Integral and hypergeometric representations for the multiple Hahn polynomials and their descendants

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Problem[1,2]

Determine polynomials satisfying certain orthogonality conditions w.r.t. the weights

$$w_j(x;\vec{\alpha},\beta,N) = \underbrace{\frac{\Gamma(\alpha_j+x+1)}{\Gamma(\alpha_j+1)}}_{v_j(x;\vec{\alpha})}\,\underbrace{\frac{\Gamma(\beta+N-x+1)}{\Gamma(x+1)\Gamma(\beta+1)\Gamma(N-x+1)}}_{v(x;\beta,N)}, \quad j=1,\dots,r,$$

supported on the discrete set

$$\Delta = \{0,\dots,N\} \subset \mathbb{N}.$$

$\longrightarrow$ assume that $\alpha_1,\dots,\alpha_r,\beta > -1$ and $|\vec{n}| \le N$ so that the system is perfect

Multiple Hahn polynomials[5]

- Type I: vectors $(Q^{(I)}_{\vec{n},1},\dots,Q^{(I)}_{\vec{n},r})$ of polynomials with $\deg Q^{(I)}_{\vec{n},j} \le n_j - 1$ and

$$\sum_{j=1}^{r}\sum_{x=0}^{N} Q^{(I)}_{\vec{n},j}(x)x^k w_j(x) = \begin{cases} 0, & k=0,\dots,|\vec{n}|-2,\\ 1, & k=|\vec{n}|-1.\end{cases}$$

- Type II: monic polynomials $Q^{(II)}_{\vec{n}}$ with $\deg Q^{(II)}_{\vec{n}} \le |\vec{n}|$ and

$$\sum_{x=0}^{N} Q^{(II)}_{\vec{n}}(x)x^k w_j(x) = 0, \quad k=0,\dots,n_j-1, \quad j=1,\dots,r.$$

$\longrightarrow$ unique solutions $(Q^{(I)}_{\vec{n},1},\dots,Q^{(I)}_{\vec{n},r})$ and $Q^{(II)}_{\vec{n}}$ (with max. degrees) as system is perfect

Integral representations[4]

- Type I multiple Hahn polynomials:

$$Q^{(I)}_{\vec{n},k}(x) = \frac{\mathcal{Q}^{(I)}_{\vec{n}}}{v_j(x)}\int_{\Sigma_k}\frac{\Gamma(x+t+1)\Gamma(t+\beta+|\vec{n}|)}{\Gamma(t+1)\Gamma(t+\beta+N+2)\prod_{j=1}^{r}(\alpha_j-t)_{n_j}}\frac{dt}{2\pi i}, \qquad \mathcal{Q}^{(I)}_{\vec{n}} \in \mathbb{R}.$$

- Type II multiple Hahn polynomials:

$$Q^{(II)}_{\vec{n}}(x) = \frac{\mathcal{Q}^{(II)}_{\vec{n}}}{v(x)}\int_{\mathcal{C}}\frac{\Gamma(s)\Gamma(s+\beta+N+1)\prod_{j=1}^{r}(\alpha_j+1-s)_{n_j}}{\Gamma(x+s+1)\Gamma(s+\beta+|\vec{n}|+1)}\frac{ds}{2\pi i}, \qquad \mathcal{Q}^{(II)}_{\vec{n}} \in \mathbb{R}.$$

The relevant contours are

- Σ_k, which only encloses the poles $\cup_{l=0}^{n_k-1}\{\alpha_k+l\}$ of the integrand, and does so exactly once (clockwise),
- $\mathcal{C}$, which encloses $[-N,0]$ exactly once (counterclockwise).

Normalization constants[4]

The normalization constants are given by

$$\mathcal{Q}^{(I)}_{\vec{n}} = (-1)^{|\vec{n}|}(N-|\vec{n}|+1)!\frac{\prod_{j=1}^{r}(\alpha_j+\beta+|\vec{n}|)_{n_j}}{(\beta+1)_{|\vec{n}|-1}},$$

$$\mathcal{Q}^{(II)}_{\vec{n}} = \frac{(-1)^{|\vec{n}|}(\beta+1)_{|\vec{n}|}}{(N-|\vec{n}|)!\prod_{j=1}^{r}(\alpha_j+\beta+|\vec{n}|+1)_{n_j}}.$$

Key ideas for the type I polynomials[4]

1. Consider the type I linear forms

$$Q^{(I)}_{\vec{n}}(x) = \sum_{j=1}^{r} Q^{(I)}_{\vec{n},j}(x)w_j(x).$$

2. Pick an appropriate φ and apply the residue theorem to write

$$Q^{(I)}_{\vec{n}}(x) = v(x)\int_{\cup_{j=1}^{r}\Sigma_i}\frac{\varphi(t)q_{|\vec{n}|}(t)}{\prod_{j=1}^{r}(\alpha_j-t)_{n_j}}\Gamma(x+t+1)\frac{dt}{2\pi i}, \qquad \deg q_{|\vec{n}|} \le |\vec{n}|.$$

3. Use the orthogonality conditions to determine $q_{|\vec{n}|}(t)$ explicitly.

Key ideas for the type II polynomials[4]

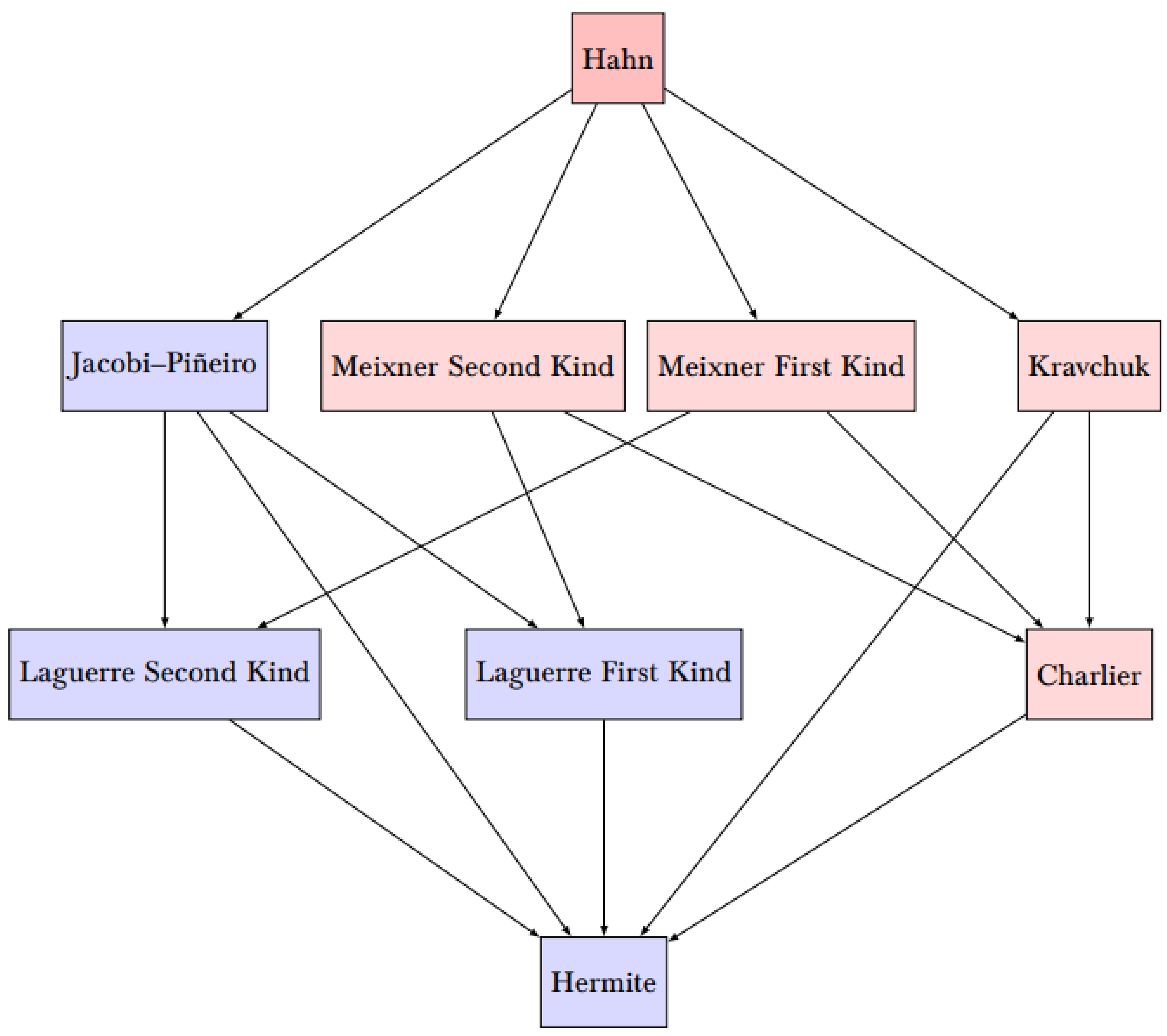
1. Consider $f = Q^{(II)}_{\vec{n}}v$ and apply a discrete variant of the Mellin transform

$$(\mathcal{M}_N f)(s) = \sum_{x=0}^{N} f(x)\Gamma(x+s).$$

2. Use the orthogonality conditions to determine $(\mathcal{M}_N f)(s)$ explicitly.
3. Apply the inverse transform

$$Q^{(II)}_{\vec{n}}(x) = \frac{1}{v(x)}\int_{\mathcal{C}}\frac{(\mathcal{M}_N f)(s)}{\Gamma(x+s+1)}\frac{ds}{2\pi i}.$$

Multiple Askey scheme[3]



$\longrightarrow$ through the appropriate limits, we also obtained integral representations for the descendants in the multiple Askey scheme

Hypergeometric representations[4]

- Type I multiple Hahn polynomials:

$$Q^{(I)}_{\vec{n},k}(x) \propto {}_{r+2}F_{r+1}\left[\begin{matrix}-n_k+1,\alpha_k+\beta+|\vec{n}|,(\alpha_k+1)\vec{1}_{r-1}-\vec{\alpha}^{*k}-\vec{n}^{*k},x+\alpha_k+1\\ \alpha_k+1,(\alpha_k+1)\vec{1}_{r-1}-\vec{\alpha}^{*k},\alpha_k+\beta+N+2\end{matrix};1\right].$$

- Type II multiple Hahn polynomials:

$$Q^{(II)}_{\vec{n}}(x) \propto \frac{\Gamma(N-x+1)}{\Gamma(\beta+N-x+1)}\,{}_{r+2}F_{r+1}\left[\begin{matrix}-|\vec{n}|-\beta,-x,\vec{\alpha}+\vec{n}+\vec{1}_r\\ -N-\beta,\vec{\alpha}+\vec{1}_r\end{matrix};1\right].$$

$\longrightarrow$ follows from the integral representations (with explicit constants) after applying the residue theorem

References

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