ker}(\mathrm{gr}(\hat{\Psi}))$. We deduce using that the ideal $\mathrm{Ker}(\hat{\Psi})$ is complete for the induced filtration, that the polynomials $H_1, \dots, H_g$ generate $\mathrm{Ker}(\hat{\Psi})$[3].

Since the inclusion $\mathbb{C}\{\Delta^\vee \cap M_\Delta\} \to \mathbb{C}[[\Delta^\vee \cap M_\Delta]]$ is an homomorphism of local rings continuous for the $\mathfrak{M}$-adic topologies which extends to the identity homomorphism between the respective completions, we have that the ring $\mathbb{C}[[\Delta^\vee \cap M_\Delta]]$ is a *faithfully flat* $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$-module[4]. The ideal J generated by $(H_1, \dots, H_g)$ on $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$ is contained in $\mathrm{Ker}(\Psi)$ and we have shown that $J\mathbb{C}[[\Delta^\vee \cap M_\Delta]] = \mathrm{Ker}(\hat{\Psi})$. The faithfully flat property implies that J coincides with the contraction of $\mathrm{Ker}(\hat{\Psi})$ in $\mathbb{C}\{\Delta^\vee \cap M_\Delta\}$[5]. Therefore we obtain that $J = \mathrm{Ker}(\Psi)$.

Let $\mathcal{U}$ be the subset of those elements in $\mathbb{C}\{\rho^\vee \cap M\}[U]$ with non zero constant term as power series. The image by Ψ_0 of a series in $\mathcal{U}$ is a unit. This implies that the localization $\mathcal{U}^{-1}\Psi_0 : \mathcal{U}^{-1}\mathbb{C}\{\rho^\vee \cap M\}[U] \to \mathbb{C}\{\rho^\vee \cap M_g\}$ is a well defined homomorphism. The same argument shows that $\mathrm{Ker}(\mathcal{U}^{-1}\Psi_0)$ is generated by $H_1, \dots, H_g$. Since $\mathcal{U} \cap \mathrm{Ker}\Psi_0 = \emptyset$ we deduce from the standard properties of the localization that $H_1, \dots, H_g$ generate $\mathrm{Ker}(\Psi_0)$. ◇

3.5.3 Simultaneous partial embedded resolution

We show that the partial embedded resolution of $Z^\Gamma \subset Z_\Delta$ built in proposition 3.6 is also a partial embedded resolution of $S \subset Z_\Delta$.

The linear subspace $\ell \subset (N_\Delta)_\mathbb{R}$ orthogonal to $\mathrm{Ker}(\varphi)$ is of dimension d and is also orthogonal to the Minkowski sum of compact edges of $\mathcal{N}(h_i)$ for $i = 1, \dots, g$.

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[3] See Proposition 12 No 9, §2, Chapitre III, of [Bbk].

[4] See Proposition 10 No 5, §3, Chapitre III of [Bbk].

[5] See Proposition 9 No 5, §3, Chapitre I of [Bbk].

Lemma 3.38. *Let Σ_0 be the smallest subdivision of Δ compatible with the Newton polyhedron of $H_1 \cdots H_g$. The cone $\sigma_0 = \Delta \cap \ell$ belongs to Σ_0. The strict transform S_{Σ_0} of S is defined on the chart Z_{σ_0} by the equations: $U_i^{-n_i} H_i = 0$ for $i = 1, \dots, g$. The intersection $S_{\Sigma_0} \cap \mathbf{O}_{\sigma_0}$ as schemes is reduced to the simple point o_{σ_0}. The germ $(S_{\Sigma_0}, o_{\sigma_0})$ is isomorphic to the germ of toric variety $Z_{\sigma_0, N_{\sigma_0}}$ at the distinguished point. If Σ is any subdivision of Δ containing the cone σ_0 and if $\sigma \in \Sigma$ with $\overset{\circ}{\sigma} \subset \overset{\circ}{\Delta}$ then $S_\Sigma \cap \mathbf{O}_\sigma \neq \emptyset$ implies that $\sigma = \sigma_0$. Moreover, if Σ' is a regular subdivision of Σ then the map $\pi_{\Sigma'} \circ \pi_\Sigma$ is an embedded pseudo-resolution of S.*

Proof. A vector $v \in \overset{\circ}{\sigma}_0$ vanish on $\mathrm{Ker}(\varphi_{\mathbb{R}})$ thus it is of the form $v = \tilde{v} \circ \varphi$ for $\tilde{v} \in N_g$ belonging to $\overset{\circ}{\rho}$ since $\tilde{v}$ vanishes only at the vertex of the cone $\rho^\vee$ (this follows from $\varphi_{\mathbb{R}}^{-1}(\rho^\vee) = \Delta^\vee + \mathrm{Ker}(\varphi_{\mathbb{R}})$ and $\overset{\circ}{\sigma}_0 \subset \overset{\circ}{\Delta}$). We deduce from this that the face defined by v on the polyhedron $\mathcal{N}(H_i)$ corresponds to the monomials of weight w such that $\langle \tilde{v}, w \rangle$ is minimal. By proposition 3.37 the symbolic restriction of H_i to this face is equal to h_i. Conversely, if h_i is the symbolic restriction of H_i to the face defined by v it follows that $v \in \ell$ and since these are compact faces we have that $v \in \overset{\circ}{\Delta}$ thus $v \in \overset{\circ}{\sigma}_0$.

The common zero locus S' of the functions $U_i^{-n_i} H_i$ for $i = 1, \dots, g$ on the chart Z_{σ_0} contains $S_{\Sigma_0} \cap Z_{\sigma_0}$. Then we deduce from the proof of lemma 3.3 that:

$$U_i^{-n_i} H_i = U_i^{-n_i} h_i + \text{ terms vanishing on the orbit } \mathbf{O}_{\sigma_0}, \text{ for } i = 1, \dots, g. \tag{3.33}$$

Since the equations $U_i^{-n_i} h_i = 0$ for $i = 1, \dots, g$, define on the chart Z_{σ_0} the strict transform $Z^\Gamma_{\Sigma_0}$ we deduce from (3.33) above that $S' \cap \mathbf{O}_{\sigma_0}$ coincides as schemes intersection with $Z^\Gamma_{\Sigma_0} \cap \mathbf{O}_{\sigma_0}$, thus it is equal to the simple point o_{σ_0} by lemma 3.5. If the germ (S', o_{σ_0}) is analytically irreducible it must coincide with the sub-germ $(S_{\Sigma_0}, o_{\sigma_0})$ since both are of the same dimension. We show this fact by proving that (S', o_{σ_0}) is isomorphic to $(Z_{\sigma_0, N_{\sigma_0}}, o_{\sigma_0})$:

We notice that the chart Z_{σ_0} is isomorphic to $\mathbf{O}_{\sigma_0} \times Z_{\sigma_0, N_{\sigma_0}}$ by (3.1). The binomials $W_i := U_i^{-n_i} h_i$ for $i = 1, \dots, g$, define a regular system of parameters at the point o_{σ_0} of the orbit $\mathbf{O}_{\sigma_0}$ therefore we can apply lemma 3.4 to the equations (3.33) to show the existence of $\phi_1, \dots, \phi_g \in \mathbb{C}\{\rho^\vee \cap M_g\}$ such that the germ (S', o_{σ_0}) is given by $W_i = \phi_i$ for $i = 1, \dots, g$.

Let Σ be any subdivision of Σ_0 containing the cone σ_0. The restriction $\pi : S_\Sigma \to S$ of π_Σ is a modification and since (S, o_Δ) is analytically irreducible the exceptional fiber is connected by the Main Theorem of Zariski (see [Mu] and [Z2]). On the other hand we have that

$$\pi^{-1}(o_\Delta) = \bigcup_{\sigma \in \Sigma, \overset{\circ}{\sigma} \subset \overset{\circ}{\Delta}} (S_\Sigma \cap \mathbf{O}_\sigma) \text{ (by (3.3))};$$

and we have shown that on the open set $S_\Sigma \cap Z_{\sigma_0}$ of S_Σ the exceptional fiber is reduced to the point o_{σ_0}, therefore the exceptional fiber $\pi^{-1}(o_\Delta)$ contains no other points (otherwise would not be a connected set).

If Σ' is a regular subdivision of Σ it follows that $\mathbf{O}_\sigma \cap S_{\Sigma'} \neq \emptyset$ if and only if $\sigma \subset \sigma_0$. Thus we can cover the strict transform with those charts Z_σ for $\sigma \subset \sigma_0$ and $\dim \sigma = \dim \sigma_0$. It follows as in the case when σ_0 is a regular cone, that the strict transform is smooth and transversal to the canonical stratification of the exceptional divisor therefore $\pi_{\Sigma'} \circ \pi_\Sigma$ is a pseudo-resolution. ◇

Theorem 3.4. *Let Σ be any subdivision of Δ containing the cone σ_0.*

1. *The strict transform S_Σ is a germ at the point o_{σ_0} isomorphic to $(Z_{\sigma_0, N_{\sigma_0}}, o_{\sigma_0})$ and the restriction $\pi_\Sigma | S_\Sigma : S_\Sigma \to S$ is the normalization map.*
2. *The morphism π_Σ is a partial embedded resolution of $S \subset Z_\Delta$.*

Proof. The first assertion follows from lemma 3.38 taking in account (3.10) (which implies that $(Z_{\sigma_0, N_{\sigma_0}}, o_{\sigma_0})$ is isomorphic to (Z_{ρ, N_g}, o_ρ)) and proposition 3.14 which implies that the integral closure of R is $\mathbb{C}\{\rho^\vee \cap M_g\}$.

By lemma 3.38 if Σ' is a regular subdivision of Σ the map $\pi_{\Sigma'} \circ \pi_\Sigma$ is a embedded pseudo-resolution of S; we show that if Σ' is a resolution of the fan Σ then the restriction $S_{\Sigma'} \to S$ is a resolution of singularities.

The germ S_Σ is parametrized by $W_i = \phi_i$ for $i = 1, \dots, g$, on the chart $Z_{\sigma_0} \cong \mathbf{O}_{\sigma_0} \times Z_{\sigma_0, N_{\sigma_0}}$ thus the restriction $S_\Sigma \to Z_{\sigma_0, N_{\sigma_0}}$ of the second projection is an isomorphism of germs. It follows that the singular locus of S_Σ lies over the singular locus of the toric variety $Z_{\sigma_0, N_{\sigma_0}}$. It is easy to see that the orbit $\mathbf{O}_\tau$ of Z_{σ_0} is the set lying over the orbit $\mathbf{O}_{\tau, N_{\sigma_0}}$ of $Z_{\sigma_0, N_{\sigma_0}}$ thus the singular locus of S_Σ is equal to $\bigcup (S_\Sigma \cap \mathbf{O}_\tau)$ for τ running through the set of non regular faces of σ_0. If Σ' is a resolution of the fan Σ and if $\sigma' \in \Sigma$ is a regular cone then $\sigma' \in \Sigma'$, thus $Z_\Sigma \to Z_{\Sigma_0}$ is an isomorphism over the points of the orbit $\mathbf{O}_{\sigma'}$ by (3.4). Therefore the restriction $S_{\Sigma'} \to S_\Sigma$ is an isomorphism outside the singular locus of S_Σ and since $S_{\Sigma'}$ is smooth this modification is a resolution of singularities of the normalization S_Σ. A forteriori the composed map $S_{\Sigma'} \to S_\Sigma$ is resolution of singularities of S. ◇

3.5.4 Relation between the partial embedded resolution procedures

We show that the partial embedded resolutions of an analytically irreducible toric quasi-ordinary germ S defined in section 3.3 and 3.5 coincide when the second is suitably chosen.

In section 3.3 we have built a partial embedded resolution π of a toric quasi-ordinary hypersurface $S \subset Z_\varrho$ which depends only on the characteristic exponents of a toric quasi-ordinary polynomial f defining the embedding. Since the germ S is analytically irreducible, the morphism π is the composition of g toroidal modifications $\pi_i : Z_i \to Z_{i-1}$ for $i = 1, \dots, g$ and g the number of characteristic exponents. In section 3.5 we have built an embedding of (S, o) as a codimension g

sub-germ of the toric variety (Z_Δ, o_Δ) and we have proved that if Σ is a subdivision of Δ compatible with certain linear subspace, the toric morphism $\pi_\Sigma : Z_\Sigma \to Z_\Delta$ is partial embedded resolution of $S \subset Z_\Delta$. Furthermore, the restriction of π (resp. of π_Σ) to the strict transform S' (resp. S_Σ) of S is the normalization map (see theorems 3.2 and 3.4).

The embedding $S \subset Z_\Delta$ defined by (3.26) extends to an embedding of the pair (S, Z_ϱ): the image of (Z_ϱ, o_ϱ) under this embedding is the sub-germ $(\mathcal{Z}, o_\Delta)$ of (Z_Δ, o_Δ) defined by the equations (see (3.32)):

$$c_j^* U_{j+1} = U_j^{n_j} - c^{(j)}_{l_1^{(j)},\dots,l_{j-1}^{(j)},0} U_1^{l_1^{(j)}} \cdots U_{j-1}^{l_{j-1}^{(j)}} - \sum_{(l_1,\dots,l_j)\neq(l_1^{(j)},\dots,l_{j-1}^{(j)},0)} c^{(j)}_{l_1,\dots,l_j} U_1^{l_1} U_2^{l_2} \cdots U_j^{l_j}, \qquad (3.34)$$

for $j = 1, \dots, g-1$. We can eliminate the variables $U_2, \dots, U_g$ in the equation:

$$U_g^{n_g} - c^{(g)}_{l_1^{(g)},\dots,l_{g-1}^{(g)},0} U_1^{l_1^{(g)}} \cdots U_{g-1}^{l_{g-1}^{(g)}} - \sum_{(l_1,\dots,l_g)\neq(l_1^{(g)},\dots,l_{j-1}^{(g)},0)} c^{(g)}_{l_1,\dots,l_g} U_1^{l_1} U_2^{l_2} \cdots U_g^{l_g} = 0, \qquad (3.35)$$

defining the embedding $S \subset \mathcal{Z}$, by using (3.34). We recover in this way a quasi-ordinary polynomial defining the embedding $S \subset Z_\varrho$.

If Σ is a *suitable* subdivision of Δ (see definition below) we prove that the strict transform $\mathcal{Z}_\Sigma$ of $\mathcal{Z}$ by the toric modification π_Σ is a *section* of the toric variety Z_Σ, transversal to the exceptional fiber of the modification π_Σ. More generally it is transversal to the orbit stratification of Z_Σ and the set of non empty intersections $\mathcal{Z}_\Sigma \cap \mathbf{O}_\sigma$ define the stratification corresponding to a natural toroidal embedding structure which is determined by Σ. In particular we obtain that the restriction $p : \mathcal{Z}_\Sigma \to \mathcal{Z}$ of π_Σ to $\mathcal{Z}_\Sigma$ is a partial embedded resolution of $S \subset \mathcal{Z}$. The main result of this section is that the partial embedded resolutions defined by π and by p are isomorphic:

Theorem 3.5. *The pair $(S_\Sigma, \mathcal{Z}_\Sigma)$ is the image of the pair (S', Z_g) by an embedding such that the following diagram commutes:*

$$\begin{tikzcd}
(S', Z_g) \arrow[r, "\cong"] \arrow[d, "\pi"'] & (S_\Sigma, \mathcal{Z}_\Sigma) \arrow[d, "p"'] \\
(S, Z_\varrho) \arrow[r, "\cong"] & (S, \mathcal{Z})
\end{tikzcd}$$

Therefore the morphism $p : \mathcal{Z}_\Sigma \to \mathcal{Z}$ is the composition of g toroidal modifications.

In the plane branch case an analogous statement (using resolution instead of partial resolution) has been announced by Goldin and Teissier without proof in [G-T]; Lejeune and Reguera have shown in that case toric resolutions of the monomial curve such that the restrictions to the strict transform of the smooth surface, which contains the re-embeded plane branch, are equal to the minimal resolution of the branch (see [LJ-R2]).

We introduce first the notion of *suitable* subdivisions of Δ. The following subsets of Δ defined for $0 \leq j < j+k \leq g$

$$\rho_j^{j+k} = \{a + \langle a, \gamma_1\rangle u_1 + \cdots + \langle a, \gamma_j\rangle u_j + n_j \langle a, \gamma_j\rangle u_{j+1} + \cdots + n_j \cdots n_{j+k-1}\langle a, \gamma_j\rangle u_{j+k} / a \in \rho\} \tag{3.36}$$

are the cones which correspond by duality to certain Minkowski sums of edges of $\mathcal{N}(H_i)$ for $i = 1, \ldots, k$. The cone ρ_0^k coincides with $\rho \times \{0\} \subset \Delta$ for $1 \leq k \leq g$.

Definition 3.12. *We say that a subdivision Σ of Δ is suitable for $\mathcal{Z}$ if it contains the set Ξ whose elements are the faces of the $2g$ cones of dimension $d+1$:*

$$\rho_j^g + \mathbb{R}_{\geq 0} u_j, \quad \rho_j^g + \rho_{j-1}^g \quad \textit{for } j = 1, \ldots, g \tag{3.37}$$

Proposition 3.39. *Let Σ a suitable subdivision of Δ. If $\sigma \in \Sigma$ and if $\overset{\circ}{\sigma} \subset \overset{\circ}{\Delta}$ then $\mathcal{Z}_\Sigma \cap \mathbf{O}_\sigma \neq \emptyset$ implies that $\sigma \in \Xi$. If $\sigma \in \Xi^{(d+1)}$ then $\mathcal{Z}_\Sigma \cap \mathbf{O}_\sigma$ is reduced to a simple point x_σ and the germ $(\mathcal{Z}_\Sigma, x_\sigma)$ is isomorphic to $(Z_{\sigma,(N_\Delta)_\sigma}, o_\sigma)$. The set $\{\mathcal{Z}_\Sigma \cap \mathbf{O}_\sigma\}_{\sigma \in \Xi}$ is the stratification associated to a toroidal embedding structure on $\mathcal{Z}_\Sigma$ which has Ξ as associated conic polyhedral complex.*

In order to prove proposition 3.39 we characterize in the lemma below some convexity properties of the the Newton polyhedra of the polynomials $H_1, \ldots, H_{g-1}$ defining the embedding $\mathcal{Z} \subset Z_\Delta$. This lemma is inspired by a result of Lejeune and Reguera in the case of sandwiched surface singularities (see Proposition 1.3 of [LJ-R]). We need some useful notations. The exponents:

$$u_{j+1}^*, \, n_j u_j^*, \, \varpi_j := \alpha^{(j)} + l_1^{(j)} u_1^* + \cdots + l_{j-1}^{(j)} u_{j-1}^*$$

are the vertices of the two dimensional face $\mathcal{T}^j$ of the polyhedron $\mathcal{N}_\Delta(H_j)$ by proposition 3.37. This face and its edges

$$\mathcal{T}_1^j := [u_{j+1}^*, n_j u_j^*], \, \mathcal{T}_2^j := [u_{j+1}^*, \varpi_j], \, \mathcal{T}_3^j := [n_j u_j^*, \varpi_j],$$

play a significant role in what follows. Any other vertex ϖ_j' of the Newton polyhedron of H_j corresponds to a monomial of weight $> n_j \gamma_j$, i.e., we have $\varpi_j' = \alpha' + l_1 u_1^* + \cdots + l_j u_j^*$ and $\alpha' + l_1 \gamma_1 + \cdots + l_j \gamma_j > n_j \gamma_j$.

Lemma 3.40. *Let $\mathcal{E}_i$ be a compact edge of $\mathcal{N}(H_i)$ for $i = 1, \ldots, g$. If $\cap_{i=1}^j \overset{\circ}{\sigma} (\mathcal{E}_i) \neq \emptyset$ for $1 \leq j \leq g$, then we have $\mathcal{E}_i = \mathcal{T}_{s(i)}^i$ where $s(i)$ is given by:*

$$\begin{cases} s(i) = 3, \; 1 \leq i \leq j \leq g-1, & \text{(A)} \\ s(i) = 3, \; 1 \leq i \leq j_0 - 1; \; s(i) = 1, \; \textit{for } j_0 \leq i \leq j \leq g-1; \; \textit{for } 1 \leq j_0 \leq g-1, & \text{(B)} \\ s(i) = 3, \; 1 \leq i \leq j_0; \; s(j_0) = 2; \; s(i) = 1, j_0 + 1 \leq i \leq j \leq g-1; \; \textit{for } 1 \leq j_0 \leq g-1, & \text{(C)} \\ s(i) = 3, \; 1 \leq i \leq j = g, & \text{(D)} \end{cases}$$

Moreover, the intersection $\cap_{i=1}^{j}\sigma(\mathcal{T}^i_{s(i)})$ *is equal to:*

$$\begin{cases} \rho_j^{j+1} & + \quad \mathbb{R}_{\geq 0}u_{j+1} & + \quad \mathbb{R}_{\geq 0}u_{j+2} + \cdots + \mathbb{R}_{\geq 0}u_g, & \textit{in the case } (\mathrm{A}) \\ \rho_{j_0}^{j+1} & + \quad \mathbb{R}_{\geq 0}u_{j_0} & + \quad \mathbb{R}_{\geq 0}u_{j+2} + \cdots + \mathbb{R}_{\geq 0}u_g, & \textit{in the case } (\mathrm{B}) \\ \rho_{j_0}^{j+1} & + \quad \rho_{j_0-1}^{j+1} & + \quad \mathbb{R}_{\geq 0}u_{j+2} + \cdots + \mathbb{R}_{\geq 0}u_g, & \textit{in the case } (\mathrm{C}) \\ \rho_g^g & & & \textit{in the case } (\mathrm{D}) \end{cases}$$

Proof. The compact faces of Newton polyhedra are determined by elements $a+v \in \Delta$ which belong to the relative interior of Δ; i.e., $a+v$ is of the form $a \in \overset{\circ}{\rho}$ and $v = \sum_{i=1}^{g} v_i u_i$ with $v_i > 0$. We calculate the values of $a+v$ on the vertices of the Newton polyhedron of H_j in terms of the weight of the corresponding monomial. We prove the lemma by induction on j, for $j = 1$ we show first that the compact edges of $\mathcal{N}(H_1)$ are exactly $\mathcal{T}_i^1$ for $i = 1, 2, 3$. We have the following:

$$\begin{aligned} &(\mathrm{i}) = \langle a+v, \varpi_1 \rangle = \langle a, \alpha^{(1)} \rangle = n_1 \langle a, \gamma_1 \rangle \\ &(\mathrm{ii}) = \langle a+v, n_1 u_1^* \rangle = n_1 v_1 \\ &(\mathrm{iii}) = \langle a+v, u_2^* \rangle = v_2 \\ &(\mathrm{iv}) = \langle a+v, \varpi_1' \rangle = \langle a+v, \alpha_1 + l_1 u_1^* \rangle = \langle a, \alpha' + l_1\gamma_1 \rangle + l_1(v_1 - \langle a, \gamma_1 \rangle) > (n_1 - l_1)\langle a, \gamma_1 \rangle + l_1 v_1 \end{aligned}$$

where the inequality on (iv) follows from (3.25) since $a \in \overset{\circ}{\rho}$. We suppose that $a+v$ determines a compact edge e_1 of $\mathcal{N}(H_1)$. Three cases appear:

- If $v_1 = \langle a, \gamma_1 \rangle$ then (iv) > (i) = (ii) thus $v_2 > n_1 \langle a, \gamma_1 \rangle$ and $\mathcal{E}_1 = \mathcal{T}_3^1$.
- If $v_1 > \langle a, \gamma_1 \rangle$ then (ii), (iv) > (i) thus $v_2 = n_1 \langle a, \gamma_1 \rangle$ and $\mathcal{E}_1 = \mathcal{T}_2^1$.
- If $v_1 < \langle a, \gamma_1 \rangle$ then (i), (iv) > (ii) thus $v_2 = n_1 v_1$ and $\mathcal{E}_1 = \mathcal{T}_1^1$.

The equality (i) = (ii) = (iii) corresponds to the two dimensional face $\mathcal{T}^1$. It follows that:

$$\sigma(\mathcal{E}_1) = \begin{cases} \rho_1^2 & + \quad \mathbb{R}_{\geq 0}u_2 & + \quad \mathbb{R}_{\geq 0}u_3 + \cdots + \mathbb{R}_{\geq 0}u_g, & \text{if } \mathcal{E}_1 = \mathcal{T}_3^1 \\ \rho_1^2 & + \quad \mathbb{R}_{\geq 0}u_1 & + \quad \mathbb{R}_{\geq 0}u_3 + \cdots + \mathbb{R}_{\geq 0}u_g, & \text{if } \mathcal{E}_1 = \mathcal{T}_2^1 \\ \rho_1^2 & + \quad \rho_0^2 & + \quad \mathbb{R}_{\geq 0}u_3 + \cdots + \mathbb{R}_{\geq 0}u_g, & \text{if } \mathcal{E}_1 = \mathcal{T}_1^1 \end{cases}$$

We suppose the result true for $j-1$. We consider a vector $a+v \in \bigcap_{i=1}^{j-1} \overset{\circ}{\sigma} (\mathcal{T}^i_{s(i)})$ determining an edge $\mathcal{E}_j$ of $\mathcal{N}(H_j)$, i.e., $a+v \in \overset{\circ}{\sigma} (\mathcal{E}_j)$. The values of $a+v$ on the vertices of $\mathcal{T}_1^j$ are:

$$\begin{aligned} &(\mathrm{ii}) = \langle a+v, n_j u_j^* \rangle = n_j v_j \\ &(\mathrm{iii}) = \langle a+v, u_{j+1}^* \rangle = v_{j+1} \end{aligned}$$

We deal first with the case (A) where $s(i) = 3$ for $1 \leq i \leq j-1$. Then $a+v \in \rho_{j-1}^j$ and it follows as before that:

$$\begin{aligned} &(\mathrm{i}) = \langle a+v, \varpi_j \rangle = n_j \langle a, \gamma_j \rangle \\ &(\mathrm{iv}) = \langle a+v, \varpi_j' \rangle = \langle a+v, \alpha_j + \textstyle\sum_{i=1}^{j} l_i u_i^* \rangle > n_j \langle a, \gamma_j \rangle + l_j(v_j - \langle a, \gamma_j \rangle) \end{aligned}$$

where the inequality is obtained from (3.25) by adding and subtracting the term $l_j \langle a, \gamma_j \rangle$. Three cases appear if v_j is $=$ (resp. $>$ or $<$) to $\langle a, \gamma_j \rangle$ and we obtain the result by arguing as in the case $j = 1$ by replacing appropriately the index 1 by j.

In any other case by induction hypothesis there exists $1 \leq j_0 \leq j-1$ such that $s(i) = 3$ for $1 \leq i < j_0$ and $s(j_0) \in \{1, 2\}$. It follows that the vector $a + v$ is of the form:

$$a + v = a + \sum_{i=1}^{j_0} \langle a, \gamma_i \rangle u_i + \sum_{i > j_0}^{g} v_i u_i$$

We bound the value of $a+v$ on a vertex of the polyhedron $\mathcal{N}(H_j)$ not lying on $\mathcal{T}_1^j$.

$$\begin{array}{rcl} \text{(iv)} & = & \langle a+v, \varpi'_j \rangle = \langle a+v, \alpha_j + \sum_{i=1}^{j} l_i u_i^* \rangle \;\; = \\ & = & \langle a, \alpha_j + \sum_{i=1}^{j} l_i \gamma_i \rangle - \sum_{i=j_0}^{j} l_i \langle a, \gamma_i \rangle + \sum_{i=j_0}^{j} l_i v_i \;\; > \\ & > & (n_j - l_j)\langle a, \gamma_j \rangle - \sum_{i=j_0}^{j-1} l_i \langle a, \gamma_i \rangle + \sum_{i=j_0}^{j} l_i v_i \;\; > \cdots > \\ & > & ((\cdots((n_j - l_j) n_{j-1} - l_{j-1}) \cdots) n_{j_0} - l_{j_0}) \, \langle a, \gamma_{j_0} \rangle + \sum_{i=j_0}^{j} l_i v_i \end{array} \tag{3.38}$$

The first inequality is given by (3.25) and the others are deduced from the inequality $n_i \gamma_i < \gamma_{i+1}$ in lemma 3.28.

In case (B) by induction hypothesis we have that $v_{j_0} = \langle a, \gamma_{j_0} \rangle + c$ for some $c > 0$ and $v_i = n_{j_0} \cdots n_{i-1} \langle a, \gamma_{j_0} \rangle$ for $j_0 < i \leq j$. In case (C) we have that

$$\left\{ \begin{array}{ccccl} n_{j_0 - 1} \langle a, \gamma_{j_0 - 1} \rangle & < & v_{j_0} & < \langle a, \gamma_{j_0} \rangle & \text{if } j_0 > 1 \\ 0 & < & v_{j_0} & < \langle a, \gamma_{j_0} \rangle & \text{if } j_0 = 1 \end{array} \right.$$

and that $v_j = n_{j_0} \cdots n_{i-1} v_{j_0}$ for $j_0 < i \leq j$. In both cases (B) and (C) when substitute the v_i on (3.38) we deduce that $\text{(iv)}, \text{(i)} > \text{(ii)}$ therefore $v_{j+1} = n_j v_j$ and $\mathcal{E}_j = \mathcal{T}_1^j$.

Finally, when $j = g$ the polynomial H_g has no term in U_{g+1}. In particular a vector $a + v \in \bigcap_{i=1}^{g-1} \overset{\circ}{\sigma} (\mathcal{T}_{s(i)}^i)$ for $s(i)$ in case (B) or (C), determines the vertex $n_g u_g^*$ of the polyhedron $\mathcal{N}(H_g)$. The only remaining case is (A) and then the condition on $a + v$ to determine a compact edge of $\mathcal{N}(H_g)$ is $v_g = \langle a, \gamma_g \rangle$; the edge is equal to $\mathcal{T}_3^g = [\varpi, n_g u_g^*]$ and $\bigcap_{i=1}^{g} = \rho_g^g$. ◇

Remark 3.41. *The cones of the form* $\cap_{i=1}^{g-1} \sigma(\mathcal{T}_{s(i)}^i)$ *defined by lemma 3.40 when* $j = g-1$ *are*

$$\left\{ \begin{array}{ll} \rho_k^g + \mathbb{R}_{\geq 0} u_k, \; \rho_k^g + \rho_{k-1}^g & \textit{for } k = 1, \ldots, g-1. \\ \rho_{g-1}^g + \mathbb{R}_{\geq 0} u_g & \end{array} \right.$$

The cones of dimension $d+1$ *in the set* Ξ *defining suitable subdivisions of* Δ *are the first* $2(g-1)$ *cones above with the cones* $\rho_g^g + \mathbb{R}_{\geq 0} u_g$ *and* $\rho_g^g + \rho_{g-1}^g$, *obtained by subdividing* $\rho_{g-1}^g + \mathbb{R}_{\geq 0} u_g$ *with* ρ_g^g.

Proof of Proposition 3.39. Let Σ' be any subdivision of Σ which is compatible with the Newton polyhedra of $H_1, \dots, H_{g-1}$ and $\sigma \in \Sigma'$ with $\overset{\circ}{\sigma}\subset\overset{\circ}{\Delta}$. By lemma 3.3 a necessary condition to have $\mathcal{Z}_{\Sigma'} \cap \mathbf{O}_\sigma \neq \emptyset$ is that σ determines a face $\mathcal{F}_i$ of dimension ≥ 1 of each polyhedron $\mathcal{N}(H_i)$ for $i = 1, \dots, g-1$. Then we have $\sigma \subset \bigcap_{i=1}^{g-1} \sigma(\mathcal{F}_i) \subset \bigcap_{i=1}^{g-1} \sigma(\mathcal{E}_i)$ for $\mathcal{E}_i$ any fixed edge of the face $\mathcal{F}_i$. The possible edges $\mathcal{E}_i$ that may appear are determined by lemma 3.40 and by duality σ is contained in the support of Ξ. By using (3.3) we deduce that if $\sigma \in \Sigma - \Xi$ then $\mathbf{O}_\sigma \cap \mathcal{Z}_\Sigma = \emptyset$.

The proof of the second assertion is analogous to the proof of lemma 3.38. Let $\sigma \in \Xi^{(d+1)}$, for instance $\sigma = \rho_j^g + \mathbb{R}_{\geq 0} u_j$ (the proof in the case $\sigma = \rho_{j-1}^g + \rho_j^g$ is analogous) for $j = 1, \dots, g$. The common zero locus $\mathcal{Z}' \subset Z_\sigma$ of the set functions $X^{-m_1} H_1, \dots, X^{-m_{g-1}} H_{g-1}$ for

$$m_i = \begin{cases} \varpi_i & \text{if } i = 1, \dots, j-1 \\ u_{i+1}^* & \text{if } i = j, \dots, g-1 \end{cases}$$

contains $\mathcal{Z}_\Sigma \cap Z_\sigma$. Each series $X^{-m_i} H_i$ is of the form

$$X^{-m_i} H_i = B_i + \text{ terms vanishing on } \mathbf{O}_\sigma$$

where

$$B_i = \begin{cases} 1 - X^{n_i u_i^* - m_i} & \text{if } i = 1, \dots, j-1 \\ c_i^* + X^{n_i u_i^* - m_i} & \text{if } i = j, \dots, g-1. \end{cases}$$

The edge $\mathcal{E}_i := [n_i u_i^*, m_i]$ is a face of the polyhedron $\mathcal{N}(H_i)$ and by lemma 3.40 we have $\sigma = \bigcap_{j=1}^{g-1} \sigma(\mathcal{E}_i)$ thus $\sigma^\perp = \bigoplus_{j=1}^{g-1} (\sigma(\mathcal{E}_i))^\perp$ since the edges $\mathcal{E}_i$ are affinely independent. Moreover, the vector $n_i u_i^* - m_i$ is primitive for the lattice M_Δ and it follows that $\sigma^\perp \cap M_\Delta = \oplus_{i=1}^{g-1} (n_i u_i^* - m_i)\mathbb{Z}$. It follows that the intersection $\mathcal{Z}' \cap \mathbf{O}_\sigma$ as schemes, defined by the equations $B_1 = \cdots = B_{g-1} = 0$, is a simple point x_σ and that $B_1, \dots, B_{g-1}$ define a regular system of parameters at the point x_σ of $\mathbf{O}_\sigma$. The germ $(\mathcal{Z}', x_\sigma)$ is analytically irreducible since it is isomorphic to $(Z_{\sigma,(N_\Delta)_\sigma}, o_\sigma)$ by lemma 3.4. It follows that it coincides with $(\mathcal{Z}_\Sigma, x_\sigma)$ since this germ is contained in $(\mathcal{Z}', x_\sigma)$ and both have the same dimension. Moreover, if τ is a face of σ then the isomorphism above induces an isomorphism between $\mathcal{Z}_\Sigma \cap \mathbf{O}_\tau$ and the orbit corresponding to τ in $Z_{\sigma,(N_\Delta)_\sigma}$. We conclude from this that $\mathcal{Z}_\Sigma$ has a toroidal embedding structure with associated c.p.c. Ξ. ◇

We recall some facts and notations about the partial embedded resolution of as an hypersurface (see theorem 3.2). Denote by (ϱ_i, N_i') the dual of the pair $(\rho \times \mathbb{R}_{\geq 0} y_i, M_i')$ where M_i' denotes the lattice $M_i \oplus y_i \mathbb{Z}$; each ϱ_i is of the form $\rho \times \mathbb{R}_{\geq 0}$, for $i = 0, \dots, g-1$. The partial embedded resolution is a composition of g toroidal modifications $\pi_i : Z_i \to Z_{i-1}$ for $Z_0 = Z_\varrho$ and $i = 1, \dots, g$. Each variety Z_i is given with a toroidal embedding structure having c.p.c. Σ_i. The c.p.c. Σ_1 is isomorphic to the subdivision of (ϱ, N_0') by the linear form $n_1(y_0 - \lambda_1) \in M_0'$. The c.p.c. Σ_j is obtained from Σ_{j-1} by adding the subdivision of the cone ϱ_{j-1} defined by $n_j(y_{j-1} - \lambda_j + \lambda_{j-1}) \in M_{j-1}'$; this subdivision has $(d+1)$-dimensional cones:

$$\sigma_j^- = \{(a, v) \in \varrho_{j-1} / 0 \leq v \leq \langle a, \lambda_j - \lambda_{j-1} \rangle\} \text{ and } \sigma_j^+ = \{(a, v) \in \varrho_{j-1} / v \geq \langle a, \lambda_j - \lambda_{j-1} \rangle\}. \quad (3.39)$$

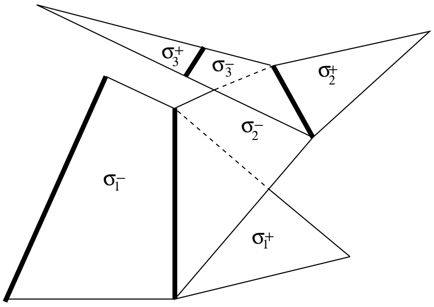

Figure 3.1: A transversal section of the convex polyhedral complex associated to a quasi-ordinary surface with three characteristic exponents

It is glued to Σ_{j-1} by identifying the face $\rho \times \{0\}$ of σ_j^- with $\sigma_{j-1}^+ \cap \sigma_{j-1}^-$ (see lemma 3.24).

Proposition 3.42. *There is an isomorphism* $\Sigma_g \cong \Xi$ *of conic polyhedral complex with integral structure.*

Proof. It is sufficient to prove that the pair $(\sigma, (N'_{j-1})_\sigma)$ is isomorphic to $(\tau, (N_\Delta)_\tau)$ when (σ, τ) is equal to $(\sigma_j^-, \rho_j^g + \rho_{j-1}^g)$ or to $(\sigma_j^+, \rho_j^g + \mathbb{R}_{\geq 0} u_j)$.

In the first case we define an homomorphism $\xi : N'_{j-1} \to N_\Delta$ by:

$$(a, v) \mapsto a + \sum_{i=1}^{j-1} \langle a, \gamma_{j-1} \rangle u_i + \sum_{i \geq j} n_j \cdots n_{i-1} w(a, v) u_i \tag{3.40}$$

where $w(a, v) = v + n_{j-1} \langle a, \gamma_{j-1} \rangle$. It follows that $\xi_\mathbb{R}(\sigma_j^-) = \rho_j^g + \rho_{j-1}^g$ since $(a, v) \in \sigma_j^-$ implies that $n_{j-1} \langle a, \gamma_{j-1} \rangle \leq w(a, v) \leq \langle a, \gamma_j \rangle$ by (3.21).

In the second case we have that $N'_{j-1} = N_j \oplus y_{j-1}^* \mathbb{Z}$ (this follows from lemma 3.17: the inclusion $N_j \hookrightarrow N'_{j-1}$ is dual to the homomorphism $M'_{j-1} \to M_j$ that maps $y_{j-1} \mapsto \lambda_j - \lambda_{j-1}$ and fixes M_{j-1}). Thus we have $(N'_{j-1})_\mathbb{R} = (N_j)_\mathbb{R} \oplus y_{j-1}^* \mathbb{R}$ and with this decomposition the cone σ_j^+ is also defined by the formula (3.39) above. Then we argue analogously, the corresponding lattice homomorphism ξ is defined by (3.40) when $w(a, v) = v + \langle a, \gamma_j \rangle$. It follows from (3.21) that $\xi_\mathbb{R}(\sigma_j^+) = \rho_j^g + \mathbb{R}_{\geq 0} u_j$. ◇

We have all the ingredients to prove theorem 3.5.

Proof of theorem 3.5. The intersection of $\mathcal{Z}$ with the torus of Z_Δ is isomorphic with Z_ϱ minus the hypersurface defined by $q_0 \dots q_{g-1} = 0$. It follows from their definition that the morphisms π and p are isomorphic over this set which is the open stratum of the stratification of Z_g and $\mathcal{Z}_\Sigma$.

Let o_τ (resp. $o_{\tau'}$) be 0-dimensional stratum of Z_g (resp. of $\mathcal{Z}_\Sigma$) associated to the cones $\tau \in \Sigma^{(d+1)}$ (resp. $\tau' \in \Xi^{(d+1)}$). If τ corresponds to τ' by the bijection established in proposition 3.42 we can extend the isomorphism from the open strata to an isomorphism $(Z_g, o_\tau) \to (\mathcal{Z}_\Sigma, o_{\tau'})$ by means of proposition 3.42 and inducing isomorphisms between the strata of dimensions $0 \leq k < d+1$

associated with corresponding faces of τ and τ'. These implies that these local isomorphism paste and provide an isomorphism $Z_g \xrightarrow{I} \mathcal{Z}_\Sigma$ which preserves the toroidal embedding structure. Since $p \circ I = \pi$ it follows that the isomorphism above is in fact an isomorphism of the pairs (S', Z_g) and $(S_\Sigma, \mathcal{Z}_\Sigma)$. ◇

3.5.5 An example

We build a example for the quasi-ordinary surface germ S defined by $f = 0$ where $f = (Y^2 - X_1^3)^2 - X_1^4 X_2 Y^2$. The polynomial $f \in \mathbb{C}\{X_1, X_2\}[Y]$ is quasi-ordinary and irreducible. The characteristic exponents and integers are $\lambda_1 = (\frac{3}{2}, 0)$, $\lambda_2 = (2, \frac{1}{2})$ and $n_1 = n_2 = 2$. The associated semigroup is $\Gamma = \mathbb{Z}^2_{\geq 0} + \gamma_1 \mathbb{Z}_{\geq 0} + \gamma_2 \mathbb{Z}_{\geq 0}$ where $\gamma_1 = (\frac{3}{2}, 0)$ and $\gamma_2 = (\frac{7}{2}, \frac{1}{2})$.

The embedding of S in $\mathbb{C}^4$ is defined by the vanishing of the polynomials:

$$H_1 := U_1^2 - X_1^3 - U_2, \quad H_2 := U_2^2 - X_1^4 X_2 U_1^2,$$

where $U_1 = Y$ and $U_2 = Y^2 - X_1^3$. We denote the coordinates of a vector in Δ with respect to the canonical basis by (v_1, v_2, w_1, w_2); (the cone ρ_0^g corresponds to $w_1 = w_2 = 0$ and we have $u_1 = (0, 0, 1, 0)$ and $u_2 = (0, 0, 0, 1)$ with the notations of the previous section). We denote by ℓ_2 the linear subspace orthogonal to the compact edge of $\mathcal{N}(H_2)$, by δ_1 the cone $\Delta \cap \ell_2 \cap \{v_1 = v_2 = 0\}$ and by δ_2 the cone $\Delta \cap \ell_2 \cap \{w_1 = 0\}$.

A suitable subdivision Σ of Δ has 4-dimensional cones:

$$\begin{aligned}
&\rho_2^2 + \delta_1 + u_2 \mathbb{R}_{\geq 0},\\
&\rho_2^2 + \delta_2 + u_2 \mathbb{R}_{\geq 0},\\
&\rho_2^2 + \rho_1^2 + \delta_1 + u_1 \mathbb{R}_{\geq 0},\\
&\rho_2^2 + \rho_1^2 + \rho_0^2 + \delta_2,\\
&\rho_1^2 + \rho_0^2 + u_1 \mathbb{R}_{\geq 0}.
\end{aligned}$$

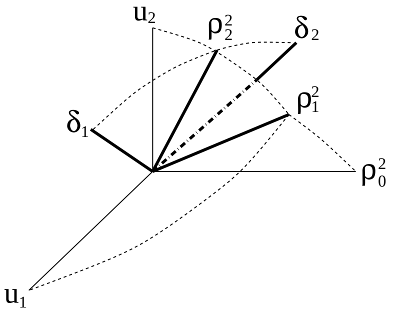

Figure 3.2: The diagram represents the suitable fan Σ

We have (see formula (3.36)):

$$\begin{aligned}
\rho_1^2 &= \mathbb{R}_{\geq 0}(2,0,3,6) + \mathbb{R}_{\geq 0}(0,1,0,0) \\
\rho_2^2 &= \mathbb{R}_{\geq 0}(2,0,3,7) + \mathbb{R}_{\geq 0}(0,2,0,1) \\
\delta_1 &= \mathbb{R}_{\geq 0}(0,0,1,1) \\
\delta_2 &= \mathbb{R}_{\geq 0}(1,0,0,2) + \mathbb{R}_{\geq 0}(0,2,0,1)
\end{aligned}$$

The cone ρ_2^2 is regular, the normalization of the quasi-ordinary surface being smooth in this example. If Σ' is any resolution of the fan Σ it follows that the cone ρ_2^2 belongs to Σ and the strict transform of S by $\pi_{\Sigma'}$ only intersects the exceptional orbit corresponding to this cone.

We build a regular cone $\sigma \supset \rho_2^2$ of dimension four, which belongs to some resolution Σ', and we compute the strict transform of S by the toric morphism on the chart Z_σ. The strategy to build σ is to find a basis of the lattice $\ell_2 \cap \mathbb{Z}^4$ and then to use the equation of the hyperplane ℓ_2 to find a basis of $\mathbb{Z}^4$.

We find in this case

$$\sigma = \mathbb{R}_{\geq 0}(2,0,3,7) + \mathbb{R}_{\geq 0}(0,2,0,1) + \mathbb{R}_{\geq 0}(1,0,2,4) + \mathbb{R}_{\geq 0}(2,1,3,8),$$

the first three vectors defining a basis of $\ell_2 \cap \mathbb{Z}^4$. The toric morphism $Z_{\Sigma'} \to \mathbb{C}^4$ on the chart is given by (see (3.2)):

$$\begin{aligned}
X_1 &= V_1^2 V_3^2 V_4 \\
X_2 &= V_2^2 V_3 \\
U_1 &= V_1^3 V_3^3 V_4^2 \\
U_2 &= V_1^3 V_2^7 V_3^8 V_4^4
\end{aligned}$$

The total transform of S is defined by

$$\begin{aligned}
&V_1^6 V_3^6 V_4^3 \,(V_4 - 1 - V_1 V_2 V_3^2 V_4) = 0 \\
&V_1^{14} V_2^2 V_3^{15} V_4^8 \,(V_3 - 1) = 0
\end{aligned}$$

The strict transform, defined by the vanishing of $V_4 - 1 - V_1 V_2 V_3^2 V_4$ and $V_3 - 1$, is clearly smooth and transversal to the exceptional divisor.

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`gonzalez@math.jussieu.fr`

Pedro Daniel González Pérez
Université de Paris 7, Institut de Mathématiques, Equipe Géométrie et Dynamique
Case 7012; 2, Place Jussieu, 75005 Paris, France.

Etude des singularités quasi-ordinaires d'hypersurfaces au moyen de la géométrie torique

Pedro D. González Pérez

Résumé. Un germe S d'hypersurface analytique complexe est *quasi-ordinaire* s'il existe un morphisme fini sur un espace affine complexe induisant un revêtement non ramifié hors d'un diviseur à croisements normaux. Cette classe de singularités, qui contient les singularités des courbes planes, apparait classiquement dans la méthode de Jung pour analyser les surfaces analytiques complexes ou algébriques et leurs paramétrisations. Comme dans le cas des courbes planes, le germe S admet des paramétrisations par des séries de puissances fractionnaires (Jung-Abhyankar) qui définissent un ensemble fini de monômes, dites *caractéristiques.* Ces monômes classifient le type topologique plongé du germe S lorsque celui-ci est analytiquement irréductible (Gau). Ce résultat généralise la caractérisation topologique des germes de courbes planes par leurs exposants caractéristiques. Notre résultat principal, qui réponds à une question de Lipman, est que les monômes caractéristiques déterminent deux résolutions plongées de S. La première est aussi une résolution plongée des hypersurfaces définies par certaines racines approchées d'un polynôme de Weierstrass adéquat définissant le germe S. La seconde est un morphisme torique construit suivant la méthode de Goldin et Teissier pour les branches planes: ce morphisme est une résolution plongée simultanée du germe S, re-plongé en utilisant ces racines approchées, et d'une spécialisation de S, qui est une variété torique affine définie par un semi-groupe de rang égal à la dimension de S. Nous montrons que ce semi-groupe, qui généralise celui d'une branche plane, est indépendant de la paramétrisation et classifie le type topologique plongée du germe. Finalement, nous faisons une comparaison de ces deux résolutions et nous montrons que la première est une restriction de la deuxième à une certaine variété lisse contenant la transformée stricte de S comme hypersurface.

Mots clés: singularités, géométrie torique, discriminants

Abstract. A germ of complex analytic hypersurface S is *quasi-ordinary* if there is a finite projection on a complex affine space defining an unramified covering outside of a normal crossing divisor. These singularities, which generalize those of plane curves, appear classically on Jung's strategy to analyse a surface singularity and its parametrizations. As in the plane curve case, the germ S admits fractional power series parametrizations (Jung-Abhyankar) which determine a finite set of monomials, called *characteristic.* These monomials classify the *embedded topology* of the hypersurface germ S when it is analytically irreducible (Gau). Gau's result generalizes the topological characterization of plane curves germs by their Puiseux characteristic exponents or by the structure of the *minimal embedded resolution.* Our main result, which gives a positive answer to a question of Lipman, is that the embedded topology of S determines two explicit embedded resolutions. The first embedded resolution that we build is also an embedded resolution of the hypersurfaces defined by certain *approximate roots* of a suitable Weierstrass polynomial defining the germ S. The second resolution is a single toric morphism built by generalizing a method of Goldin and Teissier for plane branches:

it provides a simultaneous embedded resolution of the irreducible germ S, re-embedded in a toric variety Z by using these approximate roots, and of a specialization of S, which is an affine toric variety defined by semigroup of rank equal to the dimension of S. We show that this semigroup, which generalizes the classical semigroup of a plane branch, is independent of the parametrization of S and classifies its embedded topology. We compare both resolutions and we prove that the first is the restriction of the second to a natural smooth variety containing the strict transform of S as a codimension one subvariety.

Key words: singularities, toric geometry, discriminants

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