Maxwell Orthogonal Polynomials

Ángel Álvarez Paredes[a], Ruymán Cruz Barroso[a], Francisco Marcellán Español[b]

[a]Department of Mathematical Analysis and Instituto de Matemáticas y sus Aplicaciones (IMAULL), La Laguna University, 38271 La Laguna, Tenerife, Canary Islands, Spain
[b]Department of Mathematics, Universidad Carlos III de Madrid, 28911 Leganés, Madrid, Spain

Introduction

In this study, we focus on a specific sequence of orthogonal polynomials associated with the linear functional defined by the weight function $\omega(x) = e^{-x^2}$ on the positive real semi-axis:

$$L_M[p] = \langle L_M, p\rangle = \int_0^\infty p(x)e^{-x^2}\,dx,$$

where p is any polynomial in $\mathbb{P}$, the space of polynomials with complex coefficients. This linear functional belongs to the class of semiclassical functionals, specifically of class $s = 1$, which means it satisfies a distributional Pearson equation with polynomials ϕ and ψ of degrees related to the class.

The polynomials orthogonal with respect to this functional are sometimes referred to as **Maxwell polynomials**, due to their appearance in problems involving the Maxwellian distribution in kinetic theory and statistical mechanics. Despite their applications, these polynomials have not been thoroughly explored within the framework of semiclassical orthogonal polynomials. Understanding their properties can provide valuable insights into nonlinear difference equations, integrable systems, and computational methods in applied mathematics.

This work aims to delve into the properties of the Maxwell polynomials by investigating the associated linear functional within the semiclassical framework. The primary objectives are to classify the linear functional based on its **Pearson equation**, derive structural relations for the orthogonal polynomials, set connections with **discrete Painlevé equations**, analyze the **ladder operators**, and interpret the zeros of these polynomials through an **electrostatic model**.

Pearson equation, moments and Stieltjes function

We begin by verifying that the linear functional L_M satisfies a Pearson equation of the form:

$$D(\phi L_M) = \psi L_M,$$

where $\phi(x) = x$ and $\psi(x) = 1 - 2x^2$, and D denotes the derivative operator. This confirms that L_M is a semiclassical functional of class $s = 1$.

Next, we compute the moments of L_M, $\mu_n = \langle L_M, x^n\rangle$, $n \geq 0$, using the integral definitions, leading to expressions involving the Gamma function:

$$\mu_{2n} = \frac{1}{2}\cdot\Gamma\left(n+\tfrac{1}{2}\right), \quad \mu_{2n+1} = \frac{n!}{2}, \quad n \geq 0.$$

Using the Pearson equation, we can deduce a second-order homogeneous difference equation that characterizes these moments:

$$2\mu_{n+2} - (n+1)\mu_n = 0, \quad n \geq 0, \quad \mu_0 = \frac{\sqrt{\pi}}{2}, \quad \mu_1 = \frac{1}{2}.$$

The Stieltjes function $\mathcal{S}(z)$ associated with L_M is then derived, satisfying a first-order linear differential equation, which provides a link between the moments and the functional's properties:

$$z\mathcal{S}'(z) + 2z^2\mathcal{S}(z) = 2(\mu_1 + z\mu_0).$$

Laguerre-Freud equations

Using the structure relation obtained from the Pearson equation,

$$\phi(x)P'_{n+1}(x) = (n+1)P_{n+1}(x) + \lambda_{n+1,n}P_n(x) + \lambda_{n+1,n-1}P_{n-1}(x), \quad n \geq 1,$$

where $\lambda_{n+1,n} = 2\beta_{n+1}(\alpha_{n+1}+\alpha_n)$ and $\lambda_{n+1,n-1} = 2\beta_{n+1}\beta_n$, we derive the Laguerre-Freud equations, which are nonlinear difference equations governing the recurrence coefficients α_n and β_n of the orthogonal polynomials:

$$\alpha_n^2 + \beta_{n+1} + \beta_n = n + \frac{1}{2}, \quad 4\beta_{n+1}\left[\frac{1}{2} + \beta_n + \alpha_n(\alpha_n + \alpha_{n+1})\right] = (n+1)^2, \quad n \geq 1,$$

with initial conditions

$$\beta_1 = \frac{1}{2} - \frac{1}{\pi}, \quad \beta_2 = \frac{\pi(\pi-3)}{(\pi-2)^2} \quad \text{and} \quad \alpha_1 = \frac{2}{(\pi-2)\sqrt{\pi}}.$$

These coefficients are crucial as they define the three-term recurrence relation that the monic polynomials satisfy:

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x), \quad n \geq 0.$$

By defining $g_n = \frac{n}{2} - \beta_n$, we transform these equations into a discrete Painlevé IV equation:

$$g_{n+1}^4 = \left(\tfrac{n+1}{2} - g_{n+1}\right)^2(g_{n+1}+g_n)(g_{n+2}+g_{n+1}), \quad n \geq 1,$$

with initial conditions

$$g_1 = \frac{1}{2} - \beta_1 = \frac{1}{\pi}, \quad g_2 = 1 - \beta_2 = \frac{4-\pi}{(\pi-2)^2}.$$

We finally study the asymptotic behavior of the coefficients α_n and β_n, that is the leading behavior of an asymptotic series solution for the recurrence relations discussed previously. We find that

$$\alpha_n \sim \frac{1}{6}n \quad \text{and} \quad \beta_n \sim \sqrt{\frac{2n}{3}}.$$

Ladder operators and electrostatic interpretation

The ladder operators are explicitly constructed, with the lowering operator given by:

$$L_{n+1}P_{n+1} = P_n, \quad n \geq 0, \quad \text{where} \quad L_{n+1} = A_{n+1}(x)D_x - B_{n+1}(x),$$

$$A_{n+1}(x) = \frac{\phi(x)}{2\beta_{n+1}(x-\alpha_n)+\lambda_{n+1,n}} \quad \text{and} \quad B_{n+1}(x) = \frac{n+1-2\beta_{n+1}}{2\beta_{n+1}(x-\alpha_n)+\lambda_{n+1,n}},$$

and the raising operator is also obtained.

Using these operators, we derive a second-order linear differential equation satisfied by the polynomials $P_{n+1}(x)$:

$$J(x,n)P''_{n+1}(x) + K(x,n)P'_{n+1}(x) + M(x,n)P_{n+1}(x) = 0,$$

where $J(x,n)$, $K(x,n)$, and $M(x,n)$ are explicitly defined polynomials.

The electrostatic interpretation reveals that the zeros of the Maxwell polynomials correspond to equilibrium positions of n unit-charged particles under the external potential:

$$V_n(x) = x^2 + \ln|x| - \ln\left|x + \frac{P_n^2(0)}{2h_n}\right|,$$

where h_n is the squared norm of P_n. Considering that $\frac{P_n^2(0)}{2h_n} > 0$, we have an extra charge located at $-\frac{P_n^2(0)}{2h_n} < 0$. Additionally, it can be observed that there is a negative charge at the origin, which attracts the positive ones.

Our study advances the understanding of Maxwell polynomials within the semiclassical framework. By classifying the associated linear functional and deriving the **Laguerre-Freud equations**, we shed light on the underlying structure of these polynomials. The connection to discrete Painlevé equations bridges the gap between orthogonal polynomials and integrable systems, highlighting potential areas for future research.

The construction of ladder operators and the derivation of differential equations offer valuable tools for further analytical and numerical investigations. The electrostatic interpretation not only provides a physical analogy for the zeros of the polynomials but also enriches the theoretical foundation of semiclassical orthogonal polynomials.

Main references

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