FIFTH PAINLEVÉ EQUATION AND KRAWTCHOUK POLYNOMIALS

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Motivation

It is known that a sequence of monic orthogonal polynomials $(P_n)_{n\geq 0}$ satisfies a three-term recurrence relation [1, p. 18]

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x). \tag{1}$$

Our main objective is to study how we can connect the recurrence coefficients a_n^2 and b_n with the fifth Painlevé equation, which is given by

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\alpha_5 y + \frac{\beta_5}{y}\right) + \gamma_5\frac{y}{t} + \delta_5\frac{y(y+1)}{y-1}, \tag{2}$$

where $\alpha_5, \beta_5, \gamma_5, \delta_5$ are complex parameters. We will consider the generalised monic Krawtchouk polynomials $P_n(x; t, \alpha, N)$, orthogonal with respect to the weight

$$\omega(x) = \binom{N}{x}\frac{t^x}{(1-\alpha)_x}, \quad t > 0,\ \alpha < 1,\ x \in \{0, \dots, N\},$$

and define

$$q := \frac{1}{N}\left(\frac{a_n^2}{t} + n\right), \quad p := -\frac{b_n + N + 1 + t - n - \alpha}{N}, \tag{3}$$

where a_n^2, b_n are the recurrence coefficients in (1).

The differential system

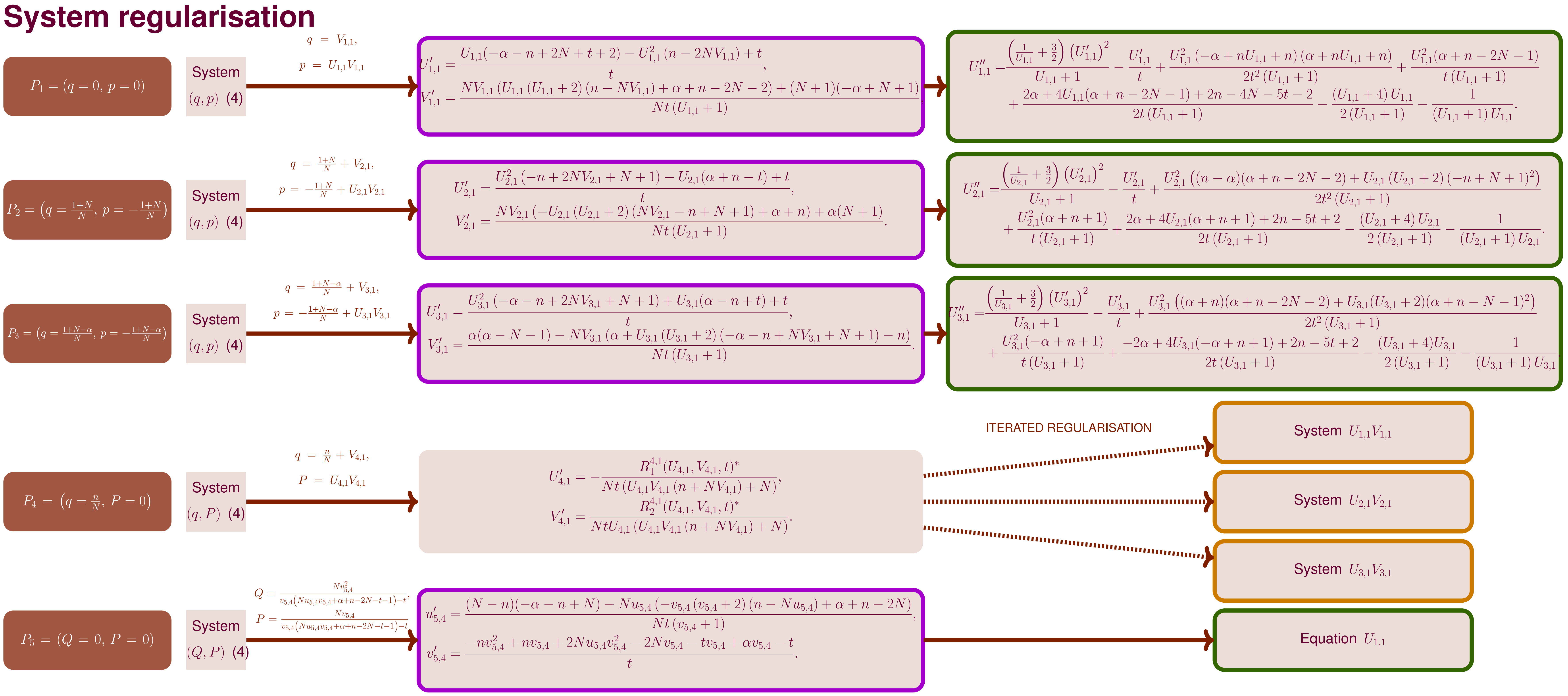
In [2] the authors prove that the auxiliary funtions defined by (3) satisfy the following differential system

$$\begin{aligned} q' &= \frac{Np^2(n-Nq) + 2Npq(n-Nq) + q(Nq(\alpha+n-2N-2) + (N+1)(-\alpha+N+1))}{Nt(p+q)},\\ p' &= \frac{2Npqt + p((N+1)(-\alpha+N+1) + Np(-\alpha+Np+2N+t+2)) + Nq^2t}{Nt(p+q)}. \end{aligned} \tag{4}$$

The regularisation procedure

- In order to take into account that Painlevé equations admit poles, we consider the differential system (4) in the complex proyective space $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, we work in coordinates (q,p), (Q,p), (q,P), (Q,P), where $Q = 1/q$, $P = 1/p$.
- We look for the points of indeterminacy (points which make both the numerator and denominator of the right hand side equal to zero in at least one equation).
- We blow up points of indeterminacy. To blow up a point of indeterminacy $P_i = (a,b)$ we apply the following changes of variable $x = a + u_iv_i = a + V_i,\ y = b + v_i = b + U_iV_i$.
- We repeat the process up to obtain regular systems in $v_i = 0,\ V_i = 0$.

System regularisation



$P_1 = (q=0,\ p=0)$ — System (q,p) (4)

$q = V_{1,1},\quad p = U_{1,1}V_{1,1}$

$$U'_{1,1} = \frac{U_{1,1}(-\alpha-n+2N+t+2) - U_{1,1}^2(n-2NV_{1,1}) + t}{t},$$
$$V'_{1,1} = \frac{NV_{1,1}\left(U_{1,1}(U_{1,1}+2)(n-NV_{1,1}) + \alpha+n-2N-2\right) + (N+1)(-\alpha+N+1)}{Nt(U_{1,1}+1)}$$

$$U''_{1,1} = \frac{\left(\frac{1}{U_{1,1}}+\frac{3}{2}\right)(U'_{1,1})^2}{U_{1,1}+1} - \frac{U'_{1,1}}{t} + \frac{U_{1,1}^2(-\alpha+nU_{1,1}+n)(\alpha+nU_{1,1}+n)}{2t^2(U_{1,1}+1)} + \frac{U_{1,1}^2(\alpha+n-2N-1)}{t(U_{1,1}+1)} + \frac{2\alpha+4U_{1,1}(\alpha+n-2N-1)+2n-4N-5t-2}{2t(U_{1,1}+1)} - \frac{(U_{1,1}+4)U_{1,1}}{2(U_{1,1}+1)} - \frac{1}{(U_{1,1}+1)U_{1,1}}.$$

$P_2 = \left(q = \frac{1+N}{N},\ p = -\frac{1+N}{N}\right)$ — System (q,p) (4)

$q = \frac{1+N}{N} + V_{2,1},\quad p = -\frac{1+N}{N} + U_{2,1}V_{2,1}$

$$U'_{2,1} = \frac{U_{2,1}^2(-n+2NV_{2,1}+N+1) - U_{2,1}(\alpha+n-t) + t}{t},$$
$$V'_{2,1} = \frac{NV_{2,1}\left(-U_{2,1}(U_{2,1}+2)(NV_{2,1}-n+N+1) + \alpha+n\right) + \alpha(N+1)}{Nt(U_{2,1}+1)}.$$

$$U''_{2,1} = \frac{\left(\frac{1}{U_{2,1}}+\frac{3}{2}\right)(U'_{2,1})^2}{U_{2,1}+1} - \frac{U'_{2,1}}{t} + \frac{U_{2,1}^2\left((n-\alpha)(\alpha+n-2N-2) + U_{2,1}(U_{2,1}+2)(-n+N+1)^2\right)}{2t^2(U_{2,1}+1)} + \frac{U_{2,1}^2(\alpha+n+1)}{t(U_{2,1}+1)} + \frac{2\alpha+4U_{2,1}(\alpha+n+1)+2n-5t+2}{2t(U_{2,1}+1)} - \frac{(U_{2,1}+4)U_{2,1}}{2(U_{2,1}+1)} - \frac{1}{(U_{2,1}+1)U_{2,1}}.$$

$P_3 = \left(q = \frac{1+N-\alpha}{N},\ p = -\frac{1+N-\alpha}{N}\right)$ — System (q,p) (4)

$q = \frac{1+N-\alpha}{N} + V_{3,1},\quad p = -\frac{1+N-\alpha}{N} + U_{3,1}V_{3,1}$

$$U'_{3,1} = \frac{U_{3,1}^2(-\alpha-n+2NV_{3,1}+N+1) + U_{3,1}(\alpha-n+t) + t}{t},$$
$$V'_{3,1} = \frac{\alpha(\alpha-N-1) - NV_{3,1}\left(\alpha + U_{3,1}(U_{3,1}+2)(-\alpha-n+NV_{3,1}+N+1) - n\right)}{Nt(U_{3,1}+1)}.$$

$$U''_{3,1} = \frac{\left(\frac{1}{U_{3,1}}+\frac{3}{2}\right)(U'_{3,1})^2}{U_{3,1}+1} - \frac{U'_{3,1}}{t} + \frac{U_{3,1}^2\left((\alpha+n)(\alpha+n-2N-2) + U_{3,1}(U_{3,1}+2)(\alpha+n-N-1)^2\right)}{2t^2(U_{3,1}+1)} + \frac{U_{3,1}^2(-\alpha+n+1)}{t(U_{3,1}+1)} + \frac{-2\alpha+4U_{3,1}(-\alpha+n+1)+2n-5t+2}{2t(U_{3,1}+1)} - \frac{(U_{3,1}+4)U_{3,1}}{2(U_{3,1}+1)} - \frac{1}{(U_{3,1}+1)U_{3,1}}.$$

ITERATED REGULARISATION

$P_4 = \left(q = \frac{n}{N},\ P = 0\right)$ — System (q,P) (4)

$q = \frac{n}{N} + V_{4,1},\quad P = U_{4,1}V_{4,1}$

$$U'_{4,1} = -\frac{R_1^{4,1}(U_{4,1}, V_{4,1}, t)^*}{Nt(U_{4,1}V_{4,1}(n+NV_{4,1})+N)},$$
$$V'_{4,1} = \frac{R_2^{4,1}(U_{4,1}, V_{4,1}, t)^*}{NtU_{4,1}(U_{4,1}V_{4,1}(n+NV_{4,1})+N)}.$$

System $U_{1,1}V_{1,1}$

System $U_{2,1}V_{2,1}$

System $U_{3,1}V_{3,1}$

$P_5 = (Q=0,\ P=0)$ — System (Q,P) (4)

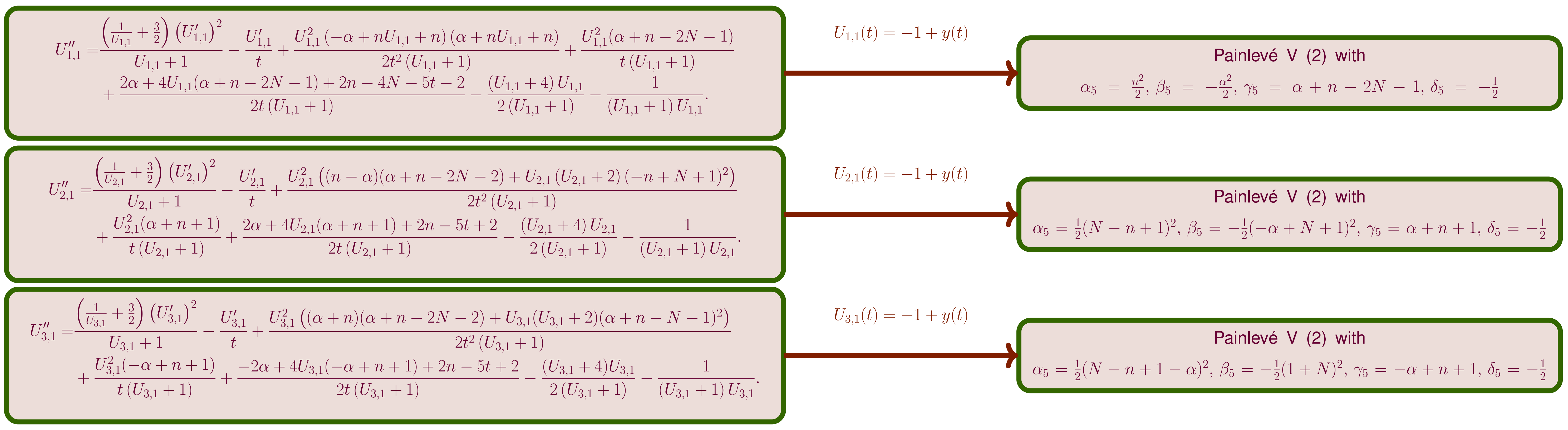
$Q = \frac{Nv_{5,4}^2}{v_{5,4}(Nu_{5,4}v_{5,4}+\alpha+n-2N-t-1)-t},\quad P = \frac{Nv_{5,4}}{v_{5,4}(Nu_{5,4}v_{5,4}+\alpha+n-2N-t-1)-t}$

$$u'_{5,4} = \frac{(N-n)(-\alpha-n+N) - Nu_{5,4}\left(-v_{5,4}(v_{5,4}+2)(n-Nu_{5,4}) + \alpha+n-2N\right)}{Nt(v_{5,4}+1)},$$
$$v'_{5,4} = \frac{-nv_{5,4}^2 + nv_{5,4} + 2Nu_{5,4}v_{5,4}^2 - 2Nv_{5,4} - tv_{5,4} + \alpha v_{5,4} - t}{t}.$$

Equation $U_{1,1}$

where $R_1^{4,1}(U_{4,1}, V_{4,1}, t)$, $R_2^{4,1}(U_{4,1}, V_{4,1}, t)$ are some known polynomials of their variables.

Painlevé connection



$$U''_{1,1} = \frac{\left(\frac{1}{U_{1,1}}+\frac{3}{2}\right)(U'_{1,1})^2}{U_{1,1}+1} - \frac{U'_{1,1}}{t} + \frac{U_{1,1}^2(-\alpha+nU_{1,1}+n)(\alpha+nU_{1,1}+n)}{2t^2(U_{1,1}+1)} + \frac{U_{1,1}^2(\alpha+n-2N-1)}{t(U_{1,1}+1)} + \frac{2\alpha+4U_{1,1}(\alpha+n-2N-1)+2n-4N-5t-2}{2t(U_{1,1}+1)} - \frac{(U_{1,1}+4)U_{1,1}}{2(U_{1,1}+1)} - \frac{1}{(U_{1,1}+1)U_{1,1}}.$$

$U_{1,1}(t) = -1 + y(t)$

Painlevé V (2) with

$\alpha_5 = \frac{n^2}{2},\ \beta_5 = -\frac{\alpha^2}{2},\ \gamma_5 = \alpha+n-2N-1,\ \delta_5 = -\frac{1}{2}$

$$U''_{2,1} = \frac{\left(\frac{1}{U_{2,1}}+\frac{3}{2}\right)(U'_{2,1})^2}{U_{2,1}+1} - \frac{U'_{2,1}}{t} + \frac{U_{2,1}^2\left((n-\alpha)(\alpha+n-2N-2) + U_{2,1}(U_{2,1}+2)(-n+N+1)^2\right)}{2t^2(U_{2,1}+1)} + \frac{U_{2,1}^2(\alpha+n+1)}{t(U_{2,1}+1)} + \frac{2\alpha+4U_{2,1}(\alpha+n+1)+2n-5t+2}{2t(U_{2,1}+1)} - \frac{(U_{2,1}+4)U_{2,1}}{2(U_{2,1}+1)} - \frac{1}{(U_{2,1}+1)U_{2,1}}.$$

$U_{2,1}(t) = -1 + y(t)$

Painlevé V (2) with

$\alpha_5 = \frac{1}{2}(N-n+1)^2,\ \beta_5 = -\frac{1}{2}(-\alpha+N+1)^2,\ \gamma_5 = \alpha+n+1,\ \delta_5 = -\frac{1}{2}$

$$U''_{3,1} = \frac{\left(\frac{1}{U_{3,1}}+\frac{3}{2}\right)(U'_{3,1})^2}{U_{3,1}+1} - \frac{U'_{3,1}}{t} + \frac{U_{3,1}^2\left((\alpha+n)(\alpha+n-2N-2) + U_{3,1}(U_{3,1}+2)(\alpha+n-N-1)^2\right)}{2t^2(U_{3,1}+1)} + \frac{U_{3,1}^2(-\alpha+n+1)}{t(U_{3,1}+1)} + \frac{-2\alpha+4U_{3,1}(-\alpha+n+1)+2n-5t+2}{2t(U_{3,1}+1)} - \frac{(U_{3,1}+4)U_{3,1}}{2(U_{3,1}+1)} - \frac{1}{(U_{3,1}+1)U_{3,1}}.$$

$U_{3,1}(t) = -1 + y(t)$

Painlevé V (2) with

$\alpha_5 = \frac{1}{2}(N-n+1-\alpha)^2,\ \beta_5 = -\frac{1}{2}(1+N)^2,\ \gamma_5 = -\alpha+n+1,\ \delta_5 = -\frac{1}{2}$

Portrait of Painlevé (1863–1933),

by Auguste Léon, 1918.

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