Spectral theory for bounded banded matrices with positive bidiagonal factorization and mixed multiple orthogonal polynomials

(joint work with Amílcar Branquinho and Ana Foulquié-Moreno)

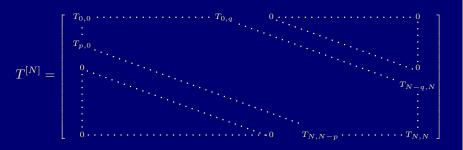
Universidad Complutense & ICMAT

January 24th, 2023

Banded matrices and orthogonality

# **Banded semi-infinite matrices**

#### **Banded matrices**



where the extreme diagonal entries are nonzero

with  $T = \lim_{N \to \infty} T^{[N]}$  (inductive limit) a banded semi-infinite matrix

### Type I recursion polynomials

$$A^{(a)}(x) = \begin{bmatrix} A_0^{(a)}(x) & A_1^{(a)}(x) \cdots \end{bmatrix}, \qquad a \in \{1, \dots, p\}$$

Left eigenvectors:

$$A^{(a)}(x)T = xA^{(a)}(x), \qquad a \in \{1, \dots, p\}$$

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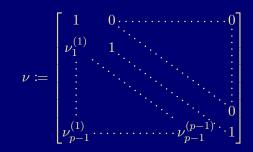
# Type I recursion polynomials Initial conditions,

 $\nu_{i}^{(i)}$  being arbitrary constants

Banded matrices and orthogonality

Type I recursion polynomials

Initial condition matrix



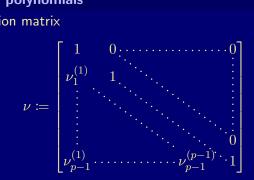
(p + q + 1)-term recursion relation,  $A_{-q}^{(a)} = \cdots = A_{-1}^{(a)} = 0$ ,  $a \in \{1, \dots, p\}$ 

$$A_{n-q}^{(a)}T_{n-q,n} + \dots + A_{n+p}^{(a)}T_{n+p,n} = xA_n^{(a)}, \qquad n \in \mathbb{N}$$

 $\blacktriangleright \ \deg A_n^{(a)} = \left\lceil \frac{n+2-a}{p} \right\rceil - 1$ 

Type I recursion polynomials

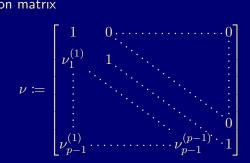
Initial condition matrix



(p + q + 1)-term recursion relation, A<sup>(a)</sup><sub>-q</sub> = ··· = A<sup>(a)</sup><sub>-1</sub> = 0, a ∈ {1, ..., p}
 A<sup>(a)</sup><sub>n-q</sub>T<sub>n-q,n</sub> + ··· + A<sup>(a)</sup><sub>n+p</sub>T<sub>n+p,n</sub> = xA<sup>(a)</sup><sub>n</sub>, n ∈ N<sub>0</sub>
 deg A<sup>(a)</sup><sub>n</sub> = [n+2-a/p] - 1

Type I recursion polynomials

Initial condition matrix



Type II recursion polynomials

$$B^{(b)}(x) = \begin{bmatrix} B_0^{(b)}(x) & B_1^{(b)}(x) \cdots \end{bmatrix}^\top, \qquad b \in \{1, \dots, q\}$$

Right eigenvectors:

$$TB^{(b)}(x) = xB^{(b)}(x), \qquad b \in \{1, \dots, q\}$$

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### Type II recursion polynomials Initial conditions,

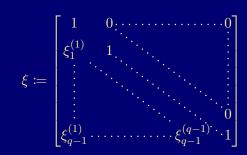
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Banded matrices and orthogonality

January 24th, 2023

Type II recursion polynomials

Initial condition matrix



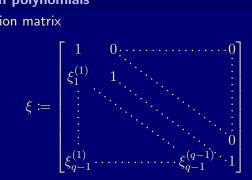
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Type II recursion polynomials

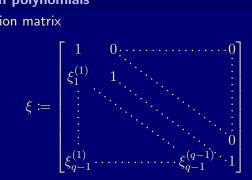
Initial condition matrix



▶ (p + q + 1)-term recursion relation, B<sup>(b)</sup><sub>-p</sub> = ··· = B<sup>(b)</sup><sub>-1</sub> = 0, b ∈ {1, ..., q}
 T<sub>n,n-p</sub>B<sup>(b)</sup><sub>n-p</sub> + ··· + T<sub>n,n+q</sub>B<sup>(b)</sup><sub>n+q</sub> = xB<sup>(b)</sup><sub>n</sub>, n ∈ {0, 1, ...}
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#### **Characteristic polynomials**

For the semi-infinite matrix T we consider the polynomials  $P_N(x)$ ,  $\deg P_N = N$ , as the characteristic polynomials of  $T^{[N-1]}$ 

$$P_N(x) := \begin{cases} 1, & N = 0\\ \det\left(xI_N - T^{[N-1]}\right), & N \in \mathbb{N} \end{cases}$$

Left and right recursion polynomials determinants

$$A_{N} \coloneqq \begin{bmatrix} A_{N}^{(1)} \cdots A_{N+p-1}^{(1)} \\ \vdots \\ A_{N}^{(p)} \cdots A_{N+p-1}^{(p)} \end{bmatrix}, \quad B_{N} \coloneqq \begin{bmatrix} B_{N}^{(1)} \cdots B_{N}^{(q)} \\ \vdots \\ B_{N+q-1}^{(1)} \cdots B_{N+q-1}^{(q)} \end{bmatrix}$$

 $\alpha_N \coloneqq (-1)^{(p-1)N} T_{p,0} \cdots T_{N+p-1,N-1}, \ \beta_N \coloneqq (-1)^{(q-1)N} T_{0,q} \cdots T_{N-1,N+q-1}$ for  $N \in \mathbb{N}$  and  $\alpha_0 = \beta_0 = 1$ 

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Recall that as the entries in the extreme diagonals do not cancel  $\alpha_N, \beta_N \neq 0$ . In terms of these objects we found the following important result:

#### Theorem

Determinantal expressions for the characteristic polynomials

$$P_N(x) = \alpha_N \det A_N(x) = \beta_N \det B_N(x)$$

Founded for q = 1, in the context of non-mixed multiple orthogonality, in: Jonathan Coussement and Walter Van Assche, *Gaussian quadrature for multiple orthogonal polynomials*, Journal of Computational and Applied Mathematics **178** (2005) 131–145.

#### Associated polynomials

 $Q_{n,N} \coloneqq \begin{vmatrix} A_{n}^{(1)} \cdots A_{n}^{(p)} \\ A_{N+1}^{(1)} \cdots A_{N+1}^{(p)} \\ \vdots & \ddots & \vdots \\ A_{N+p-1}^{(1)} \cdots A_{N+p-1}^{(p)} \end{vmatrix}, \quad R_{n,N} \coloneqq \begin{vmatrix} B_{n}^{(1)} \cdots B_{n}^{(q)} \\ B_{N+1}^{(1)} \cdots B_{N+1}^{(q)} \\ \vdots & \ddots & \vdots \\ B_{N+q-1}^{(1)} \cdots B_{N+q-1}^{(q)} \end{vmatrix}$  $Q_{N} \coloneqq \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots \end{bmatrix}, \quad Q^{\langle N \rangle} \coloneqq \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots Q_{N,N} \\ B_{N+q-1} \cdots B_{N+q-1}^{(q)} \end{vmatrix}$ 

Banded matrices and orthogonality

January 24th, 2023

1. 
$$Q_{N+1,N} = \dots = Q_{N+p-1,N} = R_{N+1,N} = \dots = R_{N+q-1,N} = 0$$
  
2.  $\alpha_N Q_{N,N} = \beta_N R_{N,N} = P_N$   
3.  $(-1)^{p-1} \alpha_{N+1} Q_{N+p,N} = (-1)^{q-1} \beta_{N+1} R_{N+q,N} = P_{N+1}$   
4.  $Q_N T = x Q_N$  and  $T R_N = x R_N$   
5.  $Q^{\langle N \rangle} T^{[N]} + [0 \dots 0 \quad T_{N+p,N} Q_{N+p,N}] = x Q^{\langle N \rangle}$ 

$${}^{\rangle}T^{[N]} + \begin{bmatrix} 0 \cdots 0 & T_{N+p,N}Q_{N+p,N} \end{bmatrix} = xQ^{\langle N \rangle}$$
$$T^{[N]}R^{\langle N \rangle} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ T_{N,N+q}R_{N+q,N} \end{bmatrix} = xR^{\langle N \rangle}.$$

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5.  $Q^{\langle N \rangle} T^{[N]} + [0 \dots 0 \quad T_{N+p,N} Q_{N+p,N}] = x Q^{\langle N \rangle}$   
 $T^{[N]} P^{\langle N \rangle} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \end{bmatrix} = x P^{\langle N \rangle}$ 

Banded matrices and orthogonality

 $\begin{bmatrix} 0\\ T_{N,N+q}R_{N+q,N} \end{bmatrix}$ 

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January 24th, 2023

### **Christoffel–Darboux**

#### Theorem (Christoffel–Darboux formulas)

**1.** For the determinantal polynomials  $Q_{n,N}$  and  $R_{n,N}$  we get the following generalized Christoffel–Darboux formula

$$\sum_{n=0}^{N} Q_{n,N}(x) R_{n,N}(y) = \frac{1}{\alpha_N \beta_N} \frac{P_{N+1}(x) P_N(y) - P_N(x) P_{N+1}(y)}{x - y}$$

**2.** The following generalized confluent Christoffel–Darboux relation is fulfilled

$$\sum_{n=0}^{N} Q_{n,N} R_{n,N} = \frac{1}{\alpha_N \beta_N} (P'_{N+1} P_N - P'_N P_{N+1})$$

January 24th, 2023 13 / 55

Assume that  $P_{N+1}$  has simple zeros at the set  $\{\lambda_k^{[N]}\}_{k=1}^{N+1}$ 

**Biorthogonal** sets of left and right eigenvectors,  $\{w_k^{\langle N \rangle}\}_{k=1}^{N+1}$ ,  $\{u_k^{\langle N \rangle}\}_{k=1}^{N+1}$ , are given by

$$w_{k}^{\langle N \rangle} = \frac{Q^{\langle N \rangle}(\lambda_{k}^{[N]})}{\beta_{N} \sum_{l=0}^{N} Q_{l,N}(\lambda_{k}^{[N]}) R_{l,N}(\lambda_{k}^{[N]})}, \quad u_{k}^{\langle N \rangle} = \beta_{N} R^{\langle N \rangle}(\lambda_{k}^{[N]})$$

The following expression holds

$$w_{k,n}^{\langle N \rangle} = \frac{\alpha_N Q_{n-1,N} \left(\lambda_k^{[N]}\right)}{P_N \left(\lambda_k^{[N]}\right) P_{N+1}' \left(\lambda_k^{[N]}\right)}, \qquad u_{k,n}^{\langle N \rangle} = \beta_N R_{n-1,N} \left(\lambda_k^{[N]}\right)$$

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 $\blacktriangleright \text{ We can write } w_{k,n}^{\langle N \rangle} = A_{n-1}^{(1)} (\lambda_k^{[N]}) \mu_{k,1}^{[N]} + \dots + A_{n-1}^{(p)} (\lambda_k^{[N]}) \mu_{k,p}^{[N]}$ 

$$\mu_{k,1}^{[N]} \coloneqq \frac{\begin{vmatrix} A_{n+1}^{(2)} \left(\lambda_k^{[N]}\right) \cdots A_{n+1}^{(p)} \left(\lambda_k^{[N]}\right) \\ \vdots & \ddots & \vdots \\ A_{n+p-1}^{(2)} \left(\lambda_k^{[N]}\right) \cdots A_{n+p-1}^{(p)} \left(\lambda_k^{[N]}\right) \end{vmatrix}}{\beta_N \sum_{l=0}^N Q_{l,N} \left(\lambda_k^{[N]}\right) R_{l,N} \left(\lambda_k^{[N]}\right)} \\ \mu_{k,2}^{[N]} \coloneqq -\frac{\begin{vmatrix} A_{n+1}^{(1)} \left(\lambda_k^{[N]}\right) & A_{n+1}^{(3)} \left(\lambda_k^{[N]}\right) \cdots A_{n+1}^{(p)} \left(\lambda_k^{[N]}\right) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n+p-1}^{(1)} \left(\lambda_k^{[N]}\right) A_{n+p-1}^{(3)} \left(\lambda_k^{[N]}\right) \cdots A_{n+p-1}^{(p)} \left(\lambda_k^{[N]}\right) \end{vmatrix}}{\beta_N \sum_{l=0}^N Q_{l,N} \left(\lambda_k^{[N]}\right) R_{l,N} \left(\lambda_k^{[N]}\right)}$$

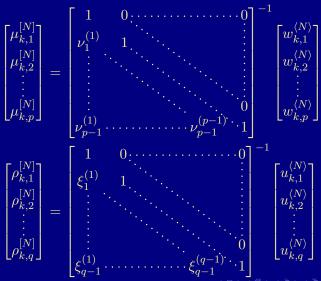
$$\mu_{k,p}^{[N]} \coloneqq (-1)^{p-1} \frac{\begin{vmatrix} A_{n+1}^{(1)} \left(\lambda_{k}^{[N]}\right) \cdots A_{n+1}^{(p-1)} \left(\lambda_{k}^{[N]}\right) \\ \vdots & \ddots & \vdots \\ A_{n+p-1}^{(1)} \left(\lambda_{k}^{[N]}\right) \cdots A_{n+p-1}^{(p-1)} \left(\lambda_{k}^{[N]}\right) \end{vmatrix}}{\beta_{N} \sum_{l=0}^{N} Q_{l,N} \left(\lambda_{k}^{[N]}\right) R_{l,N} \left(\lambda_{k}^{[N]}\right)} \overset{\square}{\Rightarrow}$$

Banded matrices and orthogonality

 $\blacktriangleright \text{ We can write } u_{k,n}^{\langle N \rangle} = B_{n-1}^{(1)} (\lambda_k^{[N]}) \rho_{k,1}^{[N]} + \dots + B_{n-1}^{(q)} (\lambda_k^{[N]}) \rho_{k,q}^{[N]}$ Christoffel numbers

$$\begin{split} \rho_{k,1}^{[N]} &\coloneqq \beta_N \begin{vmatrix} B_{N+1}^{(2)} \left(\lambda_k^{[N]}\right) \cdots B_{N+1}^{(q)} \left(\lambda_k^{[N]}\right) \\ \vdots &\vdots \\ B_{N+q-1}^{(2)} \left(\lambda_k^{[N]}\right) \cdots B_{N+q-1}^{(p)} \left(\lambda_k^{[N]}\right) \end{vmatrix} \\ \rho_{k,2}^{[N]} &\coloneqq -\beta_N \begin{vmatrix} B_{N+1}^{(1)} \left(\lambda_k^{[N]}\right) & B_{N+1}^{(3)} \left(\lambda_k^{[N]}\right) \cdots B_{N+q-1}^{(q)} \left(\lambda_k^{[N]}\right) \\ \vdots &\vdots \\ B_{N+q-1}^{(1)} \left(\lambda_k^{[N]}\right) & B_{n+q-1}^{(3)} \left(\lambda_k^{[N]}\right) \cdots B_{N+q-1}^{(p)} \left(\lambda_k^{[N]}\right) \end{aligned}$$
$$\vdots$$
$$\rho_{k,q}^{[N]} &\coloneqq (-1)^{q-1} \beta_N \begin{vmatrix} B_{N+1}^{(1)} \left(\lambda_k^{[N]}\right) \cdots B_{N+1}^{(q-1)} \left(\lambda_k^{[N]}\right) \\ \vdots \\ B_{N+q-1}^{(1)} \left(\lambda_k^{[N]}\right) \cdots B_{N+1}^{(q-1)} \left(\lambda_k^{[N]}\right) \end{vmatrix}$$

It holds that



Banded matrices and orthogonality

January 24th, 2023

Matrices U (with columns the right eigenvectors u<sub>k</sub> arranged in the standard order) and W and (with rows the left eigenvectors w<sub>k</sub> arranged in the standard order) satisfy

 $\overline{UW} = \overline{WU} = I_{N+1}$ 

In terms of the diagonal matrix  $D = ext{diag}\left(\lambda_1^{[N]}, \dots, \lambda_{N+1}^{[N]}
ight)$  we have

 $UD^nW = \left(T^{[N]}\right)^n$ 

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January 24th, 2023

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 $UD^nW = \left(T^{[N]}\right)^n$ 

#### **Step functions**

$$\psi_{b,a}^{[N]} \coloneqq \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \rho_{1,b}^{[N]} \mu_{1,a}^{[N]} + \dots + \rho_{k,b}^{[N]} \mu_{k,a}^{[N]}, & \lambda_{k+1}^{[N]} \leqslant x < \lambda_{k}^{[N]}, \ k = 1, \dots, N \\ \rho_{1,b}^{[N]} \mu_{1,a}^{[N]} + \dots + \rho_{N+1,b}^{[N]} \mu_{N+1,a}^{[N]}, & x \geqslant \lambda_{1}^{[N]} \end{cases}$$

#### Finite sums

For  $a \in \{1, \dots, p\}$  and  $b \in \{1, \dots, q\}$ , we have

$$\rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \dots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} = (\xi^{-1}I_{q,p}\nu^{-\top})_{b,a}$$

Banded matrices and orthogonality

January 24th, 2023

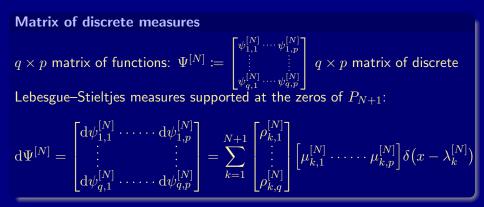
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#### **Finite sums**

For  $a \in \{1, \ldots, p\}$  and  $b \in \{1, \ldots, q\}$ , we have

$$\rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \dots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} = (\xi^{-1}I_{q,p}\nu^{-\top})_{b,a}$$



January 24th, 2023

Assume that the recursion polynomials  $P_{N+1}$  have simple zeros  $\{\lambda_k^{[N]}\}_{k=1}^{N+1}$ 

#### Theorem

The following biorthogonal relations hold

$$\sum_{a=1}^{p} \sum_{b=1}^{q} \int B_{n}^{(b)}(x) \mathrm{d}\psi_{b,a}^{[N]}(x) A_{m}^{(a)}(x) = \delta_{n,m}, \qquad n, m \in \{0, \dots, N\}$$

**Proof.** It follows from UW = I

From this biorthogonality we get the following:

#### Corollary

The following discrete type mixed multiple orthogonality for  $m \in \{1, ..., N\}$  are satisfied:

$$\sum_{a=1}^{p} \int x^{n} d\psi_{b,a}^{[N]} A_{m}^{(a)} = 0, \quad n \in \{0, \dots, \deg B_{m-1}^{(b)}\}, \quad b \in \{1, \dots, q\}$$
$$\sum_{b=1}^{q} \int B_{m}^{(b)} d\psi_{b,a}^{[N]} x^{n} = 0, \quad n \in \{0, \dots, \deg A_{m-1}^{(a)}\}, \quad a \in \{1, \dots, p\}$$

Banded matrices and orthogonality

# Positive bidiagonal factorization

### 1.

We now introduce the very important idea of **positive bidiagonal factorization (PBF)** 

 This factorization is very natural for banded matrices as all the subdiagonals may be constructed in terms of simpler bidiagonal matrices

# Positive bidiagonal factorization

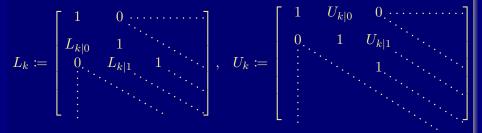
- 1. We now introduce the very important idea of **positive bidiagonal factorization (PBF)**
- 2. This factorization is very natural for banded matrices as all the subdiagonals may be constructed in terms of simpler bidiagonal matrices

# Positive bidiagonal factorization

Positive bidiagonal factorization We say that a banded matrix T admits a PBF if

 $T = L_1 \cdots L_p \Delta U_q \cdots U_1$ 

with  $\Delta = \operatorname{diag}(\Delta_0, \Delta_1, \dots)$  and bidiagonal matrices given respectively by



with  $L_{k|i}, U_{k|i}, \Delta_i > 0$ , for  $i \in \mathbb{N}_0$ 

**Totally nonnegative (TN)** All its minors are non-negative

## Invertible totally nonnegative (InTN)

All its minors are non-negative and is nonsingular

Totally positive (TP)

All its minors are positive

## Oscillatory Matrix (IITN)

A totally non negative matrix  ${\cal A}$  such that for some n, the matrix  ${\cal A}^n$  is totally positive

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**Oscillatory Matrix (IITN)** A totally non negative matrix A such that for some n, the matrix  $A^n$  is totally positive

## **Gantmacher-Krein Criterion**

A totally non negative matrix is oscillatory if and only if it is nonsingular and the elements of the first subdiagonal and superdiagonal are positive.

## **Oscillatory Jacobi Matrix**

If and only if the elements of the first subdiagonal and superdiagonal are positive, and the leading principal minors are positive

#### **Factorization** I

From Cauchy–Binet Theorem one can deduce the invariance of these sets of matrices under the usual matrix product The product of matrices in InTN is again InTN (similar statements hold for TN or oscillatory matrices)

#### Factorization II

 $\mathsf{PBF} \Rightarrow \mathsf{oscillatory}$ 

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**Factorization II** PBF  $\Rightarrow$  oscillatory

## **Eigenvalues** The eigenvalues are simple and positive

## Interlacing property

The eigenvalues strictly interlace the eigenvalues of the principal submatrix (deleting first row and column) (also last column and row)

Left and right eigenvectors  $w^{\left(k
ight)},u^{\left(k
ight)}$  to the k-th largest eigenvalue

$$U = \begin{bmatrix} u^{(1)} \cdots u^{(n)} \end{bmatrix}, \qquad W = \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(n)} \end{bmatrix}, \qquad UW = I$$

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## Sign-variation

Number of variations in the eigenvectors will lead us to interlacing properties of polynomials

#### Translations

Translations of bounded Jacobi matrices are oscillatory matrices

Banded matrices and orthogonality

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## Interlacing

Let us assume that T is oscillatory. Then:

- **1.** The polynomial  $P_{N+1}$  interlaces  $P_N$
- 2. When  $x \in \mathbb{R}$ , for the corresponding Wronskian we find  $P'_{N+1}P_N P'_NP_{N+1} > 0$ . In particular,

$$(P'_{N+1}P_N)\big|_{x=\lambda_k^{[N]}} > 0,$$
  $(P_{N+1}P'_N)\big|_{x=\lambda_k^{[N-1]}} < 0$ 

3. The confluent kernel is a positive function; i.e.,  $\alpha_N \beta_N \sum_{n=0}^N Q_{n,N}(x) R_{n,N}(x) > 0$  for  $x \in \mathbb{R}$ 

#### PBF implies oscillatory

If T has a PBF then its leading principal submatrices  $T^{[N]}$  are oscillatory

January 24th, 2023 29 / 55

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January 24th, 2023 29 / 55

#### **Darboux transformations**

Let us assume that T admits a bidiagonal factorization (not necessarily positive). For each of its truncations  $T^{[N]}$  we consider a chain of new auxiliary matrices, called Darboux transformations, given by the consecutive permutation of the triangular matrices in the factorization

$$\begin{cases} \hat{T}^{[N,+1]} = L_2^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} \\ \hat{T}^{[N,+2]} = L_3^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \\ \vdots \\ \hat{T}^{[N,+(p-1)]} = L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_{p-1}^{[N]} \\ \hat{T}^{[N,+p]} = \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]} \end{cases}$$

## **Darboux transformations**

$$\begin{split} \hat{T}^{[N,-1]} &= U_1^{[N]} L_1^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_2^{[N]} \\ \hat{T}^{[N,-2]} &= U_2^{[N]} U_1^{[N]} L_1^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_3^{[N]} \\ &\vdots \\ \hat{T}^{[N,-(q-1)]} &= U_{q-1}^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \\ \hat{T}^{[N,-q]} &= U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]} \Delta^{[N]} \end{split}$$

Banded matrices and orthogonality

January 24th, 2023

Darboux transformations are p + q + 1 banded matrices

## Theorem (PBF and Darboux transformations)

Let us assume that the PBF holds. Then,

- **1.** The Darboux transformations  $\hat{T}^{[N,+a]}$ ,  $a \in \{1, \ldots, p\}$ ,  $\hat{T}^{[N,-b]}$ ,  $b \in \{1, \ldots, q\}$  are oscillatory
  - 2. The characteristic polynomial of the Darboux transformations  $\hat{T}^{[N,+a]}$ ,  $a \in \{1, \ldots, p\}$ ,  $\hat{T}^{[N,-b]}$ ,  $b \in \{1, \ldots, q\}$  is  $P_{N+1}$
- 3. If w, u are left and right eigenvectors of  $T^{[N]}$ , respectively, then  $\hat{w} = wL_1^{[N]} \cdots L_a^{[N]}$  is a left eigenvector of  $\hat{T}^{[N,+a]}$  and  $\hat{u} = U_b^{[N]} \cdots U_1^{[N]} u$  is a right eigenvector of  $\hat{T}^{[N,-b]}$

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## Christoffel numbers are positive for PBF

$$\begin{split} \Lambda &\coloneqq \left[ \Lambda^{(1)} \cdots \cdots \Lambda^{(p)} \right] \in \mathbb{R}^{p \times p}, \qquad \Upsilon &\coloneqq \begin{bmatrix} \Upsilon^{(1)} \\ \vdots \\ \Upsilon^{(q)} \end{bmatrix} \in \mathbb{R}^{q \times q} \\ \text{re} \\ \Lambda^{(1)} &\coloneqq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \Lambda^{(k)} &\coloneqq \frac{1}{r_k} L_1^{[p-1]} \cdots L_{k-1}^{[p-1]} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ \Upsilon^{(1)} &\coloneqq \begin{bmatrix} 1 & 0 \cdots \cdots 0 \end{bmatrix}, \qquad \Upsilon^{(k)} &\coloneqq \frac{1}{s_1} \begin{bmatrix} 1 & 0 \cdots \cdots 0 \end{bmatrix} U_1^{[q-1]} \cdots U_{k-1}^{[q-1]} \end{split}$$

with

whe

$$r_k \coloneqq L_{k|0} L_{k-1|1} \cdots L_{1|k-1}, \qquad s_k \coloneqq U_{k|0} U_{k-1|1} \cdots U_{1|k-1}$$

Banded matrices and orthogonality

# Christoffel numbers are positive for PBF

#### Lemma

The matrices  $\Lambda$  and  $\Upsilon$  are positive upper and lower unitriangular matrices, respectively

Theorem (Christoffel coefficients positivity)

Let us assume that  $T\,$  has a PBF and choose the matrices of initial conditions as

$$u^{-\top} = \Lambda \mathcal{A}, \qquad \qquad \xi^{-1} = \mathcal{B} \Upsilon$$

for some upper and lower unitriangular nonnegative matrices  $\mathcal{A} \in \mathbb{R}^{p \times p}$ and  $\mathcal{B} \in \mathbb{R}^{q \times q}$ , respectively. Then,

$$\rho_{k,b}^{[N]} > 0, \quad \mu_{k,a}^{[N]} > 0, \quad k \in 1, \dots, N+1, \quad a \in \{1, \dots, p\}, \quad b \in \{1, \dots, q\}$$

# Idea of the proof I

Christoffel numbers in terms of biorthogonal families of right an left eigenvectors:

$$\left[\mu_{k,1}^{[N]}\cdots\cdots\mu_{k,p}^{[N]}\right] = \left[w_{k,1}^{\langle N\rangle}\cdots\cdots w_{k,p}^{\langle N\rangle}\right]\nu^{-\top},$$

$$\begin{bmatrix} \rho_{k,1}^{[N]} \\ \vdots \\ \rho_{k,q}^{[N]} \end{bmatrix} = \xi^{-1} \begin{bmatrix} u_{k,1}^{\langle N \rangle} \\ \vdots \\ u_{k,q}^{\langle N \rangle} \end{bmatrix}$$

Banded matrices and orthogonality

January 24th, 2023

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 $\Gamma [N]$ 

Entries of these biorthogonal right and left eigenvectors can be written as  $w_{k,a}^{\langle N \rangle} = \alpha_N \left. \frac{Q_{a-1,N}}{P_{N+1}'P_N} \right|_{x=\lambda_k^{[N]}}$  and  $u_{k,b}^{\langle N \rangle} = \beta_N R_{b-1,N} \left( \lambda_k^{[N]} \right)$ 

 $\langle N \rangle^{-}$ 

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$$\beta_N \xi^{-1} \begin{bmatrix} R_{0,N} \\ \vdots \\ R_{q-1,N} \end{bmatrix}, \qquad \frac{1}{\beta_N} \left[ Q_{0,N} \cdots Q_{p-1,N} \right] \nu^{-\top}$$

are positive vectors at the points  $x=\lambda_k^{[N]}$  ,  $k\in\{1,\ldots,N+1\}$  . In the points  $x=\lambda_k^{[N]}$  ,  $k\in\{1,\ldots,N+1\}$ 

 $1 n^{\langle N \rangle}$ 

[N]

# Idea of the proof II

Consider left and right eigenvectors with last entry normalized to 1

$$\left[ \left. \frac{Q_{0,N}}{Q_{N,N}} \right|_{x=\lambda_k^{[N]}} \quad \left. \frac{Q_{1,N}}{Q_{N,N}} \right|_{x=\lambda_1^{[N]}} \cdots \cdots 1 \right],$$

$$\begin{bmatrix} \frac{R_{0,N}}{R_{N,N}} \Big|_{x=\lambda_k^{[N]}} \\ \frac{R_{1,N}}{R_{N,N}} \Big|_{x=\lambda_k^{[N]}} \\ \vdots \\ 1 \end{bmatrix}$$

(the last entry of any eigenvector is nonzero) Recall that  $Q_{N,N} = \alpha_N^{-1} P_N$ ,  $R_{N,N} = \beta_N^{-1} P_N$  and that the first eigenvector entries are not zero; i.e.,  $\alpha_N \left. \frac{Q_{0,N}}{P_N} \right|_{x=\lambda_k^{[N]}}$ ,  $\beta_N \left. \frac{R_{0,N}}{P_N} \right|_{x=\lambda_k^{[N]}} \neq 0$ . As the last entry is positive the change sign properties described in the sign Theorem leads to

$$\left. \begin{array}{l} \alpha_N \left. \frac{Q_{0,N}}{P_N} \right|_{x=\lambda_1^{[N]}} > 0, \quad \alpha_N \left. \frac{Q_{0,N}}{P_N} \right|_{x=\lambda_2^{[N]}} < 0, \quad \alpha_N \left. \frac{Q_{0,N}}{P_N} \right|_{x=\lambda_3^{[N]}} > 0, \\ \beta_N \left. \frac{R_{0,N}}{P_N} \right|_{x=\lambda_1^{[N]}} > 0, \quad \beta_N \left. \frac{R_{0,N}}{P_N} \right|_{x=\lambda_2^{[N]}} < 0, \quad \beta_N \left. \frac{R_{0,N}}{P_N} \right|_{x=\lambda_3^{[N]}} > 0, \end{array} \right.$$

and so on, alternating the sign

Banded matrices and orthogonality

January 24th, 2023

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$$\begin{array}{c} \frac{R_{0,N}}{R_{N,N}} \\ \frac{R_{1,N}}{R_{N,N}} \\ \\ \frac{R_{1,N}}{k} \\ \\ x = \lambda_k^{[N]} \\ \vdots \\ 1 \end{array}$$

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Banded matrices and orthogonality

January 24th, 2023

As T is oscillatory and the characteristic polynomial  $P_{N+1}$  interlaces  $P_N$  we have that sgn  $P_N(\lambda_k^{[N]}) = (-1)^{k-1}$  so that

$$\alpha_N Q_{0,N}(\lambda_k^{[N]}), \beta_N R_{0,N}(\lambda_k^{[N]}) > 0, \qquad k \in \{1, \dots, N+1\}$$

# Idea of the proof III

Darboux transform  $\hat{T}^{[N,\pm 1]}$  is an oscillatory matrix with characteristic polynomial  $P_{N+1}$ . Then, a left eigenvector of  $T^{[N,+1]}$  for the eigenvalue  $\lambda_k^{[N]}$  can be chosen as

$$\left[ \alpha_N \frac{Q_{0,N}}{P_N} \Big|_{x=\lambda_k^{[N]}} \quad \alpha_N \left. \frac{Q_{1,N}}{P_N} \right|_{x=\lambda_k^{[N]}} \cdots \cdots 1 \right] L_1^{[N]} = \left[ \alpha_N \left. \frac{(Q_{0,N} + L_{1|0}Q_{1,N})}{P_N} \right|_{x=\lambda_k^{[N]}} \right|_{x=\lambda_k^{[N]}}$$

and a right eigenvector of  $T^{[N,-1]}$  for the eigenvalue  $\lambda_k^{[N]}$  can be taken as

$$U_{1}^{[N]} \begin{bmatrix} \beta_{N} \frac{R_{0,N}}{P_{N}} \Big|_{x=\lambda_{k}^{[N]}} \\ \beta_{N} \frac{R_{1,N}}{P_{N}} \Big|_{x=\lambda_{k}^{[N]}} \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \beta_{N} \frac{(R_{0,N} + U_{1|0}R_{1,N})}{P_{N}} \Big|_{x=\lambda_{k}^{[N]}} \\ \vdots \\ 1 \end{bmatrix}$$

## Idea of the proof III

Sign properties of the eigenvectors of an oscillatory matrix:

$$\alpha_N \left. \frac{\frac{1}{L_{1|0}} Q_{0,N} + Q_{1,N}}{P_N} \right|_{x=\lambda_1^{[N]}} > 0, \qquad \alpha_N \left. \frac{\frac{1}{L_{1|0}} Q_{0,N} + Q_{1,N}}{P_N} \right|_{x=\lambda_2^{[N]}} < 0$$

$$\beta_N \left. \frac{\frac{1}{U_{1|0}} R_{0,N} + R_{1,N}}{P_N} \right|_{x=\lambda_1^{[N]}} > 0, \qquad \beta_N \left. \frac{\frac{1}{U_{1|0}} R_{0,N} + R_{1,N}}{P_N} \right|_{x=\lambda_2^{[N]}} < 0$$

and so alternating sign, and recalling the sign of  ${\cal P}_N$  at the zeros of  ${\cal P}_{N+1}$  we get

$$\left. \alpha_N \left( \frac{1}{L_{1|0}} Q_{0,N} + Q_{1,N} \right) \right|_{x = \lambda_k^{[N]}}, \beta_N \left( \frac{1}{U_{1|0}} R_{0,N} + R_{1,N} \right) \right|_{x = \lambda_k^{[N]}} > 0$$

January 24th, 2023 39

Consequently, after repeating this process up to  $T^{[N,+(p-1)]}$  and  $T^{[N,-(q-1)]}$  we find that

$$\beta_N \Upsilon \begin{bmatrix} R_{0,N} \\ \vdots \\ R_{q-1,N} \end{bmatrix}, \qquad \alpha_N [Q_{0,N} \cdots Q_{p-1,N}] \Lambda$$

are positive vectors at the points  $x = \lambda_k^{[N]}$ ,  $k \in \{1, \dots, N\}$ . Therefore, if the initial condition matrices are tuned as indicated we get the result

# Second kind polynomials, resolvent and Weyl functions

From here on we assume that  $N \ge \max(p,q)$ 

Given  $r \in \mathbb{N}$ , we write  $\{e_1^{[r]}, \ldots, e_r^{[r]}\}$  for the canonical basis of  $\mathbb{R}^r$  and consider the  $r \times (N+1)$  matrix  $E_{[r]} \coloneqq [I_r \ 0_{r \times (N+1-r)}]$ . Then, we introduce the vectors  $e_a^{\nu}, e_b^{\xi} \in \mathbb{R}^{N+1}$  with

$$e_a^{\nu} \coloneqq E_{[p]}^{\top} \nu^{-\top} e_a^{[p]}, \qquad \qquad \left(e_b^{\xi}\right)^{\top} \coloneqq \left(e_b^{[q]}\right)^{\top} \xi^{-1} E_{[q]}$$

For the matrices U and W we find

$$(e_b^{\xi})^{\top} U = \left[ \rho_{1,b}^{[N]} \cdots \rho_{N+1,b}^{[N]} \right], \qquad W e_a^{\nu}$$

January 24th, 2023

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$$(e_b^{\xi})^{\top} U = \begin{bmatrix} \rho_{1,b}^{[N]} \cdots \rho_{N+1,b}^{[N]} \end{bmatrix}, \qquad W e_a^{\nu} = \begin{bmatrix} \mu_{1,a}^{[N]} \\ \vdots \\ \vdots \\ \vdots \\ \end{bmatrix}$$

January 24th, 2023

### Second kind polynomials

For  $a \in \{1, \ldots, p\}, b \in \{1, \ldots, q\}$ , and  $\pi_k^{[N]} \coloneqq \prod_{\substack{l \in \{1, \ldots, N+1\}\\ l \neq k}} (x - \lambda_l^{[N]})$ , the second kind polynomials, in terms of the adjugate, are given by

$$\begin{split} \rho_{N+1}^{(b,a)}(x) &\coloneqq (e_b^{\xi})^\top \operatorname{adj}(xI_{N+1} - T^{[N]})e_a^{\nu} \\ &= \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} \pi_k^{[N]}(x) \\ &= \int \frac{P_{N+1}(z) - P_{N+1}(x)}{z - x} \mathrm{d}\psi_{b,a}^{[N]}(x) \\ &= \alpha_{N+1} \int \frac{\det(A_{N+1}(z)) - \det(A_{N+1}(x))}{z - x} \mathrm{d}\psi_{b,a}^{[N]}(x) \\ &= \beta_{N+1} \int \frac{\det(B_{N+1}(z)) - \det(B_{N+1}(x))}{z - x} \mathrm{d}\psi_{b,a}^{[N]}(x) \end{split}$$

If T has a PBF and the initial conditions are tuned as above then  $\deg P_{N+1}^{(b,a)}=N$ 

The moments of the pq discrete measures  $d\psi_{b,a}^{[N]}$  are linked to the components of the powers of  $T^{[N]}$ :

Theorem (Discrete moments)

For  $a \in \{1, \dots, p\}, b \in \{1, \dots, q\}$ , the discrete moments we have

$$\int x^{n} \mathrm{d}\psi_{b,a}^{[N]}(x) = \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} \big(\lambda_{k}^{[N]}\big)^{n} = (e_{b}^{\xi})^{\top} \big(T^{[N]}\big)^{n} e_{a}^{\nu}$$

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**Theorem (Discrete moments)** For  $a \in \{1, ..., p\}, b \in \{1, ..., q\}$ , the discrete moments we have

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#### Proof.

We have that  $(e_b^\xi)^\top \big(T^{[N]}\big)^n e_a^\nu = (e_b^\xi)^\top U D^n W e_a^\nu$  so that

$$(e_b^{\xi})^{\top} (T^{[N]})^n e_a^{\nu} = \left[ \rho_{1,b}^{[N]} \cdots \rho_{N+1,b}^{[N]} \right] D^n \begin{bmatrix} \mu_{1,a}^{[N]} \\ \vdots \\ \mu_{N+1,a}^{[N]} \end{bmatrix}$$

Banded matrices and orthogonality January 24

January 24th, 2023 44 / 55

#### Resolvent

The resolvent matrix  $R^{[N]}(z)$  of the leading principal submatrix  $T^{[N]}$  is

$$R^{[N]}(z) \coloneqq \left(zI_{N+1} - T^{[N]}\right)^{-1} = \frac{\operatorname{adj}\left(zI_{N+1} - T^{[N]}\right)}{\det(zI_{N+1} - T^{[N]})}$$

#### Weyl's functions

For  $a \in \{1, \ldots, p\}, b \in \{1, \ldots, q\}$ , the Weyl functions are

$$\begin{split} S_{b,a}^{[N]} &\coloneqq (e_b^{\xi})^\top R^{[N]} e_a^{\nu} \\ &= \frac{P_{N+1}^{(b,a)}(z)}{P_{N+1}(z)} = \sum_{k=1}^{N+1} \frac{\rho_{k,b}^{[N]} \mu_{k,a}^{[N]}}{z - \lambda_k^{[N]}} = \int \frac{\mathrm{d}\psi_{b,a}^{[N]}(x)}{z - x} \end{split}$$

#### Interlacing

If T has a PBF for tuned initial conditions the polynomial  $P_{N+1}^{(b,a)}$  is interlaced by  $P_{N+1}$ 

#### Recursion polynomials and resolvent

$$\sum_{a=1}^{p} \int \frac{\mathrm{d}\psi_{b,a}^{[N]}(x)}{z-x} A_{n-1}^{(a)}(x) = \left(e_{b}^{\xi}\right)^{\top} R^{[N]}(z) e_{r}$$
$$\sum_{b=1}^{q} \int B_{n-1}^{(b)}(x) \frac{\mathrm{d}\psi_{b,a}^{[N]}(x)}{z-x} = e_{n}^{\top} R^{[N]}(z) e_{a}^{\nu}$$

A path to mixed Hermite-Padé

January 24th, 2023

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#### **Recursion polynomials and resolvent**

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$$\sum_{b=1}^{q} \int B_{n-1}^{(b)}(x) \frac{\mathrm{d}\psi_{b,a}^{[N]}(x)}{z-x} = e_{n}^{\top} R^{[N]}(z) e_{a}^{\nu}$$

A path to mixed Hermite-Padé

January 24th, 2023

## Helly's tools

- **1.** Helly's Selection Principle: for any uniformly bounded sequence  $\{\psi^{[N]}\}_{N=0}^{\infty}$  of non-decreasing functions defined in  $\mathbb{R}$ , there exists a convergent subsequence converging to a non-decreasing function  $\psi$  defined in  $\mathbb{R}$
- **2.** Helly's second theorem: Let us assume a uniformly bounded sequence  $\{\psi^{[N]}\}_{N=0}^{\infty}$  of non-decreasing functions on a compact interval [a, b] with limit function  $\psi$ , then for any continuous function f in [a, b] we have  $\lim_{N\to\infty} \int_a^b f(x) d\psi^{[N]}(x) = \int_a^b f(x) d\psi(x)$

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As the submatrices  $T^{[N]}$  are oscillatory, we know that  $P_{N+1}(x)$  strictly interlaces  $P_N(x)$  so that the positive sequence  $\{\lambda_1^{[N]}\}_{N=1}^{\infty}$  is a strictly increasing sequence and  $\{\lambda_{N+1}^{[N]}\}_{N=1}^{\infty}$  is a strictly decreasing sequence. As well, for bounded operators,  $||T||_{\infty} < \infty$ , we have  $||T^{[N]}||_{\infty} < ||T||_{\infty} < \infty$ . Therefore, there exists the limits  $\zeta := \lim_{N \to \infty} \lambda_{N+1}^{[N]} \ge 0$  and  $\eta := \lim_{N \to \infty} \lambda_1^{[N]} \le ||T||_{\infty}$ . We call  $[\zeta, \eta] \subseteq [0, ||T||_{\infty}]$  the true interval of orthogonality, that is the smallest interval containing all zeros of the characteristic polynomials  $P_n$ , i.e. the eigenvalues of the leading principal submatrices of T

### Theorem (Favard spectral representation)

Let us assume that

**1.** The banded matrix T is bounded and there exist  $s \ge 0$  such that T + sI has a PBF.

The sequences  $\{A_n^{(1)}, \ldots, A_n^{(p)}\}_{n=0}^{\infty}, \{B_n^{(1)}, \ldots, B_n^{(q)}\}_{n=0}^{\infty}$  of recursion polynomials are determined by the initial condition matrices  $\nu$  and  $\xi$ , respectively, such that  $\nu^{-\top} = \Lambda \mathcal{A}, \xi^{-1} = \mathcal{B}\Upsilon$ , and  $\mathcal{A} \in \mathbb{R}^{p \times p}$  is a nonnegative upper unitriangular matrices and  $\mathcal{B} \in \mathbb{R}^{q \times q}$  is a nonnegative lower unitriangular matrix.

Then, there exists pq non decreasing positive functions  $\psi_{b,a}$ ,  $a \in \{1, \ldots, p\}$  and  $b \in \{1, \ldots, q\}$  and corresponding positive Lebesgue–Stieltjes measures  $d\psi_{b,a}$  with compact support  $\Delta$  such that the following biorthogonality holds

$$\sum_{a=1}^{p} \sum_{b=1}^{q} \int_{\zeta}^{\eta} B_{l}^{(b)}(x) \mathrm{d}\psi_{b,a}(x) A_{k}^{(a)}(x) = \delta_{k,l}, \qquad k, l \in \mathbb{N}_{0}$$

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$$\sum_{a=1}^{p} \sum_{b=1}^{q} \int_{\zeta}^{\eta} B_{l}^{(b)}(x) \mathrm{d}\psi_{b,a}(x) A_{k}^{(a)}(x) = \delta_{k,l}, \qquad k, l \in \mathbb{N}_{0}$$

#### Proof.

The shift in the matrix  $T \to T + sI$  only shifts by s the eigenvalues of the truncations  $T^{[N]}$ , so that they are positive, and the dependent variable of the recursion polynomials, but do not alter the interlacing properties of the polynomials and the positivity of the corresponding Christoffel numbers. We know that the sequences  $\{\psi_{a,b}^{[N]}\}_{N=0}^{\infty}$ ,  $a \in \{1, \ldots, p\}$ ,  $b \in \{1, \ldots, q\}$  are positive, uniformly bounded and nondecreasing. Consequently, following Helly's results there exist subsequences that converge when  $N \to \infty$  to positive nondecreasing functions  $\psi_{b,a}$  with support on  $[\zeta, \eta]$  and that the discrete biorthogonal relations lead to the stated biorthogonal properties

### Corollary (Mixed multiple orthogonal relations)

In the conditions above, the mixed multiple orthogonal relations are fulfilled

$$\sum_{a=1}^{p} \int_{\zeta}^{\eta} x^{n} \mathrm{d}\psi_{b,a}(x) A_{m}^{(a)}(x) = 0, \quad n \in \{0, \dots, \deg B_{m-1}^{(b)}\}, \quad b \in \{1, \dots, q\}$$
$$\sum_{b=1}^{q} \int_{\zeta}^{\eta} B_{m}^{(b)}(x) \mathrm{d}\psi_{b,a}(x) x^{n} = 0, \quad n \in \{0, \dots, \deg A_{m-1}^{(a)}\}, \quad a \in \{1, \dots, p\}$$

Banded matrices and orthogonality

January 24th, 2023

Theorem (Spectral representation of moments and Stieltjes–Markov functions)

In the conditions above and in terms of the spectral functions  $\psi_{b,a}$ ,  $a \in \{1, \ldots, p\}$ ,  $b \in \{1, \ldots, q\}$  we find the following relations between entries of powers or the resolvent of the banded matrix and moments or the Cauchy transform of the measures, respectively:

$$(e_b^{\xi})^{\top} T^n e_a^{\nu} = \int_{\zeta}^{\eta} x^n \mathrm{d}\psi_{b,a}(x)$$
$$(e_b^{\xi})^{\top} (zI - T)^{-1} e_a^{\nu} = \int_{\zeta}^{\eta} \frac{\mathrm{d}\psi_{b,a}(x)}{z - x} \eqqcolon \hat{\psi}_{b,a}(z)$$

January 24th, 2023

### Theorem (Normal convergence of Weyl functions)

For  $a \in \{1, \ldots, p\}, b \in \{1, \ldots, q\}$ , and in the conditions above, the Weyl functions converge uniformly in compact subsets of  $\overline{\mathbb{C}} \setminus [\zeta, \eta]$  to the Stieltjes–Markov functions, i.e.,

$$S_{b,a}^{[N]}(z) = \frac{P_{N+1}^{(b,a)}(z)}{P_{N+1}(z)} \xrightarrow[N \to \infty]{} \hat{\psi}_{b,a}(z)$$

### Gauss quadrature formulas

#### **Degrees of precision**

The degrees of precision or orders  $d_{b,a}(N)$ ,  $a \in \{1, \ldots, p\}$ ,  $b \in \{1, \ldots, q\}$ , are the largest natural numbers such that

$$\left(e_{b}^{\xi}\right)^{\top}T^{n}e_{a}^{\nu}=\left(e_{b}^{\xi}\right)^{\top}\left(T^{[N]}\right)^{n}e_{a}^{\nu},\qquad 0\leqslant n\leqslant d_{b,a}(N)$$

with

$$d_{b,a}(N) = \deg A_N^{(a)} + \deg B_N^{(b)} + 1 = \left\lceil \frac{N+2-a}{p} \right\rceil + \left\lceil \frac{N+2-b}{q} \right\rceil - 1$$

**Theorem (Mixed multiple Gaussian quadrature formulas)** *The following Gauss quadrature formulas hold* 

$$\int_{\zeta}^{\eta} x^{n} \mathrm{d}\psi_{b,a}(x) = \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} (\lambda_{k}^{[N]})^{n}, \qquad 0 \leqslant n \leqslant d_{b,a}(N)$$

Here the degrees of precision  $d_{b,a}$  are optimal (for any power largest than n a positive remainder appears, an exactness is lost)