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POLONICI MATHEMATICI
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Logarithmic Jacobian ideals, quasi-ordinary hypersurfaces and equisingularity

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Dedicated to the memory of Arkadiusz Płoski

Abstract. Let $(S,0) \subset (\mathbb{C}^{d+1},0)$ be an irreducible germ of hypersurface. The germ $(S,0)$ is quasi-ordinary if $(S,0)$ has a finite projection to $(\mathbb{C}^d,0)$ which is unramified outside the coordinate hyperplanes. This implies that the normalization of S is a toric singularity. One also has a monomial variety associated to S, which is a toric singularity with the same normalization, and with possibly higher embedding dimension. Since $(S,0)$ is quasi-ordinary, the extension of the Jacobian ideal of S to the local ring of its normalization is a monomial ideal. We describe this monomial ideal by comparing it with the *logarithmic Jacobian ideals* of S and of its associated monomial variety and we give some applications.

Introduction. An equidimensional germ $(S,0)$ of complex analytic variety of dimension d is quasi-ordinary if there exists a *quasi-ordinary projection*, that is, a finite map $\pi : (S,0) \to (\mathbb{C}^d,0)$, which is unramified outside a normal crossing divisor in $(\mathbb{C}^d,0)$. This class of singularities plays an important role in the classical approach to study singularities by Jung's method (see [44, 1]).

In this paper we suppose that $(S,0)$ is an analytically irreducible hypersurface germ of dimension $d \geq 1$. The quasi-ordinary hypersurface $(S,0)$ may be defined by a quasi-ordinary polynomial $f \in \mathbb{C}\{x_1,\ldots,x_d\}[y]$, that is, a Weierstrass polynomial whose discriminant $\Delta_y f$ is of the form a monomial times a unit in the ring $\mathbb{C}\{x_1,\ldots,x_d\}$. The germ S has certain fractional power series parametrizations $y = \zeta(x_1^{1/n},\ldots,x_n^{1/n})$ for $n = \deg_y f$, which generalize the Newton–Puiseux series of plane curve singularities. These fractional power series have a finite set of *characteristic* or *distinguished mono-*

2020 *Mathematics Subject Classification*: Primary 32S25; Secondary 14M25.
Key words and phrases: Jacobian ideal, logarithmic Jacobian ideal, toric singularities, quasi-ordinary singularities, Nash modification, equisingularity.
Received 20 December 2024; revised 19 October 2025.

DOI: 10.4064/ap241220-13-11

mials, which generalize the characteristic exponents in the case of plane branches. Joseph Lipman [44] and Yih-Nan Gau [18] proved that these monomials classify the embedded topological type of $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$.

As in the case of plane branches, one has a notion of semigroup Γ associated to the quasi-ordinary singularity $(S, 0)$. The semigroup Γ determines and is determined by the characteristic monomials, and it is independent of the choice of parametrization of $(S, 0)$ (see [57, 22, 39]). The semigroup Γ defines an affine toric variety Z^{Γ}, which is a complete intersection of dimension d, called the *monomial variety* associated with S.

Quasi-ordinary hypersurface singularities are well adapted to toric geometry methods: for instance the normalization $(Z, 0) \to (S, 0)$ is a toric singularity determined by the characteristic monomials (see [21, 57]). The composition $(Z, 0) \to (\mathbb{C}^d, 0)$ of the normalization with the quasi-ordinary projection is also a quasi-ordinary projection. Recall that the singular locus of S is defined by the Jacobian ideal $\mathrm{Jac}(S)$ of S. It follows that the pull-back of the singular locus of S by the normalization map is contained in the complement of the torus of Z. This implies that the ideal $\mathrm{Jac}(S)\mathcal{O}_Z$ is a monomial ideal with respect to the toric structure of the ring $\mathcal{O}_Z$. The main goal of this paper is to provide a description of the ideal $\mathrm{Jac}(S)\mathcal{O}_Z$.

This goal is related to the study of the normalized Nash modification. Recall that the Nash modification of an algebraic (or complex analytic) variety is a canonical morphism which replaces each singular point by the limiting positions of tangent spaces at non-singular points. Since S is a complete intersection, the Nash modification of S is isomorphic to the blow up of the Jacobian ideal in $\mathcal{O}_S$ (see [45] and also [48] for a similar result in the case of Gorenstein singularities). Then, the normalized Nash modification of S, which is the Nash modification followed by the normalization, is isomorphic to the composition of the normalization map of S with the normalized blow up of $\mathrm{Jac}(S)\mathcal{O}_Z$. This follows from the properties of normalized blow ups of ideals (see [43, Propositions 3.2 and 3.3]).

The Nash modification of an affine toric variety defined over an algebraically closed field of characteristic zero is isomorphic to the blow up of a monomial ideal, called the *logarithmic Jacobian ideal* (see [26, 42, 25]). We define the logarithmic Jacobian ideal $\mathcal{J}_{\log}(V)$ of a d-dimensional variety V with toric normalization $\overline{V}$ as the image of the module of differentials Ω^d_V by the composite

$$\Omega^d_V \to \Omega^d_{\overline{V}} \to \Omega^d_{\overline{V}}(\log D) \to \mathcal{O}_{\overline{V}},$$

where $\Omega^d_{\overline{V}}(\log D)$ is the module of d-forms with logarithmic poles on the complement D of the torus of $\overline{V}$, the first map is induced by normalization, the second map is the canonical one from the toric structure of $\overline{V}$, and the third map is an isomorphism.

We describe the Jacobian ideal of an affine toric variety in Proposition 1.6. By Remark 4.9 this result implies, in the case V is the affine toric variety Z^{Γ} associated to the semigroup Γ of a quasi-ordinary hypersurface singularity $(S, 0)$, that

$$X^{\gamma_0}\, \mathcal{J}_{\log}(Z^{\Gamma}) = \mathrm{Jac}(Z^{\Gamma}), \tag{1}$$

where γ_0 is the minimal Frobenius vector of the semigroup Γ (see Proposition 2.10).

It is natural to ask if (1) can be extended to the quasi-ordinary hypersurface germ $(S, 0)$. The ideal $\mathcal{J}_{\log}(S)$ and the Frobenius vector γ_0 are determined by the characteristic monomials, hence by the embedded topological type of $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$. The ideal $\mathcal{J}_{\log}(S)$ was described by Helena Cobo and the author [8]. Our main results are Theorem 4.13, which shows the inclusion

$$X^{\gamma_0}\, \mathcal{J}_{\log}(S)\mathcal{O}_Z \subseteq \mathrm{Jac}(S)\mathcal{O}_Z, \tag{2}$$

and Corollary 4.15, which states that equality holds in (2) in the two-dimensional case.

We will apply a particular re-embedding of the germ S in an affine space of dimension $d+g$, where g is the number of characteristic monomials. Using this embedding we can build a deformation with fibers S_p, for $p \in Z^{\Gamma}$, such that if p belongs to the torus then S_p is isomorphic to the germ S, while if p is equal to the origin 0 then the special fiber S_0 is isomorphic to Z^{Γ}. In both cases the fibers have the same normalization $\overline{S}_p = Z$. There is a toric modification of $\mathbb{C}^{d+g}$ which provides an embedded resolution of S_p, for p in the torus or $p = 0$ (see [21]). In the case of plane branches this result was proved by Rebeca Goldin and Bernard Teissier [19].

In order to describe the Jacobian ideal $\mathrm{Jac}(S_p)$, we introduce a matrix $R(p)$ in terms of the Jacobian matrix of the functions defining the fibers of this deformation. The matrix $R(p)$ specializes when $p = 0$ to a matrix with integer coefficients encoding the relations between the generators of the semigroup Γ.

As an application, we give a method to compute the number $\overline{\nu}_{\mathrm{Jac}(S)}(\phi)$ associated to a function $\phi \in \mathcal{O}_S$, when the dimension of S is 2. The number $\overline{\nu}_{\mathrm{Jac}(S)}(\phi)$ is studied by Monique Lejeune-Jalabert and Bernard Teissier [43] in a more general setting. The number $\overline{\nu}_{\mathrm{Jac}(S)}(\phi)$ can be expressed in terms of ϕ and the divisorial valuations associated with the irreducible components of the pull-back of the Jacobian ideal of S in the normalized Nash modification (see [43]). By [43, Section 6], the number $(\overline{\nu}_{\mathrm{Jac}(S)}(\phi))^{-1}$ may be seen as a *Łojasiewicz exponent* associated to the ideal $\mathrm{Jac}(S)$ and the function ϕ in a compact neighborhood of $0 \in S$. The study of Łojasiewicz exponents has been one of the research interests of Arkadiusz Płoski, and he has published many articles on this topic, either alone or with collaborators (see for instance [49, 50, 51, 52, 53, 54, 55, 34, 40, 10]).

The results and proofs of this paper hold in the category of complex analytic spaces, and also in the algebroid category over an algebraically closed field of zero characteristic.

The structure of the paper is as follows. In Section 1 we introduce the basic notions of toric geometry, we review the construction of the Nash modification of an affine toric variety and we give a combinatorial description of its Jacobian ideal. In Section 2 we review the basic properties of quasi-ordinary hypersurface singularities and their associated semigroups. In Section 3 we recall the construction of the logarithmic Jacobian ideals of a toric and of a quasi-ordinary hypersurface. In Section 4 we describe the deformation of S which specializes to the monomial variety Z^Γ and then we prove the main results about the relations between the Jacobian ideal and the logarithmic Jacobian ideals. Finally, in Section 5 we give an application to the computation of the number $\overline{\nu}_{\mathrm{Jac}(S)}(\phi)$ associated to a function $\phi \in \mathcal{O}_S$.

Notation

- The result of erasing the term r_j in the sequence $r_1, \dots, r_s$ is denoted by $r_1, \dots, \hat{r}_j, \dots, r_s$.
- Let $A = (a^i_j)^{i=1,\dots,k}_{j=1,\dots,l}$ be a matrix with k rows and l columns, and with coefficients in a ring. If $1 \leq i_1 < \cdots < i_s \leq k$ (resp. if $1 \leq j_1 < \cdots < j_r \leq l$) we denote by $A^{i_1,\dots,i_s}$ (resp. by $A_{j_1,\dots,j_r}$) the submatrix

$$A^{i_1,\dots,i_s} := (a^i_j)^{i \in \{1,\dots,\hat{i}_1,\dots,\hat{i}_s,\dots,k\}}_{j \in \{1,\dots,l\}} \qquad (\text{resp. } A_{j_1,\dots,j_r} := (a^i_j)^{i \in \{1,\dots,k\}}_{j \in \{1,\dots,\hat{j}_1,\dots,\hat{j}_r,\dots,l\}}).$$

- If A is a square matrix we denote by $|A|$ its determinant. If A is a matrix we denote its rank by $\operatorname{rk} A$.
- If $\gamma, \gamma' \in \mathbb{R}^d$ we set

$$\gamma \prec \gamma' \quad \text{iff} \quad \gamma' - \gamma \in \mathbb{R}^d_{\geq 0} \text{ and } \gamma \neq \gamma'. \tag{3}$$

We write $\gamma \preceq \gamma'$ iff $\gamma' - \gamma \in \mathbb{R}^d_{\geq 0}$.

1. Nash modification of an affine toric variety

1.1. Equivariant embedding of an affine toric variety. We first give some basic definitions and notations (see [12, 61, 9]).

If $N \cong \mathbb{Z}^d$ is a lattice we denote by M its dual lattice, and by $N_{\mathbb{R}}$ (resp. $N_{\mathbb{Q}}$) the vector space spanned by N over the field $\mathbb{R}$ (resp. $\mathbb{Q}$). We use similar notations for a lattice homomorphism ϕ and the associated vector space homomorphism $\phi_{\mathbb{Q}}$ (or $\phi_{\mathbb{R}}$). We denote by

$$\langle\ ,\ \rangle : N \times M \to \mathbb{Z}, \quad (u, v) \mapsto \langle u, v \rangle,$$

the dual pairing between the lattices N and M. In what follows, a *cone* means a *rational convex polyhedral cone*: the set of nonnegative linear combinations of vectors $v_1, \dots, v_r \in N$. The cone σ is *strictly convex* if it contains no lines. The *dual* cone $\sigma^\vee$ of σ is the set $\{w \in M_\mathbb{R} \mid \langle w, u\rangle \geq 0\}$. We denote by $\mathrm{int}(\sigma)$ the *relative interior* of the cone σ.

Let Λ be a subsemigroup of finite type of the lattice M. We will assume that Λ generates M as a group, that is, $M = \mathbb{Z}\Lambda$. The $\mathbb{C}$-algebra of the semigroup, $\mathbb{C}[\Lambda] := \{\sum_{\text{finite}} a_\lambda X^\lambda \mid a_\lambda \in \mathbb{C}\}$, is of finite type and corresponds to the affine toric variety $Z^\Lambda = \operatorname{Spec} \mathbb{C}[\Lambda]$. The torus Z^M is an open dense subset of Z^Λ, which acts on Z^Λ in a way which extends the group multiplication on the torus. A closed point $p \in Z^\Lambda$ determines and is determined by the semigroup homomorphism $g_p : \Lambda \to (\mathbb{C}, \cdot)$ such that $g_p(m) = X^m(p)$ for $m \in \Lambda$, where $(\mathbb{C}, \cdot)$ represents the semigroup defined by $\mathbb{C}$ with multiplication. For instance, the unit point $\underline{1}$ of the torus Z^M corresponds to the constant homomorphism $g_{\underline{1}}$ defined by $g_{\underline{1}}(m) = 1$ for all $m \in M$. We have a bijection $\tau \mapsto \mathrm{orb}_\tau$ between the faces of σ and the orbits of the torus action which reverses the inclusion of the closures. Any affine toric variety is defined by a subsemigroup of this form.

The cone $\sigma^\vee := \mathbb{R}_{\geq 0}\Lambda \subset M_\mathbb{R}$ is rational for the lattice M, hence the semigroup $\sigma^\vee \cap M$ is of finite type. The map of $\mathbb{C}$-algebras $\mathbb{C}[\Lambda] \to \mathbb{C}[\sigma^\vee \cap M]$ induced by $\Lambda \hookrightarrow \sigma^\vee \cap M$ corresponds geometrically to the normalization map $Z^{\sigma^\vee \cap M} \to Z^\Lambda$.

From now on we assume that the cone $\sigma^\vee = \mathbb{R}_{\geq 0}\Lambda$ is strictly convex of dimension d. The zero-dimensional orbit of the torus action is then reduced to the origin of Z^Λ, which is the point $0 \in Z^\Lambda$ defined by the maximal ideal $\mathfrak{m}_\Lambda := (\Lambda \setminus \{0\})\mathbb{C}[\Lambda]$. The origin 0 of Z^Λ corresponds to the homomorphism of semigroups $g_0 : \Lambda \to (\mathbb{C}^*, 0)$ such that $g_0(m) = 0$ for all $m \in \Lambda \setminus \{0\}$ and $g_0(0) = 1$.

We denote by $\mathbb{C}\{\Lambda\}$ the local ring of germs of holomorphic functions of Z^Λ at the origin. Its completion with respect to its maximal ideal is the ring $\mathbb{C}[[\Lambda]] = \{\sum a_\lambda X^\lambda \mid a_\lambda \in \mathbb{C}\}$ of formal power series with exponents in the semigroup Λ (see [21]).

Let $\alpha_1, \dots, \alpha_m \in \Lambda \setminus \{0\}$ be a set of generators of Λ. Denote by $e_1, \dots, e_m$ the canonical basis of $\mathbb{Z}^m$. We have an exact sequence of lattice homomorphisms

$$0 \to \mathcal{L} \to \mathbb{Z}^m \xrightarrow{\psi} M \to 0, \tag{4}$$

where $\psi(e_j) = \alpha_j$ for $j = 1, \dots, m$, and $\mathcal{L}$ is the kernel of ψ. The surjective homomorphism of semigroups

$$\mathbb{Z}^m_{\geq 0} \to \Lambda, \quad e_j \mapsto \alpha_j, \quad j = 1, \dots, m, \tag{5}$$

corresponds to an equivariant embedding $Z^\Lambda \hookrightarrow \mathbb{C}^m$ given by

$$\psi_* : \mathbb{C}[U_1, \ldots, U_m] \to \mathbb{C}[\Lambda], \qquad U_j \mapsto X^{\alpha_j} \quad \text{for } j = 1, \ldots, m. \tag{6}$$

The *support* of a vector $n = \sum_{j=1}^m n_j e_j \in \mathbb{Z}^m$ is the set $\operatorname{supp}(n) = \{j \in \{1, \ldots, n\} \mid n_j \neq 0\}$. Let $n \in \mathbb{Z}^m$ be a nonzero vector in $\ker(\psi)$. The vector n decomposes in a unique way as $n = n^+ - n^-$, where $n^+, n^- \in \mathbb{Z}^m_{\geq 0}$ have disjoint supports. Notice that the vectors n^+ and n^- are nonzero, otherwise we would get a nontrivial linear combination $0 = \sum_j n_j \alpha_j$ in M, which contradicts the strict convexity of the cone $\sigma^\vee$. Since $n \in \ker \psi$, we have the relation

$$\psi(n^+) = \sum_{i=1}^m n_i^+ \alpha_i = \sum_{i=1}^m n_i^- \alpha_i = \psi(n^-). \tag{7}$$

There exists a sequence of vectors $n_1, \ldots, n_p \in \mathbb{Z}^m$ such that the ideal $\mathcal{I}^\Lambda$ of the embedding $Z^\Lambda \hookrightarrow \mathbb{C}^m$ is generated by the binomials $h_j := U^{n_j^+} - U^{n_j^-}$ for $j = 1, \ldots, p$. The vectors $n_1, \ldots, n_p$ generate $\ker(\psi)$.

We have an exact sequence

$$\mathbb{Z}^p \xrightarrow{\phi} \mathbb{Z}^m \xrightarrow{\psi} M \to 0, \tag{8}$$

in which ϕ maps the jth canonical vector to the vector n_j for $j = 1, \ldots, p$. Write $n_i = \sum_{j=1}^m \ell_j^{(i)} e_i$ for $i = 1, \ldots, p$. The *matrix of relations* associated with $n_1, \ldots, n_p$ is

$$R := \begin{pmatrix} \ell_1^{(1)} & \cdots & \ell_m^{(1)} \\ \cdots & \cdots & \cdots \\ \ell_1^{(p)} & \cdots & \ell_m^{(p)} \end{pmatrix}. \tag{9}$$

The matrix R has rank $m - d \leq p$. Notice that we can always relabel the vectors $n_1, \ldots, n_p$ in such a way that the submatrix $R^{m-d+1,\ldots,p}$ of R is of rank $m - d$.

Lemma 1.1 (see [25, Proposition 60], [11, Lemma 1.8]). *If the matrix $R^{m-d+1,\ldots,p}$ is of rank $m - d$ then for any sequence $1 \leq j_1 < \cdots < j_d \leq m$ we have*

$$\alpha_{j_1} \wedge \cdots \wedge \alpha_{j_d} = 0 \quad \textit{if and only if} \quad |R^{m-d+1,\ldots,p}_{j_1,\ldots,j_d}| = 0.$$

Proof. We consider the exact sequence induced by (8) over the field $\mathbb{Q}$. The linear subspace $L := \operatorname{Im}(\phi_{\mathbb{Q}})$ is generated by the images of the first $m - d$ canonical vectors by hypothesis. The matrix of the restriction $\phi'_{\mathbb{Q}}$ of the linear map $\phi_{\mathbb{Q}}$ to the subspace spanned by the first $m - d$ canonical vectors, with respect to the canonical basis, is equal to the transpose of $R^{m-d+1,\ldots,d}$.

The matrix P of $\psi_{\mathbb{Q}}$ with respect to the canonical basis of $\mathbb{Q}^m$ and a fixed basis of $M_{\mathbb{Q}}$ has ith column the coordinates of $\alpha_i \in M_{\mathbb{Q}}$, for $i = 1, \ldots, m$. By dualizing the sequence (8) we notice that the jth row of the matrix P defines the coordinates of a linear form $w_j \in (\mathbb{Q}^m)^*$, with respect to the dual basis, in such a way that the vector subspace L is also obtained by the intersection of the kernels of w_j for $j = 1, \ldots, d$.

We have represented the $m - d$-dimensional subspace L by equations and by generators. We deduce the assertion as a consequence of the classical relations of the Grassmann coordinates of a linear subspace and its dual Grassmann coordinates applied to L (see [37, VII, §3, Theorem 1]). ∎

A vector $\alpha_0 \in M$ is a *Frobenius vector* of the semigroup Λ if for every vector $\alpha \in \operatorname{int}(\sigma^\vee) \cap M$, the sum $\alpha_0 + \alpha$ belongs to Λ (see [5]). This notion generalizes the Frobenius number of a numerical semigroup, in the case $M = \mathbb{Z}$. For instance, if e_1, e_2 is the canonical basis of $\mathbb{Z}^2$ and then $-e_1 - e_2$ is a Frobenius vector of the semigroup $\mathbb{Z}^2_{\geq 0}$.

We say that Λ is a *complete intersection semigroup* if the ideal $\mathcal{I}^\Lambda$ is minimally generated by $m - d$ binomials. This means that Z^Λ is a complete intersection. If Λ is a complete intersection semigroup then Λ has a *minimal* Frobenius vector α_0 (see [5, Th. 3.3]). The minimality means that if α_0' is a Frobenius vector then $\alpha_0' \in \alpha_0 + \sigma^\vee$.

1.2. Nash modification of a toric singularity. In this section we review the description of the Nash modification of a complex affine toric variety Z^Λ following [25]. As an application, we obtain a description of the generators of the Jacobian ideal $\operatorname{Jac}(Z^\Lambda)$ in Proposition 1.6.

NOTATION 1.2. We consider a semigroup Λ as in Section 1.1. We denote by J (resp. $\tilde{J}$) the matrix with coefficients in $\mathbb{C}[\Lambda]$ defined from the Jacobian matrix of $(h_1, \ldots, h_p)$ by

$$J := \left(\psi_*\Big(\frac{\partial h_i}{\partial U_j}\Big)\right)_{j=1,\ldots,m}^{i=1,\ldots,p} \qquad \left(\text{resp. } \tilde{J} := \left(\psi_*\Big(U_j \cdot \frac{\partial h_i}{\partial U_j}\Big)\right)_{j=1,\ldots,m}^{i=1,\ldots,p}\right),$$

where we recall that ψ_* is defined by (6).

PROPOSITION 1.3 (see [26, 25, 42]). *The Nash modification of Z^Λ is the blow up of the ideal $\mathcal{J}_{\log}(Z^\Lambda)$ of $\mathbb{C}[\Lambda]$ generated by the images of the products $U_{j_1} \cdots U_{j_d}$ such that $\alpha_{j_1} \wedge \cdots \wedge \alpha_{j_d} \neq 0$, i.e., by the monomial ideal*

$$\mathcal{J}_{\log}(Z^\Lambda) := (\{X^{\alpha_{j_1} + \cdots + \alpha_{j_d}} \mid 1 \leq j_1 < \cdots < j_d \leq m,\ \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_d} \neq 0\}). \tag{10}$$

Proof. We suppose first that the toric singularity Z^Λ is a complete intersection. This means that we can assume $p = m - d$, and the map ϕ in the sequence (8) is injective.

Nobile proved that the Nash modification of a complete intersection X is the blow up of the Jacobian ideal of X (see [45, Theorem 1, Remark 2]).

Let us fix some $i \in \{1, \dots, m-d\}$. If $1 \leq j \leq m$ and $j \notin \operatorname{supp}(n_i)$ then $\ell_j^{(i)} = 0$. If $1 \leq j \leq m$ and if $j \in \operatorname{supp}(n_i)$ then

$$U_j \cdot \frac{\partial h_i}{\partial U_j} = \begin{cases} \ell_j^{(i)} U^{n_i^+} & \text{if } j \in \operatorname{supp}(n_i^+), \\ \ell_j^{(i)} U^{n_i^-} & \text{if } j \in \operatorname{supp}(n_i^-). \end{cases}$$

Since $\psi(n_i^+) = \psi(n_i^-)$ we get $\psi_*(U^{n_i^+}) = \psi_*(U^{n_i^-})$. It follows that the matrix $\tilde{J}$ is the result of multiplying the ith row of the matrix of relations R defined in (9) by the monomial $X^{\psi(n_i^+)}$, for $i = 1, \dots, p$.

If $1 \leq j_1 < \cdots < j_d \leq m$, then we get

$$X^{\alpha_1 + \cdots + \alpha_m} \, |J_{j_1,\dots,j_d}| = X^{\psi(n_1^+) + \cdots + \psi(n_{m-d}^+)} X^{\alpha_{j_1} + \cdots + \alpha_{j_d}} \, |R_{j_1,\dots,j_d}|. \tag{11}$$

Formula (11) implies that the Jacobian ideal of Z^Λ, which is generated by $\{|J_{j_1,\dots,j_d}|\}_{1 \leq j_1 < \cdots < j_d \leq m}$, and the ideal generated by

$$\{X^{\alpha_{j_1} + \cdots + \alpha_{j_d}} \, |R_{j_1,\dots,j_d}|\}_{1 \leq j_1 < \cdots < j_d \leq m}$$

are related by invertible ideals, hence they have isomorphic blow ups. Finally, by Lemma 1.1, the determinant $|R_{j_1,\dots,j_d}|$ vanishes if and only if $\alpha_{j_1} \wedge \cdots \wedge \alpha_{j_d} = 0$ for $1 \leq j_1 < \cdots < j_d \leq m$.

In the general case, we can assume that the submatrix $R^{m-d+1,\dots,m}$ is of maximal rank $m-d$. The Nash modification of Z^Λ is isomorphic to the blow up of the ideal $(J^{m-d+1,\dots,m}_{j_1,\dots,j_d})_{j_1,\dots,j_d}$ (see [25, Prop. 60]). Notice that if we replace $R_{j_1,\dots,j_d}$ by $R^{m-d+1,\dots,m}_{j_1,\dots,j_d}$, and $J_{j_1,\dots,j_d}$ by $J^{m-d+1,\dots,m}_{j_1,\dots,j_d}$, in (11), the resulting formula holds. Then, the conclusion follows by the previous argument using Lemma 1.1. ■

Definition 1.4. The *logarithmic Jacobian ideal* of Z^Λ is the monomial ideal $\mathcal{J}_{\log}(Z^\Lambda)$ defined by (10).

Remark 1.5. A prime characteristic version of the logarithmic Jacobian ideal of a toric variety was defined in [11]. It is also a monomial ideal whose blow up coincides with the Nash modification.

As a consequence of the proof of Proposition 1.3 we get the following description of the generators of the Jacobian ideal of the toric variety Z^Λ.

Proposition 1.6. *If* $1 \leq j_1 < \cdots < j_d \leq m$ *and* $1 \leq i_1 < \cdots < i_{m-d} \leq p$, *set*

$$m^{i_1,\dots,i_{m-d}}_{j_1,\dots,j_d} := \psi(n_{i_1}^+) + \cdots + \psi(n_{i_{m-d}}^+) - \sum_{j \in \{1,\dots,m\} \setminus \{j_1,\dots,j_d\}} \alpha_j.$$

The Jacobian ideal $\operatorname{Jac}(Z^\Lambda)$ *of the toric variety* Z^Λ *is the monomial ideal of* $\mathbb{C}[\Lambda]$ *generated by*

$$(\{X^{m^{i_1,\dots,i_{m-d}}_{j_1,\dots,j_d}} \mid \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_d} \neq 0 \text{ and } \operatorname{rk}(R^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}) = m-d\}) \subset \mathbb{C}[X^\Lambda].$$

Proof. Notice that the Jacobian ideal of Z^{Λ} is generated by the set of minors

$$\{|J^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}_{j_1,\dots,j_d}|\}^{1\le i_1<\dots<i_{m-d}\le p}_{1\le j_1<\dots<j_d\le m}.$$

By (11), we obtain the equality

$$|J^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}_{j_1,\dots,j_d}| = X^{m^{i_1,\dots,i_{m-d}}_{j_1,\dots,j_d}}\,|R^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}_{j_1,\dots,j_d}|.$$

We deduce that $|J^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}_{j_1,\dots,j_d}| = 0$ if and only if $|R^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}_{j_1,\dots,j_d}| = 0$. Notice that if $\mathrm{rk}(R^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}) < m-d$ then all the determinants $|R^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}_{j_1,\dots,j_d}|$ vanish. By Lemma 1.1, $|R^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}_{j_1,\dots,j_d}| \neq 0$ if and only if $\mathrm{rk}(R^{1,\dots,\hat{i}_1,\dots,\hat{i}_{m-d},\dots,p}) = m-d$ and $\alpha_{j_1}\wedge\dots\wedge\alpha_{j_d}\neq 0$. ■

EXAMPLE 1.7. Let $\alpha_1 = (2,0)$, $\alpha_2 = (0,2)$, $\alpha_3 = (3,0)$ and $\alpha_4 = (7,1)$. One can check that $\mathcal{L} = \ker(\psi)$ is a lattice with basis $n_1 = (3,0,-2,0)$ and $n_2 = (7,1,0,-2)$. By Proposition 2.13 below, Z^{Λ} is a complete intersection defined by the equations $h_1 = h_2 = 0$, where $h_1 = U_1^3 - U_3^2$ and $h_2 = U_1^7U_2 - U_4^2$. We have

$$\left(U_j\cdot\frac{\partial h_i}{\partial U_j}\right)_{i,j} = \begin{pmatrix} 3U_1^3 & 0 & -2U_3^2 & 0\\ 7U_1^7U_2 & U_1^7U_2 & 0 & -2U_4^2\end{pmatrix}, \qquad R = \begin{pmatrix} 3 & 0 & -2 & 0\\ 7 & 1 & 0 & -2\end{pmatrix}.$$

By Proposition 1.6, the monomial ideal of $\mathbb{C}[\Lambda]$ generated by the functions $X^{m_{1,2}}, X^{m_{1,4}}, X^{m_{2,4}}, X^{m_{3,4}}$ is the Jacobian ideal $\mathrm{Jac}(Z^{\Lambda})$ of Z^{Λ}, where

$$m_{1,2} = \alpha_3+\alpha_4,\quad m_{1,4} = 7\alpha_1+\alpha_3,\quad m_{2,4} = 6\alpha_1+\alpha_2+\alpha_3,\quad m_{3,4} = 9\alpha_1.$$

2. Quasi-ordinary hypersurface singularities.

A complex analytic germ $(S,0)$ is *quasi-ordinary* if there exists a finite projection $(S,0)\to(\mathbb{C}^d,0)$ which is a local isomorphism outside a normal crossing divisor. In the hypersurface case, there is an embedding $(S,0)\subset(\mathbb{C}^{d+1},0)$ defined by $f=0$, where $f\in\mathbb{C}\{X\}[X_{d+1}]$ is a *quasi-ordinary polynomial*, a Weierstrass polynomial with discriminant $\Delta_Y f$ of the form $\Delta_Y f = X^{\delta}\epsilon$ for a unit ϵ in the ring $\mathbb{C}\{X\}$ of convergent power series in the variables $X=(X_1,\dots,X_d)$ and $\delta\in\mathbb{Z}^d_{\ge 0}$.

We suppose in this paper that the germ $(S,0)$ is analytically irreducible. The Jung–Abhyankar theorem guarantees that the roots of a quasi-ordinary polynomial f, called *quasi-ordinary branches*, are fractional power series in the ring $\mathbb{C}\{X^{1/n}\}$, for $n=\deg f$ (see [1]). If $\{\zeta^{(l)}\}_{l=1}^{N}\subset\mathbb{C}\{X^{1/n}\}$ is the set of roots of f, the discriminant $\Delta_Y f$ of f with respect to Y is equal to $\Delta_Y f = \prod_{i\neq j}(\zeta^{(i)}-\zeta^{(j)})$, hence each factor $\zeta^{(t)}-\zeta^{(r)}$ is of the form $X^{\lambda_{t,r}}\epsilon_{t,r}$ where $\epsilon_{t,r}$ is a unit in $\mathbb{C}\{X^{1/n}\}$. The monomials $X^{\lambda_{t,r}}$ (resp. the exponents $\lambda_{t,r}$) are called *characteristic*. By Lipman [44], the characteristic exponents can be relabeled so that

$$\lambda_1 \prec \cdots \prec \lambda_g, \tag{12}$$

where the relation $\prec$ is defined in (3). The characteristic exponents determine the nested sequence of lattices

$$M_0 \subset M_1 \subset \cdots \subset M_g =: M,$$

where $M_0 := \mathbb{Z}^d$ and $M_j := M_{j-1} + \mathbb{Z}\lambda_j$ for $j = 1, \ldots, g$. We set $\lambda_0 = 0$ and $\lambda_{g+1} = +\infty$. The index n_j of the lattice M_{j-1} in the lattice M_j is strictly greater than 1 (see [44, 22]).

Lipman noticed that the support of a quasi-ordinary branch has very special properties.

Lemma 2.1 (see [18, Prop. 1.3]). *Let $\zeta = \sum c_\alpha X^\alpha \in \mathbb{C}\{X^{1/n}\}$ be a quasi-ordinary branch with characteristic exponents $\lambda_1, \ldots, \lambda_g$. If $c_\alpha \neq 0$ then there exists a unique $j \in \{1, \ldots, g+1\}$ such that $\lambda_{j-1} \preceq \alpha \not\preceq \lambda_j$ and then $\alpha \in M_{j-1}$.*

We say that the quasi-ordinary branch ζ has *well-ordered variables* if the g-tuples $(\lambda_{1,i}, \ldots, \lambda_{g,i})$, which are defined in terms of the coordinates of $\lambda_1, \ldots, \lambda_g$ with respect to the canonical basis, satisfy

$$(\lambda_{1,i}, \ldots, \lambda_{g,i}) >_{\text{lex}} (\lambda_{1,j}, \ldots, \lambda_{g,j})$$

for $1 \leq i < j \leq d$, where $>_{\text{lex}}$ denotes the lexicographic order. We can relabel the variables $X_1, \ldots, X_d$ so as to satisfy this condition.

Definition 2.2. The quasi-ordinary branch ζ is *normalized* if it has well-ordered variables, and whenever λ_1 is of the form $\lambda_1 = (\lambda_{1,1}, 0, \ldots, 0)$ then $\lambda_{1,1} > 1$.

Lipman proved that the q.o. hypersurface $(S, 0)$ can be parametrized by a normalized quasi-ordinary branch (see [18]). The embedded topological type of $(S, 0) \subset (\mathbb{C}^{d+1}, 0)$ is classified by the characteristic exponents of a normalized quasi-ordinary branch ζ parametrizing $(S, 0)$ (see [44, 18]).

2.1. Normalization of a quasi-ordinary singularity. We study quasi-ordinary hypersurface singularities by using toric geometry methods.

Notation 2.3. The semigroup $\mathbb{Z}^d_{\geq 0}$ has a minimal set of generators $e_1, \ldots, e_d$, which is a basis $\mathcal{B}$ of the lattice M_0. The dual basis of the dual lattice N_0 spans a regular cone σ in $N_{0,\mathbb{R}}$. Then $\sigma^\vee = \mathbb{R}_{\geq 0} e_1 + \cdots + \mathbb{R}_{\geq 0} e_d$ and $\mathbb{Z}^d_{\geq 0} = \sigma^\vee \cap M_0$. If $\alpha \in \sigma^\vee$ we write $\alpha = \sum_{i=1}^d \alpha_i e_i$ with respect to the basis $\mathcal{B}$. We do this in particular for the characteristic exponents $\lambda_j = \sum_{i=1}^d \lambda_{j,i} e_i$.

The $\mathbb{C}$-algebra $\mathbb{C}\{X_1, \ldots, X_d\}$ of convergent power series is isomorphic to $\mathbb{C}\{\sigma^\vee \cap M_0\}$. This isomorphism identifies the monomial $X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ with the monomial $X^\alpha \in \mathbb{C}[\sigma^\vee \cap M_0]$, where $\alpha = \sum_{k=1}^d \alpha_k e_k$.

We identify the local algebra $\mathcal{O}_S = \mathbb{C}\{X_1, \dots, X_d\}[X_{d+1}]/(f)$ of the singularity $(S, 0)$ with the ring $\mathbb{C}\{X_1, \dots, X_d\}[\zeta] = \mathbb{C}\{\sigma^\vee \cap M_0\}[\zeta]$. The normalization of a quasi-ordinary singularity (not necessarily a hypersurface) is analytically isomorphic to a toric simplicial singularity (see [59]). In the hypersurface case the normalization is determined by the characteristic exponents.

Lemma 2.4 (see [21, Prop. 14]). *The quasi-ordinary branch ζ belongs to $\mathbb{C}\{\sigma^\vee \cap M\}$. The homomorphism $\mathcal{O}_S = \mathbb{C}\{\sigma^\vee \cap M_0\}[\zeta] \to \mathbb{C}\{\sigma^\vee \cap M\}$ of $\mathbb{C}$-algebras is the inclusion of $\mathcal{O}_S$ in its integral closure $\mathcal{O}_{\overline{S}} = \mathbb{C}\{\sigma^\vee \cap M\}$ in its field of fractions.*

2.2. The semigroup of a quasi-ordinary hypersurface. Set

$$\overline{\lambda}_1 = \lambda_1 \quad \text{and} \quad \overline{\lambda}_{j+1} = n_j \overline{\lambda}_j + \lambda_{j+1} - \lambda_j \quad \text{for} \quad j = 1, \dots, g-1. \tag{13}$$

The following formula can be deduced from (13) for $j \in \{2, \dots, g\}$:

$$\begin{aligned} \overline{\lambda}_j = {} & \lambda_j + (n_{j-1} - 1)\lambda_{j-1} + n_{j-1}(n_{j-2} - 1)\lambda_{j-2} \\ & + \dots + n_{j-1} \cdots n_2 (n_1 - 1)\lambda_1. \end{aligned} \tag{14}$$

It implies that $\overline{\lambda}_1 \prec \dots \prec \overline{\lambda}_g$ by using (12). Formula (14) also implies that

$$\lambda_j \preceq \overline{\lambda}_j \quad \text{for } j = 2, \dots, g. \tag{15}$$

Definition 2.5 (see [22, 39]). The semigroup

$$\Gamma := \mathbb{Z}_{\geq 0} e_1 + \dots + \mathbb{Z}_{\geq 0} e_d + \mathbb{Z}_{\geq 0} \overline{\lambda}_1 + \dots + \mathbb{Z}_{\geq 0} \overline{\lambda}_g \subset \sigma^\vee \cap M$$

is associated to a sequence of characteristic exponents $\lambda_1, \dots, \lambda_g$ of a quasi-ordinary branch ζ. By convenience, we denote by $\gamma_1, \dots, \gamma_{d+g}$ the sequence of generators $e_1, \dots, e_d, \overline{\lambda}_1, \dots, \overline{\lambda}_g$ of the semigroup Γ. We also set $\gamma_{g+1} = \infty$.

A function $h \in \mathcal{O}_S$ has a *dominant exponent with respect to the quasi-ordinary branch* ζ if

$$h = X^{\gamma(h)} \cdot u(h) \quad \text{with } \gamma(h) \in M \text{ and } u(h) \in \mathbb{C}\{\sigma^\vee \cap M\} \text{ a unit.}$$

We say that $\gamma(h)$ is the *dominant exponent* of h. We denote by $\mathcal{C}(\zeta)$ the subset of $\mathcal{O}_S$ consisting of functions with a dominant exponent with respect to ζ. The set of dominant exponents of functions in $\mathcal{C}(\zeta)$ is equal to the semigroup Γ (see [57]). The intrinsic nature of the semigroup is more subtle.

Theorem 2.6 (see [22, 57, 23, 24]). *The semigroup Γ is an analytical invariant of the irreducible germ of quasi-ordinary hypersurface $(S, 0)$.*

Remark 2.7. If the quasi-ordinary branch ζ is normalized then $e_1, \dots, e_d$, $\overline{\lambda}_1, \dots, \overline{\lambda}_g$ is a minimal set of generators of Γ (see [22]). Notice that Γ is a subsemigroup of the lattice M. In [22] we also introduced a subsemigroup $\overline{\Gamma}$ of $\mathbb{Z}^d$ which is isomorphic to Γ. If $d = 1$ the semigroup $\overline{\Gamma}$ coincides with the classical semigroup associated to the plane branch $(S, 0) \subset (\mathbb{C}^2, 0)$. Formulas (13) and (14) are analogous to the classical formulas relating the characteristic of a plane branch and the generators of its semigroup (see [66, Th. 3.9]).

Remark 2.8. One may characterize the generators of Γ in terms of some of the *approximate roots* of the quasi-ordinary polynomial f (see [22]). Janusz Gwoździewicz and Arkadiusz Płoski gave a simplified approach to the Abhyankar–Moh theory of approximate roots in terms of intersection multiplicities (see [2, 35] and the survey [56]). Their presentation was essential in order to extend the results about approximate roots to quasi-ordinary hypersurfaces in the irreducible case in [22], and more generally in Beata Gryszka's paper [30]. In connection with these results, let us mention several irreducibility criterions for quasi-ordinary polynomials (see [31, 4, 27, 15]). The study of polar curves was also a relevant topic in the work of Arkadiusz Płoski (see for instance the survey [33]). Bunch decompositions of the polars of a quasi-ordinary hypersurface have been given in [13, 58]. A subtle factorization theorem for higher order polars in the quasi-ordinary hypersurface case was obtained by Evelia García Barroso and Janusz Gwoździewicz in [16].

Lemma 2.9 (see [22, Lemma 3.3]). *We have the following properties:*

(a) $n_j\gamma_{d+j} \prec \gamma_{d+j+1}$ *for* $j = 2, \dots, g$.
(b) *We have relations of the form*

$$n_j\gamma_{d+j} = \ell_1^{(j)}\gamma_1 + \dots + \ell_d^{(i)}\gamma_d + \ell_{d+1}^{(j)}\gamma_{d+1} + \dots + \ell_{d+j-1}^{(j)}\gamma_{d+j-1}, \tag{16}$$

where $\ell_1^{(j)}, \dots, \ell_d^{(j)} \in \mathbb{Z}_{\geq 0}$, $0 \leq \ell_{d+k}^{(j)} < n_k$ *for* $k = 1, \dots, j-1$, *and* $j = 1, \dots, g$ (*see Definition* 2.5).

As a consequence of Proposition 2.13, the semigroup Γ associated with a quasi-ordinary hypersurface is a complete intersection semigroup. This implies that Γ has a minimal Frobenius vector γ_0, which was studied by Abdallah Assi [3] (see Section 1.1 for the definition of the Frobenius vector).

Proposition 2.10 (see [3]). *The minimal Frobenius vector of the semigroup Γ is equal to*

$$\gamma_0 := \sum_{j=1}^{g}(n_j - 1)\overline{\lambda}_j - \sum_{i=1}^{d} e_j. \tag{17}$$

Remark 2.11. The *Poincaré series* $P_\Gamma(X) := \sum_{\gamma\in\Gamma} X^\gamma \in \mathbb{C}[[\Gamma]]$ of the semigroup Γ has a rational form

$$P_\Gamma(X) := \prod_{j=1}^{g}(1 - X^{n_j\overline{\lambda}_j}) \prod_{j=1}^{g}(1 - X^{\overline{\lambda}_j})^{-1} \prod_{i=1}^{d}(1 - X^{e_j})^{-1}$$

(see [24]). We observe the following symmetry property in terms of the Frobenius vector γ_0:

$$P_\Gamma(X) = (-1)^d X^{\gamma_0} P_{-\Gamma}(X).$$

Note that by Proposition 2.13, the variety Z^{Γ} is an complete intersection, hence $\mathbb{C}[\Gamma]$ is a M-graded Cohen–Macaulay Gorenstein domain (cf. [60, Chapter 1, Theorem 12.7]).

REMARK 2.12. Poincaré series associated with complete intersection semigroups are described similarly in [5, §4], where they are called *Hilbert series.* We follow here the terminology introduced by Gusein-Zade, Delgado and Campillo [32]. See also [6] for a survey of some aspects of Poincaré series of semigroups.

2.3. The associated toric variety Z^{Γ}. We consider the equivariant embedding of the toric variety $Z^{\Gamma} \subset \mathbb{C}^{d+g}$ given by $U_i = X^{\gamma_i}$ for $i = 1, \dots, d+g$, where $(U_1, \dots, U_{d+g})$ denotes a system of coordinates on $\mathbb{C}^{d+g}$.

PROPOSITION 2.13 (see [21, Prop. 38]). *The relations* (16) *define the binomials*

$$h_j := U_1^{\ell_1^{(j)}} \cdots U_{d+j-1}^{\ell_{d+j-1}^{(j)}} - U_{d+j}^{n_j} \quad \text{for } j = 1, \dots, g, \tag{18}$$

The ideal of the embedding of the toric variety $Z^{\Gamma} \subset \mathbb{C}^{d+g}$ is generated by the binomials $h_1, \dots, h_g$. The toric variety Z^{Γ} is a complete intersection.

DEFINITION 2.14. The matrix of relations associated with (18) is

$$\begin{pmatrix} \ell_1^{(1)} & \cdots & \ell_d^{(1)} & \ell_{d+1}^{(1)} & 0 & 0 & \cdots & 0 \\ \ell_1^{(2)} & \cdots & \ell_d^{(2)} & \ell_{d+1}^{(2)} & \ell_{d+2}^{(2)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \ell_1^{(g)} & \cdots & \ell_d^{(g)} & \ell_{d+1}^{(g)} & \ell_{d+2}^{(g)} & \ell_{d+3}^{(g)} & \cdots & \ell_{d+g}^{(g)} \end{pmatrix}, \tag{19}$$

where we have set $\ell_{d+j}^{(j)} := -n_j$ for $j = 1, \dots, g$.

We use some elementary properties of this matrix.

DEFINITION 2.15. For $1 \leq i \leq d$ we denote by m_i the smallest integer $1 \leq m \leq g+1$ such that $\ell_i^{(m)} \neq 0$, where by convention $m_i = g+1$ means that $\ell_i^{(1)} = \cdots = \ell_i^{(g)} = 0$.

LEMMA 2.16. *Let us take an integer $1 \leq i \leq d$.*

(a) *If $w \in \sigma^{\vee} \cap M_{m_i-1}$, then w_i is a nonnegative integer.*
(b) $m_i = \min(\{j \in \{1, \dots, g\} \mid \overline{\lambda}_{j,i} \neq 0\} \cup \{g+1\})$.
(c) *If $1 \leq j \leq g$ and $\lambda_{j,i} \neq 0$, then $n_j \overline{\lambda}_{j,i} \geq 1$.*
(d) *If $1 \leq j \leq g$ and $m_i < j$, then $\overline{\lambda}_{j,i} \geq 1$.*
(e) *If $2 \leq j \leq g$ and $0 < \overline{\lambda}_{j,i} < 1$, then $m_i = j$.*

Proof. (a) The definition of m_i implies that the lattice M_{m_i-1} is the direct sum of the sublattice spanned by $e_1, \dots, \widehat{e_i}, \dots, e_d, \lambda_1, \dots, \lambda_{m_i-1}$ and the sublattice $\mathbb{Z}e_i$. If $w \in M_{m_i-1}$ belongs to the cone $\sigma^{\vee}$ spanned by $e_1, \dots, e_d$ then $w_i \in \mathbb{Z}_{\geq 0}$.

(b) If $1 \leq j < m_i$, then the vector $\overline{\lambda}_j \in \sigma^\vee$ belongs to the lattice $M_j \subset M_{m_i-1}$ by hypothesis. By (a), $\overline{\lambda}_{j,i}$ is a nonnegative integer. By using the direct sum decomposition in (a), if $\overline{\lambda}_{j,i} \neq 0$ then the coefficient $\ell_i^{(j)}$ appearing in (16) would be nonzero, contradicting the assumption $1 \leq j < m_i$. This implies that $\overline{\lambda}_{j,i} = 0$, thus $\lambda_{j,i} = 0$ by (15). Notice also that $n_{m_i}\overline{\lambda}_{m_i} \in M_{m_i-1}$ and by definition of m_i the coefficient $\ell_i^{(m_i)}$ is nonzero. This implies that the ith coordinate $\overline{\lambda}_{m_i,i}$ is nonzero.

(c) If $j = 1$, the property holds since $n_1\overline{\lambda}_1 = n_1\lambda_1 \in M_0 = \mathbb{Z}^d$, thus $n_1\overline{\lambda}_{1,i} > 0$ if $\lambda_{1,i} \neq 0$. Suppose that the result is true for $j-1$.

If $\lambda_{j-1,i} \neq 0$, by the induction hypothesis we get $1 \leq n_{j-1}\overline{\lambda}_{j-1,i}$. Hence $1 \leq n_{j-1}\overline{\lambda}_{j-1,i} \leq \overline{\lambda}_{j,i}$ by Lemma 2.9(a).

Otherwise $\lambda_{j-1,i} = 0$. By (13), this implies that $\lambda_{s,i} = 0$ for $s = 1, \ldots, j-1$. It follows that $m_i = j$ and $\overline{\lambda}_{j,i} = \lambda_{j,i} > 0$ by (13). Since $n_j\overline{\lambda}_j \in M_{j-1}$ by (16), we deduce from (a) that $n_j\overline{\lambda}_{j,i} \geq 1$.

(d) By (b) we have $\overline{\lambda}_{m_i,i} \neq 0$. By (c) we get $n_{m_i}\overline{\lambda}_{m_i,i} \geq 1$. Since $m_i < j$, it follows from (b) that $\overline{\lambda}_{j,i} \geq n_{m_i}\overline{\lambda}_{m_i,i} \geq 1$ by (13).

(e) Notice that $\overline{\lambda}_j = n_{j-1}\overline{\lambda}_{j-1} + \lambda_j - \lambda_{j-1}$ by (13), where $0 \prec \lambda_j - \lambda_{j-1}$ by (12). If $\overline{\lambda}_{j-1,i} \neq 0$, then we get $\overline{\lambda}_{j,i} \geq n_{j-1}\overline{\lambda}_{j-1,i} \geq 1$ by (c), thus we must have $\overline{\lambda}_{j-1,i} = 0$. It follows that $\overline{\lambda}_{1,i} = \cdots = \overline{\lambda}_{j-1,i} = 0$. Then the assertion follows by (b). ■

3. The logarithmic Jacobian ideal of a quasi-ordinary hypersurface.

We introduce the logarithmic Jacobian ideal of a singularity with toric normalization following [42] in the toric case. We review the normal toric case following [46, Chapter 3] and [42, Appendix]. Then we apply this to the case of an irreducible germ of quasi-ordinary hypersurface and also to its associated monomial variety.

3.1. Logarithmic Jacobian ideals.

Let Y be the algebroid germ defined by an ideal I of $\mathbb{C}[[X_1, \ldots, X_n]]$ with analytic algebra $A = \mathbb{C}[[X_1, \ldots, X_n]]/I$. We denote by Ω_Y^1 the A-module of *Kähler differential forms* and by $d : A \to \Omega_Y^1$ its canonical derivation. As usual we denote by Ω_Y^k the A-module $\Omega_Y^k := \bigwedge_k \Omega_Y^1$. See [29, Chapter I, §1.10].

We consider the toric singularity $Z = Z^{\sigma^\vee \cap M}$ with formal local algebra $\mathcal{O}_Z = \mathbb{C}[[\sigma^\vee \cap M]]$ at the origin. We denote by D the equivariant Weil divisor defined by the sum of the orbit closures of codimension 1 in the toric variety Z. The $\mathcal{O}_Z$-module $\Omega_Z^1(\log D)$ of 1-forms of Z with logarithmic poles along D is identified with $\mathcal{O}_Z \otimes_{\mathbb{Z}} M$. We have a map of $\mathcal{O}_Z$-modules

$$\eta : \Omega_Z^1 \to \mathcal{O}_Z \otimes_{\mathbb{Z}} M \quad \text{such that} \quad dX^\gamma \mapsto X^\gamma \otimes \gamma \text{ for } \gamma \in \sigma^\vee \cap M.$$

If $u_1, \ldots, u_d$ is a basis of the lattice M we write the expansion of $\gamma \in \sigma^\vee \cap M$

as $\gamma = \sum_{i=1}^{d} \gamma_i u_i$. If $h = \sum_{\gamma \in \sigma^\vee \cap M} c_\gamma X^\gamma \in \mathcal{O}_Z$ then we have

$$\eta(dh) = \sum_{\gamma \in \sigma^\vee \cap M} c_\gamma X^\gamma \otimes \gamma = \sum_{i=1}^{d} \Big(\sum_{\gamma \in \sigma^\vee \cap M} c_\gamma \, \gamma_i X^\gamma \Big) \otimes u_i.$$

The map $\wedge^k \eta$ is a homomorphism of $\mathcal{O}_Z$-modules such that

$$\begin{aligned} \wedge^k \eta : \Omega^k_Z &\to \Omega^k_Z(\log D) = \mathcal{O}_Z \otimes_{\mathbb{Z}} \textstyle\bigwedge_k M, \\ dX^{\gamma_1} \wedge \cdots \wedge dX^{\gamma_k} &\mapsto X^{\gamma_1 + \cdots + \gamma_k} \otimes \gamma_1 \wedge \cdots \wedge \gamma_k, \end{aligned}$$

for $k = 1, \ldots, d$. Let us fix an orientation of M and assume that the basis $u_1, \ldots, u_d$ is positively oriented. We have an isomorphism $\bigwedge_d M \to \mathbb{Z}$, which sends $u_1 \wedge \cdots \wedge u_d$ to 1. This induces an identification $\Omega^d_Z(\log D) \equiv \mathcal{O}_Z$. It follows that $\eta^d(\Omega^d_Z)$ corresponds under this identification to an ideal of $\mathcal{O}_Z$, which is independent of the basis of M chosen. This ideal is called the *logarithmic Jacobian ideal* of Z in [42]. It is a monomial ideal generated by $(X^{\beta_{j_1} + \cdots + \beta_{j_d}})^{1 \leq j_1 < \cdots < j_d \leq r}_{\beta_{j_1} \wedge \cdots \wedge \beta_{j_d} \neq 0}$, where $\{\beta_1, \ldots, \beta_r\}$ is a generating system of the semigroup $\sigma^\vee \cap M$.

DEFINITION 3.1. Let W be an analytically irreducible algebroid germ of singularity of dimension d such that its normalization Z is a toric singularity. The *logarithmic Jacobian ideal* $\mathcal{J}_{\log}(W)\mathcal{O}_Z$ is the ideal of $\mathcal{O}_Z$ generated by the image of the module Ω^d_W of differentials by the composite τ of the maps

$$\Omega^d_W \to \Omega^d_Z \xrightarrow{\wedge_d \eta} \Omega^d_Z(\log D) \equiv \mathcal{O}_Z, \tag{20}$$

where $\Omega^d_W \to \Omega^d_Z$ is the map induced by normalization.

3.2. The logarithmic Jacobian ideal of the quasi-ordinary hypersurface $(S, 0)$. We consider an analytically irreducible quasi-ordinary hypersurface $(S, 0)$ germ as in Section 2 and we describe its logarithmic Jacobian ideal using the fact that its normalization is a toric singularitiy.

DEFINITION 3.2. We let $\mathcal{J}_g$ be the monomial ideal of $\mathcal{O}_Z$ generated by monomials X^α for α in the set

$$\Xi_g := \Big\{ \sum_{j=1}^{d} e_j \Big\} \cup \big\{ e_1 + \cdots + \widehat{e_i} + \cdots + e_d + \lambda_{m_i} \;\big|\; i \in \{1, \ldots, d\}, m_i \in \{1, \ldots, g\} \big\}. \tag{21}$$

We denote by $\mathcal{I}_g$ the monomial ideal of $\mathcal{O}_Z$ generated by X^α for α in the set

$$\{\gamma_{i_1} + \cdots + \gamma_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq d + g,\ \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_d} \neq 0\} \tag{22}$$

(see Definition 2.5).

THEOREM 3.3 (see [8, Prop. 9.2]). *The ideal $\mathcal{J}_{\log}(S)\mathcal{O}_Z$ is equal to the monomial ideal $\mathcal{J}_g$.*

Proof. We have the inclusion $\iota : \mathcal{O}_S \to \mathcal{O}_{\overline{S}}$ corresponding to the normalization map, where $\mathcal{O}_{\overline{S}} = \mathbb{C}[[\sigma^\vee \cap M]]$ by Lemma 2.4. We consider a fixed quasi-ordinary branch $\zeta = \sum_\alpha c_\alpha X^\alpha$ parametrizing $(S, 0)$. We denote by x_i the image of the coordinate X_i in $\mathcal{O}_S$ for $i = 1, \dots, d+1$. By Lemma 2.4, the function $\iota(x_{d+1}) = \zeta$ belongs to $\mathbb{C}\{\sigma^\vee \cap M\}$ and $\iota(x_i) = X^{e_i}$ for $i = 1, \dots, d$ (see Notation 2.3).

We analyze the image of $dx_{j_1} \wedge \cdots \wedge dx_{j_d}$ by the composite τ of the maps (20) for $W = S$. Since $dx_{j_1} \wedge \cdots \wedge dx_{j_d}$ generate the $\mathcal{O}_S$-module Ω^d_S it follows that $\tau(dx_{j_1} \wedge \cdots \wedge dx_{j_d})$ for $1 \leq j_1 < \cdots < j_d \leq d+1$ generate the ideal $\tau(\Omega^d_S)$ of $\mathbb{C}\{\sigma^\vee \cap M\}$.

If $(j_1, \dots, j_d) = (1, \dots, d)$, then $\tau(dx_1 \wedge \cdots \wedge dx_d)$ is equal to $X^{e_1+\cdots+e_d}$ times a nonzero constant, hence $X^{e_1+\cdots+e_d}$ belongs to $\mathcal{J}_{\log}(S)\mathcal{O}_Z$.

Otherwise, we have $j_d = d+1$ and $1 \leq j_1 < \cdots < j_{d-1} \leq d$. We denote by i the integer such that $\{j_1, \dots, j_{d-1}, i\} = \{1, \dots, d\}$. Let us consider the index m_i introduced in Definition 2.15. We decompose the quasi-ordinary branch $\zeta = \sum_\alpha c_\alpha X^\alpha$ as $\zeta = \zeta_i^- + \zeta_i^+$, where $\zeta_i^+ = \sum_{\lambda_{m_i} \preceq \alpha} c_\alpha X^\alpha$ and $\zeta_i^- = \zeta - \zeta_i^+$ (see Lemma 2.1).

By definition we have $\zeta_i^- = \sum_{\lambda_{m_i} \not\preceq \alpha} c_\alpha X^\alpha$. Let $\alpha \in M$ be an element appearing in the support of the series ζ_i^-. By Lemma 2.1, α belongs to the lattice $M_{m_i - 1}$. In addition, if we expand $\alpha = \sum_{s=1}^d \alpha_s e_s$ in terms of the canonical basis $\mathcal{B}$, then the coefficient α_i is a nonnegative integer by Lemma 2.16(a). We write

$$\begin{aligned}\tau(dx_{j_1} \wedge \cdots \wedge dx_{j_d}) = {} & \wedge_d \eta(dX^{e_{j_1}} \wedge \cdots \wedge dX^{e_{j_{d-1}}} \wedge d\zeta_i^-) \\ & + \wedge_d \eta(dX^{e_{j_1}} \wedge \cdots \wedge dX^{e_{j_{d-1}}} \wedge d\zeta_i^+)\end{aligned}$$

Then the term

$$\wedge_d \eta(dX^{e_{j_1}} \wedge \cdots \wedge dX^{e_{j_{d-1}}} \wedge d\zeta_i^-) = \sum_{\lambda_{m_i} \not\preceq \alpha} c_\alpha \cdot e_{j_1} \wedge \cdots \wedge e_{j_{d-1}} \wedge \alpha \cdot X^{e_{j_1}+\cdots+e_{j_{d-1}}+\alpha}$$

is divisible by $X^{e_1+\cdots+e_d}$. We deduce that

$$\wedge_d \eta(dX^{e_{j_1}} \wedge \cdots \wedge dX^{e_{j_{d-1}}} \wedge d\zeta_i^+) = \sum_{\lambda_{m_i} \preceq \alpha} c_\lambda \cdot e_{j_1} \wedge \cdots \wedge e_{j_{d-1}} \wedge \alpha \cdot X^{e_{j_1}+\cdots+e_{j_{d-1}}+\alpha}$$

is the product of the monomial $X^{e_{j_1}+\cdots+e_{j_{d-1}}+\lambda_{m_i}}$ times a unit, since

$$c_{\lambda_{m_i}} \cdot e_{j_1} \wedge \cdots \wedge e_{j_{d-1}} \wedge \lambda_{m_i} \neq 0.$$

This implies that $\mathcal{J}_{\log}(S)\mathcal{O}_Z$ is equal to the monomial ideal $\mathcal{J}_g$ (see Definition 3.2). ∎

Let $\Lambda \subset M$ be a subsemigroup generated by $\alpha_1, \dots, \alpha_m \in M$ such that $\mathbb{Z}\Lambda = M$ and $\mathbb{R}_{\geq 0}\Lambda = \sigma^\vee$ as in Section 1.1. Then, the canonical map $Z := Z^{\sigma^\vee \cap M} \to Z^\Lambda$ is the normalization of Z^Λ. Notice that the logarithmic Jacobian ideal $\mathcal{J}_{\log}(Z^\Lambda)\mathcal{O}_Z$ is the extension of the ideal of $\mathcal{O}_{Z^\Lambda}$ defined in Proposition 1.3. The following result is a consequence of the definitions and the previous discussion in the toric case.

PROPOSITION 3.4 (see also [42]). *The ideal* $\mathcal{J}_{\log}(Z^{\Lambda})\mathcal{O}_Z$ *is the monomial ideal of* $Z^{\sigma^\vee \cap M}$ *defined by the monomials*

$$\{X^{\alpha_{j_1}+\cdots+\alpha_{j_d}} \mid 1 \leq j_1 < \cdots < j_d \leq m \text{ and } \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_d} \neq 0\}.$$

We apply Proposition 3.4 to the particular case of Z^{Γ} taking into account Definitions 2.5 and 3.2.

COROLLARY 3.5. *The logarithmic Jacobian ideal* $\mathcal{J}_{\log}(Z^{\Gamma})\mathcal{O}_Z$ *is equal to the monomial ideal* $\mathcal{I}_g$.

REMARK 3.6. Lejeune-Jalabert and Reguera [42] have shown that the logarithmic Jacobian ideal of a normal affine toric surface plays a significant role in the rational expression of the associated *geometric motivic Poincaré series*. In a joint work with Helena Cobo [7, 8], we have shown similar results in the cases of affine toric varieties and of irreducible quasi-ordinary hypersurface singularities.

4. Description of the Jacobian ideal.

In this section we give some relations between the Jacobian ideal of the quasi-ordinary hypersurface S and the logarithmic Jacobian ideals of S and of its associated monomial variety Z^{Γ}. We need to introduce first a deformation of S which has generic fiber isomorphic to S and specializes to Z^{Γ}.

4.1. Overweight deformations.

In [20] we construct an embedding

$$(S, 0) \hookrightarrow (\mathbb{C}^{d+g}, 0)$$

together with a deformation $\mathcal{S}$ of S over Z^{Γ}, with generic fiber $S_p \subset \mathbb{C}^{d+g}$ isomorphic to S and special fiber $S_0 = Z^{\Gamma} \subset \mathbb{C}^{d+g}$. This deformation is equisingular in the sense that there exists a toric modification $\pi : W \to \mathbb{C}^{d+g}$, characterized by the properties of the semigroup Γ, that provides a simultaneous normalization of $(S_p, 0)$, and the composite of π with any toric resolution of the singularities of W is an embedded resolution of $(S_p, 0)$ for $p = 0$ or p in the torus of Z^{Γ} (see [21]). This result generalizes one of Teissier and Goldin [19] for plane branches. See also Teissier's [64, 65] development of this approach in terms of valuations.

Let us describe this deformation. First, we have the exact sequence (4) associated to the map

$$\psi : \mathbb{Z}^{n+g} \to M, \quad \psi(u_i) = \gamma_i \quad \text{for } i = 1, \ldots, d+g,$$

which sends the elements of the canonical basis $u_1, \ldots, u_{d+g}$ of $\mathbb{Z}^{n+g}$ to the generators of the semigroup Γ (see (5) and Definition 2.5).

We consider the map $\psi^* : \mathbb{C}[U_1, \ldots, U_{d+g}] \to \mathbb{C}[\Gamma]$ as in (6). We define the *weight* of a monomial $U^{\alpha} \in \mathbb{C}[U_1, \ldots, U_{d+g}]$ by $\psi(\alpha) \in \Gamma$. This defines a $\mathit{\Gamma}$-grading of $\mathbb{C}[U_1, \dots, U_{d+g}]$. Notice that the binomials h_j defined in (18) are weighted homogeneous of degree $n_j \overline{\lambda}_j$ for $j = 1, \dots, g$.

Definition 4.1. An *overweight deformation* of the binomials h_j in (18), for $j = 1, \dots, g$, is of the form

$$H_j = h_j + c_j \cdot t^{\overline{\lambda}_{j+1} - n_j \overline{\lambda}_j} \cdot U_{d+j+1} + \sum_{n_j \overline{\lambda}_j \prec \psi(\alpha^{(j)})} c_{\alpha^{(j)}} \cdot t^{\psi(\alpha^{(j)}) - n_j \overline{\lambda}_j} \cdot U_1^{\alpha_1^{(j)}} \cdots U_{d+j}^{\alpha_{d+j}^{(j)}}, \tag{23}$$

where $\alpha^{(j)} = \sum_{i=1}^{d+i} \alpha_i^{(j)} u_i$ and the coefficients satisfy $c_{\alpha^{(j)}} \in \mathbb{C}$, $c_j \in \mathbb{C}^*$ for $j = 1, \dots, g-1$, and $c_g = 0$. Notice that H_j is an element of the ring $\mathbb{C}[t^{\mathit{\Gamma}}][[U_1, \dots, U_{d+g}]]$ for $j = 1, \dots, g$.

The terms added to h_j in (23) have weight higher than $n_j \overline{\lambda}_j$. That is why we say that H_j is an overweight deformation of h_j. We use Teissier's terminology [65].

If p is a closed point in $Z^{\mathit{\Gamma}}$ we can evaluate the monomials of the form t^γ, for $\gamma \in \mathit{\Gamma}$, appearing in the expansion (23). We obtain the series

$$H_{j,p} = h_j + c_j \cdot t^{\overline{\lambda}_{j+1} - n_j \overline{\lambda}_j}(p) \cdot U_{d+j+1} + \sum_{\alpha^{(j)}} c_{\alpha^{(j)}} \cdot t^{\psi(\alpha^{(j)}) - n_j \overline{\lambda}_j}(p) \cdot U_1^{\alpha_1^{(j)}} \cdots U_{d+i}^{\alpha_{d+i}^{(j)}}. \tag{24}$$

Note that $H_{j,p} \in \mathbb{C}[[U_1, \dots, U_{d+g}]]$. The ideal $(H_{1,p}, \dots, H_{g,p})$ defines a germ S_p at the origin of $\mathbb{C}^{d+g}$, which may be seen as the fiber of a family $\mathcal{S}$ defined by $(H_1, \dots, H_g)$ over the point $p \in Z^{\mathit{\Gamma}}$. Next, we describe some of the fibers S_p of such a family, when p is the origin 0 of the toric variety $Z^{\mathit{\Gamma}}$ or when p belongs to the torus of $Z^{\mathit{\Gamma}}$.

Proposition 4.2 (see [20, Props. 3.15, 3.17] and [21, Prop. 39]). *Let $(S, 0)$ be a germ of quasi-ordinary singularity with associated semigroup $\mathit{\Gamma}$. There exists an overweight deformation $H_1, \dots, H_g$ of $h_1, \dots, h_g$ such that:*

(a) *If p belongs to the torus of $Z^{\mathit{\Gamma}}$ then the germ $(S_p, 0)$ is analytically isomorphic to $(S, 0)$.*
(b) *If $p = 0$ then $(S_0, 0) = (Z^{\mathit{\Gamma}}, 0) \subset (\mathbb{C}^{d+g}, 0)$.*

If p belongs to the torus of $Z^{\mathit{\Gamma}}$, or if $p = 0 \in Z^{\mathit{\Gamma}}$, we have a map

$$\iota_p : \mathbb{C}\{U_1, \dots, U_{d+g}\} \to \mathbb{C}\{\sigma^\vee \cap M\}, \tag{25}$$

defined as the composition of the canonical maps $\mathbb{C}\{U_1, \dots, U_{d+g}\} \to \mathcal{O}_{S_p}$ with the inclusion of $\mathcal{O}_{S_p}$ in its integral closure $\mathcal{O}_{\overline{S}_p}$, and then

$$\mathcal{O}_{\overline{S}_p} = \mathbb{C}\{\sigma^\vee \cap M\}. \tag{26}$$

For p in the torus, (26) is consequence of Lemma 2.4 and Proposition 4.2. See [21] for the case $p = 0$ of (26).

Remark 4.3. Take an overweight deformation $H_1, \dots, H_g$ of $h_1, \dots, h_g$ and a point p in the torus of Z^{Γ}. Then

$$t^{\overline{\lambda}_{j+1} - n_j \overline{\lambda}_j}(p) \neq 0 \quad \text{for } j = 1, \dots, g-1.$$

We can use this fact to eliminate recursively the variables $U_{d+g}, \dots, U_{d+2}$ in (23). We set $X_i := U_i$ for $i = 1, \dots, d+1$ and we obtain the equation of the embedding $(S_p, 0) \subset (\mathbb{C}^{d+1}, 0)$ with coordinates $(X_1, \dots, X_{d+1})$. By [21] the restriction to S_p of the projection

$$(X_1, \dots, X_{d+1}) \to (X_1, \dots, X_d)$$

is a quasi-ordinary projection, and S_p is parametrized by a quasi-ordinary branch $X_{d+1} = \zeta_p$, which has characteristic monomials $\lambda_1, \dots, \lambda_g$. It follows that $\mathcal{J}_{\log}(S_p) = \mathcal{J}_g$ for all p in the torus of Z^{Γ} (see Definition 3.2). In particular, if $\underline{1}$ is the unit point of the torus Z^M, we can set $(S, 0) := (S_{\underline{1}}, 0)$, and then the statement of Proposition 4.2 holds for this deformation.

4.2. The extension of the matrix of relations. We consider an overweight deformation $H_1, \dots, H_g$ of the binomials $h_1, \dots, h_g$ defining the toric variety Z^{Γ}. It defines a deformation $\mathcal{S}$ with fibers S_p for $p \in Z^{\Gamma}$.

Notation 4.4. We denote by $J(p)$ (resp. $\tilde{J}(p)$) the matrix with coefficients in the ring $\mathbb{C}\{\sigma^{\vee} \cap M\}$, which is defined from the Jacobian matrix of $(H_{1,p}, \dots, H_{g,p})$ by the formulas

$$J(p) := \left(\iota\left(\frac{\partial H_{j,p}}{\partial U_i} \right) \right)_{i=1,\dots,d+g}^{j=1,\dots,g} \quad \left(\text{resp. } \tilde{J}(p) := \left(\iota\left(U_i \frac{\partial H_{j,p}}{\partial U_i} \right) \right)_{i=1,\dots,d+g}^{j=1,\dots,g} \right),$$

where ι is the map defined by (25). We denote by $R(p)$ the matrix obtained from $\tilde{J}(p)$ by factoring out from the entries of the jth row the term $X^{n_j \overline{\lambda}_j}$ for $j = 1, \dots, g$. We denote the entry $L_i^{(j)}(p)$ of the matrix $R(p)$ in row j and in column i simply by $L_i^{(j)}$ to avoid cumbersome notation.

Lemma 4.5. *The matrix $R(0)$ has integer coefficients and coincides with the matrix of relations* (19) *associated with the presentation of the semigroup Γ.*

Proof. By Proposition 4.2 the terms appearing in H_j have weight greater than or equal to $n_j \overline{\lambda}_j$, with equality precisely for those terms appearing in h_j. This implies that $X^{n_j \overline{\lambda}_j}$ divides the entries of the jth row of $\tilde{J}(p)$ for

$j = 1, \ldots, g$. It follows that the matrix $R(p)$ is of the form

$$(27)\quad \begin{pmatrix} L_1^{(1)} & \cdots & L_{d+1}^{(1)} & \epsilon_1 X^{\overline{\lambda}_2 - n_1\overline{\lambda}_1} & 0 & 0 & 0 \\ L_1^{(2)} & \cdots & L_{d+1}^{(2)} & L_{d+2}^{(2)} & \epsilon_2 X^{\overline{\lambda}_3 - n_2\overline{\lambda}_2} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_1^{(g-1)} & \cdots & L_{d+1}^{(g-1)} & L_{d+2}^{(g-1)} & L_{d+3}^{(g-1)} & \cdots & \epsilon_{g-1} X^{\overline{\lambda}_g - n_{g-1}\overline{\lambda}_{g-1}} \\ L_1^{(g)} & \cdots & L_{d+1}^{(g)} & L_{d+2}^{(g)} & L_{d+3}^{(g)} & \cdots & L_{d+g}^{(g)} \end{pmatrix},$$

where $L_i^{(j)} \in \mathbb{C}\{\sigma^\vee \cap M\}$ and the coefficient

$$(28)\qquad \epsilon_j := c_j \cdot t^{\overline{\lambda}_{j+1} - n_j\overline{\lambda}_j}(p)$$

vanishes when $p = 0$, while it is nonzero if p belongs to the torus of Z^Γ.

The coefficient $L_i^{(j)}$ has an expansion of the form

$$(29)\quad L_i^{(j)} = \ell_i^{(j)} + \sum_{n_j\overline{\lambda}_j \prec \psi(\alpha^{(j)}),\, \alpha_i^{(j)} \geq 1} c_{\alpha^{(j)}} \cdot t^{\psi(\alpha^{(j)}) - n_j\overline{\lambda}_j}(p) \cdot X^{\psi(\alpha^{(j)}) - n_j\overline{\lambda}_j}$$

for $j = 1, \ldots, g$ and $i = 1, \ldots, d + j$. Notice that the condition $\alpha_i^{(j)} \geq 1$ characterizes when the partial derivative $\frac{\partial}{\partial U_i}(U_1^{\alpha_1^{(j)}} \cdots U_{d+i}^{\alpha_{d+i}^{(j)}})$ is nonzero. The weight condition $n_j\overline{\lambda}_j \prec \psi(\alpha^{(j)})$ implies that $t^{\psi(\alpha^{(j)}) - n_j\overline{\lambda}_j}(0) = 0$. ■

Definition 4.6. We say that the matrix $R(p)$ is the *extension of the matrix of relations* $R(0)$.

The following lemma is a consequence of the proof of Lemma 4.5.

Lemma 4.7. *If* $1 \leq j < m_i$ *then the monomial* X^{e_i} *divides* $L_i^{(j)}$ *for* $i = 1, \ldots, d$ *(see Notation* 2.15*).*

Proof. Consider the expansion (29) of $L_i^{(j)}$. Take $\alpha^{(j)} \in \mathbb{Z}^{d+g}_{\geq 0}$ such that $n_j\overline{\lambda}_j \prec \psi(\alpha^{(j)})$ and $\alpha_i^{(j)} \geq 1$. By Lemma 2.16(b), we have $\overline{\lambda}_{j,i} = 0$. Then the condition $\alpha_i^{(j)} \geq 1$ implies that $e_i \prec \psi(\alpha^{(j)}) - n_j\overline{\lambda}_j$, that is, X^{e_i} divides $X^{\psi(\alpha^{(j)}) - n_j\overline{\lambda}_j}$ in the ring $\mathbb{C}\{\sigma^\vee \cap M\}$. This implies the assertion. ■

4.3. Description of the image of the Jacobian ideal of S_p.

Next, we provide a description of the Jacobian ideal of S_p in terms of the matrix $R(p)$. The approach is inspired by Proposition 1.6 in the toric case.

Proposition 4.8. *The extension* $\mathrm{Jac}(S_p)\mathcal{O}_Z$ *of the Jacobian ideal of* S_p *in the local algebra of the normalization* Z *of* S_p *is generated by*

$$\{X^{\gamma_0 + \gamma_{i_1} + \cdots + \gamma_{i_d}} \cdot |R(p)_{i_1,\ldots,i_d}| \mid 1 \leq i_1 < \cdots < i_d \leq d + g\}.$$

Proof. The Jacobian ideal of S_p can be expressed in terms of the generators $H_{1,p}, \ldots, H_{g,p}$ of the defining ideal of S_p. It is the ideal of the local algebra $\mathcal{O}_{S_p}$ defined by the vanishing of the minors of order g of the Jacobian matrix $\big(\frac{\partial H_{j,p}}{\partial U_i}\big)$. It follows that the ideal $\mathrm{Jac}(S_p)\mathcal{O}_Z$ is defined by the minors $|J(p)_{i_1,\dots,i_d}|$ of order g of the matrix $J(p)$ for $1 \le i_1 < \dots < i_d \le d+g$ (see Notation 4.4). Then we argue as in the proof of (11) to obtain

$$X^{\gamma_1+\dots+\gamma_{d+g}}|J_{i_1,\dots,i_d}(p)| = X^{n_1\overline{\lambda}_1+\dots+n_g\overline{\lambda}_g}X^{\gamma_{i_1}+\dots+\gamma_{i_d}}|R(p)_{i_1,\dots,i_d}|. \tag{30}$$

The assertion follows by taking into account the description of the Frobenius vector γ_0 (see Proposition 2.10). ∎

Remark 4.9. By Lemma 1.1, if $p = 0$ is the origin of Z^{Γ} we have

$$|R(0)_{i_1,\dots,i_d}| = 0 \quad \text{if and only if} \quad \gamma_{i_1}\wedge\dots\wedge\gamma_{i_d} \neq 0. \tag{31}$$

We obtain the following formula, which connects the Jacobian ideal and the logarithmic Jacobian ideal:

$$\mathrm{Jac}(S_0)\mathcal{O}_Z = X^{\gamma_0}\cdot\mathcal{J}_{\log}(S_0)\mathcal{O}_Z = X^{\gamma_0}\cdot\mathcal{I}_g.$$

It is a particular case of Proposition 1.6 (see Remark 3.4 and Definition 3.2).

Remark 4.10. Let us apply the previous discussion to the case of plane curve singularities, which was one of the favorite research topics of Arkadiusz Płoski. We consider the case when S is a plane branch, that is, when $d = 1$. Assume that the line $X_1 = 0$ is not tangent to S. The algebra of the normalization of S is of the form $\mathcal{O}_Z = \mathbb{C}\{\sigma^\vee\cap M\}$, where $\sigma^\vee\cap M = \frac{1}{n}\mathbb{Z}_{\ge 0}$ and n is the multiplicity of S. This implies that $\gamma_1,\dots,\gamma_{g+1}\subset\frac{1}{n}\mathbb{Z}_{\ge 0}$ is the minimal system of generators of the semigroup Γ (see Definition 2.5).

Since $d = 1$, $\mathcal{O}_Z$ is a discrete valuation ring, and if $t := X^{1/n}$ then $\mathcal{O}_Z = \mathbb{C}\{t\}$. We denote by ν_t its t-adic valuation. Let us denote by $\overline{\beta}_0,\dots,\overline{\beta}_g \in \mathbb{N}$ the minimal system of generators of the *classical semigroup* of the branch S, defined by taking intersection multiplicities of S with other curves not containing S as a component (see [66, 47]). We have $\nu_t(X^{\gamma_i}) = n\gamma_i = \overline{\beta}_{i-1}$ for $i = 1,\dots,g+1$. The *multiplicity of the Jacobian ideal* of the branch S is defined by

$$e(\mathrm{Jac}(S)) = \nu_t(\mathrm{Jac}(S)\mathcal{O}_Z).$$

Let us explain how our method provides a formula for $e(\mathrm{Jac}(S_p))$ for $p\in Z^{\Gamma}$.

In this case, $R(p)$ is a $g\times(g+1)$ matrix. By Proposition 4.8, the ideal $\mathrm{Jac}(S_p)\mathcal{O}_Z$ is generated by $\{X^{\gamma_0+\gamma_i}\cdot|R(p)_i| \mid i = 1,\dots,g+1\}$. By definition we can expand the minor $|R(p)_i|$ as a series in $\mathbb{C}\{t\}$ with constant term $|R(0)_i|$. By (31) the minor $|R(0)_i|$ does not vanish since $\gamma_i\neq 0$, for $i = 1,\dots,g+1$. Notice also that $\min_{i=1,\dots,g+1}\{\gamma_0+\gamma_i\} = \gamma_0+\gamma_1$. It follows that $\mathrm{Jac}(S_p)\mathcal{O}_Z = (X^{\gamma_0+\gamma_1})\mathcal{O}_Z$. We deduce the equality

$$e(\mathrm{Jac}(S_p)) = \nu_t(X^{\gamma_0+\gamma_1}) = n(\gamma_0+\gamma_1) \overset{(17)}{=} \sum_{j=1}^{g}(n_j-1)\overline{\beta}_j. \tag{32}$$

This holds in particular when $p = \underline{1}$ is the unit point of the torus of Z^{Γ}, and when $p = 0$. These points define the fibers $S_{\underline{1}} = S$ and $S_0 = Z^{\Gamma}$.

A particular case of a result of Teissier [63, Prop. II.1.2] in the case of plane branches provides the relation

$$\mu(S) = e(\mathrm{Jac}(S)) - n + 1, \tag{33}$$

where $\mu(S)$ is the *Milnor number* and n is the multiplicity of the branch S. See [41, 28] for the case of complete intersection curves. Formula (32) can be deduced from (33) by using the expression of the Milnor number of a branch in terms of the minimal system of generators of its semigroup [66, Chapter II.3]; see also [14, Section 15]. Let us mention the contributions of Arkadiusz Płoski and Evelia García Barroso on the Milnor number of curves in arbitrary characteristic (see for instance [17]).

We come back to the general case of a quasi-ordinary hypersurface S of dimension d.

PROPOSITION 4.11. *For any point p in the torus of Z^Γ we have*

$$X^{\gamma_0}(\mathcal{I}_g + \mathcal{J}_g) \subset \mathrm{Jac}(S_p)\mathcal{O}_Z.$$

Proof. It is enough to prove the inclusions $X^{\gamma_0} \cdot \mathcal{I}_g \subset \mathrm{Jac}(S_p)\mathcal{O}_Z$ and $X^{\gamma_0} \cdot \mathcal{J}_g\mathcal{O}_Z \subset \mathrm{Jac}(S_p)\mathcal{O}_Z$.

To prove $X^{\gamma_0} \cdot \mathcal{I}_g \subset \mathrm{Jac}(S_p)\mathcal{O}_Z$, take $1 \leq i_1 < \cdots < i_d \leq d + g$ such that $\gamma_{i_1} \wedge \cdots \wedge \gamma_{i_d} \neq 0$. We obtain a generator $X^{\gamma_{i_1}+\cdots+\gamma_{i_d}}$ of the ideal $\mathcal{I}_g$ (see Definition 3.2). Then

$$X^{\gamma_0+\gamma_{i_1}+\cdots+\gamma_{i_d}} \cdot |R(p)_{i_1,\ldots,i_d}| \in \mathrm{Jac}(S_p)\mathcal{O}_Z$$

by Proposition 4.8. Notice that $|R(p)_{i_1,\ldots,i_d}|$ is a unit of $\mathcal{O}_Z = \mathbb{C}[[\sigma^\vee \cap M]]$, since it can be expanded as a power series with constant term equal to $|R(0)_{i_1,\ldots,i_d}|$, which is nonzero by hypothesis and (31). Thus, the monomial $X^{\gamma_0+\gamma_{i_1}+\cdots+\gamma_{i_d}}$ belongs to $\mathrm{Jac}(S_p)\mathcal{O}_Z$.

To prove $X^{\gamma_0} \cdot \mathcal{J}_g \subset \mathrm{Jac}(S_p)\mathcal{O}_Z$, notice that by the previous case $X^{\gamma_0+\cdots+\gamma_d}$ belongs to $\mathrm{Jac}(S_p)\mathcal{O}_Z$. By Definition 3.2, it is enough to prove that for any vector α of the form

$$\alpha = e_1 + \cdots + \widehat{e_i} + \cdots + e_d + \lambda_{m_i} \quad \text{with} \quad i \in \{1, \ldots, d\},\ m_i \in \{1, \ldots, g\},$$

the monomial $X^{\gamma_0+\alpha}$ belongs to $\mathrm{Jac}(S_p)\mathcal{O}_Z$.

If $m_i = 1$ then $e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_d \wedge \lambda_1 \neq 0$. Since $\lambda_1 = \overline{\lambda}_1$ it follows that $X^{e_1+\cdots+\widehat{e_i}+\cdots+e_d+\lambda_1}$ belongs to $\mathcal{I}_g$ and the assertion follows by the previous case (see Definition 2.5).

We assume that $1 < m_i \leq g$ and we set

$$\delta := \gamma_0 + \gamma_1 + \cdots + \widehat{\gamma_i} + \cdots + \gamma_d + \gamma_{d+1}. \tag{34}$$

We use the fact that

$$X^\delta |R(p)_{1,\ldots,\hat{\imath},\ldots,d+1}| \in \mathrm{Jac}(S_p)\mathcal{O}_Z \tag{35}$$

by Proposition 4.8. We expand the minor $|R(p)_{1,\dots,\hat{i},\dots,d+1}|$ along its first column to get

$$|R(p)_{1,\dots,\hat{i},\dots,d+1}| = \sum_{r=1}^{g} (-1)^{r+1} L_i^{(r)} |R(p)^r_{1,\dots,d+1}|.$$

If $1 \leq r < m_i$, then X^{e_i} divides $L_i^{(r)}$ by Lemma 4.7. It follows that the term $X^{\delta} L_i^{(r)} |R(p)^r_{1,\dots,d+1}|$ is divisible by the monomial $X^{\gamma_0+\gamma_1+\dots+\gamma_d} \in \operatorname{Jac}(S_p)\mathcal{O}_Z$, and thus

$$\sum_{i=1}^{m_i-1} X^{\delta} L_i^{(r)} |R(p)^r_{1,\dots,d+1}| \in \operatorname{Jac}(S_p)\mathcal{O}_Z. \tag{36}$$

If $m_i \leq r \leq g$, then we use the structure of the matrix $R(p)$ (see (27)). The matrix $R(p)^r_{1,\dots,d+1}$ is block lower triangular. It has two blocks on the diagonal,

$$A_r(p) := R(p)^{r,\dots,g}_{1,\dots,d+1,d+r+1,\dots,d+g} \quad \text{and} \quad B_r(p) = R(p)^{1,\dots,r}_{1,\dots,d+r},$$

where

$$A_r(p) = \begin{pmatrix} \epsilon_1 X^{\overline{\lambda}_2 - n_1 \overline{\lambda}_{j_1}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ * & * & \dots & \epsilon_{r-1} X^{\overline{\lambda}_r - n_{r-1}\overline{\lambda}_{r-1}} \end{pmatrix},$$

and

$$B_r(0) = \begin{pmatrix} n_{r+1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ * & * & \dots & n_g \end{pmatrix}. \tag{37}$$

Notice that the coefficients (28) are nonzero if p belongs to the torus of Z^{Γ}. Then the equality

$$\overline{\lambda}_2 - n_1\overline{\lambda}_1 + \dots + \overline{\lambda}_r - n_{r-1}\overline{\lambda}_{r-1} \overset{(13)}{=} \lambda_2 - \lambda_1 + \dots + \lambda_r - \lambda_{r-1} = \lambda_r - \lambda_1$$

implies that the determinant $|A_r(p)|$ is of the form $X^{\lambda_r - \lambda_1}$ times a unit.

The determinant $|B_r(p)|$ is a power series in the ring $\mathbb{C}[[\sigma^{\vee} \cap M]]$ with nonzero constant term $|B_r(0)| = n_{r+1} \cdots n_g$ by (37). By using (12), there exists a unit $\epsilon \in \mathbb{C}[[\sigma^{\vee} \cap M]]$ such that

$$\sum_{r=m_i}^{g} (-1)^{r+1} L_i^{(r)} |R(p)^r_{1,\dots,d+1}| = X^{\lambda_{m_i} - \lambda_1} \cdot \epsilon.$$

By (35) and (36) we deduce that $X^{\delta + \lambda_{m_i} - \lambda_1} \cdot \epsilon \in \operatorname{Jac}(S_p)\mathcal{O}_Z$. This implies the assertion, since by (34) and Definition 2.5 we have

$$\begin{aligned}\delta + \lambda_{m_i} - \lambda_1 &= \gamma_0 + \gamma_1 + \cdots + \widehat{\gamma_i} + \cdots + \gamma_d + \gamma_{d+1} + \lambda_{m_i} - \lambda_1 \\ &= \gamma_0 + e_1 + \cdots + \widehat{e_i} + \cdots + e_d + \lambda_{m_i}. \ \blacksquare\end{aligned}$$

Next, we prove the following inclusion of the monomial ideals in Definition 3.2.

PROPOSITION 4.12. *We have* $\mathcal{I}_g \subset \mathcal{J}_g$.

Proof. Take $1 \leq j_1 < \cdots < j_k \leq d$ and $1 \leq i_{k+1} < \cdots < i_d \leq d$ such that

$$\overline{\lambda}_{j_1} \wedge \cdots \wedge \overline{\lambda}_{j_k} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_{i_d} \neq 0. \tag{38}$$

Set $\alpha = \sum_{\ell=1}^{k} \overline{\lambda}_{j_\ell} + \sum_{\ell=k+1}^{d} e_{i_\ell}$. It is enough to prove that for any α of this form there exists an element $\beta \in \Xi_g$ (see (21)) such that $\beta \preceq \alpha$. For instance, if $k = 0$ then $\alpha = \sum_{\ell=1}^{d} e_\ell \in \Xi_g$ and we take $\beta = \alpha$. Up to relabeling, we may assume that $i_\ell = \ell$ for $\ell \in \{k+1, \ldots, d\}$ and then

$$\alpha = \sum_{\ell=1}^{k} \overline{\lambda}_{j_\ell} + \sum_{\ell=k+1}^{d} e_\ell$$

(we do not assume here that the quasi-ordinary branch ζ has well-ordered variables). If $k = 1$ then $\alpha = \overline{\lambda}_{j_1} + e_2 + \cdots + e_d$, and $\lambda_{j_1,1} \neq 0$ by (38). By Lemma 2.16(b), we have $m_1 \leq j_1$. Then we can take $\beta := \overline{\lambda}_{m_1} + e_2 + \cdots + e_d \in \Xi_g$, and $\beta \preceq \alpha$.

Next, assume that $k > 1$. Condition (38) means that the matrix

$$C := (\overline{\lambda}_{j_\ell, i})_{i=1,\ldots,k}^{\ell=1,\ldots,k}$$

has rank k.

CLAIM. *If* $i \in \{1, \ldots, k\}$ *then*

$$m_i \leq j_k, \tag{39}$$

and equality holds in (39) *for at most one index* $i_0 \in \{1, \ldots, k\}$.

Notice that $m_i > j_k$ implies that the ith column of the matrix C is zero, and this contradicts the condition $|C| \neq 0$ (see Lemma 2.16(b)). If $m_{i_0} = j_k$, the ith column vector of the matrix C is of the form $(0, \ldots, 0, \overline{\lambda}_{j_k, i_0})$ with $\overline{\lambda}_{j_k, i_0} \neq 0$. Notice that we cannot have two columns of this form since $|C| \neq 0$. This ends the proof of the claim.

- If the inequality (39) is strict for all $i \in \{1, \ldots, k\}$, then by Lemma 2.16(d), the inequality

$$\overline{\lambda}_{j_k, i} \geq 1 \tag{40}$$

holds for $i = 1, \ldots, k$. Then $\beta = e_1 + \cdots + e_d$ belongs to Ξ_g and $\beta \preceq \alpha$. This ends the proof in this case.

• If equality holds in (39) for one index i_0, we suppose up to relabeling that $i_0 = k$. By Lemma 2.16(d), the inequality (40) holds for $i = 1, \dots, k-1$.

We distinguish two subcases.

If $\overline{\lambda}_{j_k,k} \geq 1$, then we can take $\beta = e_1 + \cdots + e_d \in \Xi_g$, and $\beta \preceq \alpha$ as in the previous case.

Otherwise, we have $0 < \overline{\lambda}_{j_k,k} < 1$. Then we can take $\beta = \lambda_{j_k} + e_1 + \cdots + e_{k-1} + e_{k+1} + \cdots + e_d \in \Xi_g$, since $m_k = j_k$. We claim that $\beta \preceq \alpha$. It is equivalent to prove that

$$\lambda_{j_k} + \sum_{\ell=1}^{k-1} e_\ell \preceq \sum_{\ell=1}^{k} \overline{\lambda}_{j_\ell}. \tag{41}$$

Notice that the kth column vector of C is $(0, \dots, 0, \overline{\lambda}_{j_k,k})$. This implies that the first minor of order $k-1$ of C is nonzero. As in the previous case, we use this fact to prove that if $i \in \{1, \dots, k-1\}$ then

$$m_i \leq j_{k-1}, \tag{42}$$

with equality for at most one index i_1. We then distinguish two subcases:

If the inequality in (42) is strict for all $i \in \{1, \dots, k-1\}$ then $\overline{\lambda}_{j_{k-1},i} \geq 1$ by Lemma 2.16(d). This implies that $e_1 + \cdots + e_{k-1} \preceq \overline{\lambda}_{j_{k-1}}$ and (41) holds since $\lambda_{j_k} \preceq \overline{\lambda}_{j_k}$.

Otherwise, we have equality in (42) for some index i_1. We may suppose that $i_1 = k-1$. If $\overline{\lambda}_{j_{k-1},k-1} \geq 1$ then (41) holds since (42) implies that $\overline{\lambda}_{j_{k-1},i} \geq 1$ for $i \in \{1, \dots, k-2\}$ by Lemma 2.16(d).

It remains to handle the case $0 < \overline{\lambda}_{j_{k-1},k-1} < 1$. We set $j = j_k$ and $p = j_{k-1}$ for simplicity. Using (14) we compute

$$\begin{aligned} \overline{\lambda}_j - \lambda_j + \overline{\lambda}_p = {} & \sum_{\ell=1}^{p-1} (n_\ell - 1) n_{\ell+1} \cdots n_{p-1} \cdot (1 + n_p \cdots n_{j-1}) \lambda_\ell \\ & + (1 + (n_p - 1) n_{p+1} \cdots n_{j-1})\, \lambda_p \\ & + \sum_{\ell=p+1}^{j-1} (n_\ell - 1) n_{\ell+1} \cdots n_{j-1} \lambda_\ell. \end{aligned}$$

Notice that the coefficient $a_p := 1 + (n_p - 1) n_{p+1} \cdots n_{j-1}$ of λ_p in this formula satisfies $a_p > n_p$, since $a_p - n_p = (n_{p+1} \cdots n_{j-1} - 1)(n_p - 1) \geq 1$.

Take $i \in \{1, \dots, k-1\}$. Then

$$\overline{\lambda}_{j,i} - \lambda_{j,i} + \overline{\lambda}_{p,i} \geq a_p\, \lambda_{p,i} > n_p\, \lambda_{p,i} = n_{j_{k-1}} \lambda_{j_{k-1},i}, \tag{43}$$

where the last equality is just rewriting.

If $i \in \{1, \dots, k-2\}$, then we have a strict inequality in (42), and by Lemma 2.16(d) we get $\overline{\lambda}_{j_{k-1},i} \geq 1$. In addition, by Lemma 2.16(c), the condition $0 < \overline{\lambda}_{j_{k-1},k-1} < 1$ implies $\overline{\lambda}_{j_{k-1},k-1} = \lambda_{j_{k-1},k-1}$ and $n_{j_{k-1}} \lambda_{j_{k-1},k-1} \geq 1$.

By combining these inequalities with (43) we get

$$\overline{\lambda}_{j_k,i} - \lambda_{j_k,i} + \overline{\lambda}_{j_{k-1},i} \geq 1 \quad \text{for } i \in \{1, \dots, k-1\}. \tag{44}$$

This means that

$$\lambda_{j_k} + e_1 + \cdots + e_{k-1} \preceq \overline{\lambda}_{j_k} + \overline{\lambda}_{j_{k-1}} \preceq \overline{\lambda}_{j_k} + \overline{\lambda}_{j_{k-1}} + \cdots + \overline{\lambda}_{j_1}.$$

Therefore, (41) holds and this ends the proof. ■

As a corollary of Theorem 3.3 and Propositions 4.11 and 4.12 we deduce the following result.

THEOREM 4.13. *For any point p in the torus of Z^{Γ} we have*

$$X^{\gamma_0}\, \mathcal{J}_{\log}(S_p) \subset \mathrm{Jac}(S_p)\mathcal{O}_Z.$$

Next, we study the case of quasi-ordinary surfaces.

PROPOSITION 4.14. *Assume that $d = 2$ and S is parametrized by a normalized quasi-ordinary branch ζ (see Definition 2.2). Take a point p in the torus of Z^{Γ}. Let I be the ideal of $\mathcal{O}_Z$ generated by*

$$\{X^{\gamma_{i_1}+\gamma_{i_2}} \cdot |R(p)_{i_1,i_2}| \mid 1 \leq i_1 < \cdots < i_2 \leq 2+g\}.$$

Then $I \subset \mathcal{J}_g$.

Proof. Since ζ is normalized, we have

$$\lambda_{1,1} \neq 0 \quad \text{and} \quad \text{if } \lambda_{1,2} = 0 \text{ then } \lambda_{1,1} > 1. \tag{45}$$

If $1 \leq i_1 < i_2 \leq 2+g$ and if $\gamma_{i_1} \wedge \gamma_{i_2} \neq 0$ then $X^{\gamma_{i_1}+\gamma_{i_2}} \cdot |R(p)_{i_1,i_2}| \in \mathcal{I}_g$. Here we use $\mathcal{I}_g \subset \mathcal{J}_g$ by Proposition 4.12.

By the hypothesis (45) and (12), if $\gamma_{i_1} = \overline{\lambda}_{j_1}$ and $\gamma_{i_2} = e_2$ then we get $\overline{\lambda}_{j_1} \wedge e_2 \neq 0$. Thus, if $\gamma_{i_1} \wedge \gamma_{i_2} = 0$ then there are two cases:

(a) $i_1 = 1$ and $i_2 = 2 + j_1$ with $1 \leq j_1 \leq g$ and $e_1 \wedge \overline{\lambda}_{j_1} = 0$.
(b) $i_1 = 2 + j_1$, $i_2 = 2 + j_2$ with $1 \leq j_1 < j_2 \leq g$ and $\overline{\lambda}_{j_1} \wedge \overline{\lambda}_{j_2} = 0$.

Case (a). The case $j_1 = 1$ was considered in the proof of Proposition 4.11. It is enough to consider the case $j_1 > 1$. Recall the definition of the index m_2 introduced in Definition 2.15. By the hypothesis $e_1 \wedge \overline{\lambda}_1 = 0$, we get $e_1 \wedge \lambda_1 = 0$, thus $j_1 < m_2$. We expand the minor $|R(p)_{1,2+j_1}|$ along its first column to get

$$|R(p)_{1,2+j_1}| = \sum_{j=1}^{g} L_2^{(j)} |R(p)^{j}_{1,2,2+j_1}|.$$

We distinguish two subcases:

If $1 \leq j < m_2$ then X^{e_2} divides the coefficient $L_2^{(j)}$ (see Lemma 4.7). It follows that $X^{e_1+e_2}$ divides the term $X^{\gamma_1+\gamma_{2+j_1}} L_2^{(j)} |R(p)^{j}_{1,2,2+j_1}|$, hence this term belongs to $\mathcal{I}_g$. This ends the proof of this case if $m_2 = g + 1$.

If $m_2 \leq j \leq g$, the matrix $|R(p)^{j}_{1,2,2+j_1}|$ is block lower triangular. It has three blocks on the diagonal, A_j, B_j and C_j, where

$$A_j = R(p)^{j_1,\dots,g}_{1,2,2+j_1,\dots,2+g}, \quad B_j = R(p)^{1,\dots,j_1-1,j,\dots,g}_{1,2,\dots,2+j_1,2+j+1,\dots,2+g}, \quad C_j = R(p)^{1,\dots,j}_{1,\dots,2+j}.$$

In particular, the matrix B_j is diagonal of the form

$$B_j = \begin{pmatrix} \epsilon_{j_1} X^{\overline{\lambda}_{j_1+1} - n_{j_1}\overline{\lambda}_{j_1}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ * & * & \dots & \epsilon_{j-1} X^{\overline{\lambda}_{j} - n_{j-1}\overline{\lambda}_{j-1}} \end{pmatrix}. \tag{46}$$

Notice that $\sum_{s=j_1}^{j-1} \overline{\lambda}_{s+1} - n_s \overline{\lambda}_s = \sum_{s=j_1}^{j-1} \lambda_{s+1} - \lambda_s = \lambda_j - \lambda_{j_1}$. We deduce that $X^{\lambda_j - \lambda_{j_1}}$ divides the minor $|R(p)^{j}_{1,2,2+j_1}|$. It follows that $X^{e_1+\overline{\lambda}_{j_1}} |R(p)^{j}_{1,2,2+j_1}|$ is divisible by $X^{e_1+\lambda_j}$, since $\lambda_{j_1} \preceq \overline{\lambda}_{j_1}$. Since $m_2 \leq j$ we have $e_1 \wedge \lambda_j \neq 0$, hence $X^{e_1+\lambda_j}$ belongs to $\mathcal{J}_g$. This ends the proof in this case.

Case (b). By hypothesis $\lambda_{j_1,1} \neq 0$, and since $j_1 < j_2$, we get $\overline{\lambda}_{j_2,1} \geq 1$.

Assume first that $\lambda_{j_1,2} \neq 0$. By Lemma 2.16(c) we get $n_{j_1}\overline{\lambda}_{j_1,2} \geq 1$. Since $j_2 > j_1$, we obtain $\overline{\lambda}_{j_2,2} \geq \overline{\lambda}_{j_1,2} \geq 1$ by definition (13) and relation (12). We have shown that $e_1 + e_2 \prec \overline{\lambda}_{j_2}$, therefore $X^{\overline{\lambda}_{j_1}+\overline{\lambda}_{j_2}} |R(p)_{2+j_1,2+j_2}|$ belongs to $\mathcal{I}_g$, since it is divisible by $X^{e_1+e_2} \in \mathcal{I}_g$.

Otherwise we must have $\lambda_{j_1,2} = 0$ and then $\lambda_{j_2,2} = 0$, since $\overline{\lambda}_{j_1} \wedge \overline{\lambda}_{j_2} = 0$, and by definition of m_2 we also have $j_1 < j_2 < m_2$. We expand the minor $|R(p)_{2+j_1,2+j_2}|$ along its second column to get

$$|R(p)_{2+j_1,2+j_2}| = \sum_{j=1}^{g} L_2^{(j)} |R(p)^{j}_{2,2+j_1,2+j_2}|.$$

We distinguish two subcases:

If $1 \leq j < m_2$ we have $\lambda_{j,2} = 0$ by Lemma 2.16(b). Then the monomial X^{e_2} divides the coefficient $L_2^{(j)}$ (see Lemma 4.7). Since $\lambda_{j_1,1} \neq 0$, we get $\overline{\lambda}_{j_2,1} \geq 1$ and thus $e_1 \prec \overline{\lambda}_{j_2}$. The monomial $X^{e_1+e_2} \in \mathcal{I}_g$ divides the term $X^{\overline{\lambda}_{j_1}+\overline{\lambda}_{j_2}} L_2^{(j)} |R(p)^{j}_{2,2+j_1,2+j_2}|$, hence this term belongs to $\mathcal{I}_g$. This ends the proof when $m_2 = g+1$.

If $m_2 \leq j \leq g$, the matrix $R(p)^{j}_{2,2+j_1,2+j_2}$ is block lower triangular. It has three blocks on the diagonal, A_j, B_j and C_j, where

$$A_j = R(p)^{j_2,\dots,g}_{2,2+j_1,2+j_2,\dots,2+g}, \; B_j = R(p)^{1,\dots,j_2-1,j,\dots,g}_{1,\dots,2+j_2,2+j+1,\dots,2+g}, \; C_j = R(p)^{1,\dots,j}_{1,\dots,2+j}.$$

In particular, B_j is diagonal of the form (46) with j_1 replaced by j_2. Therefore, the minor $|B_j|$ is equal to the monomial $X^{\lambda_j - \lambda_{j_2}}$ times a unit. As in the previous case, the term $X^{\overline{\lambda}_{j_1}+\overline{\lambda}_{j_2}} |R(p)^{j}_{2,2+j_1,2+j_2}|$ is divisible by $X^{e_1+\lambda_j}$, since $\lambda_{j_2} \preceq \overline{\lambda}_{j_2}$. Since $m_2 \leq j$ we have $e_1 \wedge \lambda_j \neq 0$, hence $X^{e_1+\lambda_j}$ belongs to $\mathcal{J}_g$. This ends the proof in this case. ∎

The following corollary is an immediate consequence of Propositions 4.8 and 4.14, and Theorems 3.3 and 4.13.

Corollary 4.15. *Assume that the dimension of the quasi-ordinary hypersurface S is* 2. *Then*

$$\mathrm{Jac}(S_p)\mathcal{O}_Z = X^{\gamma_0}\mathcal{J}_{\log}(S_p)$$

for any point p in the torus of Z^Γ.

Remark 4.16. It is an open question whether the inclusion in Theorem 4.13 is an equality when the dimension of S is greater than 2.

Corollary 4.17. *Let S be an analytically irreducible quasi-ordinary hypersurface singularity of dimension* 2. *The composite of the normalization map of S with the normalized blow up of the monomial ideal $\mathcal{J}_{\log}(S)$ is the normalized Nash modification of S.*

Proof. Since S is a complete intersection, the Nash modification of S is isomorphic to the blow up of the Jacobian ideal in $\mathcal{O}_S$ (see [45]). Then the normalized Nash modification of S is equal to the composition of the normalization map with the normalized blow up of $\mathrm{Jac}(S)\mathcal{O}_Z$ (see [43, Propositions 3.2 and 3.3]). Now the statement follows from Corollary 4.15, since the ideals $\mathrm{Jac}(S)\mathcal{O}_Z$ and $\mathcal{J}_{\log}(S)\mathcal{O}_Z$ are related by invertible monomial ideals, hence they have isomorphic blow ups. ■

Example 4.18. Let $(S,0)$ be a quasi-ordinary hypersurface parametrized by a quasi-ordinary branch ζ with characteristic exponents

$$\lambda_1 = (3/2, 0), \quad \lambda_2 = (7/4, 0), \quad \lambda_3 = (2, 1/2).$$

The characteristic integers are $n_1 = n_2 = n_3 = 2$. The generators of the semigroup Γ are

$$e_1 = (1,0) \quad, e_2 = (0,1), \quad \overline{\lambda}_1 = (3/2, 0), \quad \overline{\lambda}_2 = (13/4, 0), \quad \overline{\lambda}_3 = (27/4, 1/2).$$

Notice that the group M generated by Γ is equal to the lattice with basis $v_1 := \frac{1}{4}e_1$, $v_2 := \frac{1}{2}e_2$. The saturation of Γ in M is the semigroup $\sigma^\vee \cap M = \mathbb{Z}_{\geq 0}v_1 + \mathbb{Z}_{\geq 0}v_2$. This implies that the normalization $Z = Z^{\sigma^\vee \cap M}$ of S is smooth, and $\mathcal{O}_Z$ is isomorphic to the ring of convergent power series $\mathbb{C}\{z_1, z_2\}$ where $z_1 = X^{v_1}$ and $z_2 = X^{v_2}$. For simplicity, we write the coordinates of the generators of the semigroup Γ (resp. of the characteristic exponents) with respect to the basis v_1, v_2. We obtain

$$e_1 = (4,0), \quad e_2 = (0,2), \quad \overline{\lambda}_1 = (6,0), \quad \overline{\lambda}_2 = (13,0), \quad \overline{\lambda}_3 = (27,1),$$

resp.

$$\lambda_1 = (6,0), \quad \lambda_2 = (7,0), \quad \lambda_3 = (8,1).$$

The minimal Frobenius vector with respect to this basis is $\gamma_0 = (42, -1)$. The following relations hold:

$$2\overline{\lambda}_1 = 3e_1, \quad 2\overline{\lambda}_2 = 5e_1 + \overline{\lambda}_1, \quad 2\overline{\lambda}_3 = 12e_1 + e_2 + \overline{\lambda}_1.$$

The matrix of relations is

$$R_0 = \begin{pmatrix} 3 & 0 & -2 & 0 & 0 \\ 5 & 0 & 1 & -2 & 0 \\ 12 & 1 & 1 & 0 & -2 \end{pmatrix}.$$

By Definition 3.2 and Proposition 4.12 we get

$$\mathcal{I}_3 = (X^{e_1+e_2}, X^{e_2+\overline{\lambda}_1}, X^{e_1+\overline{\lambda}_3}) \subset \mathcal{J}_3 = (X^{e_1+e_2}, X^{e_2+\lambda_1}, X^{e_1+\lambda_3}) \subset \mathcal{O}_Z.$$

By Corollary 4.11, if p is a point in the torus of Z^{Γ}, then $\mathrm{Jac}(S_p)\mathcal{O}_Z$ is the monomial ideal $X^{\gamma_0}\mathcal{J}_3$. By Remark 4.9, if $p = 0$ then $\mathrm{Jac}(S_0)\mathcal{O}_Z$ is the monomial ideal $X^{\gamma_0}\mathcal{I}_3$.

5. An application. In this section we give a method to compute the number $\overline{\nu}_{\mathrm{Jac}(S)}(\phi)$ associated to a function $\phi \in \mathcal{O}_S$ and the Jacobian ideal $\mathrm{Jac}(S)$ when S is an irreducible quasi-ordinary hypersurface of dimension 2.

We first recall the definition of the numbers $\overline{\nu}_I(a)$ and $\overline{\nu}_{\mathcal{I}}(\mathcal{J})$ associated to an element $a \in A$ and a pair of proper ideals $\mathcal{I}, \mathcal{J}$ of a ring A, given by Lejeune-Jalabert and Teissier [43]. These numbers have many interesting applications in analytic geometry, in particular in connection with the Łojasiewicz exponents and analytic arcs (see [43, 62, 36]). We show next how to describe the invariant $\overline{\nu}_{\mathcal{I}}(\mathcal{J})$ explicitly when $A = \mathbb{C}\{\Lambda\}$ is the ring of germs of functions at the origin of an affine toric variety Z^{Λ}, and $\mathcal{I}$ and $\mathcal{J}$ are a pair of monomial ideals of A.

5.1. The number $\overline{\nu}_I(a)$. Let A be a ring and $\mathcal{I} \neq A$ be an ideal. Let us consider first the order function $\nu_{\mathcal{I}} : A \to \mathbb{Z}$ associated with the $\mathcal{I}$-adic filtration $A = \mathcal{I}^0 \supset \mathcal{I} \supset \mathcal{I}^2 \supset \cdots$ of A, which is defined by

$$\nu_{\mathcal{I}}(a) := \sup\{n \mid a \in \mathcal{I}^n\} \in \mathbb{Z}_{\geq 0}.$$

If $\mathcal{J}$ is an ideal of A we set $\nu_{\mathcal{I}}(\mathcal{J}) := \sup\{n \mid \mathcal{J} \subset \mathcal{I}^n\}$. We also have an order function $\overline{\nu}_{\mathcal{I}} : A \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ defined by

$$\overline{\nu}_I(a) := \lim_{k\to\infty} \frac{\nu_I(a^k)}{k}.$$

If $\mathcal{J}$ is an ideal of A then we set $\overline{\nu}_{\mathcal{I}}(\mathcal{J}) := \lim_{k\to\infty} \frac{\nu_I(\mathcal{J}^k)}{k}$. If $\mathcal{J}$ is generated by $a_1, \ldots, a_s$ then $\overline{\nu}_{\mathcal{I}}(\mathcal{J}) := \min_{i=1,\ldots,s} \overline{\nu}_I(a_i)$. These numbers belong to $\mathbb{R}_{\geq 0} \cup \{\infty\}$ (see [43]).

5.2. Normalized blow up of a monomial ideal in a toric variety. The *Newton polyhedron* of $\phi = \sum c_\alpha X^\alpha \in \mathbb{C}\{\sigma^\vee \cap M\}$ is the convex hull $\mathcal{N}(\phi)$ of the union of the Minkowski sum of the sets $\alpha + \sigma^\vee$ for α such that $c_\alpha \neq 0$. Similarly, the *Newton polyhedron* of a monomial ideal $\mathcal{I} = (X^u / u \in I)$ of $\mathbb{C}\{\sigma^\vee \cap M\}$ is the convex hull $\mathcal{N}(\mathcal{I})$ of the Minkowski sum of the sets $I + \sigma^\vee$. If $\mathcal{N}$ is the Newton polyhedron of a function or of a monomial ideal we denote by $\mathrm{ord}_{\mathcal{N}}$ the support function of the polyhedron $\mathcal{N}$, which is defined by

$$\mathrm{ord}_{\mathcal{N}} : \sigma \to \mathbb{R}, \qquad \nu \mapsto \inf_{\omega \in \mathcal{N}} \langle \nu, \omega \rangle.$$

The *dual fan* $\Sigma(\mathcal{I})$ associated to the polyhedron $\mathcal{N}(\mathcal{I})$ consists of the cones

$$\sigma(\mathcal{F}) := \{\eta \in \sigma \mid \langle \eta, v \rangle = \mathrm{ord}_I(\eta),\ \forall v \in \mathcal{F}\},$$

for $\mathcal{F}$ running through the faces of $\mathcal{N}(\mathcal{I})$.

If $\Sigma = \Sigma(\mathcal{I})$, the associated toric modification $\pi_\Sigma : Z_\Sigma \to Z_\sigma$ is the *normalized blow up* of Z_σ centered at the monomial ideal $\mathcal{I}$ of $\mathbb{C}\{\sigma^\vee \cap M\}$ (see [38, Chapter 1] or [42]).

Let $\{D_k\}_{k \in K(\mathcal{I})}$ be the irreducible components of the exceptional divisor of π_Σ which is defined by the ideal sheaf $\mathcal{I}\mathcal{O}_{Z_\Sigma}$. Each component D_k is an invariant divisor for the torus action, and the divisorial valuation ν_{D_k} defined by D_k is a monomial valuation of the form

$$\nu_{D_k}\Big(\sum a_m X^m\Big) := \inf_{a_m \neq 0} \langle n_k, m \rangle,$$

where n_k is a primitive vector for the lattice N in an edge ρ of Σ such that, if τ is the unique face of σ with $\rho \subset \mathrm{int}(\tau)$, then the orbit orb_τ is contained in the zero locus of $\mathcal{I}$ in Z^Λ. Then we have

$$\nu_{D_k}(\mathcal{I}\mathcal{O}_{Z_\Sigma}) = \min_{u \in I} \nu_{D_k}(X^u) = \mathrm{ord}_{\mathcal{I}}(n_k). \tag{47}$$

See [42, 38, 23] for more details.

5.3. The number $\nu_I(\phi)$ in the toric case. Suppose from now on that $A = \mathbb{C}\{\Lambda\}$ and $\mathcal{I} = (X^u \mid u \in I)$, $\phi \in \mathbb{C}\{\Lambda\}$ and $\mathcal{J} = (X^u \mid u \in J)$ are proper monomial ideals of $\mathbb{C}\{\Lambda\}$, for Λ as in Section 1.1. By [43, Prop. 0.20] we have

$$\overline{\nu}_{\mathcal{I}}(\phi) = \overline{\nu}_{\mathcal{I}\mathbb{C}\{\sigma^\vee \cap M\}}(\phi) \quad \text{and} \quad \overline{\nu}_{\mathcal{I}}(\mathcal{J}) = \overline{\nu}_{\mathcal{I}\mathbb{C}\{\sigma^\vee \cap M\}}(\mathcal{J}\mathbb{C}\{\sigma^\vee \cap M\}).$$

In order to describe the numbers $\overline{\nu}_{\mathcal{I}}(\phi)$ and $\overline{\nu}_{\mathcal{I}}(\mathcal{J})$ we can assume that $\Lambda = \sigma^\vee \cap M$.

PROPOSITION 5.1. *With the previous notations we have*

$$\overline{\nu}_{\mathcal{I}}(\phi) = \min_{k \in K(\mathcal{I})} \frac{\mathrm{ord}_{\mathcal{N}(\phi)}(n_k)}{\mathrm{ord}_{\mathcal{I}}(n_k)} \quad \textit{and} \quad \overline{\nu}_{\mathcal{I}}(\mathcal{J}) = \min_{k \in K(\mathcal{I})} \frac{\mathrm{ord}_{\mathcal{J}}(n_k)}{\mathrm{ord}_{\mathcal{I}}(n_k)}.$$

Proof. If $k \in K(\mathcal{I})$ then $\nu_{D_k}(\mathcal{I}\mathcal{O}_{Z_\Sigma}) = \mathrm{ord}_{\mathcal{I}}(n_k)$ by (47). By the results of [43, §4.1], if $\phi \in \mathbb{C}\{\sigma^\vee \cap M\}$ then

$$\overline{\nu}_{\mathcal{I}}(\phi) = \min_{k \in K(\mathcal{I})} \frac{\nu_{D_k}(\phi)}{\nu_{D_k}(\mathcal{I}\mathcal{O}_{Z_\Sigma})}. \tag{48}$$

Since ν_{D_k} is the monomial valuation defined by the primitive integral vector n_k, we get $\nu_{D_k}(\phi) = \mathrm{ord}_{\mathcal{N}(\phi)}(n_k)$.

If $\mathcal{J} = (X^{m_1}, \dots, X^{m_r})$ we have $\nu_{D_k}(X^{m_i}) = \langle n_k, m_i \rangle$. We get

$$\begin{aligned} \overline{\nu}_{\mathcal{I}}(\mathcal{J}) &= \min_{i=1,\dots,r} \min_{k \in K(\mathcal{I})} \langle n_k, m_i \rangle (\mathrm{ord}_{\mathcal{I}}(n_k))^{-1} \\ &= \min_{k \in K(\mathcal{I})} \min_{i=1,\dots,r} \langle n_k, m_i \rangle (\mathrm{ord}_{\mathcal{I}}(n_k))^{-1} \\ &= \min_{k \in K(\mathcal{I})} \frac{\mathrm{ord}_{\mathcal{J}}(n_k)}{\mathrm{ord}_{\mathcal{I}}(n_k)}. \ \blacksquare \end{aligned}$$

5.4. Application to the case of a quasi-ordinary hypersurface S of dimension 2. Let $(S,0) \subset (\mathbb{C}^3, 0)$ denote an irreducible germ of quasi-ordinary hypersurface of dimension 2 parametrized by a normalized quasi-ordinary branch ζ. Denote by Z the normalization of S. It is the germ of affine toric variety $Z^{\sigma^\vee \cap M}$ at its origin. We use the notations of Section 2.

By Corollary 4.15 and Theorem 3.3 the ideal $\mathrm{Jac}(S)\mathcal{O}_Z = X^{\gamma_0}\mathcal{J}_g$ is a monomial ideal determined by the characteristic exponents of ζ. By [43, Prop. 0.20] we have $\overline{\nu}_{\mathrm{Jac}(S)}(\phi) = \overline{\nu}_{\mathrm{Jac}(S)\mathcal{O}_Z}(\phi)$. Then we can apply Proposition 5.1 to determine $\overline{\nu}_{\mathrm{Jac}(S)}(\phi)$ from the Newton polygons of ϕ and of the monomial ideal $\mathcal{J}_g$ of $\mathbb{C}\{\sigma^\vee \cap M\}$.

Acknowledgements. The author thanks Evelia García Barroso, Antoni Rangachev and Bernard Teissier and the referee for their comments, which helped to improve the presentation of the paper.

Funding. This work is supported by the Spanish grants of MCIN (PID2024-156181NB-C32 and PID2020-114750GB-C32/AEI/10.13039/501100011033).

References

[1] S. S. Abhyankar, *On the ramification of algebraic functions*, Amer. J. Math. 77 (1955), 575–592.

[2] S. S. Abhyankar and T. T. Moh, *Newton-Puiseux expansion and generalized Tschirnhausen transformation. I, II*, J. Reine Angew. Math. 260 (1973), 47–83; ibid. 261 (1973), 29–54.

[3] A. Assi, *The Frobenius vector of a free affine semigroup*, J. Algebra Appl. 11 (2012), 1250065, 10.

[4] A. Assi, *Irreducibility criterion for quasi-ordinary polynomials*, J. Singul. 4 (2012), 23–34.

[5] A. Assi, P. A. García-Sánchez, and I. Ojeda, *Frobenius vectors, Hilbert series and gluings of affine semigroups*, J. Commutative Algebra 7 (2015), 317–335.

[6] A. Campillo and R. Melgar, *Poincaré series of semigroups*, arXiv:2410.17818v1 (2024).

[7] H. Cobo Pablos and P. D. González Pérez, *Motivic Poincaré series, toric singularities and logarithmic Jacobian ideals*, J. Algebraic Geom. 21 (2012), 495–529.

[8] H. Cobo Pablos and P. D. González Pérez, *Geometric motivic Poincaré series of quasi-ordinary sigularities*, Math. Proc. Cambridge Philos. Soc. 149 (2010), 49–74.

[9] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric Varieties*, Grad. Stud. Math. 124, Amer. Math. Soc., Providence, RI, 2011.

[10] A. B. de Felipe, E. R. García Barroso, J. Gwoździewicz, and A. Płoski, *Łojasiewicz exponents and Farey sequences*, Rev. Mat. Complut. 29 (2016), 719–724.

[11] D. Duarte, J. Jeffries, and L. Núñez Betancourt, *Nash blowups of toric varieties in prime characteristic*, Collect. Math. 75 (2024), 629–637.

[12] W. Fulton, *Introduction to Toric Varieties*, Ann. Math. Stud. 131, Princeton Univ. Press, Princeton, NJ, 1993.

[13] E. R. García Barroso and P. D. González-Pérez, *Decomposition in bunches of the critical locus of a quasi-ordinary map*, Compos. Math. 141 (2005), 461–486.

[14] E. R. García Barroso, P. D. González Pérez, and P. Popescu-Pampu, *Lotuses as computational architectures*, in: Algebraic and Topological Interplay of Algebraic Varieties: A Tribute to the Work of E. Artal and A. Melle, Contemp. Math., Amer. Math. Soc., to appear; arXiv:2502.17102.

[15] E. R. García Barroso and J. Gwoździewicz, *Quasi-ordinary singularities: tree model, discriminant, and irreducibility*, Int. Math. Res. Notices 2015, 5783–5805.

[16] E. R. García Barroso and J. Gwoździewicz, *Higher order polars of quasi-ordinary singularities*, Int. Math. Res. Notices 2022, 1045–1080.

[17] E. R. García Barroso and A. Płoski, *On the Milnor formula in arbitrary characteristic*, in: Singularities, Algebraic Geometry, Commutative Algebra, and Related Topics, Springer, 2018, 119–133.

[18] Y.-N. Gau, *Embedded topological classification of quasi-ordinary singularities* (with an appendix by J. Lipman), Mem. Amer. Math. Soc. 74 (1988), 109–129.

[19] R. Goldin and B. Teissier, *Resolving singularities of plane analytic branches with one toric morphism*, in: Resolution of Singularities (Obergurgl, 1997), Progr. Math. 181, Birkhäuser, 2000, 315–340.

[20] P. D. González Pérez, *Quasi-ordinary singularities via toric geometry*, Ph.D. thesis, Universidad de La Laguna, 2000.

[21] P. D. González Pérez, *Toric embedded resolutions of quasi-ordinary hypersurface singularities*, Ann. Inst. Fourier (Grenoble) 53 (2003), 1819–1881.

[22] P. D. González Pérez, *The semigroup of a quasi-ordinary hypersurface*, J. Inst. Math. Jussieu 2 (2003), 383–399.

[23] P. D. González Pérez and G. Gonzalez-Springberg, *Analytical invariants of quasi-ordinary hypersurface singularities associated to divisorial valuations*, Kodai Math. J. 27 (2004), 164–173.

[24] P. D. González Pérez and F. Hernando, *Quasi-ordinary singularities, essential divisors and Poincaré series*, J. London Math. Soc. (2) 79 (2009), 780–802.

[25] P. D. González Pérez and B. Teissier, *Toric geometry and the Semple–Nash modification*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 108 (2014), 1–48.

[26] G. Gonzalez Sprinberg, *Transformé de Nash et éventail de dimension 2*, C. R. Acad. Sci. Paris Sér. A-B 284 (1977), A69–A71.

[27] M. González Villa, *Newton process and semigroups of irreducible quasi-ordinary power series*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 108 (2014), 259–279.

[28] G.-M. Greuel, *Der Gauss–Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten*, Math. Ann. 214 (1975), 235–266.

[29] G.-M. Greuel, C. Lossen, and E. Shustin, *Introduction to Singularities and Deformations*, Springer Monogr. Math., Springer, Berlin, 2007.

[30] B. Gryszka, *Approximate roots of quasi-ordinary polynomials*, Dissertationes Math. 572 (2022), 48 pp.

[31] B. Gryszka and J. Gwoździewicz, *On the Abhyankar–Moh irreducibility criterion for quasi-ordinary polynomials*, Colloq. Math. 164 (2021), 149–160.

[32] S. M. Gusein-Zade, F. Delgado, and A. Campillo, *On the monodromy of a plane curve singularity and the Poincaré series of its ring of functions*, Funct. Anal. Appl. 33 (1999), 56–57.

[33] J. Gwoździewicz, A. Lenarcik, and A. Płoski, *Polar invariants of plane curve singularities: intersection theoretical approach*, Demonstratio Math. 43 (2010), 303–323.

[34] J. Gwoździewicz and A. Płoski, *Łojasiewicz exponents and singularities at infinity of polynomials in two complex variables*, Colloq. Math. 103 (2005), 47–60.

[35] J. Gwoździewicz and A. Płoski, *On the approximate roots of polynomials*, Ann. Polon. Math. 60 (1995), 199–210.

[36] M. Hickel, *Fonction de Artin et germes de courbes tracées sur un germe d'espace analytique*, Amer. J. Math. 115 (1993), 1299–1334.

[37] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry. Vol. I*, Cambridge Math. Library, Cambridge Univ. Press, Cambridge, 1994.

[38] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Math. 339, Springer, Berlin, 1973.

[39] K. Kiyek and M. Micus, *Semigroup of a quasiordinary singularity*, in: Topics in Algebra, Part 2 (Warszawa, 1988), PWN–Polish Sci. Publ., 1990, 149–156.

[40] T. Krasiński, G. Oleksik, and A. Płoski, *The Łojasiewicz exponent of an isolated weighted homogeneous surface singularity*, Proc. Amer. Math. Soc. 137 (2009), 3387–3397.

[41] Dũng Tráng Lê, *Computation of the Milnor number of an isolated singularity of a complete intersection*, Funktsional. Anal. i Prilozhen. 8 (1974), no. 2, 45–49 (in Russian).

[42] M. Lejeune-Jalabert and A. J. Reguera, *The Denef–Loeser series for toric surface singularities*, Rev. Mat. Iberoamer. 19 (2003), 581–612.

[43] M. Lejeune-Jalabert et B. Teissier, *Clôture intégrale des idéaux et équisingularité* (avec un appendice de J.-J. Risler), Ann. Fac. Sci. Toulouse Math. (6) 17 (2008), 781–859.

[44] J. Lipman, *Topological invariants of quasi-ordinary singularities*, Mem. Amer. Math. Soc. 74 (1988), 1–107.

[45] A. Nobile, *Some properties of the Nash blowing-up*, Pacific J. Math. 60 (1975), 297–305.

[46] T. Oda, *Convex Bodies and Algebraic Geometry*, Ergeb. Math. Grenzgeb. (3) 15, Springer, Berlin, 1988.

[47] G. Pfister and A. Płoski, *Plane Algebroid Curves in Arbitrary Characteristic*, IMPAN Lect. Notes 4, Inst. Math., Polish Acad. Sci., Warszawa, 2022.

[48] R. Piene, *Ideals associated to a desingularization*, in: Algebraic Geometry (Copenhagen, 1978), Lecture Notes in Math. 732, Springer, 1979, 503–517.

[49] A. Płoski, *On the growth of proper polynomial mappings*, Ann. Polon. Math. 45 (1985), 297–309.

[50] A. Płoski, *Multiplicity and the Łojasiewicz exponent*, in: Singularities (Warszawa, 1985), Banach Center Publ. 20, PWN–Polish Sci. Publ., 1988, 353–364.

[51] A. Płoski, *Sur l'exposant d'une application analytique. I*, Bull. Polish Acad. Sci. Math. 32 (1984), 669–673.

[52] A. Płoski, *Sur l'exposant d'une application analytique. II*, Bull. Polish Acad. Sci. Math. 33 (1985), 123–127.

[53] A. Płoski, *A note on the Łojasiewicz exponent at infinity*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. 17 (1994), 11–15.

[54] A. Płoski, *Discriminant and the Łojasiewicz exponent*, Univ. Iagel. Acta Math. 46 (2008), 87–93.
[55] A. Płoski, *Semicontinuity of the Łojasiewicz exponent*, Univ. Iagel. Acta Math. 48 (2010), 103–110.
[56] P. Popescu-Pampu, *Approximate roots*, in: Valuation Theory and Its Applications, Vol. II (Saskatoon, SK, 1999), Fields Inst. Comm. 33, Amer. Math. Soc., 2003, 285–321.
[57] P. Popescu-Pampu, *On the analytical invariance of the semigroups of a quasi-ordinary hypersurface singularity*, Duke Math. J. 124 (2004), 67–104.
[58] P. Popescu-Pampu, *Sur le contact d'une hypersurface quasi-ordinaire avec ses hypersurfaces polaires*, J. Inst. Math. Jussieu 3 (2004), 105–138.
[59] P. Popescu-Pampu, *On higher dimensional Hirzebruch–Jung singularities*, Rev. Mat. Complut. 18 (2005), 209–232.
[60] R. P. Stanley, *Combinatorics and Commutative Algebra*, 2nd ed., Progr. Math. 41, Birkhäuser Boston, Boston, MA, 1996.
[61] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lecture Ser. 8, Amer. Math. Soc., Providence, RI, 1996.
[62] B. Teissier, *Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces*, Invent. Math. 40 (1977), 267–292.
[63] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, in: Singularités à Cargèse (Cargèse, 1972), Astérisque 7–8 (1973), 285–362.
[64] B. Teissier, *Valuations, deformations, and toric geometry*, in: Valuation Theory and Its Applications, Vol. II (Saskatoon, SK, 1999), Fields Inst. Comm. 33, Amer. Math. Soc., 2003, 361–459.
[65] B. Teissier, *Overweight deformations of affine toric varieties and local uniformization*, in: Valuation Theory in Interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., 2014, 474–565.
[66] O. Zariski, *The Moduli Problem for Plane Branches*, Univ. Lecture Ser. 39, Amer. Math. Soc., Providence, RI, 2006.

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