Spectral theory for bounded banded matrices with positive bidiagonal factorization and mixed multiple orthogonal polynomials

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Banded semi-infinite matrices

Banded matrices

$$T^{[N]} = \begin{bmatrix} T_{0,0} & \cdots & T_{0,q} & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ T_{p,0} & & & & \ddots & 0 \\ 0 & \ddots & & & & T_{N-q,N} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & 0 & T_{N,N-p} & \cdots & T_{N,N} \end{bmatrix}$$

where the extreme diagonal entries are nonzero

with $T = \lim_{N\to\infty} T^{[N]}$ (inductive limit) a banded semi-infinite matrix

Recursion polynomials

Type I recursion polynomials

$$A^{(a)}(x) = \begin{bmatrix} A_0^{(a)}(x) & A_1^{(a)}(x) \cdots \end{bmatrix}, \qquad a \in \{1, \dots, p\}$$

- Left eigenvectors:

$$A^{(a)}(x)T = xA^{(a)}(x), \qquad a \in \{1, \dots, p\}$$

Recursion polynomials

Type I recursion polynomials

- Initial conditions,

$$\begin{cases} A_0^{(1)} = 1, \\ A_1^{(1)} = \nu_1^{(1)}, \\ \quad\vdots \\ A_{p-1}^{(1)} = \nu_{p-1}^{(1)}, \end{cases} \qquad \begin{cases} A_0^{(2)} = 0, \\ A_1^{(2)} = 1, \\ A_2^{(2)} = \nu_2^{(2)}, \\ \quad\vdots \\ A_{p-1}^{(2)} = \nu_{p-1}^{(2)}, \end{cases} \qquad \dots \qquad \begin{cases} A_0^{(p)} = 0, \\ \quad\vdots \\ A_{p-2}^{(p)} = 0, \\ A_{p-1}^{(p)} = 1, \end{cases}$$

$\nu_j^{(i)}$ being arbitrary constants

Recursion polynomials

Type I recursion polynomials

- Initial condition matrix

$$\nu := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \nu_1^{(1)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \nu_{p-1}^{(1)} & \cdots & \nu_{p-1}^{(p-1)} & 1 \end{bmatrix}$$

- $(p+q+1)$-term recursion relation, $A_{-q}^{(a)} = \cdots = A_{-1}^{(a)} = 0$, $a \in \{1, \ldots, p\}$

$$A_{n-q}^{(a)} T_{n-q,n} + \cdots + A_{n+p}^{(a)} T_{n+p,n} = x A_n^{(a)}, \qquad n \in \mathbb{N}_0$$

- $\deg A_n^{(a)} = \left\lceil \frac{n+2-a}{p} \right\rceil - 1$

Recursion polynomials

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Recursion polynomials

Type II recursion polynomials

$$B^{(b)}(x) = \left[B_0^{(b)}(x) \quad B_1^{(b)}(x) \cdots \right]^\top, \qquad\qquad b \in \{1, \ldots, q\}$$

- Right eigenvectors:

$$TB^{(b)}(x) = xB^{(b)}(x), \qquad\qquad b \in \{1, \ldots, q\}$$

Recursion polynomials

Type II recursion polynomials

- Initial conditions,

$$\begin{cases} B_0^{(1)} = 1, \\ B_1^{(1)} = \xi_1^{(1)}, \\ \qquad \vdots \\ B_{q-1}^{(1)} = \xi_{q-1}^{(1)}, \end{cases} \qquad \begin{cases} B_0^{(2)} = 0, \\ B_1^{(2)} = 1, \\ B_2^{(2)} = \xi_2^{(2)}, \\ \qquad \vdots \\ B_{p-1}^{(2)} = \xi_{q-1}^{(2)}, \end{cases} \qquad \dots \qquad \begin{cases} B_0^{(q)} = 0, \\ \qquad \vdots \\ B_{q-2}^{(q)} = 0, \\ B_{q-1}^{(q)} = 1, \end{cases}$$

$\xi_j^{(i)}$ being arbitrary constants

Recursion polynomials

Type II recursion polynomials

- Initial condition matrix

$$\xi := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \xi_1^{(1)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \xi_{q-1}^{(1)} & \cdots & \xi_{q-1}^{(q-1)} & 1 \end{bmatrix}$$

- $(p+q+1)$-term recursion relation, $B_{-p}^{(b)} = \cdots = B_{-1}^{(b)} = 0$, $b \in \{1, \ldots, q\}$

$$T_{n,n-p} B_{n-p}^{(b)} + \cdots + T_{n,n+q} B_{n+q}^{(b)} = x B_n^{(b)}, \qquad n \in \{0, 1, \ldots\}$$

- $\deg B_n^{(b)} = \left\lceil \frac{n+2-b}{q} \right\rceil - 1$

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- $\deg B_n^{(b)} = \left\lceil \frac{n+2-b}{q} \right\rceil - 1$

Recursion polynomials

Characteristic polynomials

For the semi-infinite matrix T we consider the polynomials $P_N(x)$, $\deg P_N = N$, as the characteristic polynomials of $T^{[N-1]}$

$$P_N(x) := \begin{cases} 1, & N = 0 \\ \det\left(xI_N - T^{[N-1]}\right), & N \in \mathbb{N} \end{cases}$$

Left and right recursion polynomials determinants

$$A_N := \begin{bmatrix} A_N^{(1)} & \cdots\cdots & A_{N+p-1}^{(1)} \\ \vdots & & \vdots \\ A_N^{(p)} & \cdots\cdots & A_{N+p-1}^{(p)} \end{bmatrix}, \quad B_N := \begin{bmatrix} B_N^{(1)} & \cdots\cdots\cdots & B_N^{(q)} \\ \vdots & & \vdots \\ B_{N+q-1}^{(1)} & \cdots\cdots & B_{N+q-1}^{(q)} \end{bmatrix}$$

$\alpha_N := (-1)^{(p-1)N} T_{p,0} \cdots T_{N+p-1,N-1}$, $\beta_N := (-1)^{(q-1)N} T_{0,q} \cdots T_{N-1,N+q-1}$ for $N \in \mathbb{N}$ and $\alpha_0 = \beta_0 = 1$

Recursion polynomials

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$\alpha_N := (-1)^{(p-1)N} T_{p,0} \cdots T_{N+p-1,N-1}$, $\beta_N := (-1)^{(q-1)N} T_{0,q} \cdots T_{N-1,N+q-1}$ for $N \in \mathbb{N}$ and $\alpha_0 = \beta_0 = 1$

Recursion polynomials

Recall that as the entries in the extreme diagonals do not cancel $\alpha_N, \beta_N \neq 0$. In terms of these objects we found the following important result:

Theorem

Determinantal expressions for the characteristic polynomials

$$P_N(x) = \alpha_N \det A_N(x) = \beta_N \det B_N(x)$$

Founded for $q = 1$, in the context of non-mixed multiple orthogonality, in:

Jonathan Coussement and Walter Van Assche, *Gaussian quadrature for multiple orthogonal polynomials*, Journal of Computational and Applied Mathematics **178** (2005) 131–145.

Associated polynomials

Associated polynomials

$$Q_{n,N} := \begin{vmatrix} A_n^{(1)} & \cdots & A_n^{(p)} \\ A_{N+1}^{(1)} & \cdots & A_{N+1}^{(p)} \\ \vdots & \ddots & \vdots \\ A_{N+p-1}^{(1)} & \cdots & A_{N+p-1}^{(p)} \end{vmatrix}, \quad R_{n,N} := \begin{vmatrix} B_n^{(1)} & \cdots & B_n^{(q)} \\ B_{N+1}^{(1)} & \cdots & B_{N+1}^{(q)} \\ \vdots & \ddots & \vdots \\ B_{N+q-1}^{(1)} & \cdots & B_{N+q-1}^{(q)} \end{vmatrix}$$

$$Q_N := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots \end{bmatrix}, \; Q^{\langle N\rangle} := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots & Q_{N,N} \end{bmatrix}$$
$$R_N := \begin{bmatrix} R_{0,N} & R_{1,N} \cdots \end{bmatrix}^\top, \; R^{\langle N\rangle} := \begin{bmatrix} R_{0,N} & R_{1,N} \cdots & R_{N,N} \end{bmatrix}^\top$$

Associated polynomials

1. $Q_{N+1,N} = \cdots = Q_{N+p-1,N} = R_{N+1,N} = \cdots = R_{N+q-1,N} = 0$
2. $\alpha_N Q_{N,N} = \beta_N R_{N,N} = P_N$
3. $(-1)^{p-1}\alpha_{N+1} Q_{N+p,N} = (-1)^{q-1}\beta_{N+1} R_{N+q,N} = P_{N+1}$
4. $Q_N T = x Q_N$ and $T R_N = x R_N$
5.

$$Q^{\langle N\rangle} T^{[N]} + \begin{bmatrix} 0 \cdots \cdots 0 & T_{N+p,N} Q_{N+p,N} \end{bmatrix} = x Q^{\langle N\rangle}$$

$$T^{[N]} R^{\langle N\rangle} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ T_{N,N+q} R_{N+q,N} \end{bmatrix} = x R^{\langle N\rangle}.$$

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Christoffel–Darboux

Theorem (Christoffel–Darboux formulas)

1. *For the determinantal polynomials $Q_{n,N}$ and $R_{n,N}$ we get the following generalized Christoffel–Darboux formula*

$$\sum_{n=0}^{N} Q_{n,N}(x)R_{n,N}(y) = \frac{1}{\alpha_N\beta_N}\frac{P_{N+1}(x)P_N(y) - P_N(x)P_{N+1}(y)}{x-y}$$

2. *The following generalized confluent Christoffel–Darboux relation is fulfilled*

$$\sum_{n=0}^{N} Q_{n,N}R_{n,N} = \frac{1}{\alpha_N\beta_N}\left(P'_{N+1}P_N - P'_N P_{N+1}\right)$$

Spectral properties

Assume that P_{N+1} has simple zeros at the set $\left\{\lambda_k^{[N]}\right\}_{k=1}^{N+1}$

- Biorthogonal sets of left and right eigenvectors, $\left\{w_k^{\langle N\rangle}\right\}_{k=1}^{N+1}$, $\left\{u_k^{\langle N\rangle}\right\}_{k=1}^{N+1}$, are given by

$$w_k^{\langle N\rangle} = \frac{Q^{\langle N\rangle}\big(\lambda_k^{[N]}\big)}{\beta_N \sum_{l=0}^{N} Q_{l,N}\big(\lambda_k^{[N]}\big) R_{l,N}\big(\lambda_k^{[N]}\big)}, \quad u_k^{\langle N\rangle} = \beta_N R^{\langle N\rangle}\big(\lambda_k^{[N]}\big)$$

- The following expression holds

$$w_{k,n}^{\langle N\rangle} = \frac{\alpha_N Q_{n-1,N}\big(\lambda_k^{[N]}\big)}{P_N\big(\lambda_k^{[N]}\big) P'_{N+1}\big(\lambda_k^{[N]}\big)}, \qquad u_{k,n}^{\langle N\rangle} = \beta_N R_{n-1,N}\big(\lambda_k^{[N]}\big)$$

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Spectral properties

- We can write $w_{k,n}^{\langle N\rangle} = A_{n-1}^{(1)}(\lambda_k^{[N]})\mu_{k,1}^{[N]} + \cdots + A_{n-1}^{(p)}(\lambda_k^{[N]})\mu_{k,p}^{[N]}$
  Christoffel numbers

$$\mu_{k,1}^{[N]} := \frac{\begin{vmatrix} A_{n+1}^{(2)}\left(\lambda_k^{[N]}\right) & \cdots\cdots\cdots & A_{n+1}^{(p)}\left(\lambda_k^{[N]}\right) \\ \vdots & \ddots & \vdots \\ A_{n+p-1}^{(2)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & A_{n+p-1}^{(p)}\left(\lambda_k^{[N]}\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda_k^{[N]}\right) R_{l,N}\left(\lambda_k^{[N]}\right)}$$

$$\mu_{k,2}^{[N]} := -\frac{\begin{vmatrix} A_{n+1}^{(1)}\left(\lambda_k^{[N]}\right) & A_{n+1}^{(3)}\left(\lambda_k^{[N]}\right) & \cdots\cdots\cdots & A_{n+1}^{(p)}\left(\lambda_k^{[N]}\right) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n+p-1}^{(1)}\left(\lambda_k^{[N]}\right) & A_{n+p-1}^{(3)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & A_{n+p-1}^{(p)}\left(\lambda_k^{[N]}\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda_k^{[N]}\right) R_{l,N}\left(\lambda_k^{[N]}\right)}$$

$$\vdots$$

$$\mu_{k,p}^{[N]} := (-1)^{p-1}\frac{\begin{vmatrix} A_{n+1}^{(1)}\left(\lambda_k^{[N]}\right) & \cdots\cdots\cdots & A_{n+1}^{(p-1)}\left(\lambda_k^{[N]}\right) \\ \vdots & \ddots & \vdots \\ A_{n+p-1}^{(1)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & A_{n+p-1}^{(p-1)}\left(\lambda_k^{[N]}\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda_k^{[N]}\right) R_{l,N}\left(\lambda_k^{[N]}\right)}$$

Spectral properties

- We can write $u_{k,n}^{\langle N\rangle} = B^{(1)}_{n-1}(\lambda_k^{[N]})\rho^{[N]}_{k,1} + \cdots + B^{(q)}_{n-1}(\lambda_k^{[N]})\rho^{[N]}_{k,q}$
  Christoffel numbers

$$\rho^{[N]}_{k,1} := \beta_N \begin{vmatrix} B^{(2)}_{N+1}\left(\lambda_k^{[N]}\right) \cdots\cdots & B^{(q)}_{N+1}\left(\lambda_k^{[N]}\right) \\ \vdots & \vdots \\ B^{(2)}_{N+q-1}\left(\lambda_k^{[N]}\right) \cdots & B^{(p)}_{N+q-1}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

$$\rho^{[N]}_{k,2} := -\beta_N \begin{vmatrix} B^{(1)}_{N+1}\left(\lambda_k^{[N]}\right) & B^{(3)}_{N+1}\left(\lambda_k^{[N]}\right) \cdots\cdots & B^{(q)}_{N+1}\left(\lambda_k^{[N]}\right) \\ \vdots & \vdots & \vdots \\ B^{(1)}_{N+q-1}\left(\lambda_k^{[N]}\right) & B^{(3)}_{n+q-1}\left(\lambda_k^{[N]}\right) \cdots & B^{(p)}_{N+q-1}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

$$\vdots$$

$$\rho^{[N]}_{k,q} := (-1)^{q-1}\beta_N \begin{vmatrix} B^{(1)}_{N+1}\left(\lambda_k^{[N]}\right) \cdots\cdots & B^{(q-1)}_{N+1}\left(\lambda_k^{[N]}\right) \\ \vdots & \vdots \\ B^{(1)}_{N+q-1}\left(\lambda_k^{[N]}\right) \cdots & B^{(q-1)}_{N+q-1}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

Spectral properties

- It holds that

$$\begin{bmatrix} \mu_{k,1}^{[N]} \\ \mu_{k,2}^{[N]} \\ \vdots \\ \mu_{k,p}^{[N]} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \nu_1^{(1)} & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \nu_{p-1}^{(1)} & \cdots & \cdots & \nu_{p-1}^{(p-1)} & 1 \end{bmatrix}^{-1} \begin{bmatrix} w_{k,1}^{\langle N\rangle} \\ w_{k,2}^{\langle N\rangle} \\ \vdots \\ w_{k,p}^{\langle N\rangle} \end{bmatrix}$$

$$\begin{bmatrix} \rho_{k,1}^{[N]} \\ \rho_{k,2}^{[N]} \\ \vdots \\ \rho_{k,q}^{[N]} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \xi_1^{(1)} & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \xi_{q-1}^{(1)} & \cdots & \cdots & \xi_{q-1}^{(q-1)} & 1 \end{bmatrix}^{-1} \begin{bmatrix} u_{k,1}^{\langle N\rangle} \\ u_{k,2}^{\langle N\rangle} \\ \vdots \\ u_{k,q}^{\langle N\rangle} \end{bmatrix}$$

Spectral properties

- Matrices U (with columns the right eigenvectors u_k arranged in the standard order) and W and (with rows the left eigenvectors w_k arranged in the standard order) satisfy

$$UW = WU = I_{N+1}$$

- In terms of the diagonal matrix $D = \operatorname{diag}\left(\lambda_1^{[N]}, \dots, \lambda_{N+1}^{[N]}\right)$ we have

$$UD^nW = \left(T^{[N]}\right)^n$$

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Orthogonality

Step functions

$$\psi_{b,a}^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{k,b}^{[N]}\mu_{k,a}^{[N]}, & \lambda_{k+1}^{[N]} \leqslant x < \lambda_k^{[N]},\ k = 1, \ldots, N \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]}, & x \geqslant \lambda_1^{[N]} \end{cases}$$

Finite sums

For $a \in \{1, \ldots, p\}$ and $b \in \{1, \ldots, q\}$, we have

$$\rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} = (\xi^{-1} I_{q,p} \nu^{-\top})_{b,a}$$

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Finite sums

For $a \in \{1, \dots, p\}$ and $b \in \{1, \dots, q\}$, we have

$$\rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} = (\xi^{-1} I_{q,p} \nu^{-\top})_{b,a}$$

Orthogonality

Matrix of discrete measures

$q\times p$ matrix of functions: $\Psi^{[N]} := \begin{bmatrix}\psi^{[N]}_{1,1} & \cdots & \psi^{[N]}_{1,p}\\ \vdots & & \vdots\\ \psi^{[N]}_{q,1} & \cdots & \psi^{[N]}_{q,p}\end{bmatrix}$ $q\times p$ matrix of discrete Lebesgue–Stieltjes measures supported at the zeros of P_{N+1}:

$$\mathrm{d}\Psi^{[N]} = \begin{bmatrix}\mathrm{d}\psi^{[N]}_{1,1} & \cdots\cdots & \mathrm{d}\psi^{[N]}_{1,p}\\ \vdots & & \vdots\\ \mathrm{d}\psi^{[N]}_{q,1} & \cdots\cdots & \mathrm{d}\psi^{[N]}_{q,p}\end{bmatrix} = \sum_{k=1}^{N+1}\begin{bmatrix}\rho^{[N]}_{k,1}\\ \vdots\\ \rho^{[N]}_{k,q}\end{bmatrix}\begin{bmatrix}\mu^{[N]}_{k,1} & \cdots\cdots & \mu^{[N]}_{k,p}\end{bmatrix}\delta\big(x-\lambda^{[N]}_k\big)$$

Orthogonality

Assume that the recursion polynomials P_{N+1} have simple zeros $\left\{\lambda_k^{[N]}\right\}_{k=1}^{N+1}$

Theorem

The following biorthogonal relations hold

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int B_n^{(b)}(x)\mathrm{d}\psi_{b,a}^{[N]}(x)A_m^{(a)}(x)=\delta_{n,m}, \qquad n,m\in\{0,\dots,N\}$$

Proof.

It follows from $UW=I$ □

Orthogonality

From this biorthogonality we get the following:

Corollary

The following discrete type mixed multiple orthogonality for $m \in \{1, \dots, N\}$ *are satisfied:*

$$\sum_{a=1}^{p} \int x^n \mathrm{d}\psi_{b,a}^{[N]} A_m^{(a)} = 0, \quad n \in \{0, \dots, \deg B_{m-1}^{(b)}\}, \quad b \in \{1, \dots, q\}$$

$$\sum_{b=1}^{q} \int B_m^{(b)} \mathrm{d}\psi_{b,a}^{[N]} x^n = 0, \quad n \in \{0, \dots, \deg A_{m-1}^{(a)}\}, \quad a \in \{1, \dots, p\}$$

Positive bidiagonal factorization

1. We now introduce the very important idea of **positive bidiagonal factorization (PBF)**
2. This factorization is very natural for banded matrices as all the subdiagonals may be constructed in terms of simpler bidiagonal matrices

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Positive bidiagonal factorization

Positive bidiagonal factorization

We say that a banded matrix T admits a PBF if

$$T = L_1 \cdots L_p \Delta U_q \cdots U_1$$

with $\Delta = \operatorname{diag}(\Delta_0, \Delta_1, \dots)$ and bidiagonal matrices given respectively by

$$L_k := \begin{bmatrix} 1 & 0 & \cdots & \\ L_{k|0} & 1 & \ddots & \\ 0 & L_{k|1} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad U_k := \begin{bmatrix} 1 & U_{k|0} & 0 & \cdots \\ 0 & 1 & U_{k|1} & \ddots \\ \vdots & \ddots & 1 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

with $L_{k|i}, U_{k|i}, \Delta_i > 0$, for $i \in \mathbb{N}_0$

Oscillatory Matrices

Totally nonnegative (TN)

All its minors are non-negative

Invertible totally nonnegative (InTN)

All its minors are non-negative and is nonsingular

Totally positive (TP)

All its minors are positive

Oscillatory Matrix (IITN)

A totally non negative matrix A such that for some n, the matrix A^n is totally positive

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Gantmacher-Krein Criterion

A totally non negative matrix is oscillatory if and only if it is nonsingular and the elements of the first subdiagonal and superdiagonal are positive.

Oscillatory Jacobi Matrix

If and only if the elements of the first subdiagonal and superdiagonal are positive, and the leading principal minors are positive

Factorization I

From Cauchy–Binet Theorem one can deduce the invariance of these sets of matrices under the usual matrix product
The product of matrices in InTN is again InTN (similar statements hold for TN or oscillatory matrices)

Factorization II

PBF $\Rightarrow$ oscillatory

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Eigenvalues

The eigenvalues are simple and positive

Interlacing property

The eigenvalues strictly interlace the eigenvalues of the principal submatrix (deleting first row and column) (also last column and row)

Left and right eigenvectors $w^{(k)}, u^{(k)}$ to the k-th largest eigenvalue

$$U = \left[u^{(1)} \cdots\cdots u^{(n)}\right], \qquad W = \begin{bmatrix} w^{(1)} \\ \vdots \\ \vdots \\ w^{(n)} \end{bmatrix}, \qquad UW = I$$

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Sign-variation

Number of variations in the eigenvectors will lead us to interlacing properties of polynomials

Translations

Translations of bounded Jacobi matrices are oscillatory matrices

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Interlacing

Let us assume that T is oscillatory. Then:

1. The polynomial P_{N+1} interlaces P_N
2. When $x \in \mathbb{R}$, for the corresponding Wronskian we find $P'_{N+1}P_N - P'_N P_{N+1} > 0$. In particular,
$$\left.(P'_{N+1}P_N)\right|_{x=\lambda_k^{[N]}} > 0, \qquad \left.(P_{N+1}P'_N)\right|_{x=\lambda_k^{[N-1]}} < 0$$
3. The confluent kernel is a positive function; i.e., $\alpha_N\beta_N \sum_{n=0}^{N} Q_{n,N}(x)R_{n,N}(x) > 0$ for $x \in \mathbb{R}$

PBF implies oscillatory

If T has a PBF then its leading principal submatrices $T^{[N]}$ are oscillatory

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Darboux transformations

Darboux transformations

Let us assume that T admits a bidiagonal factorization (not necessarily positive). For each of its truncations $T^{[N]}$ we consider a chain of new auxiliary matrices, called Darboux transformations, given by the consecutive permutation of the triangular matrices in the factorization

$$\begin{cases} \hat{T}^{[N,+1]} = L_2^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} \\ \hat{T}^{[N,+2]} = L_3^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \\ \qquad \vdots \\ \hat{T}^{[N,+(p-1)]} = L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_{p-1}^{[N]} \\ \hat{T}^{[N,+p]} = \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]} \end{cases}$$

Darboux transformations

Darboux transformations

$$\begin{cases} \hat{T}^{[N,-1]} = U_1^{[N]} L_1^{[N]} \dots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \dots U_2^{[N]} \\ \hat{T}^{[N,-2]} = U_2^{[N]} U_1^{[N]} L_1^{[N]} \dots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \dots U_3^{[N]} \\ \vdots \\ \hat{T}^{[N,-(q-1)]} = U_{q-1}^{[N]} \dots U_1^{[N]} L_1^{[N]} L_2^{[N]} \dots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \\ \hat{T}^{[N,-q]} = U_q^{[N]} \dots U_1^{[N]} L_1^{[N]} L_2^{[N]} \dots L_p^{[N]} \Delta^{[N]} \end{cases}$$

Darboux transformations

Darboux transformations are $p+q+1$ banded matrices

Theorem (PBF and Darboux transformations)

Let us assume that the PBF holds. Then,

1. *The Darboux transformations* $\hat{T}^{[N,+a]}$, $a \in \{1,\dots,p\}$, $\hat{T}^{[N,-b]}$, $b \in \{1,\dots,q\}$ *are oscillatory*
2. *The characteristic polynomial of the Darboux transformations* $\hat{T}^{[N,+a]}$, $a \in \{1,\dots,p\}$, $\hat{T}^{[N,-b]}$, $b \in \{1,\dots,q\}$ *is* P_{N+1}
3. *If* w, u *are left and right eigenvectors of* $T^{[N]}$*, respectively, then* $\hat{w} = wL_1^{[N]} \cdots L_a^{[N]}$ *is a left eigenvector of* $\hat{T}^{[N,+a]}$ *and* $\hat{u} = U_b^{[N]} \cdots U_1^{[N]} u$ *is a right eigenvector of* $\hat{T}^{[N,-b]}$

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Christoffel numbers are positive for PBF

$$\Lambda := \begin{bmatrix}\Lambda^{(1)} & \cdots\cdots & \Lambda^{(p)}\end{bmatrix} \in \mathbb{R}^{p\times p}, \qquad \Upsilon := \begin{bmatrix}\Upsilon^{(1)} \\ \vdots \\ \vdots \\ \Upsilon^{(q)}\end{bmatrix} \in \mathbb{R}^{q\times q}$$

where

$$\Lambda^{(1)} := \begin{bmatrix}1\\0\\ \vdots \\ \vdots \\ 0\end{bmatrix}, \qquad \Lambda^{(k)} := \frac{1}{r_k} L_1^{[p-1]} \cdots L_{k-1}^{[p-1]} \begin{bmatrix}1\\0\\ \vdots \\ \vdots \\ 0\end{bmatrix}$$

$$\Upsilon^{(1)} := \begin{bmatrix}1 & 0 \cdots\cdots 0\end{bmatrix}, \qquad \Upsilon^{(k)} := \frac{1}{s_k}\begin{bmatrix}1 & 0 \cdots\cdots 0\end{bmatrix} U_1^{[q-1]} \cdots U_{k-1}^{[q-1]}$$

with

$$r_k := L_{k|0} L_{k-1|1} \cdots L_{1|k-1}, \qquad s_k := U_{k|0} U_{k-1|1} \cdots U_{1|k-1}$$

Christoffel numbers are positive for PBF

Lemma

The matrices Λ and Υ are positive upper and lower unitriangular matrices, respectively

Theorem (Christoffel coefficients positivity)

Let us assume that T has a PBF and choose the matrices of initial conditions as

$$\nu^{-\top} = \Lambda\mathcal{A}, \qquad \xi^{-1} = \mathcal{B}\Upsilon$$

for some upper and lower unitriangular nonnegative matrices $\mathcal{A} \in \mathbb{R}^{p\times p}$ and $\mathcal{B} \in \mathbb{R}^{q\times q}$, respectively. Then,

$$\rho_{k,b}^{[N]} > 0, \quad \mu_{k,a}^{[N]} > 0, \quad k \in 1, \dots, N+1, \quad a \in \{1, \dots, p\}, \quad b \in \{1, \dots, q\}$$

Idea of the proof I

Christoffel numbers in terms of biorthogonal families of right an left eigenvectors:

$$\begin{bmatrix} \mu^{[N]}_{k,1} & \cdots\cdots & \mu^{[N]}_{k,p} \end{bmatrix} = \begin{bmatrix} w^{\langle N \rangle}_{k,1} & \cdots\cdots & w^{\langle N \rangle}_{k,p} \end{bmatrix} \nu^{-\top}, \qquad \begin{bmatrix} \rho^{[N]}_{k,1} \\ \vdots \\ \vdots \\ \rho^{[N]}_{k,q} \end{bmatrix} = \xi^{-1} \begin{bmatrix} u^{\langle N \rangle}_{k,1} \\ \vdots \\ \vdots \\ u^{\langle N \rangle}_{k,q} \end{bmatrix}$$

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Entries of these biorthogonal right and left eigenvectors can be written as $w_{k,a}^{\langle N\rangle} = \alpha_N \left.\frac{Q_{a-1,N}}{P'_{N+1}P_N}\right|_{x=\lambda_k^{[N]}}$ and $u_{k,b}^{\langle N\rangle} = \beta_N R_{b-1,N}\big(\lambda_k^{[N]}\big)$

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CD+interlacing leads to the fact that the Christoffel numbers are positive if and only if

$$\beta_N\xi^{-1}\begin{bmatrix}R_{0,N}\\ \vdots \\ R_{q-1,N}\end{bmatrix}, \qquad \frac{1}{\beta_N}\begin{bmatrix}Q_{0,N} & \cdots\cdots & Q_{p-1,N}\end{bmatrix}\nu^{-\top}$$

are positive vectors at the points $x=\lambda^{[N]}_k$, $k\in\{1,\dots,N+1\}$

Idea of the proof II

Consider left and right eigenvectors with last entry normalized to 1

$$\begin{bmatrix} \left.\dfrac{Q_{0,N}}{Q_{N,N}}\right|_{x=\lambda_k^{[N]}} & \left.\dfrac{Q_{1,N}}{Q_{N,N}}\right|_{x=\lambda_1^{[N]}} & \cdots\cdots & 1 \end{bmatrix}, \qquad \begin{bmatrix} \left.\dfrac{R_{0,N}}{R_{N,N}}\right|_{x=\lambda_k^{[N]}} \\ \left.\dfrac{R_{1,N}}{R_{N,N}}\right|_{x=\lambda_k^{[N]}} \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

(the last entry of any eigenvector is nonzero) Recall that $Q_{N,N} = \alpha_N^{-1} P_N$, $R_{N,N} = \beta_N^{-1} P_N$ and that the first eigenvector entries are not zero; i.e., $\alpha_N \left.\frac{Q_{0,N}}{P_N}\right|_{x=\lambda_k^{[N]}}, \beta_N \left.\frac{R_{0,N}}{P_N}\right|_{x=\lambda_k^{[N]}} \neq 0$. As the last entry is positive the change sign properties described in the sign Theorem leads to

$$\alpha_N \left.\frac{Q_{0,N}}{P_N}\right|_{x=\lambda_1^{[N]}} > 0, \quad \alpha_N \left.\frac{Q_{0,N}}{P_N}\right|_{x=\lambda_2^{[N]}} < 0, \quad \alpha_N \left.\frac{Q_{0,N}}{P_N}\right|_{x=\lambda_3^{[N]}} > 0,$$

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Idea of the proof II

As T is oscillatory and the characteristic polynomial P_{N+1} interlaces P_N we have that $\operatorname{sgn} P_N\big(\lambda_k^{[N]}\big) = (-1)^{k-1}$ so that

$$\alpha_N Q_{0,N}\big(\lambda_k^{[N]}\big), \beta_N R_{0,N}\big(\lambda_k^{[N]}\big) > 0, \qquad k \in \{1, \dots, N+1\}$$

Idea of the proof III

Darboux transform $\hat{T}^{[N,\pm 1]}$ is an oscillatory matrix with characteristic polynomial P_{N+1}. Then, a left eigenvector of $T^{[N,+1]}$ for the eigenvalue $\lambda_k^{[N]}$ can be chosen as

$$\begin{bmatrix} \alpha_N \frac{Q_{0,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} & \alpha_N \frac{Q_{1,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} & \cdots & \cdots & 1 \end{bmatrix} L_1^{[N]} = \begin{bmatrix} \alpha_N \frac{(Q_{0,N}+L_{1|0}Q_{1,N})}{P_N}\Big|_{x=\lambda_k^{[N]}}$$

and a right eigenvector of $T^{[N,-1]}$ for the eigenvalue $\lambda_k^{[N]}$ can be taken as

$$U_1^{[N]} \begin{bmatrix} \beta_N \frac{R_{0,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} \\ \beta_N \frac{R_{1,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \beta_N \frac{(R_{0,N}+U_{1|0}R_{1,N})}{P_N}\Big|_{x=\lambda_k^{[N]}} \\ \vdots \\ 1 \end{bmatrix}$$

Idea of the proof III

Sign properties of the eigenvectors of an oscillatory matrix:

$$\alpha_N \left.\frac{\frac{1}{L_{1|0}}Q_{0,N}+Q_{1,N}}{P_N}\right|_{x=\lambda_1^{[N]}}>0, \qquad \alpha_N \left.\frac{\frac{1}{L_{1|0}}Q_{0,N}+Q_{1,N}}{P_N}\right|_{x=\lambda_2^{[N]}}<0$$

$$\beta_N \left.\frac{\frac{1}{U_{1|0}}R_{0,N}+R_{1,N}}{P_N}\right|_{x=\lambda_1^{[N]}}>0, \qquad \beta_N \left.\frac{\frac{1}{U_{1|0}}R_{0,N}+R_{1,N}}{P_N}\right|_{x=\lambda_2^{[N]}}<0$$

and so alternating sign, and recalling the sign of P_N at the zeros of P_{N+1} we get

$$\alpha_N \left.\left(\frac{1}{L_{1|0}}Q_{0,N}+Q_{1,N}\right)\right|_{x=\lambda_k^{[N]}}, \beta_N \left.\left(\frac{1}{U_{1|0}}R_{0,N}+R_{1,N}\right)\right|_{x=\lambda_k^{[N]}}>0$$

Idea of the proof IV

Consequently, after repeating this process up to $T^{[N,+(p-1)]}$ and $T^{[N,-(q-1)]}$ we find that

$$\beta_N \Upsilon \begin{bmatrix} R_{0,N} \\ \vdots \\ R_{q-1,N} \end{bmatrix}, \qquad \alpha_N \begin{bmatrix} Q_{0,N} & \cdots & \cdots & Q_{p-1,N} \end{bmatrix} \Lambda$$

are positive vectors at the points $x = \lambda_k^{[N]}$, $k \in \{1, \dots, N\}$. Therefore, if the initial condition matrices are tuned as indicated we get the result

Second kind polynomials, resolvent and Weyl functions

From here on we assume that $N \geqslant \max(p,q)$

Given $r \in \mathbb{N}$, we write $\left\{e_1^{[r]}, \ldots, e_r^{[r]}\right\}$ for the canonical basis of $\mathbb{R}^r$ and consider the $r \times (N+1)$ matrix $E_{[r]} := \left[I_r \; 0_{r\times(N+1-r)}\right]$. Then, we introduce the vectors $e_a^\nu, e_b^\xi \in \mathbb{R}^{N+1}$ with

$$e_a^\nu := E_{[p]}^\top \nu^{-\top} e_a^{[p]}, \qquad \left(e_b^\xi\right)^\top := \left(e_b^{[q]}\right)^\top \xi^{-1} E_{[q]}$$

For the matrices U and W we find

$$(e_b^\xi)^\top U = \left[\rho_{1,b}^{[N]} \cdots\cdots \rho_{N+1,b}^{[N]}\right], \qquad W e_a^\nu = \begin{bmatrix} \mu_{1,a}^{[N]} \\ \vdots \\ \mu_{N+1,a}^{[N]} \end{bmatrix}$$

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$$e^{\nu}_a := E^{\top}_{[p]} \nu^{-\top} e^{[p]}_a, \qquad \left(e^{\xi}_b\right)^{\top} := \left(e^{[q]}_b\right)^{\top} \xi^{-1} E_{[q]}$$

For the matrices U and W we find

$$(e^{\xi}_b)^{\top} U = \left[\rho^{[N]}_{1,b} \cdots\cdots \rho^{[N]}_{N+1,b} \right], \qquad W e^{\nu}_a = \begin{bmatrix} \mu^{[N]}_{1,a} \\ \vdots \\ \vdots \\ \mu^{[N]}_{N+1,a} \end{bmatrix}$$

Second kind polynomials, resolvent and Weyl functions

Second kind polynomials

For $a \in \{1, \dots, p\}, b \in \{1, \dots, q\}$, and $\pi_k^{[N]} := \prod_{\substack{l \in \{1,\dots,N+1\} \\ l \neq k}} (x - \lambda_l^{[N]})$, the second kind polynomials, in terms of the adjugate, are given by

$$
\begin{aligned}
P_{N+1}^{(b,a)}(x) &:= (e_b^{\xi})^\top \operatorname{adj}(xI_{N+1} - T^{[N]}) e_a^{\nu} \\
&= \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} \pi_k^{[N]}(x) \\
&= \int \frac{P_{N+1}(z) - P_{N+1}(x)}{z - x} \mathrm{d}\psi_{b,a}^{[N]}(x) \\
&= \alpha_{N+1} \int \frac{\det(A_{N+1}(z)) - \det(A_{N+1}(x))}{z - x} \mathrm{d}\psi_{b,a}^{[N]}(x) \\
&= \beta_{N+1} \int \frac{\det(B_{N+1}(z)) - \det(B_{N+1}(x))}{z - x} \mathrm{d}\psi_{b,a}^{[N]}(x)
\end{aligned}
$$

Second kind polynomials, resolvent and Weyl functions

If T has a PBF and the initial conditions are tuned as above then $\deg P^{(b,a)}_{N+1} = N$

The moments of the pq discrete measures $\mathrm{d}\psi^{[N]}_{b,a}$ are linked to the components of the powers of $T^{[N]}$:

Theorem (Discrete moments)

For $a \in \{1, \dots, p\}, b \in \{1, \dots, q\}$, the discrete moments we have

$$\int x^n \mathrm{d}\psi^{[N]}_{b,a}(x) = \sum_{k=1}^{N+1} \rho^{[N]}_{k,b}\mu^{[N]}_{k,a}\big(\lambda^{[N]}_k\big)^n = (e^{\xi}_b)^\top \big(T^{[N]}\big)^n e^{\nu}_a$$

Second kind polynomials, resolvent and Weyl functions

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Second kind polynomials, resolvent and Weyl functions

Proof.

We have that $(e_b^\xi)^\top (T^{[N]})^n e_a^\nu = (e_b^\xi)^\top U D^n W e_a^\nu$ so that

$$(e_b^\xi)^\top (T^{[N]})^n e_a^\nu = \begin{bmatrix} \rho_{1,b}^{[N]} & \cdots & \cdots & \rho_{N+1,b}^{[N]} \end{bmatrix} D^n \begin{bmatrix} \mu_{1,a}^{[N]} \\ \vdots \\ \vdots \\ \mu_{N+1,a}^{[N]} \end{bmatrix}$$

□

Second kind polynomials, resolvent and Weyl functions

Resolvent

The resolvent matrix $R^{[N]}(z)$ of the leading principal submatrix $T^{[N]}$ is

$$R^{[N]}(z) \coloneqq \left(zI_{N+1} - T^{[N]}\right)^{-1} = \frac{\operatorname{adj}\left(zI_{N+1} - T^{[N]}\right)}{\det(zI_{N+1} - T^{[N]})}$$

Weyl's functions

For $a \in \{1, \dots, p\}, b \in \{1, \dots, q\}$, the Weyl functions are

$$\begin{aligned} S^{[N]}_{b,a} &\coloneqq (e^{\xi}_b)^\top R^{[N]} e^{\nu}_a \\ &= \frac{P^{(b,a)}_{N+1}(z)}{P_{N+1}(z)} = \sum_{k=1}^{N+1} \frac{\rho^{[N]}_{k,b}\mu^{[N]}_{k,a}}{z - \lambda^{[N]}_k} = \int \frac{\mathrm{d}\psi^{[N]}_{b,a}(x)}{z - x} \end{aligned}$$

Second kind polynomials, resolvent and Weyl functions

Interlacing

If T has a PBF for tuned initial conditions the polynomial $P^{(b,a)}_{N+1}$ is interlaced by P_{N+1}

Recursion polynomials and resolvent

$$\sum_{a=1}^{p}\int\frac{\mathrm{d}\psi^{[N]}_{b,a}(x)}{z-x}A^{(a)}_{n-1}(x)=\left(e^{\xi}_{b}\right)^{\top}R^{[N]}(z)e_{n}$$

$$\sum_{b=1}^{q}\int B^{(b)}_{n-1}(x)\frac{\mathrm{d}\psi^{[N]}_{b,a}(x)}{z-x}=e_{n}^{\top}R^{[N]}(z)e^{\nu}_{a}$$

A path to mixed Hermite–Padé

Second kind polynomials, resolvent and Weyl functions

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A path to mixed Hermite–Padé

Helly's tools

1. **Helly's Selection Principle**: for any uniformly bounded sequence $\{\psi^{[N]}\}_{N=0}^{\infty}$ of non-decreasing functions defined in $\mathbb{R}$, there exists a convergent subsequence converging to a non-decreasing function ψ defined in $\mathbb{R}$
2. **Helly's second theorem**: Let us assume a uniformly bounded sequence $\{\psi^{[N]}\}_{N=0}^{\infty}$ of non-decreasing functions on a compact interval $[a,b]$ with limit function ψ, then for any continuous function f in $[a,b]$ we have $\lim_{N\to\infty}\int_a^b f(x)\mathrm{d}\psi^{[N]}(x) = \int_a^b f(x)\mathrm{d}\psi(x)$

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Favard theorem

As the submatrices $T^{[N]}$ are oscillatory, we know that $P_{N+1}(x)$ strictly interlaces $P_N(x)$ so that the positive sequence $\{\lambda_1^{[N]}\}_{N=1}^{\infty}$ is a strictly increasing sequence and $\{\lambda_{N+1}^{[N]}\}_{N=1}^{\infty}$ is a strictly decreasing sequence. As well, for bounded operators, $\|T\|_\infty < \infty$, we have $\|T^{[N]}\|_\infty < \|T\|_\infty < \infty$. Therefore, there exists the limits $\zeta := \lim_{N\to\infty} \lambda_{N+1}^{[N]} \geqslant 0$ and $\eta := \lim_{N\to\infty} \lambda_1^{[N]} \leqslant \|T\|_\infty$. We call $[\zeta, \eta] \subseteq [0, \|T\|_\infty]$ the true interval of orthogonality, that is the smallest interval containing all zeros of the characteristic polynomials P_n, i.e. the eigenvalues of the leading principal submatrices of T

Favard theorem

Theorem (Favard spectral representation)

Let us assume that

1. *The banded matrix T is bounded and there exist $s \geqslant 0$ such that $T + sI$ has a PBF.*
2. *The sequences $\left\{A_n^{(1)}, \dots, A_n^{(p)}\right\}_{n=0}^{\infty}$, $\left\{B_n^{(1)}, \dots, B_n^{(q)}\right\}_{n=0}^{\infty}$ of recursion polynomials are determined by the initial condition matrices ν and ξ,respectively, such that $\nu^{-\top} = \Lambda\mathcal{A}$, $\xi^{-1} = \mathcal{B}\Upsilon$, and $\mathcal{A} \in \mathbb{R}^{p\times p}$ is a nonnegative upper unitriangular matrices and $\mathcal{B} \in \mathbb{R}^{q\times q}$ is a nonnegative lower unitriangular matrix.*

Then, there exists pq non decreasing positive functions $\psi_{b,a}$, $a \in \{1, \dots, p\}$ and $b \in \{1, \dots, q\}$ and corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\psi_{b,a}$ with compact support Δ such that the following biorthogonality holds

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int_{\zeta}^{\eta} B_l^{(b)}(x)\mathrm{d}\psi_{b,a}(x)A_k^{(a)}(x) = \delta_{k,l}, \qquad k, l \in \mathbb{N}_0$$

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Favard theorem

Proof.

The shift in the matrix $T \to T + sI$ only shifts by s the eigenvalues of the truncations $T^{[N]}$, so that they are positive, and the dependent variable of the recursion polynomials, but do not alter the interlacing properties of the polynomials and the positivity of the corresponding Christoffel numbers. We know that the sequences $\{\psi^{[N]}_{a,b}\}_{N=0}^{\infty}$, $a \in \{1, \dots, p\}$, $b \in \{1, \dots, q\}$ are positive, uniformly bounded and nondecreasing. Consequently, following Helly's results there exist subsequences that converge when $N \to \infty$ to positive nondecreasing functions $\psi_{b,a}$ with support on $[\zeta, \eta]$ and that the discrete biorthogonal relations lead to the stated biorthogonal properties □

Favard theorem

Corollary (Mixed multiple orthogonal relations)

In the conditions above, the mixed multiple orthogonal relations are fulfilled

$$\sum_{a=1}^{p}\int_{\zeta}^{\eta} x^n \mathrm{d}\psi_{b,a}(x)A^{(a)}_m(x) = 0, \quad n \in \big\{0,\dots,\deg B^{(b)}_{m-1}\big\}, \quad b \in \{1,\dots,q\}$$

$$\sum_{b=1}^{q}\int_{\zeta}^{\eta} B^{(b)}_m(x)\mathrm{d}\psi_{b,a}(x)x^n = 0, \quad n \in \big\{0,\dots,\deg A^{(a)}_{m-1}\big\}, \quad a \in \{1,\dots,p\}$$

Favard theorem

Theorem (Spectral representation of moments and Stieltjes–Markov functions)

In the conditions above and in terms of the spectral functions $\psi_{b,a}$, $a \in \{1,\dots,p\}$, $b \in \{1,\dots,q\}$ *we find the following relations between entries of powers or the resolvent of the banded matrix and moments or the Cauchy transform of the measures, respectively:*

$$(e_b^{\xi})^\top T^n e_a^{\nu} = \int_{\zeta}^{\eta} x^n \mathrm{d}\psi_{b,a}(x)$$

$$(e_b^{\xi})^\top (zI - T)^{-1} e_a^{\nu} = \int_{\zeta}^{\eta} \frac{\mathrm{d}\psi_{b,a}(x)}{z - x} =: \hat{\psi}_{b,a}(z)$$

Favard theorem

Theorem (Normal convergence of Weyl functions)

For $a \in \{1, \ldots, p\}, b \in \{1, \ldots, q\}$, and in the conditions above, the Weyl functions converge uniformly in compact subsets of $\bar{\mathbb{C}} \setminus [\zeta, \eta]$ to the Stieltjes–Markov functions, i.e.,

$$S_{b,a}^{[N]}(z) = \frac{P_{N+1}^{(b,a)}(z)}{P_{N+1}(z)} \xrightarrow[N\to\infty]{} \hat{\psi}_{b,a}(z)$$

Gauss quadrature formulas

Degrees of precision

The degrees of precision or orders $d_{b,a}(N)$, $a \in \{1,\dots,p\}$, $b \in \{1,\dots,q\}$, are the largest natural numbers such that

$$\left(e_b^{\xi}\right)^{\top} T^n e_a^{\nu} = \left(e_b^{\xi}\right)^{\top} \left(T^{[N]}\right)^n e_a^{\nu}, \qquad 0 \leqslant n \leqslant d_{b,a}(N)$$

with

$$d_{b,a}(N) = \deg A_N^{(a)} + \deg B_N^{(b)} + 1 = \left\lceil \frac{N+2-a}{p} \right\rceil + \left\lceil \frac{N+2-b}{q} \right\rceil - 1$$

Gauss quadrature formulas

Theorem (Mixed multiple Gaussian quadrature formulas)

The following Gauss quadrature formulas hold

$$\int_\zeta^\eta x^n \mathrm{d}\psi_{b,a}(x) = \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} \big(\lambda_k^{[N]}\big)^n, \qquad 0 \leqslant n \leqslant d_{b,a}(N)$$

Here the degrees of precision $d_{b,a}$ are optimal (for any power largest than n a positive remainder appears, an exactness is lost)