Ladder operators for generalized Zernike or disk polynomials

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Abstract

The aim of this work is to report on several ladder operators for generalized Zernike polynomials which are orthogonal polynomials on the unit disk $\mathbf{D} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leqslant 1\}$ with respect to the weight function $W_\mu(x,y) = (1-x^2-y^2)^\mu$ where $\mu > -1$. These polynomials can be expressed in terms of the univariate Jacobi polynomials and, thus, we start by deducing several ladder operators for the Jacobi polynomials. Due to the symmetry of the disk and the weight function W_μ, it turns out that it is more convenient to use complex variables $z = x+iy$ and $\bar{z} = x - iy$.

Generalized Zernike or disk polynomials

Generalized Zernike polynomials: *For $n \geqslant 0$ and $0 \leqslant j \leqslant \frac{n}{2}$, the generalized Zernike polynomials are defined as*

$$\begin{aligned} P^{n,\mu}_{j,1}(x,y) &= P^{(\mu,n-2j)}_j(2r^2-1)\, r^{n-2j} \cos(n-2j)\theta, \\ P^{n,\mu}_{j,2}(x,y) &= P^{(\mu,n-2j)}_j(2r^2-1)\, r^{n-2j} \sin(n-2j)\theta, \end{aligned} \qquad (1)$$

where $(x,y) = (r\cos\theta, r\sin\theta)$, $0 \leqslant r \leqslant 1$, $0 \leqslant \theta < 2\pi$, and $P^{(\alpha,\beta)}_n(t)$ denotes the classical Jacobi polynomial of degree n.

- **Domain:** The unit disk in $\mathbb{R}^2$:
$$\mathbf{D} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leqslant 1\}$$
- **Weight function:** For $\mu > -1$,
$$W_\mu(x,y) = (1-x^2-y^2)^\mu, \quad (x,y) \in \mathbf{D}.$$
- **Inner product:**
$$\langle P, Q\rangle_\mu = b_\mu \int\int_{\mathbf{D}} P(x,y)\, Q(x,y)\, W_\mu(x,y)\, dxdy,$$
where
$$b_\mu = \left(\int\int_{\mathbf{D}} W_\mu(x,y)\, dxdy\right)^{-1} = \frac{\mu+1}{\pi}.$$
- **Orthogonality:** For $n \geqslant 0$ and $0 \leqslant j \leqslant \frac{n}{2}$,
$$\langle P^{n,\mu}_{j,\nu}, P^{m,\mu}_{k,\eta}\rangle_\mu = H^{n,\mu}_j\, \delta_{n,m}\, \delta_{j,k}\, \delta_{\nu,\eta},$$
where
$$H^{n,\mu}_j = \frac{(\mu+1)_j\,(n-j)!\,(n-j+\mu+1)}{j!\,(n+2)_{n-j}\,(n+\mu+1)} \begin{cases} \times 2, & n \neq 2j, \\ \times 1, & n = 2j. \end{cases}$$

Complex generalized Zernike polynomials

Complex Generalized Zernike polynomials: *For $k, j \geqslant 0$, the complex generalized Zernike polynomials are defined as*

$$Q^\mu_{k,j}(z,\bar{z}) = \frac{(\mu+1)_{k+j}}{(\mu+1)_k\,(\mu+1)_j} \sum_{n=0}^{\min\{k,j\}} \frac{(-k)_n\,(-j)_n}{n!\,(-\mu-k-j)_n} z^{k-n}\,\bar{z}^{j-n}. \qquad (2)$$

where $z = x+iy$, $\bar{z} = x-iy$. The polynomials (2) are normalized by $Q^\mu_{k,j}(1,1) = 1$.

- From (2), we immediately get that the polynomials $Q^\mu_{k,j}(z,\bar{z})$ are invariant under the simultaneous permutations of the variables $\{z,\bar{z}\}$ and the subindices $\{k,j\}$, that is,
$$Q^\mu_{k,j}(z,\bar{z}) = Q^\mu_{j,k}(\bar{z},z). \qquad (3)$$
- The complex generalized Zernike polynomials can be written in terms of the Jacobi polynomials as follows,
$$Q^\mu_{k,j}(z,\bar{z}) = \frac{j!}{(\mu+1)_j} z^{k-j}\, P^{(\mu,k-j)}_j(2z\bar{z}-1), \quad k \geqslant j, \qquad (4)$$
and
$$Q^\mu_{k,j}(z,\bar{z}) = \frac{k!}{(\mu+1)_k} \bar{z}^{j-k}\, P^{(\mu,j-k)}_k(2z\bar{z}-1) \quad j \geqslant k. \qquad (5)$$
- **Weight function:** For $\mu > -1$,
$$w_\mu(z) = (1-z\bar{z})^\mu, \quad \mu > -1, \quad z \in \mathbf{D}.$$
- **Orthogonality:**
$$b_\mu \int_{\mathbf{D}} Q^\mu_{k,j}(z,\bar{z})\, \overline{Q^\mu_{m,\ell}(z,\bar{z})}\, w_\mu(z)\, dz = h^\mu_{k,j}\, \delta_{k,m}\, \delta_{j,\ell}, \qquad (6)$$
where
$$h^\mu_{k,j} = \frac{\mu+1}{\mu+k+j+1} \frac{k!\,j!}{(\mu+1)_k\,(\mu+1)_j}. \qquad (7)$$
- By (4),
$$\mathrm{Re}\{Q^\mu_{n-j,j}(z,\bar{z})\} = \frac{j!}{(\mu+1)_j} P^{n,\mu}_{j,1}(x,y), \; 0 \leqslant j \leqslant \frac{n}{2},$$
$$\mathrm{Im}\{Q^\mu_{n-j,j}(z,\bar{z})\} = \frac{j!}{(\mu+1)_j} P^{n,\mu}_{j,2}(x,y), \; 0 \leqslant j \leqslant \frac{n}{2},$$

Type I ladder operators

- $Q^\mu_{k,j}(z,\bar{z}) \to Q^{\mu-1}_{k+1,j}(z,\bar{z})$:
$$\left\{(1-z\bar{z})\frac{\partial}{\partial \bar{z}} - \mu\, z\right\} Q^\mu_{k,j}(z,\bar{z}) = -\mu\, Q^{\mu-1}_{k+1,j}(z,\bar{z}), \quad \mu > 0. \qquad (8)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^{\mu-1}_{k,j+1}(z,\bar{z})$:
$$\left\{(1-z\bar{z})\frac{\partial}{\partial z} - \mu\, \bar{z}\right\} Q^\mu_{k,j}(z,\bar{z}) = -\mu\, Q^{\mu-1}_{k,j+1}(z,\bar{z}), \quad \mu > 0. \qquad (9)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^{\mu-1}_{k+1,j+1}(z,\bar{z})$:
$$\begin{aligned} \left\{(1-z\bar{z})\bar{z}\frac{\partial}{\partial \bar{z}} + (k+1)(1-z\bar{z}) - \mu z\bar{z}\right\} Q^\mu_{k,j}(z,\bar{z}) &= -\mu\, Q^{\mu-1}_{k+1,j+1}(z,\bar{z}), \\ \left\{(1-z\bar{z})z\frac{\partial}{\partial z} + (j+1)(1-z\bar{z}) - \mu z\bar{z}\right\} Q^\mu_{k,j}(z,\bar{z}) &= -\mu\, Q^{\mu-1}_{k+1,j+1}(z,\bar{z}). \end{aligned} \qquad (10)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^{\mu-1}_{k,j}(z,\bar{z})$: For $\mu > 0$,
$$\begin{aligned} \left\{(1-z\bar{z})\, z\frac{\partial}{\partial z} - k\,(1-z\bar{z}) - \mu\right\} Q^\mu_{k,j}(z,\bar{z}) &= -\mu\, Q^{\mu-1}_{k,j}(z,\bar{z}), \\ \left\{(1-z\bar{z})\, \bar{z}\frac{\partial}{\partial \bar{z}} - j\,(1-z\bar{z}) - \mu\right\} Q^\mu_{k,j}(z,\bar{z}) &= -\mu\, Q^{\mu-1}_{k,j}(z,\bar{z}). \end{aligned} \qquad (11)$$

Type II ladder operators

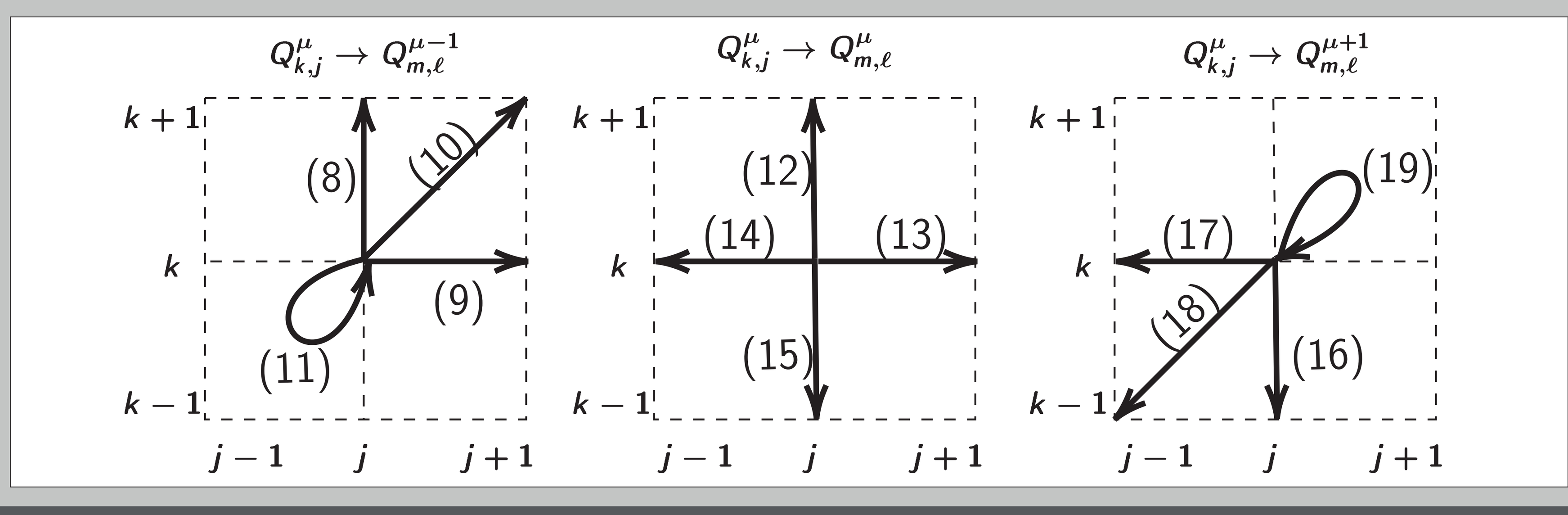
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^\mu_{k+1,j}(z,\bar{z})$: For $\mu > -1$,
$$\left\{(1-z\bar{z})\frac{\partial}{\partial \bar{z}} - (k+\mu+1)z\right\} Q^\mu_{k,j}(z,\bar{z}) = -(k+\mu+1)\, Q^\mu_{k+1,j}(z,\bar{z}). \qquad (12)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^\mu_{k,j+1}(z,\bar{z})$: For $\mu > -1$,
$$\left\{(1-z\bar{z})\frac{\partial}{\partial z} - (j+\mu+1)\bar{z}\right\} Q^\mu_{k,j}(z,\bar{z}) = -(j+\mu+1)\, Q^\mu_{k,j+1}(z,\bar{z}). \qquad (13)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^\mu_{k-1,j}(z,\bar{z})$:
$$\left\{(1-z\bar{z})\frac{\partial}{\partial z} + k\bar{z}\right\} Q^\mu_{k,j}(z,\bar{z}) = k\, Q^\mu_{k-1,j}(z,\bar{z}), \quad \mu > -1. \qquad (14)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^\mu_{k,j-1}(z,\bar{z})$:
$$\left\{(1-z\bar{z})\frac{\partial}{\partial \bar{z}} + jz\right\} Q^\mu_{k,j}(z,\bar{z}) = j\, Q^\mu_{k,j-1}(z,\bar{z}), \quad \mu > -1. \qquad (15)$$

Type III ladder operators

- $Q^\mu_{k,j}(z,\bar{z}) \to Q^{\mu+1}_{k-1,j}(z,\bar{z})$:
$$\frac{\partial}{\partial z} Q^\mu_{k,j}(z,\bar{z}) = \frac{k\,(j+\mu+1)}{\mu+1} Q^{\mu+1}_{k-1,j}(z,\bar{z}), \quad \mu > -1. \qquad (16)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^{\mu+1}_{k,j-1}(z,\bar{z})$:
$$\frac{\partial}{\partial \bar{z}} Q^\mu_{k,j}(z,\bar{z}) = \frac{j\,(k+\mu+1)}{\mu+1} Q^{\mu+1}_{k,j-1}(z,\bar{z}), \quad \mu > -1. \qquad (17)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^{\mu+1}_{k-1,j-1}(z,\bar{z})$:
$$\begin{aligned} \left\{z\frac{\partial}{\partial z} - k\right\} Q^\mu_{k,j}(z,\bar{z}) &= \frac{k\,j}{\mu+1} Q^{\mu+1}_{k-1,j-1}(z,\bar{z}) \quad \mu > -1, \\ \left\{\bar{z}\frac{\partial}{\partial \bar{z}} - j\right\} Q^\mu_{k,j}(z,\bar{z}) &= \frac{k\,j}{\mu+1} Q^{\mu+1}_{k-1,j-1}(z,\bar{z}), \quad \mu > -1. \end{aligned} \qquad (18)$$
- $Q^\mu_{k,j}(z,\bar{z}) \to Q^{\mu+1}_{k,j}(z,\bar{z})$: For $\mu > -1$,
$$\begin{aligned} \left\{z\frac{\partial}{\partial z} + j + \mu + 1\right\} Q^\mu_{k,j}(z,\bar{z}) &= \frac{(k+\mu+1)\,(j+\mu+1)}{\mu+1} Q^{\mu+1}_{k,j}(z,\bar{z}), \\ \left\{\bar{z}\frac{\partial}{\partial \bar{z}} + k + \mu + 1\right\} Q^\mu_{k,j}(z,\bar{z}) &= \frac{(k+\mu+1)\,(j+\mu+1)}{\mu+1} Q^{\mu+1}_{k,j}(z,\bar{z}). \end{aligned} \qquad (19)$$

Summary of ladder operators

- Illustration of how the ladder operators increase or decrease the parameters in $Q^\mu_{k,j}(z,\bar{z})$.



A note on Sobolev orthogonality

- Observe that 8 and 16 do not depend on the degrees k or j. Moreover, these ladder operators make sense for $\mu = 0$ and $\mu = -1$, respectively. We can use them to prove the following result.

Definition. *Define the polynomials*

$$\begin{aligned} &Q^{-1}_{k,0}(z,\bar{z}) = z^k, \quad k \geqslant 0, \quad Q^{-1}_{0,j}(z,\bar{z}) = \bar{z}^j, \quad j \geqslant 0, \\ &Q^{-1}_{k,j}(z,\bar{z}) = (1-z\bar{z})\frac{\partial}{\partial \bar{z}} Q^0_{k,j-1}(z,\bar{z}), \quad k,j \geqslant 1. \end{aligned} \qquad (20)$$

Lemma. *The polynomials defined in (20) satisfy*

$$\frac{\partial}{\partial z} Q^{-1}_{k,j}(z,\bar{z}) = -Q^0_{k-1,j}(z,\bar{z}), \quad k,j \geqslant 1.$$

Proof. *We compute,*

$$\frac{\partial}{\partial z} Q^{-1}_{k,j}(z,\bar{z}) = \left\{(1-z\bar{z})\frac{\partial}{\partial z} - \bar{z}\right\} Q^1_{k-1,j-1}(z,\bar{z}).$$

Our result follows from (9).

Proposition. *The polynomials defined in (20) constitute a mutually orthogonal polynomial system with respect to $(\cdot,\cdot)_1$, where*

$$(f,g)_1 = \frac{\lambda}{\pi}\int_{\mathbf{D}} \frac{\partial f}{\partial z}(z,\bar{z})\, \overline{\frac{\partial g}{\partial z}(z,\bar{z})}\, dz + \frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta}, e^{-i\theta})\, \overline{g(e^{i\theta}, e^{-i\theta})}\, d\theta, \quad \lambda > 0.$$

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