

POSITIVE BIDIAGONAL FACTORIZATIONS: COMBINATORIAL STRUCTURE, SPECTRAL THEORY, AND MATRIX CONTINUED FRACTIONS

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ABSTRACT. An ordered positive bidiagonal factorization (PBF) is more than a representation of a banded matrix: the combinatorics of its factor word controls a cyclic spectral geometry, two complementary sparse recurrences, matrix continued fractions, and integrable dynamics. Starting from

$$T = L_1 \cdots L_p U_q \cdots U_1,$$

with arbitrary positive integers p, q , we prove that every cyclic permutation of the prescribed factors again admits a normalized PBF. Under the stated one-sided maximal-degree hypotheses, consecutive cyclic products are connected by degree-one matrix Christoffel transformations with determinant cx , $c \neq 0$; a complete circuit gives multiplication by x , up to a constant triangular terminal gauge. Fixed row and column Christoffel multipliers generate a finite rectangle whenever its boundary chains are weakly normal, automatically so under total positivity of the scalar moment matrix.

Directed positive refactorization has an exact combinatorial classification. Lower Christoffel words determine the minimum-height factor patterns, while prefix dominance identifies the maximal interval of cyclic products that can reach them. This produces two complementary sparse matrices. The minimum-height matrix $\mathcal{H}_{(a)}$ has only two block displacements and $\gcd(p, q)$ states per height; the prescribed-order matrix $\mathcal{H}$ retains every original factor coefficient. Height grouping makes $\mathcal{H}$ block tridiagonal and yields a factor-resolved matrix continued fraction, with explicit Padé-type contact, denominator factorization, backward evaluation, complexity, and residual bounds. A second grouping gives a q -state first-return recursion. For bounded nonnegative factors the convergents are entrywise monotone, their denominators are nonsingular M -matrices, and they approach the least nonnegative solution. Under the bounded Favard identification, a rectangular resolvent projection is the mixed-type Weyl matrix, so all its Stieltjes transforms are approximated simultaneously.

The matrices $\mathcal{H}_{(a)}$ recover the cyclic resolvents and have finite nonzero spectra on a $(p + q)$ -ray star. A planar-network expansion proves coefficientwise nonnegativity of their radial Hankel minors and hence produces positive radial Stieltjes representations. Their explicit triangular moment factorizations give scalar weak normality in the coprime case and a block Gauss–Borel realization in general. For fixed compactly supported radial representatives, the Fourier components lift to finite complex measures on the compact star exactly under the corresponding atom-at-zero and negative fractional-moment conditions.

The same combinatorial construction selects an integrable system. For $M = p + q$, the factor-compatible arithmetic-support hierarchy is

$$\partial_t \mathcal{L} = [(\mathcal{L}^{M^\ell})_+, \mathcal{L}], \quad \mathcal{L} = \mathcal{S}^q + c\mathcal{S}^{-p}.$$

When p, q are coprime, $\mathcal{H}_{(a)}$ realizes this hierarchy; for $q = 1$, its first flow is the Narita–Itoh–Bogoyavlensky lattice. Radially deformed star moments give exact tau-determinant solutions. The sparse Lax matrix is a common integrable M -th root of the cyclic Toda family in the (q, p) -bigraded setting, while the Christoffel transfers are discrete Bäcklund directions in Sato-compatible gauges.

For the mixed Piñeiro system we obtain explicit positive factors, an exact factor-resolved fraction, and a rational $(3, 2)$ example with certified numerical convergence; its first real flow has global positive coefficientwise solutions expressed by Kummer determinants. For the Jacobi-like extension we derive beta-convolution measures and an explicit hypergeometric Weyl matrix; in the coprime intersection of the PBF and AT regions, the first real flow gives analogous global positive ${}_qF_q$ -determinant solutions. For unbounded factorizations the algebraic identities, formal approximation, and finite algorithm remain valid, while analytic finite-section convergence requires the stated closed-realization, core, stability, and spectral-identification hypotheses.

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1. INTRODUCTION

Continued fractions provide a fundamental link between orthogonal polynomials, spectral theory, and rational approximation. For a Jacobi matrix, the three-term recurrence produces a Stieltjes continued fraction for the transform of the orthogonality measure. Its convergents are Padé approximants, their denominators are the orthogonal polynomials, and their poles are the eigenvalues of finite Jacobi truncations. A continued fraction therefore converts local recurrence coefficients into information about the measure, the spectrum, polynomial zeros, and rational approximation.

The step-line recurrence in ordinary multiple orthogonality is lower Hessenberg rather than tridiagonal. Its p -branched continued fractions encode the additional subdiagonals and simultaneously approximate a vector of Stieltjes transforms. The production/output formalism of Deutsch–Ferrari–Rinaldi gives a path interpretation of lower-Hessenberg recurrences [21]; its relation with multiple orthogonality and branched Jacobi fractions is developed in [43, 46]. When the recurrence has p lower bidiagonal factors and one upper factor, resolving those factors replaces direct p -Łukasiewicz paths by weighted p -Dyck paths and yields the p -branched Stieltjes fraction [32, 9, 23, 50, 5].

A bidiagonal factorization, however, carries more than continued-fraction data. Its ordered factors define a chain of Darboux–Christoffel transformations, while their positivity controls moment and spectral properties that are not visible from the entries of their product alone. The spectral and approximation-theoretic roles of the factors are developed here simultaneously.

Mixed-type multiple orthogonality has several lower and upper recurrence diagonals, and its Weyl object is a rectangular matrix. This leads to the three questions addressed here. Consider a semi-infinite recurrence matrix with a prescribed bidiagonal factorization

$$T = L_1 \cdots L_p U_q \cdots U_1,$$

where the L_a are lower bidiagonal and the U_b are upper bidiagonal. When their nontrivial entries are positive, this is called a positive bidiagonal factorization, or PBF. The questions are: Does positive bidiagonal factorization survive when the factors are moved cyclically? Can the factor data be reorganized into a sparse recurrence carrying its own moment and spectral structure? And can the prescribed factor order be converted into a continued fraction for the resolvent that retains the individual factor coefficients, rather than only their sums of products in T ? All three questions have affirmative answers for arbitrary p and q . The main results are grouped accordingly.

The decisive organizing mechanism is combinatorial. Words in the two factor types are interpreted as rational lattice paths. Christoffel words identify the positive refactorizations of minimum height span, while prefix dominance determines which cyclic products can reach them [8]. A second path construction, based on planar networks and nonintersecting paths, proves the Hankel positivity underlying the radial Stieltjes measures [35, 25]. Thus combinatorics first constructs the sparse recurrences and then supplies their moment positivity.

- (1) *Cyclic permutation preserves the positive factorization and transports the spectral data.* A local refactorization of an upper–lower pair shows that every cyclic permutation of the prescribed bidiagonal factors again admits a PBF (Lemma 2.2 and Theorem 2.3). Thus one PBF generates a complete cyclic family of positive banded matrices. This is an algebraic result and does not require boundedness. Under the corresponding bounded or unbounded Favard hypotheses, every member of this family has positive mixed-type spectral data [10, 11, 14].

When suitable one-sided maximal-degree data are fixed at two consecutive members of the cyclic family, moving one bidiagonal factor induces a matrix Christoffel transformation of degree one. Its determinant is a nonzero multiple of x , and one complete circuit through all factors multiplies the matrix of measures by x , up to a constant triangular change of initial conditions (Theorem 6.18 and Corollary 6.20). Fixing instead the standard row and column Christoffel transformations produces a finite two-sided family of transformed moment systems. Its existence requires weak normality of the two boundary chains; this condition is automatic when the scalar step-line moment matrix is totally positive (Theorem 6.29).

- (2) *Selected cyclic products yield sparse recurrences with spectral data on a star.* For an explicitly determined interval of cyclic products, a finite sequence of exchanges reaches a distinguished positive pattern of minimum

height span, generally before the terminal lower–upper PBF is reached (Theorem 2.8). This factorization is retained because it produces, directly from its bidiagonal coefficients, a sparse semi-infinite matrix $\mathcal{H}_{(a)}$ with only two nonzero block diagonals. If $d = \gcd(p, q)$, each height contains d coordinates and the two block displacements are

$$+\frac{q}{d} \quad \text{and} \quad -\frac{p}{d}.$$

When p and q are coprime, $\mathcal{H}_{(a)}$ is scalar and its only nonzero entries are

$$(\mathcal{H}_{(a)})_{h,h+q} = 1, \quad (\mathcal{H}_{(a)})_{h,h-p} = c_h^{(a)} > 0.$$

The matrices $\mathcal{H}_{(a)}$ are not merely auxiliary linearizations. They are recurrence matrices of explicitly constructed moment-functional systems; their radial moments are Stieltjes moment sequences, and the nonzero spectra of their finite truncations lie on a $(p + q)$ -ray star. For fixed compactly supported radial representing measures, a separate entrywise criterion determines when the corresponding Fourier components admit finite complex representing measures supported on the corresponding compact star. Furthermore, a projected resolvent of $\mathcal{H}_{(a)}$ recovers the resolvent of the corresponding cyclic product $T_{(a)}$ (Corollary 3.11). The sparse recurrence therefore retains the spectral information of $T_{(a)}$ while exposing an additional star structure. When p and q are coprime, it also defines an arithmetic-support difference operator. Section 9 proves that this operator carries an infinite hierarchy of local commuting Lax flows. The star moment matrix reconstructs the evolving coefficients by a four-minor τ -quotient, so the PBF supplies an exact solution rather than only initial data. The $(p + q)$ -th power evolves all cyclic products of the refactorized factors by synchronized Toda equations, and the Christoffel transfers commute with these flows as discrete Bäcklund transformations. On the real first-time axis, the Piñeiro case gives global positive coefficientwise solutions expressed by determinants of Kummer functions; the intersection of the Jacobi-like positive-factorization chamber with the requisite AT range gives analogous global positive coefficientwise hypergeometric determinant solutions.

- (3) *The original bidiagonal factorization yields a matrix continued fraction for the Weyl matrix.* The construction in the preceding item replaces the original bidiagonal factors by positively refactorized ones. A second construction starts directly from $T = L_1 \cdots L_p U_q \cdots U_1$ and leaves these factors unchanged. It resolves one application of T into its $p + q$ successive bidiagonal steps, so that every coefficient of every factor remains visible. Introducing one intermediate coordinate after each factor, ordering these coordinates by an integer height, and grouping consecutive heights turns the resulting sparse matrix into a block-tridiagonal matrix. Its Schur complements give the matrix continued fraction of Theorem 3.20. A different grouping of the same coordinates gives the q -state recursion of Theorem 4.2; when $q = 1$, this recursion reduces to the classical p -branched Stieltjes fraction.

Finite height-block truncations give rational approximants for which the Padé-type order of contact, denominator factorization, backward evaluation algorithm, and a posteriori error bound are obtained explicitly. For bounded nonnegative factors, these convergents are entrywise monotone, their Schur-complement denominators are nonsingular M -matrices, and their limit is the least nonnegative solution of the continued-fraction equation. The coefficientwise approximation and the finite algorithm remain valid for unbounded factorizations. Operator-resolvent convergence then requires a specified closed realization, the stated core hypothesis, and uniform finite-section stability.

In the bounded setting, under the moment identification, a rectangular projection of the resolvent of this sparse matrix recovers the Weyl matrix of the mixed-type system. The finite continued fractions therefore approximate simultaneously all the Stieltjes transforms in the underlying matrix of measures. In the unbounded setting, convergence to the projected resolvent and identification of that resolvent with the Cauchy transform of a chosen Favard measure are separate assertions. The two associated block output matrices recover the coefficient arrays of both mixed-type polynomial families (Theorems 5.1 and 5.7).

We next place these results in the context of the state of the art. The relevant literature concerns sparse recurrences and spectral problems on star-like sets, vector and matrix continued fractions for banded operators, matrix-analytic methods for block recursions, and mixed-type multiple orthogonality. We review these four directions in turn and indicate precisely where the present constructions differ from their closest precedents.

Two-diagonal lower-Hessenberg matrices, corresponding here to $q = 1$, occur in multiple orthogonality on star-like sets [20, 33]; rotational spectral patterns for finite matrices with two separated diagonals were studied in [40]. The family $\mathcal{H}_{(a)}$ constructed above is their two-sided mixed-type extension. For a Christoffel factorization, the general factorization theorem of [12] identifies its cyclic products with successive Christoffel shifts. The present positive refactorization theorem proves, for bounded and unbounded factors alike, that the complete cyclic orbit remains in the PBF class. The spectral consequences use two distinct parts of the mixed-type theory. The Favard theorem, established first for bounded matrices and subsequently extended to unbounded matrices under a coherence hypothesis for the finite PBFs, supplies positive measures [10, 14]. In the bounded case these measures are compactly supported. In the genuinely unbounded case, the coherence hypothesis for the shifted finite PBFs in the general theorem is automatic here: every leading principal submatrix inherits the leading submatrices of the same semi-infinite factors, with zero shift. Hence the boundary matrices occurring in the coherence condition are independent of the truncation size once their fixed leading blocks are present. Separately, the BTP criterion gives strong normality when ν^{-T} is upper ΔTP and ξ^{-1} is lower ΔTP [11]. The cited paper uses the word *normality* for this stronger maximal-degree property; it is called *strong normality* here. For general bidiagonal factorizations the cyclic resolvent identities remain valid. Positivity is the additional hypothesis used to obtain positive spectral data.

Vector and matrix continued fractions for banded operators have a substantial literature. Relevant precedents include the vector continued fractions and Favard theory of Kaliaguine [28], the genetic-sum representation of Aptekarev–Kaliaguine–Van Iseghem [4], the matrix continued fractions of Sorokin–Van Iseghem [48], matrix Schur algorithms [17], and noncommutative convergence questions [45]. López-García and Prokhorov related the Jacobi–Perron algorithm for Hessenberg resolvents to weighted Łukasiewicz paths and also treated a partial p -Dyck subfamily [36].

The closest operator-theoretic precedent is [30], where $q \times p$ generating matrices for two-sided paths and resolvent blocks of banded difference operators are represented by matrix continued fractions. There the path weights are the entries of the banded operator and the rational approximants use its principal truncations. Here each recurrence step is first resolved into the prescribed bidiagonal steps, and the convergents truncate the height-ordered cyclic matrix rather than T . Consequently the present fraction keeps the factor coefficients, the cyclic Darboux products, and factor positivity visible. The two approaches share a Schur-complement mechanism but start from different data and retain different structure.

After the height grouping, the resulting block-tridiagonal matrices also resemble the transition matrices of quasi-birth-and-death processes and the matrices treated by matrix-analytic methods [41, 31]. In that theory, the block transition probabilities are given data and the central minimal nonnegative object is the G -matrix of the Markov process. Here the blocks are constructed from a prescribed bidiagonal factorization, retain its individual coefficients, and the least nonnegative solution is a factor-resolved tail resolvent. Thus the parallel extends to the matrix recursions and their minimal nonnegative solutions, whereas the input data and the interpreted object are different.

The mixed-type framework has antecedents in the matrix-functional setting of Sorokin–Van Iseghem [47]. Daems and Kuijlaars later introduced the terminology of mixed-type multiple orthogonality and related its Christoffel–Darboux kernel to nonintersecting Brownian motions [18]. The Gauss–Borel formulation of the two mixed-type polynomial families and their multi-component 2D Toda dynamics was developed in [3]. Under the Favard hypotheses used below, a rectangular projection of $(I - tT)^{-1}$ is generated by a $q \times p$ matrix of measures. The continued fraction therefore gives simultaneous rational approximants to all entries of its Weyl matrix. Two normalized block output matrices recover the coefficient arrays of the right and left mixed-type polynomial systems from powers of T and T^T . The cyclic matrix records the intermediate factor positions and serves a different purpose.

The Piñeiro and Jacobi-like systems illustrate what the general construction adds once a PBF is available. Their factor and special-function formulas are imported as inputs from [13, 38, 37]; the Piñeiro factors and their positivity chamber also have the self-contained alternative derivation in [38, Section 7 and Appendix A]. The general construction is independent of those formulas. The Piñeiro section contains one complete rational example, including its matrix measure, banded product, bidiagonal factors, finite fraction, stochastic normalization, and Weyl

approximation. The Jacobi-like section gives the general beta-convolution measure, its hypergeometric Weyl matrix, and their rotational lift to the star. In that family the upper Christoffel transforms can leave the initial special-function family, although cyclic closure preserves their PBFs.

The paper is organized as follows. Section 2 proves positive adjacent refactorization, shows that every cyclic product has a terminal lower–upper PBF, and identifies those cyclic products for which a finite initial sequence of the same exchanges reaches a minimum-height factorization. It defines directly the sparse matrix for the prescribed order and the family $\mathcal{H}_{(a)}$ obtained from these minimum-height factorizations. It then introduces the auxiliary cyclic matrix and records its Darboux resolvent blocks. Section 3 proves the two direct placement rules, decomposes the polynomial solutions of $\mathcal{H}_{(a)}$ into cyclic-product recurrences, and derives the block continued fraction together with its approximation and convergence properties. Section 4 gives the branched and q -state recursions for the original two-run order. Section 5 identifies the rectangular Weyl matrix inside the cyclic resolvent and develops its finite rational approximation in the bounded and unbounded settings. Section 6 constructs the full star moment systems associated with $\mathcal{H}_{(a)}$, proves coefficientwise nonnegativity of the radial Hankel minors and positive Stieltjes representations, establishes scalar weak normality in the coprime case and a uniform block Gauss–Borel realization in general, proves finite ray localization, and identifies the finite rectangle of simultaneous row and column moment shifts. The condensed Piñeiro and Jacobi-like applications occupy Sections 7 and 8. Section 9 identifies the integrable hierarchies selected by the coprime sparse recurrences, reconstructs their solutions from the star moments, proves compatibility with the discrete Christoffel transformations, and gives explicit hypergeometric determinant solutions for the Piñeiro and Jacobi-like specializations.

2. POSITIVE REFACTORIZATION AND THE TWO SPARSE MATRICES

This section constructs the two sparse matrices used in the rest of the article. The first retains the prescribed factors and all their coefficients. The second, denoted by $\mathcal{H}_{(a)}$, is obtained from a minimum-height factorization of each selected cyclic product $T_{(a)}$, $a \in \mathcal{A}_{\text{cyc}}$. The distinction between this factorization and the terminal lower–upper PBF of the same product is made explicit in Subsection 2.2.

The section is organized as follows. Subsection 2.1 proves that a positive upper–lower pair can be interchanged while preserving normalized bidiagonal form and positivity. Subsection 2.2 first identifies the selected cyclic products and then uses these exchanges to construct their minimum-height factorizations. Subsection 2.3 then defines the two sparse matrices directly. Only after these constructions are complete does Subsection 2.4 introduce the cyclic matrix. That matrix will be used to prove the identities connecting the sparse matrices with T , the cyclic products in the displayed range, and their resolvents. The two sparse matrices themselves are assembled directly from their bidiagonal coefficients.

Before beginning the construction, we fix two terminological conventions used throughout the article. Our terminology for total positivity follows Fallat and Johnson [22]. A real matrix is *totally nonnegative* (TN) if all its minors are nonnegative, and *totally positive* (TP) if all its minors are positive. For classical background, see also [29, 44]. Pinkus uses TP and STP for the classes called TN and TP here, respectively; statements cited from that monograph are expressed in our TN–TP terminology. A nontrivial triangular matrix cannot be TP in this sense. Following Fallat and Johnson, we write ΔTP for a triangular TN matrix whose minors are positive whenever they are not forced to vanish solely by triangularity. Throughout the article, *triangular totally positive* means ΔTP .

Definition 2.1 (Weak and strong normality along a step-line). A position of a prescribed mixed-type step-line is called *weakly normal* if each of its right and left mixed-type orthogonality problems has a one-dimensional solution space; equivalently, each polynomial vector exists and is unique up to multiplication by a nonzero constant. It is called *strongly normal* if, in addition, every component whose prescribed polynomial space is nontrivial attains its maximal allowed degree. A step-line is weakly, respectively strongly, normal through order N when this property holds at every one of its first N positions.

2.1. Positive adjacent refactorization. Let $p, q \in \mathbb{N}$, and put

$$M := p + q.$$

The pair (p, q) remains fixed throughout the general construction. Once defined, $\mathcal{C}_{p,q}$ and $\mathcal{H}_{p,q}$ are abbreviated to $\mathcal{C}$ and $\mathcal{H}$; their subscripts are restored for special values of (p, q) and for the explicit families. All matrices below act algebraically on $\mathcal{E} := \mathbb{C}^{\mathbb{N}_0}$, the vector space of all complex sequences $x = (x_n)_{n \geq 0}$, without a support condition. For each $n \in \mathbb{N}_0$, let e_n denote the sequence having a 1 in coordinate n and 0 in every other coordinate. Write $\mathbf{1} = (1, 1, \dots) \in \mathcal{E}$ for the constant sequence of ones and, for $r \in \mathbb{N}$, $\mathbf{1}_r = (1, \dots, 1) \in \mathbb{C}^r$. Every row of a bidiagonal factor has at most two nonzero entries, and every finite product of the factors is banded. Hence each coordinate of the image of any $x \in \mathcal{E}$ is a finite sum, so no convergence condition is needed for this algebraic action. Formal resolvents are understood coefficient by coefficient. In the analytic statements, the matrices are restricted to the indicated ℓ^s spaces and are explicitly assumed to be bounded there. For $1 \leq s \leq \infty$, the notation $\|\cdot\|_s$ denotes the ℓ^s -norm for vectors and the corresponding subordinate norm for matrices and operators; the relevant coordinate spaces are determined by the domain and codomain displayed in each statement. For a semi-infinite matrix $X = [X_{i,j}]_{i,j \in \mathbb{N}_0}$ and $N \in \mathbb{N}_0$, write

$$X^{[N]} := [X_{i,j}]_{0 \leq i,j \leq N}$$

for its leading principal truncation, which has order $N + 1$. Let $L_1, \dots, L_p$ be lower bidiagonal matrices and $U_1, \dots, U_q$ upper bidiagonal matrices acting on $\mathcal{E}$, with entries

$$\begin{aligned} (L_a)_{n,n} &= 1, & (L_a)_{n+1,n} &= \ell_n^{(a)}, & a &\in \{1, \dots, p\}, & n &\in \mathbb{N}_0, \\ (U_b)_{n,n} &= u_n^{(b)}, & (U_b)_{n,n+1} &= 1, & b &\in \{1, \dots, q\}, & n &\in \mathbb{N}_0. \end{aligned}$$

All other entries vanish. Put $A := L_1 \cdots L_p$ and $B := U_q \cdots U_1$. Consider the banded matrix

$$(2.1) \quad T := AB = L_1 \cdots L_p U_q \cdots U_1.$$

Enumerate the bidiagonal factors in their order of appearance in this product by

$$F_a := L_a, \quad 1 \leq a \leq p, \quad F_{p+b} := U_{q-b+1}, \quad 1 \leq b \leq q.$$

Then $T = F_1 F_2 \cdots F_M$. For $a \in \{1, \dots, M\}$, define the cyclic product starting with F_a by

$$T_{(a)} := F_a F_{a+1} \cdots F_M F_1 \cdots F_{a-1}.$$

Thus $T_{(1)} = T$. The product $T_{(a)}$ is obtained by moving $F_1, \dots, F_{a-1}$ from the beginning of the product to the end, without changing their relative order. Such a change of the initial factor is called a cyclic permutation of the factors. The matrix T has at most p strict subdiagonals and q strict superdiagonals. The displayed factorization is called nonnegative when every $\ell_n^{(a)}$ and $u_n^{(b)}$ is nonnegative, and positive when all of them are strictly positive. The fixed diagonal entries of the lower factors and the fixed superdiagonal entries of the upper factors are the normalizing entries of the factorization.

The entries of T combine contributions from all $p + q$ factors. The two sparse matrices introduced below instead record one factor at a time and retain the factor from which each coefficient comes.

Finite bidiagonal factorizations are a basic tool in the classical theory of totally nonnegative matrices; see, among others, [24, 44, 22]; the related finite qd and LR transformations are reviewed in [27]. Those results concern factorization and parametrization of a fixed finite totally nonnegative matrix. The result below instead gives an explicit normalized semi-infinite refactorization $UL = \tilde{L}\tilde{U}$ that preserves strict positivity coefficient by coefficient. Its iteration will imply that every cyclic permutation of an arbitrary prescribed PBF again admits a PBF.

Lemma 2.2 (Positive adjacent refactorization). *Let U and L be normalized upper and lower bidiagonal matrices, respectively, with*

$$U_{n,n} = u_n > 0, \quad U_{n,n+1} = 1, \quad L_{n,n} = 1, \quad L_{n+1,n} = \ell_n > 0.$$

Then there is a unique normalized positive pair $\tilde{L}, \tilde{U}$ such that

$$UL = \tilde{L}\tilde{U}.$$

Its nontrivial entries are determined recursively by

$$(2.2) \quad \tilde{u}_0 = u_0 + \ell_0, \quad \tilde{\ell}_n = \frac{u_{n+1}\ell_n}{\tilde{u}_n}, \quad \tilde{u}_{n+1} = u_{n+1} + \ell_{n+1} - \tilde{\ell}_n, \quad n \in \mathbb{N}_0.$$

More precisely,

$$(2.3) \quad \tilde{u}_n > \ell_n, \quad 0 < \tilde{\ell}_n < u_{n+1}, \quad n \in \mathbb{N}_0.$$

If the two original coefficient sequences are bounded, then so are the refactorized sequences.

Proof. The product UL has first superdiagonal equal to 1, diagonal entries $u_n + \ell_n$, and first subdiagonal entries $u_{n+1}\ell_n$. On the other hand, $\tilde{L}\tilde{U}$ has first superdiagonal equal to 1, diagonal entries $\tilde{u}_n + \tilde{\ell}_{n-1}$, with $\tilde{\ell}_{-1} := 0$, and first subdiagonal entries $\tilde{\ell}_n\tilde{u}_n$. Equality of the zeroth diagonal entry, followed successively by equality of the n -th subdiagonal and the $(n+1)$ -st diagonal entries, gives exactly (2.2). This also proves uniqueness.

It remains to prove that no denominator vanishes and that every new entry is positive. The initial value satisfies $\tilde{u}_0 = u_0 + \ell_0 > \ell_0$. If $\tilde{u}_n > \ell_n$, then

$$0 < \tilde{\ell}_n = \frac{u_{n+1}\ell_n}{\tilde{u}_n} < u_{n+1},$$

and consequently

$$\tilde{u}_{n+1} = u_{n+1} + \ell_{n+1} - \tilde{\ell}_n > \ell_{n+1}.$$

Induction proves (2.3) and the strict positivity of the two refactorized factors.

Finally, if $u_n, \ell_n \leq \Gamma$ for all n , then $\tilde{\ell}_n < u_{n+1} \leq \Gamma$, while $\tilde{u}_0 \leq 2\Gamma$ and $\tilde{u}_{n+1} < u_{n+1} + \ell_{n+1} \leq 2\Gamma$. Thus both new coefficient sequences are bounded. $\square$

The preceding local exchange can be iterated to restore the lower–upper order after any cyclic permutation of the factors.

Theorem 2.3 (Cyclic closure of positive bidiagonal factorizations). *Suppose that the factorization (2.1) is positive. Every cyclic product $T_{(a)}$, with $a \in \{1, \dots, M\}$, admits a normalized positive bidiagonal factorization with p lower factors followed by q upper factors. The resulting lower–upper factors are independent of the sequence of adjacent refactorizations. If the coefficients of the original factors are bounded, the coefficients in every one of these cyclic factorizations are bounded.*

Proof. Fix a cyclic product and read its factors from left to right. Whenever an upper factor occurs immediately before a lower factor, apply Lemma 2.2 to replace that pair by a normalized positive lower–upper pair with the same product. Each interchange decreases by one the number of upper–lower inversions in the factor order. After finitely many interchanges no such inversion remains, so all p lower factors precede all q upper factors. Every intermediate product equals the original cyclic product and every factor remains positive.

We prove independence of the exchange sequence by induction on the number of upper–lower inversions. There is nothing to prove when this number is zero. If two exchange sequences begin with the same exchange, the induction hypothesis applies after that common first step. If they begin with different exchanges, the two occurrences of UL are disjoint, because two such occurrences cannot overlap. The corresponding refactorizations act on disjoint factor pairs and therefore commute. Applying both exchanges in either order gives the same factor list, now with fewer inversions, so the induction hypothesis shows that the two sequences have the same terminal factors.

Boundedness is preserved at each interchange by the last assertion of Lemma 2.2; only finitely many interchanges are required. $\square$

Remark 2.4 (Boundary correction for separately truncated factors). The refactorization identity concerns semi-infinite matrices. If the factors are truncated separately before multiplication, the last diagonal entry acquires a boundary correction. Indeed, if P_N denotes the coordinate projection onto the indices $0, \dots, N$, then in general $P_N U L P_N \neq (P_N U P_N)(P_N L P_N)$, because the product on the right omits the contribution $U_{N,N+1} L_{N+1,N} = \ell_N$ to its last diagonal entry. When all lower factors precede all upper factors, every leading principal submatrix does factor as the product of the corresponding leading submatrices, as proved in Lemma 6.14.

2.2. Selected cyclic products and their minimum-height factorizations. Each matrix $T_{(a)}$ is obtained by choosing a different first factor in the prescribed product and reading the factors cyclically from that point. Starting from this cyclic order, each adjacent exchange $UL = \widetilde{L}\widetilde{U}$ preserves the product and positivity but replaces the two factors involved. Continuing these exchanges until no upper factor precedes a lower factor gives the terminal lower–upper PBF of $T_{(a)}$ for every a , by Theorem 2.3.

For the indices selected below, a finite initial sequence of exchanges reaches a pattern of minimum height span. We call a factorization with such a factor-type pattern a *minimum-height factorization*. We retain the factorization obtained at that stage. Its factors are generally interlaced; they coincide with the terminal PBF only when this pattern is already lower–upper. Theorem 2.8 proves that the resulting factors are independent of the exchange sequence. They still have product $T_{(a)}$, but now construct the two-block-diagonal matrix $\mathcal{H}_{(a)}$, whose projected resolvent recovers that of $T_{(a)}$ and whose powers define the star moment system of Section 6. We now determine the admissible indices.

Put

$$d := \gcd(p, q), \quad P := \frac{p}{d}, \quad Q := \frac{q}{d}, \quad R := P + Q = \frac{M}{d}.$$

Definition 2.5 (Factor-position heights). Let $\sigma = (\sigma_1, \dots, \sigma_M)$ be an ordered sequence containing p symbols L and q symbols U . Set

$$\eta_0 := 0, \quad \eta_k := \eta_{k-1} + \begin{cases} Q, & \sigma_k = L, \\ -P, & \sigma_k = U, \end{cases} \quad k \in \{1, \dots, M\}.$$

The integers $\eta_0, \dots, \eta_M$ are the *factor-position heights* of σ . Since $pQ = qP$, one has $\eta_M = \eta_0 = 0$. The *height span* of σ is

$$\max_{0 \leq k \leq M} \eta_k - \min_{0 \leq k \leq M} \eta_k.$$

Definition 2.6 (Selected cyclic products). Define

$$c_- := \left\lceil \frac{P}{Q} \right\rceil, \quad c_+ := \left\lceil \frac{Q}{P} \right\rceil, \\ a_- := p - c_- + 1, \quad a_+ := p + 1 + \min\{q - 1, c_+\}, \quad \mathcal{A}_{\text{cyc}} := \{a_-, a_- + 1, \dots, a_+\}.$$

The matrices $T_{(a)}$ with $a \in \mathcal{A}_{\text{cyc}}$ are called the selected cyclic products.

Lemma 2.7 (Factor pattern of minimum height span). For $k \in \{0, \dots, M\}$, let r_k be the remainder of kQ upon division by R , and define $\sigma_{\min} = (\sigma_{\min,1}, \dots, \sigma_{\min,M})$ by

$$\sigma_{\min,k+1} = \begin{cases} L, & r_k < P, \\ U, & r_k \geq P, \end{cases} \quad k \in \{0, \dots, M-1\}.$$

The sequence $\sigma_{\min}$ contains p symbols L and q symbols U . Its factor-position heights satisfy $\eta_k = r_k$, and its height span is $R - 1$, the minimum possible. Moreover, it begins with c_- symbols L and ends with c_+ symbols U .

There are exactly R distinct factor-type sequences with height span $R - 1$, namely the distinct cyclic rotations of $\sigma_{\min}$. Among them, $\sigma_{\min}$ is the unique sequence whose heights all belong to $\{0, \dots, R - 1\}$. If $N_L(\rho; k)$ denotes the number of letters L among the first k positions of a minimizing sequence ρ , then

$$N_L(\sigma_{\min}; k) \geq N_L(\rho; k), \quad k \in \{0, \dots, M\}.$$

The first R symbols of $\sigma_{\min}$ form the lower Christoffel word with P letters L and Q letters U , with L horizontal and U vertical; hence $\sigma_{\min}$ is the d -th power of that primitive word [8].

Proof. Let $\eta_0, \dots, \eta_M$ be the factor-position heights of $\sigma_{\min}$. We prove by induction that $\eta_k = r_k$. This is clear for $k = 0$. If $r_k < P$, then position $k + 1$ is lower and $\eta_{k+1} = r_k + Q < R$. If $r_k \geq P$, then that position is upper and $\eta_{k+1} = r_k - P = r_k + Q - R$. In either case $\eta_{k+1} = r_{k+1}$.

Since $\gcd(Q, R) = 1$, the residues $r_0, \dots, r_{R-1}$ are a permutation of $0, \dots, R-1$. Hence one block of R positions contains exactly P lower and Q upper positions. Repetition over the d residue cycles gives exactly $p = dP$ lower and $q = dQ$ upper positions. The sequence begins with $c_- = \lceil P/Q \rceil$ lower positions. Reading the residue cycle backwards from zero shows that it ends with $c_+ = \lceil Q/P \rceil$ upper positions.

The factor-position heights are the residues r_k , so their span is $R-1$. To prove minimality, observe that the two increments $+Q$ and $-P$ are congruent to Q modulo $R = P + Q$. Therefore the first R factor-position heights of any sequence represent all residue classes modulo R . Any R integers with distinct residues modulo R have span at least $R-1$.

Let ρ be any sequence whose height span is $R-1$, and denote its heights by θ_k . The integers $\theta_0, \dots, \theta_{R-1}$ have all the residues modulo R and lie in an interval of length $R-1$. Since $\theta_0 = 0$, that interval is

$$\{-j, -j+1, \dots, R-1-j\}$$

for a unique $j \in \{0, \dots, R-1\}$. For every k , the height θ_k is therefore the unique representative of r_k in this interval. Thus the increments, and hence the letters of ρ , are uniquely determined by j . Choosing the unique position s with $r_s = j$ shows that ρ is the cyclic rotation of $\sigma_{\min}$ beginning at position $s+1$. Conversely, every cyclic rotation has the same height span. The primitive block has length R , because P and Q are coprime; hence there are exactly R distinct rotations. The value $j = 0$ gives $\sigma_{\min}$, and it is the unique rotation whose heights are all nonnegative.

Finally, both possible height increments are congruent to Q modulo R , so $\theta_k \equiv r_k \pmod{R}$. The representative of r_k in the interval above is either r_k or $r_k - R$; consequently $\theta_k \leq r_k$. For any factor-type sequence,

$$N_L(\rho; k) = \frac{\theta_k + Pk}{R}.$$

Since the corresponding height of $\sigma_{\min}$ is r_k , the asserted prefix dominance follows. The residue construction of the first block is precisely the lower Christoffel construction, and the periodicity of r_k gives its d -fold repetition. $\square$

Figure 1 shows the cyclic factor-type orders of the selected products and the common minimum-height pattern reached after a suitable finite sequence of adjacent refactorizations. We retain the factorization obtained at this stage in order to construct $\mathcal{H}_{(a)}$. Further adjacent refactorizations are possible and eventually produce the terminal lower-upper PBF.

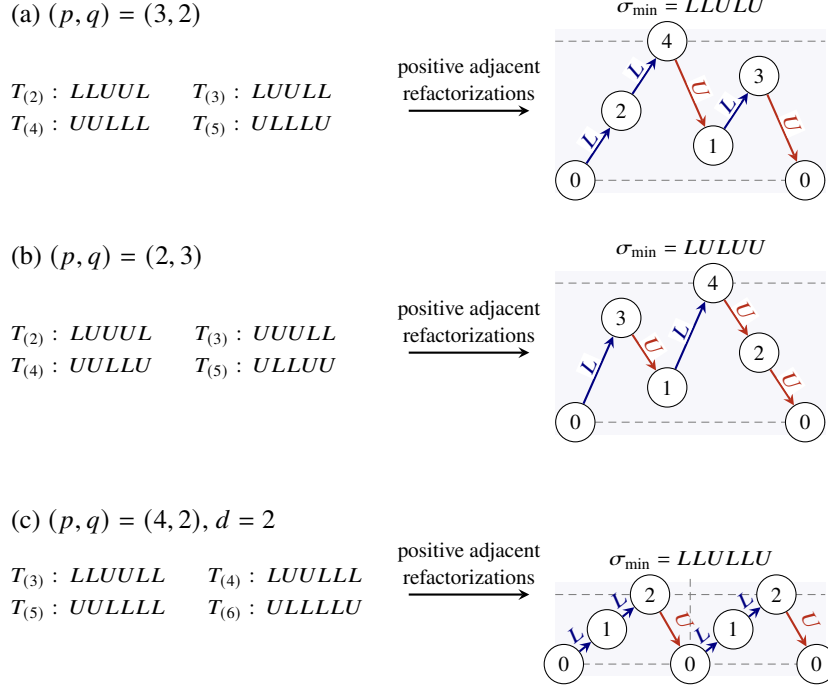


FIGURE 1. Selected cyclic products and their minimum-height refactorizations. The words on the left are the factor-type orders in the products $T_{(a)}$; the path on the right belongs to the resulting minimum-height factorization. Blue L -steps have increment $+Q$, and red U -steps have increment $-P$. Panel (c) displays the repeated residue cycle when $d > 1$.

Theorem 2.8 (Positive refactorization into the minimum-height pattern). *For $a \in \{1, \dots, M\}$, the following conditions are equivalent:*

- (i) $a \in \mathcal{A}_{\text{cyc}}$;
- (ii) *the cyclically ordered factors of $T_{(a)}$ can be transformed, by finitely many directed adjacent refactorizations $UL = \widetilde{L}\widetilde{U}$, into normalized positive bidiagonal factors whose lower–upper type sequence is $\sigma_{\min}$.*

When these conditions hold, the resulting factors $\widetilde{F}_1^{(a)}, \dots, \widetilde{F}_M^{(a)}$ are independent of the sequence of adjacent refactorizations and satisfy

$$(2.4) \quad T_{(a)} = \widetilde{F}_1^{(a)} \cdots \widetilde{F}_M^{(a)}.$$

If the original factor coefficients are bounded, then the coefficients of the resulting factors are bounded. Moreover, this choice is maximal among the minimum-height patterns: if the factor-type sequence of a cyclic product can be transformed by directed exchanges into any sequence of height span $R - 1$, then it can be transformed into $\sigma_{\min}$.

Proof. By Lemma 2.7, $\sigma_{\min}$ begins with c_- lower positions and ends with c_+ upper positions. We determine which cyclic factor orders can be transformed into this pattern by directed exchanges.

We first record the reachability criterion used below. For a factor-type sequence ρ , let $N_L(\rho; k)$ be the number of lower positions in its first k entries. If $\widehat{\rho}$ is obtained from ρ by one directed exchange $UL \mapsto LU$, then $N_L(\widehat{\rho}; k) \geq N_L(\rho; k)$ for every k . Hence a necessary condition for transforming ρ into a target sequence τ is

$$N_L(\tau; k) \geq N_L(\rho; k), \quad k \in \{0, \dots, M\}.$$

It is also sufficient. Indeed, let $i_1 < \dots < i_p$ and $j_1 < \dots < j_p$ be the positions of the lower factors in ρ and τ , respectively. The prefix inequalities are equivalent to $j_m \leq i_m$ for every m . Proceed by induction on m . Before stage m , the first $m - 1$ lower factors occupy the positions $j_1, \dots, j_{m-1}$, while the m -th lower factor still occupies position

i_m : the preceding moves cross only upper factors lying to its left and do not change the positions of the later lower factors. Since all earlier target positions are strictly smaller than j_m , every position from j_m through $i_m - 1$ is upper. Successive directed exchanges $UL \mapsto LU$ therefore move the m -th lower factor from i_m to j_m without disturbing the factors already placed. After the p stages the target sequence is reached.

Consider first $a \in \{1, \dots, p\}$. The factor types in $T_{(a)}$ occur as

$$L^{p-a+1}U^qL^{a-1}.$$

Put $x = p - a + 1$. If $x > c_-$, the first $c_- + 1$ positions of the initial sequence are lower, whereas $\sigma_{\min}$ has only c_- lower positions in that initial segment; reachability therefore fails. Conversely, assume $x \leq c_-$. Up to position x , both sequences are lower. From position $x + 1$ through position $x + q$, the initial arrangement has exactly x lower positions, while $N_L(\sigma_{\min}; k) \geq \min\{k, c_-\} \geq x$. After position $x + q$, any arrangement containing only q upper symbols has at least $k - q$ lower symbols in its first k positions, which is exactly the number in the initial arrangement. Hence the prefix condition is equivalent to $x \leq c_-$, or $a \geq a_-$.

Now let $a = p + b$, with $b \in \{1, \dots, q\}$. The cyclic product begins with the type sequence

$$U^{q-b+1}L^pU^{b-1}.$$

Put $z = b - 1$, the length of its final upper group. If $z > c_+$, the suffix of length z of $\sigma_{\min}$ contains a lower symbol, whereas the corresponding suffix of the initial arrangement contains only upper symbols. At position $M - z$, the initial sequence has already accumulated all p lower symbols but $\sigma_{\min}$ has not, so the prefix condition fails. Conversely, let $z \leq c_+$. Before the initial lower group begins, its prefix contains no lower symbol. During that lower group, the final z positions of $\sigma_{\min}$ are upper, so its first k positions contain at most $q - z$ upper symbols and therefore at least as many lower symbols as the initial arrangement. After the initial lower group ends, the omitted suffix of $\sigma_{\min}$ lies inside its final upper group, so its prefix contains all p lower symbols. Thus the prefix condition is equivalent to $z = b - 1 \leq c_+$. Since $b \leq q$, this is equivalent to $a \leq a_+$. The two cases give exactly $\mathcal{A}_{\text{cyc}}$.

For such an index a , choose any sequence of adjacent exchanges taking the initial type sequence to $\sigma_{\min}$, and apply Lemma 2.2 at every exchange. The product remains $T_{(a)}$, and all factors remain normalized and positive. The final assertion of Lemma 2.2 preserves uniform boundedness of the coefficient sequences at each exchange. Hence, if the original sequences are bounded, the finitely many exchanges produce bounded factor coefficients.

Every directed exchange decreases by one the number of inversions consisting of an upper position followed by a lower position. Therefore every exchange sequence with the prescribed initial and final type sequences has the same length and is automatically reduced.

It remains to prove independence of the chosen exchange sequence. Give temporary labels to the factors in order to follow them through the calculation. Every upper–lower pair whose relative order changes is exchanged exactly once, and no other pair is exchanged. The exchanges involving a fixed factor have a forced order. Thus every reduced exchange sequence is a linear extension of the same finite partially ordered set of required crossings. Two crossings that share an upper factor are ordered by the unchanged relative order of the lower factors that it crosses; the same argument applies to two crossings that share a lower factor. Hence two incomparable crossings involve disjoint factor pairs. Any two linear extensions differ by successive interchanges of adjacent incomparable crossings, and the corresponding bidiagonal refactorizations commute. Consequently all reduced exchange sequences give the same final matrices.

Finally, let ρ be any minimum-height target reachable from the type sequence of a cyclic product. By Lemma 2.7,

$$N_L(\sigma_{\min}; k) \geq N_L(\rho; k)$$

for every k . The reachability criterion and the inequalities from the initial sequence to ρ therefore show that the same initial sequence can be transformed into $\sigma_{\min}$. This proves the final maximality assertion. $\square$

For the remainder of the construction, fix $a \in \mathcal{A}_{\text{cyc}}$. Use the minimum-height factorization (2.4). The matrix built from these factors will be denoted by $\mathcal{H}_{(a)}$. The index (a) records the selected cyclic product $T_{(a)}$. Its entries are taken from the new factors $\tilde{F}_1^{(a)}, \dots, \tilde{F}_M^{(a)}$, not from the original factors with their labels permuted. They coincide with the terminal lower–upper PBF factors only when the minimum-height pattern itself is lower–upper. In

particular, $p+1 \in \mathcal{A}_{\text{cyc}}$ and $T_{(p+1)} = BA$. The next subsection constructs $\mathcal{H}_{(a)}$ from these factors, and Theorem 3.9 proves that it has exactly two nonzero block diagonals.

2.3. Direct construction of the two sparse matrices. We now define the two matrices directly from the bidiagonal coefficients; the cyclic matrix will enter later in the resolvent identities.

Definition 2.9 (The two sparse matrices). Retain the prescribed order

$$F_1, \dots, F_M = L_1, \dots, L_p, U_q, \dots, U_1$$

and define

$$\omega_j = (j-1)Q, \quad j \in \{1, \dots, p\}, \quad \omega_{p+b} = (q-b+1)P, \quad b \in \{1, \dots, q\}.$$

For $h \in \mathbb{N}_0$, let

$$I_h := \{j \in \{1, \dots, M\} : h - \omega_j \in R\mathbb{N}_0\}, \quad n_j(h) := \frac{h - \omega_j}{R} \quad (j \in I_h).$$

The rows and columns of the first sparse matrix are indexed by the pairs (h, j) with $j \in I_h$, ordered first by h and then by j . For any factor-position index $k \in \{1, \dots, M\}$, write $k^+ = k+1$ when $k < M$ and $k^+ = 1$ when $k = M$. Define $\mathcal{H}$ by

$$(2.5) \quad \mathcal{H}_{(h,j), (h+Q, j^+)} = 1,$$

and by the following coefficient entries:

$$(2.6) \quad \mathcal{H}_{(h,j), (h-P, j^+)} = \ell_{n_j(h)-1}^{(j)}, \quad j \in \{1, \dots, p\}, \quad n_j(h) \geq 1,$$

$$(2.7) \quad \mathcal{H}_{(h, p+b), (h-P, (p+b)^+)} = u_{n_{p+b}(h)}^{(q-b+1)}, \quad b \in \{1, \dots, q\}.$$

All entries not specified in (2.5)–(2.7) are zero. A term with a negative height is absent. This defines $\mathcal{H}$ directly from the original factor coefficients, including its finite initial part. When $d = 1$, each nonempty I_h contains one factor position; when $d > 1$, the same formula gives finite blocks indexed by the elements of I_h .

For the second matrix, use the selected index a fixed after Theorem 2.8 and write

$$T_{(a)} = \tilde{F}_1^{(a)} \cdots \tilde{F}_M^{(a)}.$$

For $k \in \{0, \dots, R-1\}$, let r_k be the remainder of kQ upon division by R . For $j \in \{1, \dots, M\}$, let

$$\tilde{\omega}_j := r_{(j-1) \bmod R},$$

and put

$$I_h^{(a)} := \{j \in \{1, \dots, M\} : h - \tilde{\omega}_j \in R\mathbb{N}_0\}, \quad \tilde{n}_j(h) := \frac{h - \tilde{\omega}_j}{R}.$$

The rows and columns of $\mathcal{H}_{(a)}$ are indexed by (h, j) , with $j \in I_h^{(a)}$, in the same order as above. For every such pair, place the normalizing entry of $\tilde{F}_j^{(a)}$, which is equal to 1, at

$$(\mathcal{H}_{(a)})_{(h,j), (h+Q, j^+)} = 1.$$

If $\tilde{F}_j^{(a)}$ is lower bidiagonal and $\tilde{n}_j(h) \geq 1$, place its subdiagonal entry at

$$(\mathcal{H}_{(a)})_{(h,j), (h-P, j^+)} = (\tilde{F}_j^{(a)})_{\tilde{n}_j(h), \tilde{n}_j(h)-1}.$$

If $\tilde{F}_j^{(a)}$ is upper bidiagonal, place its diagonal entry at

$$(\mathcal{H}_{(a)})_{(h,j), (h-P, j^+)} = (\tilde{F}_j^{(a)})_{\tilde{n}_j(h), \tilde{n}_j(h)}.$$

All other entries are zero. Here every $I_h^{(a)}$ has exactly d elements. Thus the definition gives two block diagonals from the first row; when $d = 1$, it gives directly

$$(\mathcal{H}_{(a)})_{h,h+q} = 1, \quad (\mathcal{H}_{(a)})_{h,h-p} = c_h^{(a)} > 0 \quad (h \geq p).$$

Section 3 proves these placement rules in detail and derives the block continued fraction. It extends the factor-position heights of Definition 2.5 to the coordinate labels and uses them to verify the two target indices and to relate the directly constructed matrices to the products T and $T_{(a)}$.

2.4. Auxiliary cyclic realization. The two sparse matrices have already been defined from the factor coefficients. We now introduce an auxiliary matrix that will prove their relation with the original product and its cyclic permutations. Enlarge the space to the direct sum $\mathcal{E}^{\oplus M} = \mathcal{E} \oplus \cdots \oplus \mathcal{E}$, with $M = p + q$ ordered summands. The a -th copy of $\mathcal{E}$ means the a -th summand in this direct sum.

Recall the ordered factors $F_1, \dots, F_M$ introduced in Subsection 2.1, so that $T = F_1 F_2 \cdots F_M$. Introduce the cyclic matrix

$$\mathcal{C}_{p,q} := \begin{bmatrix} 0 & F_1 & 0 & \cdots & 0 \\ 0 & 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & F_{M-1} \\ F_M & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

For $a \in \{1, \dots, M\}$, let $\Pi_a : \mathcal{E}^{\oplus M} \rightarrow \mathcal{E}$ be the coordinate projection, so that $\Pi_a(x^{(1)}, \dots, x^{(M)}) = x^{(a)}$. Its transpose $\Pi_a^\top$ embeds $\mathcal{E}$ into the a -th summand. Thus Π_1 selects the first block row of the displayed cyclic matrix and $\Pi_1^\top$ selects its first block column.

Lemma 2.10 (The cyclic matrix and the factor products). *One complete circuit through the factor positions gives*

$$\mathcal{C}^M = \text{diag}(T_{(1)}, \dots, T_{(M)}).$$

In particular,

$$\Pi_1 \mathcal{C}^M \Pi_1^\top = T.$$

Consequently, as an identity of formal power series in τ ,

$$\Pi_1 (I - \tau \mathcal{C})^{-1} \Pi_1^\top = (I - \tau^M T)^{-1}.$$

Proof. The nonzero blocks of $\mathcal{C}$ are

$$(\mathcal{C})_{a,a+1} = F_a, \quad a \in \{1, \dots, M-1\},$$

and

$$(\mathcal{C})_{M,1} = F_M.$$

Reading the block indices cyclically, induction on s gives

$$(\mathcal{C}^s)_{a,a+s} = F_a F_{a+1} \cdots F_{a+s-1}.$$

For $s \in \{1, \dots, M-1\}$, this is the only nonzero block in row a , and it is not on the main block diagonal. Therefore

$$\Pi_1 \mathcal{C}^s \Pi_1^\top = 0, \quad s \in \{1, \dots, M-1\}.$$

After M steps the block position returns to its starting point, and

$$(\mathcal{C}^M)_{a,a} = F_a F_{a+1} \cdots F_M F_1 \cdots F_{a-1}.$$

These are precisely $T_{(1)}, \dots, T_{(M)}$, which proves the first assertion and its $(1, 1)$ -block identity. Repeating complete circuits shows, for $n \geq 0$ and $s \in \{0, \dots, M-1\}$, that

$$\Pi_1 \mathcal{C}^{Mn+s} \Pi_1^\top = \begin{cases} T^n, & s = 0, \\ 0, & s \neq 0. \end{cases}$$

Substitution in the formal Neumann series now gives

$$\Pi_1 (I - \tau \mathcal{C})^{-1} \Pi_1^\top = \sum_{m \geq 0} \tau^m \Pi_1 \mathcal{C}^m \Pi_1^\top = \sum_{n \geq 0} \tau^{Mn} T^n = (I - \tau^M T)^{-1}.$$

□

2.5. Darboux interpretation and off-diagonal resolvent blocks. The cyclic products $T_{(a)}$ introduced above also form the Darboux orbit of the prescribed factorization: passing from $T_{(a)}$ to $T_{(a+1)}$ moves the first factor to the right end. By Lemma 2.10, these products occur as the diagonal blocks of $\mathcal{C}^M$.

When the prescribed factors are Christoffel connection matrices, the matrices $T_{(a)}$ are the recurrence matrices of the corresponding transformed systems [12]. Their precise spectral realization, including the bounded and unbounded Favard cases, is developed in Section 6.

The diagonal blocks describe complete circuits through the factors. The off-diagonal blocks of the cyclic resolvent describe the intervening partial products. We now make this statement precise. For $a, b \in \{1, \dots, M\}$, let $s(a, b) \in \{0, \dots, M-1\}$ be the unique cyclic distance determined by

$$b \equiv a + s(a, b) \pmod{M},$$

where factor positions are represented by $1, \dots, M$. Define

$$F_{a \rightarrow b} := F_a F_{a+1} \cdots F_{a+s(a,b)-1},$$

with cyclic indices; when $a = b$, this is the empty product and hence the identity. The next proposition records both the diagonal and off-diagonal resolvent blocks in terms of these objects.

Proposition 2.11 (Resolvent blocks for the cyclic Darboux transforms). *For $a, b \in \{1, \dots, M\}$, the partial cyclic product defined above satisfies*

$$(2.8) \quad T_{(a)} F_{a \rightarrow b} = F_{a \rightarrow b} T_{(b)}.$$

Moreover, as an identity of formal power series in τ ,

$$(2.9) \quad \Pi_a (I - \tau \mathcal{C})^{-1} \Pi_b^\top = \tau^{s(a,b)} F_{a \rightarrow b} (I - \tau^M T_{(b)})^{-1}.$$

Equivalently,

$$(2.10) \quad \Pi_a (I - \tau \mathcal{C})^{-1} \Pi_b^\top = \tau^{s(a,b)} (I - \tau^M T_{(a)})^{-1} F_{a \rightarrow b}.$$

In particular,

$$(2.11) \quad \Pi_a (I - \tau \mathcal{C})^{-1} \Pi_a^\top = (I - \tau^M T_{(a)})^{-1}.$$

Proof. The identity (2.8) follows by expanding both sides. Indeed, the left-hand side is

$$F_a \cdots F_{a+s(a,b)-1} F_{a+s(a,b)} \cdots F_{a-1} F_a \cdots F_{a+s(a,b)-1},$$

whereas the right-hand side places $F_{a \rightarrow b}$ in front of the cyclic product starting at b . Since $b \equiv a + s(a, b) \pmod{M}$, the two products coincide.

For the resolvent identity, expand the resolvent into its Neumann series. A term contributing to the (a, b) -block must have length $s(a, b) + Mn$, with $n \in \mathbb{N}_0$. The first $s(a, b)$ elementary steps contribute $F_{a \rightarrow b}$. Each subsequent full cycle, starting and ending at the b -th copy, contributes one power of $T_{(b)}$. Therefore

$$\Pi_a (I - \tau \mathcal{C})^{-1} \Pi_b^\top = \sum_{n \geq 0} \tau^{s(a,b)+Mn} F_{a \rightarrow b} (T_{(b)})^n,$$

which gives (2.9). The equivalent form (2.10) follows from the intertwining relation (2.8). The diagonal case $a = b$ gives (2.11). $\square$

3. PROOF OF THE SPARSE FORMS AND BLOCK CONTINUED FRACTIONS

The matrices $\mathcal{H}$ and $\mathcal{H}_{(a)}$ were defined directly from the bidiagonal coefficients in Subsection 2.3. This section proves those definitions have the asserted two-diagonal form and relates them to the auxiliary cyclic matrices. A coordinate is labeled by the factor being applied and its row index. The integer height assigned to that label shows that every nonzero entry has displacement $+Q$ or $-P$.

After the two sparse forms have been proved, consecutive heights are grouped into finite blocks. This makes $\mathcal{H}$ block tridiagonal, and ordinary finite-dimensional Schur complementation produces the continued fraction for its resolvent.

The first three subsections verify the direct placement rules for the prescribed order and for $\mathcal{H}_{(a)}$, and describe the polynomial recurrences contained in the latter matrix. The remaining subsections construct the continued fraction from the prescribed factor order, then establish its finite approximation, convergence, positivity, backward evaluation, and residual estimates. Notation without the cyclic subscript (a) refers to the prescribed order $L_1 \cdots L_p U_q \cdots U_1$ and hence to the original factor coefficients.

3.1. Verification of the direct coefficient placement. We verify the direct construction of Subsection 2.3. To retain the information carried by the individual bidiagonal factors, one application of their product is resolved into $M = p + q$ elementary stages. A coordinate is therefore specified by the factor position $a \in \{1, \dots, M\}$ and the row index $n \in \mathbb{N}_0$; the pair (a, n) is the coordinate label used in the proof.

Recall the factor-position heights $\eta_0, \dots, \eta_M$ from Definition 2.5. Extend them to the coordinate labels by defining

$$h(a, n) := Rn + \eta_{a-1}.$$

Thus the factor position determines the residue of the height modulo R , whereas n determines its multiple of R . This choice makes every nonzero entry of an individual bidiagonal factor produce one of the two height displacements $+Q$ and $-P$. For the prescribed two-run order, $\eta_{a-1} = \omega_a$; for a selected refactorization, $\eta_{a-1} = r_{a-1}$.

The height is integer-valued; the word “rational” below refers to the coprime pair of increments $+Q$ and $-P$, or equivalently to their rational slope. The type of a transition is determined by its position in the bidiagonal factor, not by its numerical value. The *normalizing entries* are the fixed diagonal entries of the lower factors and the fixed superdiagonal entries of the upper factors; by the factor normalization specified above, these entries are identically equal to 1. They will be represented as upward steps. The *coefficient entries* are the subdiagonal entries $\ell_n^{(a)}$ of the lower factors and the diagonal entries $u_n^{(b)}$ of the upper factors; they will be represented as downward steps carrying those coefficients. In particular, if a coefficient $\ell_n^{(a)}$ or $u_n^{(b)}$ happens to equal 1, it is still a coefficient entry and still gives a downward step.

Minors of weighted path matrices admit a combinatorial reading through the Lindström–Gessel–Viennot lemma [35, 25]; the determinant factorizations below are instead obtained from Schur complements, which produce directly the ordered pivot blocks required by the backward recursion, and all path arguments here are one-particle expansions.

Throughout the path interpretation the convention is the same as in the Hessenberg case of [9]: a matrix entry is read in row-to-column orientation. Thus an entry $A_{i,j}$ is regarded as an edge from row state i to column state j . This is a path-reading convention only; all algebraic identities for the cyclic linearization and for resolvents remain in the usual matrix notation. With this convention, a subdiagonal entry of a lower bidiagonal factor gives a downward edge, and a superdiagonal entry of an upper bidiagonal factor gives an upward edge.

Recall from Subsection 2.2 that

$$d := \gcd(p, q), \quad P := \frac{p}{d}, \quad Q := \frac{q}{d}, \quad R := P + Q = \frac{M}{d}.$$

The division by $d = \gcd(p, q)$ removes the common factor of the two unscaled height increments

$$+q, \quad -p.$$

Indeed,

$$q = dQ, \quad p = dP.$$

Thus every edge in the reduced height graph has increment

$$+Q, \quad -P.$$

When $d > 1$, the integer d appears as the multiplicity of factor labels over the same reduced height. Thus P and Q determine the two reduced height steps, while d measures how many factor labels may project to the same height residue.

For example, if $(p, q) = (4, 2)$, then $d = 2$, $P = 2$, and $Q = 1$. The unreduced steps $+2$ and -4 become

$$+1, \quad -2,$$

but with two labels over each reduced residue class.

Use the ordered factors $F_1, \dots, F_M$ defined in Subsection 2.1. A coordinate state of $\mathcal{G}^{\oplus M}$ is written

$$e_{(a,n)} := \Pi_a^\top e_n, \quad a \in \{1, \dots, M\}, \quad n \in \mathbb{N}_0.$$

Thus the label (a, n) records the summand associated with the factor F_a and the coordinate n inside that summand. The complete set of labeled coordinate states is denoted by

$$\mathcal{V}_{p,q} := \{(a, n) : a \in \{1, \dots, M\}, n \in \mathbb{N}_0\}.$$

For any set $\mathcal{S}$ of such labels, the notation $\text{span } \mathcal{S}$ means

$$\text{span } \mathcal{S} := \text{span}\{e_{(a,n)} : (a, n) \in \mathcal{S}\}.$$

For the prescribed two-run order, write the state height as

$$h(a, n) = Rn + \omega_a,$$

where $\omega_a = h(a, 0)$ is the height of the factor position a . The numbers ω_a are determined by requiring the normalizing entries to give rises of height $+Q$.

For the lower factors, the normalizing entries are the diagonal entries of $L_1, \dots, L_p$. They do not change n . Hence the height shifts must increase by Q :

$$\omega_1 = 0, \quad \omega_a = (a - 1)Q, \quad a \in \{1, \dots, p + 1\}.$$

For the upper factors, the normalizing entries are the superdiagonal entries of $U_q, \dots, U_1$. In the row-to-column path convention, such an entry gives an edge from index n to index $n + 1$. Hence the term Rn increases by $R = P + Q$, and the height shift must decrease by P in order to make the total increment equal to $+Q$. Thus, for $b \in \{1, \dots, q\}$,

$$(3.1) \quad \omega_{p+b} = (q - b + 1)P.$$

For the common endpoint $a = p + 1$, the two descriptions give the same value: from the lower side $\omega_{p+1} = pQ$, while from the upper side $\omega_{p+1} = qP$. These coincide because $pQ = qP = pq/d$.

The pairs

$$(h(a, n); a)$$

are height states with an additional factor label. The label a is kept because it identifies the bidiagonal factor containing the coefficient of the transition. If the height alone determines a , the label can be suppressed. If not, it must be retained.

Proposition 3.1 (The two possible height increments). *After relabeling the cyclic linearization by the height states $(h(a, n); a)$, every nonzero elementary transition has height increment either $+Q$ or $-P$. More precisely:*

- (i) for $a \in \{1, \dots, p\}$, the fixed diagonal entry of L_a gives a transition of height $+Q$, while the subdiagonal entry $\ell_n^{(a)}$ gives a coefficient transition of height $-P$;
- (ii) for $b \in \{1, \dots, q\}$, the factor in cyclic position $p + b$ is U_{q-b+1} ; its fixed superdiagonal entry gives a transition of height $+Q$, while its diagonal entry $u_n^{(q-b+1)}$ gives a coefficient transition of height $-P$.

The classification is positional: statements (i) and (ii) remain unchanged when one of the coefficients $\ell_n^{(a)}$ or $u_n^{(b)}$ has the numerical value 1.

Proof. For the lower factors, the diagonal entry of L_a gives an edge from index n at F_a to index n at F_{a+1} . Therefore n is unchanged, and the height increment is

$$\omega_{a+1} - \omega_a = Q.$$

The subdiagonal entry of L_a gives an edge from row index n to column index $n - 1$. Thus the term Rn decreases by R , while the height shift increases by Q . The total height increment is

$$Q - R = Q - (P + Q) = -P.$$

For the upper factors, the fixed superdiagonal entry gives an edge from row index n to column index $n + 1$. The term Rn therefore increases by $R = P + Q$, while the height shift decreases by P . The total height increment is

$$R - P = Q.$$

The diagonal entry $u_n^{(b)}$ preserves e_n . Hence only the height shift changes, and it changes by $-P$. Therefore the diagonal entry gives a coefficient transition of height $-P$. $\square$

Proposition 3.1 determines the two possible height increments. The next question is whether the height alone identifies the factor that contains an edge weight. This is true in the coprime case, apart from finitely many missing heights near the origin, but factor labels remain necessary when $\gcd(p, q) > 1$.

Proposition 3.2 (Projection to the reduced height graph). *Assume first that $d = 1$. Then $P = p$, $Q = q$, and $R = M = p + q$. The height shifts*

$$\omega_1, \dots, \omega_M$$

are pairwise distinct modulo M . Hence, whenever a height belongs to the image of the map $(a, n) \mapsto h(a, n)$, that height determines the factor label a . In this case the labeled height graph projects to a scalar rational path graph with steps

$$+q, \quad -p,$$

apart from finitely many missing heights near the origin.

If $d > 1$, different factor labels may give the same height modulo R . Then the height alone does not determine the current factor, and the labels must be retained.

Proof. When $d = 1$, formula (3.1) gives, for every factor position $a \in \{1, \dots, M\}$,

$$\omega_a \equiv (a - 1)q \pmod{M}.$$

For $1 \leq a \leq p + 1$, this follows from $\omega_a = (a - 1)q$. For an upper-factor position $a = p + b$, formula (3.1) gives

$$(p + b - 1)q - \omega_{p+b} = (p + b - 1)q - (q - b + 1)p = (b - 1)M,$$

which proves the same congruence. Moreover, $\gcd(q, M) = \gcd(q, p + q) = \gcd(p, q) = 1$. Multiplication by q therefore permutes the residue classes modulo M , so the M shifts $\omega_1, \dots, \omega_M$ are pairwise distinct modulo M . Thus two states with the same height must have the same factor label and the same index n .

However, the restriction $n \geq 0$ may remove some small heights from the image. For example, height 1 is missing when $(p, q) = (2, 3)$. Away from this finite set of missing initial heights, the projected height graph is the usual rational-step graph.

If $d > 1$, there are $M = dR$ factor labels but only R residue classes modulo R . Thus different labels may have the same height residue, and the label is necessary to know which bidiagonal coefficient is attached to the transition. $\square$

The preceding congruence constructs the prescribed-order matrix directly from the factor coefficients.

Corollary 3.3 (Direct placement of the prescribed factor coefficients). *Assume that $d = 1$, so that $M = p + q$. For every height h , let $k(h) \in \{0, \dots, M - 1\}$ be the unique integer satisfying*

$$k(h)q \equiv h \pmod{M},$$

and put

$$a(h) := k(h) + 1, \quad n(h) := \frac{h - \omega_{a(h)}}{M}.$$

The height h occurs in the prescribed ordering precisely when $n(h) \geq 0$. At every such height there is an entry equal to 1 joining h to $h + q$. When the indicated coefficient exists, the entry joining h to $h - p$ is

$$c_h^{\text{pre}} := \begin{cases} \ell_{n(h)-1}^{(a(h))}, & a(h) \in \{1, \dots, p\}, \quad n(h) \geq 1, \\ u_{n(h)}^{(q-b+1)}, & a(h) = p + b, \quad b \in \{1, \dots, q\}. \end{cases}$$

Consequently, after omitting the finitely many heights for which $n(h) < 0$, the two nonzero diagonals of $\mathcal{H}$ are obtained by reading the original factor coefficients in the periodic order determined by $k(h)q \equiv h \pmod{M}$. In particular, every height $h \geq pq$ occurs.

Proof. Proposition 3.2 gives $\omega_a \equiv (a - 1)q \pmod{M}$, so the congruence selects the unique factor position whose states can have height h . Solving $h = Mn + \omega_a$ gives $n = n(h)$. A lower factor has its coefficient entry in row n , column $n - 1$, with value $\ell_{n-1}^{(a)}$; an upper factor in position $p + b$ has its coefficient entry on row and column n , with value $u_n^{(q-b+1)}$. Proposition 3.1 gives the two target heights. Finally, $\max_a \omega_a = pq$, so $n(h) \geq 0$ for every selected factor when $h \geq pq$. $\square$

For example, during one period the factors selected by increasing height are

(p, q)	$h \equiv 0$	$h \equiv 1$	$h \equiv 2$	$h \equiv 3$	$h \equiv 4$	$(\text{mod } 5).$
$(3, 2)$	L_1	U_2	L_2	U_1	L_3	
$(2, 3)$	L_1	U_3	U_1	L_2	U_2	

The formula for $n(h)$ automatically removes an occurrence that would require a negative row index; for instance, height 1 is absent in the initial part of both examples. After this finite initial part, the same five-factor order repeats with the next coefficient of each factor.

When $d > 1$, the same rule is block-valued. For a height residue $\rho \in \{0, \dots, R - 1\}$, the congruence

$$(a - 1)Q \equiv \rho \pmod{R}$$

selects exactly d factor positions. For each selected position one solves $h = Rn + \omega_a$, discards negative values of n , and places the corresponding bidiagonal coefficient in the block from height h to height $h - P$; the normalizing entries fill the block from h to $h + Q$. Thus the block matrix is also obtained directly from the periodic ordering of the original coefficients. The coordinate-state notation keeps track of the d rows and columns within each block.

For the continued fraction it is useful to group several consecutive heights into one block. Before doing so, the exact-height ordering reveals a more rigid recurrence structure that will also be used in the spectral interpretation. For $h \in \mathbb{N}_0$, put

$$\mathcal{F}_h := \{(a, n) : h(a, n) = h\}, \quad \mathcal{E}_h := \text{span } \mathcal{F}_h.$$

Empty spaces $\mathcal{E}_h$ are allowed in the finite initial part of the decomposition.

Definition 3.4 (Height ordering used in the proof). Recall that $e_{(a,n)} = \Pi_a^\top e_n$ denotes the coordinate vector corresponding to coordinate n in factor copy a . Enumerate these vectors as

$$e_{(a_0, n_0)}, e_{(a_1, n_1)}, e_{(a_2, n_2)}, \dots$$

by requiring

$$h(a_j, n_j) < h(a_k, n_k) \implies j < k,$$

and, when $h(a_j, n_j) = h(a_k, n_k)$, by placing the smaller factor label first.

Let $\mathcal{P}_h$ be the permutation matrix determined by

$$\mathcal{P}_h e_{(a_j, n_j)} = e_j, \quad j \in \mathbb{N}_0.$$

Proposition 3.5 (Agreement with the auxiliary cyclic matrix). *The matrix $\mathcal{H}_{p,q}$ defined directly in Subsection 2.3 satisfies*

$$\mathcal{H}_{p,q} = \mathcal{P}_h \mathcal{C}_{p,q} \mathcal{P}_h^\top.$$

Proof. Fix a coordinate label (a, n) , and write

$$h = Rn + \omega_a.$$

The nonzero entries in the corresponding row of $\mathcal{C}_{p,q}$ are precisely the nonzero entries in row n of the factor F_a , and their column labels belong to the next factor position a^+ .

Suppose first that $a \in \{1, \dots, p\}$, so that $F_a = L_a$. The diagonal entry of L_a is equal to 1. It joins (a, n) to (a^+, n) , whose height is $h + Q$. If $n \geq 1$, the other nonzero entry in row n is

$$(L_a)_{n, n-1} = \ell_{n-1}^{(a)}.$$

It joins (a, n) to $(a^+, n-1)$, whose height is $h - P$. Since $n = n_a(h)$, these are exactly the entries in (2.5) and (2.6).

Now suppose that $a = p + b$, so that $F_a = U_{q-b+1}$. The fixed superdiagonal entry of this factor is equal to 1. It joins (a, n) to $(a^+, n+1)$, whose height is again $h + Q$. Its diagonal entry is

$$(U_{q-b+1})_{n, n} = u_n^{(q-b+1)}.$$

It joins (a, n) to (a^+, n) , whose height is $h - P$. Since $n = n_{p+b}(h)$, these are exactly the entries in (2.5) and (2.7).

There are no further entries because every factor is bidiagonal. The permutation $\mathcal{P}_h$ merely places the labels (a, n) in the order used in the direct definition of $\mathcal{H}_{p,q}$. Therefore $\mathcal{P}_h \mathcal{C}_{p,q} \mathcal{P}_h^\top$ and $\mathcal{H}_{p,q}$ agree entry by entry. $\square$

Theorem 3.6 (Two-diagonal form of the prescribed matrix). *Group together the coordinate labels having the same height. Relative to the resulting sequence of coordinate spaces*

$$\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \dots,$$

the directly defined matrix $\mathcal{H}$ has only two nonzero block diagonals:

$$(3.2) \quad [\mathcal{H}]_{\mathcal{E}_h \times \mathcal{E}_k} = 0 \quad \text{unless} \quad k - h \in \{Q, -P\}.$$

The block with $k = h + Q$ consists of normalizing entries, whereas the block with $k = h - P$ consists of the individual coefficient entries of the bidiagonal factors. Moreover,

$$0 \leq \dim \mathcal{E}_h \leq d,$$

and $\dim \mathcal{E}_h = d$ whenever $h \geq \max_{1 \leq a \leq M} \omega_a = pQ$.

If $d = 1$, every sufficiently large exact height carries one state, and the scalar tail of $\mathcal{H}$ is a two-diagonal band matrix: its only nonzero diagonals are the Q -th superdiagonal and the P -th subdiagonal. In particular, for $q = 1$ this is the two-diagonal lower-Hessenberg matrix associated with the usual star-like p -orthogonality. For arbitrary p and q , (3.2) is its two-sided block analogue.

Proof. Proposition 3.1 states that every nonzero transition has increment $+Q$ or $-P$, and it also identifies which type of factor entry produces each increment. This proves (3.2) and the assertion about its two block diagonals.

For $a \leq p + 1$, the definition gives $\omega_a = (a - 1)Q$. If $a = p + b$, with $b \in \{1, \dots, q\}$, then the upper-factor formula for the shift gives

$$(p + b - 1)Q - \omega_{p+b} = (b - 1)(P + Q) = (b - 1)R.$$

Consequently, $\omega_a \equiv (a - 1)Q \pmod{R}$ for every factor label a . Since $\gcd(Q, R) = 1$ and $M = dR$, every residue class modulo R occurs for exactly d factor labels. For fixed h , each such label determines at most one index,

$$n = \frac{h - \omega_a}{R}.$$

Hence $\dim \mathcal{E}_h \leq d$. If $h \geq \max_a \omega_a$, all the resulting indices are nonnegative, so all d labels occur and $\dim \mathcal{E}_h = d$. When $d = 1$, the exact-height blocks are therefore scalar after the finite initial part, and the two increments become the stated scalar diagonal offsets. $\square$

The rotational symmetry of this two-diagonal matrix is already present before height ordering. Let ζ be an M -th root of unity and set

$$\mathcal{D}_\zeta := \text{diag}(I, \zeta I, \dots, \zeta^{M-1} I)$$

on the M factor copies. Write

$$\widehat{\mathcal{D}}_\zeta := \mathcal{P}_h \mathcal{D}_\zeta \mathcal{P}_h^\top$$

for the same diagonal transformation in the ordering by height.

Proposition 3.7 (Rotational symmetry and cyclic products). *Let ζ be an M -th root of unity. Then*

$$(3.3) \quad \mathcal{D}_\zeta^{-1} \mathcal{C} \mathcal{D}_\zeta = \zeta \mathcal{C}, \quad \widehat{\mathcal{D}}_\zeta^{-1} \mathcal{H} \widehat{\mathcal{D}}_\zeta = \zeta \mathcal{H},$$

Moreover,

$$(3.4) \quad \mathcal{H}^M = \mathcal{P}_h \text{diag}(T_{(1)}, \dots, T_{(M)}) \mathcal{P}_h^\top.$$

Consequently, every finite principal submatrix of $\mathcal{H}$ has spectrum invariant under multiplication by ζ . In particular, this holds for the leading principal submatrices obtained by retaining the first finitely many height blocks. Equation (3.4) expresses $\mathcal{H}^M$, up to the permutation $\mathcal{P}_h$, as the direct sum of the cyclic products $T_{(1)}, \dots, T_{(M)}$.

Proof. In the cyclic matrix, the block in row a and column $a + 1$ is F_a , with the factor-copy indices read modulo M . Conjugation by $\mathcal{D}_\zeta$ multiplies each of these blocks by ζ . For the block joining the last copy to the first, this follows from $\zeta^M = 1$. Hence

$$\mathcal{D}_\zeta^{-1} \mathcal{C} \mathcal{D}_\zeta = \zeta \mathcal{C}.$$

Conjugating this identity by $\mathcal{P}_h$ gives the second identity in (3.3).

Lemma 2.10 gives

$$\mathcal{C}^M = \text{diag}(T_{(1)}, \dots, T_{(M)}).$$

Conjugation by $\mathcal{P}_h$ proves (3.4).

Finally, $\widehat{\mathcal{D}}_\zeta$ is diagonal in the coordinate ordering by height. If S is any finite set of coordinate indices, restriction of the second identity in (3.3) gives

$$\left(\widehat{\mathcal{D}}_\zeta|_S\right)^{-1} \mathcal{H}_{S \times S} \left(\widehat{\mathcal{D}}_\zeta|_S\right) = \zeta \mathcal{H}_{S \times S}.$$

Thus $\mathcal{H}_{S \times S}$ and $\zeta \mathcal{H}_{S \times S}$ are similar, which proves the asserted spectral invariance. $\square$

This symmetry has a discrete Fourier interpretation. Put

$$\zeta_M := \exp\left(\frac{2\pi i}{M}\right).$$

The diagonal matrices $\mathcal{D}_{\zeta_M^j}$, $j \in \{0, \dots, M-1\}$, implement the characters of the cyclic group $\mathbb{Z}/M\mathbb{Z}$ on the factor-copy index. The discrete Fourier transform on that index converts this diagonal character action into cyclic translation. Accordingly,

$$\frac{1}{M} \sum_{j=0}^{M-1} \zeta_M^{j(n-s)} = \begin{cases} 1, & n \equiv s \pmod{M}, \\ 0, & n \not\equiv s \pmod{M}, \end{cases}$$

is the Fourier projection onto the congruence class s . For the prescribed-order Weyl matrix, the zero Fourier mode produces the rotational lift in Theorem 5.11. The full star moment system of $\mathcal{H}_{(a)}$ assembles all congruence classes. Its component corresponding to $T_{(a)}$ is the transformed system $d\Psi^{(a)}$; for $a = p + 1$, this is $x d\Psi$. The remaining classes give the Fourier components studied in Proposition 6.5. Thus the rotational spectral symmetry,

the decomposition by powers modulo M , and the Fourier components of the measures on the star are different forms of the same discrete Fourier decomposition.

For positive scalar double-band matrices, rotationally arranged spectra are also obtained in [40]. In the Hessenberg case $q = 1$, the recurrence

$$\lambda P_h(\lambda) = P_{h+1}(\lambda) + c_h P_{h-p}(\lambda)$$

and its orthogonality on a star-like set are classical; see [20, 33]. Theorem 3.6 shows that a mixed-type bidiagonal factorization produces the two-sided version, with Q forward levels and P backward levels and with all intervening diagonals equal to zero. The continued fraction below is obtained by grouping this exact-height matrix into neighboring finite blocks, which provides a Schur-complement evaluation of its resolvent.

Remark 3.8 (Terminology and rational Dyck paths). The term “rational Dyck path” is avoided because it already denotes other path families in rational Catalan combinatorics [26, 19, 6]. Here the paths are simply weighted paths with increments $+Q$ and $-P$, and the factor label is retained to identify the bidiagonal factor supplying each coefficient. When $q = 1$, these increments become $+1$ and $-p$, and the construction reduces to the standard weighted p -Dyck path model for the p -branched Stieltjes fraction [23, 50, 5].

3.2. Verification of the two-block-diagonal form of $\mathcal{H}_{(a)}$. We now verify the direct construction of $\mathcal{H}_{(a)}$. Let

$$T_{(a)} = \tilde{F}_1^{(a)} \cdots \tilde{F}_M^{(a)}$$

be the factorization supplied by Theorem 2.8, and let $\mathcal{G}_{(a)}$ be the cyclic matrix having $\tilde{F}_k^{(a)}$ in block position $(k, k+1)$, with the block indices read cyclically. In the refactorized factor arrangement, read the remainders r_k periodically modulo R . The height reached before the factor in position j is then r_{j-1} . Accordingly, the height of its coordinate state is

$$h^{(a)}(j, n) := Rn + r_{j-1}.$$

This is the definition $h(j, n) = Rn + h_{j-1}$ from Subsection 3.1, specialized to the minimum-height factorization.

For $h \geq 0$, let $\mathcal{G}_h^{(a)}$ be the span of the coordinate labels having height h in this order. These are precisely the rows and columns indexed by $I_h^{(a)}$ in the direct definition. Ordering the auxiliary cyclic matrix by these spaces gives the directly defined matrix $\mathcal{H}_{(a)}$, by the same entry-by-entry verification as in Proposition 3.5. Recall that a monomial matrix is a square matrix with exactly one nonzero entry in each row and each column; it is positive when all those entries are positive.

Theorem 3.9 (The two block diagonals of $\mathcal{H}_{(a)}$). *For every $h \in \mathbb{N}_0$, there are exactly d coordinate states of height h ; equivalently,*

$$\dim \mathcal{G}_h^{(a)} = d.$$

With the row-to-column convention fixed above, the only possible transitions from height h are the following:

- (i) *Every state of height h has one transition of weight 1 to height $h + Q$. These transitions give a bijection between the d states of heights h and $h + Q$. Consequently, the block*

$$[\mathcal{H}_{(a)}]_{\mathcal{G}_h^{(a)} \times \mathcal{G}_{h+Q}^{(a)}}$$

has exactly one nonzero entry in every row and every column, and all these entries are equal to 1.

- (ii) *If $h \geq P$, every state of height h has one transition with a strictly positive factor coefficient to height $h - P$. These transitions give a bijection between the d states of heights h and $h - P$. Consequently, the block*

$$[\mathcal{H}_{(a)}]_{\mathcal{G}_h^{(a)} \times \mathcal{G}_{h-P}^{(a)}}$$

has exactly one strictly positive entry in every row and every column. If $h < P$, no such transition exists.

- (iii) *All other blocks are zero.*

Proof. Write the height uniquely as

$$h = Rn + \rho, \quad n \in \mathbb{N}_0, \quad \rho \in \{0, \dots, R-1\}.$$

In this factor arrangement, every residue $\rho \in \{0, \dots, R-1\}$ occurs exactly d times among the $M = dR$ factor positions. Therefore there are exactly d positions k for which

$$r_{k-1} = \rho.$$

For each of them, the coordinate state (k, n) has height

$$Rn + r_{k-1} = Rn + \rho = h.$$

Conversely, every coordinate state of height h must be obtained in this way. Hence $\dim \mathcal{E}_h^{(a)} = d$.

We now examine the two nonzero entries in the row of each bidiagonal factor. The index $k+1$ is read cyclically when $k = M$.

If $\tilde{F}_k^{(a)}$ is lower bidiagonal, then $r_{k-1} < P$ and

$$r_k = r_{k-1} + Q.$$

Its fixed diagonal entry, which is equal to 1, gives

$$(k, n) \mapsto (k+1, n).$$

The height of the target is

$$Rn + r_k = Rn + r_{k-1} + Q = h + Q.$$

Its strictly positive subdiagonal coefficient gives, when $n \geq 1$,

$$(k, n) \mapsto (k+1, n-1),$$

whose target has height

$$R(n-1) + r_k = h + Q - R = h - P.$$

If $\tilde{F}_k^{(a)}$ is upper bidiagonal, then $r_{k-1} \geq P$ and

$$r_k = r_{k-1} + Q - R = r_{k-1} - P.$$

Its fixed superdiagonal entry, which is equal to 1, gives

$$(k, n) \mapsto (k+1, n+1),$$

and the target has height

$$R(n+1) + r_k = h + Q.$$

Its strictly positive diagonal coefficient gives

$$(k, n) \mapsto (k+1, n),$$

whose target has height

$$Rn + r_k = h - P.$$

Thus every state has exactly one weight-1 transition to height $h + Q$. A transition to height $h - P$ exists for every state exactly when $h \geq P$: at a lower-factor position this is equivalent to $n \geq 1$, while every state belonging to an upper-factor position already has height at least P .

Finally, advancing the factor position k cyclically is bijective. Hence the transitions just described match the d source states bijectively with the d target states. Since a bidiagonal factor has no other nonzero entries, no other height transitions occur. $\square$

Corollary 3.10 (Coprime double-band matrix). *If $d = 1$, the height determines the factor position and*

$$(\mathcal{H}_{(a)})_{h, h+q} = 1 \quad (h \geq 0), \quad (\mathcal{H}_{(a)})_{h, h-p} = c_h^{(a)} > 0 \quad (h \geq p),$$

with every other entry equal to zero.

Proof. When $d = 1$, one has $P = p$, $Q = q$, and $\dim \mathcal{E}_h^{(a)} = 1$ for every $h \geq 0$. Hence the permutation block from $\mathcal{E}_h^{(a)}$ to $\mathcal{E}_{h+Q}^{(a)}$ is the scalar 1, while the positive monomial block from $\mathcal{E}_h^{(a)}$ to $\mathcal{E}_{h-P}^{(a)}$ is a positive scalar, denoted by $c_h^{(a)}$. The asserted entries and the vanishing of all the others follow from Theorem 3.9. $\square$

Corollary 3.11 (Two-block-diagonal recurrences and continued fractions along the constructed cyclic family). *For every $a \in \mathcal{A}_{\text{cyc}}$, let $\mathcal{H}_{(a)}$ be the height-ordered matrix obtained from the refactorization of $T_{(a)}$ in Theorem 2.8. Then $\mathcal{H}_{(a)}$ has only the two block displacements $+Q$ and $-P$, as described in Theorem 3.9.*

Let R_a denote the coordinate projection onto all coordinates belonging to the first factor position in this refactorization. Then, as a formal power-series identity,

$$(3.5) \quad R_a(I - \tau \mathcal{H}_{(a)})^{-1} R_a^\top = (I - \tau^M T_{(a)})^{-1}.$$

If the matrices involved define bounded operators, the same identity holds on the common operator-resolvent domain.

Consequently, grouping consecutive heights of $\mathcal{H}_{(a)}$ into finite blocks and applying Schur complementation gives a factor-resolved height-block continued fraction whose R_a -projection recovers the resolvent of $T_{(a)}$ through (3.5). Whenever $T_{(a)}$ is supplied with mixed-type Favard data, the corresponding rectangular projection gives simultaneous rational approximants to its Weyl matrix.

Proof. Theorem 3.9 gives the two stated block displacements. Before the height permutation, the matrix $\mathcal{H}_{(a)}$ is the cyclic linearization of the minimum-height factorization of $T_{(a)}$. Applying Lemma 2.10 to those factors and then conjugating by the height permutation gives (3.5). The block continued fraction now follows from Theorem 3.20, and the Weyl statement follows from the rectangular projection in Proposition 5.4, applied to the same cyclic product $T_{(a)}$ with its Favard normalization matrices. $\square$

The entries $c_h^{(a)}$ in the preceding corollary can be read directly from the refactorized bidiagonal matrices.

Corollary 3.12 (Direct placement in $\mathcal{H}_{(a)}$). *Write $h = Rn + \rho$, with $\rho \in \{0, \dots, R-1\}$, and let $k(\rho) \in \{0, \dots, R-1\}$ be determined by*

$$k(\rho)Q \equiv \rho \pmod{R}.$$

The factor positions associated with height h are

$$a_j(h) := k(\rho) + 1 + jR, \quad j \in \{0, \dots, d-1\}.$$

For each of these positions, the coefficient placed in the block from height h to height $h - P$ is

$$(3.6) \quad \begin{cases} (\tilde{F}_{a_j(h)}^{(a)})_{n,n-1}, & \tilde{F}_{a_j(h)}^{(a)} \text{ is lower bidiagonal,} \\ (\tilde{F}_{a_j(h)}^{(a)})_{n,n}, & \tilde{F}_{a_j(h)}^{(a)} \text{ is upper bidiagonal.} \end{cases}$$

The first case occurs only when $n \geq 1$; together with the second case, it supplies all d entries of the positive monomial block whenever $h \geq P$. The block from height h to height $h + Q$ is the permutation block formed by the normalizing entries. Hence $\mathcal{H}_{(a)}$ is obtained directly from the entries of $\tilde{F}_1^{(a)}, \dots, \tilde{F}_M^{(a)}$ by this periodic modular rule.

Proof. The congruence selects exactly the positions for which $r_{a-1} = \rho$. At each such position the coordinate index is the quotient n in $h = Rn + \rho$. The two alternatives in (3.6) are precisely the coefficient entries of a lower or upper bidiagonal factor. Their target height is $h - P$, while the normalizing entry has target height $h + Q$, by Theorem 3.9. $\square$

In particular, when $d = 1$, one reads a single factor at each height and the scalar coefficient in Corollary 3.10 is

$$c_h^{(a)} = \begin{cases} (\tilde{F}_{a(h)}^{(a)})_{n,n-1}, & \tilde{F}_{a(h)}^{(a)} \text{ is lower bidiagonal,} \\ (\tilde{F}_{a(h)}^{(a)})_{n,n}, & \tilde{F}_{a(h)}^{(a)} \text{ is upper bidiagonal,} \end{cases} \quad h = (p+q)n + \rho,$$

where $a(h) = k(\rho) + 1$. Definition 2.9 determines the factor position $a(h)$ directly from the residue of h . Thus, once the adjacent refactorization has produced the factors in the star arrangement, the entries of $\mathcal{H}_{(a)}$ are read from those factors by the displayed modular rule.

The adjacent exchanges preserve $T_{(a)}$, although they generally change its individual bidiagonal factors. For $a \in \{1, \dots, M\}$, put

$$C_a := F_1 \cdots F_{a-1}, \quad D_a := F_a \cdots F_M.$$

Then

$$T = C_a D_a, \quad T_{(a)} = D_a C_a.$$

All these products are banded and are therefore well defined entry by entry.

Theorem 3.13 (Resolvent transport along the cyclic family). *For every $a \in \{1, \dots, M\}$, the following identities hold as formal power series in t :*

$$(3.7) \quad (I - tT)^{-1} = I + tC_a(I - tT_{(a)})^{-1}D_a,$$

$$(3.8) \quad (I - tT_{(a)})^{-1} = I + tD_a(I - tT)^{-1}C_a,$$

$$(I - tT)^{-1}C_a = C_a(I - tT_{(a)})^{-1},$$

$$D_a(I - tT)^{-1} = (I - tT_{(a)})^{-1}D_a.$$

For $a \in \mathcal{A}_{\text{cyc}}$, let R_a be the coordinate projection in (3.5). With $t = \tau^M$,

$$(3.9) \quad (I - \tau^M T)^{-1} = I + \tau^M C_a R_a (I - \tau \mathcal{H}_{(a)})^{-1} R_a^\top D_a.$$

Consequently,

$$(3.10) \quad E_{[q]}(I - \tau^M T)^{-1} E_{[p]}^\top = E_{[q]} E_{[p]}^\top + \tau^M E_{[q]} C_a R_a (I - \tau \mathcal{H}_{(a)})^{-1} R_a^\top D_a E_{[p]}^\top.$$

Thus every selected cyclic product gives a factor-resolved continued fraction for the same rectangular resolvent of T .

If the displayed matrices define bounded operators on the chosen sequence space, the same formulas hold as operator identities wherever the corresponding inverses exist.

Proof. For $n \geq 1$,

$$(C_a D_a)^n = C_a (D_a C_a)^{n-1} D_a,$$

which gives (3.7) after summing the formal Neumann series. Interchanging C_a and D_a gives (3.8). The coefficient identities

$$(C_a D_a)^n C_a = C_a (D_a C_a)^n, \quad D_a (C_a D_a)^n = (D_a C_a)^n D_a$$

prove the two intertwining formulas. Substitution of (3.5) into (3.7), followed by the rectangular coordinate projections, proves (3.9) and (3.10). For bounded operators the same identities follow by multiplying the corresponding inverse operators. $\square$

The resolvents of consecutive cyclic products are connected by one factor. If

$$K_a := F_{a+1} \cdots F_M F_1 \cdots F_{a-1},$$

with indices read modulo M , then

$$T_{(a)} = F_a K_a, \quad T_{(a+1)} = K_a F_a,$$

and therefore

$$(I - tT_{(a)})^{-1} F_a = F_a (I - tT_{(a+1)})^{-1}.$$

Here $a + 1$ is read cyclically. This is the one-factor Darboux relation joining consecutive members of the cyclic family.

3.3. Polynomial recurrences contained in $\mathcal{H}_{(a)}$. The matrix $\mathcal{H}_{(a)}$ is linked to the original factorization at the level of recurrence equations. One complete circuit through its factor copies recovers a cyclic product of the refactorized factors. The next results make this statement precise for right and left polynomial solutions of those equations. The discussion in this subsection is algebraic; orthogonality and representing measures enter in Section 6.

For $k \in \{1, \dots, M\}$, set

$$\tilde{T}_{(a;k)} := \tilde{F}_k^{(a)} \tilde{F}_{k+1}^{(a)} \dots \tilde{F}_M^{(a)} \tilde{F}_1^{(a)} \dots \tilde{F}_{k-1}^{(a)}.$$

These are the cyclic products of the refactorization of $T_{(a)}$, and $\tilde{T}_{(a;1)} = T_{(a)}$. Let a right vector on the M factor copies be written as $\mathbf{X} = (X_1^\top, \dots, X_M^\top)^\top$, and write a left vector as $\mathbf{Y} = (Y_1, \dots, Y_M)$.

Proposition 3.14 (Polynomial solutions along a complete factor circuit). *Write*

$$\mathbf{X}(\lambda) = (X_1(\lambda)^\top, \dots, X_M(\lambda)^\top)^\top, \quad \mathbf{Y}(\lambda) = (Y_1(\lambda), \dots, Y_M(\lambda)),$$

and adopt the cyclic convention

$$\mathbf{X}_{M+1} = \mathbf{X}_1, \quad \mathbf{Y}_{M+1} = \mathbf{Y}_1.$$

Then

$$\mathcal{C}_{(a)} \mathbf{X}(\lambda) = \lambda \mathbf{X}(\lambda)$$

if and only if

$$\tilde{F}_k^{(a)} \mathbf{X}_{k+1}(\lambda) = \lambda \mathbf{X}_k(\lambda), \quad k \in \{1, \dots, M\}.$$

In this case, each component satisfies

$$(3.11) \quad \tilde{T}_{(a;k)} \mathbf{X}_k(\lambda) = \lambda^M \mathbf{X}_k(\lambda).$$

Similarly,

$$\mathbf{Y}(\lambda) \mathcal{C}_{(a)} = \lambda \mathbf{Y}(\lambda)$$

if and only if

$$\mathbf{Y}_k(\lambda) \tilde{F}_k^{(a)} = \lambda \mathbf{Y}_{k+1}(\lambda), \quad k \in \{1, \dots, M\}.$$

In this case,

$$\mathbf{Y}_k(\lambda) \tilde{T}_{(a;k)} = \lambda^M \mathbf{Y}_k(\lambda).$$

Proof. By the block definition of $\mathcal{C}_{(a)}$, its k -th block row contains only the factor $\tilde{F}_k^{(a)}$, in block column $k+1$, with the indices read cyclically. Therefore the k -th block component of $\mathcal{C}_{(a)} \mathbf{X}(\lambda) = \lambda \mathbf{X}(\lambda)$ is precisely

$$\tilde{F}_k^{(a)} \mathbf{X}_{k+1}(\lambda) = \lambda \mathbf{X}_k(\lambda).$$

This proves the first equivalence.

Starting with the equation of index k and substituting successively the equations for the following $M-1$ factor positions gives

$$\tilde{F}_k^{(a)} \tilde{F}_{k+1}^{(a)} \dots \tilde{F}_{k+M-1}^{(a)} \mathbf{X}_k(\lambda) = \lambda^M \mathbf{X}_k(\lambda),$$

where all factor indices are read modulo M . The product on the left is $\tilde{T}_{(a;k)}$, which proves (3.11).

The left eigenvalue equation is treated in the same way. Its k -th block component is

$$\mathbf{Y}_k(\lambda) \tilde{F}_k^{(a)} = \lambda \mathbf{Y}_{k+1}(\lambda).$$

Composing these identities once around the complete factor circuit and using the cyclic convention gives

$$\mathbf{Y}_k(\lambda) \tilde{T}_{(a;k)} = \lambda^M \mathbf{Y}_k(\lambda),$$

as claimed. □

The rotational symmetry separates the powers of λ according to their residues modulo M . To state this precisely, choose ordered right boundary states

$$(\alpha_r, n_r^R), \quad r \in \{1, \dots, q\},$$

and ordered left boundary states

$$(\beta_s, n_s^L), \quad s \in \{1, \dots, p\},$$

where the first component is the factor-copy label and the second is the coordinate within that copy. Normalize the corresponding polynomial solutions by

$$(X_{\alpha_r, j}(\lambda))_{n_r^R} = \delta_{rj}, \quad (Y_{i, \beta_s}(\lambda))_{n_s^L} = \delta_{is}.$$

For an integer ℓ , let $[\ell]_M \in \{0, \dots, M-1\}$ denote its residue modulo M .

Theorem 3.15 (Fourier decomposition into cyclic-product recurrences). *Assume that the normalized right and left polynomial solutions described above exist and are unique. Their blocks on factor copy k have the forms*

$$(3.12) \quad X_{k,j}(\lambda) = \lambda^{[k-\alpha_j]_M} V_{k,j}(\lambda^M),$$

$$(3.13) \quad Y_{i,k}(\lambda) = \lambda^{[\beta_i-k]_M} W_{i,k}(\lambda^M).$$

With $x = \lambda^M$, the radial polynomial sequences satisfy

$$(3.14) \quad \tilde{T}_{(a;k)} V_{k,j}(x) = x V_{k,j}(x), \quad W_{i,k}(x) \tilde{T}_{(a;k)} = x W_{i,k}(x).$$

Moreover, the individual refactorized factors connect consecutive radial systems; the copy index $k+1$ is read cyclically:

$$(3.15) \quad \tilde{F}_k^{(a)} V_{k+1,j} = \begin{cases} x V_{k,j}, & k \equiv \alpha_j - 1 \pmod{M}, \\ V_{k,j}, & \text{otherwise,} \end{cases}$$

and

$$(3.16) \quad W_{i,k} \tilde{F}_k^{(a)} = \begin{cases} x W_{i,k+1}, & k \equiv \beta_i \pmod{M}, \\ W_{i,k+1}, & \text{otherwise.} \end{cases}$$

Thus the polynomial solution of the cyclic recurrence of the refactorized factors contains, as its Fourier components, polynomial solutions of all the cyclic products $\tilde{T}_{(a;1)}, \dots, \tilde{T}_{(a;M)}$.

Proof. Let

$$\mathcal{D}_\zeta = \text{diag}(I, \zeta I, \dots, \zeta^{M-1} I), \quad \zeta^M = 1.$$

The proof of Proposition 3.7, applied to the cyclic matrix of the refactorized factors, gives

$$\mathcal{D}_\zeta^{-1} \mathcal{C}_{(a)} \mathcal{D}_\zeta = \zeta \mathcal{C}_{(a)}.$$

Hence $\mathcal{D}_\zeta X(\lambda)$ is a right polynomial solution at $\zeta\lambda$. Restoring the identity normalization at the right boundary and using uniqueness gives

$$X(\zeta\lambda) = \mathcal{D}_\zeta X(\lambda) \text{diag}(\zeta^{1-\alpha_1}, \dots, \zeta^{1-\alpha_q}).$$

Thus every power occurring in $X_{k,j}$ is congruent to $k - \alpha_j$ modulo M , which proves (3.12). The left relation

$$Y(\zeta\lambda) = \text{diag}(\zeta^{\beta_1-1}, \dots, \zeta^{\beta_p-1}) Y(\lambda) \mathcal{D}_\zeta^{-1}$$

proves (3.13).

Substitution in Proposition 3.14 proves (3.14). In the right factor equation, the residues $[k - \alpha_j]_M + 1$ and $[k + 1 - \alpha_j]_M$ agree as integers unless the second residue wraps from M to 0. In that case their powers differ by M , leaving the factor $\lambda^M = x$. This is precisely (3.15). The same calculation in the left factor equation proves (3.16). $\square$

The refactorization is a factorization of $T_{(a)} = D_a C_a$, whereas the original matrix is $T = C_a D_a$. The corresponding polynomial solutions are related without any spectral hypothesis.

Suppose that polynomial vectors $\mathbf{P}_T(x)$ and $\mathbf{Q}_T(x)$ satisfy $T\mathbf{P}_T(x) = x\mathbf{P}_T(x)$, $\mathbf{Q}_T(x)T = x\mathbf{Q}_T(x)$. Define

$$\mathbf{P}^{(a)}(x) := D_a \mathbf{P}_T(x), \quad \mathbf{Q}^{(a)}(x) := \mathbf{Q}_T(x) C_a.$$

These products are well defined coordinatewise because C_a and D_a are banded.

Proposition 3.16 (Polynomial transport to a cyclic product in the constructed range). *The polynomial vectors defined above satisfy $T_{(a)}\mathbf{P}^{(a)} = x\mathbf{P}^{(a)}$, $\mathbf{Q}^{(a)}T_{(a)} = x\mathbf{Q}^{(a)}$.*

Proof. The right recurrence $C_a D_a \mathbf{P}_T = x\mathbf{P}_T$ gives

$$D_a C_a (D_a \mathbf{P}_T) = D_a (C_a D_a) \mathbf{P}_T = x D_a \mathbf{P}_T.$$

This proves the right identity. Similarly,

$$(\mathbf{Q}_T C_a) D_a C_a = \mathbf{Q}_T (C_a D_a) C_a = x \mathbf{Q}_T C_a,$$

which proves the left identity. □

Consequently, $D_a \mathbf{P}_T$ and $\mathbf{Q}_T C_a$ are polynomial solutions for $T_{(a)}$. If their right and left initial-condition blocks are constant and nonsingular, uniqueness of the recurrence problems identifies them, up to constant nonsingular changes of initial conditions, with the radial system at the factor copy for which $\tilde{T}_{(a;1)} = T_{(a)}$. Without this additional condition no constant-gauge identification is asserted. The other radial systems are obtained by the explicit factor connections (3.15) and (3.16). Thus the polynomial solutions of the recurrence $\mathcal{H}_{(a)}$ combine, through their residue classes modulo M , the recurrence solutions of the cyclic products of the refactorization. These identities give the recurrence solutions algebraically. Under the Favard–Christoffel hypotheses, the radial systems with the prescribed admissible initial conditions are the mixed-type orthogonal polynomials for the transform associated with $T_{(a)}$. For $a = p + 1$, this is the familiar transform $x \, d\Psi$. What remains, and is treated in Section 6, is the moment problem that assembles all cyclic components into the full recurrence $\mathcal{H}_{(a)}$ on the star.

For reference, Table 1 collects the notation used for the state heights, the two blockings, and the original and reordered coordinate projections.

TABLE 1. Principal notation for the height and boundary-state constructions.

Symbol	Meaning
d, P, Q, R, w	$d = \gcd(p, q)$, reduced steps $+Q$ and $-P$, $R = P + Q$, and height-block width $w = \max(P, Q)$.
$h(j, n), h^{(a)}(j, n)$	Heights in the prescribed and minimum-height factor orders.
$\mathbf{v}$	Positive eigenvector used only for the diagonal normalization.
$\mathcal{F}_h, \mathcal{E}_h, \mathcal{E}_h^{(a)}$	Labeled states of height h and their coordinate spaces in the prescribed and minimum-height orders.
$\Pi_a, \mathbf{E}_{[r]}$	Projection onto the a -th copy of the original sequence space; projection onto its first r coordinates.
$\mathbf{Q}_r^{(a)}$	The same r -coordinate projection after height reordering.
$\mathbf{J}_j$	Zero-filling injection of a block column into the coordinates of the height block $\mathcal{V}_j$.
$\mathcal{V}_m, \mathcal{V}_{[m, N]}$	Height block of width w ; union of the height blocks $\mathcal{V}_m, \dots, \mathcal{V}_N$.
$\mathcal{H}^{(m)}, \mathcal{H}^{[m, N]}$	Semi-infinite tail beginning at $\mathcal{V}_m$; finite principal matrix on blocks $m, \dots, N$.
$\mathbf{A}_m, \mathbf{B}_m, \mathbf{C}_m, \Phi_m$	Diagonal and rectangular neighboring-block couplings for the width- w blocking; its leading tail-resolvent block.
$\mathcal{U}_m, \mathbf{E}_m, \mathbf{G}_m$	Group of q boundary states over Q heights; its coordinate isometry; its $q \times q$ tail resolvent.
$\mathbf{A}_m, \mathbf{B}_m, \mathbf{D}_m^{(\eta)}$	The $q \times q$ internal, upward, and depth- η return matrices in the boundary-state recursion.

3.4. The matrix continued fraction obtained from height blocks. Before constructing the continued fraction, the correspondence is summarized. One application of the original recurrence matrix T is decomposed into $M = p + q$ elementary factor steps. Each elementary step is one edge of the labeled height graph. The fixed normalizing entries are the upward edges, whereas the coefficient entries are the downward edges and carry the corresponding bidiagonal coefficients. Thus:

- a bidiagonal factor corresponds to an elementary step;
- a coefficient entry corresponds to a downward edge carrying that coefficient as its weight;
- one complete cycle

$$L_1 \cdots L_p U_q \cdots U_1$$

corresponds to one complete sequence of $M = p + q$ elementary steps.

The continued fraction is obtained by applying Schur complements to finite sections of the resolvent matrix of this height graph. Since the jumps are generally $+Q$ and $-P$, the graph is first grouped into finite height blocks; this is what turns the reordered cyclic linearization into a block-tridiagonal matrix. The infinite tail identity is then obtained by coefficientwise stabilization of these finite calculations.

The rational-step height graph has elementary steps $+Q$ and $-P$. Unless $P = Q = 1$, this graph is not tridiagonal with respect to individual heights: one elementary transition may jump over several height levels. Thus, if the basis were ordered by single heights, the corresponding matrix would not be tridiagonal.

To recover a tridiagonal structure, group consecutive heights into blocks large enough so that one elementary transition can move at most to a neighboring block. Put

$$w := \max(P, Q).$$

For $m \in \mathbb{N}_0$, define the m -th height block by

$$\mathcal{V}_m := \{(a, n) : mw \leq h(a, n) < (m+1)w\}.$$

Here

$$|\mathcal{V}_m| := \#\{(a, n) : mw \leq h(a, n) < (m+1)w\}$$

is the number of labeled states in the block, and hence the dimension of its coordinate space. In particular, $I_{|\mathcal{V}_m|}$ denotes the identity matrix of that order. The congruence

$$\omega_a \equiv (a-1)Q \pmod{R}, \quad a \in \{1, \dots, M\},$$

follows from $\omega_a = (a-1)Q$ at the lower-factor positions. For an upper-factor position $a = p+b$, formula (3.1) and $pQ = qP$ give

$$(p+b-1)Q - \omega_{p+b} = (b-1)(P+Q) = (b-1)R,$$

which proves the same congruence. Because $\gcd(Q, R) = 1$, each residue class modulo R occurs for exactly d factor labels. Hence every height supports at most d labeled states. Since a block contains w consecutive heights,

$$0 \leq |\mathcal{V}_m| \leq dw = \max\{p, q\}.$$

In fact, no block is empty. Starting from the state $(1, 0)$ and following only normalizing entries produces a state at every height in $\mathbb{Q}\mathbb{N}_0$. Every interval of $w \geq Q$ consecutive heights contains a multiple of Q , and hence contains at least one labeled state. Therefore

$$1 \leq |\mathcal{V}_m| \leq \max\{p, q\}, \quad m \in \mathbb{N}_0.$$

Moreover, if $mw \geq \max_{1 \leq a \leq M} \omega_a$, then all d labels belonging to each height in the block occur with a nonnegative sequence index, and therefore

$$|\mathcal{V}_m| = \max\{p, q\}.$$

Since $\max_a \omega_a = pQ$ and $w \geq Q$, this equality holds, in particular, for every $m \geq p$. Thus only finitely many initial blocks can have smaller dimension. Grouping the basis of the cyclic linearization in this way makes its matrix block tridiagonal, so the standard finite-dimensional Schur-complement recursion applies to every finite section.

The consecutive-height blocks retain the exact-height order of Definition 3.4. In block notation this order is

$$\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots,$$

with increasing exact height inside each block and with the factor label retained whenever a height carries more than one state. This displays the already defined matrix $\mathcal{H}_{p,q}$ in finite neighboring blocks.

Example 3.17 (The height pattern for $(p, q) = (2, 3)$). For $(p, q) = (2, 3)$, one has

$$d = 1, \quad P = 2, \quad Q = 3, \quad R = 5, \quad w = 3.$$

The first labeled states, ordered by height and then grouped into intervals of width w , are listed in Table 2. The state at height 1 is absent because of the restriction $n \geq 0$. The normalizing edges of height +3 and the

TABLE 2. The first height blocks for $(p, q) = (2, 3)$. The columns give the block number, its height interval, and the corresponding labeled state set.

m	height interval	$\mathcal{V}_m$
0	$\{0, 1, 2\}$	$\{(1, 0), (5, 0)\}$
1	$\{3, 4, 5\}$	$\{(2, 0), (4, 0), (1, 1)\}$
2	$\{6, 7, 8\}$	$\{(3, 0), (5, 1), (2, 1)\}$
3	$\{9, 10, 11\}$	$\{(4, 1), (1, 2), (3, 1)\}$

coefficient edges of height -2 connect only the same or adjacent blocks. Therefore Table 2 already displays the block-tridiagonal structure. The general finite-dimensional Schur-complement recursion below, followed by coefficientwise stabilization, therefore applies to this pair (p, q) .

Figure 2 shows the same construction for $(p, q) = (3, 2)$, where the two groupings used in the article have different widths. The lower row of the figure anticipates the groups of Q consecutive heights used in the q -state recursion of Subsection 4.2.

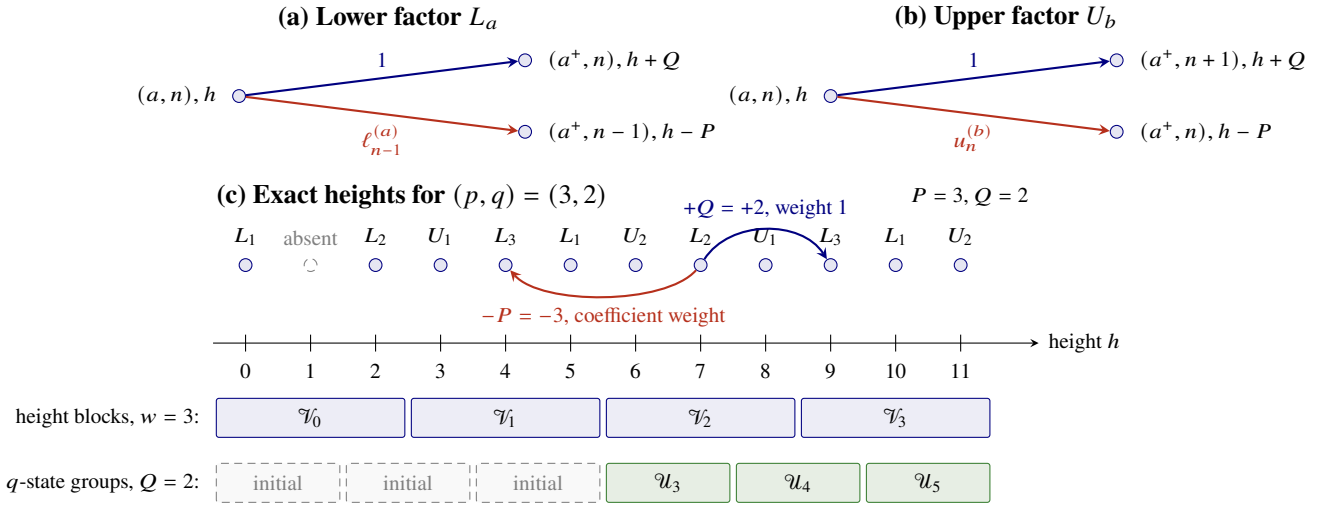


FIGURE 2. States, heights, and the two groupings. Panels (a) and (b) show that every nonzero bidiagonal entry produces one of the two height displacements $+Q$ and $-P$. Panel (c) shows the beginning of the exact-height order for $(p, q) = (3, 2)$. Blocks of width $w = \max(P, Q)$ give the block-tridiagonal matrix used in the Schur continued fraction; groups of width Q , complete from $m = p$ onward, give the q -state recursion.

Lemma 3.18 (Block-tridiagonal height form). *Relative to the decomposition*

$$\text{span } \mathcal{V}_{p,q} = \bigoplus_{m \geq 0} \text{span } \mathcal{V}_m,$$

the matrix $\mathcal{H}$ has block-tridiagonal form

$$\mathcal{H} = \begin{bmatrix} A_0 & B_0 & 0 & \cdots \\ C_0 & A_1 & B_1 & \ddots \\ 0 & C_1 & A_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Every nonzero entry of the blocks is either 1, or one of the individual bidiagonal coefficients $\ell_n^{(a)}, u_n^{(b)}$.

Proof. Every elementary transition changes height by either $+Q$ or $-P$. Since

$$P \leq w, \quad Q \leq w,$$

a transition starting from a state in $\mathcal{V}_m$ can end only in $\mathcal{V}_{m-1}$, $\mathcal{V}_m$, or $\mathcal{V}_{m+1}$. Hence, in the block ordering, only the diagonal, upper, and lower block diagonals can be nonzero. The assertion about the entries follows from Proposition 3.1, because the height reordering does not change the actual coefficients of the elementary transitions. $\square$

The sizes of the three block types will be used repeatedly. With the row-to-column convention fixed above,

$$A_m \in \mathbb{C}^{|\mathcal{V}_m| \times |\mathcal{V}_m|}, \quad B_m \in \mathbb{C}^{|\mathcal{V}_m| \times |\mathcal{V}_{m+1}|}, \quad C_m \in \mathbb{C}^{|\mathcal{V}_{m+1}| \times |\mathcal{V}_m|}.$$

Thus the coupling blocks B_m and C_m are generally rectangular, although every Schur complement below is taken in a square finite-dimensional matrix.

The variable τ marks one elementary factor step of the cyclic linearization. Thus a path of length ℓ in the enlarged graph contributes a factor τ^ℓ . Since one complete multiplication by the original matrix T corresponds to $M = p + q$ elementary factor steps, the variable for the original recurrence matrix will later be

$$t = \tau^M.$$

Equivalently, in the Weyl or Stieltjes transform variable z , use

$$z^{-1} = \tau^M.$$

Thus τ counts individual factor steps. After projection back to the first copy of the sequence space, the variables for the original operator are $t = \tau^M$ and $z = t^{-1}$. Here τ is only the elementary-step variable; it is unrelated to the Christoffel determinants or the integrable tau functions introduced later.

Definition 3.19 (Coordinate submatrices, compressions, and finite sections). Let K be a matrix written in the labeled-state basis. For two sets of states $\mathcal{A}$ and $\mathcal{B}$, finite or infinite, write

$$[K]_{\mathcal{A} \times \mathcal{B}}$$

for the matrix obtained by retaining the rows indexed by $\mathcal{A}$ and the columns indexed by $\mathcal{B}$. If $\mathcal{A} = \mathcal{B} = \mathcal{S}$, the principal compression of K to $\mathcal{S}$ is the principal submatrix $[K]_{\mathcal{S} \times \mathcal{S}}$. Thus a compression simply deletes all rows and columns outside $\mathcal{S}$; the coordinate subspace spanned by $\mathcal{S}$ may therefore fail to be invariant under K .

For the height blocks introduced above, set

$$\mathcal{V}_{\geq m} := \bigcup_{k \geq m} \mathcal{V}_k, \quad \mathcal{V}_{[m, N]} := \bigcup_{k=m}^N \mathcal{V}_k \quad (m \leq N),$$

and use the notation

$$\mathcal{H}^{(m)} := [\mathcal{H}]_{\mathcal{V}_{\geq m} \times \mathcal{V}_{\geq m}}, \quad \mathcal{H}^{[m, N]} := [\mathcal{H}]_{\mathcal{V}_{[m, N]} \times \mathcal{V}_{[m, N]}}.$$

The first matrix is semi-infinite and acts, coordinatewise, on sequences indexed by $\mathcal{V}_{\geq m}$. The second is called the *finite section* from block m through block N : it is simply the finite principal matrix obtained by truncating the tail after block N . It acts on the finite-dimensional coordinate space

$$\mathbb{C}^{\mathcal{V}_{[m, N]}} = \text{span}\{e_{(a, n)} : (a, n) \in \mathcal{V}_{[m, N]}\} = \bigoplus_{k=m}^N \text{span } \mathcal{V}_k, \quad \dim \mathbb{C}^{\mathcal{V}_{[m, N]}} = \sum_{k=m}^N |\mathcal{V}_k|.$$

Consequently, angle brackets always indicate an infinite tail, whereas square brackets with two endpoints indicate the finite principal submatrix that retains the stated range of blocks. In both cases the endpoints refer to height-block indices, not to the original sequence indices. In particular, $\mathcal{H}^{[0, N]}$ retains the blocks $0, \dots, N$. This notation will be used consistently below. If the full matrix carries a family label, the compression is taken after fixing the family, as in $(\mathcal{H}_{p, q}^P)^{(m)}$ or $(\mathcal{H}_{p, q}^J)^{[m, N]}$.

Until boundedness is imposed in Subsection 3.5, every infinite resolvent in this subsection is understood through its formal Neumann series. This is well-defined coefficient by coefficient because each state has only finitely many outgoing edges, so only finitely many paths of a fixed length begin at a given state. Define the initial block of its resolvent by

$$\Phi_m(\tau) := [(I - \tau \mathcal{H}^{(m)})^{-1}]_{\mathcal{V}_m \times \mathcal{V}_m}.$$

This matrix records all paths which start in the block $\mathcal{V}_m$, remain in the tail

$$\mathcal{V}_m \cup \mathcal{V}_{m+1} \cup \dots,$$

and return to $\mathcal{V}_m$. The entries of $\Phi_m(\tau)$ are therefore generating functions for returns between labeled states of the same height block.

For $N \geq m$, define also the resolvent block of the finite section by

$$\Phi_m^{[N]}(\tau) := \left[\left(I - \tau \mathcal{H}^{[m,N]} \right)^{-1} \right]_{\mathcal{V}_m \times \mathcal{V}_m}.$$

The matrix inverted here is square and finite-dimensional. These finite resolvent blocks will be used to derive the recursion for Φ_m , and all Schur complements below are finite-dimensional.

Theorem 3.20 (Matrix continued fraction from height blocks). *For every $m \in \mathbb{N}_0$, the resolvent blocks Φ_m satisfy*

$$\Phi_m(\tau) = \left(I_{|\mathcal{V}_m|} - \tau \mathbf{A}_m - \tau^2 \mathbf{B}_m \Phi_{m+1}(\tau) \mathbf{C}_m \right)^{-1}.$$

Consequently, $\Phi_0(\tau)$ is given by an iterated matrix continued fraction whose coefficients are the individual bidiagonal entries of

$$L_1, \dots, L_p, U_q, \dots, U_1.$$

Proof. Fix $N > m$ and separate the first height block of the finite section from its remaining blocks. With respect to the finite-dimensional decomposition

$$\text{span } \mathcal{V}_m \oplus \bigoplus_{k=m+1}^N \text{span } \mathcal{V}_k,$$

the square matrix $I - \tau \mathcal{H}^{[m,N]}$ has the block form

$$I - \tau \mathcal{H}^{[m,N]} = \begin{bmatrix} I_{|\mathcal{V}_m|} - \tau \mathbf{A}_m & -\tau \mathbf{B}_m \\ -\tau \mathbf{C}_m & I - \tau \mathcal{H}^{[m+1,N]} \end{bmatrix}.$$

Here $\mathbf{A}_m$ describes transitions staying inside $\mathcal{V}_m$, $\mathbf{B}_m$ describes transitions from $\mathcal{V}_m$ to $\mathcal{V}_{m+1}$, and $\mathbf{C}_m$ describes transitions from $\mathcal{V}_{m+1}$ back to $\mathcal{V}_m$.

All four blocks have finite dimensions. The lower-right block has constant term I , so it is invertible as a finite matrix over $\mathbb{C}[[\tau]]$. Applying the ordinary Schur-complement formula gives

$$\Phi_m^{[N]}(\tau) = \left(I_{|\mathcal{V}_m|} - \tau \mathbf{A}_m - \tau^2 \mathbf{B}_m \left[(I - \tau \mathcal{H}^{[m+1,N]})^{-1} \right]_{\mathcal{V}_{m+1} \times \mathcal{V}_{m+1}} \mathbf{C}_m \right)^{-1}.$$

By the definition of the finite resolvent block on the next tail, this is

$$\Phi_m^{[N]}(\tau) = \left(I_{|\mathcal{V}_m|} - \tau \mathbf{A}_m - \tau^2 \mathbf{B}_m \Phi_{m+1}^{[N]}(\tau) \mathbf{C}_m \right)^{-1}.$$

This calculation involves only the finite square matrix $I - \tau \mathcal{H}^{[m,N]}$.

It remains to pass to the infinite tail. The coefficient of τ^j in Φ_m is the sum of all length- j paths that begin and end in $\mathcal{V}_m$ and stay in the tail. Since the height-block matrix is block tridiagonal, a path that reaches the first deleted block $\mathcal{V}_{N+1}$ and returns to $\mathcal{V}_m$ has length at least $2(N - m + 1)$. Consequently,

$$[\tau^j] \Phi_m^{[N]}(\tau) = [\tau^j] \Phi_m(\tau) \quad \text{for } j < 2(N - m + 1).$$

The analogous statement holds for the tail beginning at $\mathcal{V}_{m+1}$. Hence both finite resolvent blocks in the preceding recursion converge coefficientwise to their corresponding formal tail resolvent blocks as $N \rightarrow \infty$. Finally, the coefficient of degree j in the inverse of a matrix power series with constant term I depends only on its coefficients through degree j . Taking the coefficientwise limit in the finite Schur-complement identity therefore gives

$$\Phi_m(\tau) = \left(I_{|\mathcal{V}_m|} - \tau \mathbf{A}_m - \tau^2 \mathbf{B}_m \Phi_{m+1}(\tau) \mathbf{C}_m \right)^{-1}.$$

The factor τ^2 has a simple meaning. To leave the block $\mathcal{V}_m$, enter the next tail, and return to $\mathcal{V}_m$, a path must use one elementary step from $\mathcal{V}_m$ to $\mathcal{V}_{m+1}$, represented by $\tau \mathbf{B}_m$, and one elementary step back from $\mathcal{V}_{m+1}$ to $\mathcal{V}_m$, represented by $\tau \mathbf{C}_m$. The excursions inside the next tail are summed by $\Phi_{m+1}(\tau)$. Thus the product

$$\tau \mathbf{B}_m \Phi_{m+1}(\tau) \tau \mathbf{C}_m = \tau^2 \mathbf{B}_m \Phi_{m+1}(\tau) \mathbf{C}_m$$

is the contribution of all paths that leave $\mathcal{V}_m$, make an arbitrary excursion in the next tail, and return to $\mathcal{V}_m$. $\square$

Remark 3.21 (The variables τ , t , and z). The matrix continued fraction above is written in the elementary-step variable τ . Each application of $\mathcal{H}$ corresponds to one bidiagonal factor.

The original recurrence matrix T , however, is recovered only after one full cycle of M elementary steps. Hence, after projection onto the first copy of the cyclic linearization, only powers τ^{Mn} survive. This is why the moment variable for the original matrix is $t = \tau^M$. In the Weyl matrix one uses the resolvent

$$(zI - T)^{-1} = \frac{1}{z}(I - z^{-1}T)^{-1}.$$

Thus the corresponding relation with the elementary-step variable is $z^{-1} = \tau^M$. The matrix continued fraction is therefore first constructed in the variable τ , and its rectangular projection onto the first copy becomes a function of $t = \tau^M$, or equivalently of z through $z^{-1} = \tau^M$.

Remark 3.22 (The case $q = 1$ and the scalar tails). When $q = 1$, the labeled height graph becomes the ordinary weighted p -Dyck path graph associated with the bidiagonal factors. The direct production graph of T is instead the contracted p -Łukasiewicz graph. The height-block resolvents Φ_m remain matrix-valued when the blocks of width p are retained. What becomes scalar is the finer resolvent obtained by cutting the graph at one height at a time. These single-height tails, denoted by $\widehat{G}_k$ below, satisfy the p -branched Stieltjes recursion; the precise reduction is proved in Subsection 4.1.

Remark 3.23 (Relation with branched continued fractions). The construction above extends the branched continued fractions of the Hessenberg case $q = 1$ to arbitrary (p, q) [43, 46, 32, 9]. For $q = 1$, the recurrence matrix is the scalar lower-Hessenberg production matrix. Its direct paths are p -Łukasiewicz paths and give the branched Jacobi fraction; the additional bidiagonal factorization unfolds them into weighted p -Dyck paths and gives the branched Stieltjes fraction used here. For general (p, q) , the same recurrence matrix T , grouped into blocks of size q , is lower block Hessenberg. Section 5 introduces its block output array and proves the corresponding mixed-type production/output identities, giving a block extension of the scalar formalism.

The production matrix is T , whereas the sparse matrix $\mathcal{H}$ records the individual bidiagonal factors before they are multiplied. The reduced steps $+Q$ and $-P$, together with the labels attached to the cyclic factor positions, prevent a scalar first-return recursion in general. The powers of $\mathcal{H}$ still enumerate the labeled paths, but the first-return generating functions are matrices indexed by boundary states. When T comes from a Gauss–Borel factorization of a mixed-type moment matrix, Theorem 5.1 makes this correspondence precise: two block output matrices recover the inverses of the complete coefficient arrays of the two mixed-type polynomial vectors.

3.5. Finite convergents and denominator factorization. The Schur-complement recursion gives the convergent $\Phi_m^{[N]}$ when the height graph is cut after block N . This subsection determines its denominator factorization and the first coefficient that can be lost at the truncation boundary. The following two subsections use these algebraic results to prove analytic convergence and positivity, and then to formulate the Padé-type approximation, backward evaluation, and residual error estimates. Only the block-tridiagonal structure is used, so the results apply to every factorization considered above.

For $0 \leq m \leq N$, the finite coordinate space is

$$\mathbb{C}^{\mathcal{V}_{[m,N]}} = \bigoplus_{k=m}^N \text{span } \mathcal{V}_k.$$

Thus, by Definition 3.19, $\mathcal{H}^{[m,N]}$ is obtained from $\mathcal{H}$ by retaining exactly the rows and columns in those blocks. The convergent for the tail beginning at block m , truncated after block N , was defined above by

$$\Phi_m^{[N]}(\tau) := \left[\left(I - \tau \mathcal{H}^{[m,N]} \right)^{-1} \right]_{\mathcal{V}_m \times \mathcal{V}_m}.$$

The subscript m records the first retained block, and the bracketed superscript $[N]$ records the last one. The matrix $\Phi_m^{[N]}$ is the $\mathcal{V}_m \times \mathcal{V}_m$ block of the resolvent of the finite matrix $\mathcal{H}^{[m,N]}$. Schur complementation from the last block upwards gives

$$(3.17) \quad \Phi_N^{[N]}(\tau) = (I - \tau \mathbf{A}_N)^{-1},$$

$$(3.18) \quad \Phi_m^{[N]}(\tau) = \left(I - \tau \mathbf{A}_m - \tau^2 \mathbf{B}_m \Phi_{m+1}^{[N]}(\tau) \mathbf{C}_m \right)^{-1}, \quad 0 \leq m < N.$$

Thus $\Phi_0^{[N]}$ is obtained by evaluating the matrix continued fraction from the terminal value (3.17) back to the initial height block.

The matrices inverted during this backward evaluation are the successive Schur-complement pivots. Recording them separately identifies a common denominator of the convergent and relates its possible poles to the spectrum of the finite height section. For fixed $0 \leq m \leq N$, whenever the required inverses exist, define these pivots backwards by

$$(3.19) \quad \mathbf{D}_N^{[N]}(\tau) := I - \tau \mathbf{A}_N,$$

$$(3.20) \quad \mathbf{D}_k^{[N]}(\tau) := I - \tau \mathbf{A}_k - \tau^2 \mathbf{B}_k (\mathbf{D}_{k+1}^{[N]}(\tau))^{-1} \mathbf{C}_k, \quad m \leq k < N.$$

The superscript $[N]$ records that the recursion starts at the terminal block N .

Proposition 3.24 (Schur-complement denominators and determinant factorization). *Fix $0 \leq m \leq N$, and assume that the denominator matrices defined above exist. Then $\Phi_k^{[N]} = (\mathbf{D}_k^{[N]})^{-1}$, and*

$$(3.21) \quad \det(I - \tau \mathcal{H}^{[m,N]}) = \prod_{k=m}^N \det \mathbf{D}_k^{[N]}(\tau).$$

Consequently, every entry of $\Phi_m^{[N]}$ is rational in τ ; its poles are contained among the reciprocals of the nonzero eigenvalues of $\mathcal{H}^{[m,N]}$.

Proof. Eliminate the height blocks in the order $\mathcal{V}_N, \mathcal{V}_{N-1}, \dots, \mathcal{V}_m$. The last diagonal block of $I - \tau \mathcal{H}^{[m,N]}$ is $I - \tau \mathbf{A}_N = \mathbf{D}_N^{[N]}$. After this block has been eliminated, its contribution to the preceding diagonal block is

$$(-\tau \mathbf{B}_{N-1})(\mathbf{D}_N^{[N]})^{-1}(-\tau \mathbf{C}_{N-1}) = \tau^2 \mathbf{B}_{N-1}(\mathbf{D}_N^{[N]})^{-1} \mathbf{C}_{N-1}.$$

The new diagonal block at level $N-1$ is therefore $\mathbf{D}_{N-1}^{[N]}$. Eliminating successively the blocks $\mathcal{V}_{N-1}, \dots, \mathcal{V}_{m+1}$ shows inductively that the pivot block at level k is exactly the matrix $\mathbf{D}_k^{[N]}$ defined in (3.20).

The leading block of the inverse of the remaining tail $\mathcal{V}_k \oplus \dots \oplus \mathcal{V}_N$ is the inverse of its first Schur complement. Hence $\Phi_k^{[N]} = (\mathbf{D}_k^{[N]})^{-1}$. These block eliminations also give the block factorization

$$I - \tau \mathcal{H}^{[m,N]} = \mathcal{F}_{m,N}^{\text{up}}(\tau) \text{diag} \left(\mathbf{D}_m^{[N]}(\tau), \dots, \mathbf{D}_N^{[N]}(\tau) \right) \mathcal{F}_{m,N}^{\text{low}}(\tau),$$

where $\mathcal{F}_{m,N}^{\text{up}}$ and $\mathcal{F}_{m,N}^{\text{low}}$ are respectively upper and lower block unitriangular. Their determinants equal 1. Taking determinants leaves only the determinants of the pivot blocks and therefore proves (3.21). Cramer's rule applied to the finite matrix $I - \tau \mathcal{H}^{[m,N]}$ now shows that every entry of the inverse, and in particular every entry of $\Phi_m^{[N]}$, is rational in τ . Its scalar denominator divides $\det(I - \tau \mathcal{H}^{[m,N]})$. The zeros of this determinant are the reciprocals of the nonzero eigenvalues of the finite height section. An entry may have cancellations between its numerator and denominator, so the set of its poles is contained in, but need not equal, that reciprocal spectrum. $\square$

The determinant factorization describes the rational function $\Phi_m^{[N]}$, but not how accurately it approximates the tail resolvent Φ_m . The path expansion identifies the coefficient at the earliest degree at which a return through the omitted block can occur. That coefficient may vanish because the required block path need not lift to a path between individual height states; the precise structural condition is recorded after the proposition.

Proposition 3.25 (Coefficient at the first possible truncation-error degree). *As a formal power series at the origin,*

$$(3.22) \quad \Phi_m(\tau) - \Phi_m^{[N]}(\tau) = \tau^{2(N-m+1)} \mathbf{B}_m \mathbf{B}_{m+1} \cdots \mathbf{B}_N \mathbf{C}_N \mathbf{C}_{N-1} \cdots \mathbf{C}_m + O\left(\tau^{2(N-m+1)+1}\right), \quad \tau \rightarrow 0.$$

If the displayed block product is nonzero, then $\Phi_m - \Phi_m^{[N]}$ has a zero of exact order $2(N - m + 1)$ at $\tau = 0$.

Proof. The coefficient of τ^k in Φ_m is the $\mathcal{V}_m \times \mathcal{V}_m$ block of $(\mathcal{H}^{(m)})^k$. It is the sum of the weights of all length- k paths that start and finish in $\mathcal{V}_m$ and remain in the tail. The corresponding coefficient of $\Phi_m^{[N]}$ is the same sum restricted to paths that never leave $\mathcal{V}_m \cup \cdots \cup \mathcal{V}_N$.

It follows that a term in the difference must come from a path that visits $\mathcal{V}_{N+1}$. Since $\mathcal{H}$ is block tridiagonal, the block index changes by at most one at each step. The path must therefore make at least $N - m + 1$ upward block transitions to reach $\mathcal{V}_{N+1}$, and at least $N - m + 1$ downward block transitions to return to $\mathcal{V}_m$. No deleted path can have length smaller than $2(N - m + 1)$.

At length $2(N - m + 1)$, there is no time for a horizontal step or a reversal before reaching the omitted block $\mathcal{V}_{N+1}$. Hence there is only one possible sequence of block indices:

$$m, m + 1, \dots, N, N + 1, N, \dots, m + 1, m.$$

With the convention that matrix products are read in the chronological order of the block transitions, the outward segment has weight $\mathbf{B}_m \mathbf{B}_{m+1} \cdots \mathbf{B}_N$, and the returning segment has weight $\mathbf{C}_N \mathbf{C}_{N-1} \cdots \mathbf{C}_m$. Their product is therefore the coefficient displayed in (3.22). Every other deleted path has at least one further step and contributes only to powers $\tau^{2(N-m+1)+1}$ and higher. This proves the expansion. If the displayed product is nonzero, the coefficient of $\tau^{2(N-m+1)}$ cannot vanish, so the difference has a zero of exact order $2(N - m + 1)$ at $\tau = 0$. $\square$

Remark 3.26 (When the first possible omitted coefficient vanishes). Put $s = N - m + 1$. The coefficient displayed in Proposition 3.25 follows the block-index path

$$m, m + 1, \dots, m + s, m + s - 1, \dots, m.$$

Its nonvanishing requires the following structural condition, independently of the values of the coefficient entries:

$$s |P - Q| < w, \quad w = \max(P, Q).$$

Indeed, if the product is nonzero, at least one labeled path follows the displayed sequence of blocks. Write its initial height as $mw + x$, with $0 \leq x < w$. If $P \geq Q$, each of the s rises of size Q must cross one block boundary, which is possible exactly when $x \geq s(P - Q)$. If instead $Q > P$, the rises cross automatically, whereas the s falls of size P cross successively exactly when $x + s(Q - P) < w$. The existence of such an integer x forces the displayed inequality.

Conversely, suppose that all coefficient entries are strictly positive, that $s|P - Q| < w$, and that

$$(m + 1)w > (p - 1)Q.$$

It remains to lift a corresponding reduced-height path to the labeled graph. If $Q \geq P$, choose $x = 0$. The required initial state exists because following normalizing entries from $(1, 0)$ produces a state at every height in $Q\mathbb{N}_0$. If instead $P > Q$, choose $x = P - 1$, so the initial height is $h_0 = (m + 1)P - 1$. Every residue modulo R occurs among the shifts ω_a . A matching shift different from pQ is at most $(p - 1)Q$, and hence at most h_0 by the additional hypothesis. If the only matching shift is pQ , then $h_0 \equiv pQ \pmod{R}$ and $h_0 \geq (p - 1)Q$; since $pQ - (p - 1)Q = Q < R$, these relations also force $h_0 \geq pQ$. Thus the initial height again carries a labeled state.

All outward rises use normalizing entries and therefore exist. The sources of the returning falls lie in the blocks $\mathcal{V}_{m+1}, \dots, \mathcal{V}_{m+s}$, so their heights are at least $(m + 1)w > (p - 1)Q$. A lower-factor state with sequence index $n = 0$ has height $\omega_a = (a - 1)Q \leq (p - 1)Q$; hence no returning fall can be lost through a nonexistent lower-factor subdiagonal entry. Upper-factor falls use diagonal entries and also exist at $n = 0$. Consequently, the reduced-height path lifts to a labeled path, and strict positivity prevents cancellation in the block product. This proves sufficiency. Without the additional inequality, it remains possible for a returning fall to start at a lower-factor state with $n = 0$, and the height condition alone need not be sufficient.

For both $(p, q) = (3, 2)$ and $(p, q) = (2, 3)$, one has $w = 3$ and $|P - Q| = 1$, so the reduced-height condition leaves only $s = 1, 2$ as possible. At $m = 0$, however, the additional inequality above fails. Direct inspection at $m = N = 0$ in both examples shows that every omitted return of length 2 or 3 has zero weight. In block notation, the length-two pattern is $0, 1, 0$. The three possible length-three patterns that visit the omitted block are

$$0, 0, 1, 0, \quad 0, 1, 1, 0, \quad 0, 1, 0, 0.$$

The first and third contain the vanishing length-two return as a subproduct, and direct multiplication shows that the middle pattern also has zero weight. On the other hand, the length-four pattern $0, 1, 2, 1, 0$ has nonzero weight. Equivalently, the displayed block product is zero for $s = 1$ and nonzero for $s = 2$, but the explicit length-three check is needed when the retained section ends at $N = 0$. Thus, when $m = N = 0$, $\Phi_0 - \Phi_0^{[0]}$ has a zero of exact order 4 at $\tau = 0$ in both systems. For $s \geq 3$, the unconditional necessary condition fails, so the displayed coefficient vanishes and the actual first omitted term occurs at a higher degree. Proposition 3.25 identifies the coefficient at the first possible degree and gives exactness when this coefficient is nonzero.

3.6. Analytic convergence and positivity. The preceding proposition is a statement about formal power series. Analytic convergence on the chosen sequence space follows when the height-ordered operator is bounded, equivalently when the scalar entries in the given bidiagonal factorization are uniformly bounded. In the nonnegative case, a positive eigenvector gives a second useful normalization in which the infinity norm is known exactly.

Define the uniform size of the nontrivial factor entries by

$$\Gamma_{\text{fac}} := \sup \left\{ |\ell_n^{(a)}|, |u_n^{(b)}| : n \geq 0, 1 \leq a \leq p, 1 \leq b \leq q \right\} \in [0, \infty].$$

Lemma 3.27 (Boundedness in terms of the bidiagonal coefficients). *The following conditions are equivalent:*

- (i) $\Gamma_{\text{fac}} < \infty$;
- (ii) $\mathcal{C}$ is bounded on $\ell^s(\{1, \dots, M\} \times \mathbb{N}_0)$ for one, and hence for every, $1 \leq s \leq \infty$;
- (iii) $\mathcal{H}$ is bounded on $\ell^s(\mathcal{V}_{p,q})$ for one, and hence for every, $1 \leq s \leq \infty$.

If the bidiagonal factorization is nonnegative, these conditions are also equivalent to boundedness of T on $\ell^s(\mathbb{N}_0)$ for one, and hence for every, $1 \leq s \leq \infty$. When these conditions hold,

$$\Gamma_{\text{fac}} \leq \|\mathcal{H}\|_s = \|\mathcal{C}\|_s \leq 1 + \Gamma_{\text{fac}}, \quad 1 \leq s \leq \infty,$$

and the original recurrence matrix is bounded on ℓ^s , with

$$\|T\|_s \leq \|\mathcal{H}\|_s^M \leq (1 + \Gamma_{\text{fac}})^M.$$

Proof. In the cyclic ordering, each row and each column contains at most two nonzero entries: one has modulus 1, and the other is one of the bidiagonal coefficients. Hence $\|\mathcal{C}\|_\infty \leq 1 + \Gamma_{\text{fac}}$ and $\|\mathcal{C}\|_1 \leq 1 + \Gamma_{\text{fac}}$. Interpolation between these two bounds gives $\|\mathcal{C}\|_s \leq 1 + \Gamma_{\text{fac}}$ for every $1 \leq s \leq \infty$. Height ordering is a coordinate permutation and is therefore an isometry on each of these sequence spaces, so $\|\mathcal{H}\|_s = \|\mathcal{C}\|_s$.

Conversely, every number $\ell_n^{(a)}$ and $u_n^{(b)}$ occurs as an entry of $\mathcal{C}$, and therefore also as an entry of $\mathcal{H}$. Apply the operator to the corresponding coordinate vector. Since that vector has ℓ^s -norm one, the modulus of the selected output coordinate does not exceed the operator norm. Taking the supremum over all factor entries gives $\Gamma_{\text{fac}} \leq \|\mathcal{H}\|_s$. This proves the equivalence and both bounds for the linearization.

Finally, Lemma 2.10 gives $T = \Pi_1 \mathcal{C}^M \Pi_1^\top$. The coordinate projection and embedding have norm one, and hence

$$\|T\|_s \leq \|\mathcal{C}\|_s^M = \|\mathcal{H}\|_s^M.$$

This proves in particular that boundedness of the factors implies boundedness of T , without a sign assumption.

Suppose conversely that the factorization is nonnegative and that T is bounded on one ℓ^s space. In the matrix product defining T , choose the fixed normalizing superdiagonal entry in every upper factor, the entry $\ell_n^{(a)}$ in L_a , and

the fixed normalizing diagonal entry in every other lower factor. This path contributes $\ell_n^{(a)}$ to the entry $T_{n+1,n+q}$. Since all other contributions are nonnegative,

$$0 \leq \ell_n^{(a)} \leq T_{n+1,n+q} \leq \|T\|_s.$$

Similarly, for $n \geq q - b$, choose the diagonal entry $u_n^{(b)}$ in U_b , the fixed normalizing superdiagonal entry in every other upper factor, and the fixed normalizing diagonal entry in every lower factor. This contributes $u_n^{(b)}$ to $T_{n-q+b,n+b-1}$, and hence

$$0 \leq u_n^{(b)} \leq T_{n-q+b,n+b-1} \leq \|T\|_s.$$

For each b , the omitted indices $0 \leq n < q - b$ form a finite set. Consequently all factor coefficients are uniformly bounded, so $\Gamma_{\text{fac}} < \infty$. The equivalence already proved then gives boundedness on every ℓ^s space. $\square$

Thus the boundedness condition used below can be checked before the height blocks are assembled: it is exactly $\Gamma_{\text{fac}} < \infty$. The converse statement involving only the product T is false for signed factorizations, because cancellation in the product can conceal unbounded individual factors. For example, take $p = q = 1$, put

$$\ell_{2k} = k + 1, \quad \ell_{2k+1} = \frac{1}{k + 1}, \quad u_0 = -1, \quad u_n = -\ell_{n-1} \quad (n \geq 1).$$

Then the coefficients of both factors are unbounded, whereas $T = LU$ has unit superdiagonal, diagonal $(-1, 0, 0, \dots)$, and subdiagonal entries $-1, -1, -2, -1, -3/2, -1, -4/3, -1, \dots$; hence T is bounded on every ℓ^s . The continued fraction resolves those individual factors, so its analytic convergence must retain control of them. For a nonnegative bidiagonal factorization, the last part of the lemma shows instead that boundedness may be checked on the given product T itself. The diagonal normalization below gives a second direct route: after normalization the height-ordered matrix is row-stochastic and hence automatically bounded.

Definition 3.28 (Diagonal normalization associated with a positive eigenvector). Let $\mathbf{v}$ be a strictly positive vector satisfying $T\mathbf{v} = \mathbf{v}$. Put

$$\mathbf{v}^{[r]} := F_{r+1} \cdots F_M \mathbf{v}, \quad 0 \leq r < M, \quad \mathbf{v}^{[M]} := \mathbf{v},$$

so that $\mathbf{v}^{[0]} = T\mathbf{v} = \mathbf{v}$. Assume that every intermediate vector $\mathbf{v}^{[r]}$ is strictly positive, and let $\mathcal{D}_r := \text{diag}(\mathbf{v}^{[r]})$. The normalized factors are

$$\widehat{F}_r := \mathcal{D}_{r-1}^{-1} F_r \mathcal{D}_r, \quad 1 \leq r \leq M.$$

The associated normalized cyclic matrix is defined blockwise by

$$(\widehat{\mathcal{C}})_{r,r+1} = \widehat{F}_r \quad (1 \leq r < M), \quad (\widehat{\mathcal{C}})_{M,1} = \widehat{F}_M,$$

with every other block equal to zero. Its height-ordered form is

$$\widehat{\mathcal{H}} := \mathcal{P}_h \widehat{\mathcal{C}} \mathcal{P}_h^\top,$$

where $\mathcal{P}_h$ is the height-ordering permutation introduced above. If instead $T\mathbf{v} = \lambda\mathbf{v}$ with $\lambda > 0$, the same construction is applied to $\lambda^{-1}T$, for example by replacing one factor by its multiple λ^{-1} . This auxiliary rescaling changes the fixed normalization of that one factor, but it does not change its bidiagonal sparsity or any cyclic resolvent identity. The height increments depend only on the positions of the two bidiagonal entries, so the height blocks are unchanged; the rescaled entry is simply retained as an edge weight. The resolvent of T is then recovered from that of $\lambda^{-1}T$ by replacing its moment variable t with λt . If $t = \tau^M$, the corresponding cyclic variable therefore satisfies $\widehat{\tau}^M = \lambda\tau^M$.

Proposition 3.29 (Properties of the diagonal normalization). *Use the notation of Definition 3.28, and assume in addition that all factors $F_1, \dots, F_M$ are nonnegative. Then every $\widehat{F}_r$ is a nonnegative row-stochastic bidiagonal matrix. The normalized matrices are related to the original ones by diagonal conjugation; this is an operator similarity whenever the diagonal maps and their inverses are bounded. Moreover,*

$$\|\widehat{\mathcal{H}}\|_\infty = 1, \quad \|\widehat{\mathcal{H}}\|_2 \leq \sqrt{2}.$$

Proof. The strict positivity of the intermediate vectors makes every $\mathcal{D}_r$ invertible. Since diagonal multiplication by positive matrices preserves entrywise signs, every entry of $\widehat{F}_r$ is nonnegative.

By construction,

$$F_r \mathbf{v}^{[r]} = F_r F_{r+1} \cdots F_M \mathbf{v} = \mathbf{v}^{[r-1]}.$$

Consequently,

$$\widehat{F}_r \mathbf{1} = \mathcal{D}_{r-1}^{-1} F_r \mathcal{D}_r \mathbf{1} = \mathcal{D}_{r-1}^{-1} F_r \mathbf{v}^{[r]} = \mathcal{D}_{r-1}^{-1} \mathbf{v}^{[r-1]} = \mathbf{1}.$$

Hence every row of $\widehat{F}_r$ sums to 1. Diagonal multiplication cannot create new nonzero entries, so $\widehat{F}_r$ is still bidiagonal. This proves that it is a nonnegative row-stochastic bidiagonal factor.

At the r -th position in the factor cycle, the diagonal change of coordinates is $\mathcal{D}_r$. If $\mathcal{D} = \text{diag}(\mathcal{D}_0, \dots, \mathcal{D}_{M-1})$, checking one factor transition at a time gives $\widehat{\mathcal{C}} = \mathcal{D}^{-1} \mathcal{C} \mathcal{D}$. The last-to-first transition uses $\mathcal{D}_M = \mathcal{D}_0$, because $\mathbf{v}^{[M]} = \mathbf{v}^{[0]} = \mathbf{v}$. Height ordering applies the same permutation to the rows and columns and therefore carries this identity to the height-ordered matrices:

$$\widehat{\mathcal{H}} = (\mathcal{P}_h \mathcal{D} \mathcal{P}_h^\top)^{-1} \mathcal{H} (\mathcal{P}_h \mathcal{D} \mathcal{P}_h^\top).$$

It is an operator similarity precisely when $\mathcal{D}$ and $\mathcal{D}^{-1}$ act boundedly on the chosen sequence space.

Every row of the normalized cyclic operator sums to 1, and a permutation preserves row sums. Hence $\|\widehat{\mathcal{H}}\|_\infty = 1$. Every column contains at most two nonzero entries. Each of them lies in $[0, 1]$, since it belongs to a stochastic row, so the column sum is at most 2. Thus $\|\widehat{\mathcal{H}}\|_1 \leq 2$, and the standard interpolation estimate gives

$$\|\widehat{\mathcal{H}}\|_2^2 \leq \|\widehat{\mathcal{H}}\|_1 \|\widehat{\mathcal{H}}\|_\infty \leq 2.$$

□

Remark 3.30 (Finite resolvents under diagonal normalization). Every fixed finite section is related by a finite-dimensional diagonal similarity. Let $\mathcal{D}_h := \mathcal{P}_h \mathcal{D} \mathcal{P}_h^\top$, and let $\mathcal{D}_{h,[m,N]}$ be its restriction to the coordinates in $\mathcal{V}_{[m,N]}$. Then

$$\widehat{\mathcal{H}}^{[m,N]} = \mathcal{D}_{h,[m,N]}^{-1} \mathcal{H}^{[m,N]} \mathcal{D}_{h,[m,N]},$$

and therefore

$$\left(I - \tau \widehat{\mathcal{H}}^{[m,N]} \right)^{-1} = \mathcal{D}_{h,[m,N]}^{-1} \left(I - \tau \mathcal{H}^{[m,N]} \right)^{-1} \mathcal{D}_{h,[m,N]}.$$

Thus, for fixed input and output coordinates, the normalized and unnormalized finite resolvent entries differ by fixed positive diagonal factors, independent of N once those coordinates are retained. The same relation holds coefficientwise for the formal infinite resolvents. Consequently, convergence of normalized finite entries transfers to the corresponding unnormalized entries after this fixed rescaling. A bounded operator similarity on the whole sequence space additionally requires boundedness of $\mathcal{D}_h$ and $\mathcal{D}_h^{-1}$. The Piñeiro experiment below is carried out directly with $\widehat{\mathcal{H}}$.

The preceding lemma translates the operator hypothesis into the scalar data supplied by the factorization. The theorem is stated with the actual operator norm because that gives the sharper estimate; the directly computable bound $\|\mathcal{H}\|_s \leq 1 + \Gamma_{\text{fac}}$ may always be substituted. The shortest-path argument below gives both the exact coefficient range and an error bound given by a geometric series.

Theorem 3.31 (Finite convergents and a norm error bound). *Let $0 \leq m \leq N$. The following identity holds for the formal power series at the origin:*

$$(3.23) \quad \Phi_m(\tau) - \Phi_m^{[N]}(\tau) = \mathcal{O}\left(\tau^{2(N-m+1)}\right) \quad (\tau \rightarrow 0).$$

Equivalently, the two power series have identical coefficients through degree $2(N - m + 1) - 1$.

Suppose in addition that $\mathcal{H}$ defines a bounded operator on $\ell^s(\mathcal{V}_{p,q})$, for some $1 \leq s \leq \infty$. If $\theta := |\tau| \|\mathcal{H}\|_s < 1$, then

$$(3.24) \quad \left\| \Phi_m(\tau) - \Phi_m^{[N]}(\tau) \right\|_s \leq \frac{2\theta^{2(N-m+1)}}{1-\theta}.$$

In particular, for every fixed m , the convergents $\Phi_m^{[N]}$ converge locally uniformly to the tail resolvent Φ_m in $|\tau| < \|\mathcal{H}\|_s^{-1}$. Consequently, a convergence disc expressed only in the entries of the given factorization is

$$|\tau| < \frac{1}{1 + \Gamma_{\text{fac}}}.$$

More precisely, if $\eta := |\tau|(1 + \Gamma_{\text{fac}}) < 1$, then

$$\left\| \Phi_m(\tau) - \Phi_m^{[N]}(\tau) \right\|_s \leq \frac{2\eta^{2(N-m+1)}}{1-\eta}.$$

Proof. Consider first a coefficient of degree k in the two resolvent expansions. It is the corresponding block of $(\mathcal{H}^{(m)})^k$ or of $(\mathcal{H}^{[m,N]})^k$. Since the operator is block tridiagonal, a product contributing to the $\mathcal{V}_m \times \mathcal{V}_m$ block changes the block index by at most one at each multiplication.

A product that is present in the infinite tail but absent from the finite compression must visit $\mathcal{V}_{N+1}$. Starting at $\mathcal{V}_m$, reaching $\mathcal{V}_{N+1}$ requires at least $N - m + 1$ outward block transitions. Returning to $\mathcal{V}_m$ requires at least the same number of inward transitions. Consequently, such a product has length at least $2(N - m + 1)$. The coefficients of all lower degrees therefore agree, which proves (3.23).

For the norm estimate, expand both resolvents in Neumann series. Principal compression does not increase the subordinate norm, so $\|\mathcal{H}^{[m,N]}\|_s \leq \|\mathcal{H}\|_s$. The equality of the coefficients below degree $2(N - m + 1)$ gives

$$\left\| \Phi_m(\tau) - \Phi_m^{[N]}(\tau) \right\|_s \leq 2 \sum_{k=2(N-m+1)}^{\infty} |\tau|^k \|\mathcal{H}\|_s^k = \frac{2\theta^{2(N-m+1)}}{1-\theta}.$$

The bound is uniform when θ ranges over a compact subinterval of $[0, 1)$, and local uniform convergence follows. By Lemma 3.27, $\theta \leq \eta$. Since $x \mapsto 2x^{2(N-m+1)}/(1-x)$ is increasing on $[0, 1)$, replacing θ by η gives the final estimate stated in the theorem. $\square$

The coefficient bound provides a convergence disc directly from the input factorization, but it need not exhaust the resolvent set. Beyond that disc, convergence follows if the inverses of the finite sections are uniformly bounded. The same stability principle also applies to a closed unbounded realization of the height matrix. In that case the finite sequences must be a core, so that vectors in the operator domain and their images can both be approximated by finite sequences. The theorem below assumes uniform stability of the explicit finite matrices assembled from the bidiagonal factors. This is the standard stability condition in the finite-section method for infinite matrices; see [34].

Theorem 3.32 (Convergence under a uniform finite-section resolvent bound). *Fix m and $1 \leq s < \infty$, and put $X_m := \ell^s(\mathcal{V}_{\geq m})$. Let $\mathcal{H}_m$ be a closed, densely defined realization on X_m of the matrix $\mathcal{H}^{(m)}$, and assume that $c_{00}(\mathcal{V}_{\geq m})$ is a core for $\mathcal{H}_m$. Let K be a compact subset of*

$$\Omega_m := \{\tau \in \mathbb{C} : I - \tau \mathcal{H}_m : D(\mathcal{H}_m) \longrightarrow X_m \text{ is bijective with bounded inverse}\}.$$

Suppose that the finite-section inverses are uniformly bounded on K :

$$(3.25) \quad \sup_{\substack{N \geq m \\ \tau \in K}} \left\| \left(I - \tau \mathcal{H}^{[m,N]} \right)^{-1} \right\|_s < \infty.$$

Here the norm in (3.25) is the induced operator norm on $\ell^s(\mathcal{V}_{[m,N]})$. For $\tau \in K$, put

$$\Phi_m(\tau) := \left[(I - \tau \mathcal{H}_m)^{-1} \right]_{\mathcal{V}_m \times \mathcal{V}_m}.$$

Then

$$(3.26) \quad \sup_{\tau \in K} \left\| \Phi_m(\tau) - \Phi_m^{[N]}(\tau) \right\|_s \longrightarrow 0.$$

More generally, for any fixed finite state sets $I, J \subset \mathcal{V}_{\geq m}$, one has, once N contains both sets,

$$(3.27) \quad \sup_{\tau \in K} \left\| \left[\left(I - \tau \mathcal{H}^{[m,N]} \right)^{-1} \right]_{I \times J} - \left[(I - \tau \mathcal{H}_m)^{-1} \right]_{I \times J} \right\|_s \longrightarrow 0.$$

The norm in (3.26) is the induced operator norm on the fixed space $\ell^s(\mathcal{V}_m)$; in (3.27) it is the induced norm from $\ell^s(J)$ to $\ell^s(I)$. Thus $\Phi_m^{[N]} \rightarrow \Phi_m$ locally uniformly on every compact part of Ω_m where the finite-section inverses are uniformly bounded. In particular, the assertion applies to the bounded realization $\mathcal{H}_m = \mathcal{H}^{(m)}$ considered above.

Proof. Let $\mathcal{P}_N : X_m \rightarrow \ell^s(\mathcal{V}_{[m,N]})$ be the coordinate projection, and let $\mathcal{J}_N$ extend a finite sequence by zero. Fix a finitely supported vector u , and set

$$x(\tau) := (I - \tau \mathcal{H}_m)^{-1} u, \quad \tau \in K.$$

The set $\{x(\tau) : \tau \in K\}$ is compact in the graph norm $\|x\|_s + \|\mathcal{H}_m x\|_s$. Indeed, the resolvent is continuous in operator norm on Ω_m , and, for $\tau \neq 0$,

$$\mathcal{H}_m(I - \tau \mathcal{H}_m)^{-1} = \tau^{-1}((I - \tau \mathcal{H}_m)^{-1} - I).$$

If $0 \in K$, the definition of Ω_m forces $D(\mathcal{H}_m) = X_m$; the closed graph theorem then makes $\mathcal{H}_m$ bounded, and the same conclusion follows directly.

Because the finite sequences form a core, a compact set in the graph norm can be approximated uniformly by finitely many finite sequences. Thus, for every $\varepsilon > 0$, there are $y_1, \dots, y_r \in c_{00}(\mathcal{V}_{\geq m})$ such that, for every $\tau \in K$, one of these vectors satisfies

$$\|x(\tau) - y_j\|_s + \|\mathcal{H}_m(x(\tau) - y_j)\|_s < \varepsilon.$$

Choose N large enough that all the y_j are supported in $\mathcal{V}_{[m,N]}$. Since the matrix is banded, increasing N , if necessary, also ensures

$$\mathcal{H}^{[m,N]} y_j = \mathcal{P}_N \mathcal{H}_m y_j \quad (1 \leq j \leq r).$$

Let

$$x_N(\tau) := \left(I - \tau \mathcal{H}^{[m,N]} \right)^{-1} \mathcal{P}_N u.$$

For a vector y_j chosen as above, the two resolvent equations give

$$\left(I - \tau \mathcal{H}^{[m,N]} \right) (x_N(\tau) - y_j) = \mathcal{P}_N (I - \tau \mathcal{H}_m) (x(\tau) - y_j).$$

Put C_K equal to the supremum in (3.25) and $R_K := \sup_{\tau \in K} |\tau|$. It follows that

$$\|x_N(\tau) - y_j\|_s \leq C_K(1 + R_K)\varepsilon.$$

Consequently,

$$\sup_{\tau \in K} \|\mathcal{J}_N x_N(\tau) - x(\tau)\|_s \leq (1 + C_K(1 + R_K))\varepsilon.$$

As ε is arbitrary, the zero-extended finite resolvent columns converge uniformly on K for every finitely supported input vector. Take the inputs successively in the coordinate basis indexed by J , and project the outputs onto the coordinates in I . Since both sets are finite, convergence of these columns is equivalent to convergence in the induced matrix norm. This proves (3.27); taking $I = J = \mathcal{V}_m$ gives (3.26). $\square$

Recall from Definition 3.19 that $\mathcal{V}_{\geq m} = \bigcup_{k \geq m} \mathcal{V}_k$. The sets in this union are pairwise disjoint because they correspond to disjoint height intervals. The argument takes place on $\ell^2(\mathcal{V}_{\geq m})$; its orthonormal basis is formed by the labeled states in the height blocks $\mathcal{V}_m, \mathcal{V}_{m+1}, \dots$. On this space, the numerical range gives a concrete region where the bounds in Theorem 3.32 hold uniformly. For a bounded operator A on this space, its numerical range is

$$W(A) := \{\langle Ax, x \rangle : x \in \ell^2(\mathcal{V}_{\geq m}), \|x\|_2 = 1\}.$$

Corollary 3.33 (A numerical-range convergence domain). *Assume that $\mathcal{H}^{(m)}$ is bounded on $\ell^2(\mathcal{V}_{\geq m})$. Then, for fixed m , the convergents $\Phi_m^{[N]}(\tau)$ converge locally uniformly to $\Phi_m(\tau)$ as $N \rightarrow \infty$ on the set*

$$\tau = 0 \quad \text{or} \quad \tau^{-1} \notin \overline{W(\mathcal{H}^{(m)})}.$$

Proof. The numerical range of every principal compression is contained in the numerical range of the full tail. If a compact set of values $z = \tau^{-1}$ stays at distance $\delta > 0$ from its closure, then the full operator $zI - \mathcal{H}^{(m)}$ and every finite compression are invertible, by the standard numerical-range resolvent estimate, and

$$\left\| (zI - \mathcal{H}^{[m,N]})^{-1} \right\|_2 \leq \frac{1}{\delta}$$

uniformly in N . Since $I - \tau\mathcal{H} = \tau(zI - \mathcal{H})$, for $\tau \neq 0$, it follows that

$$\left\| (I - \tau\mathcal{H}^{[m,N]})^{-1} \right\|_2 \leq \frac{1}{|\tau|\delta}.$$

On a compact subset of the stated domain that does not contain 0, $|\tau|$ is bounded away from zero, so this is the uniform bound required by Theorem 3.32. A neighborhood of $\tau = 0$ is covered directly by Theorem 3.31. $\square$

When the entries are nonnegative, comparison of path weights yields more than norm convergence. It enlarges the basic disc from the operator norm to the spectral radius and gives monotonicity of the convergents together with the M -matrix property of the denominator blocks.

Definition 3.34 (Z -matrices and nonsingular M -matrices). A real square matrix is a Z -matrix if all its off-diagonal entries are nonpositive. It is a nonsingular M -matrix if it is a Z -matrix, is invertible, and its inverse is entrywise nonnegative.

This is the standard inverse-positivity characterization of a nonsingular M -matrix; see [7].

Theorem 3.35 (Convergence and monotonicity for nonnegative coefficients when $|\tau|\rho(\mathcal{H}) < 1$). *Assume that $\mathcal{H}$ is a bounded operator on $\ell^s(\mathcal{V}_{p,q})$, for some $1 \leq s \leq \infty$, and that all its entries are nonnegative. Let $\varrho := \rho(\mathcal{H})$ be its spectral radius on $\ell^s(\mathcal{V}_{p,q})$. If $|\tau|\varrho < 1$, then the convergents $\Phi_m^{[N]}$ converge locally uniformly to Φ_m . More precisely, choose any $\gamma > \varrho$ such that $|\tau|\gamma < 1$, and put*

$$C_\gamma := \sup_{k \geq 0} \frac{\|\mathcal{H}^k\|_s}{\gamma^k} < \infty.$$

Then

$$(3.28) \quad \left\| \Phi_m(\tau) - \Phi_m^{[N]}(\tau) \right\|_s \leq C_\gamma \frac{(|\tau|\gamma)^{2(N-m+1)}}{1 - |\tau|\gamma}.$$

For real $0 \leq \tau < \varrho^{-1}$, every finite convergent is nonnegative and, for fixed m ,

$$(3.29) \quad 0 \leq \Phi_m^{[N]}(\tau) \leq \Phi_m^{[N+1]}(\tau) \leq \Phi_m(\tau)$$

entrywise. In the same real range, the denominator matrices in the backward recursion, defined in (3.19)–(3.20), are nonsingular M -matrices in the sense of Definition 3.34.

Proof. First consider the constant C_γ . By the spectral-radius formula, $\lim_{k \rightarrow \infty} \|\mathcal{H}^k\|_s^{1/k} = \varrho$. Since $\gamma > \varrho$, one has $\|\mathcal{H}^k\|_s \leq \gamma^k$ for all sufficiently large k , up to a fixed multiplicative constant. Enlarging that constant to cover the finitely many remaining powers gives $C_\gamma < \infty$.

Next compare the infinite and finite coefficients. Every entry of a power of a nonnegative matrix is a sum of nonnegative path weights. Every path contained in the finite section is also a path of the infinite tail, with unchanged weight. Hence, after embedding the finite block in the infinite sequence space,

$$0 \leq [(\mathcal{H}^{[m,N]})^k]_{\mathcal{V}_m \times \mathcal{V}_m} \leq [(\mathcal{H}^{(m)})^k]_{\mathcal{V}_m \times \mathcal{V}_m}$$

entrywise. For nonnegative matrices, entrywise domination implies domination of every induced ℓ^s -norm: for an arbitrary vector x , apply the two matrices to $|x|$. The norm of the right-hand block is therefore bounded by $\|\mathcal{H}^k\|_s$. By Proposition 3.25, the difference of the two series has no term below degree $2(N - m + 1)$. Therefore

$$\left\| \Phi_m - \Phi_m^{[N]} \right\|_s \leq \sum_{k=2(N-m+1)}^{\infty} |\tau|^k \|\mathcal{H}^k\|_s \leq C_\gamma \sum_{k=2(N-m+1)}^{\infty} (|\tau|\gamma)^k,$$

which proves (3.28).

For real $\tau \geq 0$, all terms of the finite and infinite Neumann series are nonnegative. Passing from the cut at N to the cut at $N + 1$ retains every old path and adds the paths that visit the new terminal block. Passing to the infinite tail adds all remaining paths. This gives each of the entrywise inequalities in (3.29). The estimate just proved also shows that the increasing limit is Φ_m , locally uniformly for $|\tau|\varrho < 1$.

The matrix $I - \tau\mathcal{H}^{[m,N]}$ is a Z-matrix in the sense of Definition 3.34, and its inverse is a convergent nonnegative Neumann series, because the path comparison above also gives $\rho(\mathcal{H}^{[m,N]}) \leq \varrho$. Indeed, after zero extension, every power of the compression is entrywise dominated by the corresponding power of the full nonnegative operator. The preceding ℓ^s -norm comparison and the spectral-radius formula then give the stated inequality. Thus $\tau\rho(\mathcal{H}^{[m,N]}) < 1$, and the finite resolvent is entrywise nonnegative. This proves that $I - \tau\mathcal{H}^{[m,N]}$ is a nonsingular M -matrix.

For any $k \in \{m, \dots, N\}$, the matrix $D_k^{[N]}$ is the Schur complement obtained after eliminating the blocks $k + 1, \dots, N$. Its off-diagonal entries are nonpositive: the original diagonal block is a Z-matrix and the correction $\tau^2 B_k (D_{k+1}^{[N]})^{-1} C_k$ is nonnegative. Its inverse is $\Phi_k^{[N]}$, the leading block of a nonnegative finite resolvent. Hence $D_k^{[N]}$ is a nonsingular M -matrix for every level k , as asserted. $\square$

Corollary 3.36 (Convergence after row-stochastic normalization). *Under the hypotheses of Proposition 3.29, form the tail resolvents $\widehat{\Phi}_m$ and their finite convergents $\widehat{\Phi}_m^{[N]}$ from $\widehat{\mathcal{H}}$. Then, for $|\tau| < 1$,*

$$(3.30) \quad \left\| \widehat{\Phi}_m(\tau) - \widehat{\Phi}_m^{[N]}(\tau) \right\|_\infty \leq \frac{2|\tau|^{2(N-m+1)}}{1 - |\tau|}.$$

Consequently, for every fixed m , the convergents tend locally uniformly to $\widehat{\Phi}_m$ throughout the unit disc under the stated stochastic-normalization hypotheses. For real $0 \leq \tau < 1$, the convergence is entrywise monotone and every denominator in the backward recursion is a nonsingular M -matrix.

The same normalized operator is bounded on ℓ^2 . Hence the convergence also holds locally uniformly when

$$\tau = 0 \quad \text{or} \quad \tau^{-1} \notin \overline{W(\widehat{\mathcal{H}}^{(m)})}.$$

In this second domain, the uniform bound on all finite-section inverses is a consequence of the numerical-range estimate rather than a separate hypothesis.

Proof. Proposition 3.29 gives $\|\widehat{\mathcal{H}}\|_\infty = 1$. Applying Theorem 3.31 in ℓ^∞ gives (3.30). Moreover, $\widehat{\mathcal{H}}\mathbf{1} = \mathbf{1}$, so its spectral radius on ℓ^∞ is one. The entrywise monotonicity and the M -matrix assertion now follow from Theorem 3.35.

The same proposition gives $\|\widehat{\mathcal{H}}\|_2 \leq \sqrt{2}$. Thus Corollary 3.33 applies and proves the last assertion, including the uniform finite-section resolvent bound used in its proof. $\square$

3.7. Padé-type approximation, backward evaluation, and residual control. The coefficient agreement proved above is now translated into approximation language. The term “Padé-type” is used because the denominator is fixed by the finite height section rather than chosen through a separate left or right matrix-denominator normalization.

Definition 3.37 (Padé-type order). Let $\mathcal{F}(\tau)$ be a matrix formal power series. A rational matrix $\mathcal{R}(\tau)$ is called a Padé-type approximant of order r at the origin if $\mathcal{F}(\tau) - \mathcal{R}(\tau) = O(\tau^{r+1})$ as $\tau \rightarrow 0$. The qualifier “type” records that the denominator is prescribed by the finite height section; no left or right matrix-denominator normalization is imposed. Similarly, if $\mathcal{F}(z) = \sum_{n \geq 0} M_n z^{-n-1}$, a rational matrix $\mathcal{R}(z)$ has moment order r at infinity when $\mathcal{F}(z) - \mathcal{R}(z) = O(z^{-r-1})$ as $z \rightarrow \infty$; equivalently, its first r moments agree with those of $\mathcal{F}$.

Corollary 3.38 (Padé-type approximation by the convergents). *The convergent $\Phi_m^{[N]}$ is a Padé-type approximant of order $2(N - m + 1) - 1$ to Φ_m . A common scalar denominator for its entries is $\det(I - \tau \mathcal{H}^{[m,N]})$, and this denominator factors according to (3.21).*

Proof. The approximation order is Proposition 3.25; the denominator statement follows from Proposition 3.24. $\square$

The block-tridiagonal structure used to obtain the denominator also makes the approximants inexpensive to evaluate: the full finite section is never inverted at once. Instead, the continued fraction requires one bounded-size matrix inversion per height block. Put $w_0 := \max\{p, q\}$, the eventual order of every height block.

Corollary 3.39 (Cost of backward evaluation). *Every height block has cardinality at most $\max\{p, q\}$, with equality after finitely many initial blocks. Consequently, $\Phi_0^{[N]}$ can be evaluated from (3.17)–(3.18) using $N + 1$ inversions of matrices of order at most w_0 . With standard dense matrix algorithms, each inversion requires $O(w_0^3)$ arithmetic operations and $O(w_0^2)$ storage as $w_0 \rightarrow \infty$. Across the $N + 1$ height blocks, the complete backward evaluation requires $O((N + 1)w_0^3)$ operations as $N + w_0 \rightarrow \infty$, and $O(w_0^2)$ working storage as $w_0 \rightarrow \infty$. For fixed p, q , the operation count is linear in N . These estimates assume that the bidiagonal coefficients are generated as needed.*

Proof. The cardinality statement was proved when the blocks $\mathcal{V}_m$ were defined. The operation and storage bounds now follow directly from the backward recursion. $\square$

The estimates obtained from the Neumann series above depend on a norm bound for $\mathcal{H}$ and can be conservative. A practical stopping criterion should instead measure the coupling that remains across the truncation boundary. The finite resolvent column needed for this residual is obtained recursively from level m to level N , after computing the convergents $\Phi_k^{[N]}$ from level N down to level m . For every j , let J_j be the zero-filling injection of $\mathbb{C}^{\mathcal{V}_j}$ into the height-ordered sequence space: $J_j y$ equals y on the coordinates in $\mathcal{V}_j$ and is zero on every other height block. When only the tail beginning at $\mathcal{V}_m$ is considered, the same symbol denotes its restriction to that tail. For $m \leq k \leq N$, define the corresponding finite resolvent column by

$$X_k^{[m,N]}(\tau) := \left[\left(I - \tau \mathcal{H}^{[m,N]} \right)^{-1} \right]_{\mathcal{V}_k \times \mathcal{V}_m}.$$

Proposition 3.40 (A posteriori estimate from the truncation boundary). *Assume that the finite tail matrices*

$$I - \tau \mathcal{H}^{[k,N]}, \quad m \leq k \leq N,$$

are invertible. Then the finite resolvent column defined above can be recovered from the backward recursion without another large inversion:

$$(3.31) \quad X_m^{[m,N]} = \Phi_m^{[N]},$$

$$(3.32) \quad X_{k+1}^{[m,N]} = \tau \Phi_{k+1}^{[N]} C_k X_k^{[m,N]}, \quad m \leq k < N.$$

If, in addition, $I - \tau \mathcal{H}^{(m)}$ is invertible on $\ell^s(\mathcal{V}_{\geq m})$, for some $1 \leq s \leq \infty$, the exact error identity is

$$(3.33) \quad \Phi_m - \Phi_m^{[N]} = \tau \left[\left(I - \tau \mathcal{H}^{(m)} \right)^{-1} \right]_{\mathcal{V}_m \times \mathcal{V}_{N+1}} C_N X_N^{[m,N]}.$$

In particular, if $\theta = |\tau| \|\mathcal{H}\|_s < 1$, then

$$(3.34) \quad \left\| \Phi_m - \Phi_m^{[N]} \right\|_s \leq \frac{|\tau|}{1 - \theta} \left\| C_N X_N^{[m,N]} \right\|_s.$$

Proof. The first block of the finite resolvent column is the leading block of the finite resolvent, and hence it is $\Phi_m^{[N]}$. This proves (3.31).

After the blocks $\mathcal{V}_{k+2}, \dots, \mathcal{V}_N$ have been eliminated, the block row at level $k + 1$ of the finite resolvent equation reads

$$D_{k+1}^{[N]} X_{k+1}^{[m,N]} - \tau C_k X_k^{[m,N]} = 0.$$

Multiplication by $(D_{k+1}^{[N]})^{-1} = \Phi_{k+1}^{[N]}$ gives (3.32). Thus the tail resolvents are computed from level N down to level m , and the resolvent column is then computed from level m up to level N .

Extend the finite resolvent column by zero on $\mathcal{V}_{N+1}, \mathcal{V}_{N+2}, \dots$, and call the extended column $\tilde{X}^{[m,N]}$. Its block equations coincide with the infinite resolvent equation through level N . At level $N+1$, the infinite operator still contains the coupling from $\mathcal{V}_N$, while the extended column has a zero $\mathcal{V}_{N+1}$ -block. Therefore

$$(I - \tau \mathcal{H}^{(m)}) \tilde{X}^{[m,N]} = J_m - \tau J_{N+1} C_N X_N^{[m,N]},$$

where J_j is the zero-filling coordinate injection defined above.

The corresponding column of the infinite resolvent is $X = (I - \tau \mathcal{H}^{(m)})^{-1} J_m$. Subtracting the preceding equation from its resolvent equation and applying the infinite inverse gives

$$X - \tilde{X}^{[m,N]} = \tau (I - \tau \mathcal{H}^{(m)})^{-1} J_{N+1} C_N X_N^{[m,N]}.$$

The $\mathcal{V}_m$ -block of this identity is (3.33).

Finally, the Neumann-series estimate gives

$$\left\| (I - \tau \mathcal{H}^{(m)})^{-1} \right\|_s \leq \frac{1}{1 - \theta},$$

because the norm of the tail compression does not exceed the norm used in the hypothesis. Apply this estimate to the exact identity; the zero-extension maps J_j and the coordinate projections have norm 1. This proves (3.34). Every quantity on the right-hand side is produced by these two recursions and the known norm bound, so the estimate is a computable stopping criterion. $\square$

Classical Pincherle theorems identify the value of a scalar continued fraction through a distinguished solution of its associated recurrence. In the scalar three-term setting, a nonzero solution (y_n) is called *minimal*, or *recessive*, if $y_n/z_n \rightarrow 0$ for one, and hence every, linearly independent solution (z_n) . Matrix extensions usually use invertible square matrices in the corresponding transfer recurrence [1]. The height blocks here may have different sizes and the coupling blocks may be singular. Nonnegativity gives an alternative that does not require those blocks to be inverted: the continued fraction selects the least entrywise nonnegative solution of the recurrence below. For fixed m and every τ for which the tail resolvent exists, write

$$X_k(\tau) := \left[(I - \tau \mathcal{H}^{(m)})^{-1} \right]_{\mathcal{V}_k \times \mathcal{V}_m}, \quad k \geq m.$$

Theorem 3.41 (Least nonnegative solution and a matrix Pincherle principle). *Assume that $\mathcal{H}$ is a nonnegative bounded operator on $\ell^s(\mathcal{V}_{p,q})$, for some $1 \leq s \leq \infty$, and let $0 \leq \tau < \rho(\mathcal{H})^{-1}$, where the spectral radius is taken on this space. Fix m . The blocks defined above have dimensions $X_k(\tau) \in \mathbb{C}^{|\mathcal{V}_k| \times |\mathcal{V}_m|}$, so every product in the recurrence below has the stated rectangular dimensions. Then $(X_k)_{k \geq m}$ is the least entrywise nonnegative solution of*

$$(3.35) \quad (I - \tau A_m) X_m - \tau B_m X_{m+1} = I,$$

$$(3.36) \quad -\tau C_{k-1} X_{k-1} + (I - \tau A_k) X_k - \tau B_k X_{k+1} = 0, \quad k > m.$$

Moreover,

$$(3.37) \quad X_m = \Phi_m, \quad X_{m+1} = \tau \Phi_{m+1} C_m \Phi_m.$$

For each fixed $k \geq m$, the corresponding finite blocks with the terminal condition $X_{N+1}^{[m,N]} = 0$ increase entrywise to X_k as $N \rightarrow \infty$.

Proof. The block rows of

$$(I - \tau \mathcal{H}^{(m)}) \begin{bmatrix} X_m \\ X_{m+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \end{bmatrix}$$

are exactly (3.35) and (3.36). The first identity in (3.37) is the definition of Φ_m . For the second identity, split the tail after $\mathcal{V}_m$. The lower-left block of the inverse of the resulting 2×2 block operator is

$$\tau(I - \tau\mathcal{H}^{(m+1)})^{-1}\mathbf{C}_m\Phi_m.$$

Projecting its output onto the first block of the smaller tail replaces the resolvent by its leading block Φ_{m+1} . Hence $X_{m+1} = \tau\Phi_{m+1}\mathbf{C}_m\Phi_m$.

It remains to prove the least-solution assertion. If $(Y_k)_{k \geq m}$ is any other nonnegative solution, the boundary equation and the recurrence can be written jointly as $Y = J_m + \tau\mathcal{H}^{(m)}Y$, where J_m is the block column with I in its first position. This identity and its finite iterations are legitimate coordinatewise without assuming that the entries of Y form a bounded operator: each row of the banded matrix $\mathcal{H}^{(m)}$ contains only finitely many nonzero entries. Iterating this identity $r + 1$ times gives

$$Y = \sum_{j=0}^r \tau^j (\mathcal{H}^{(m)})^j J_m + \tau^{r+1} (\mathcal{H}^{(m)})^{r+1} Y.$$

The last term is nonnegative, so Y dominates every partial Neumann sum. After zero extension from the tail space to $\ell^s(\mathcal{V}_{p,q})$, every power of $\mathcal{H}^{(m)}$ is entrywise dominated by the corresponding power of $\mathcal{H}$. Monotonicity of the induced ℓ^s -norm under entrywise domination and the spectral-radius formula therefore give $\rho(\mathcal{H}^{(m)}) \leq \rho(\mathcal{H})$. For $\tau = 0$ the assertion is immediate. If $\tau > 0$, choose γ with $\rho(\mathcal{H}) < \gamma < \tau^{-1}$. The spectral-radius formula, applied as in the proof of Theorem 3.35, gives a constant C_γ such that

$$\|(\mathcal{H}^{(m)})^j\|_s \leq C_\gamma \gamma^j, \quad j \in \mathbb{N}_0.$$

Therefore the Neumann series converges in operator norm on the tail space, and its first block column is X . Letting r tend to infinity yields $Y \geq X$, so X is the least nonnegative solution.

Finally, the terminal condition $X_{N+1}^{[m,N]} = 0$ removes exactly the paths that cross the boundary after $\mathcal{V}_N$. The finite resolvent column therefore sums the paths confined to the blocks $m, \dots, N$. Increasing N retains every old path and adds new paths with nonnegative weights. Theorem 3.35 identifies their increasing limit with the infinite resolvent column. This is the matrix Pincherle principle appropriate here: the least nonnegative solution is selected by finite truncations satisfying $X_{N+1}^{[m,N]} = 0$, without requiring the generally rectangular coupling blocks B_k, C_k to be invertible. $\square$

4. BRANCHED AND q -STATE MATRIX CONTINUED FRACTIONS

This section relates the matrix continued fraction of Theorem 3.20 to branched first-return recursions. Subsection 4.1 treats the case $q = 1$: replacing the height blocks by single-height blocks recovers the scalar p -branched Stieltjes fraction with the original bidiagonal entries as its weights. Restricting the $(p + 1)$ -step contraction to the first factor copy gives the direct p -Łukasiewicz production graph of T and hence its branched Jacobi fraction; the remaining factor copies give the analogous graphs of the cyclic products. The scalar Stieltjes fraction appears before this contraction, at the level of the individual factors. Subsection 4.2 keeps instead q boundary states, formed from $Q = q/\gcd(p, q)$ consecutive reduced heights and the $\gcd(p, q)$ factor labels over them. Its main result is a $q \times q$ matrix first-return recursion valid for arbitrary p and q ; the scalar fraction and the two-state fraction are derived below as special cases.

4.1. Reduction to the branched Stieltjes fraction. When $q = 1$, the preceding construction contains the p -branched Stieltjes continued fraction associated with the displayed bidiagonal factorization of the $(p + 2)$ -term Hessenberg recurrence. A general Hessenberg recurrence gives a branched Jacobi fraction; the Stieltjes form used here comes from the additional factorization. The matrix continued fraction of Theorem 3.20 is constructed after grouping heights into blocks of width $w = \max(P, Q)$. When $q = 1$, one has $w = p$. If this block grouping is kept, the Schur complements in every finite truncation are still matrix-valued. The scalar branched fraction is obtained by using the extra simplification specific to $q = 1$: the height itself already determines the factor label, and the Schur-complement decomposition may be refined from height blocks to single-height tails.

Assume $q = 1$. Then

$$d = 1, \quad P = p, \quad Q = 1, \quad R = p + 1.$$

The ordered factors are

$$F_1 = L_1, \dots, F_p = L_p, \quad F_{p+1} = U_1.$$

The height shifts are

$$\omega_1 = 0, \quad \omega_2 = 1, \quad \dots, \quad \omega_{p+1} = p.$$

Thus

$$h(a, n) = (p + 1)n + a - 1, \quad a \in \{1, \dots, p + 1\}.$$

Hence the map

$$(a, n) \mapsto (p + 1)n + a - 1$$

is a bijection from the labeled height states onto $\mathbb{N}_0$. In this case the height serves as the label, since it determines both the factor position and the sequence index.

Define the scalar weights

$$(4.1) \quad \gamma_{(p+1)n} = u_n^{(1)}, \quad \gamma_{(p+1)n+a} = \ell_n^{(a)}, \quad a \in \{1, \dots, p\}, \quad n \in \mathbb{N}_0.$$

Under the height identification, the normalizing entries give rises

$$h \mapsto h + 1,$$

whereas the coefficient entries give falls

$$h + p \mapsto h$$

with weight γ_h . Indeed, for $a \in \{1, \dots, p\}$ and $n \in \mathbb{N}_0$, the coefficient entry of L_a sends $(a, n + 1)$ to $(a + 1, n)$. The source and target heights are

$$(p + 1)(n + 1) + a - 1 = (p + 1)n + a + p$$

and

$$(p + 1)n + a,$$

respectively. Thus this transition is the fall from height $h + p$ to height h , with $h = (p + 1)n + a$, and its weight is $\ell_n^{(a)} = \gamma_h$. Similarly, if $h = (p + 1)n$, the diagonal entry of U_1 sends $(p + 1, n)$ to $(1, n)$, which is the fall from height $h + p$ to height h with weight $u_n^{(1)} = \gamma_h$. Thus the projected graph is exactly the factor-resolved weighted p -Dyck path graph, with rises $+1$ and falls $-p$.

The relation between the scalar tails and the resolvents introduced above is now made explicit. After the identification of labeled states with heights, write $e_k := e_{(a,n)}$ for the unique basis state satisfying $h(a, n) = k$. In the notation of Definition 3.19, define the single-height tail by

$$\mathcal{H}_{p,1}^{(k)} := [\mathcal{H}_{p,1}]_{\{j:j \geq k\} \times \{j:j \geq k\}}.$$

Thus it is obtained by deleting all rows and columns whose height is less than k . Here the angle brackets again indicate an infinite tail; the index is a single height because every height contains one state when $q = 1$. Define, with the inverse again interpreted as a formal Neumann series,

$$(4.2) \quad \widehat{G}_k(\tau) := e_k^\top \left(I - \tau \mathcal{H}_{p,1}^{(k)} \right)^{-1} e_k.$$

Equivalently, $\widehat{G}_k(\tau)$ is the diagonal resolvent entry of the cyclic linearized operator restricted to the tail of heights not below k . It generates precisely the paths that start at height k , end at height k , and never go below k .

Every closed path in this tail contains, for each coefficient fall, exactly p normalizing rises. Hence its length in the elementary-step variable is a multiple of $p + 1$. Therefore there is a unique formal power series $G_k(t)$ such that

$$\widehat{G}_k(\tau) = G_k(\tau^{p+1}).$$

Thus the standard branched-Stieltjes variable is

$$t = \tau^{p+1}.$$

In this normalization, t marks one coefficient fall, equivalently one complete circuit through the $p + 1$ factor positions. On the first factor copy this circuit represents one application of T ; on the other factor copies it represents the corresponding cyclic products.

This definition also clarifies the relation with the block tails $\Phi_m(\tau)$ of Theorem 3.20. For $q = 1$, the height blocks have width p . Under the bijection between labeled states and heights,

$$\mathcal{V}_m \longleftrightarrow \{mp, mp + 1, \dots, mp + p - 1\},$$

with both sides ordered by increasing height. Consequently,

$$G_{mp}(\tau^{p+1}) = (\Phi_m(\tau))_{mp, mp}.$$

For a general height k , however, G_k is not simply a diagonal entry of the block tail beginning at mp , because Φ_m allows paths to move inside the whole block $\mathcal{V}_m$, possibly below k . The scalar branched fraction is obtained by replacing the block tails Φ_m with the single-height tail resolvents (4.2).

Theorem 4.1 (The case $q = 1$ and the branched Stieltjes recursion). *Assume $q = 1$. Then the height model underlying the matrix continued fraction reduces, after regrouping the states into single-height tails, to the scalar factor-resolved p -Dyck path graph. The scalar tail functions satisfy the p -branched Stieltjes recursion*

$$G_k(t) = 1 + \gamma_k t G_k(t) G_{k+1}(t) \cdots G_{k+p}(t),$$

or, equivalently,

$$(4.3) \quad G_k(t) = \frac{1}{1 - \gamma_k t G_{k+1}(t) \cdots G_{k+p}(t)}.$$

In particular, $G_0(t)$ is the p -branched Stieltjes series with weights $\gamma_0, \gamma_1, \dots$, where the weights are given by (4.1).

Proof. The height identification above shows that, when $q = 1$, the labeled height graph is the scalar graph with elementary transitions

$$h \mapsto h + 1, \quad h + p \mapsto h.$$

The first transition has weight 1, while the second has weight γ_h . Therefore this is precisely the factor-resolved weighted p -Dyck path graph. Restricting the $(p + 1)$ -step contraction to the first factor copy gives the p -Łukasiewicz production graph of T ; restriction to any other factor copy gives the production graph of the corresponding cyclic product.

A nonempty path counted by $G_k(t)$ has a unique first return to height k . This return is a fall from height $k + p$. In the part of the path preceding that fall, mark the last rise from each of the heights $k, k + 1, \dots, k + p - 1$. These marked rises occur successively:

$$k \longrightarrow k + 1 \longrightarrow \cdots \longrightarrow k + p.$$

The first fall returning to height k then goes from height $k + p$ to height k . This fall has weight γ_k and contributes one power of t . Because this is a first-return decomposition, the path has no excursion based at k before the first marked rise: any nonempty such excursion would already return to k earlier.

Between two consecutive marked rises the path contains an arbitrary excursion based at the current height, but never below that height. If it went below, it would later have to use another rise from the preceding height, contrary to the choice of the marked rise. After the last marked rise, the path may make an arbitrary excursion based at $k + p$ before the returning fall; this gives $G_{k+p}(t)$ and is not a segment between two marked rises. After the fall from $k + p$ to k , the remainder is an arbitrary excursion based at k , giving $G_k(t)$. Hence the scalar factors which appear are

$$G_{k+1}(t), \quad G_{k+2}(t), \quad \dots, \quad G_{k+p}(t), \quad G_k(t).$$

Since these are scalar series, their order is immaterial. Therefore the first-return decomposition gives

$$G_k(t) = 1 + \gamma_k t G_k(t) G_{k+1}(t) \cdots G_{k+p}(t),$$

where the term 1 corresponds to the empty path. Solving for $G_k(t)$ gives (4.3). $\square$

4.2. A q -state first-return recursion. The reduction in Subsection 4.1 shows that the case $q = 1$ permits a finer grouping than the general height-block matrix continued fraction: after passing to single-height tails, one obtains the scalar p -branched Stieltjes fraction. The corresponding grouping with q boundary states for arbitrary q is described next. The outcome is a matrix-valued first-return recursion, equivalently a matrix-valued branched continued fraction with q boundary states, obtained from $(I - \tau\mathcal{H})^{-1}$. When $Q < P$, it groups Q consecutive heights instead of the P consecutive heights used in Theorem 3.20; when $Q \geq P$, the two groupings coincide once every admissible factor label is present. It is called the q -state recursion because each tail begins with q boundary states.

Recall that

$$d = \gcd(p, q), \quad P = \frac{p}{d}, \quad Q = \frac{q}{d}, \quad R = P + Q.$$

The elementary height steps are $+Q$ for the normalizing entries and $-P$ for the coefficient entries. The height-block continued fraction of Theorem 3.20 groups heights in intervals of width $\max(P, Q)$, in order to obtain a block-tridiagonal matrix. Here the heights are grouped in intervals of width Q . This produces $q = dQ$ boundary states: Q consecutive reduced heights, with d possible factor labels at each height.

For small heights, some labeled states are absent because $n \geq 0$. Choose an index m_* such that every admissible label is present for $m \geq m_*$. From (3.1) and from the lower-side formula $\omega_a = (a - 1)Q$, the largest height shift is

$$\max_{1 \leq a \leq M} \omega_a = pQ = qP.$$

Hence one may take

$$m_* = p.$$

We use this uniform choice throughout. In the case $q = 1$, the height map is already a bijection onto $\mathbb{N}_0$, so one may and will take $m_* = 0$. Indeed, if $m \geq p$ and

$$h \in \{mQ, mQ + 1, \dots, (m + 1)Q - 1\},$$

then $h \geq pQ$. For any factor label a satisfying $h \equiv \omega_a \pmod{R}$, the integer

$$n = \frac{h - \omega_a}{R}$$

is nonnegative because $h \geq \max_a \omega_a$. Thus no factor label allowed by the residue class is lost at the boundary $n = 0$.

It remains to count the labels over each reduced height. Consecutive offsets satisfy

$$\omega_{a+1} \equiv \omega_a + Q \pmod{R},$$

also across the upper-factor part, because $-P \equiv Q \pmod{R}$. Therefore

$$\omega_a \equiv (a - 1)Q \pmod{R}.$$

Since $\gcd(Q, R) = 1$ and $M = dR$, the list of M offsets contains every residue class modulo R exactly d times. Consequently, for every $m \geq m_*$, each reduced height in the interval

$$mQ, mQ + 1, \dots, (m + 1)Q - 1$$

carries exactly d factor labels. From this point on set

$$\mathcal{U}_m := \text{span}\{e_{(a,n)} : mQ \leq h(a,n) < (m + 1)Q\}, \quad m \geq m_*.$$

Then

$$\mathcal{U}_m \simeq \mathbb{C}^Q \otimes \mathbb{C}^d, \quad \dim \mathcal{U}_m = q.$$

Write the reduced-height coordinate as

$$c \in \{0, \dots, Q - 1\}$$

and the remaining coordinate as one of the d factor labels above that height. Thus a boundary state is the pair formed by this coordinate and the corresponding factor label. Let

$$E_m : \mathbb{C}^q \longrightarrow \ell^2(\mathcal{V}_{p,q})$$

be the coordinate isometry whose range is $\mathcal{U}_m$, with the columns ordered by the boundary-state basis just described. Thus

$$E_m^\top E_m = I_q, \quad E_m E_m^\top \text{ is the coordinate projection onto } \mathcal{U}_m.$$

To distinguish this Q -height grouping from the height blocks of width $w = \max\{P, Q\}$ used earlier, define

$$\mathcal{H}^{(m)Q} := [\mathcal{H}]_{\{(a,n):h(a,n) \geq mQ\} \times \{(a,n):h(a,n) \geq mQ\}}.$$

Equivalently, this compression retains the coordinate spaces $\mathcal{U}_m, \mathcal{U}_{m+1}, \dots$. The subscript Q records that the cutoff is the height mQ , rather than the left endpoint mw of the earlier block $\mathcal{V}_m$. The q -state tail resolvent is

$$(4.4) \quad \mathbf{G}_m(\tau) := E_m^\top \left(I - \tau \mathcal{H}^{(m)Q} \right)^{-1} E_m \in \mathbb{C}[[\tau]]^{q \times q}, \quad m \geq m_*.$$

Thus the scalar tails of the case $q = 1$ are replaced by matrix-valued tails with q boundary states.

Write

$$(4.5) \quad P = \kappa Q + \sigma, \quad \kappa \in \mathbb{N}_0, \quad \sigma \in \{0, \dots, Q-1\}.$$

A fall of size $-P$ from reduced-height coordinate c in $\mathcal{U}_{m+\eta}$ lands at height

$$mQ + (\eta - \kappa)Q + c - \sigma.$$

Therefore it can land in $\mathcal{U}_m$ only for

$$\eta = \kappa \quad \text{or} \quad \eta = \kappa + 1.$$

More precisely, a fall of depth κ uses the reduced-height coordinates $c \geq \sigma$, whereas a fall of depth $\kappa + 1$ uses the coordinates $c < \sigma$. If $\sigma = 0$, only the depth κ case occurs. If $\kappa = 0$, the falls of depth zero are internal transitions inside $\mathcal{U}_m$.

Using the source-row and target-column convention fixed above, define

$$(4.6) \quad \mathbf{A}_m := E_m^\top \mathcal{H} E_m, \quad \mathbf{B}_m := E_m^\top \mathcal{H} E_{m+1}, \quad \mathbf{D}_m^{(\eta)} := E_{m+\eta}^\top \mathcal{H} E_m, \quad \eta \in \{\kappa, \kappa + 1\}.$$

All three matrices are square of order q :

$$\mathbf{A}_m, \mathbf{B}_m, \mathbf{D}_m^{(\eta)} \in \mathbb{C}^{q \times q}.$$

Here $\mathbf{A}_m$ contains the one-step transitions that remain inside $\mathcal{U}_m$. The block $\mathbf{B}_m$ contains the normalizing upward transitions from $\mathcal{U}_m$ to $\mathcal{U}_{m+1}$. The block $\mathbf{D}_m^{(\eta)}$ contains the coefficient falls from $\mathcal{U}_{m+\eta}$ back to $\mathcal{U}_m$. When the corresponding subset of boundary states is empty, this block is the zero matrix. The nonzero entries of $\mathbf{B}_m$ are equal to 1 for the canonical factor normalization fixed at the beginning of Subsection 3.1, whereas the nonzero entries of $\mathbf{D}_m^{(\eta)}$ are precisely the bidiagonal coefficients $\ell_n^{(a)}$ and $u_n^{(b)}$ selected by the factor label. After a diagonal or eigenvalue normalization, the same definition applies but the nonzero entries of $\mathbf{B}_m$ need not equal 1.

The diagonal block $\mathbf{A}_m$ can be nonzero only when $\kappa = 0$. To see this, recall that the only transitions inside or below one of the spaces $\mathcal{U}_j$, each formed from Q consecutive heights, which are not normalizing rises are the coefficient falls of size $-P$. If $\kappa \geq 1$, then $P \geq Q$, and every such fall starting in $\mathcal{U}_m$ lands in some $\mathcal{U}_j$ with $j < m$; it cannot remain in $\mathcal{U}_m$. The normalizing transitions go from $\mathcal{U}_m$ to $\mathcal{U}_{m+1}$. Therefore

$$\mathbf{A}_m = 0, \quad \kappa \geq 1.$$

When $\kappa = 0$, the falls with reduced-height coordinate $c \geq \sigma$ have depth zero and are precisely the internal coefficient transitions recorded by $\mathbf{A}_m$.

The next identity is the matrix extension of the scalar branched Stieltjes recursion. For $q = 1$, all its blocks are scalars and the ordered product of the tail matrices $\mathbf{G}_j$ becomes the product of scalar tails in the p -branched continued fraction, after the change of variable $t = \tau^{p+1}$. For $q > 1$, the scalar tails are replaced by $q \times q$ resolvent blocks, and their order in the product is essential. Put

$$\mathcal{J}_+ := \{\eta \in \{\kappa, \kappa + 1\} : \eta \geq 1\},$$

the set of positive return depths.

Theorem 4.2 (The q -state matrix extension of the branched Stieltjes recursion). *For every $m \geq m_*$, the q -state tails (4.4) satisfy the following identity of matrix-valued formal power series:*

$$(4.7) \quad \mathbf{G}_m(\tau) = \left(I_q - \tau \mathbf{A}_m - \sum_{\eta \in \mathcal{I}_+} \tau^{\eta+1} \mathbf{B}_m \mathbf{G}_{m+1}(\tau) \mathbf{B}_{m+1} \mathbf{G}_{m+2}(\tau) \cdots \mathbf{B}_{m+\eta-1} \mathbf{G}_{m+\eta}(\tau) \mathbf{D}_m^{(\eta)} \right)^{-1}.$$

When $\eta = 1$, the alternating product in the summand stops after $\mathbf{B}_m \mathbf{G}_{m+1}(\tau)$; there is no intermediate $\mathbf{B}$ -factor. If $\sigma = 0$, then $\mathbf{D}_m^{(\kappa+1)} = 0$. If $\kappa = 0$, the depth-zero falls are included in $\mathbf{A}_m$, and this is why $0 \notin \mathcal{I}_+$.

Proof. A path counted by $\mathbf{G}_m$ starts in $\mathcal{U}_m$, stays in the tail

$$\mathcal{U}_m \oplus \mathcal{U}_{m+1} \oplus \mathcal{U}_{m+2} \oplus \cdots,$$

and ends in $\mathcal{U}_m$. Decompose such paths into primitive returns, where a primitive return is a nonempty path whose first return to $\mathcal{U}_m$ occurs at its last step.

A primitive return has two possible types. It may be a one-step transition that stays inside $\mathcal{U}_m$, and the corresponding contribution is $\tau \mathbf{A}_m$. Otherwise, it must first leave $\mathcal{U}_m$ by the upward normalizing transition

$$\mathcal{U}_m \longrightarrow \mathcal{U}_{m+1},$$

which contributes $\tau \mathbf{B}_m$. There is no excursion before this first step: an internal one-step transition would already be a return to $\mathcal{U}_m$, while a downward transition would leave the retained tail. Either possibility is incompatible with a primitive return of the second type.

Since the only upward block move is from $\mathcal{U}_j$ to $\mathcal{U}_{j+1}$, a primitive return whose final fall comes from $\mathcal{U}_{m+\eta}$ must pass successively through

$$\mathcal{U}_{m+1}, \mathcal{U}_{m+2}, \dots, \mathcal{U}_{m+\eta}.$$

The initial passage from $\mathcal{U}_m$ is its last one before the final fall, because a primitive return cannot visit $\mathcal{U}_m$ earlier. For each $r \in \{1, \dots, \eta - 1\}$, also mark the last upward passage from $\mathcal{U}_{m+r}$ to $\mathcal{U}_{m+r+1}$ before the final fall. After the marked passage from the preceding block, the path cannot visit $\mathcal{U}_{m+r-1}$ again; otherwise the marked passage from that block would not have been its last one. This remains true when a coefficient fall jumps across several groups. If such a fall lands below $\mathcal{U}_{m+r-1}$, then, because every upward move advances by exactly one group, the path must later cross $\mathcal{U}_{m+r-1} \rightarrow \mathcal{U}_{m+r}$ in order to reach the final falling state, contradicting the choice of the last marked passage. The segment before the next marked passage is therefore an arbitrary return to $\mathcal{U}_{m+r}$ that stays in the tail beginning there. It is summed by $\mathbf{G}_{m+r}(\tau)$.

After the last marked rise into $\mathcal{U}_{m+\eta}$, the segment before the final coefficient fall is an arbitrary return based at $\mathcal{U}_{m+\eta}$ that stays in its tail; it contributes $\mathbf{G}_{m+\eta}(\tau)$. Thus this final segment is accounted for separately rather than as a segment between two marked rises. The final fall contributes $\tau \mathbf{D}_m^{(\eta)}$. This last-exit decomposition is unique. Hence the contribution of all primitive returns whose final fall has depth $\eta \geq 1$ is

$$\tau^{\eta+1} \mathbf{B}_m \mathbf{G}_{m+1}(\tau) \mathbf{B}_{m+1} \mathbf{G}_{m+2}(\tau) \cdots \mathbf{B}_{m+\eta-1} \mathbf{G}_{m+\eta}(\tau) \mathbf{D}_m^{(\eta)}.$$

The exponent $\eta + 1$ counts the η upward normalizing steps and the final coefficient fall.

By (4.5), a single fall can return from $\mathcal{U}_{m+\eta}$ to $\mathcal{U}_m$ only for $\eta = \kappa$ or $\eta = \kappa + 1$. Therefore the generating matrix of primitive nonempty returns is

$$\tau \mathbf{A}_m + \sum_{\eta \in \mathcal{I}_+} \tau^{\eta+1} \mathbf{B}_m \mathbf{G}_{m+1}(\tau) \mathbf{B}_{m+1} \mathbf{G}_{m+2}(\tau) \cdots \mathbf{B}_{m+\eta-1} \mathbf{G}_{m+\eta}(\tau) \mathbf{D}_m^{(\eta)}.$$

An arbitrary return to $\mathcal{U}_m$ is a concatenation of primitive returns. Summing the resulting noncommutative geometric series gives (4.7). $\square$

The next corollary verifies the reduction to the scalar p -branched Stieltjes fraction when $q = 1$. The case $q = 2$ then gives the first matrix-valued extension.

Corollary 4.3 (Recovery of the scalar branched fraction). *For $q = 1$, take $m_* = 0$. Then the q -state recursion in Theorem 4.2 is the scalar p -branched Stieltjes fraction of Theorem 4.1.*

Proof. If $q = 1$, then

$$d = 1, \quad Q = 1, \quad P = p.$$

Hence $\kappa = p$, $\sigma = 0$, and each boundary-state block has dimension one. Moreover $\mathbf{A}_m = 0$, $\mathbf{B}_m = 1$, and $\mathbf{D}_m^{(p)} = \gamma_m$, where γ_m is the weight of the fall $m + p \rightarrow m$. Formula (4.7) becomes

$$\widehat{G}_m(\tau) = \frac{1}{1 - \gamma_m \tau^{p+1} \widehat{G}_{m+1}(\tau) \cdots \widehat{G}_{m+p}(\tau)}.$$

By (4.2), these are precisely the single-height resolvents introduced above. Substituting $\widehat{G}_j(\tau) = G_j(t)$ and setting $t = \tau^{p+1}$ gives exactly the scalar p -branched Stieltjes recursion. $\square$

Corollary 4.4 (The two-state case). *For $q = 2$, Theorem 4.2 gives a 2×2 matrix-valued branched continued fraction with two boundary states. If $p = 2r + 1$, then*

$$d = 1, \quad Q = 2, \quad P = 2r + 1,$$

and the two return depths are r and $r + 1$. If $p = 2r$, then

$$d = 2, \quad Q = 1, \quad P = r,$$

and the two boundary states are the two factor labels over each reduced height; there is a single return depth r . In the endpoint case $p = 1$, which corresponds to $r = 0$ in the first alternative, the depth-zero falls are internal transitions and are included in $\mathbf{A}_m$. Thus only the depth-one term occurs in the sum in (4.7).

Proof. If $p = 2r + 1$, then $\gcd(p, 2) = 1$, so $P = 2r + 1$ and $Q = 2$. The Euclidean division $P = rQ + 1$ gives the two possible depths r and $r + 1$, and each group contains $Q = 2$ reduced heights with one label over each. When $r = 0$, the explanation following (4.6) places the depth-zero falls in $\mathbf{A}_m$, so they do not appear in the sum over positive depths. If $p = 2r$, then $\gcd(p, 2) = 2$, so $P = r$ and $Q = 1$. Each group then contains one reduced height with two labels over it. Since the remainder in the division of P by Q is zero, only the depth r occurs. $\square$

Remark 4.5 (Relation between the q -state and height-block recursions). The grouping into q boundary states and the grouping into height blocks give two first-return descriptions of resolvent blocks of $\mathcal{H}$. The former is proved by the path decomposition in Theorem 4.2; the latter is obtained from the finite-dimensional Schur complements in Theorem 3.20. Theorem 3.20 uses height blocks $\mathcal{V}_m$ large enough to make the operator block tridiagonal for arbitrary (p, q) . Theorem 4.2 uses instead the grouping

$$\mathcal{U}_m \simeq \mathbb{C}^q,$$

which keeps together the q boundary states associated with a group of Q consecutive heights. If $Q < P$, the two constructions group the height states differently. If $Q \geq P$, equivalently $q \geq p$, then $w = Q$, and, for every $m \geq m_*$, the spaces $\mathcal{U}_m$ and $\mathcal{V}_m$ contain exactly the same labeled states. With the same ordering of those states, their coefficient blocks and tail recursions coincide. Thus the boundary-state construction is genuinely different from the height-block construction only when $q < p$; the scalar branched fraction for $q = 1 < p$ is the basic instance.

Remark 4.6 (The q -state tails and the rectangular Weyl matrix). The matrices $\mathbf{G}_m(\tau)$ are principal $q \times q$ resolvent blocks of tails beginning at $m \geq m_*$, where every reduced height carries all d factor labels. The rectangular moment series at the origin is a different projection: it observes the first factor copy through $\mathbf{Q}_q$ and $\mathbf{Q}_p$, and the selected states may lie in the finite initial part below m_* . Hence it is not, in general, one of the matrices $\mathbf{G}_m$.

Remark 5.6 gives an explicit finite-window extraction using the width- w height-block tail Φ_{N+1} . If the remote tail is instead evaluated through the q -state recursion, it must still be coupled to the initial coordinates by a finite Schur-complement calculation and, when $P > Q$, through the finite boundary slab crossed by the downward transitions.

5. RECTANGULAR RESOLVENTS AND WEYL APPROXIMATION

This section answers two approximation-theoretic questions. How can the rectangular moment series and Weyl matrix of T be recovered from the factor-resolved cyclic linearization? And what do the finite matrix continued fractions approximate, both when the PBF is bounded and when it is unbounded? The first question connects the cyclic resolvent with mixed-type multiple orthogonality; the second determines the precise analytic or asymptotic meaning of its finite truncations.

Subsection 5.1 proves that rectangular projections of the cyclic resolvent recover the moment series of T and of all its cyclic Darboux transforms. It also identifies the two block output matrices associated with the mixed-type recurrence and clarifies their relation with the scalar production-matrix formalism.

Subsection 5.2 applies the mixed-type spectral Favard theory developed in [10, 14]. In the bounded setting it identifies the relevant rectangular resolvent block with the Stieltjes transform and transfers the convergence and Padé-type results to finite rectangular approximants. In the unbounded setting the Poincaré moment approximation of the measure-defined Weyl matrix is separated from analytic finite-section convergence: the latter targets the projected resolvent of a specified closed realization and gives the measure-defined Weyl matrix only under an additional spectral identification.

5.1. Rectangular projections of the cyclic resolvent. After projection onto a fixed copy, only complete cycles of length M contribute, and each complete cycle gives one power of the corresponding Darboux transform of T .

For $r \in \mathbb{N}$, let

$$\mathbf{E}_{[r]} := \begin{bmatrix} I_r & 0_{r \times \infty} \end{bmatrix}$$

be the projection onto the first r coordinates of the original sequence space. The rectangular moment series associated with T is the $q \times p$ matrix of formal power series

$$\mathbf{G}_{p,q}(t) := \mathbf{E}_{[q]}(I - tT)^{-1}\mathbf{E}_{[p]}^T = \sum_{n=0}^{\infty} \mathbf{E}_{[q]}T^n\mathbf{E}_{[p]}^T t^n,$$

whose coefficients $\mathbf{E}_{[q]}T^n\mathbf{E}_{[p]}^T$ are the rectangular moments entering the spectral Favard theorem for mixed-type multiple orthogonality. More generally, for the cyclic Darboux transforms $T_{(a)}$, set

$$\mathbf{G}_{p,q}^{(a)}(t) := \mathbf{E}_{[q]}(I - tT_{(a)})^{-1}\mathbf{E}_{[p]}^T, \quad a \in \{1, \dots, M\}.$$

In the production/output formalism of Deutsch–Ferrari–Rinaldi [21], a unit-lower-Hessenberg matrix Π is called the production matrix. Its output matrix is $\mathbb{O}(\Pi)$, with entries

$$\mathbb{O}(\Pi)_{n,k} = (\Pi^n)_{0,k}.$$

Thus $\mathbb{O}(\Pi)_{n,k}$ is the total weight of the n -step paths from state 0 to state k . Since Π is unit lower Hessenberg, $\mathbb{O}(\Pi)_{n,k} = 0$ for $k > n$ and $\mathbb{O}(\Pi)_{n,n} = 1$; hence the output matrix $\mathbb{O}(\Pi)$, not the production matrix Π , is lower unitriangular. In the polynomial setting, the rows of $\mathbb{O}(\Pi)^{-1}$ contain the coefficients of the associated monic polynomial sequence, and multiplication by the variable is represented by Π . Its connection with multiple orthogonality and branched continued fractions is developed in [43, 46].

The same distinction applies in the mixed-type setting. When $q > 1$, the recurrence matrix T is not lower Hessenberg in scalar coordinates, but it is lower block Hessenberg after consecutive coordinates are grouped into blocks of size q . Similarly, T^T is lower block Hessenberg for blocks of size p . Applying the output construction to these two block production matrices gives two output matrices. They will be identified below with $\mathcal{L}^{-1}$ and $\mathcal{U}^{-T}$; the latter are output matrices, not production matrices.

For a semi-infinite matrix S , an integer $r \geq 1$, and a nonsingular $r \times r$ matrix C , define its normalized block output matrix by

$$\mathbb{O}_r(S; C) := \begin{bmatrix} CE_{[r]} \\ CE_{[r]}S \\ CE_{[r]}S^2 \\ \vdots \end{bmatrix}.$$

Its n -th block row, numbered from zero, is $CE_{[r]}S^n$. When $r = 1$ and $C = 1$, this is the usual output matrix of a production matrix: its (n, k) -entry is $(S^n)_{0,k}$. Deleting the first block row of $\mathbb{O}_r(S; C)$ gives $\mathbb{O}_r(S; C)S$, which is the block analogue of the defining row-production identity. For later use, write

$$\chi_{[r]}(x) := [I_r \quad xI_r \quad x^2I_r \quad \cdots]^\top,$$

and let $\Lambda_{[r]}$ be the block shift characterized by $\Lambda_{[r]}\chi_{[r]}(x) = x\chi_{[r]}(x)$.

We use the Gauss–Borel formulation of mixed-type multiple orthogonality developed in [3]. Let μ be a $q \times p$ matrix of measures whose step-line moment matrix has the Gauss–Borel factorization

$$\mathcal{M} = \mathcal{L}^{-1}\mathcal{U}^{-1},$$

where $\mathcal{L}$ is lower unitriangular and $\mathcal{U}$ is upper triangular. Suppose that the associated recurrence matrix has the two representations

$$T = \mathcal{L}\Lambda_{[q]}\mathcal{L}^{-1} = \mathcal{U}^{-1}\Lambda_{[p]}^\top\mathcal{U}.$$

The notation is chosen to agree with the initial-condition matrices introduced in [10, Section 3]. There, ξ collects the initial values of the q right recursion-polynomial sequences, whereas ν collects those of the p left recursion-polynomial sequences. In the present Gauss–Borel realization these matrices are no longer independent data: they are the leading blocks of the two triangular coefficient matrices. Indeed, since

$$\mathbf{B}(x) = \mathcal{L}\chi_{[q]}(x), \quad \mathbf{A}(x) = \chi_{[p]}(x)^\top\mathcal{U},$$

the matrix $\mathbf{B}(x) \in \mathbb{C}[x]^{\mathbb{N}_0 \times q}$ is semi-infinite by q , whereas $\mathbf{A}(x) \in \mathbb{C}[x]^{p \times \mathbb{N}_0}$ is p by semi-infinite. Thus the corresponding initial-condition matrices are

$$\xi := \mathbf{E}_{[q]}\mathcal{L}\mathbf{E}_{[q]}^\top, \quad \nu^\top := \mathbf{E}_{[p]}\mathcal{U}\mathbf{E}_{[p]}^\top.$$

The transpose in the second identity reflects the row convention for the left polynomial family. Consequently, the factors ξ^{-1} and ν^{-1} in the normalized output matrices remove precisely these initial conditions.

Theorem 5.1 (Output matrices of the mixed-type recurrence). *Under the preceding hypotheses,*

$$(5.1) \quad \mathbb{O}_q(T; \xi^{-1}) = \mathcal{L}^{-1}, \quad \mathbb{O}_p(T^\top; \nu^{-1}) = \mathcal{U}^{-1}.$$

Consequently,

$$(5.2) \quad \mathcal{M} = \mathbb{O}_q(T; \xi^{-1})\mathbb{O}_p(T^\top; \nu^{-1})^\top.$$

Moreover, the complete coefficient arrays of the two step-line mixed-type polynomial matrices $\mathbf{B}$ and $\mathbf{A}$ are respectively

$$\mathcal{L} = \mathbb{O}_q(T; \xi^{-1})^{-1}, \quad \mathcal{U}^\top = \mathbb{O}_p(T^\top; \nu^{-1})^{-1}.$$

Proof. The first recurrence representation gives

$$\Lambda_{[q]}\mathcal{L}^{-1} = \mathcal{L}^{-1}T.$$

The leading block of the inverse of a lower triangular matrix is the inverse of its leading block. Hence

$$\mathbf{E}_{[q]}\mathcal{L}^{-1} = \xi^{-1}\mathbf{E}_{[q]}.$$

The n -th block row of $\mathcal{L}^{-1}$ is therefore

$$\mathbf{E}_{[q]}\Lambda_{[q]}^n\mathcal{L}^{-1} = \mathbf{E}_{[q]}\mathcal{L}^{-1}T^n = \xi^{-1}\mathbf{E}_{[q]}T^n.$$

Stacking these block rows proves the first identity in (5.1).

Transposing the second recurrence representation gives

$$\Lambda_{[p]} \mathcal{U}^{-\top} = \mathcal{U}^{-\top} T^\top.$$

Since the leading $p \times p$ block of $\mathcal{U}^{-\top}$ is ν^{-1} , triangularity gives

$$E_{[p]} \mathcal{U}^{-\top} = \nu^{-1} E_{[p]}.$$

Repeating the preceding block-row argument yields

$$E_{[p]} \Lambda_{[p]}^n \mathcal{U}^{-\top} = \nu^{-1} E_{[p]} (T^\top)^n,$$

which proves the second identity. Equation (5.2) is now the Gauss–Borel factorization itself. Finally, $\mathcal{L}$ and $\mathcal{U}^\top$ are the coefficient matrices in the displayed definitions of $\mathbf{B}$ and $\mathbf{A}$, so the last two identities follow by inversion. $\square$

Proposition 5.2 (The recurrence and the initial data determine the moment functional). *Fix a semi-infinite banded matrix T and nonsingular matrices $\xi \in \mathbb{C}^{q \times q}$ and $\nu \in \mathbb{C}^{p \times p}$. There is a unique $q \times p$ matrix $\mathfrak{F}_{T;\xi,\nu}$ of linear functionals on $\mathbb{C}[x]$ whose moments satisfy*

$$(5.3) \quad \langle \mathfrak{F}_{T;\xi,\nu}, x^n \rangle = \xi^{-1} E_{[q]} T^n E_{[p]}^\top \nu^{-\top}, \quad n \in \mathbb{N}_0.$$

Thus the triple (T, ξ, ν) determines the moment functional. If a matrix of measures represents the moments in (5.3), it represents this unique functional. In an indeterminate moment problem, several matrices of measures may represent that functional. Conversely, suppose that the right and left polynomial solutions of the recurrence for T , normalized by ξ and ν , form a weakly normal mixed-type system for a functional $\mathfrak{G}$. Then $\mathfrak{G} = \mathfrak{F}_{T;\xi,\nu}$.

Proof. The monomials form a basis of $\mathbb{C}[x]$, and (5.3) specifies the value of every entry of the functional on each monomial. For $f(x) = \sum_{n=0}^N f_n x^n$, linearity therefore forces

$$\langle \mathfrak{F}_{T;\xi,\nu}, f \rangle = \sum_{n=0}^N f_n \xi^{-1} E_{[q]} T^n E_{[p]}^\top \nu^{-\top},$$

which also proves existence. For the converse, the spectral reconstruction identity for a weakly normal mixed-type recurrence gives

$$\langle \mathfrak{G}, x^n \rangle = \xi^{-1} E_{[q]} T^n E_{[p]}^\top \nu^{-\top}, \quad n \in \mathbb{N}_0;$$

see [10, Proposition 9.4 and Remark 9.5] and its unbounded extension [14]. The two functionals therefore agree on every monomial and hence on $\mathbb{C}[x]$. $\square$

For $q = 1$ with monic normalization, $\xi = 1$, and the first identity in (5.1) is exactly Sokal’s Hessenberg production/output identity [46]: T is the production matrix, $\mathcal{O}_1(T; 1) = \mathcal{L}^{-1}$ is its output matrix, and $\mathcal{L}$ is the coefficient matrix of the monic type-II polynomial sequence. Thus it is T , not either triangular matrix, that plays the role of production matrix. In the mixed-type problem, applying the construction to T , grouped into blocks of size q , and to $T^\top$, grouped into blocks of size p , gives the two output matrices associated with the two polynomial families.

Remark 5.3 (Two finite approximations of the Weyl matrix). The spectral construction in [10, Proposition 4.1] uses polynomial left and right eigenvectors of a principal truncation $T^{[K]}$. Its determinantal kernel reduces to a quotient involving two consecutive characteristic polynomials and controls the Christoffel numbers of that truncation, and hence gives a finite spectral approximation of the Weyl matrix.

The continued-fraction convergents studied here provide a second finite approximation of the same Weyl matrix. They retain the height blocks $0, \dots, N$ of the sparse matrix $\mathcal{H}$, rather than a principal truncation of T . Thus both procedures have the same infinite spectral target and use different finite matrices.

For $a \in \{1, \dots, M\}$ and $r \in \mathbb{N}$, let $Q_r^{(a)}$ denote the coordinate projection onto the states

$$(a, 0), (a, 1), \dots, (a, r-1)$$

in the coordinate ordering used for $\mathcal{H}$, and write $\mathbf{Q}_r := \mathbf{Q}_r^{(1)}$. These coordinates need not be consecutive; $\mathbf{Q}_r^{(a)}$ selects them through their labels, and $(\mathbf{Q}_r^{(a)})^\top$ embeds vectors into those coordinates. Whenever a finite compression contains all the selected states, the same symbol denotes the restriction of this projection to the retained coordinate space, and its transpose denotes the corresponding finite-dimensional embedding. Proposition 3.5 identifies $\mathcal{H}$ with the matrix of $\mathcal{C}$ in this ordering. Hence the projection Π_a onto the a -th copy followed by $\mathbf{E}_{[r]}$ becomes $\mathbf{Q}_r^{(a)}$.

Proposition 5.4 (Rectangular projection of the matrix continued fraction). *Let $t = \tau^M$. Then*

$$(5.4) \quad \mathbf{G}_{p,q}(t) = \mathbf{Q}_q(I - \tau\mathcal{H})^{-1}\mathbf{Q}_p^\top.$$

Thus the rectangular moment series of T is a rectangular block of $(I - \tau\mathcal{H})^{-1}$, whose tail blocks give the matrix continued fraction in Theorem 3.20.

Proof. By Lemma 2.10, $\Pi_1(I - \tau\mathcal{C})^{-1}\Pi_1^\top = (I - \tau^M T)^{-1}$. Taking the first q rows and the first p columns, and reading the coordinate projections in the height-ordered basis, gives (5.4) with $t = \tau^M$. $\square$

The first-copy projection recovers the moment series of the original recurrence matrix. Observing another position in the factor cycle gives the moment series of the corresponding cyclic Darboux transform, with the partial product between the two factor positions appearing explicitly in the rectangular projected resolvent blocks.

Corollary 5.5 (Rectangular moment series for the cyclic Darboux transforms). *Let $a, b \in \{1, \dots, M\}$, and let $s(a, b)$ and $F_{a \rightarrow b}$ be as in Proposition 2.11. Then, with $t = \tau^M$, for $r, s \in \mathbb{N}$,*

$$\begin{aligned} \mathbf{Q}_r^{(a)}(I - \tau\mathcal{H})^{-1}(\mathbf{Q}_s^{(b)})^\top &= \tau^{s(a,b)}\mathbf{E}_{[r]}F_{a \rightarrow b}(I - tT_{(b)})^{-1}\mathbf{E}_{[s]}^\top \\ &= \tau^{s(a,b)}\mathbf{E}_{[r]}(I - tT_{(a)})^{-1}F_{a \rightarrow b}\mathbf{E}_{[s]}^\top. \end{aligned}$$

In particular, for $a = b$, $r = q$, $s = p$,

$$\mathbf{Q}_q^{(a)}(I - \tau\mathcal{H})^{-1}(\mathbf{Q}_p^{(a)})^\top = \mathbf{G}_{p,q}^{(a)}(t).$$

Thus the cyclic resolvent $(I - \tau\mathcal{H})^{-1}$ contains the rectangular moment series of all cyclic Darboux transforms $T_{(1)}, \dots, T_{(M)}$, together with the partial products $F_{a \rightarrow b}$ connecting two factor positions.

Proof. Apply Proposition 2.11 and take the first r rows and first s columns; the two forms correspond to (2.9) and (2.10). $\square$

Remark 5.6 (Coupling the tail to the initial height blocks). The coordinate projections above generally involve initial states distributed among several height blocks. Use the state set $\mathcal{V}_{[0,N]} = \mathcal{V}_0 \cup \dots \cup \mathcal{V}_N$, with N large enough so that the states selected by $\mathbf{Q}_r^{(a)}$ and $\mathbf{Q}_s^{(b)}$ lie in $\mathcal{V}_{[0,N]}$. For $K > N$, apply finite Schur complements to $I - \tau\mathcal{H}^{[0,K]}$, eliminating the blocks $\mathcal{V}_K, \dots, \mathcal{V}_{N+1}$. The corresponding projected block stabilizes coefficientwise as $K \rightarrow \infty$, and its limit is a submatrix of

$$\left(I_{|\mathcal{V}_{[0,N]}|} - \tau[\mathcal{H}]_{\mathcal{V}_{[0,N]} \times \mathcal{V}_{[0,N]}} - \tau^2[\mathcal{H}]_{\mathcal{V}_{[0,N]} \times \mathcal{V}_{N+1}} \Phi_{N+1}(\tau)[\mathcal{H}]_{\mathcal{V}_{N+1} \times \mathcal{V}_{[0,N]}} \right)^{-1}.$$

Here Φ_{N+1} is the coefficientwise limit of the finite terminal blocks $\Phi_{N+1}^{[K]}$. Thus the blocks beyond $\mathcal{V}_{[0,N]}$ are represented by Φ_{N+1} , while the initial blocks determine which cyclic Darboux transform and which initial coordinates are observed. In particular, when the q -state tails beginning at $m \geq m_*$ from Subsection 4.2 are used, these initial blocks connect the resolvent blocks $\mathbf{G}_m$ to the rectangular projection at the origin.

5.2. Spectral representation and convergence of the rectangular approximants. We first treat the bounded case, in which the operator resolvent can be identified directly with a Stieltjes transform. By Lemma 3.27, boundedness of $\mathcal{H}$ is equivalent to uniform boundedness of the scalar bidiagonal coefficients. For a nonnegative factorization, it is also equivalent to boundedness of T . The unbounded case, where the Weyl matrix is defined from the Favard measure rather than from an operator resolvent, is treated at the end of the subsection.

Let $\xi \in \mathbb{C}^{q \times q}$ and $\nu \in \mathbb{C}^{p \times p}$ be nonsingular. Whenever T is a bounded operator, define its normalized rectangular Weyl matrix by

$$(5.5) \quad S(z) := \xi^{-1} E_{[q]} (zI - T)^{-1} E_{[p]}^T \nu^{-T}, \quad z \in \rho(T).$$

Theorem 5.7 (Factor-resolved formula for the rectangular Weyl matrix). *Assume that $\mathcal{H}$ is bounded on $\ell^2(\mathcal{V}_{p,q})$. If $z^{-1} = \tau^M$ and $|\tau| < \|\mathcal{H}\|_2^{-1}$, then*

$$(5.6) \quad S(z) = \frac{1}{z} \xi^{-1} Q_q (I - \tau \mathcal{H})^{-1} Q_p^T \nu^{-T}.$$

The right-hand side is independent of the choice of the M -th root τ .

Proof. Lemma 3.27 implies that T is bounded, and T is a coordinate compression of $\mathcal{H}^M$. Hence

$$\|T\|_2 \leq \|\mathcal{H}\|_2^M < |z|,$$

so $z \in \rho(T)$ and (5.5) is defined. The Neumann series for the resolvent of $\mathcal{H}$ converges in operator norm. Proposition 5.4 shows that the projected powers vanish unless their degree is divisible by M , and that the surviving powers satisfy

$$Q_q \mathcal{H}^{Mn} Q_p^T = E_{[q]} T^n E_{[p]}^T.$$

Thus Proposition 5.4, with $z^{-1} = \tau^M$, gives

$$E_{[q]} (zI - T)^{-1} E_{[p]}^T = \frac{1}{z} Q_q (I - \tau \mathcal{H})^{-1} Q_p^T.$$

Multiplication by ξ^{-1} and ν^{-T} proves (5.6). The projected Neumann series contains only powers τ^{Mn} , so its value does not depend on the chosen M -th root of z^{-1} . $\square$

By Theorem 3.20, the projected resolvent in (5.6) is evaluated by the height-block matrix continued fraction. Hence that continued fraction represents the rectangular Weyl matrix S .

We next identify S with a matrix Stieltjes transform. This step uses only the following moment identity from mixed-type spectral theory. Suppose that a compactly supported $q \times p$ matrix of finite measures $d\Psi$ satisfies

$$(5.7) \quad \xi^{-1} E_{[q]} T^n E_{[p]}^T \nu^{-T} = \int x^n d\Psi(x), \quad n \in \mathbb{N}_0.$$

Sufficient Favard hypotheses for the existence of $d\Psi$, ξ , and ν are given in [10, Proposition 9.4 and Remark 9.5].

Proposition 5.8 (Stieltjes representation of the Weyl matrix). *Assume that T is bounded and that $d\Psi$ satisfies (5.7). Then*

$$S(z) = \int \frac{d\Psi(x)}{z - x}$$

entry by entry on the unbounded connected component of the common analytic domain.

Proof. Choose R_Ψ such that every entry of $d\Psi$ is supported in $\{x : |x| \leq R_\Psi\}$. For $|z| > \max\{\|T\|_2, R_\Psi\}$, the Neumann series and (5.7) give, entry by entry,

$$S(z) = \sum_{n=0}^{\infty} \frac{\xi^{-1} E_{[q]} T^n E_{[p]}^T \nu^{-T}}{z^{n+1}} = \int \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}} d\Psi(x) = \int \frac{d\Psi(x)}{z - x}.$$

Compact support justifies the interchange of sum and integral. Analytic continuation proves the stated identity on the unbounded component. $\square$

Remark 5.9 (Domains of analytic convergence). The norm disc in Theorem 5.7 is only a sufficient domain. Theorem 3.31 gives locally uniform convergence of the finite tails there, with the estimate (3.24). Theorem 3.32 extends convergence to compact subsets of the resolvent set on which the finite-truncation inverses are uniformly bounded, and Corollary 3.33 gives a domain in terms of the numerical range. When $\mathcal{H}$ is nonnegative, Theorem 3.35 gives entrywise monotone convergence on

$$0 \leq \tau < \rho(\mathcal{H})^{-1}.$$

The preceding Weyl matrix is naturally the radial form of a Cauchy transform on a star. The cyclic linearization gives this representation directly from the radial moment series. The average over the rays below is the rotationally invariant discrete Fourier projection described after Proposition 3.7. Recall that $\zeta_M = \exp(2\pi i/M)$, and put

$$\Sigma_M := \bigcup_{j=0}^{M-1} \zeta_M^j [0, \infty).$$

When the diagonal normalization of Definition 3.28 is used, let $\mathcal{D}_h = \mathcal{P}_h \mathcal{D} \mathcal{P}_h^\top$ and define the finite diagonal matrices on the selected coordinates by

$$\mathcal{D}_r^{(a)} := \mathbf{Q}_r^{(a)} \mathcal{D}_h (\mathbf{Q}_r^{(a)})^\top.$$

For every finite truncation containing these coordinates, diagonal conjugation gives the exact identity

$$(5.8) \quad \mathbf{Q}_q^{(a)} (I - \tau \mathcal{H}^{[0, N]})^{-1} (\mathbf{Q}_p^{(a)})^\top = \mathcal{D}_q^{(a)} \mathbf{Q}_q^{(a)} (I - \tau \widehat{\mathcal{H}}^{[0, N]})^{-1} (\mathbf{Q}_p^{(a)})^\top (\mathcal{D}_p^{(a)})^{-1}.$$

Thus diagonal normalization changes a selected resolvent block only by the two explicitly displayed coordinate factors.

Definition 5.10 (Rotational lift of the matrix of measures). Assume that $d\Psi$ is supported on $[0, \infty)$. Its rotational lift to Σ_M is the $q \times p$ matrix of measures $d\widehat{\Psi}$ defined, entry by entry, by

$$(5.9) \quad \int_{\Sigma_M} f(\lambda) d\widehat{\Psi}(\lambda) := \frac{1}{M} \sum_{j=0}^{M-1} \int_{[0, \infty)} f(\zeta_M^j x^{1/M}) d\Psi(x)$$

for every continuous compactly supported scalar function f on Σ_M . Here $x^{1/M}$ denotes the nonnegative root.

For the measure $d\Psi$ and its rotational lift, write

$$\mathbf{S}_\Psi(z) := \int_{[0, \infty)} \frac{d\Psi(x)}{z - x}, \quad \widehat{\mathbf{S}}(\lambda) := \int_{\Sigma_M} \frac{d\widehat{\Psi}(\zeta)}{\lambda - \zeta}.$$

Theorem 5.11 (Rotational lift of the Weyl matrix). Let $d\Psi$ be a $q \times p$ matrix of finite measures supported on $[0, \infty)$, with all moments finite, and let $d\widehat{\Psi}$ be its rotational lift. Then

$$(5.10) \quad \widehat{\mathbf{S}}(\lambda) = \lambda^{M-1} \mathbf{S}_\Psi(\lambda^M), \quad \lambda \in \mathbb{C} \setminus \Sigma_M.$$

Moreover,

$$(5.11) \quad \int_{\Sigma_M} \zeta^n d\widehat{\Psi}(\zeta) = \begin{cases} \int_{[0, \infty)} x^k d\Psi(x), & n = Mk, \\ 0, & M \nmid n. \end{cases}$$

Proof. For $r \geq 0$ and $\lambda^M \neq r^M$, the partial-fraction identity

$$\frac{1}{M} \sum_{j=0}^{M-1} \frac{1}{\lambda - \zeta_M^j r} = \frac{\lambda^{M-1}}{\lambda^M - r^M}$$

follows by taking the logarithmic derivative of $\lambda^M - r^M$. Apply it entry by entry in (5.9), with $r = x^{1/M}$. This proves (5.10). Finally,

$$\frac{1}{M} \sum_{j=0}^{M-1} \zeta_M^{jn} = \begin{cases} 1, & M \mid n, \\ 0, & M \nmid n, \end{cases}$$

and substitution in the definition of the lifted measure proves (5.11). $\square$

Theorem 5.11 applies to finite measures with all moments. The following operator-resolvent representation adds the boundedness hypothesis.

Proposition 5.12 (Resolvent representation of the star Weyl matrix). *Assume that $\mathcal{H}$ is bounded, that $d\Psi$ is compactly supported, and that (5.7) holds. For all sufficiently large $|\lambda|$,*

$$(5.12) \quad \widehat{S}(\lambda) = \xi^{-1} E_{[q]} \Pi_1 (\lambda I - \mathcal{C})^{-1} \Pi_1^T E_{[p]}^T v^{-T}$$

$$(5.13) \quad = \frac{1}{\lambda} \xi^{-1} Q_q (I - \lambda^{-1} \mathcal{H})^{-1} Q_p^T v^{-T}.$$

Proof. Proposition 5.8 identifies S_Ψ with S , and Theorem 5.7 represents S by the projected resolvent of $\mathcal{H}$. It remains to use

$$\Pi_1 (\lambda I - \mathcal{C})^{-1} \Pi_1^T = \lambda^{M-1} (\lambda^M I - T)^{-1},$$

which follows from Lemma 2.10. Taking the first q rows and p columns gives (5.12); conjugating by the height permutation gives (5.13). $\square$

Since all three members of (5.12)–(5.13) are analytic, the equalities extend from large $|\lambda|$ to the unbounded connected component of their common domain. In particular, the factor-resolved continued fraction represents the rotationally symmetric $q \times p$ Weyl matrix without choosing an M -th root in the λ -variable.

Remark 5.13 (Diagonal normalization and spectral identification). For every finite truncation, (5.8) gives the exact rescaling of the projected resolvent. If the normalized finite resolvents converge, the same identity gives a limit for the original projected entries. A specified operator realization identifies this limit with its projected resolvent. A representing measure identifies the same limit with its Cauchy transform only when the corresponding resolvent–Cauchy-transform equality has been established separately. Stochastic normalization supplies convergence under its stated hypotheses.

6. MOMENT SYSTEMS FOR THE RECURRENCE $\mathcal{H}_{(a)}$ AND CYCLIC CHRISTOFFEL FAMILIES

Fix $a \in \mathcal{A}_{\text{cyc}}$. The terminal lower–upper PBF of $T_{(a)}$ supplies its Favard data; when the prescribed factors are Christoffel connection matrices, these are the transformed data of Theorem 6.24, including $T_{(p+1)} = BA$ with measure $x d\Psi$. The minimum-height factorization of the same product constructs $\mathcal{H}_{(a)}$. This section builds its moment system, relates its polynomial solutions to the cyclic-product recurrences, and derives its radial measures and finite star spectra. The common product and projected resolvent link the two factorizations; the moment functional of $\mathcal{H}_{(a)}$ is constructed from its powers.

The moments of the system associated with $\mathcal{H}_{(a)}$ are defined directly from powers of the matrix $\mathcal{H}_{(a)}$. Its rotational symmetry separates them into congruence classes modulo $M = p + q$. The resulting moment matrix has an explicit triangular factorization and the matrix $\mathcal{H}_{(a)}$ is exactly its recurrence matrix. The nonzero moments in each congruence class form a positive Stieltjes moment sequence on the radial variable. For fixed compactly supported radial representing measures, the atom-at-zero and negative fractional-moment conditions characterize, entry by entry, representation of the corresponding Fourier components by finite complex measures supported on the corresponding compact M -ray star. In parallel, the cyclic products are identified with the recurrence matrices along their cyclic orbit. The Christoffel description is developed at three levels. First, the PBF itself determines matrix Christoffel transformations adapted to its individual factors and to the initial conditions chosen at consecutive cyclic products. Second, the fixed multipliers $K_q(x)$ and $K_p(x)^T$ give a more explicit rectangle when its two one-sided

boundary chains are weakly normal. Third, total positivity of the scalar moment matrix makes this boundary normality automatic.

Subsection 6.1 proves weak normality at every scalar order in the coprime case. In general, grouping the $d = \gcd(p, q)$ states of each height gives a block moment matrix with invertible triangular factors whose recurrence is exactly $\mathcal{H}_{(a)}$. It also locates the spectra of the finite truncations of $\mathcal{H}_{(a)}$ on the $p + q$ rays. For fixed compactly supported radial representing measures, it records the criterion for lifting the algebraic moment functionals to finite complex measures supported on the corresponding compact star. Strong normality additionally requires every nontrivial component to attain its prescribed maximal degree and is established under the separate maximal-degree hypotheses stated in the individual results and applications. Subsection 6.2 constructs the factor-adapted Christoffel transformations along the cyclic PBF orbit and explains the role of triangular changes of initial conditions. It proves that a complete factor circuit gives multiplication by x , up to a constant triangular normalization at the endpoint, and records the direct transport of the polynomial recurrence solutions through the bidiagonal factors. Subsection 6.3 then studies the special rectangle obtained by multiplying the matrix of measures with the fixed matrices $K_q(x)$ and $K_p(x)^\top$. Under weak normality of the two boundary chains, the bidiagonal connections propagate weak normality throughout the rectangle and identify the measure, moment matrix, recurrence matrix, and both polynomial families at every vertex. If the scalar moment matrix is totally positive, all the shifted moment matrices are totally positive submatrices, so the boundary hypothesis is automatic. Strong normality still requires the separate maximal-degree conditions stated in the individual results and applications. Subsection 6.4 returns to rational approximation and derives finite approximants, convergence estimates, and Padé-type orders for the Weyl matrices on the line and on the star. Subsection 6.5 states separately the Poincaré moment approximation for a measure-defined Weyl matrix and the conditional operator-resolvent convergence of stable finite sections.

6.1. Moment functionals for $\mathcal{H}_{(a)}$, normality, and finite spectra. Theorem 5.11 averages equally over the M rays and therefore retains only the Fourier mode that is invariant under rotation. The matrix $\mathcal{H}_{(a)}$ contains more information: an entry joining factor positions a and b can be nonzero only in powers congruent to $b - a \pmod{M}$. This congruence selects a Fourier character on the star. We first encode these moments as linear functionals on the polynomial space $\mathbb{C}[\lambda]$. For a fixed compactly supported radial representing measure, representation of the corresponding Fourier component by a finite complex measure supported on the corresponding compact star is characterized by the subsequent lifting criterion.

Definition 6.1 (Moment functionals of the matrix $\mathcal{H}_{(a)}$). Order the coordinate states of $\mathcal{H}_{(a)}$ by increasing height in the factor order used to construct it, using the convention of Definition 3.4. Let $b_0, \dots, b_{q-1}$ be its first q coordinate states and $c_0, \dots, c_{p-1}$ its first p coordinate states. Let $E_{[q]}^{(a)}$ and $E_{[p]}^{(a)}$ be the coordinate projections onto $(b_0, \dots, b_{q-1})$ and $(c_0, \dots, c_{p-1})$, respectively. For $j \in \{0, \dots, q-1\}$ and $i \in \{0, \dots, p-1\}$, define the linear functional

$$\widehat{\Psi}_{j,i} : \mathbb{C}[\lambda] \longrightarrow \mathbb{C}$$

by prescribing its moments

$$\langle \widehat{\Psi}_{j,i}, \lambda^n \rangle := \left(E_{[q]}^{(a)} (\mathcal{H}_{(a)})^n (E_{[p]}^{(a)})^\top \right)_{j,i}, \quad n \in \mathbb{N}_0.$$

Thus, if $f(\lambda) = \sum_{n=0}^N f_n \lambda^n$, then

$$\langle \widehat{\Psi}_{j,i}, f \rangle = \sum_{n=0}^N f_n \langle \widehat{\Psi}_{j,i}, \lambda^n \rangle.$$

The $q \times p$ matrix formed by these functionals is denoted by $\widehat{\Psi}$.

If $a(b_j)$ and $a(c_i)$ are the factor-position labels of the two states, put

$$s_{j,i} \equiv a(c_i) - a(b_j) \pmod{M}, \quad 0 \leq s_{j,i} < M.$$

Every path from b_j to c_i has length congruent to $s_{j,i}$ modulo M . Consequently,

$$\langle \widehat{\Psi}_{j,i}, \lambda^n \rangle = 0 \quad \text{if} \quad n \not\equiv s_{j,i} \pmod{M}.$$

Thus each entry retains one Fourier character of the rotational symmetry. For later use, retain its nonzero moments in the radial functional $\mathcal{R}_{j,i}$ defined by

$$\mathcal{R}_{j,i}(x^k) := \left(\mathbf{E}_{[q]}^{(a)}(\mathcal{H}_{(a)})^{s_{j,i}+Mk}(\mathbf{E}_{[p]}^{(a)})^\top \right)_{j,i}, \quad k \in \mathbb{N}_0.$$

Before lifting these moments to the star, we determine their positivity on the radial variable $x = \lambda^M$. For factor-copy labels $a, b \in \{1, \dots, M\}$, let

$$s(a, b) \in \{0, \dots, M-1\}, \quad b \equiv a + s(a, b) \pmod{M},$$

and denote by $\widetilde{F}_{a \rightarrow b}$ the product of the $s(a, b)$ consecutive refactorized factors beginning with $\widetilde{F}_a$; when $a = b$, this is the identity. The intertwining identity

$$\widetilde{T}_{(a)} \widetilde{F}_{a \rightarrow b} = \widetilde{F}_{a \rightarrow b} \widetilde{T}_{(b)}$$

shows that the two expressions

$$m_k^{a,b} := \left(\widetilde{F}_{a \rightarrow b} (\widetilde{T}_{(b)})^k \right)_{0,0} = \left((\widetilde{T}_{(a)})^k \widetilde{F}_{a \rightarrow b} \right)_{0,0}$$

coincide.

Theorem 6.2 (Positive measures for all radial moment sequences). *For every pair $a, b \in \{1, \dots, M\}$, every finite minor of the Hankel matrix*

$$\left[m_{r+c}^{a,b} \right]_{r,c \geq 0}$$

is a polynomial with nonnegative integer coefficients in the nontrivial entries of the refactorized bidiagonal factors. For strictly positive numerical factors, the leading Hankel determinants and their one-step shifts satisfy

$$(6.1) \quad \det \left[m_{r+c}^{a,b} \right]_{r,c=0}^{k-1} > 0, \quad \det \left[m_{r+c+1}^{a,b} \right]_{r,c=0}^{k-1} > 0$$

for every $k \geq 1$. Consequently, there is a positive measure $d\rho^{a,b}$ on $[0, \infty)$, with infinite support, such that

$$m_k^{a,b} = \int_{[0, \infty)} x^k d\rho^{a,b}(x), \quad k \in \mathbb{N}_0.$$

If the refactorized factor coefficients are uniformly bounded, every representing measure is compactly supported and the measure is unique.

Proof. Construct a periodic directed network with vertices

$$(t, n) \in \mathbb{Z} \times \mathbb{N}_0.$$

The layer between times t and $t+1$ carries the refactorized factor whose copy label is congruent to $a+t$ modulo M . If that factor is $\widetilde{F}_c$, draw an edge from (t, n) to $(t+1, m)$, with weight $(\widetilde{F}_c)_{n,m}$, whenever this matrix entry is nonzero. A lower-factor layer has the changes $m-n = 0, -1$, and an upper-factor layer has the changes $m-n = 0, +1$. The network is therefore acyclic and planar in the upper half-plane. Matrix multiplication identifies an entry of a consecutive factor product with the total weight of the corresponding paths. Although the periodic network is infinite, fixed sources and targets involve only the finite subnetwork between their time coordinates and below the largest height reachable in that interval. Thus the finite-network form of the Lindström–Gessel–Viennot lemma applies to every minor considered below. The nested family used to prove strict positivity is shown schematically in Figure 3 for three paths.

Fix strictly increasing index sets

$$I = \{i_1 < \dots < i_k\}, \quad J = \{j_1 < \dots < j_k\},$$

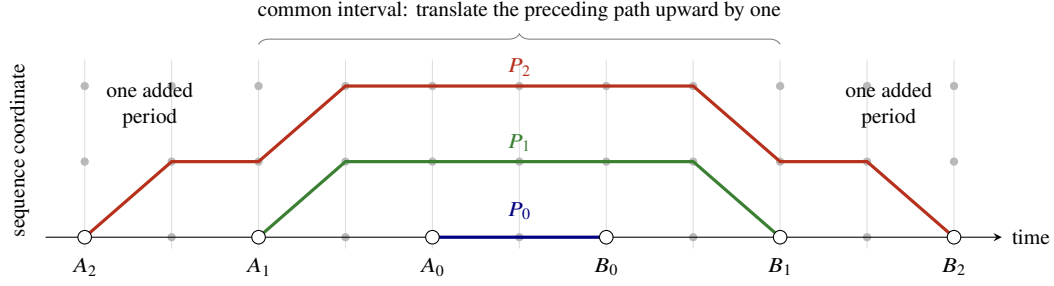


FIGURE 3. The nested vertex-disjoint paths used in the LGV argument, shown for $k = 3$. On the interval where P_r and P_{r-1} coexist, P_r is the vertical translate of P_{r-1} by one coordinate. One period is added at each end, where the path rises from and returns to the lower boundary. Thus P_r is disjoint from every earlier path and supplies a strictly positive monomial in the corresponding Hankel determinant.

and put $s = s(a, b)$. Choose sources and targets

$$A_r = (-Mi_r, 0), \quad B_c = (s + Mj_c, 0).$$

Periodicity of the factors shows that the total weight of paths from A_r to B_c is $m_{i_r+j_c}^{a,b}$. Along the lower boundary the vertices occur in the order

$$A_k, \dots, A_1, B_1, \dots, B_k.$$

Thus a vertex-disjoint path family can only connect A_r to B_r . The Lindström–Gessel–Viennot lemma [35, 25] gives

$$\det \left[m_{i_r+j_c}^{a,b} \right]_{r,c=1}^k = \sum_{\mathcal{P}} \text{wt}(\mathcal{P}),$$

where the sum runs over these vertex-disjoint families. Every summand is a monomial with a nonnegative integer coefficient. Since I and J were arbitrary, this proves the first assertion. The determinant may vanish when no such family exists, so strict positivity is established separately for the two families of determinants needed below.

We next exhibit a positive family for each determinant in (6.1). For the first determinant, take the sources $(-Mr, 0)$ and targets $(s + Mr, 0)$, with $r \in \{0, \dots, k-1\}$. Denote the path with these endpoints by P_r . The path P_0 stays at coordinate 0 and uses only diagonal entries.

Suppose that $P_0, \dots, P_{r-1}$ have been constructed. On the time interval common to P_r and P_{r-1} , define P_r to be the vertical translate of P_{r-1} by one coordinate. Prefix this translated path by one additional period: use diagonal edges at coordinate 0 until an upper-factor layer, take its superdiagonal edge from coordinate 0 to coordinate 1, and then use diagonal edges at coordinate 1 until the common interval begins. Suffix it by one additional period in the analogous way, using diagonal edges at coordinate 1 until a lower-factor layer, its subdiagonal edge from coordinate 1 to coordinate 0, and then diagonal edges at coordinate 0. Every period contains both an upper and a lower factor, so these choices are always available and have strictly positive weight.

On the common interval, P_r lies one coordinate above P_{r-1} , whereas all previously constructed paths lie on or below P_{r-1} . Outside that interval none of the earlier paths is present. Thus P_r is disjoint from every one of $P_0, \dots, P_{r-1}$. Induction gives the required k vertex-disjoint paths of strictly positive weight.

For the shifted determinant, use the sources $(-M(r+1), 0)$ and the targets $(s + Mr, 0)$. The innermost path stays at coordinate 0 from time $-M$ to time s . Each subsequent path is obtained from the preceding one by exactly the same three operations: add a left period, translate the whole common part upward by one coordinate, and add a right period. The preceding argument again proves disjointness from all earlier paths. Both determinants in (6.1) are therefore strictly positive.

The Hankel matrix and its one-step shift are positive definite. The Stieltjes moment criterion [2] gives a positive representing measure on $[0, \infty)$. Strict positivity at every order forces infinite support. If the factors are uniformly

bounded, the partial product $\widetilde{F}_{a \rightarrow b}$ and the cyclic product $\widetilde{T}_{(b)}$ are bounded. Hence there are constants $C, R > 0$ such that $m_k^{a,b} \leq CR^k$ for every k . Fix $R_1 > R$. If a representing measure assigned positive mass to $[R_1, \infty)$, then

$$m_k^{a,b} \geq R_1^k \rho^{a,b}([R_1, \infty)),$$

which contradicts $m_k^{a,b} \leq CR^k$ for large k . Thus every representing measure is supported in $[0, R]$. Polynomial density on this compact interval then gives uniqueness. $\square$

Corollary 6.3 (Stieltjes moment sequences throughout the prescribed cyclic orbit). *Let $F_1, \dots, F_M$ be the prescribed positive bidiagonal factors and, for $a, b \in \{1, \dots, M\}$, define*

$$\mu_k^{a,b} := \left(F_{a \rightarrow b} (T_{(b)})^k \right)_{0,0}, \quad k \in \mathbb{N}_0.$$

Every finite minor of the Hankel matrix $\left[\mu_{r+c}^{a,b} \right]_{r,c \geq 0}$ is a polynomial with nonnegative integer coefficients in the nontrivial entries of the prescribed factors. Its leading Hankel determinants and those of its one-step shift are strictly positive. Consequently, every sequence $(\mu_k^{a,b})_{k \geq 0}$ is represented by a positive Stieltjes measure with infinite support. In particular, this holds for the moments

$$(T^k)_{0,0} \quad \text{and} \quad ((T_{(a)})^k)_{0,0}$$

of the original product and all its cyclic products.

Proof. The proof of Theorem 6.2 uses periodicity, positivity, and the lower or upper bidiagonal form of each factor. The same periodic planar network argument applies to the prescribed factors $F_1, \dots, F_M$. Taking $a = b = 1$ gives $F_{1 \rightarrow 1} = I$ and $T_{(1)} = T$; taking $a = b$ gives the stated moments of every cyclic product. $\square$

Remark 6.4. The same argument applies to any periodic order of positive lower and upper bidiagonal factors. For $q = 1$, it reduces to the coefficientwise Hankel total positivity underlying the branched Stieltjes fractions [43]. It extends to arbitrary p and q because the planar network retains the elementary factor steps instead of contracting them into the entries of the banded product.

For the boundary states in Definition 6.1, one has

$$\mathcal{R}_{j,i}(x^k) = m_k^{a(b_j), a(c_i)}.$$

Indeed, the first q and p states have heights below Q and P , respectively, and therefore all have sequence coordinate zero. Theorem 6.2 therefore supplies a positive radial measure for every entry of the moment matrix of $\mathcal{H}_{(a)}$. If the refactorized coefficients are uniformly bounded, that measure is unique and compactly supported. The next result characterizes the additional integrability condition after a compactly supported radial representative has been fixed. Figure 4 summarizes the lift and its selection of one congruence class of moments.

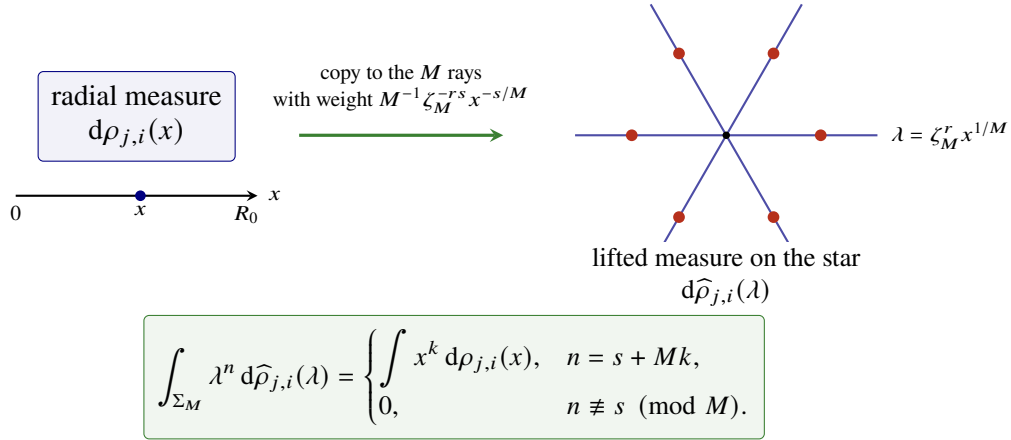


FIGURE 4. Fourier lift of a radial measure to the M -ray star. A point x is sent to its M roots $\zeta_M^r x^{1/M}$, weighted by the s -th character of the cyclic rotation. The root-of-unity average retains exactly the moments whose degrees are congruent to s modulo M . Six rays are drawn for illustration; the construction uses $M = p + q$ rays.

Proposition 6.5 (A measure on one ray variable and its Fourier lift to the star). *Fix (j, i) and put $s = s_{j,i}$. Suppose that $\mathcal{R}_{j,i}$ is represented by a finite complex measure $d\rho_{j,i}$ supported on $[0, R_0]$. The linear functional $\widehat{\Psi}_{j,i}$ is represented by a finite complex measure on $\Sigma_M^{[R_0]}$ if and only if, when $s > 0$, $\rho_{j,i}(\{0\}) = 0$ and*

$$(6.2) \quad \int_{(0, R_0]} x^{-s/M} d|\rho_{j,i}|(x) < \infty.$$

When these conditions hold, set $\Sigma_M^{[R_0]} = \bigcup_{r=0}^{M-1} \zeta_M^r [0, R_0^{1/M}]$ and define one such representing measure by

$$(6.3) \quad \int_{\Sigma_M^{[R_0]}} f(\lambda) d\widehat{\rho}_{j,i}(\lambda) := \frac{1}{M} \sum_{r=0}^{M-1} \zeta_M^{-rs} \int_{(0, R_0]} f(\zeta_M^r x^{1/M}) x^{-s/M} d\rho_{j,i}(x).$$

When $s = 0$, an atom of $\rho_{j,i}$ at the origin is included in the right-hand side in the evident way. Then $d\widehat{\rho}_{j,i}$ is a finite complex measure and

$$(6.4) \quad \int_{\Sigma_M^{[R_0]}} \lambda^n d\widehat{\rho}_{j,i}(\lambda) = \begin{cases} \mathcal{R}_{j,i}(x^k), & n = s + Mk, \\ 0, & n \not\equiv s \pmod{M}. \end{cases}$$

Its Cauchy transform is

$$(6.5) \quad \int_{\Sigma_M^{[R_0]}} \frac{d\widehat{\rho}_{j,i}(\zeta)}{\lambda - \zeta} = \lambda^{M-s-1} \int_{[0, R_0]} \frac{d\rho_{j,i}(x)}{\lambda^M - x}, \quad \lambda \in \mathbb{C} \setminus \Sigma_M^{[R_0]}.$$

Proof. We first prove sufficiency. Condition (6.2) shows that every term in (6.3) has finite total variation. Thus the formula defines a finite complex measure. When $s > 0$, the hypothesis $\rho_{j,i}(\{0\}) = 0$ ensures that no mass is placed where the factor $x^{-s/M}$ is singular. Moreover, for every $n \geq 0$,

$$x^{(n-s)/M} = x^{n/M} x^{-s/M}$$

is integrable with respect to $|\rho_{j,i}|$, because $x^{n/M}$ is bounded on $[0, R_0]$ and the negative fractional moment is finite. When $s = 0$, the same integrability follows directly from the finiteness and compact support of $\rho_{j,i}$. For $n \geq 0$, the root-of-unity average is

$$\frac{1}{M} \sum_{r=0}^{M-1} \zeta_M^{r(n-s)} = \begin{cases} 1, & n \equiv s \pmod{M}, \\ 0, & n \not\equiv s \pmod{M}. \end{cases}$$

If $n = s + Mk$, the remaining power of x in (6.3) is exactly x^k . This proves (6.4). Finally, for $r = x^{1/M}$ and $|\lambda| > R_0^{1/M}$, expansion in powers of r/λ gives

$$\frac{1}{M} \sum_{h=0}^{M-1} \frac{\zeta_M^{-hs}}{\lambda - \zeta_M^h r} = \frac{r^s \lambda^{M-s-1}}{\lambda^M - r^M}.$$

Multiplication by $x^{-s/M} = r^{-s}$ and integration prove (6.5); analytic continuation gives the identity throughout $\mathbb{C} \setminus \Sigma_M^{[R_0]}$.

For necessity, let $d\eta$ be any finite complex measure on $\Sigma_M^{[R_0]}$ representing $\widehat{\Psi}_{j,i}$. Replace it by its s -th Fourier component under rotations:

$$\int f(\lambda) d\eta_s(\lambda) := \frac{1}{M} \sum_{r=0}^{M-1} \zeta_M^{-rs} \int f(\zeta_M^r \lambda) d\eta(\lambda).$$

This is again a finite measure and has the same moments as η , because the prescribed moments vanish outside degrees congruent to s modulo M . Push the measure $\lambda^s d\eta_s(\lambda)$ forward under $\lambda \mapsto \lambda^M$, and call the resulting measure $d\rho'$. For every $k \geq 0$,

$$\int_{[0, R_0]} x^k d\rho'(x) = \int_{\Sigma_M^{[R_0]}} \lambda^{s+Mk} d\eta_s(\lambda) = \mathcal{R}_{j,i}(x^k).$$

Finite complex measures on the compact interval $[0, R_0]$ are determined by their moments. Hence $\rho' = \rho_{j,i}$. If $s > 0$, multiplication by λ^s removes any atom at the origin, so $\rho_{j,i}(\{0\}) = 0$. Moreover, the total-variation inequality for a pushforward gives

$$\int_{(0, R_0]} x^{-s/M} d|\rho_{j,i}|(x) \leq \int_{\Sigma_M^{[R_0] \setminus \{0\}}} |\lambda|^{-s} |\lambda|^s d|\eta_s|(\lambda) \leq \|\eta_s\| < \infty.$$

This proves necessity. For $s = 0$, the same construction requires no negative moment and retains a possible atom at the origin. $\square$

Proposition 6.6 (Obstruction to positive scalar representations). *Let $s \in \{1, \dots, M-1\}$. A nonzero functional whose moments vanish outside degrees congruent to s modulo M cannot be represented by a finite positive scalar measure on the star.*

Proof. The moment of degree zero vanishes. If a positive measure represented the functional, that moment would be its total mass. The measure would therefore be zero, contrary to the assumption that the functional is nonzero. $\square$

The support of every lifted entry lies on the same M rays as the rotationally invariant lift. Since $s_{j,i} \equiv a(c_i) - a(b_j) \pmod{M}$, the phases on each ray separate into a left and a right diagonal unitary factor. The additional factor $x^{-s_{j,i}/M}$ in (6.3) shows that support reaching the origin adds a negative fractional-moment condition to the full moment problem on the star. Positivity belongs to the radial measures on $[0, \infty)$; the nonzero Fourier components on the star are necessarily complex.

For nonsingular triangular matrices $\xi \in \mathbb{C}^{q \times q}$ and $\nu \in \mathbb{C}^{p \times p}$, normalize

$$\widehat{\Psi}^{\xi, \nu} := \xi^{-1} \widehat{\Psi} \nu^{-\top}.$$

Form its scalar step-line moment matrix by

$$\mathcal{M}_{qu+j, pv+i}^{\xi, \nu} := \left(\langle \widehat{\Psi}^{\xi, \nu}, \lambda^{u+v} \rangle \right)_{j,i}.$$

Write $\mathcal{M} := \mathcal{M}^{I_q, I_p}$ for the corresponding unnormalized scalar moment matrix.

Recall the normality terminology fixed in Definition 2.1. For the scalar moment matrix above, the row positions are ordered as $qu + j$, with $j \in \{0, \dots, q-1\}$, and the column positions as $pv + i$, with $i \in \{0, \dots, p-1\}$. Hence its leading $N \times N$ block determines the row and column multi-indices at the N -th position of the prescribed step-line.

For the scalar step-line moment matrix used here, nonvanishing of the leading minors of orders $1, \dots, N$ is equivalent to weak normality through order N . The maximal-degree condition in strong normality requires additional information. When $d > 1$, the argument below first proves weak normality at orders $d, 2d, 3d, \dots$, because those are precisely the cuts that retain all d coordinate states having the last included height.

Terminology in the literature is not uniform. The word *normality* in [11] denotes the maximal-degree property and therefore corresponds to strong normality in Definition 2.1. In the hypergeometric setting of [37], normality denotes existence and uniqueness, while the AT property established there also gives the maximal degrees and hence strong normality.

The factorization used in both normality statements can be written before any normalization is chosen. Define two scalar path arrays by

$$\mathcal{A}_{qu+j,k} := ((\mathcal{H}_{(a)})^u)_{j,k}, \quad \mathcal{B}_{pv+i,k} := ((\mathcal{H}_{(a)})^v)_{k,i},$$

where $j \in \{0, \dots, q-1\}$ and $i \in \{0, \dots, p-1\}$ refer to the first q and p scalar states in exact-height order. Then the unnormalized scalar moment matrix satisfies

$$(6.6) \quad \mathcal{M} = \mathcal{A}\mathcal{B}^\top.$$

Indeed, its $(qu+j, pv+i)$ -entry is

$$\sum_{k \geq 0} ((\mathcal{H}_{(a)})^u)_{j,k} ((\mathcal{H}_{(a)})^v)_{k,i} = ((\mathcal{H}_{(a)})^{u+v})_{j,i}.$$

For fixed u and v , the sum is finite because $\mathcal{H}_{(a)}$ is banded.

Proposition 6.7 (Weak normality after retaining complete height groups). *For every $K \geq 1$, the leading principal minor of order dK of the unnormalized scalar moment matrix is nonzero. Equivalently, weak normality holds after every truncation that retains all the coordinate states of heights $0, \dots, K-1$.*

Proof. Use the factorization (6.6). The first $q = dQ$ row-boundary states consist of the d states at each of the heights $0, \dots, Q-1$.

After u successive upward steps, their endpoints are precisely the d states at each of the heights

$$uQ, \dots, (u+1)Q-1.$$

No other path of u steps can end at a greater height. Therefore, when the indices are grouped according to their common height, $\mathcal{A}$ is block lower triangular, with blocks of order d . Each diagonal block is a permutation matrix: between a boundary state and its maximal endpoint there is a unique path consisting entirely of upward steps, and every upward edge has weight 1.

Similarly, the first $p = dP$ column-boundary states consist of the d states at each of the heights

$$0, \dots, P-1.$$

A path of v steps ending at one of these states can start no higher than the initial point of the unique path consisting entirely of downward steps. These initial points are precisely the states at the heights

$$vP, \dots, (v+1)P-1.$$

Hence $\mathcal{B}$ is also block lower triangular. Its diagonal blocks are positive monomial matrices, because they are supplied by the unique all-down paths. In particular, every diagonal block of $\mathcal{A}$ and $\mathcal{B}$ is invertible.

Fix $K \geq 1$. The first dK indices comprise all the states of heights $0, \dots, K-1$. Block lower triangularity shows that the first dK rows of both path arrays vanish in every column with index at least dK . Therefore the leading $dK \times dK$ block of

$$\mathcal{M} = \mathcal{A}\mathcal{B}^\top$$

is the product of the corresponding leading blocks:

$$\mathcal{M}^{[dK-1]} = \mathcal{A}^{[dK-1]} (\mathcal{B}^{[dK-1]})^\top.$$

Both factors are block triangular with invertible diagonal blocks. Consequently,

$$\det \mathcal{M}^{[dK-1]} = \det \mathcal{A}^{[dK-1]} \det \mathcal{B}^{[dK-1]} \neq 0.$$

□

Theorem 6.8 (Weak normality at every scalar order in the coprime case). *Assume that $d = \gcd(p, q) = 1$, and let ξ and ν be lower unitriangular. Every leading principal minor of $\mathcal{M}^{\xi, \nu}$ is nonzero. Hence the prescribed step-line is weakly normal at every scalar order.*

Proof. Start from the path-array factorization (6.6).

Since $d = 1$, every height contains a single coordinate state. In increasing exact-height order, the row of $\mathcal{A}$ numbered $qu + j$ records paths of u steps beginning at height j . Such a path can end no higher than $j + uq = qu + j$, and equality is attained only by the all-up path, whose weight is one. Hence $\mathcal{A}$ is scalar lower unitriangular. Applying ξ^{-1} to each block of q consecutive rows preserves this scalar lower-triangular structure and its nonzero diagonal. Denote the resulting array by $\mathcal{A}^\xi$.

The array $\mathcal{B}$ is also scalar lower triangular. Its row numbered $p\nu + i$ records, in transposed path orientation, paths of ν steps ending at the boundary state of height i . Since one step can decrease height by at most p , a path beginning at height k can reach that state only if

$$k \leq i + \nu p = p\nu + i.$$

Thus the entries in columns numbered above the row index vanish. Equality requires every step to be downward, so the diagonal entry is the strictly positive weight of the unique all-down path.

Right normalization of the moment matrix by $\nu^{-\top}$ replaces $\mathcal{B}$ by an array $\mathcal{B}^\nu$ obtained by applying ν^{-1} to each block of p consecutive rows. Because ν^{-1} is lower triangular, $\mathcal{B}^\nu$ remains scalar lower triangular. The positive diagonal blocks described above and the unit diagonal of ν^{-1} show that every diagonal entry of $\mathcal{B}^\nu$ is nonzero.

The normalized moment factorization is

$$\mathcal{M}^{\xi, \nu} = \mathcal{A}^\xi (\mathcal{B}^\nu)^\top.$$

Because $\mathcal{A}^\xi$ is scalar lower triangular, its first N rows vanish in every column numbered at least N . Therefore the leading $N \times N$ block of the displayed product is the product of the leading blocks of $\mathcal{A}^\xi$ and $(\mathcal{B}^\nu)^\top$. Both determinants are nonzero. By Definition 2.1, this proves weak normality at every scalar order. □

Remark 6.9 (Weak and strong normality for the sparse recurrence). The preceding theorem establishes weak normality. Strong normality may fail even if ξ and ν are chosen to be Pascal-type totally positive matrices. For example, take $p = 2$ and $q = 3$. With the usual lower triangular normalization, the first right polynomial vector is $\mathbf{B}_0 = (1, 0, 0)$. The row of the two-diagonal recurrence at height 0 gives

$$\mathbf{B}_3(x) = x\mathbf{B}_0(x) = (x, 0, 0).$$

At this step-line position the second and third components have nontrivial constant polynomial spaces, but they vanish. Thus this index pair is weakly normal and not strongly normal.

The BTP criterion of [11] applies to the cyclic products that have a PBF. The sparse matrix $\mathcal{H}_{(a)}$ has a zero main diagonal and zero diagonals between its two extreme nonzero diagonals, so it lies outside the BTP class.

Group the scalar coordinate states of $\mathcal{H}_{(a)}$ according to their common height. Each group has dimension d , and the path arrays $\mathcal{A}$ and $\mathcal{B}$ become semi-infinite lower triangular block matrices with blocks of order d . Set

$$\mathcal{L}_\star := \mathcal{A}^{-1}, \quad \mathcal{U}_\star := \mathcal{B}^{-\top}.$$

Let $\Lambda_{[Q]}^{(d)}$ be the block shift whose nonzero blocks are $d \times d$ identity matrices in block positions $(k, k + Q)$, and define $\Lambda_{[P]}^{(d)}$ analogously.

Theorem 6.10 (The matrix $\mathcal{H}_{(a)}$ as a block mixed-type recurrence). *With the preceding notation,*

$$\mathcal{M} = \mathcal{L}_\star^{-1} \mathcal{U}_\star^{-1}, \quad \mathcal{L}_\star^{-1} = \mathcal{A}, \quad \mathcal{U}_\star^{-\top} = \mathcal{B}$$

is a triangular factorization of the moment matrix by invertible lower and upper block matrices. Equivalently, after a diagonal-block normalization, it is a block Gauss–Borel factorization with the same moment matrix. Moreover,

$$(6.7) \quad \mathcal{H}_{(a)} = \mathcal{L}_\star \Lambda_{[Q]}^{(d)} \mathcal{L}_\star^{-1} = \mathcal{U}_\star^{-1} (\Lambda_{[P]}^{(d)})^\top \mathcal{U}_\star.$$

Proof. The proof of Proposition 6.7 shows that $\mathcal{A}$ and $\mathcal{B}$, after grouping equal-height states, are lower triangular block matrices with invertible $d \times d$ diagonal blocks. They are therefore invertible as formal triangular block matrices. The path-array identity gives

$$\mathcal{M} = \mathcal{A} \mathcal{B}^\top = \mathcal{L}_\star^{-1} \mathcal{U}_\star^{-1}.$$

The block row of $\mathcal{A}$ numbered $Qu + \alpha$, with $\alpha \in \{0, \dots, Q-1\}$, is the block row of $(\mathcal{H}_{(a)})^u$ beginning at height α . Deleting the first Q block rows therefore advances the power by one:

$$\Lambda_{[Q]}^{(d)} \mathcal{A} = \mathcal{A} \mathcal{H}_{(a)}.$$

Substitution of $\mathcal{A} = \mathcal{L}_\star^{-1}$ gives the first representation in (6.7).

Similarly, the block row of $\mathcal{B}$ numbered $Pv + \beta$, with $\beta \in \{0, \dots, P-1\}$, is the corresponding block row of $((\mathcal{H}_{(a)})^\top)^\nu$. Hence

$$\Lambda_{[P]}^{(d)} \mathcal{B} = \mathcal{B} (\mathcal{H}_{(a)})^\top.$$

Transposition and substitution of $\mathcal{B} = \mathcal{U}_\star^{-\top}$ give the second representation. These are the right and left recurrence identities of the polynomial systems associated with the block Gauss–Borel factors. Flattening the blocks recovers the scalar moment functionals already defined. A representing matrix of measures, when it exists, changes the realization of those functionals but not these recurrence identities. $\square$

The preceding factorization also makes precise the relation between the rotational lift of the Favard measure of $T_{(a)}$ and the matrix of star functionals $\widehat{\Psi}$. Define the right and left polynomial arrays of the star recurrence by

$$\mathbf{B}_\star(\lambda) := \mathcal{L}_\star \chi_{[q]}(\lambda), \quad \mathbf{A}_\star(\lambda) := \chi_{[p]}(\lambda)^\top \mathcal{U}_\star.$$

Let $\mathbf{R}_a$ be the factor-copy projection used in Corollary 3.11, and put

$$\mathbf{Q}_{r,\star}^{(a)} := \mathbf{E}_{[r]} \mathbf{R}_a, \quad \mathbf{B}_a(\lambda) := \mathbf{Q}_{q,\star}^{(a)} \mathbf{B}_\star(\lambda), \quad \mathbf{A}_a(\lambda) := \mathbf{A}_\star(\lambda) (\mathbf{Q}_{p,\star}^{(a)})^\top.$$

Thus $\mathbf{B}_a$ and $\mathbf{A}_a$ are the polynomial matrices attached to the first q and p coordinates, respectively, of the factor copy whose complete circuit gives $T_{(a)}$.

Proposition 6.11 (The rotational lift inside the star moment system). *Suppose that the matrix of functionals $\widehat{\Psi}$ is represented on the star by a $q \times p$ matrix of finite measures $d\widehat{\Psi}_\star$. Suppose also that $T_{(a)}$ has Favard data $d\Psi^{(a)}, \xi^{(a)}, \nu^{(a)}$, with $d\Psi^{(a)}$ supported on $[0, \infty)$, and let $d\widehat{\Psi}^{(a)}$ be the rotational lift of $d\Psi^{(a)}$ given by Definition 5.10. Then, for every $n \in \mathbb{N}_0$,*

$$(6.8) \quad \int_{\Sigma_M} \lambda^n d\widehat{\Psi}^{(a)}(\lambda) = (\xi^{(a)})^{-1} \int_{\Sigma_M} \mathbf{B}_a(\lambda) \lambda^n d\widehat{\Psi}_\star(\lambda) \mathbf{A}_a(\lambda) (\nu^{(a)})^{-\top}.$$

Consequently, the rotational lift is the matrix spectral measure of the star system observed from the coordinates belonging to that factor copy. If the measures are compactly supported, the moment identity determines them and one has the measure identity

$$d\widehat{\Psi}^{(a)}(\lambda) = (\xi^{(a)})^{-1} \mathbf{B}_a(\lambda) d\widehat{\Psi}_\star(\lambda) \mathbf{A}_a(\lambda) (\nu^{(a)})^{-\top}.$$

Proof. The moment factorization in Theorem 6.10 gives

$$\int_{\Sigma_M} \mathbf{B}_\star(\lambda) d\widehat{\Psi}_\star(\lambda) \mathbf{A}_\star(\lambda) = I.$$

The two recurrence identities in (6.7) imply

$$\mathcal{H}_{(a)} \mathbf{B}_\star(\lambda) = \lambda \mathbf{B}_\star(\lambda), \quad \mathbf{A}_\star(\lambda) \mathcal{H}_{(a)} = \lambda \mathbf{A}_\star(\lambda).$$

It follows that

$$\int_{\Sigma_M} \mathbf{B}_\star(\lambda) \lambda^n d\widehat{\Psi}_\star(\lambda) \mathbf{A}_\star(\lambda) = (\mathcal{H}_{(a)})^n.$$

Selecting the indicated factor-copy coordinates therefore gives

$$\begin{aligned} \int_{\Sigma_M} \mathbf{B}_a(\lambda) \lambda^n d\widehat{\Psi}_\star(\lambda) \mathbf{A}_a(\lambda) \\ = \mathbf{Q}_{q,\star}^{(a)} (\mathcal{H}_{(a)})^n (\mathbf{Q}_{p,\star}^{(a)})^\top. \end{aligned}$$

By (3.5), the last member is zero when $M \nmid n$, while for $n = Mk$ it equals

$$\mathbf{E}_{[q]}(T_{(a)})^k \mathbf{E}_{[p]}^\top.$$

After multiplication by the normalization matrices, these are precisely the moments of the rotational lift of $d\Psi^{(a)}$. This proves (6.8). For compact support, polynomial moments determine finite measures on the finite union of rays Σ_M , so the measure identity follows entry by entry. $\square$

Thus $\mathcal{H}_{(a)}$ is the recurrence matrix of the block moment system constructed above, with Q right and P left boundary components of size d . When $d = 1$, this is an ordinary weakly normal mixed-type system along the full scalar step-line. When $d > 1$, the result gives block normality after complete groups of d consecutive states. Proposition 6.5 characterizes, under its compact-support hypothesis, which of these functionals admit finite representing measures supported on the corresponding compact star.

Remark 6.12 (Linear functionals and measures on the star). For $\gcd(p, q) = 1$, Theorem 6.10 gives a recurrence realization for the matrix of linear functionals $\widehat{\Psi}$. Theorem 6.2 proves directly from the PBF that every radial moment sequence occurring in this matrix is a Stieltjes moment sequence, and therefore supplies a positive radial measure for every boundary pair. When the chosen radial measure is compactly supported, Proposition 6.5 gives a finite complex measure supported on the corresponding compact star for the Fourier component if and only if, in nonzero mode, the radial measure has no atom at the origin and satisfies the negative fractional-moment condition. No such equivalence for an arbitrary noncompact representative is asserted here; in an indeterminate radial moment problem the moments alone do not select that representative. The lifted measures are complex in every nonzero Fourier mode, as Proposition 6.6 shows. Verification of the compactly supported lifting condition is family-dependent. The weak normality result in the coprime case, the block recurrence (6.7), and the finite spectral results below are established at the level of the moment functionals. When $\gcd(p, q) > 1$, the uniform conclusion is the block Gauss–Borel realization in Theorem 6.10, with cuts taken after complete groups of equal-height states. Strong normality requires additional information, as explained in Remark 6.9.

Theorem 6.13 (Spectra of finite truncations of $\mathcal{H}_{(a)}$). *Let K_N be the finite principal matrix obtained from $\mathcal{H}_{(a)}$ by retaining all coordinate states of heights $0, \dots, N$ in this order. Thus K_N has order $d(N + 1)$. There are an integer $r_N \in \{0, \dots, M - 1\}$ and a polynomial $\mathcal{P}_N$, with $\mathcal{P}_N(0) \neq 0$, such that*

$$\det(\lambda I - K_N) = \lambda^{r_N} \mathcal{P}_N(\lambda^M), \quad r_N = d(N + 1) - M \left\lfloor \frac{d(N + 1)}{M} \right\rfloor.$$

Moreover,

$$\deg \mathcal{P}_N = \left\lfloor \frac{d(N + 1)}{M} \right\rfloor = \left\lfloor \frac{N + 1}{P + Q} \right\rfloor.$$

Every zero of $\mathcal{P}_N$ is positive. Hence every nonzero pole of the finite resolvent of $\mathcal{H}_{(a)}$ lies on one of the $M = p + q$ rays through the M -th roots of unity. More precisely, order the retained states by factor copy and let $C_{a,N}$ be the diagonal block of K_N^M on the a -th copy. If that copy contains $n_{a,N}$ states and $m_N = \lfloor d(N+1)/M \rfloor$, then

$$\det(yI - C_{a,N}) = y^{n_{a,N} - m_N} \mathcal{P}_N(y).$$

Thus $\mathcal{P}_N$ is the characteristic polynomial of every diagonal block of order m_N in the M -th power, and its zeros are the nonzero radial spectral parameters.

Proof. Put

$$D_N := d(N+1) = \dim K_N.$$

The proof has three steps.

Rotational form. Every nonzero entry advances the factor-copy label by one modulo M . If D multiplies the coordinates in copy a by ζ_M^a , then

$$D^{-1}K_N D = \zeta_M K_N.$$

Consequently, the nonzero terms in the characteristic polynomial have degrees congruent to D_N modulo M . Let r_N be the remainder of D_N on division by M . At this stage the symmetry gives

$$\det(\lambda I - K_N) = \lambda^{r_N} \mathcal{R}_N(\lambda^M)$$

for some polynomial $\mathcal{R}_N$; it does not yet exclude an additional factor of λ^M in $\mathcal{R}_N(\lambda^M)$.

The M -th power. Order the coordinates by factor copy. The finite truncation of $\mathcal{H}_{(a)}$ is then a cyclic block matrix whose block from one copy to the next is a leading rectangular submatrix of one of the positive bidiagonal factors. Every such rectangular block is totally nonnegative. Hence the M -th power is block diagonal, and each square diagonal block is a product of compatible rectangular totally nonnegative matrices. Repeated Cauchy–Binet shows that these cyclic products are totally nonnegative, so their eigenvalues are real and nonnegative [22, 44].

Put $m_N = \lfloor D_N/M \rfloor$ and write $N = sR + t$, where $s \in \mathbb{N}_0$ and $t \in \{0, \dots, R-1\}$. The states in factor copy a have heights

$$\tilde{\omega}_a, \tilde{\omega}_a + R, \tilde{\omega}_a + 2R, \dots$$

Consequently, the number $n_{a,N}$ of retained states in that copy is $s+1$ when $\tilde{\omega}_a \leq t$, and is s otherwise. Because every residue in $\{0, \dots, R-1\}$ occurs exactly d times among $\tilde{\omega}_1, \dots, \tilde{\omega}_M$, precisely $d(t+1)$ copies have size $s+1$.

If $t < R-1$, then $m_N = s$ and $r_N = d(t+1)$. If $t = R-1$, then every copy has size $s+1 = m_N$ and $r_N = 0$. Thus every factor copy contains either m_N or $m_N + 1$ retained states, exactly r_N copies have the larger size, and at least one copy has size m_N . Choose one such copy. In the repeated Cauchy–Binet expansion of the determinant of its $m_N \times m_N$ cyclic product, select the first m_N indices at every intermediate copy. This term is the product of leading square minors of positive bidiagonal factors and is strictly positive; all other terms are nonnegative. This diagonal block of K_N^M is therefore nonsingular.

Common nonzero spectrum. The nonsingular block just constructed is totally nonnegative. Its m_N eigenvalues are therefore positive. Repeated use of the finite-dimensional AB/BA identity shows that all M cyclic blocks of K_N^M have these same m_N nonzero eigenvalues, counting algebraic multiplicity; any additional eigenvalues of a larger block are zero. Hence K_N^M has exactly Mm_N nonzero eigenvalues.

By the spectral mapping theorem, the eigenvalues of K_N^M , with algebraic multiplicity, are the M -th powers of the eigenvalues of K_N . Therefore K_N itself has exactly Mm_N nonzero eigenvalues. Its zero eigenvalue has algebraic multiplicity

$$D_N - Mm_N = r_N.$$

Consequently $\mathcal{R}_N(0) \neq 0$. Set $\mathcal{P}_N = \mathcal{R}_N$; this proves both the asserted factor λ^{r_N} and the formula $\deg \mathcal{P}_N = m_N$.

The block $C_{a,N}$ has order $n_{a,N} \in \{m_N, m_N + 1\}$; by the preceding argument its nonzero eigenvalues are the same m_N positive numbers and its remaining eigenvalues are zero. Since both characteristic polynomials are monic, this proves

$$\det(yI - C_{a,N}) = y^{n_{a,N} - m_N} \mathcal{P}_N(y).$$

The block $C_{a,N}$ is totally nonnegative, so all its eigenvalues are real and nonnegative. The displayed characteristic-polynomial identity shows that the zeros of $\mathcal{P}_N$ are exactly its nonzero eigenvalues. They are therefore positive. Finally, the spectral mapping theorem implies that every nonzero eigenvalue λ of K_N satisfies $\lambda^M > 0$, which proves the claimed ray localization. $\square$

Proposition 5.12 identifies the Weyl matrix observed from the first factor copy in the bounded case. The next corollary shows that a global PBF also supplies positive Favard data at every factor position, whether or not the factors are bounded and whether or not their origin is a Christoffel transformation.

Lemma 6.14 (Compatible truncations of a global PBF). *Let*

$$S = K_1 \cdots K_p V_q \cdots V_1$$

be a semi-infinite product in which the K_a are lower bidiagonal and the V_b are upper bidiagonal. If $S^{[N]}$, $K_a^{[N]}$, and $V_b^{[N]}$ denote their leading principal submatrices of order $N + 1$, then

$$(6.9) \quad S^{[N]} = K_1^{[N]} \cdots K_p^{[N]} V_q^{[N]} \cdots V_1^{[N]}.$$

Consequently, if the semi-infinite factorization is positive, all its leading principal submatrices have PBFs with zero scalar shift. These PBFs are coherent in the sense of the unbounded Favard theorem: the finite boundary matrices are independent of N once their fixed number of coordinates is present.

Proof. An entry of the product is a sum over intermediate coordinate indices. Starting from a row index at most N , the lower factors can only decrease the current index or leave it fixed, so the index is still at most N when the upper factors begin. The upper factors can only increase the index or leave it fixed. If the index then became larger than N , it could not return to a final column index at most N . Hence every contributing intermediate index also lies in $\{0, \dots, N\}$, which proves (6.9).

If the factors are positive, their leading submatrices give the asserted finite PBFs. The normalization convention used in [14] places one positive diagonal matrix between the lower and upper factors and makes every upper factor unitriangular. It is equivalent to the convention used here. Indeed, put $D_0 = I$ and choose positive diagonal matrices $D_1, \dots, D_q$ recursively so that

$$\bar{U}_b := D_b^{-1} U_b D_{b-1}$$

has unit diagonal. Equivalently, the diagonal of D_b is the entrywise product of the diagonals of U_b and D_{b-1} . Since $U_b D_{b-1} = D_b \bar{U}_b$, successive cancellation gives

$$U_q \cdots U_1 = D_q \bar{U}_q \cdots \bar{U}_1.$$

Thus D_q is precisely the positive diagonal factor required by that normalization, and all the new superdiagonal entries are positive.

The boundary matrices of [14, Definition 4.1] use only fixed upper-left portions of these factors. Because the finite factors are the leading submatrices of one fixed semi-infinite factorization, those portions do not depend on N once $N \geq \max\{p, q\} - 1$. Thus the coherence condition in that definition is satisfied: the same initial-condition matrices can be used for every N . $\square$

Corollary 6.15 (Favard data for the complete cyclic orbit). *Assume that the factorization (2.1) is positive. For every $a \in \{1, \dots, M\}$, there exist nonsingular initial-condition matrices $\xi_F^{(a)}$ and $\nu_F^{(a)}$, two mixed-type polynomial systems, and an entrywise positive $q \times p$ matrix of finite measures $d\Psi_F^{(a)}$ such that*

$$(6.10) \quad (\xi_F^{(a)})^{-1} E_{[q]}(T_{(a)})^n E_{[p]}^T (\nu_F^{(a)})^{-T} = \int x^n d\Psi_F^{(a)}(x), \quad n \in \mathbb{N}_0.$$

For every a for which $T_{(a)}$ is bounded on $\ell^2(\mathbb{N}_0)$, the measures may be chosen compactly supported and the corresponding diagonal projection of the common cyclic resolvent has this rectangular Stieltjes representation. If $T_{(a)}$ is unbounded on $\ell^2(\mathbb{N}_0)$, the moment identity remains valid. An operator-resolvent representation then requires the additional analytic hypotheses stated below.

Proof. Theorem 2.3 gives a normalized positive bidiagonal factorization in standard lower–upper order for every $T_{(a)}$. Lemma 6.14 shows that its leading principal submatrices have coherent PBFs with zero shift and the same initial-condition matrices. Fix a . If $T_{(a)}$ is bounded on $\ell^2(\mathbb{N}_0)$, the last part of Lemma 3.27, applied to its nonnegative factorization, shows that the refactorized bidiagonal coefficients are uniformly bounded. The bounded Favard theorem [10] then gives an entrywise positive matrix of compactly supported measures. If $T_{(a)}$ is unbounded on $\ell^2(\mathbb{N}_0)$, the unbounded Favard theorem [14] applies to the coherent family just described and gives an entrywise positive matrix of finite measures. In both cases one obtains the moment identity (6.10). In the bounded case, Proposition 2.11 also identifies the corresponding diagonal block of the cyclic resolvent with the resolvent of $T_{(a)}$. $\square$

Remark 6.16 (Normality of the cyclic Favard systems). By Theorem 2.3, every cyclic product has a PBF and is therefore BTP. The maximal-degree criterion of [11] then gives strong normality when $\nu^{-\top}$, the inverse of the left initial block $\nu^\top$, is upper Δ TP, and ξ^{-1} , the inverse of the right initial block ξ , is lower Δ TP. Thus the Favard theorem supplies the measures in both the bounded and unbounded cases, while the BTP theorem supplies the maximal-degree condition and hence strong normality, in the terminology of Definition 2.1.

6.2. Christoffel transformations adapted to the cyclic PBF. The Favard data supplied at a cyclic product depend on the two triangular initial-condition matrices. This freedom is part of the Christoffel transformation: after a factor is moved cyclically, its natural matrix polynomial need not be the fixed shift matrix $K_q(x)$ or $K_p(x)^\top$. The next proposition records exactly how a change of initial conditions modifies the moments while leaving the recurrence matrix unchanged.

Proposition 6.17 (Moment functionals for different initial conditions). *Let $\mathcal{T}$ be one of the semi-infinite (p, q) -banded recurrence matrices considered above, admitting a positive bidiagonal factorization. Let two pairs of nonsingular lower triangular initial-condition matrices reconstruct from the same matrix $\mathcal{T}$ the block moments*

$$W_n = \xi^{-1} E_{[q]} \mathcal{T}^n E_{[p]}^\top \nu^{-\top}, \quad \widetilde{W}_n = \widetilde{\xi}^{-1} E_{[q]} \mathcal{T}^n E_{[p]}^\top \widetilde{\nu}^{-\top}.$$

Then

$$(6.11) \quad \widetilde{W}_n = G W_n H, \quad G := \widetilde{\xi}^{-1} \xi, \quad H := \nu^\top \widetilde{\nu}^{-\top}.$$

The matrix G is nonsingular lower triangular and H is nonsingular upper triangular.

Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be the corresponding scalar step-line moment matrices. If $\mathcal{G}_L = \text{diag}(G, G, \dots)$ and $\mathcal{G}_R = \text{diag}(H, H, \dots)$, then

$$\widetilde{\mathcal{M}} = \mathcal{G}_L \mathcal{M} \mathcal{G}_R.$$

For every $N \in \mathbb{N}_0$,

$$(6.12) \quad \det \widetilde{\mathcal{M}}^{[N]} = \det \mathcal{G}_L^{[N]} \det \mathcal{M}^{[N]} \det \mathcal{G}_R^{[N]}.$$

Consequently, $\mathcal{M}$ is weakly normal at every scalar order if and only if $\widetilde{\mathcal{M}}$ is. This triangular gauge statement concerns weak normality only; it does not in general preserve the maximal component degrees required for strong normality.

Proof. Equation (6.11) follows by inserting the definitions and cancelling $\xi \xi^{-1}$ and $\nu^{-\top} \nu^\top$. Repeating this identity in every row and column block gives $\widetilde{\mathcal{M}} = \mathcal{G}_L \mathcal{M} \mathcal{G}_R$. Products and inverses of nonsingular lower triangular matrices are lower triangular, so G is lower triangular; transposition gives that H is upper triangular. Because $\mathcal{G}_L$ is lower triangular and $\mathcal{G}_R$ is upper triangular, their first $N + 1$ rows and columns do not receive contributions from indices outside the leading $(N + 1) \times (N + 1)$ blocks. Taking determinants therefore proves (6.12). The two triangular determinants are nonzero. $\square$

For the remainder of this subsection, the initial conditions are part of the polynomial data. More precisely, a right polynomial array $\mathbf{B}$ for a recurrence matrix $\mathcal{T}$, with right initial-condition matrix ξ , is the solution of $\mathcal{T} \mathbf{B}(x) = x \mathbf{B}(x)$ satisfying $E_{[q]} \mathbf{B}(x) = \xi$. Similarly, a left polynomial array $\mathbf{A}$, with left initial-condition matrix

ν , satisfies $A(x)\mathcal{T} = xA(x)$ and $A(x)E_{[p]}^\top = \nu^\top$. Once the nonsingular matrices ξ and ν have been fixed, the two recurrence equations determine their polynomial arrays uniquely. Strong normality means in addition that these arrays attain the prescribed maximal step-line degrees.

Consider a PBF $T_0 = F_1 \cdots F_{M-1}U$, where the last factor U is normalized upper bidiagonal, and set $D := F_1 \cdots F_{M-1}$ and $T_1 := UD$. Hence $T_0 = DU$, while T_1 is the cyclic product obtained by moving U to the left end.

Fix weakly normal biorthogonal data $(B_0, A_0, d\Psi_0)$ for T_0 , and assume that the right array B_0 attains its prescribed maximal step-line degrees. Let ξ_0, ν_0 be its nonsingular lower triangular initial-condition matrices, so that $E_{[q]}B_0 = \xi_0$ and $A_0E_{[p]}^\top = \nu_0^\top$. By Theorem 2.3, T_1 also has a PBF and is therefore BTP. Hence the maximal-degree criterion recalled in Remark 6.16 guarantees the existence of admissible right initial conditions; in particular, it is enough to choose ξ_1 so that ξ_1^{-1} is lower Δ TP. Fix such a matrix ξ_1 , and let B_1 be the unique solution of

$$T_1 B_1(x) = x B_1(x), \quad E_{[q]} B_1(x) = \xi_1,$$

which then has the prescribed maximal step-line degrees.

Theorem 6.18 (Christoffel transformation associated with an upper cyclic factor). *Under the preceding assumptions, the following statements hold.*

(i) *There is a unique $q \times q$ matrix polynomial $S(x)$ such that*

$$(6.13) \quad UB_0(x) = B_1(x)S(x).$$

It is given by

$$S(x) = \xi_1^{-1} E_{[q]} UB_0(x).$$

Moreover, S is regular and has degree one, its leading coefficient has rank one, and $\det S(x) = cx$ for some $c \neq 0$.

(ii) *Set*

$$A_1 := A_0 U^{-1}, \quad \nu_1^\top := A_1 E_{[p]}^\top, \quad d\Psi_1(x) := S(x) d\Psi_0(x).$$

Then ν_1 is nonsingular lower triangular, $A_1(x)T_1 = xA_1(x)$, and

$$\int B_1(x) d\Psi_1(x) A_1(x) = I.$$

The array A_1 belongs componentwise to the prescribed left step-line polynomial spaces, while B_1 attains its prescribed maximal degrees. Consequently, $(B_1, A_1, d\Psi_1)$ is a weakly normal biorthogonal system with recurrence matrix T_1 and initial-condition matrices ξ_1, ν_1 .

If, in addition, A_1 attains its prescribed maximal step-line degrees, then the transformed system is strongly normal. In particular, this additional conclusion holds if $\nu_0^{-\top}$ is upper Δ TP.

(iii) *For any other pair of nonsingular lower triangular initial-condition matrices $(\tilde{\xi}_1, \tilde{\nu}_1)$, the moment functional reconstructed from the same recurrence matrix T_1 is represented, whenever $d\Psi_1$ is a representing matrix of measures, by*

$$d\tilde{\Psi}_1 = \tilde{\xi}_1^{-1} \xi_1 d\Psi_1 \nu_1^\top \tilde{\nu}_1^{-\top}.$$

Such a triangular change preserves weak normality. Strong normality of the regauged polynomial arrays must be imposed or verified separately.

Proof. The eigenvalue equation $T_0 B_0(x) = x B_0(x)$, together with $T_0 = DU$ and $T_1 = UD$, gives $T_1(UB_0) = UDUB_0 = UT_0 B_0 = xUB_0$. Thus UB_0 is a right polynomial solution of the recurrence associated with T_1 .

A right polynomial solution is determined by its first q rows. Applying $E_{[q]}$ to (6.13) and using $E_{[q]}B_1 = \xi_1$ gives necessarily

$$S(x) = \xi_1^{-1} E_{[q]} UB_0(x).$$

With this choice, the two sides of (6.13) have the same first q rows and satisfy the same recurrence for T_1 . Uniqueness of the polynomial solution therefore proves the identity in every row.

We next prove that S is regular. If $\det S$ were identically zero, there would be a nonzero vector $\mathbf{v}(x) \in \mathbb{C}(x)^q$ such that $S(x)\mathbf{v}(x) = 0$. Equation (6.13) would then imply $U\mathbf{B}_0(x)\mathbf{v}(x) = 0$. Applying D and using $T_0 = DU$, one would obtain $x\mathbf{B}_0(x)\mathbf{v}(x) = 0$, and hence $\mathbf{B}_0(x)\mathbf{v}(x) = 0$. Its first q rows form the nonsingular matrix ξ_0 , so $\mathbf{v}(x) = 0$, a contradiction. Therefore $\det S \neq 0$.

Write $\mathbf{B}_{0,n}(x)$ for the n -th row of $\mathbf{B}_0(x)$, and denote its s -th component by $B_{0,n}^{(s)}(x)$, where $s \in \{0, \dots, q-1\}$. If $n = qm + j$, with $j \in \{0, \dots, q-1\}$, the maximal step-line degrees are

$$\deg B_{0,n}^{(s)} = \begin{cases} m, & 0 \leq s \leq j, \\ m-1, & j < s \leq q-1, \end{cases}$$

with the usual convention that a negative upper bound means that the corresponding polynomial vanishes. In particular, $\mathbf{B}_{0,0}, \dots, \mathbf{B}_{0,q-1}$ are constant, whereas $\mathbf{B}_{0,q}$ has degree one and its coefficient of x is nonzero only in its first component.

Since U is normalized upper bidiagonal,

$$(U\mathbf{B}_0)_n = u_n \mathbf{B}_{0,n} + \mathbf{B}_{0,n+1}.$$

Therefore the rows $(U\mathbf{B}_0)_0, \dots, (U\mathbf{B}_0)_{q-2}$ are constant, while $(U\mathbf{B}_0)_{q-1}$ has degree one. Its coefficient of x is nonzero by the assumed maximal-degree property of $\mathbf{B}_0$. Consequently,

$$\mathbf{E}_{[q]} U\mathbf{B}_0(x) = C_0 + xC_1,$$

where C_1 has exactly one nonzero row and hence rank one. Finally,

$$S(x) = \xi_1^{-1} \mathbf{E}_{[q]} U\mathbf{B}_0(x) = S_0 + xS_1.$$

Because ξ_1^{-1} is nonsingular, $\text{rank } S_1 = \text{rank } C_1 = 1$. By multilinearity of the determinant, $\deg \det S \leq 1$.

It remains to locate the zero of $\det S$. The matrix $D = F_1 \cdots F_{M-1}$ is the product of the remaining bidiagonal factors. Since U is the last upper factor of the PBF, the first row of D is supported in the first q coordinates and its restriction to those coordinates is nonzero. From $D\mathbf{B}_1(x)S(x) = DUB_0(x) = x\mathbf{B}_0(x)$, evaluation at $x = 0$ gives $D\mathbf{B}_1(0)S(0) = 0$. Since the first q rows of $\mathbf{B}_1(0)$ form the nonsingular matrix ξ_1 , the first row of $D\mathbf{B}_1(0)$ is nonzero and provides a nonzero left null vector of $S(0)$. Thus $\det S(0) = 0$. Since S is regular and $\deg \det S \leq 1$, it follows that $\det S(x) = cx$ with $c \neq 0$.

For the left polynomial solution, $A_0T_0 = xA_0$ and $T_0 = DU$ imply $A_0D = xA_0U^{-1}$. Therefore $A_1T_1 = A_0U^{-1}UD = A_0D = xA_0U^{-1} = xA_1$.

Finally, using $\mathbf{B}_1S = U\mathbf{B}_0$ and $A_1 = A_0U^{-1}$, one obtains

$$\int \mathbf{B}_1 d\Psi_1 A_1 = \int \mathbf{B}_1 S d\Psi_0 A_0 U^{-1} = U \left(\int \mathbf{B}_0 d\Psi_0 A_0 \right) U^{-1} = I.$$

It remains to record exactly what the triangular transformation implies for the left polynomial array. Put

$$U_{[p]} := \mathbf{E}_{[p]} U \mathbf{E}_{[p]}^T.$$

Since U^{-1} is upper triangular, its first p columns are supported in the first p rows, and hence

$$U^{-1} \mathbf{E}_{[p]}^T = \mathbf{E}_{[p]}^T U_{[p]}^{-1}.$$

Therefore

$$\mathbf{v}_1^T = A_0 U^{-1} \mathbf{E}_{[p]}^T = \mathbf{v}_0^T U_{[p]}^{-1}.$$

Both factors on the right are nonsingular upper triangular, so $\mathbf{v}_1$ is nonsingular lower triangular.

Moreover, for every $n \geq 0$,

$$A_{1,n} = \sum_{k=0}^n A_{0,k} (U^{-1})_{k,n}.$$

For each component, the maximal degree allowed at position k is no larger than the maximal degree allowed at position n whenever $k \leq n$. Thus $\mathbf{A}_{1,n}$ belongs to the prescribed left polynomial space. This triangular combination need not, however, preserve a nonzero maximal-degree coefficient.

The biorthonormality identity proved above makes every finite initial segment of each transformed family linearly independent. Since the polynomials lie in the corresponding step-line flags, they form dual bases of those flags. Hence the transformed system is weakly normal. The right array $\mathbf{B}_1$ has maximal degrees by the choice of ξ_1 .

Finally, if $\nu_0^{-\top}$ is upper Δ TP, then

$$\nu_1^{-\top} = U_{[p]} \nu_0^{-\top}.$$

The matrix $U_{[p]}$ is upper bidiagonal and totally nonnegative, with positive diagonal. For every upper-triangularly admissible minor with row and column sets I, J , the $K = I$ term in the Cauchy–Binet sum is $\det U_{[p]}[I, I] \det \nu_0^{-\top}[I, J] > 0$, while all other terms are nonnegative. Thus $\nu_1^{-\top}$ is upper Δ TP. Since T_1 is BTP and ξ_1^{-1} is lower Δ TP, the maximal-degree criterion recalled in Remark 6.16 gives strong normality. The final gauge statement is Proposition 6.17 applied to the two pairs of lower triangular initial conditions. This completes the proof. $\square$

Corollary 6.19 (Christoffel transformation associated with a lower cyclic factor). *Let $T_0 = LF_2 \cdots F_M$, where L is normalized lower bidiagonal, and put $D := F_2 \cdots F_M$ and $T_1 := DL$. Fix weakly normal biorthogonal data $(\mathbf{B}_0, \mathbf{A}_0, d\Psi_0)$ for T_0 , with nonsingular lower triangular initial-condition matrices ξ_0, ν_0 , and assume that the left array $\mathbf{A}_0$ attains its prescribed maximal step-line degrees. Choose a nonsingular lower triangular left initial-condition matrix ν_1 for which the corresponding left polynomial array $\mathbf{A}_1$ for T_1 has its prescribed maximal step-line degrees.*

Then there is a regular $p \times p$ matrix polynomial $R(x)$ of degree one, with rank-one leading coefficient and $\det R(x) = c'x$, $c' \neq 0$, such that

$$\mathbf{A}_0(x)L = R(x)\mathbf{A}_1(x), \quad d\Psi_1(x) = d\Psi_0(x)R(x).$$

Define

$$\mathbf{B}_1 := L^{-1}\mathbf{B}_0, \quad \xi_1 := E_{[q]}\mathbf{B}_1.$$

Then ξ_1 is nonsingular lower triangular, and $(\mathbf{B}_1, \mathbf{A}_1, d\Psi_1)$ is a weakly normal biorthogonal system with recurrence matrix T_1 . The left array $\mathbf{A}_1$ has maximal step-line degrees by construction.

If, in addition, $\mathbf{B}_1$ attains its prescribed maximal step-line degrees, then the transformed system is strongly normal. In particular, this additional conclusion holds if ξ_0^{-1} is lower Δ TP. Different nonsingular lower triangular initial conditions change this functional by the constant transformations of Proposition 6.17 and preserve weak normality; strong normality must be verified separately after such a change.

Proof. Transpose the argument of Theorem 6.18. The connection polynomial now acts on the right of the matrix of measures. Under transposition, the nonzero subdiagonal entries of L become a nonunit superdiagonal; the proof of regularity, the rank-one leading coefficient, and the simple zero of the determinant uses only their nonvanishing, so the same argument applies with these scalar factors retained. The identities

$$T_1 L^{-1} = L^{-1} T_0, \quad \mathbf{A}_0 L = R \mathbf{A}_1$$

give the transformed recurrence equations, and

$$\int \mathbf{B}_1 d\Psi_1 \mathbf{A}_1 = L^{-1} \left(\int \mathbf{B}_0 d\Psi_0 \mathbf{A}_0 \right) L = I.$$

Since L^{-1} is lower triangular, $\mathbf{B}_{1,n}$ is a finite linear combination of $\mathbf{B}_{0,0}, \dots, \mathbf{B}_{0,n}$; hence it belongs to its prescribed right polynomial space. Its leading $q \times q$ block is the product of two nonsingular lower triangular matrices, so ξ_1 is nonsingular lower triangular. Biorthonormality then gives weak normality. Triangularity alone does not prevent cancellation of a maximal right degree, so strong normality requires the additional hypothesis stated in the corollary.

For the sufficient condition, put $L_{[q]} := E_{[q]} L E_{[q]}^T$. The initial blocks satisfy $\xi_1 = L_{[q]}^{-1} \xi_0$, and hence

$$\xi_1^{-1} = \xi_0^{-1} L_{[q]}.$$

If ξ_0^{-1} is lower Δ TP, then for every lower-triangularly admissible minor with row and column sets I, J , the $K = J$ term in the Cauchy–Binet sum for $\xi_0^{-1} L_{[q]}$ is strictly positive and all other terms are nonnegative. Thus ξ_1^{-1} is lower Δ TP. The BTP criterion therefore supplies the missing maximal right degrees. $\square$

Corollary 6.20 (A complete factor circuit). *Recall that $T = AB$, where $A = L_1 \cdots L_p$ and $B = U_q \cdots U_1$. Set $a_0 := 1$ and $a_j := M - j + 1$ for $j \in \{1, \dots, q\}$. Moving successively the rightmost upper factor to the left carries $T_{(a_{j-1})}$ into $T_{(a_j)}$. In particular, $T_{(a_0)} = T$ and $T_{(a_q)} = T_{(p+1)} = BA$.*

Fix a weakly normal biorthogonal system $(\mathbf{B}^{(a_0)}, \mathbf{A}^{(a_0)}, d\Psi)$ at $T_{(a_0)} = T$, and assume that its right array attains the prescribed maximal step-line degrees. For each $j \in \{1, \dots, q\}$, choose a nonsingular lower triangular right initial-condition matrix $\xi^{(a_j)}$ for which the corresponding right polynomial array $\mathbf{B}^{(a_j)}$ has the prescribed maximal step-line degrees. Transport the left arrays and moment functionals successively by the upper-factor transformations of Theorem 6.18. Let $S_j(x)$ be the connection matrix associated by that theorem with the transfer from $T_{(a_{j-1})}$ to $T_{(a_j)}$. Then the following statements hold.

- (i) *There is a nonsingular constant lower triangular matrix H such that*

$$S_q(x) \cdots S_1(x) = xH.$$

The matrix H is the lower triangular endpoint-normalization factor.

- (ii) *The composition of the q Christoffel transformations gives the representative $xH d\Psi$ at $T_{(a_q)} = BA$. Replacing the endpoint right initial condition by $\tilde{\xi}^{(a_q)} = \xi^{(a_q)} H$ applies the constant left gauge H^{-1} to the functional, and the endpoint representative becomes $x d\Psi$.*
- (iii) *The circuit obtained by moving successively the lower factors has the transposed form: one chooses a left-maximal array at every vertex and transports the right arrays and functionals. Its connection matrices act on the right and their product is xK , where K is a nonsingular constant upper triangular matrix.*

Proof. Put $P(x) := S_q(x) \cdots S_1(x)$. At the j -th transfer, Theorem 6.18 relates the right polynomial arrays at $T_{(a_{j-1})}$ and $T_{(a_j)}$. Its one-sided conclusion is sufficient for iteration: the transported system remains weakly normal, while the right array at the next vertex is maximal by construction. Composing these q relations gives $B\mathbf{B}^{(a_0)}(x) = \mathbf{B}^{(a_q)}(x)P(x)$.

The first q rows of $B\mathbf{B}^{(a_0)}(x)$ have degree at most one. Indeed, B is a product of q upper bidiagonal matrices, so these rows involve only the first $2q$ rows of $\mathbf{B}^{(a_0)}$, whose maximal step-line degrees are at most one. The first q rows of $\mathbf{B}^{(a_q)}$ form the nonsingular constant matrix $\xi^{(a_q)}$. It follows that $\deg P \leq 1$.

The eigenvalue equation for $\mathbf{B}^{(a_0)}$ gives $A(B\mathbf{B}^{(a_0)}(x)) = T\mathbf{B}^{(a_0)}(x) = x\mathbf{B}^{(a_0)}(x)$. Evaluating at $x = 0$ and using the invertibility of the lower triangular matrix A yields $B\mathbf{B}^{(a_0)}(0) = 0$. The endpoint connection identity then gives $\xi^{(a_q)} P(0) = 0$. Since $\xi^{(a_q)}$ is nonsingular, $P(0) = 0$. Therefore $P(x) = xH$ for some constant matrix H .

For every $j \in \{1, \dots, q\}$, Theorem 6.18 gives $\det S_j(x) = c_j x$, with $c_j \neq 0$. Hence $\det P(x) = (\prod_{j=1}^q c_j) x^q$. On the other hand, $P(x) = xH$ gives $\det P(x) = x^q \det H$. Thus $\det H = \prod_{j=1}^q c_j \neq 0$, and H is nonsingular. To see the triangular form explicitly, put $A_{[q]} := E_{[q]} A E_{[q]}^T$. Since $A(B\mathbf{B}^{(a_0)}) = x\mathbf{B}^{(a_0)}$, the endpoint connection identity and $P = xH$ give, after cancelling x ,

$$\mathbf{B}^{(a_q)} H = A^{-1} \mathbf{B}^{(a_0)}.$$

Taking the first q rows yields

$$H = (\xi^{(a_q)})^{-1} A_{[q]}^{-1} \xi^{(a_0)}.$$

All three factors are lower triangular, so H is lower triangular. This proves (i).

At each transfer the matrix of measures is multiplied on the left by the corresponding connection matrix. After the q transfers it is therefore $P(x) d\Psi(x) = xH d\Psi(x)$. The formula above gives $\tilde{\xi}^{(a_q)} = \xi^{(a_q)} H = A_{[q]}^{-1} \xi^{(a_0)}$, which is

nonsingular lower triangular. Moreover,

$$(\widehat{\xi}^{(a_q)})^{-1} \xi^{(a_q)} = H^{-1}.$$

Proposition 6.17 therefore applies the constant left gauge H^{-1} to $xH d\Psi$ and produces the representative $x d\Psi$. This proves (ii).

Applying the same argument to the left polynomial arrays and the lower bidiagonal factors proves (iii). $\square$

The connection matrices depend covariantly on the chosen initial conditions. Indeed, if $\mathbf{B}_j$ is replaced by $\mathbf{B}_j^\# = \mathbf{B}_j C_j$, with C_j a constant nonsingular lower triangular matrix, then

$$S_j^\#(x) = C_j^{-1} S_j(x) C_{j-1}.$$

Thus all intermediate gauges cancel in a complete circuit; only the two endpoint gauges change the constant matrix H .

Corollary 6.21 (Comparison of endpoint functionals). *Consider two factor-transfer constructions of the preceding type, starting from the same strongly normal biorthogonal system with recurrence matrix T_{in} and moment functional Ψ_{in} . At every upper, respectively lower, step choose the right-, respectively left-maximal data required by Theorem 6.18 and Corollary 6.19. The intermediate recurrence matrices and their lower triangular initial-condition matrices may be different. Suppose that both constructions end at the same recurrence matrix T_{out} .*

Let Ψ_1 and Ψ_2 be the two endpoint moment functionals, and let (ξ_1, v_1) and (ξ_2, v_2) be their respective endpoint initial-condition matrices. Then

$$\Psi_2 = G \Psi_1 H, \quad G = \xi_2^{-1} \xi_1, \quad H = v_1^\top v_2^{-\top}.$$

The matrix G is nonsingular lower triangular and H is nonsingular upper triangular. If the two constructions use the same initial-condition matrices at T_{out} , then $G = I_q$, $H = I_p$, and $\Psi_1 = \Psi_2$.

Proof. For $i \in \{1, 2\}$, reconstruction from the endpoint recurrence and initial conditions gives

$$\langle \Psi_i, x^n \rangle = \xi_i^{-1} E_{[q]} T_{\text{out}}^n E_{[p]}^\top v_i^{-\top}, \quad n \in \mathbb{N}_0.$$

Substitution shows on every monomial that $\Psi_2 = (\xi_2^{-1} \xi_1) \Psi_1 (v_1^\top v_2^{-\top})$, and hence proves the displayed functional identity. The standard lower triangular orientation of the initial-condition matrices makes G lower triangular and H upper triangular. If the endpoint matrices coincide, both constant transformations are the identity matrices. $\square$

The recurrence solutions can also be transported directly through the bidiagonal factors. Under the Christoffel hypotheses introduced in the next subsection, the transported solutions belong to the standard step-line polynomial spaces and carry the normalization of the transformed mixed-type orthogonal system. For the first cyclic step, write

$$T = F_1 G_1, \quad G_1 = F_2 \cdots F_M, \quad T_{(2)} = G_1 F_1.$$

If $\mathbf{B}(x)$ and $\mathbf{A}(x)$ are respectively right and left polynomial solutions of the recurrence equations for T , then

$$(6.14) \quad \widetilde{\mathbf{B}}^{(2)}(x) = F_1^{-1} \mathbf{B}(x) = \frac{1}{x} G_1 \mathbf{B}(x), \quad \widetilde{\mathbf{A}}^{(2)}(x) = \mathbf{A}(x) F_1.$$

Indeed, $T F_1 = F_1 T_{(2)}$. Therefore

$$T_{(2)} F_1^{-1} \mathbf{B} = F_1^{-1} T \mathbf{B} = x F_1^{-1} \mathbf{B},$$

and the analogous calculation on the left proves the two recurrence equations. The identity $G_1 \mathbf{B} = x F_1^{-1} \mathbf{B}$ follows from $T = F_1 G_1$. Since $F_1 = L_1$ is lower unitriangular, F_1^{-1} acts on each component by a finite triangular sum, so the first vector in (6.14) is polynomial.

At a general factor position, define

$$G_a := F_{a+1} \cdots F_M F_1 \cdots F_{a-1}, \quad T_{(a)} = F_a G_a, \quad T_{(a+1)} = G_a F_a,$$

with cyclic indices. The same intertwining calculation shows that $G_a \mathbf{B}^{(a)}$ and $\mathbf{A}^{(a)} F_a$ satisfy the right and left recurrence equations for $T_{(a+1)}$. When the factors form the weakly normal Christoffel chain of Theorem 6.24, the

nonvanishing Christoffel determinants ensure existence and divisibility, while the initial blocks $\xi^{(a+1)}$ and $\nu^{(a+1)}$ select the standard normalization and identify the transformed measures. For a general factorization, the same calculation gives the transport of recurrence solutions.

This observation has different content in the two applications below. For the Piñeiro system, each Christoffel step only shifts a power parameter, so (6.14) becomes a contiguous relation between already known hypergeometric formulas. In the Jacobi-like family, the lower steps are explicit but the upper steps require the upper bidiagonal factors, whose entries are ratios of moment minors. Once those minors have been evaluated, the complete cyclic chain of polynomial systems becomes explicit.

Remark 6.22. This factor-adapted construction requires no separate nonvanishing condition for the fixed moment shifts: its matrix polynomial is determined from the two cyclic recurrence systems. The fixed shift matrices considered next give a more explicit statement, but their one-sided transforms must exist.

6.3. The two-sided Christoffel rectangle. This subsection answers the following question. Starting from a matrix of measures and its bidiagonally factorized recurrence matrix, can the elementary row and column Christoffel transformations be performed simultaneously while retaining positive bidiagonal recurrence factorizations?

For $b \in \{0, \dots, q\}$ and $a \in \{0, \dots, p\}$, the vertices of the rectangle will be represented by

$$d\Psi_{(b,a)}(x) = K_q(x)^b d\Psi(x) (K_p(x)^a)^\top.$$

Its scalar moment matrix is obtained from the original moment matrix $\mathcal{M}$ by shifting the row index by b and the column index by a . The main result shows that weak normality on the two boundary chains propagates to every vertex. At each vertex it identifies the transformed measure, the shifted moment matrix, the recurrence matrix, its positive bidiagonal factors, and the two mixed-type polynomial families. The proof reduces to the positive refactorization attached to one elementary square. Figure 5 displays both the full rectangle and that elementary square.

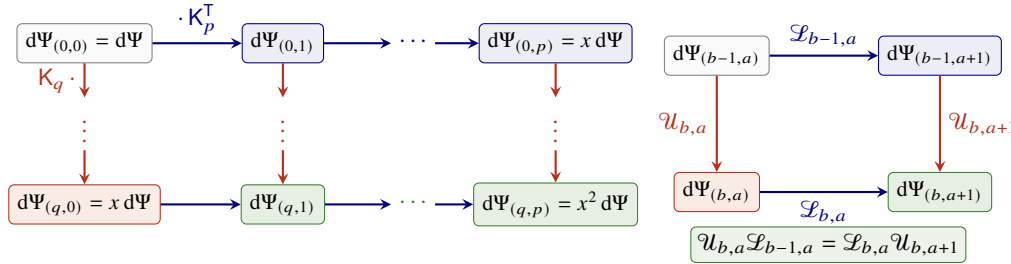


FIGURE 5. The two-sided Christoffel rectangle and its elementary commuting square. Horizontal arrows are column-side Christoffel transformations and downward arrows are row-side Christoffel transformations; they are shown in blue and red, respectively, while green marks vertices reached in both directions. A factorization path from $(q, 0)$ to $(0, p)$ traverses horizontal arrows forward and vertical arrows backward.

6.3.1. Boundary chains and shifted moment matrices. Corollary 6.15 gives positive spectral data for every cyclic product. We now describe the more specific situation in which the prescribed bidiagonal factors are the connection matrices of two one-sided Christoffel chains. For $r \geq 1$, define

$$K_r(x) := \begin{bmatrix} 0_{(r-1) \times 1} & I_{r-1} \\ x & 0_{1 \times (r-1)} \end{bmatrix}, \quad K_r(x)^r = x I_r.$$

Starting from a $q \times p$ matrix of measures $d\Psi$, define the successive right Christoffel transforms by $d\Psi_{(0,a)}(x) := d\Psi(x) (K_p(x)^a)^\top$, for $a \in \{0, \dots, p\}$, and the successive left Christoffel transforms by $d\Psi_{(b,0)}(x) := K_q(x)^b d\Psi(x)$, for $b \in \{0, \dots, q\}$. Thus $d\Psi_{(0,0)} = d\Psi$, and the two chains have the common endpoint $d\Psi_{(0,p)} = d\Psi_{(q,0)} = x d\Psi$.

The chains are weakly normal when all their step-line orthogonality problems exist and are unique up to nonzero constants. At each degree, the Christoffel formula is obtained from a finite linear system expressing the required cancellation at $x = 0$. The determinants of these systems are the Christoffel τ -determinants. The Christoffel factorization theorem [12] states that weak normality of the two chains is equivalent to the nonvanishing of the corresponding τ -determinants at every degree.

Definition 6.23 (One-step Christoffel connection matrices). Let $d\Psi_{0,0}$ be a matrix of measures with a weakly normal mixed-type system $(A_{0,0}, B_{0,0})$ and fixed initial conditions. Suppose that the two elementary Christoffel transforms

$$d\Psi_{0,1} = d\Psi_{0,0} K_p^T, \quad d\Psi_{1,0} = K_q d\Psi_{0,0}$$

are weakly normal, and equip their polynomial arrays with the initial conditions obtained from the corresponding one-step Christoffel formulas. The normalized lower bidiagonal matrix L characterized by

$$A_{0,0}L = K_p^T A_{0,1}, \quad LB_{0,1} = B_{0,0}$$

is called the one-step column-side Christoffel connection matrix. The normalized upper bidiagonal matrix U characterized by

$$UB_{0,0} = B_{1,0}K_q, \quad A_{1,0}U = A_{0,0}$$

is called the one-step row-side Christoffel connection matrix. The same terminology is used at every successive vertex of either one-sided chain.

Denote by $T_{(0,a)}$ and $T_{(b,0)}$ the recurrence matrices of the systems represented by $d\Psi_{(0,a)}$ and $d\Psi_{(b,0)}$, respectively. Their initial-condition matrices are obtained successively from those at $(0,0)$ by the one-step Christoffel formulas.

For $a \in \{1, \dots, M\}$, set

$$d\Psi^{(a)} := \begin{cases} d\Psi_{(0,a-1)}, & 1 \leq a \leq p+1, \\ d\Psi_{(M+1-a,0)}, & p+2 \leq a \leq M. \end{cases}$$

Let $\mathcal{M}^{(a)}$ be its moment matrix and let $\xi^{(a)}, \nu^{(a)}$ be the initial-condition matrices transported by the corresponding one-step Christoffel transformations.

Theorem 6.24 (Recurrence matrices under cyclic Christoffel shifts). *Let $d\Psi$ be a $q \times p$ matrix of measures with all moments finite, and suppose that its moment matrix is weakly normal along the step-line and has recurrence matrix T with initial-condition matrices ξ, ν . Suppose that both one-sided chains defined above are weakly normal. Assume, in the sense of Definition 6.23, that L_a is the normalized column-side connection matrix from $d\Psi_{(0,a-1)}$ to $d\Psi_{(0,a)}$, for every $a \in \{1, \dots, p\}$, and that U_b is the normalized row-side connection matrix from $d\Psi_{(b-1,0)}$ to $d\Psi_{(b,0)}$, for every $b \in \{1, \dots, q\}$. Then, for every $a \in \{1, \dots, M\}$, $T_{(a)}$ is the recurrence matrix of the transformed system with data $(d\Psi^{(a)}, \xi^{(a)}, \nu^{(a)})$. At the first position,*

$$d\Psi^{(1)} = d\Psi, \quad \xi^{(1)} = \xi, \quad \nu^{(1)} = \nu.$$

At the common endpoint of the row and column Christoffel chains,

$$T_{(p+1)} = BA, \quad d\Psi^{(p+1)} = x d\Psi.$$

For every $a \in \{1, \dots, M\}$, these data satisfy

$$(6.15) \quad (\xi^{(a)})^{-1} E_{[q]} (T_{(a)})^n E_{[p]}^T (\nu^{(a)})^{-T} = \int x^n d\Psi^{(a)}(x), \quad n \in \mathbb{N}_0.$$

The right and left polynomial systems at that position are the Christoffel transforms determined by the transported initial-condition matrices $\xi^{(a)}$ and $\nu^{(a)}$. These conclusions hold for both bounded and unbounded factors.

Proof. The weak normality hypothesis ensures that the two one-sided transformed systems and their transported initial conditions are defined at every step-line degree. The one-step Christoffel connection formulas give the intertwining identities

$$T_{(0,a-1)}L_a = L_aT_{(0,a)}, \quad U_bT_{(b-1,0)} = T_{(b,0)}U_b.$$

Starting from $T = L_1 \cdots L_p U_q \cdots U_1$, induction therefore yields

$$\begin{aligned} T_{(0,a)} &= L_{a+1} \cdots L_p U_q \cdots U_1 L_1 \cdots L_a, & 0 \leq a \leq p, \\ T_{(b,0)} &= U_b \cdots U_1 L_1 \cdots L_p U_q \cdots U_{b+1}, & 1 \leq b \leq q. \end{aligned}$$

Since $d\Psi_{(0,p)} = d\Psi_{(q,0)} = x d\Psi$, these two lists join at their common endpoint. Comparison with the definition of $T_{(a)}$ now gives

$$T_{(a+1)} = T_{(0,a)} \quad (0 \leq a \leq p), \quad T_{(M+1-b)} = T_{(b,0)} \quad (1 \leq b \leq q).$$

Thus the cyclic products are exactly the recurrence matrices of the transformed systems listed above. At each one-step transformation the Christoffel formula transports the recurrence solutions, their initial conditions, and the represented functional. Starting from $(T, \xi, \nu, d\Psi)$ and iterating around the two boundary chains therefore gives (6.15). Equivalently, Proposition 5.2 identifies $d\Psi^{(a)}$ with the unique functional determined by $T_{(a)}, \xi^{(a)}, \nu^{(a)}$. The spectral representations are valid in both the bounded and unbounded settings under the hypotheses of [10, 14]. $\square$

The initial conditions throughout the rectangle are fixed by transport. Starting from those at the vertex $(0, 0)$, the one-sided Christoffel formulas determine the initial conditions at the boundary vertices and then at every interior vertex. Thus the polynomial arrays at different vertices are not independently regauged. In the statements below, a normalized connection matrix always refers to this transported choice of initial conditions, together with the bidiagonal normalization fixed in Section 2.

For a recurrence matrix S with initial-condition matrices ξ_S and ν_S , let $\mathcal{R}(S; \xi_S, \nu_S)$ denote the scalar moment matrix reconstructed from the functional in Proposition 5.2. Explicitly, for $u, v \in \mathbb{N}_0$, $j \in \{0, \dots, q-1\}$, and $i \in \{0, \dots, p-1\}$,

$$(6.16) \quad \mathcal{R}(S; \xi_S, \nu_S)_{qu+j, pv+i} = \left[\xi_S^{-1} E_{[q]} S^{u+v} E_{[p]}^\top \nu_S^{-\top} \right]_{j+1, i+1}.$$

Lemma 6.25 (Measures representing row and column moment shifts). *For $b \in \{0, \dots, q\}$ and $a \in \{0, \dots, p\}$, define the shifted scalar matrix*

$$\mathcal{M}^{(b,a)} = [\mathcal{M}_{r+b, s+a}]_{r,s \geq 0}.$$

If $\mathcal{M}$ is represented by a $q \times p$ matrix of measures $d\Psi$, then $\mathcal{M}^{(b,a)}$ is represented by

$$(6.17) \quad K_q(x)^b d\Psi(x) (K_p(x)^a)^\top.$$

Proof. Left multiplication by $K_q(x)$ advances the scalar step-line row index by one, including the factor x when the component index passes from q back to 1. Likewise, right multiplication by $K_p(x)^\top$ advances the scalar column index by one. Iterating these two identities shows, entry by entry, that the moments of (6.17) are $\mathcal{M}_{r+b, s+a}$, as claimed. $\square$

6.3.2. The elementary square and construction of the rectangle.

Lemma 6.26 (Permutability of one row and one column Christoffel step). *Under the hypotheses and with the notation of Definition 6.23, let*

$$UL = L'U'$$

be its normalized positive lower–upper refactorization. Then

$$d\Psi_{1,1} = K_q d\Psi_{0,0} K_p^\top$$

has a weakly normal mixed-type system. Its polynomial arrays are

$$(6.18) \quad B_{1,1} = (L')^{-1} B_{1,0}, \quad A_{1,1} = A_{0,1} (U')^{-1},$$

and the outgoing connection relations are

$$(6.19) \quad \begin{aligned} A_{1,0} L' &= K_p^\top A_{1,1}, & L' B_{1,1} &= B_{1,0}, \\ U' B_{0,1} &= B_{1,1} K_q, & A_{1,1} U' &= A_{0,1}. \end{aligned}$$

Proof. First we verify the degrees. Since L' is normalized lower bidiagonal, every finite leading block of $(L')^{-1}$ is lower unitriangular. Hence, for every n ,

$$\mathbf{B}_{1,1,n} = \mathbf{B}_{1,0,n} + \sum_{k < n} c_{n,k} \mathbf{B}_{1,0,k}$$

for suitable scalars $c_{n,k}$. The column-side shift from $(1, 0)$ to $(1, 1)$ does not change the prescribed right step-line polynomial space. Since the degree bounds in that flag are nondecreasing with n , the displayed triangular combination shows that $\mathbf{B}_{1,1,n}$ belongs to the required space. No maximal-degree conclusion is needed here.

Likewise, U' is upper bidiagonal with a nonzero diagonal. If its n -th diagonal entry is u'_n , then the n -th column of the second formula in (6.18) gives

$$\mathbf{A}_{1,1,n} = \frac{1}{u'_n} \mathbf{A}_{0,1,n} + \sum_{k < n} d_{k,n} \mathbf{A}_{0,1,k}.$$

The row-side shift from $(0, 1)$ to $(1, 1)$ does not change the prescribed left step-line polynomial space. The same monotonicity of the degree bounds shows that $\mathbf{A}_{1,1,n}$ belongs to that space. The triangular combination need not preserve a nonzero maximal-degree coefficient, and weak normality does not require it.

The initial blocks remain nonsingular with the standard lower triangular orientation. Indeed, if $\xi_{1,0}$ and $\nu_{0,1}$ denote the boundary initial-condition matrices and

$$L'_{[q]} = E_{[q]} L' E_{[q]}^T, \quad U'_{[p]} = E_{[p]} U' E_{[p]}^T,$$

then triangularity gives

$$E_{[q]} \mathbf{B}_{1,1} = (L'_{[q]})^{-1} \xi_{1,0}, \quad \mathbf{A}_{1,1} E_{[p]}^T = \nu_{0,1}^T (U'_{[p]})^{-1}.$$

Two relations in (6.19) follow immediately from (6.18). For the other two, use the incoming relations:

$$UL\mathbf{B}_{0,1} = U\mathbf{B}_{0,0} = \mathbf{B}_{1,0}\mathbf{K}_q.$$

Substitution of $UL = L'U'$, followed by multiplication on the left by $(L')^{-1}$, gives $U'\mathbf{B}_{0,1} = \mathbf{B}_{1,1}\mathbf{K}_q$. Similarly,

$$\mathbf{A}_{1,0}UL = \mathbf{A}_{0,0}L = \mathbf{K}_p^T \mathbf{A}_{0,1}.$$

Substitution of $UL = L'U'$, followed by multiplication on the right by $(U')^{-1}$, gives $\mathbf{A}_{1,0}L' = \mathbf{K}_p^T \mathbf{A}_{1,1}$.

It remains to verify the orthogonality directly. Put

$$\mathcal{F} := \int \mathbf{B}_{1,1}(x) d\Psi_{1,0}(x) \mathbf{A}_{1,0}(x).$$

Since $L'\mathbf{B}_{1,1} = \mathbf{B}_{1,0}$ and the system at $(1, 0)$ is biorthonormal,

$$L'\mathcal{F} = \int \mathbf{B}_{1,0}(x) d\Psi_{1,0}(x) \mathbf{A}_{1,0}(x) = I.$$

Thus $\mathcal{F} = (L')^{-1}$, and

$$\begin{aligned} \int \mathbf{B}_{1,1}(x) d\Psi_{1,1}(x) \mathbf{A}_{1,1}(x) &= \int \mathbf{B}_{1,1}(x) d\Psi_{1,0}(x) \mathbf{K}_p(x)^T \mathbf{A}_{1,1}(x) \\ &= \mathcal{F} L' = I. \end{aligned}$$

The two arrays are therefore biorthonormal for $d\Psi_{1,1}$. The degree statements proved at the beginning of the argument show that they belong to the prescribed step-line polynomial spaces. Hence the system at $(1, 1)$ is weakly normal. $\square$

Corollary 6.27 (The elementary τ -determinants). *In the setting of Lemma 6.26, the Christoffel τ -determinants on the outgoing column and row sides are*

$$\tau_n^{\text{col}} = A_{1,0,n}^{(1)}(0), \quad \tau_n^{\text{row}} = B_{0,1,n}^{(1)}(0),$$

and are nonzero for every $n \in \mathbb{N}_0$. Equivalently, the first row of the upper Gauss–Borel coefficient matrix at $(1, 0)$ and the first column of the lower Gauss–Borel coefficient matrix at $(0, 1)$ have no zero entries. The superscripts col

and row refer to the outgoing edges controlled by these determinants: the evaluation at $(1, 0)$ controls the column step to $(1, 1)$, whereas the evaluation at $(0, 1)$ controls the row step to $(1, 1)$.

Proof. Both $K_p^\top$ and K_q have the single simple determinantal zero $x = 0$. The relevant left null vector of $K_p(0)^\top$, and the relevant right null vector of $K_q(0)$, are in both cases the first coordinate vector. Hence the general determinants of [39] reduce to the two scalar evaluations displayed above.

Write $L'_{n+1,n} = \ell'_n > 0$, $U'_{n,n} = u'_n > 0$, and $U'_{n,n+1} = 1$. Evaluating the first and third relations in (6.19) at $x = 0$, and selecting the first polynomial component, gives

$$A_{1,0,n}^{(1)}(0) + \ell'_n A_{1,0,n+1}^{(1)}(0) = 0, \quad u'_n B_{0,1,n}^{(1)}(0) + B_{0,1,n+1}^{(1)}(0) = 0.$$

The initial evaluations are nonzero because the initial-condition matrices are triangular and nonsingular. Iterating the two preceding identities gives, for every $n \geq 1$,

$$A_{1,0,n}^{(1)}(0) = \frac{(-1)^n A_{1,0,0}^{(1)}(0)}{\prod_{k=0}^{n-1} \ell'_k}, \quad B_{0,1,n}^{(1)}(0) = (-1)^n \left(\prod_{k=0}^{n-1} u'_k \right) B_{0,1,0}^{(1)}(0).$$

Thus the two sequences alternate in sign and have no zero entries. Notice that this conclusion uses only the connection identities, the nonzero initial evaluations, and the strict positivity of the bidiagonal coefficients; no total-positivity assumption on the moment matrix is used. The same calculation applies to an edge on either boundary whenever its prescribed factor is already known to be the normalized Christoffel connection matrix. Finally, $\mathbf{B}(0) = \mathcal{L} \mathbf{E}_{[q]}^\top$ and $\mathbf{A}(0) = \mathbf{E}_{[p]} \mathcal{U}$, so these evaluations are the entries of the stated column and row. $\square$

Proposition 6.28 (Positive bidiagonal factors on the moment-shift rectangle). *There is a unique collection of matrices $\mathcal{U}_{b,a}$ and $\mathcal{L}_{b,a}$, indexed by the edges of the moment-shift rectangle, that satisfies the boundary conditions*

$$\mathcal{U}_{b,0} = U_b, \quad b \in \{1, \dots, q\}, \quad \mathcal{L}_{0,a} = L_{a+1}, \quad a \in \{0, \dots, p-1\}.$$

At every elementary square these matrices satisfy

$$(6.20) \quad \mathcal{U}_{b,a} \mathcal{L}_{b-1,a} = \mathcal{L}_{b,a} \mathcal{U}_{b,a+1}, \quad b \in \{1, \dots, q\}, \quad a \in \{0, \dots, p-1\}.$$

Every $\mathcal{U}_{b,a}$ is normalized upper bidiagonal and every $\mathcal{L}_{b,a}$ is normalized lower bidiagonal. All their nontrivial entries are strictly positive.

Proof. Suppose that row $b-1$ has already been constructed. At the beginning of row b , the factor $\mathcal{U}_{b,0} = U_b$ is known. For $a = 0$, the left-hand side of (6.20) is therefore a known normalized positive upper–lower product. Applying Lemma 2.2 determines a unique normalized positive lower–upper pair $\mathcal{L}_{b,0} \mathcal{U}_{b,1}$. The newly obtained upper factor is then used at $a = 1$, and the argument continues until the end of the row. Repeating this procedure for $b = 1, \dots, q$ constructs the entire rectangle. The same lemma proves at every step the asserted normalization, positivity, and uniqueness. $\square$

6.3.3. Recurrences and polynomial systems at the vertices.

Theorem 6.29 (Moment systems on the two-sided Christoffel rectangle). *Suppose that the prescribed PBF is the Christoffel factorization of a weakly normal system represented by $d\Psi$, with recurrence matrix T and initial-condition matrices ξ, ν . Suppose also that the two one-sided Christoffel chains along the lower and left boundaries of the rectangle are weakly normal, or equivalently that their Christoffel τ -determinants do not vanish.*

For every vertex (b, a) , the matrices $\mathcal{L}_{b,a}$ and $\mathcal{U}_{b,a}$ are the normalized column-side and row-side connection matrices on the corresponding edges. The right and left recurrence polynomials and their initial-condition matrices obtained by transporting (T, ξ, ν) along a path from $(0, 0)$ to (b, a) are independent of that path. Denote them by $\mathbf{B}_{(b,a)}, \mathbf{A}_{(b,a)}, \xi_{(b,a)}, \nu_{(b,a)}$, and denote their recurrence matrix by $T_{(b,a)}$. Their reconstructed moment matrix is

$$(6.21) \quad \mathcal{R}(T_{(b,a)}; \xi_{(b,a)}, \nu_{(b,a)}) = \mathcal{M}^{(b,a)} = [\mathcal{M}_{r+b, s+a}]_{r, s \geq 0}.$$

The matrix of measures

$$(6.22) \quad d\Psi_{(b,a)}(x) := K_q(x)^b d\Psi(x) (K_p(x)^a)^\top$$

represents the reconstructed moment matrix. In particular, for every $n \in \mathbb{N}_0$,

$$(6.23) \quad \xi_{(b,a)}^{-1} E_{[q]} T_{(b,a)}^n E_{[p]}^\top \nu_{(b,a)}^{-\top} = \int x^n d\Psi_{(b,a)}(x).$$

The right and left polynomial solutions of the recurrence for $T_{(b,a)}$, normalized by $\xi_{(b,a)}$ and $\nu_{(b,a)}$, are the mixed-type multiple orthogonal polynomials for $d\Psi_{(b,a)}$.

Proof. We first identify the systems at the vertices and the factors on the interior edges. The systems on the two boundary chains and their connection factors are known by hypothesis. We fill the rectangle one elementary square at a time, from left to right in each row and then from the bottom row to the top. Suppose that the systems at the lower-left, lower-right, and upper-left vertices of a square have already been constructed. In the notation of Lemma 6.26, their measures are

$$d\Psi_{0,0}, \quad d\Psi_{0,1} = d\Psi_{0,0} K_p^\top, \quad d\Psi_{1,0} = K_q d\Psi_{0,0}.$$

The two known incoming edge factors form the upper-lower product on the left-hand side of (6.20). Its unique positive lower-upper refactorization is the pair of outgoing edge factors constructed in Proposition 6.28. Lemma 6.26 then constructs the two polynomial arrays at the fourth vertex, proves their biorthogonality for

$$K_q d\Psi_{0,0} K_p^\top,$$

and proves all four connection identities on the square. This also proves weak normality at the fourth vertex. Induction therefore gives, at every vertex,

$$(6.24) \quad d\Psi_{(b,a)}(x) = K_q(x)^b d\Psi(x) (K_p(x)^a)^\top,$$

and identifies every factor in Proposition 6.28 as the corresponding normalized Christoffel connection matrix. Corollary 6.27 shows at the same time that all elementary τ -determinants on the new edges are nonzero.

Any two monotone paths from $(0, 0)$ to a fixed vertex (b, a) are connected by elementary-square moves. The four connection identities in Lemma 6.26 show that replacing either two-edge route across one square by the other produces the same normalized polynomial arrays and the same transported initial conditions. Hence $B_{(b,a)}$, $A_{(b,a)}$, $\xi_{(b,a)}$, and $\nu_{(b,a)}$ are independent of the chosen path. Their recurrence equations determine $T_{(b,a)}$ uniquely.

It remains to identify the scalar moment matrix reconstructed from these recurrence data. By Lemma 6.25, the moments of (6.24) form precisely

$$\mathcal{M}^{(b,a)} = [\mathcal{M}_{r+b, s+a}]_{r, s \geq 0}.$$

On the other hand, Proposition 5.2, applied to the normalized endpoint system, reconstructs those same moments from $T_{(b,a)}$, $\xi_{(b,a)}$, $\nu_{(b,a)}$. This proves (6.21); inserting the definition (6.16) gives (6.23). Hence (6.22) is a representing matrix of measures for the reconstructed functional, and the endpoint arrays are its two mixed-type multiple orthogonal polynomial families. $\square$

Corollary 6.30 (Path factorizations of the Darboux product). *For a monotone path γ from $(q, 0)$ to $(0, p)$, let $\mathcal{F}(\gamma)$ be the ordered product obtained by assigning $\mathcal{U}_{b,a}$ to each step $(b, a) \mapsto (b-1, a)$ and $\mathcal{L}_{b,a}$ to every step $(b, a) \mapsto (b, a+1)$. Then*

$$\mathcal{F}(\gamma) = BA = U_q \cdots U_1 L_1 \cdots L_p,$$

so $\mathcal{F}(\gamma)$ is independent of γ . If γ passes through (b, a) , reading its factors cyclically from that vertex gives the recurrence matrix $T_{(b,a)}$ of Theorem 6.29; in particular, the cut product is also independent of γ . The $\binom{p+q}{p}$ monotone paths therefore give positive bidiagonal factorizations of BA with every interlacing of the lower and upper factor types that preserves their relative orders.

Proof. For the boundary path consisting first of the q upward steps and then of the p rightward steps, the edge factors are $U_q, \dots, U_1, L_1, \dots, L_p$, and their product is BA . Any two monotone paths are connected by elementary-square moves. Each move replaces $\mathcal{U}_{b,a}\mathcal{L}_{b-1,a}$ by $\mathcal{L}_{b,a}\mathcal{U}_{b,a+1}$, or conversely, and therefore leaves the product unchanged by (6.20).

Now cut a path through (b, a) at that vertex and denote its cyclic product by $T_{(b,a)}^\gamma$. Composing the right connection identities around the path makes the horizontal connections telescope, while the q vertical connections contribute $K_q(x)^q = xI_q$. Hence

$$T_{(b,a)}^\gamma \mathbf{B}_{(b,a)}(x) = x\mathbf{B}_{(b,a)}(x).$$

Composing the left connection identities in the opposite order gives

$$\mathbf{A}_{(b,a)}(x)T_{(b,a)}^\gamma = x\mathbf{A}_{(b,a)}(x),$$

because $K_p(x)^p = xI_p$. Thus the cut product is the recurrence matrix of the normalized weakly normal system at (b, a) . The uniqueness of that recurrence matrix gives $T_{(b,a)}^\gamma = T_{(b,a)}$, independently of γ . $\square$

Corollary 6.31 (Minimum-height factorizations represented in the rectangle). *Fix $a \in \mathcal{A}_{\text{cyc}} = \{a_-, \dots, a_+\}$. The positive refactorization used to construct $\mathcal{H}_{(a)}$ is obtained by cutting one of the paths in Corollary 6.30 at the vertex immediately preceding the first factor of the cyclic product $T_{(a)}$. The cyclic products of the resulting factorization are the recurrence matrices $T_{(b,a)}$ of the simultaneously shifted systems encountered at the successive vertices of that path.*

Proof. Start with the boundary path and mark the vertex immediately preceding the first factor of $T_{(a)}$. Perform the adjacent exchanges used in Theorem 2.8, without moving a factor across the marked cut. Each exchange replaces the two edges of one route through an elementary square by the other two edges and is therefore exactly (6.20). Uniqueness in Lemma 2.2 identifies the resulting edge factors with those used to construct $\mathcal{H}_{(a)}$. The claim about their cyclic products now follows from Corollary 6.30. $\square$

6.3.4. Total positivity and the Piñeiro and Jacobi-like systems.

Corollary 6.32 (The totally positive moment case). *In the setting of Theorem 6.29, assume that the scalar step-line moment matrix $\mathcal{M}$ is totally positive in the TN–TP terminology fixed above. Then, for every $b \in \{0, \dots, q\}$ and $a \in \{0, \dots, p\}$, the shifted matrix*

$$\mathcal{M}^{(b,a)} = [\mathcal{M}_{r+b, s+a}]_{r,s \geq 0}$$

is totally positive. Consequently, the two boundary chains and every interior vertex are weakly normal, all their Christoffel τ -determinants are nonzero, and every conclusion of Theorem 6.29 holds without separately assuming weak normality of the boundary chains.

Proof. Every finite minor of $\mathcal{M}^{(b,a)}$ is a finite minor of $\mathcal{M}$, obtained by translating its row indices by b and its column indices by a . It is therefore strictly positive. In particular, all leading principal minors of every shifted matrix are positive, which is equivalent to weak normality along its step-line. The one-step existence criterion of [39] identifies this weak normality with the nonvanishing of the corresponding Christoffel τ -determinants. The hypotheses of Theorem 6.29 are therefore satisfied. $\square$

Remark 6.33 (Explicit formulas on the boundary chains). The preceding theorem identifies the moment functional, recurrence matrix, and two mixed-type polynomial families at every vertex of the rectangle; in particular, it establishes the full bilateral Christoffel transformation. Closed determinant formulas expressing the transformed polynomials in terms of the original ones are presently available for a polynomial perturbation on one side of the rectangular matrix of measures; in the present notation they cover the boundary chains $d\Psi_{(b,0)}$, with $b \in \{0, \dots, q\}$, and $d\Psi_{(0,a)}$, with $a \in \{0, \dots, p\}$; see [39]. At an interior vertex the polynomials are obtained here by transporting the recurrence and its initial conditions through the bidiagonal relations along a path. Finally, the moment identity determines the polynomial functional; in an indeterminate moment problem, several matrices of measures may represent the same functional.

Remark 6.34 (Direction of the transformations along a path). Each vertex $d\Psi_{(b,a)}$ is defined by applying the displayed row and column Christoffel multipliers to $d\Psi$. A path from $(q, 0)$ to $(0, p)$, however, traverses these transformations in two different directions. A step $(b, a) \mapsto (b, a + 1)$ is an elementary column-side Christoffel transformation, whereas $(b, a) \mapsto (b - 1, a)$ reverses an elementary row-side Christoffel transformation and is therefore a Geronimus step. The functionals and the initial conditions at both endpoints have already been fixed by $d\Psi_{(b,a)}$. The uniqueness of the normalized recurrence solutions therefore determines the Geronimus parameter and the connecting factor.

Corollary 6.35 (The Piñeiro moment-shift rectangle). *For the Piñeiro matrix*

$$d\Psi_{j,i}(x) = x^{\alpha_i + \beta_j} dx, \quad 0 < x < 1,$$

assume

$$\alpha_1 < \cdots < \alpha_p < \alpha_1 + 1, \quad \beta_1 < \cdots < \beta_q < \beta_1 + 1, \quad \alpha_1 + \beta_1 + 1 > 0.$$

Then the scalar moment matrix is totally positive. Every shifted system in the rectangle is therefore weakly normal, and the cyclic product obtained by cutting any path at a vertex has precisely the corresponding Piñeiro measure and its transported initial-condition matrices. Moreover, the row and column power systems at every vertex are AT systems; hence all these transformed systems are strongly normal.

Proof. With

$$x_{qu+j} = u + \beta_{j+1}, \quad y_{pv+i} = v + \alpha_{i+1} + 1,$$

the moment matrix is the Cauchy matrix $\mathcal{M}_{r,s} = 1/(x_r + y_s)$. The parameter inequalities make both sequences strictly increasing and every sum positive, so every minor is a positive Cauchy determinant. Weak normality and the identification of the shifted measures follow from Corollary 6.32, and the cut-product identification follows from Corollary 6.30. The left and right Christoffel multiplications replace the exponent lists by the affine cyclic shifts described in [13] and used independently in [38, Section 7 and Appendix A]. The displayed strict inequalities are preserved by those shifts. Hence the two transformed power vectors remain AT systems. The mixed AT theorem then gives the maximal component degrees and strong normality. $\square$

Corollary 6.36 (The Jacobi-like moment-shift rectangle). *In the positive factorization chamber of [38, Theorem 9.4], the scalar Jacobi-like moment matrix is totally positive. Hence every simultaneous row and column moment shift in the rectangle is weakly normal, and the cyclic product obtained by cutting any path at a vertex is the recurrence matrix of the corresponding shifted Jacobi-like measure. A transformed system is strongly normal whenever its two transformed weight vectors satisfy the AT hypotheses of [37]. Thus the AT range gives strong normality, while the larger positive-factorization chamber gives the PBF and weak normality conclusions.*

Proof. The proof of the cited theorem factors the scalar moment matrix as $\mathcal{M}^J = \mathcal{C}\mathcal{G}$, where $\mathcal{C}$ is block diagonal with nonsingular totally nonnegative finite blocks, while $\mathcal{G}$ is totally positive in the TN–TP terminology fixed above. Fix finite row and column sets I, J of the same cardinality. The rows of $\mathcal{C}$ indexed by I are linearly independent, so some column set K satisfies $\det \mathcal{C}_{I,K} > 0$. Since $\det \mathcal{G}_{K,J} > 0$, this is a strictly positive term in the Cauchy–Binet expansion of $\det \mathcal{M}_{I,J}^J$. Only finitely many column sets occur because $\mathcal{C}$ is block diagonal with finite blocks, and every other term is nonnegative. Thus all minors of $\mathcal{M}^J$ are positive. The moment-shift and recurrence identifications follow from Corollaries 6.32 and 6.30. $\square$

6.4. Weyl approximation on the line and on the star. The preceding results identify both the radial Weyl matrix $S(z)$ and its star lift $\widehat{S}(\lambda)$ in the bounded case. Choose N_0 so that all states selected by Q_q and Q_p belong to $\mathcal{V}_{[0, N_0]}$. For $N \geq N_0$, define

$$(6.25) \quad S^{[N]}(z) := \frac{1}{z} \xi^{-1} Q_q \left(I - \tau \mathcal{H}^{[0, N]} \right)^{-1} Q_p^\top \nu^{-\top}, \quad \tau^M = z^{-1}.$$

Here $\mathcal{H}^{[0, N]}$ is the finite principal submatrix from Definition 3.19, obtained by retaining the height blocks $0, \dots, N$. Thus the superscript $[N]$ on $S^{[N]}$ records the terminal height block. This definition uses only a finite matrix and is

therefore meaningful even if $\mathcal{H}$ is unbounded. Under the hypotheses of Theorem 5.7, the convergence results of Subsection 3.5 apply to these rectangular approximants.

The corresponding approximation on the star is obtained without choosing a root:

$$\widehat{\mathbf{S}}^{[N]}(\lambda) := \frac{1}{\lambda} \xi^{-1} \mathbf{Q}_q \left(I - \lambda^{-1} \mathcal{H}^{[0,N]} \right)^{-1} \mathbf{Q}_p^\top v^{-\top} = \lambda^{M-1} \mathbf{S}^{[N]}(\lambda^M).$$

Thus the same backward Schur recursion computes rational approximants either radially, in the variable $z = \lambda^M$, or directly on the star, in the variable λ . In the latter form their poles are eigenvalues of the finite two-diagonal height matrix itself, before taking M -th powers.

In the next corollary, $\|\cdot\|_2$ denotes the subordinate norm between the indicated Euclidean coordinate spaces, also for rectangular matrices.

Corollary 6.37 (Convergence to the rectangular Weyl matrix). *Assume the hypotheses of Theorem 5.7. With the notation above, $\mathbf{S}^{[N]}(z)$ is independent of the chosen M -th root τ . If*

$$\theta_z := \|\mathcal{H}\|_2 |z|^{-1/M} < 1,$$

then

$$(6.26) \quad \left\| \mathbf{S}(z) - \mathbf{S}^{[N]}(z) \right\|_2 \leq \frac{2}{|z|} \|\xi^{-1}\|_2 \|v^{-\top}\|_2 \frac{\theta_z^{2(N-N_0+1)}}{1 - \theta_z}.$$

Consequently, the rectangular approximants $\mathbf{S}^{[N]}(z)$ converge locally uniformly to $\mathbf{S}(z)$ in $|z| > \|\mathcal{H}\|_2^M$.

If $\mathcal{H}$ is nonnegative, set $\varrho := \rho_{\ell^2}(\mathcal{H})$. For every $\gamma > \varrho$ satisfying $\gamma|z|^{-1/M} < 1$, define

$$C_{\gamma,2} := \sup_{k \geq 0} \frac{\|\mathcal{H}^k\|_2}{\gamma^k} < \infty.$$

Then

$$(6.27) \quad \left\| \mathbf{S}(z) - \mathbf{S}^{[N]}(z) \right\|_2 \leq \frac{C_{\gamma,2}}{|z|} \|\xi^{-1}\|_2 \|v^{-\top}\|_2 \frac{(\gamma|z|^{-1/M})^{2(N-N_0+1)}}{1 - \gamma|z|^{-1/M}}.$$

Consequently, convergence holds throughout $|z| > \varrho^M$.

Proof. The factor position advances by one at every multiplication by $\mathcal{H}$, and the principal height compression only deletes edges; it does not alter their factor-position labels. Hence every surviving path from the first position back to the first position has length divisible by M . The rectangular block in (6.25) therefore depends on τ only through $\tau^M = z^{-1}$.

A product contributing to the difference between the infinite and finite rectangular resolvents must leave $\mathcal{V}_{[0,N]}$. Both selected endpoints lie in blocks with index at most N_0 . Such a product must make at least $N - N_0 + 1$ outward block transitions and the same number of inward transitions. Thus the coefficients of the two rectangular resolvents agree below degree $2(N - N_0 + 1)$. Repeating the Neumann-series estimate in the proof of Theorem 3.31 yields

$$\left\| \mathbf{Q}_q \left[(I - \tau \mathcal{H})^{-1} - (I - \tau \mathcal{H}^{[0,N]})^{-1} \right] \mathbf{Q}_p^\top \right\|_2 \leq \frac{2\theta_z^{2(N-N_0+1)}}{1 - \theta_z}.$$

Multiplication by $z^{-1}\xi^{-1}$ and $v^{-\top}$ gives (6.26).

For nonnegative $\mathcal{H}$, replace the norm estimate by the pathwise spectral-radius estimate (3.28). The same endpoint argument starts the omitted paths at degree $2(N - N_0 + 1)$, and gives (6.27). $\square$

Besides analytic convergence, the approximant $\mathbf{S}^{[N]}$ reproduces an exact range of moment coefficients by the path argument in Proposition 3.25. Expressed in the variable z , this becomes a rectangular Padé-type approximation at infinity. Put

$$\text{ord}_N := \left\lceil \frac{2(N - N_0 + 1)}{M} \right\rceil.$$

Corollary 6.38 (Rectangular Padé-type approximation at infinity). *Assume the hypotheses of Theorem 5.7. Then the matrix $\mathbf{S}^{[N]}(z)$ is a rational $q \times p$ matrix, independent of the chosen M -th root of z^{-1} , and*

$$(6.28) \quad \mathbf{S}(z) - \mathbf{S}^{[N]}(z) = \mathcal{O}\left(z^{-\text{ord}_N - 1}\right), \quad z \rightarrow \infty.$$

Thus $\mathbf{S}^{[N]}$ is a rectangular Padé-type approximant of moment order ord_N : it reproduces $\int x^n d\Psi(x)$ for $0 \leq n < \text{ord}_N$. Its poles are contained among the M -th powers of the nonzero eigenvalues of $\mathcal{H}^{[0,N]}$, after cancellation.

Proof. The proof of Corollary 6.37 shows that the projected resolvents agree in every power τ^k with $k < 2(N - N_0 + 1)$. Only powers divisible by M survive the projection onto one copy of the cyclic linearization. Consequently, the coefficients of $\tau^{Mn} = z^{-n}$ agree for

$$Mn < 2(N - N_0 + 1),$$

which is equivalent to $0 \leq n < \text{ord}_N$. The additional factor z^{-1} in (6.25) gives (6.28).

Proposition 3.24 places the poles of the finite resolvent at reciprocals of the nonzero eigenvalues of the height compression in the variable τ . Since the projected rational matrix is independent of the chosen M -th root τ of z^{-1} , these poles occur in M -cycles and map to the M -th powers of the corresponding eigenvalues in the variable $z = \tau^{-M}$. $\square$

6.5. Unbounded positive bidiagonal factorizations. The preceding norm estimates concern bounded realizations. In the nonnegative PBF setting, Lemma 3.27 identifies this condition with boundedness of the original matrix T . For unbounded T , the formal continued fraction, every finite convergent, and the moment identities remain well defined. Analytic finite-section convergence requires a specified closed realization with the stated core and stability properties, and its target is the projected resolvent of that realization. The spectral Weyl matrix is defined separately from a chosen Favard measure; equality of these two analytic objects is an additional identification.

Definition 6.39 (Spectral Weyl matrix). Let $d\Psi$ be a $q \times p$ matrix of entrywise nonnegative finite measures supported on $\mathbb{R}$, and assume that all its moments are finite. Its spectral Weyl matrix is

$$\mathbf{S}_\Psi(z) := \int_{\mathbb{R}} \frac{d\Psi(x)}{z - x}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where the integral is taken entry by entry. If $d\Psi$ is compactly supported, satisfies (5.7), and $\mathcal{H}$ is bounded, then Proposition 5.8 gives $\mathbf{S}_\Psi = \mathbf{S}$ on their common domain. In an indeterminate moment problem, $\mathbf{S}_\Psi$ depends on the chosen representing measure; the moment identity alone does not identify it with a projected operator resolvent.

Let $\mathcal{H}_0$ be a closed, densely defined realization of $\mathcal{H}$ on $X_0 = \ell^2(\mathcal{V}_{\geq 0})$, assume that $c_{00}(\mathcal{V}_{\geq 0})$ is a core, and let $K \Subset \Omega_0 \setminus \{0\}$ be a compact set on which the finite sections satisfy (3.25). Define

$$\mathbf{R}_{\mathcal{H}_0}(\tau) := \tau^M \xi^{-1} \mathbf{Q}_q (I - \tau \mathcal{H}_0)^{-1} \mathbf{Q}_p^T \nu^{-T}.$$

The finite-window assertion (3.27), with $m = 0$, gives

$$(6.29) \quad \sup_{\tau \in K} \left\| \mathbf{S}^{[N]}(\tau^{-M}) - \mathbf{R}_{\mathcal{H}_0}(\tau) \right\|_2 \rightarrow 0.$$

If

$$K^{-M} := \{\tau^{-M} : \tau \in K\} \subset \mathbb{C} \setminus \mathbb{R}$$

and the chosen Favard measure additionally satisfies

$$(6.30) \quad \mathbf{S}_\Psi(\tau^{-M}) = \mathbf{R}_{\mathcal{H}_0}(\tau), \quad \tau \in K,$$

then the same convergence is to the measure-defined spectral Weyl matrix.

For measures with unbounded support, approximation at infinity is formulated in the Poincaré sense. The following elementary remainder formula gives the corresponding Padé-type statement.

Proposition 6.40 (Moment expansion for an unbounded spectral measure). *Let $d\Psi$ satisfy Definition 6.39, and put $\mu_n := \int x^n d\Psi(x)$. For every integer $r \geq 1$,*

$$(6.31) \quad S_\Psi(z) = \sum_{n=0}^{r-1} \frac{\mu_n}{z^{n+1}} + O(|z|^{-r-1}), \quad z \longrightarrow \infty,$$

entry by entry, uniformly in every closed region satisfying

$$\text{dist}(z, \mathbb{R}) \geq \delta|z| \quad (\delta > 0).$$

Consequently, if the moment identity in (5.7) holds without the compact-support assumption, then, for every fixed N , the finite height approximant $S^{[N]}$ in (6.25) still satisfies

$$S_\Psi(z) - S^{[N]}(z) = O(|z|^{-\text{ord}_N - 1}), \quad z \longrightarrow \infty,$$

in the same regions. Both Landau estimates are entrywise. In the first, the implied constant may depend on δ , r , and $d\Psi$; in the second, it may depend on δ , N , and the fixed spectral data, but in neither case does it depend on z .

Proof. The scalar identity

$$\frac{1}{z-x} = \sum_{n=0}^{r-1} \frac{x^n}{z^{n+1}} + \frac{x^r}{z^r(z-x)}$$

gives, entry by entry,

$$S_\Psi(z) - \sum_{n=0}^{r-1} \frac{\mu_n}{z^{n+1}} = \frac{1}{z^r} \int_{\mathbb{R}} \frac{x^r}{z-x} d\Psi(x).$$

In the stated region, $|z-x| \geq \delta|z|$ for $x \in \mathbb{R}$. Since the measures are positive, finiteness of the even moments implies finiteness of every absolute moment. The absolute value of each entry of the remainder is therefore bounded by its finite absolute r -th moment divided by $\delta|z|^{r+1}$. This proves (6.31). The path argument in the proof of Corollary 6.38 shows algebraically, without a boundedness assumption, that $S^{[N]}$ has the same first ord_N moment coefficients. The matrix $S^{[N]}$ is rational and analytic at infinity. Consequently, for fixed N , its Laurent expansion may be truncated after $z^{-\text{ord}_N}$, with a rational remainder that is $O(|z|^{-\text{ord}_N - 1})$ for all sufficiently large $|z|$. Subtracting this truncated expansion from (6.31), with $r = \text{ord}_N$, proves the last claim. $\square$

The mixed-type Favard theorem, first established in the bounded setting and then extended to unbounded matrices, supplies a positive spectral measure when the finite shifted PBFs form a coherent family with fixed initial-condition matrices [10, 14]; Δ TP initial conditions give the strong-normality conclusion described in [11]. These results justify the existence of the measure-defined spectral Weyl matrix used above. Independently, (6.29) shows that uniformly stable height sections converge to the projected resolvent $R_{\mathcal{H}_0}$ of a specified closed realization. This limit is the spectral Weyl matrix S_Ψ only under the additional identification (6.30). The formal moment-matching statement holds independently of these analytic hypotheses.

Remark 6.41 (Weyl matrices of the cyclic Darboux transforms). When the PBF is the mixed-type Christoffel factorization and its τ -determinants do not vanish, Theorem 6.24 supplies the transformed measure, spectral data, and normalization matrices of every cyclic Darboux transform. Its spectral Weyl matrix is defined from that measure by

$$S^{(a)}(z) := \int_{\mathbb{R}} \frac{d\Psi^{(a)}(x)}{z-x}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

If $T_{(a)}$ has a specified closed operator realization and its normalized projected operator resolvent equals this Cauchy transform—in particular under the bounded hypotheses used above—then Corollary 5.5 gives, for $z^{-1} = \tau^M$,

$$S^{(a)}(z) = \frac{1}{z} (\xi^{(a)})^{-1} Q_q^{(a)} (I - \tau \mathcal{H})^{-1} (Q_p^{(a)})^\top (\nu^{(a)})^{-\top}.$$

Thus, after replacing Q_q, Q_p, ξ, ν by $Q_q^{(a)}, Q_p^{(a)}, \xi^{(a)}, \nu^{(a)}$, the continued-fraction representation derived from $(I - \tau\mathcal{H})^{-1}$ gives the Weyl matrices of all cyclic Darboux transforms. For an unbounded PBF, the Favard measures and their spectral Weyl matrices are defined by the measure integral. The displayed resolvent formula additionally requires the stated operator realization and its identification with that integral. For a general bidiagonal factorization, the projected-resolvent identities retain their formal power-series meaning.

7. THE MIXED-TYPE PIÑEIRO WEYL FRACTION

The general construction is now specialized to the mixed-type Piñeiro system of [13, 38]. The cited works provide the polynomial forms and bidiagonal coefficients used here; [38, Section 7 and Appendix A] gives an independent, self-contained derivation of the factor entries and the exact PBF chamber. The rational example below evaluates these formulas completely: the measure, every required factor entry, the product matrix, and the finite resolvents are displayed and checked directly. The positive endpoint vector then turns the resulting fraction into convergent Padé-type approximants to the explicit Piñeiro Weyl matrix throughout $|z| > 1$. The canonical Piñeiro bidiagonal factors may be unbounded on the standard sequence space. Their Favard representation and every finite continued fraction remain explicit, while analytic convergence follows from the stochastic diagonal normalization used below. Subsection 7.1 starts from its matrix of power weights and their moments, recalls the positive-factor chamber, and identifies the factorization used as input. Subsection 7.2 gives the resulting height-block continued fraction and tests backward evaluation, convergence, and residual error in a rational (3, 2) example. It also gives a two-state recursion with two different return depths, genuinely distinct from the height-block recursion. Subsection 7.3 specializes the q -state first-return recursion, and Subsection 7.4 identifies the rectangular Weyl matrix extracted from $(I - \tau\mathcal{H}_{p,q}^P)^{-1}$ and proves the stochastic convergence statement just described.

7.1. Measures, explicit bidiagonal factorization, and positivity. Consider a concrete matrix of measures for which the bidiagonal coefficients entering the preceding constructions can be written explicitly. This example is the mixed-type Piñeiro specialization of the Jacobi-like systems [37], and it will be used throughout this section. It is useful for two reasons. First, it gives a nontrivial family where the continued fractions in the individual bidiagonal coefficients contain no unevaluated determinants or moments. Second, under the explicit parameter conditions given below, all bidiagonal coefficients are positive. The resulting continued fraction therefore has positive coefficients and represents the corresponding mixed-type Weyl matrix whenever Theorem 5.7 and Proposition 5.8 apply.

The cyclic Christoffel transforms remain within the Piñeiro special-function family. Multiplying a row or a column of the matrix weight by x increments the corresponding power parameter, and the polynomial transport in (6.14) reduces to the associated contiguous hypergeometric identities. The role of the Piñeiro section is instead to provide a determinant-free test of the cyclic matrix, the two continued-fraction recursions, the Weyl projection, and the positivity and convergence results.

Let

$$\alpha = (\alpha_1, \dots, \alpha_p), \quad \beta = (\beta_1, \dots, \beta_q),$$

and consider the mixed-type Piñeiro matrix of measures on $(0, 1)$

$$(7.1) \quad d\mu^P(x) = \left[x^{\alpha_i + \beta_j} \right]_{\substack{j \in \{1, \dots, q\} \\ i \in \{1, \dots, p\}}} dx.$$

The moment entries are therefore

$$\int_0^1 x^n d\mu_{j,i}^P(x) = \frac{1}{n + \alpha_i + \beta_j + 1},$$

whenever the exponents are integrable at the origin. This is the degeneration of the Jacobi-like matrix in which both sides of the matrix of measures are power vectors.

Throughout this section, assume that $\alpha_i + \beta_j > -1$ for every $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. Assume also that the step-line multi-indices of the original system and of all the cyclic Christoffel transforms used below are weakly normal. Equivalently, the corresponding leading principal minors of the moment matrices are nonzero. This

hypothesis guarantees the existence of the Gauss–Borel factorizations, the recurrence matrix, and the Christoffel connections. It also excludes the zeros of the denominators in the explicit formulas below. The parameter conditions (7.3) give a concrete sufficient condition for weak normality. In that chamber the two power systems are AT systems, so the corresponding step-line problems are in fact strongly normal in the sense of Definition 2.1.

The step-line recurrence matrix has the bidiagonal factorization

$$(7.2) \quad T^P = L_1^P \cdots L_p^P U_q^P \cdots U_1^P.$$

The determinant-free Gamma–Pochhammer formulas for all its nontrivial entries are proved in [37, Corollary 7.5 and equations (133)–(136)]; see also the Christoffel-factorization theorem in [12]. The example below evaluates them at rational parameters and displays the factors, the resulting recurrence matrix, and a finite continued fraction. Write

$$\ell_N^{P,(a)} := (L_a^P)_{N+1,N}, \quad u_N^{P,(b)} := (U_b^P)_{N,N}$$

for the nontrivial entries used in the computation below.

The factorization is positive in the chamber

$$(7.3) \quad \alpha_1 < \cdots < \alpha_p < \alpha_1 + 1, \quad \beta_1 < \cdots < \beta_q < \beta_1 + 1, \quad \alpha_1 + \beta_1 + 1 > 0,$$

as follows from the endpoint and factor-positivity results in [38]. Those results use the auxiliary normalization $\alpha_1, \beta_1 > -1$, which causes no loss here. Indeed, the opposite translation

$$\alpha_i \mapsto \alpha_i + c, \quad \beta_j \mapsto \beta_j - c$$

leaves the measures and the factor quotients unchanged. Since $\alpha_1 + \beta_1 + 1 > 0$, the interval $(-1 - \alpha_1, \beta_1 + 1)$ is nonempty, and any c in this interval makes both translated first parameters greater than -1 . Therefore, in (7.3), all entries

$$\ell_N^{P,(k)} > 0, \quad u_N^{P,(b)} > 0, \quad N \in \mathbb{N}_0,$$

are positive. If B_N^P is the N -th right step-line form, then

$$v_N^P := B_N^P(1) = \sum_{j=1}^q B_N^{(j),P}(1), \quad N \in \mathbb{N}_0.$$

The endpoint theorem gives $v_N^P > 0$, and evaluation of the recurrence at $x = 1$ gives

$$T^P v^P = v^P.$$

The terminating component formulas in [13, 38] express every v_N^P as a finite Gamma–Pochhammer sum. This formula supplies the positive eigenvector used in the stochastic normalization below. Since each factor is nonnegative and has a fixed positive normalizing entry in every row, all intermediate factor images of v^P are strictly positive.

7.2. The height-block fraction and an exact rational example. The general construction of Sections 2–4 applies directly to (7.2). The rational example follows the chain

$$\text{bidiagonal factors} \longrightarrow T^P \longrightarrow \text{step-line recurrence} \longrightarrow \text{finite matrix continued fraction}.$$

Positivity and the endpoint vector then yield numerical convergence. For a semi-infinite matrix F , write

$$\text{pr}_5(F) := [F_{n,k}]_{n,k=0}^4.$$

In the prescribed factor order all lower factors precede all upper factors. Consequently, a path contributing to this leading principal block cannot leave the indices 0, 1, 2, 3, 4 and later return, and hence

$$\text{pr}_5(T) = \text{pr}_5(L_1) \cdots \text{pr}_5(L_p) \text{pr}_5(U_q) \cdots \text{pr}_5(U_1).$$

This permits the finite factorization and its coefficient sequences to be displayed together.

Example 7.1 (Backward evaluation for a rational $(3, 2)$ Piñero system). Take

$$\alpha = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & \frac{1}{3} \end{bmatrix}.$$

These parameters satisfy (7.3). In this case the matrix of measures is

$$d\mu^P(x) = \begin{bmatrix} x^{-1/3} & 1 & x^{1/3} \\ 1 & x^{1/3} & x^{2/3} \end{bmatrix} dx, \quad 0 < x < 1.$$

Thus every moment, factor coefficient, and finite continued-fraction block is rational. Evaluation of the explicit factor formulas cited above gives

$$\begin{aligned} \text{pr}_5(L_1^P) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & 1 & 0 & 0 & 0 \\ 0 & \frac{8}{35} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{12} & 1 & 0 \\ 0 & 0 & 0 & \frac{7}{22} & 1 \end{bmatrix}, & \text{pr}_5(L_2^P) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{20} & 1 & 0 & 0 & 0 \\ 0 & \frac{5}{28} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{15} & 1 & 0 \\ 0 & 0 & 0 & \frac{40}{143} & 1 \end{bmatrix}, \\ \text{pr}_5(L_3^P) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{30} & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{55} & 1 & 0 \\ 0 & 0 & 0 & \frac{45}{182} & 1 \end{bmatrix}, & \text{pr}_5(U_2^P) &= \begin{bmatrix} \frac{3}{5} & 1 & 0 & 0 & 0 \\ 0 & \frac{5}{21} & 1 & 0 & 0 \\ 0 & 0 & \frac{7}{12} & 1 & 0 \\ 0 & 0 & 0 & \frac{8}{55} & 1 \\ 0 & 0 & 0 & 0 & \frac{15}{26} \end{bmatrix}, \\ \text{pr}_5(U_1^P) &= \begin{bmatrix} \frac{2}{3} & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & 1 & 0 & 0 \\ 0 & 0 & \frac{5}{28} & 1 & 0 \\ 0 & 0 & 0 & \frac{35}{66} & 1 \\ 0 & 0 & 0 & 0 & \frac{8}{65} \end{bmatrix}. \end{aligned}$$

Every entry of the five finite bidiagonal factors is now explicit. Direct multiplication gives

$$\text{pr}_5(T^P) = \text{pr}_5(L_1^P) \text{pr}_5(L_2^P) \text{pr}_5(L_3^P) \text{pr}_5(U_2^P) \text{pr}_5(U_1^P) = \begin{bmatrix} \frac{2}{5} & \frac{6}{5} & 1 & 0 & 0 \\ \frac{1}{15} & \frac{12}{35} & \frac{7}{12} & 1 & 0 \\ \frac{1}{100} & \frac{19}{175} & \frac{43}{120} & \frac{183}{110} & 1 \\ \frac{1}{5040} & \frac{17}{2940} & \frac{13}{352} & \frac{1051}{3080} & \frac{123}{260} \\ 0 & \frac{1}{2310} & \frac{389}{56628} & \frac{1008}{7865} & \frac{5282}{15015} \end{bmatrix}.$$

The zero pattern displays the complete $(p, q) = (3, 2)$ band: $T_{n,k}^P = 0$ outside $n - 3 \leq k \leq n + 2$. In particular, the first three entries on the outer superdiagonal are 1. The lower left corner also shows the band boundary: $T_{3,0}^P = 1/5040 > 0$ but $T_{4,0}^P = 0$, while $T_{4,1}^P = 1/2310 > 0$. For example, the first row is the concrete recurrence

$$xB_0^P(x) = \frac{2}{5}B_0^P(x) + \frac{6}{5}B_1^P(x) + B_2^P(x).$$

Thus the upper and lower recurrence coefficients are recovered from the bidiagonal factors rather than inserted as independent data. Let v_N be the value at $x = 1$ of the N -th right step-line form, and put $\mathbf{v} = (v_N)_{N \geq 0}$. Its first values are

$$v_0 = 1, \quad v_1 = \frac{1}{3}, \quad v_2 = \frac{1}{5}, \quad v_3 = \frac{1}{28}, \quad v_4 = \frac{1}{44}, \quad v_5 = \frac{7}{2145}.$$

The recurrence at $x = 1$ gives $T^P \mathbf{v} = \mathbf{v}$, and the ordered positive factorization gives positive intermediate vectors. The diagonal normalization of Definition 3.28, with the properties proved in Proposition 3.29, therefore converts all five factors into row-stochastic bidiagonal matrices. Hence $\|\mathcal{H}_{3,2}^P\|_\infty = 1$.

For $(p, q) = (3, 2)$, the reduced height parameters are $P = 3$, $Q = 2$, $R = 5$, and the block width is 3. With the factor order $(L_1, L_2, L_3, U_2, U_1)$, the first height blocks are

$$\begin{aligned}\mathcal{V}_0 &= ((1, 0), (2, 0)), & \mathcal{V}_1 &= ((5, 0), (3, 0), (1, 1)), \\ \mathcal{V}_2 &= ((4, 0), (2, 1), (5, 1)), & \mathcal{V}_3 &= ((3, 1), (1, 2), (4, 1)).\end{aligned}$$

Before stochastic normalization, the cut after $\mathcal{V}_3$ is the following explicit block-tridiagonal matrix. The horizontal and vertical lines separate the coordinates belonging to $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$, in that order:

$$(\mathcal{H}_{3,2}^P)^{[0,3]} = \begin{bmatrix} \mathbf{A}_0^P & \mathbf{B}_0^P & 0 & 0 \\ \mathbf{C}_0^P & \mathbf{A}_1^P & \mathbf{B}_1^P & 0 \\ 0 & \mathbf{C}_1^P & \mathbf{A}_2^P & \mathbf{B}_2^P \\ 0 & 0 & \mathbf{C}_2^P & \mathbf{A}_3^P \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \frac{2}{3} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{3}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{20} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{5} & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{30} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{35} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{21} & 0 & 0 & 0 \end{bmatrix}.$$

Thus the individual factor coefficients occur as the nonzero transitions of the height matrix. Define the Schur denominators of the displayed finite continued fraction backwards by

$$\begin{aligned}\mathbf{D}_3(\tau) &= I_3 - \tau \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{D}_2(\tau) &= I_3 - \tau \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \tau^2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{D}_3(\tau)^{-1} \begin{bmatrix} \frac{1}{30} & 0 & 0 \\ 0 & \frac{8}{35} & 0 \\ 0 & 0 & \frac{5}{21} \end{bmatrix}, \\ \mathbf{D}_1(\tau) &= I_3 - \tau \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \tau^2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{D}_2(\tau)^{-1} \begin{bmatrix} \frac{3}{5} & 0 & 0 \\ 0 & \frac{1}{20} & 0 \\ 0 & 0 & \frac{3}{5} \end{bmatrix}, \\ \mathbf{D}_0(\tau) &= I_2 - \tau \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \tau^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{D}_1(\tau)^{-1} \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \\ 0 & \frac{1}{12} \end{bmatrix}.\end{aligned}$$

The concrete finite fraction is

$$\Phi_0^{P,[3]}(\tau) = \mathbf{D}_0(\tau)^{-1}.$$

Thus every block entering $\Phi_0^{P,[3]}$ is visible. Direct inversion of the displayed 11×11 truncation and backward evaluation of the continued fraction give the same exact value:

$$\left[\left(I_{11} - \frac{1}{2} (\mathcal{H}_{3,2}^P)^{[0,3]} \right)^{-1} \right]_{\mathcal{V}_0 \times \mathcal{V}_0} = \Phi_0^{P,[3]} \left(\frac{1}{2} \right) = \begin{bmatrix} \frac{14857}{14670} & \frac{414}{815} \\ \frac{187}{7335} & \frac{828}{815} \end{bmatrix}.$$

This identity is the finite calculation behind the general Schur-complement recursion: the left-hand side uses the complete truncated height matrix, whereas the middle expression evaluates only its successive diagonal blocks. This small cut displays the block structure and the finite fraction explicitly. In this example the states selected by Q_2 and Q_3 lie in the height blocks 0, 1, and 3, so the minimal index in the general Weyl approximation is $N_0 = 3$. Consequently, the cut $N = 3$ has $\text{ord}_3 = \lceil 2/5 \rceil = 1$ and reproduces only the zeroth moment. The deeper cuts in Table 3 illustrate the increasing contact order together with analytic convergence.

The same four-block display in the stochastic basis is

$$(\widehat{\mathcal{H}}_{3,2}^P)^{[0,3]} = \begin{bmatrix} \begin{array}{c|c|c|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \end{array} & \begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 \end{array} & \begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \\ \hline \begin{array}{c|c|c|c} 0 & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} & \begin{array}{c|c|c|c} 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{2}{7} & 0 & 0 \\ 0 & 0 & \frac{4}{7} & 0 \end{array} & \begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \end{bmatrix}.$$

The semi-infinite matrix $\widehat{\mathcal{H}}_{3,2}^P$ is row-stochastic, but the displayed principal truncation is only substochastic at its terminal boundary. Its last two rows sum to $2/7$ and $4/7$, because their outgoing normalizing transitions enter $\mathcal{V}_4$ and are deleted by the cut. The full operator includes these transitions and satisfies $\|\widehat{\mathcal{H}}_{3,2}^P\|_\infty = 1$. Its finite continued fraction is equally explicit. With backward denominators

$$\begin{aligned} \widehat{D}_3(\tau) &= I_3 - \tau \begin{bmatrix} 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \widehat{D}_2(\tau) &= I_3 - \tau \begin{bmatrix} 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \tau^2 \begin{bmatrix} 0 & 0 & 0 \\ \frac{4}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \widehat{D}_3(\tau)^{-1} \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{2}{7} & 0 \\ 0 & 0 & \frac{4}{7} \end{bmatrix}, \\ \widehat{D}_1(\tau) &= I_3 - \tau \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \tau^2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{3}{4} & 0 \end{bmatrix} \widehat{D}_2(\tau)^{-1} \begin{bmatrix} \frac{3}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \\ \widehat{D}_0(\tau) &= I_2 - \tau \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \tau^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \widehat{D}_1(\tau)^{-1} \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \end{aligned}$$

one has

$$\widehat{\Phi}_0^{[3]}(\tau) = \widehat{D}_0(\tau)^{-1}.$$

At the same test point,

$$\left[\left(I_{11} - \frac{1}{2} (\widehat{\mathcal{H}}_{3,2}^P)^{[0,3]} \right)^{-1} \right]_{\mathcal{V}_0 \times \mathcal{V}_0} = \widehat{\Phi}_0^{[3]} \left(\frac{1}{2} \right) = \begin{bmatrix} \frac{14857}{14670} & \frac{414}{815} \\ \frac{187}{7335} & \frac{828}{815} \end{bmatrix}.$$

The equality of this leading resolvent block with the unnormalized one is exact: the zeroth coordinate of every intermediate vector $\mathbf{v}^{[r]}$ is 1, and both states of $\mathcal{V}_0$ have sequence index 0; hence the height-ordered diagonal

normalization restricts to I_2 on $\mathcal{V}_0$. Finite diagonal conjugation therefore forces the two leading blocks to agree. The stochastic calculation instead displays the normalized transition probabilities and verifies the corresponding backward denominators, with every block substituted. To make the notation in the numerical comparison explicit, set

$$\begin{aligned}\widehat{\Phi}_m(\tau) &:= \left[\left(I - \tau(\widehat{\mathcal{H}}_{3,2}^P)^{(m)} \right)^{-1} \right]_{\mathcal{V}_m \times \mathcal{V}_m}, \\ \widehat{\Phi}_m^{[N]}(\tau) &:= \left[\left(I - \tau(\widehat{\mathcal{H}}_{3,2}^P)^{[m,N]} \right)^{-1} \right]_{\mathcal{V}_m \times \mathcal{V}_m}.\end{aligned}$$

For the infinite tail, Theorem 3.20 gives, at the first level,

$$\widehat{\Phi}_0(\tau) = \left(I_2 - \tau \widehat{\mathbf{A}}_0^P - \tau^2 \widehat{\mathbf{B}}_0^P \widehat{\Phi}_1(\tau) \widehat{\mathbf{C}}_0^P \right)^{-1}.$$

The subsequent levels are obtained by continuing the same explicit backward recursion. Moreover, $\|\widehat{\mathcal{H}}_{3,2}^P\|_\infty = 1$. All resolvents, errors, and bounds in the remainder of this example are computed directly for the normalized operator $\widehat{\mathcal{H}}_{3,2}^P$; finite truncations are related to the unnormalized ones by the diagonal conjugations above.

At $\tau = 0.75$, cutting after block 220 gives the reference

$$\widehat{\Phi}_0^{[220]}(0.75) = \begin{bmatrix} 1.111994065342 & 0.856167208327 \\ 0.149325420456 & 1.141556277770 \end{bmatrix}.$$

Table 3 compares shallower cuts with this reference. The observed error is measured in the matrix infinity norm. The a priori column uses (3.24) with $\theta = 0.75$, while the residual column uses (3.34).

The factors were generated exactly and converted to double precision for the linear algebra. Direct inversion and backward block evaluation were used independently to check every displayed value.

TABLE 3. Convergence and certified error bounds for the rational (3, 2) Piñeiro system at $\tau = 0.75$. The columns give the truncation block N , the finite-section dimension, the observed absolute matrix infinity-norm difference from the truncation after block 220, the a priori bound, and the a posteriori residual bound.

N	dim	observed error	a priori	residual
2	8	8.09×10^{-3}	1.42	2.51×10^{-1}
4	14	1.91×10^{-4}	4.51×10^{-1}	2.49×10^{-2}
8	26	6.78×10^{-8}	4.51×10^{-2}	4.92×10^{-4}
12	38	2.86×10^{-11}	4.52×10^{-3}	1.07×10^{-5}
32	98	roundoff	4.54×10^{-8}	6.10×10^{-14}
64	194	roundoff	4.58×10^{-16}	5.80×10^{-27}
192	578	roundoff	4.75×10^{-48}	1.16×10^{-78}

The direct inversion and backward evaluation agree to double-precision roundoff in every row. The last printed digits depend on the order of the floating-point operations and are not used as reproducibility targets. At $N = 32$ and beyond in the table, the observed difference from the reference is at the level of floating-point roundoff; the theoretical and residual bounds continue to decrease, certifying convergence beyond the precision visible in the observed-error column.

7.3. Branched and q -state tails. The second grouping retains the q labeled states at the lower boundary of each tail. These *boundary states* form the space $\mathcal{U}_m$. This grouping is also explicit for the mixed-type Piñeiro family with arbitrary p and q . Let

$$P = \kappa Q + \sigma, \quad \kappa \in \mathbb{N}_0, \quad \sigma \in \{0, \dots, Q-1\},$$

and, for $m \geq m_* = p$, form the $q \times q$ matrices

$$\mathbf{A}_m^P, \quad \mathbf{B}_m^P, \quad \mathbf{D}_m^{P,(\eta)}, \quad \eta \in \{\kappa, \kappa+1\},$$

from $\mathcal{H}_{p,q}^P$ by the projections in (4.6). The nonzero entries of $\mathbf{B}_m^P$ are equal to 1. This statement refers to the unnormalized Piñeiro factors. After a diagonal normalization, the corresponding rise entries remain positive but need not equal 1, as the matrix in Example 7.1 shows. Every nonzero coefficient entry of $\mathbf{A}_m^P$ or $\mathbf{D}_m^{P,(\eta)}$ is one of the factor coefficients

$$\ell_N^{P,(a)} \quad \text{or} \quad u_N^{P,(b)}$$

of (7.2). As in Subsection 4.2, the superscript $\langle m \rangle_Q$ denotes principal compression to the tail formed by the spaces $\mathcal{U}_m, \mathcal{U}_{m+1}, \dots$, each formed from Q consecutive heights. For $m \geq p$, define the Piñeiro q -state tail by

$$\mathbf{G}_m^P(\tau) := E_m^\top \left(I - \tau (\mathcal{H}_{p,q}^P)^{\langle m \rangle_Q} \right)^{-1} E_m.$$

Theorem 4.2, applied to the Piñeiro factorization, now gives the q -state recursion without repeating its general formula. Its possible return depths are κ and $\kappa+1$, as determined by the Euclidean division of P by Q . If $\sigma = 0$, the second depth is absent; if $\kappa = 0$, the depth-zero falls are contained in $\mathbf{A}_m^P$. The weights retain the complete dependence on α and β through the explicit factor sequences. They are strictly positive under (7.3).

Example 7.2 (The two return depths in the rational $(3, 2)$ system). Return to the rational system in Example 7.1. Here

$$d = 1, \quad P = 3, \quad Q = 2, \quad P = 1 \cdot Q + 1.$$

Thus $\kappa = 1$, $\sigma = 1$, and both possible return depths occur. The first complete boundary-state space, ordered by increasing height, is

$$\mathcal{U}_3 = ((4, 0), (2, 1)).$$

The pairs record the factor label and the nonnegative sequence index. In this basis, and in the corresponding increasing-height bases of $\mathcal{U}_4$ and $\mathcal{U}_5$, direct substitution of the factor data used in Example 7.1 gives

$$\mathbf{A}_3^P = 0, \quad \mathbf{B}_3^P = \mathbf{B}_4^P = I_2, \quad \mathbf{D}_3^{P,(1)} = \begin{bmatrix} 0 & 0 \\ \frac{1}{30} & 0 \end{bmatrix}, \quad \mathbf{D}_3^{P,(2)} = \begin{bmatrix} 0 & \frac{8}{35} \\ 0 & 0 \end{bmatrix}.$$

The next two pairs of nonzero return coefficients are

m	3	4	5
$(\mathbf{D}_m^{P,(1)})_{2,1}$	$\frac{1}{30}$	$\frac{5}{21}$	$\frac{5}{28}$
$(\mathbf{D}_m^{P,(2)})_{1,2}$	$\frac{8}{35}$	$\frac{5}{28}$	$\frac{1}{7}$

In particular, the first tail satisfies

$$\mathbf{G}_3^P(\tau) = \left(I_2 - \tau^2 \mathbf{G}_4^P(\tau) \mathbf{D}_3^{P,(1)} - \tau^3 \mathbf{G}_4^P(\tau) \mathbf{G}_5^P(\tau) \mathbf{D}_3^{P,(2)} \right)^{-1}.$$

Since $Q < P$, each $\mathcal{U}_m$ groups two consecutive heights, whereas the block-tridiagonal construction groups three. Thus the two recursions use genuinely different groupings in this example.

For $q = 1$, Corollary 4.3 reduces this specialization of Theorem 4.2 to the Jacobi–Piñeiro p -branched Stieltjes fraction, with scalar weights given by the same coefficients $u_N^{P,(1)}$ and $\ell_N^{P,(a)}$.

7.4. The rectangular Weyl matrix. It remains to identify the rectangular Weyl matrix represented by the general mixed-type Piñero continued fraction. The moments of (7.1) are

$$\int_0^1 x^n d\mu^P(x) = \left[\frac{1}{n + \alpha_i + \beta_j + 1} \right]_{\substack{j \in \{1, \dots, q\} \\ i \in \{1, \dots, p\}}}, \quad n \in \mathbb{N}_0.$$

For $|z| > 1$, the corresponding Stieltjes matrix is

$$(7.4) \quad S_{p,q}^P(z) = \int_0^1 \frac{d\mu^P(x)}{z - x} = \left[\frac{1}{z(\alpha_i + \beta_j + 1)} {}_2F_1 \left(\begin{matrix} 1, \alpha_i + \beta_j + 1 \\ \alpha_i + \beta_j + 2 \end{matrix}; \frac{1}{z} \right) \right]_{\substack{j \in \{1, \dots, q\} \\ i \in \{1, \dots, p\}}}.$$

The weak-normality hypothesis gives the step-line Gauss–Borel factorization

$$\mathcal{M}^P = (\mathcal{L}^P)^{-1} (\mathcal{U}^P)^{-1},$$

and hence the two finite normalizations

$$\xi^P := E_{[q]} \mathcal{L}^P E_{[q]}^T, \quad (\nu^P)^T := E_{[p]} \mathcal{U}^P E_{[p]}^T.$$

Proposition 7.3 (Piñero moment normalization). *For every $n \in \mathbb{N}_0$, the matrices just defined satisfy*

$$(\xi^P)^{-1} E_{[q]} (T^P)^n E_{[p]}^T (\nu^P)^{-T} = \int_0^1 x^n d\mu^P(x).$$

Proof. This is Theorem 5.1 applied to the Piñero moment matrix, with the normalization determined by its leading Gauss–Borel blocks. $\square$

Proposition 7.3 verifies the moment hypothesis of Proposition 5.8 with the concrete matrix of measures (7.1). Its support is contained in $[0, 1]$.

The preceding normalization also gives an explicit rotational lift of the Piñero matrix of measures and identifies its Cauchy transform with a projected resolvent of the sparse matrix associated with the prescribed factor order. For the two compactly supported applications below, write $\Sigma_M^{[1]} := \bigcup_{k=0}^{M-1} \zeta_M^k [0, 1]$.

Proposition 7.4 (Rotational lift of the Piñero measure and its Weyl matrix). *The rotational lift of $d\mu^P$ to $\Sigma_M^{[1]}$ is the positive $q \times p$ matrix of measures determined entrywise by*

$$(7.5) \quad \int_{\Sigma_M^{[1]}} f(\lambda) d\widehat{\mu}_{j,i}^P(\lambda) = \frac{1}{M} \sum_{k=0}^{M-1} \int_0^1 f(\zeta_M^k x^{1/M}) x^{\alpha_i + \beta_j} dx.$$

Its moments and Weyl matrix are

$$(7.6) \quad \int_{\Sigma_M^{[1]}} \lambda^n d\widehat{\mu}_{j,i}^P(\lambda) = \begin{cases} \frac{1}{r + \alpha_i + \beta_j + 1}, & n = Mr, \\ 0, & M \nmid n, \end{cases}$$

$$(7.7) \quad \widehat{S}_{p,q}^P(\lambda) = \lambda^{M-1} S_{p,q}^P(\lambda^M), \quad \lambda \in \mathbb{C} \setminus \Sigma_M^{[1]}.$$

This rotational lift is the rotationally invariant component of the cyclic spectral system. The factor-resolved representation and convergence of its Weyl matrix are stated after the endpoint normalization in Corollary 7.5. If $\gcd(p, q) = 1$, the recurrence for the complete matrix of moment functionals is instead $\mathcal{H}_{(a)}$ of Theorem 6.10; in exact-height order, only its q -th superdiagonal and p -th subdiagonal are nonzero. If $\gcd(p, q) > 1$, every height contains $d = \gcd(p, q)$ coordinate states and the matrix $\mathcal{H}_{(a)}$ has the two monomial block diagonals described in Theorem 3.9. Grouping those states gives the block mixed-type recurrence of Theorem 6.10, with normality at cuts formed by complete height groups.

At every cyclic factor position, the corresponding Favard system is the usual Piñero Christoffel transform: one of the power parameters is increased by one according to its position in the right or left Christoffel chain. Thus all

the transformed matrix measures and polynomial systems are again explicit Piñeiro systems, and their rotational lifts are obtained from (7.5) with the shifted parameter.

Proof. Equation (7.5) is the specialization of Definition 5.10. Summing the M -th roots of unity gives (7.6), and Theorem 5.11 gives (7.7). The assertions about the matrix $\mathcal{H}_{(a)}$ follow from Theorems 3.9 and 6.10; Corollary 3.10 gives the scalar form when $d = 1$. Finally, Theorem 6.24 identifies the other cyclic positions with the successive Christoffel transforms. For the power matrix (7.1), these transforms increment the corresponding exponent, so the family is preserved. $\square$

Proposition 7.4 produces an explicit rotationally invariant positive matrix of measures on the star and its Weyl matrix from the original interval system. This positive lift represents the invariant Fourier component; the remaining character components are the generally complex measures described above. In the rational $(3, 2)$ example, $\gcd(p, q) = 1$. Its prescribed-order exact-height matrix is scalar two-diagonal after a finite initial part, whereas $\mathcal{H}_{(a)}$ has that scalar two-diagonal form from height 0.

In the positive chamber, the endpoint vector and stochastic normalization yield the following global convergence statement.

Corollary 7.5 (Piñeiro Weyl convergence in the positive chamber). *Assume (7.3). Apply Definition 3.28 to $\mathbf{v}^P$, and denote the resulting height-ordered diagonal matrix and stochastic height matrix by $\mathcal{D}_h^P$ and $\widehat{\mathcal{H}}_{p,q}^P$, respectively. For $z^{-1} = \tau^{p+q}$ and $|\tau| < 1$,*

$$(7.8) \quad \mathbf{S}_{p,q}^P(z) = \frac{1}{z} (\xi^P)^{-1} \mathbf{Q}_q \mathcal{D}_h^P \left(I - \tau \widehat{\mathcal{H}}_{p,q}^P \right)^{-1} (\mathcal{D}_h^P)^{-1} \mathbf{Q}_p^T (\mathbf{v}^P)^{-T}, \quad z^{-1} = \tau^{p+q}, \quad |\tau| < 1.$$

The right-hand side is independent of the chosen $(p + q)$ -th root τ of z^{-1} , because the two coordinate projections select paths that complete whole factor cycles. The corresponding finite rectangular projections converge locally uniformly to $\mathbf{S}_{p,q}^P(z)$ for $|z| > 1$, with the error estimate of (3.30) before projection. For real $0 \leq \tau < 1$, equivalently $z = \tau^{-(p+q)} > 1$, the normalized tail blocks increase entrywise and their Schur denominators are nonsingular M -matrices.

Proof. The endpoint vector verifies the hypotheses of Proposition 3.29, so $\|\widehat{\mathcal{H}}_{p,q}^P\|_\infty = 1$ and Corollary 3.36 applies. The finite diagonal conjugation of Remark 3.30, followed by the coordinate projections, gives the finite version of (7.8); only finitely many entries of $\mathcal{D}_h^P$ occur beside those projections. Passing to the limit and using Proposition 7.3 identifies the result with the Stieltjes matrix (7.4). $\square$

Thus the family realizes the complete general construction: its explicit Pochhammer factors determine the finite fractions, and their rectangular projections converge to the explicit $q \times p$ Weyl matrix. The cyclic polynomial systems are the Piñeiro formulas with shifted power parameters and give a direct verification of the Darboux transport.

8. JACOBI-LIKE EXTENSION OF THE PIÑEIRO WEYL FRACTION

The mixed-type Piñeiro system of [13, 38] is the special case of the broader Jacobi-like family of [37] in which all Gamma factors cancel. In the complete-cancellation specialization, this relation holds for the recurrence matrix, the rectangular Weyl matrix, and its continued-fraction approximants. The weight and factor formulas of [37] provide the input for the construction. This section identifies the beta-convolution measure, computes its hypergeometric Weyl matrix, and gives its rotational lift to the star. When the parameters additionally belong to the positive-factorization chamber stated below, the canonical PBF yields the factor-resolved continued fraction and its finite Weyl approximants.

In the Jacobi-like family the lower factors have closed Gamma–Pochhammer entries, whereas the upper factors depend on moment minors. Consequently, the transformed systems can leave the original special-function family, whereas the Piñeiro cyclic chain is described by direct parameter shifts. Within that positive-factorization chamber, the essential input from [38, Theorem 9.4] is stronger than positivity of the original recurrence: it proves weak normality throughout the cyclic Christoffel chain. The present paper turns that simultaneous weak normality into

positive Favard systems for the complete Darboux orbit and into their rotationally symmetric systems on $p + q$ rays. The canonical Jacobi-like factors can be unbounded on the standard sequence space in part of the positive region. The unbounded Favard theorem still provides the matrix of measures and the weakly normal Christoffel chain; boundedness enters only in the unnormalized operator-resolvent representation. The subsection below derives the beta-type convolution weights, Gamma-quotient moments, and generalized hypergeometric expression for the Weyl matrix and its rotational lift. Cancellation of all Gamma quotients recovers the Piñeiro measure and Weyl matrix.

The standing condition $a_1 > -1$ is a convenient sufficient condition for the beta-convolution construction. In the complete-cancellation case it is slightly stronger than the full Piñeiro integrability condition $\alpha_1 + \beta_1 + 1 > 0$.

8.1. Beta-convolution weights and the Weyl matrix. We use the vector notation

$$\Gamma(\mathbf{c}) := \prod_{h=1}^r \Gamma(c_h), \quad (\mathbf{c})_n := \prod_{h=1}^r (c_h)_n, \quad (c)_n := \frac{\Gamma(c+n)}{\Gamma(c)}, \quad \mathbf{c} = (c_1, \dots, c_r).$$

The generalized hypergeometric function is normalized by

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!},$$

in its disk of convergence, with the usual terminating or analytically continued interpretation when applicable. The Meijer function appearing below is taken in the Mellin–Barnes normalization

$$G_{q,q}^{q,0} \left(x \left| \begin{matrix} \mathbf{b} + \mathbf{e}_j^{(q)} \\ \mathbf{a} \end{matrix} \right. \right) := \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s\mathbf{1}_q + \mathbf{a})}{\Gamma(s\mathbf{1}_q + \mathbf{b} + \mathbf{e}_j^{(q)})} x^{-s} ds,$$

where $\mathcal{L}$ is an admissible Mellin–Barnes contour. Equivalently, this normalization is characterized by the Mellin transform

$$\int_0^1 x^{s-1} G_{q,q}^{q,0} \left(x \left| \begin{matrix} \mathbf{b} + \mathbf{e}_j^{(q)} \\ \mathbf{a} \end{matrix} \right. \right) dx = \frac{\Gamma(s\mathbf{1}_q + \mathbf{a})}{\Gamma(s\mathbf{1}_q + \mathbf{b} + \mathbf{e}_j^{(q)})}$$

throughout the common fundamental strip.

Let

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p), \quad \mathbf{a} = (a_1, \dots, a_q), \quad \mathbf{b} = (b_1, \dots, b_q),$$

and assume

$$(8.1) \quad \alpha_1 < \cdots < \alpha_p < \alpha_1 + 1, \quad a_1 < \cdots < a_q, \quad b_1 < \cdots < b_q, \quad a_h \leq b_h.$$

Assume also that $a_1 > -1$ and $\alpha_i + a_h > -1$ for all i, h . Define the row weights by their Mellin transforms

$$\int_0^1 x^{s-1} w_j^{\mathbf{J}}(x) dx = \frac{1}{s + b_j} \frac{\Gamma(s\mathbf{1}_q + \mathbf{a})}{\Gamma(s\mathbf{1}_q + \mathbf{b})}, \quad 1 \leq j \leq q.$$

When all inequalities $a_h < b_h$ are strict, these are positive Mellin convolutions of beta-type kernels. In the non-strict case, common Gamma factors can be cancelled. Indeed, $a_i = b_j$ implies $j \leq i$, since $j > i$ would give $a_i = b_j > b_i$. Deleting this common value preserves the componentwise inequalities between the remaining ordered lists: before position j nothing changes, between j and i each surviving a -parameter is paired with the next larger b -parameter, and after position i both indices shift by one. Repeating the cancellation leaves strict inequalities among all surviving pairs. Hence the remaining Gamma quotient is again the Mellin transform of a positive beta-type convolution. If all factors cancel, then $w_j^{\mathbf{J}}(x) = x^{b_j}$. Thus the row weights remain positive. In standard Meijer notation,

$$w_j^{\mathbf{J}}(x) = G_{q,q}^{q,0} \left(x \left| \begin{matrix} \mathbf{b} + \mathbf{e}_j^{(q)} \\ \mathbf{a} \end{matrix} \right. \right), \quad 0 < x < 1.$$

Here $\mathbf{e}_j^{(q)} \in \mathbb{Z}^q$ has a 1 in position j and 0 in every other position.

The Jacobi-like matrix of measures is

$$d\mu^J(x) = \left[x^{\alpha_i} w_j^J(x) \right]_{\substack{1 \leq j \leq q \\ 1 \leq i \leq p}} dx.$$

Its moments are explicit:

$$(8.2) \quad \int_0^1 x^n d\mu_{j,i}^J(x) = \frac{1}{n + \alpha_i + b_j + 1} \frac{\Gamma((n + \alpha_i + 1)\mathbf{1}_q + \mathbf{a})}{\Gamma((n + \alpha_i + 1)\mathbf{1}_q + \mathbf{b})}, \quad n \in \mathbb{N}_0.$$

Consequently the Weyl matrix can be written without an unevaluated integral. Put

$$\tilde{\mathbf{a}}_i := (\alpha_i + 1)\mathbf{1}_q + \mathbf{a} = (\tilde{a}_{i,1}, \dots, \tilde{a}_{i,q}), \quad \tilde{\mathbf{b}}_i := (\alpha_i + 1)\mathbf{1}_q + \mathbf{b} = (\tilde{b}_{i,1}, \dots, \tilde{b}_{i,q}), \quad c_{j,i}^J := \frac{1}{\tilde{b}_{i,j}} \frac{\Gamma(\tilde{\mathbf{a}}_i)}{\Gamma(\tilde{\mathbf{b}}_i)}.$$

For $|z| > 1$, expansion of $(z - x)^{-1}$ and (8.2) give

$$(8.3) \quad \left(\mathbf{S}_{p,q}^J(z) \right)_{j,i} := \int_0^1 \frac{x^{\alpha_i} w_j^J(x)}{z - x} dx = \frac{c_{j,i}^J}{z} {}_{q+1}F_q \left(\begin{matrix} 1, \tilde{\mathbf{a}}_i \\ \tilde{\mathbf{b}}_i + \mathbf{e}_j^{(q)} \end{matrix}; \frac{1}{z} \right).$$

Indeed,

$$\frac{1}{n + \tilde{b}_{i,j}} = \frac{1}{\tilde{b}_{i,j}} \frac{(\tilde{b}_{i,j})_n}{(\tilde{b}_{i,j} + 1)_n},$$

and the remaining Gamma quotients in (8.2) become the Pochhammer quotients in (8.3). The identity $(1)_n/n! = 1$ supplies the first upper hypergeometric parameter.

The Jacobi-like data likewise determine both a concrete rotational lift of the matrix of measures and the corresponding change of variable in its Weyl matrix.

Proposition 8.1 (Rotational lift of the Jacobi-like measure and its Weyl matrix). *The positive $q \times p$ matrix of measures on $\Sigma_M^{[1]}$ is defined by*

$$(8.4) \quad \int_{\Sigma_M^{[1]}} f(\lambda) d\widehat{\mu}_{j,i}^J(\lambda) = \frac{1}{M} \sum_{k=0}^{M-1} \int_0^1 f(\zeta_M^k x^{1/M}) x^{\alpha_i} w_j^J(x) dx.$$

Its moments vanish outside multiples of M ; for $n = Mr$,

$$(8.5) \quad \int_{\Sigma_M^{[1]}} \lambda^{Mr} d\widehat{\mu}_{j,i}^J(\lambda) = \frac{1}{r + \alpha_i + b_j + 1} \frac{\Gamma((r + \alpha_i + 1)\mathbf{1}_q + \mathbf{a})}{\Gamma((r + \alpha_i + 1)\mathbf{1}_q + \mathbf{b})}.$$

Its Weyl matrix is the explicit hypergeometric matrix

$$(8.6) \quad \widehat{\mathbf{S}}_{p,q}^J(\lambda) = \lambda^{M-1} \mathbf{S}_{p,q}^J(\lambda^M), \quad \lambda \in \mathbb{C} \setminus \Sigma_M^{[1]}.$$

For $|\lambda| > 1$, the entries on the right are given by the convergent hypergeometric series in (8.3). Elsewhere in $\mathbb{C} \setminus \Sigma_M^{[1]}$, the right-hand side denotes the single-valued analytic continuation determined by the Stieltjes integral from infinity. This rotational lift is the rotationally invariant component of the cyclic spectral system.

Proof. Positivity of the radial weights was proved above. Equation (8.4) is therefore a positive specialization of Definition 5.10; the root-of-unity sum and (8.2) give (8.5). Finally, Theorem 5.11 and (8.3) give (8.6) on the stated domain. $\square$

We now impose the additional hypotheses needed for the PBF conclusions and assume in addition that $q \geq 2$. Let $r_j \in \{0, \dots, q\}$ be the largest integer such that every member of the initial string $b_1, \dots, b_{r_j}$ occurs in the $\mathbf{a}$ -list. Delete those values from both lists and denote the remaining increasing $\mathbf{a}$ -list by $\widehat{a}_1 < \dots < \widehat{a}_m$, where $m = q - r_j$. In addition to (8.1) and the preceding integrability conditions, assume

$$(8.7) \quad a_q < b_1 + 1, \quad \widehat{a}_{m-1} \leq b_{r_j+1} \quad \text{when } m \geq 2.$$

Corollary 8.2 (Jacobi-like PBF realization of the Weyl matrices). *Under (8.7), form the scalar step-line moment matrix*

$$\mathcal{M}_{qu+j, pv+i}^J := \int_0^1 x^{u+v} d\mu_{j+1, i+1}^J(x),$$

where $u, v \in \mathbb{N}_0$, $0 \leq j < q$, and $0 \leq i < p$. It has a Gauss–Borel factorization

$$\mathcal{M}^J = (\mathcal{L}^J)^{-1} (\mathcal{U}^J)^{-1},$$

and its recurrence matrix has the canonical positive bidiagonal factorization

$$(8.8) \quad T^J = L_1^J \cdots L_p^J U_q^J \cdots U_1^J, \quad T^J = \mathcal{L}^J \Lambda_{[q]} (\mathcal{L}^J)^{-1} = (\mathcal{U}^J)^{-1} \Lambda_{[p]}^T \mathcal{U}^J.$$

With

$$\xi^J := E_{[q]} \mathcal{L}^J E_{[q]}^T, \quad (\nu^J)^T := E_{[p]} \mathcal{U}^J E_{[p]}^T,$$

one has

$$(8.9) \quad (\xi^J)^{-1} E_{[q]} (T^J)^n E_{[p]}^T (\nu^J)^{-T} = \int_0^1 x^n d\mu^J(x), \quad n \in \mathbb{N}_0.$$

Let $\mathcal{H}_{\text{ord}}^J$ be the prescribed-order height matrix formed from (8.8). If it is bounded, then, whenever $z^{-1} = \tau^M$ and $|\tau| < \|\mathcal{H}_{\text{ord}}^J\|_2^{-1}$,

$$S_{p,q}^J(z) = \frac{1}{z} (\xi^J)^{-1} Q_q (I - \tau \mathcal{H}_{\text{ord}}^J)^{-1} Q_p^T (\nu^J)^{-T}.$$

The height-block Schur recursion therefore gives its factor-resolved matrix continued fraction and finite Weyl approximants. Without boundedness, the same identity holds coefficientwise as a formal Laurent series at infinity. Under the closed-realization, core, and uniform-stability hypotheses, analytic finite-section convergence is to the corresponding projected operator resolvent. Identifying that limit with the Weyl matrix of a chosen Favard measure requires the additional equality (6.30).

For each $a \in \mathcal{A}_{\text{cyc}}$, let $\mathcal{H}_{(a)}^J$ be the minimum-height matrix obtained from the positive refactorization of $T_{(a)}^J$. After the transported normalizations, its factor-copy moment functional belonging to $T_{(a)}^J$ agrees, at the level of moments, with the rotational lift of the corresponding shifted Favard measure. If $\gcd(p, q) = 1$, $\mathcal{H}_{(a)}^J$ is the scalar recurrence of the complete matrix of star moment functionals. If $d = \gcd(p, q) > 1$, it is instead the block mixed-type recurrence, with normality asserted at cuts containing complete groups of the d equal-height coordinates.

Proof. The positive-factorization theorem [38, Theorem 9.4] and Corollary 6.36 give the Gauss–Borel and positive bidiagonal factorizations. Theorem 5.1 gives (8.9), while Proposition 5.8 and Theorem 5.7 identify the explicit Weyl matrix and give the factor-resolved formula. The assertions for the selected products follow from Corollary 6.15, Theorem 6.24, (3.5), and Theorems 3.9 and 6.10. $\square$

Thus Proposition 8.1 supplies the rotational measure and its Weyl matrix under the basic beta-convolution hypotheses, whereas Corollary 8.2 supplies the recurrence and factor-resolved conclusions only under the additional positive-factorization chamber. The other cyclic products have their own shifted positive Favard data, and their upper Christoffel transforms need not remain Jacobi-like.

9. INTEGRABLE DEFORMATIONS ENCODED BY THE PBF

The sparse recurrences constructed above do more than provide static spectral models. A scalar two-term difference operator with exponents q and $-p$ carries a hierarchy of commuting Lax flows for arbitrary positive p, q . When p and q are coprime, $\mathcal{H}_{(a)}$ is precisely the semi-infinite matrix of such an operator: the PBF selects its initial values, while the star moment functionals give its exact determinant solution. The same construction evolves all cyclic products of the refactorized factors with common Toda times. Moreover, the continuous flows commute with the discrete Christoffel transformations generated by the factors.

The case $q = 1$ includes the Narita–Itoh–Bogoyavlensky lattice. Moment solutions and Miura-type transformations for that one-sided setting have been studied in [49, 42]. After two general algebraic results, the PBF realization below uses the two-sided factorization with arbitrary coprime p and q , and simultaneously retains its complete cyclic family.

9.1. The arithmetic-support hierarchy. The next two algebraic results do not require p and q to be coprime. Let $p, q \geq 1$, put

$$d := \gcd(p, q), \quad M := p + q, \quad R := \frac{M}{d},$$

and let $\mathcal{S}$ be the forward shift on scalar sequences, $(\mathcal{S}f)_h = f_{h+1}$. Let $c = (c_h)_{h \in \mathbb{Z}}$ be a scalar sequence, with $c_h = 0$ for $h < p$, and define

$$\mathcal{L} := \mathcal{S}^q + c\mathcal{S}^{-p}$$

Multiplication is determined by $\mathcal{S}^j c = c \cdot_{+j} \mathcal{S}^j$. For a difference operator X , write X_+ for the sum of its nonnegative powers of $\mathcal{S}$, and define explicitly $\text{res}_{\mathcal{S}} X := [\mathcal{S}^0]X$, the coefficient of $\mathcal{S}^0$.

For $d > 1$, the minimal positive powers that generically preserve the two-term support are $\mathcal{L}^{R\ell}$: all exponents in $\mathcal{L}^n$ are congruent to $-np$ modulo M , and the required tangency occurs when $M \mid np$. We deliberately retain below the subsequence

$$\mathcal{L}^{M\ell} = \mathcal{L}^{R(d\ell)},$$

because its first residue expansion has $M = p + q$ positions and therefore matches the M bidiagonal factors. Thus, when $d > 1$, the hierarchy displayed below is the factor-compatible commuting subhierarchy; when $d = 1$, one has $R = M$ and no support-preserving positive flows are omitted.

Theorem 9.1 (The factor-compatible sparse arithmetic-support hierarchy). *For every $\ell \geq 1$, the equation*

$$(9.1) \quad \frac{\partial \mathcal{L}}{\partial t_\ell} = [(\mathcal{L}^{M\ell})_+, \mathcal{L}]$$

defines a formal flow that preserves the two exponents q and $-p$, and these formal flows commute. On the Banach space

$$\mathcal{X}_\infty := \{c \in \ell^\infty(\mathbb{N}_0; \mathbb{C}) : c_h = 0 \text{ for } 0 \leq h < p\},$$

with each sequence extended by zero to negative indices, the right-hand side is locally Lipschitz; hence, for bounded initial data, each equation defines a unique local C^1 flow. If

$$\varrho_h^{(\ell)} := \left(\text{res}_{\mathcal{S}} \mathcal{L}^{M\ell} \right)_h,$$

then

$$(9.2) \quad \frac{\partial c_h}{\partial t_\ell} = c_h \left(\varrho_h^{(\ell)} - \varrho_{h-p}^{(\ell)} \right).$$

Consequently, the boundary values $c_h = 0$, $h < p$, remain zero. Along every real solution, strict positivity of the remaining coefficients is preserved throughout its interval of existence.

Proof. Work first in the bilateral difference-operator algebra and put

$$X_\ell := (\mathcal{L}^{M\ell})_+, \quad Y_\ell := (\mathcal{L}^{M\ell})_-.$$

A monomial in $\mathcal{L}^{M\ell}$ containing k upward shifts has exponent

$$kq - (M\ell - k)p = M(k - \ell p),$$

so every exponent is a multiple of M . Since $[X_\ell + Y_\ell, \mathcal{L}] = 0$,

$$[X_\ell, \mathcal{L}] = -[Y_\ell, \mathcal{L}].$$

The left-hand side has no exponent below $-p$, whereas the right-hand side has no exponent above $q - M = -p$. Hence both sides are supported only at $\mathcal{S}^{-p}$; in particular, the coefficient of $\mathcal{S}^q$ vanishes. Differentiating X_ℓ , taking nonnegative parts, and using $[\mathcal{L}^{Mk}, \mathcal{L}^{M\ell}] = 0$ gives the zero-curvature identity

$$\partial_{t_k} X_\ell - \partial_{t_\ell} X_k - [X_k, X_\ell] = 0,$$

which proves that the formal flows commute.

The identities are polynomial in finitely many shifts of the coefficients. Their restriction to nonnegative indices produces no boundary term: a path that starts at a nonnegative height and visits a negative height must first use a downward edge from some height $h < p$, whose weight is $c_h = 0$.

Therefore, in the coefficient of $\mathcal{S}^{-p}$ in the commutator, only the coefficient of $\mathcal{S}^0$ in $(\mathcal{L}^{M\ell})_+$ contributes. Since

$$\varrho^{(\ell)} c \mathcal{S}^{-p} - c \mathcal{S}^{-p} \varrho^{(\ell)} = c(\varrho^{(\ell)} - \mathcal{S}^{-p}(\varrho^{(\ell)})) \mathcal{S}^{-p},$$

equation (9.2) follows.

For fixed ℓ , each $\varrho_h^{(\ell)}$ is a polynomial in a fixed finite set of shifts of c , uniformly in h . Thus the right-hand side of (9.2) defines a locally Lipschitz map $\mathcal{X}_\infty \rightarrow \mathcal{X}_\infty$. The Banach-space Picard–Lindelöf theorem gives the asserted local flow. Finally,

$$c_h(t) = c_h(0) \exp\left(\int_0^t (\varrho_h^{(\ell)} - \varrho_{h-p}^{(\ell)})(s) ds\right)$$

proves preservation of the boundary values and of strict positivity along real solutions. $\square$

General arithmetic-support hierarchies are classified in [16]; the preceding two-term proof is included to keep the support-preservation argument used here self-contained.

For the first flow of this factor-compatible subhierarchy, the residue has a direct path expansion. Let $\mathcal{P}_{p,q}$ consist of the sequences $\epsilon = (\epsilon_1, \dots, \epsilon_M)$, with $\epsilon_j \in \{q, -p\}$, containing p occurrences of q and q occurrences of $-p$. Put $\xi_0 = 0$ and $\xi_j = \epsilon_1 + \dots + \epsilon_j$. Identifying $+q$ with L and $-p$ with U identifies $\mathcal{P}_{p,q}$ with the set of all factor-type words of Definition 2.5. If η_j^{fac} denotes the factor-position height there, then $\xi_j = d\eta_j^{\text{fac}}$, where $d = \gcd(p, q)$. Thus the two statistics coincide in the coprime case. The sum below runs over all such words, not only over $\sigma_{\min}$ or its rotations.

Proposition 9.2 (The first factor-compatible nonlinear flow). *The polynomial in the first flow is*

$$(9.3) \quad \varrho_h^{(1)} = \sum_{\epsilon \in \mathcal{P}_{p,q}} \prod_{\substack{1 \leq j \leq M \\ \epsilon_j = -p}} c_{h+\xi_{j-1}}.$$

Before the boundary values are imposed, it is homogeneous of degree q and has positive integer coefficients. If $q = 1$, then

$$(9.4) \quad \frac{\partial c_h}{\partial t_1} = c_h \left(\sum_{j=1}^p c_{h+j} - \sum_{j=1}^p c_{h-j} \right),$$

which is the Narita–Itoh–Bogoyavlensky lattice.

Proof. A term of exponent zero in $\mathcal{L}^M$ uses p shifts $+q$ and q shifts $-p$. A downward shift in position j contributes the coefficient at height $h + \xi_{j-1}$, which gives (9.3). When $q = 1$, placing the unique downward shift in each of the $p + 1$ possible positions gives $\varrho_h^{(1)} = \sum_{j=0}^p c_{h+j}$. Substitution in (9.2) proves (9.4); see also [51]. $\square$

9.2. Exact determinant solution from the star moments. We now return to the PBF realization within this subsection. Assume $\gcd(p, q) = 1$, fix $a \in \mathcal{A}_{\text{cyc}}$, set $c_h = c_h^{(a)}$ for $h \geq p$ and $c_h = 0$ for $h < p$, and write

$$\mathcal{L}_{(a)}(\mathbf{0}) := \mathcal{S}^q + c^{(a)} \mathcal{S}^{-p}.$$

By Corollary 3.10, its semi-infinite matrix is $\mathcal{H}_{(a)}$.

Let $\widehat{\Psi}_{(a)}$ be the matrix of star moment functionals whose recurrence is $\mathcal{H}_{(a)}$, as constructed in Section 6. Put

$$E_t(\lambda) := \exp\left(\sum_{\ell \geq 1} t_\ell \lambda^{M\ell}\right).$$

All formal assertions below are understood coefficientwise after restriction to an arbitrary finite set of active times. Define the radial deformation of the functionals coefficientwise by

$$(9.5) \quad \left\langle \widehat{\Psi}_{(a)}(t), f \right\rangle := \left\langle \widehat{\Psi}_{(a)}, f(\lambda) E_t(\lambda) \right\rangle, \quad f \in \mathbb{C}[\lambda].$$

When representing measures exist and the displayed exponential moments are finite, this is equivalently

$$d\widehat{\Psi}_{(a)}(\lambda; t) = E_t(\lambda) d\widehat{\Psi}_{(a)}(\lambda).$$

For an analytic interpretation, fix $L < \infty$, set $t_\ell = 0$ for $\ell > L$, and fix a connected open neighbourhood $D \subset \mathbb{C}^L$ of $\mathbf{0}$ on which all pairings in (9.5) exist and depend holomorphically on t . This is automatic when the corresponding height operator is bounded, and also when the relevant representing measures are compactly supported. The deformation is radial: it depends on λ only through $x = \lambda^M$ and hence preserves every Fourier character of the star functional.

Let $\widehat{\mathcal{M}}_{(a)}(t)$ be the scalar step-line moment matrix of (9.5). Thus, for $r = qu + j$, $s = pv + i$,

$$(9.6) \quad \left(\widehat{\mathcal{M}}_{(a)}(t)\right)_{r,s} = \left\langle \widehat{\Psi}_{(a),j,i}, \lambda^{u+v} \exp\left(\sum_{\ell \geq 1} t_\ell \lambda^{M\ell}\right) \right\rangle.$$

Set $\tau_0^{(a)}(t) = 1$ and, for $N \geq 1$,

$$\tau_N^{(a)}(t) := \det\left(\widehat{\mathcal{M}}_{(a)}(t)^{[N-1]}\right).$$

These are the integrable tau functions of the time-deformed star moment matrix. They should be distinguished from the elementary Christoffel τ -determinants of Corollary 6.27, which test the existence of the discrete one-step connections. Both families arise from moment determinants, but they play different roles.

Theorem 9.3 (Determinant solution selected by the PBF). *Assume $\gcd(p, q) = 1$. In every fixed finite set of times, let $\mathcal{L}_{(a)}(t)$ be the unique coefficientwise formal solution of (9.1) with initial operator $\mathcal{L}_{(a)}(\mathbf{0})$ defined above, whose semi-infinite matrix is $\mathcal{H}_{(a)}$. This formal solution is reconstructed coefficientwise from (9.6). Its nontrivial coefficients are*

$$(9.7) \quad c_h^{(a)}(t) = \frac{\tau_{h-p}^{(a)}(t) \tau_{h+1}^{(a)}(t)}{\tau_{h-p+1}^{(a)}(t) \tau_h^{(a)}(t)}, \quad h \geq p.$$

For the analytic interpretation, let D be a domain satisfying the hypotheses stated above. For each $h \geq p$, let D_h be the connected component containing $\mathbf{0}$ of

$$\left\{ t \in D : \tau_N^{(a)}(t) \neq 0, \ 1 \leq N \leq h+1 \right\}.$$

Then the finite Gauss–Borel construction through row h exists holomorphically on D_h , and (9.7) holds there. In particular, every fixed coefficient $c_h^{(a)}$ is analytic in a neighbourhood of $\mathbf{0}$, which may depend on h . If all leading principal minors $\tau_N^{(a)}$, $N \geq 1$, remain nonzero throughout D , then the full recurrence is coefficientwise holomorphic on D .

Proof. Theorem 6.8 gives $\tau_N^{(a)}(\mathbf{0}) \neq 0$ for every N , so the formal Gauss–Borel factorization below exists and is unique. In the analytic setting, ordinary finite-dimensional Gaussian elimination shows that the factors through row h depend holomorphically on t throughout D_h , since precisely the required pivots remain nonzero. No convergence

of an infinite triangular factorization is needed for this coefficientwise assertion. Let $\Lambda_{[q]}$ and $\Lambda_{[p]}$ be the two step-line block shifts. Equation (9.6) gives the equivalent moment evolutions

$$\begin{aligned}\widehat{\mathcal{M}}_{(a)}(\mathbf{t}) &= \exp\left(\sum_{\ell \geq 1} t_\ell \Lambda_{[q]}^{M_\ell}\right) \widehat{\mathcal{M}}_{(a)}(\mathbf{0}), \\ \widehat{\mathcal{M}}_{(a)}(\mathbf{t}) &= \widehat{\mathcal{M}}_{(a)}(\mathbf{0}) \exp\left(\sum_{\ell \geq 1} t_\ell ((\Lambda_{[p]})^\top)^{M_\ell}\right).\end{aligned}$$

Factor the moment matrix as $\widehat{\mathcal{M}}_{(a)} = \mathcal{L}^{-1} \mathcal{U}^{-1}$, with $\mathcal{L}$ unit lower triangular and $\mathcal{U}$ upper triangular. The corresponding recurrence is

$$\mathcal{L}_{(a)}(\mathbf{t}) = \mathcal{L} \Lambda_{[q]} \mathcal{L}^{-1} = \mathcal{U}^{-1} (\Lambda_{[p]})^\top \mathcal{U}.$$

Differentiating the first moment evolution and separating strictly lower and upper triangular parts gives $(\partial_{t_\ell} \mathcal{L}) \mathcal{L}^{-1} = -(\mathcal{L}_{(a)}^{M_\ell})_-$. Therefore

$$\frac{\partial \mathcal{L}_{(a)}}{\partial t_\ell} = [(\mathcal{L}_{(a)}^{M_\ell})_+, \mathcal{L}_{(a)}].$$

The recurrence produced by the moment factorization belongs to the fixed band $[-p, q]$. In that band the Lax right-hand side is a finite-range polynomial vector field, so its formal initial-value problem is unique coefficientwise. Theorem 9.1 shows that the two-term submanifold is invariant. Since the recurrence from the moment factorization has initial operator $\mathcal{L}_{(a)}(\mathbf{0})$ (equivalently, initial semi-infinite matrix $\mathcal{H}_{(a)}$), uniqueness forces it to remain on that submanifold. It is therefore the unique formal solution with the prescribed initial value.

The diagonal pivots of the Gauss–Borel factorization satisfy

$$\mathcal{U}_{n,n} = \frac{\tau_n^{(a)}}{\tau_{n+1}^{(a)}}.$$

In the second recurrence representation, the entry in position $(h, h-p)$ has only the diagonal contribution

$$c_h^{(a)} = \frac{\mathcal{U}_{h-p, h-p}}{\mathcal{U}_{h,h}}.$$

Substituting the pivot formula proves (9.7). □

The theorem distinguishes the time-dependent solution from the static PBF formulas. The PBF coefficients displayed in the Piñeiro and Jacobi-like sections determine, through the selected minimum-height refactorizations, the initial coefficients $c_h^{(a)}(\mathbf{0})$. Formula (9.7) supplies their canonical formal evolution and, on the domains specified in the theorem, their coefficientwise analytic evolution.

9.3. Discrete Christoffel transformations and synchronized Toda flows. The continuous deformation above is compatible with the discrete transformations generated by the bidiagonal factors. The compatibility is most transparent at the level of the polynomial arrays.

Proposition 9.4 (A Christoffel transfer commutes with the Toda times). *Let*

$$E_t(x) := \exp\left(\sum_{k \geq 1} t_k x^k\right), \quad d\Psi_j(x; \mathbf{t}) = E_t(x) d\Psi_j(x; \mathbf{0}), \quad j \in \{0, 1\},$$

where $d\Psi_0(x; \mathbf{0})$ and $d\Psi_1(x; \mathbf{0})$ are two consecutive functionals related by a fixed row- or column-side Christoffel multiplication. Assume that both deformed systems are weakly normal as formal series. For the analytic statement, assume instead a common domain on which the two deformed moment matrices are entrywise holomorphic and all leading minors required for their Gauss–Borel factorizations remain nonzero. Let $T_j(\mathbf{t})$ and $\mathbf{B}_j(x; \mathbf{t})$ be their recurrence matrices and right polynomial arrays in Sato-compatible Gauss–Borel normalizations. These normalizations are chosen by fixed constant gauges at $\mathbf{t} = \mathbf{0}$; their numerical initial blocks are not assumed to remain

constant in time. Let $F(\mathbf{t})$ be the normalized bidiagonal connection matrix and $K(x)$ the fixed matrix polynomial in the corresponding right-array relation, after orienting the labels 0, 1 so that

$$F(\mathbf{t})\mathbf{B}_0(x; \mathbf{t}) = \mathbf{B}_1(x; \mathbf{t})K(x).$$

For a column-side step, this orientation takes system 0 to be the transformed system and system 1 the preceding one; then $F = L$ and $K = I$. A row-side step uses the forward orientation displayed above. For every $\ell \geq 1$,

$$(9.8) \quad \frac{\partial F}{\partial t_\ell} = (T_1^\ell)_+ F - F(T_0^\ell)_+.$$

Consequently, the Sato-compatible normalized connections around every elementary square of the Christoffel rectangle satisfy the same qd refactorization throughout that common time domain.

Proof. Since $E_t(x)$ is scalar, it commutes with every fixed row- or column-side Christoffel multiplier. Hence the Christoffel transform of the deformed functional is the deformation of the Christoffel-transformed functional, and the matrix polynomial $K(x)$ is independent of $\mathbf{t}$. Weak normality and the chosen Sato-compatible Gauss–Borel normalizations give the unique connection

$$F\mathbf{B}_0 = \mathbf{B}_1K$$

throughout the common time domain.

In this normalization the Sato equations are

$$\partial_{t_\ell} \mathbf{B}_j = -(T_j^\ell)_- \mathbf{B}_j.$$

Differentiating the connection gives

$$\left(\frac{\partial F}{\partial t_\ell} - F(T_0^\ell)_- + (T_1^\ell)_- F \right) \mathbf{B}_0 = 0.$$

Since $\mathbf{B}_0$ is a polynomial basis, its coefficient matrix is invertible lower triangular and it can be cancelled. The recurrence equations also give $T_1^\ell F = FT_0^\ell$. Splitting both sides into their upper and strictly lower triangular parts yields

$$\frac{\partial F}{\partial t_\ell} = (T_1^\ell)_+ F - F(T_0^\ell)_+.$$

Around an elementary square the fixed row and column Christoffel multipliers act on opposite sides and hence commute; the common scalar deformation commutes with both. The two routes therefore give the same deformed moment functional. Uniqueness of the chosen Gauss–Borel factorizations and of the normalized connections gives the time-dependent qd identity. $\square$

Recall that the cyclic products of the refactorized factors belonging to $\mathcal{H}_{(a)}$ are denoted by $\tilde{T}_{(a;1)}, \dots, \tilde{T}_{(a;M)}$.

Theorem 9.5 (One sparse Lax operator for the cyclic Toda family). *Assume $\gcd(p, q) = 1$, and let $\mathcal{L}_{(a)}(\mathbf{t})$ be the PBF-selected solution of Theorem 9.3. Then every cyclic block satisfies*

$$(9.9) \quad \frac{\partial \tilde{T}_{(a;r)}}{\partial t_\ell} = \left[(\tilde{T}_{(a;r)}^\ell)_+, \tilde{T}_{(a;r)} \right], \quad r \in \{1, \dots, M\}.$$

The factors connecting consecutive blocks satisfy the auxiliary equations

$$(9.10) \quad \frac{\partial \tilde{F}_r^{(a)}}{\partial t_\ell} = (\tilde{T}_{(a;r)}^\ell)_+ \tilde{F}_r^{(a)} - \tilde{F}_r^{(a)} (\tilde{T}_{(a;r+1)}^\ell)_+,$$

where the indices are read cyclically and the orientation is fixed by $\tilde{T}_{(a;r)} \tilde{F}_r^{(a)} = \tilde{F}_r^{(a)} \tilde{T}_{(a;r+1)}$. Thus the semi-infinite matrix $\mathcal{H}_{(a)}(\mathbf{t})$ of $\mathcal{L}_{(a)}(\mathbf{t})$ is a common integrable M -th root of the cyclic Toda family. If, in addition, the factor-copy polynomial arrays are taken in the Sato-compatible Gauss–Borel normalizations of Proposition 9.4, so that the off-diagonal blocks $\tilde{F}_r^{(a)}$ are the corresponding Christoffel connections, then the factor transfers are discrete Bäcklund directions and (9.10) is precisely (9.8). Under other time-dependent initial-condition gauges, the compatibility is gauge-covariant and contains the corresponding gauge terms.

Proof. Let $\mathcal{P}_a$ denote the fixed permutation from factor-copy order to height order, so that

$$\mathcal{H}_{(a)} = \mathcal{P}_a \mathcal{C}_{(a)} \mathcal{P}_a^{-1}.$$

In factor-copy order,

$$\mathcal{P}_a^{-1} \mathcal{H}_{(a)}^M \mathcal{P}_a = \mathcal{C}_{(a)}^M = \text{diag}(\tilde{T}_{(a;1)}, \dots, \tilde{T}_{(a;M)}).$$

The coordinates in one factor copy occur at heights separated by M . Hence nonnegative height displacement is equivalent to an upper triangular entry inside that copy, and

$$\mathcal{P}_a^{-1} (\mathcal{H}_{(a)}^M)_+ \mathcal{P}_a = \text{diag}((\tilde{T}_{(a;1)}^\ell)_+, \dots, (\tilde{T}_{(a;M)}^\ell)_+).$$

Conjugating the Lax equation for $\mathcal{H}_{(a)}$ by $\mathcal{P}_a^{-1}$ gives the corresponding equation for $\mathcal{C}_{(a)}$. Its cyclic off-diagonal blocks are the factors $\tilde{F}_r^{(a)}$ and give (9.10). Taking the M -th power and then the diagonal factor-copy blocks gives (9.9). Under the additional normalization hypothesis in the statement, Proposition 9.4 identifies these auxiliary equations with the continuous–discrete Christoffel compatibility; the statement for other gauges is the usual change-of-gauge form of the same identity. $\square$

The band of each cyclic product extends from the p -th subdiagonal to the q -th superdiagonal, so its equations belong to the (q, p) -bigraded Toda setting [15]. The new feature here is the factor-resolved sparse root: one scalar sequence $c_h^{(a)}(t)$ evolves the entire cyclic family and retains the individual bidiagonal connections between its members.

9.4. Hypergeometric determinant solutions. The determinant solution becomes explicit once the cyclic spectral data are known. Fix two factor copies r, s , let $\tilde{F}_{r \rightarrow s}$ be their consecutive partial product, and let $(\mathbf{B}^{(s)}, \mathbf{A}^{(s)}, d\Psi^{(s)})$ be biorthogonal data for $\tilde{T}_{(a;s)}$. The radial moments of Section 6 have the spectral representation

$$(9.11) \quad m_k^{r,s} = \int (e_0^\top \tilde{F}_{r \rightarrow s} \mathbf{B}^{(s)}(x)) x^k d\Psi^{(s)}(x) (\mathbf{A}^{(s)}(x) e_0).$$

Denote the scalar measure occurring on the right-hand side by

$$d\sigma^{r,s}(x) := (e_0^\top \tilde{F}_{r \rightarrow s} \mathbf{B}^{(s)}(x)) d\Psi^{(s)}(x) (\mathbf{A}^{(s)}(x) e_0).$$

It is a finite signed or complex measure whenever the displayed spectral integral is defined; it is not positive in general.

Proposition 9.6 (Radial deformation in cyclic spectral coordinates). *Equation (9.11) says that $d\sigma^{r,s}$ and the positive radial measure $d\rho^{r,s}$ of Theorem 6.2 represent the same moment functional. If $d\sigma^{r,s}$ is positive on $[0, \infty)$ and the corresponding Stieltjes moment problem is determinate, then $d\sigma^{r,s} = d\rho^{r,s}$. The same equality follows without positivity whenever both measures are compactly supported on the real line. Under the first radial time $t = t_1$, the formal deformed moments are*

$$(9.12) \quad m_k^{r,s}(t) = \int (e_0^\top \tilde{F}_{r \rightarrow s} \mathbf{B}^{(s)}(x)) x^k e^{tx} d\Psi^{(s)}(x) (\mathbf{A}^{(s)}(x) e_0).$$

Thus every entry of the deformed star moment matrix is a finite polynomial dressing of the deformed moments of a cyclic Favard system. In the compactly supported case the identity is entire in $t \in \mathbb{C}$.

Proof. Biorthogonality and either recurrence equation give

$$(\tilde{T}_{(a;s)})^k = \int \mathbf{B}^{(s)}(x) x^k d\Psi^{(s)}(x) \mathbf{A}^{(s)}(x).$$

Multiplication on the left by $e_0^\top \tilde{F}_{r \rightarrow s}$ and on the right by e_0 gives (9.11). Its moments are $(\tilde{F}_{r \rightarrow s} (\tilde{T}_{(a;s)})^k)_{0,0}$, so it is the moment functional of the radial measure constructed in Theorem 6.2. This proves equality of the two moment functionals. Stieltjes determinacy gives equality of the measures when the dressed measure is positive; compact support on the real line gives equality for finite signed or complex measures by polynomial density. Multiplication by e^{tx} , interpreted coefficientwise in t , gives (9.12); compact support makes this an entire identity. $\square$

For the Piñeiro system, the first-time deformation of the scalar step-line moment matrix is completely explicit. With $r = qu + j$, $s = pv + i$, put $x_r = u + \beta_{j+1}$ and $y_s = v + \alpha_{i+1} + 1$. Under (7.3),

$$x_r + y_s \geq \alpha_1 + \beta_1 + 1 > 0.$$

Thus the following integral converges for every r, s and defines an entire function of t :

$$(9.13) \quad \mathcal{M}_{r,s}^P(t) = \int_0^1 x^{x_r+y_s-1} e^{tx} dx = \frac{1}{x_r + y_s} {}_1F_1 \left(\begin{matrix} x_r + y_s \\ x_r + y_s + 1 \end{matrix}; t \right).$$

Theorem 9.7 (Global positive coefficientwise solutions for the Piñeiro system). *Under (7.3), the deformed Piñeiro moment matrix in (9.13) is totally positive for every real t . All moment factorizations and normalized Christoffel connections in its finite rectangle therefore exist and remain positive for every real t ; these assertions hold for arbitrary p, q . If, in addition, $\gcd(p, q) = 1$, the selected star refactorizations give a global positive coefficientwise solution of the first scalar flow in (9.1), and their cyclic products satisfy (9.9) with $\ell = 1$. More generally, replacing e^{tx} by $\exp(\sum_{\ell=1}^L t_\ell x^\ell)$, with finitely many real times, gives, in the coprime case, global positive coefficientwise solutions of the corresponding commuting scalar flows.*

When $\gcd(p, q) = 1$, every coefficient $c_h^{(a)}(t)$ of the first-time deformation can be obtained by finitely many determinants and rational operations from the Kummer entries in (9.13). The initial coefficients $c_h^{(a)}(0)$ are those of the selected minimum-height refactorizations of the Piñeiro PBF. For the rational system in Example 7.1, they are rational.

Proof. For finite increasing row and column sets $I = \{r_1 < \dots < r_N\}$ and $J = \{s_1 < \dots < s_N\}$, expand the two determinants below. Although an individual function $z^{y_{s_v}-1}$ need not be integrable at the origin, every monomial in the finite expansion has, in each variable, an exponent of the form $x_{r_u} + y_{s_v} - 1 > -1$. The expansion is therefore absolutely integrable, and Andréief's identity gives

$$\det [\mathcal{M}_{r_u, s_v}^P(t)]_{u,v=1}^N = \frac{1}{N!} \int_{(0,1)^N} \det [z_k^{x_{r_u}}]_{u,k=1}^N \det [z_k^{y_{s_v}-1}]_{v,k=1}^N \prod_{k=1}^N e^{tz_k} dz_k.$$

The inequalities in (7.3) imply that the exponent sequences are strictly increasing. Indeed, successive differences in (x_r) are either $\beta_{j+2} - \beta_{j+1} > 0$ or $1 + \beta_1 - \beta_q > 0$, and the corresponding differences in (y_s) are either $\alpha_{i+2} - \alpha_{i+1} > 0$ or $1 + \alpha_1 - \alpha_p > 0$. On $0 < z_1 < \dots < z_N < 1$, the two generalized Vandermonde determinants have the same strictly positive sign because both exponent sequences are strictly increasing. The remaining factor is positive for every real t , so the minor is strictly positive. The same proof applies with e^{tz_k} replaced by $\exp(\sum_{\ell=1}^L t_\ell z_k^\ell)$, which proves the multi-time assertion.

Total positivity gives the Gauss–Borel factorization at every vertex of the finite rectangle for every real t . Choose there the Sato-compatible Gauss–Borel normalizations of Proposition 9.4; the resulting Christoffel connection matrices have entries that are ratios of minors of (9.13). Starting from the chosen cyclic product, the finite directed sequence of adjacent qd refactorizations then constructs the factors of $\mathcal{H}_{(a)}(t)$; each of their entries is a rational expression in those Christoffel coefficients and remains strictly positive.

The connections satisfy Proposition 9.4, so their cyclic matrix satisfies the factor-copy Lax equation. If $\gcd(p, q) = 1$, conjugating by the fixed height permutation gives (9.1) for $\mathcal{H}_{(a)}(t)$. In that coprime case, at $t = 0$ this is the prescribed sparse matrix, so uniqueness of the formal initial-value problem identifies it with the determinant solution of Theorem 9.3. Since the construction exists and stays positive for every real t , the coefficientwise solution is global on the real first-time axis. The same finite sequence of determinants and rational operations proves the asserted Kummer description. $\square$

In the coprime case, these solutions are not finite-soliton solutions in general. The Piñeiro measure is absolutely continuous on $(0, 1)$. On the first-time axis its τ -functions are determinants of Kummer moments of this continuous spectral measure; for several times the entries are the corresponding exponential-polynomial moments. When only finitely many times are nonzero, these tau functions are entire functions of those complex times. Total positivity

excludes their zeros for all real values of those times, so every coefficient $c_h^{(a)}(t)$ is positive and real analytic there. For complex times, each fixed coefficient is meromorphic, with possible poles only at zeros of the two denominator tau functions in (9.7). Thus the Piñeiro construction supplies global positive coefficientwise solutions, represented by continuous-spectrum determinants and selected explicitly by the original PBF.

The Jacobi-like system has an analogous explicit deformation. If $s = n + \alpha_i + 1$, then

$$(9.14) \quad \int_0^1 x^n e^{tx} d\mu_{j,i}^J(x) = \left(\int_0^1 x^n d\mu_{j,i}^J(x) \right) {}_qF_q \left(\begin{matrix} s\mathbf{1}_q + \mathbf{a} \\ s\mathbf{1}_q + \mathbf{b} + \mathbf{e}_j^{(q)} \end{matrix}; t \right).$$

Indeed, expanding e^{tx} and using (8.2) gives, for $k \geq 0$,

$$\frac{\int_0^1 x^{n+k} d\mu_{j,i}^J(x)}{\int_0^1 x^n d\mu_{j,i}^J(x)} = \frac{(s + b_j)_k}{(s + b_j + 1)_k} \frac{(s\mathbf{1}_q + \mathbf{a})_k}{(s\mathbf{1}_q + \mathbf{b})_k}.$$

The factor $(s + b_j)_k$ cancels the j -th component of the second denominator. The remaining lower vector is $s\mathbf{1}_q + \mathbf{b} + \mathbf{e}_j^{(q)}$, which gives exactly the ${}_qF_q$ in (9.14).

At every vertex for which the two weight systems at $t = 0$ satisfy the AT hypotheses, their deformations remain AT for every real t . Indeed, multiplying all weights in a linear form by the common factor $e^{tx} > 0$ does not change its zeros or their multiplicities. Assume now that $q \geq 2$, that the parameters lie in the Jacobi-like positive-factorization chamber (8.7) and that, at every vertex required by the selected Christoffel rectangle, both transformed weight systems satisfy the AT hypotheses. When $\gcd(p, q) = 1$, the same argument gives global positive coefficientwise determinant solutions, built from the ${}_qF_q$ entries in (9.14) and their corresponding row and column shifts. Indeed, the AT hypotheses prevent the required determinants from vanishing; continuity then preserves the positive signs of the PBF coefficients from $t = 0$ along the real axis.

In the larger Jacobi-like PBF chamber with $\gcd(p, q) = 1$, Theorem 9.3 still gives the formal solution. Since every deformed moment entry is entire, for each fixed h it also gives a coefficient $c_h^{(a)}$ analytic in a neighbourhood of $\mathbf{0}$ on which the finitely many minors through order $h + 1$ do not vanish. This neighbourhood may depend on h ; neither a common neighbourhood for all coefficients nor global nonvanishing of all deformed moment minors is asserted.

The radial deformation (9.5) is a common-time reduction of the multi-component 2D Toda deformation associated with mixed-type moment matrices [3]. Its role here is to evolve the PBF coefficients while synchronizing the complete cyclic family through Theorem 9.5. For $d = \gcd(p, q) > 1$, the scalar two-term operator still carries the factor-compatible support-preserving commuting subhierarchy above and decomposes into d invariant residue classes. What fails is its identification with $\mathcal{H}_{(a)}$, which has d coordinate states at each height and block displacements $+Q$ and $-P$. A PBF-adapted matrix tau-function description of this block problem, and its relation to the independent times of the full multi-component 2D Toda hierarchy, is left open.

10. CONCLUSIONS

The article shows that a positive bidiagonal factorization is an organizing datum in its own right, finer than the banded matrix obtained by multiplying the factors. Its principal conclusions fall into the same three groups as the results stated in the Introduction.

- (1) Every cyclic permutation of the prescribed factors again admits a normalized PBF in lower–upper order, for bounded and unbounded factors alike. With suitable one-sided maximal-degree data at consecutive cyclic products, each cyclic transfer determines a degree-one matrix Christoffel polynomial with determinant cx , $c \neq 0$, and the transformed functional has the next cyclic product as recurrence matrix. Different nonsingular lower triangular initial conditions produce constant lower/upper triangular changes of the functional and preserve weak normality; maximal component degrees require separate admissibility. A complete factor circuit gives

multiplication by x , up to the terminal normalization. For the fixed multipliers $K_q(x)$ and $K_p(x)^T$, weak normality of the two boundary chains gives the finite rectangle of simultaneous moment shifts. Total positivity of the scalar moment matrix makes this additional boundary hypothesis automatic.

- (2) Exactly the cyclic products $T_{(a)}$, $a \in \mathcal{A}_{\text{cyc}} = \{a_-, \dots, a_+\}$, can reach the minimum-height factor pattern through the directed positive adjacent exchanges used here. Their associated sparse matrices $\mathcal{H}_{(a)}$ have only the block displacements $+Q$ and $-P$, with $d = \gcd(p, q)$ coordinates at each height; in the coprime case only the q -th superdiagonal and the p -th subdiagonal remain. Their projected resolvents recover the resolvents of the cyclic products, their finite nonzero spectra lie on a $(p + q)$ -ray star, and their explicit block moment factorizations give positive radial Stieltjes moment sequences. For fixed compactly supported radial representing measures, the Fourier components admit finite complex measures supported on the corresponding compact star precisely when the corresponding atom-at-zero and negative fractional-moment conditions of Proposition 6.5 hold entrywise. In the coprime case, each $\mathcal{H}_{(a)}$ also carries a local arithmetic-support Lax hierarchy. A radial exponential deformation of the star moments gives the evolving coefficients by an explicit quotient of four leading minors. The $(p + q)$ -th power of the sparse Lax matrix synchronizes the Toda flows of the cyclic products of the refactorized factors, and the Christoffel transfers commute with the continuous times. On the real first-time axis, the Piñeiro chamber produces global positive coefficientwise solutions expressed by Kummer determinants; the intersection of the Jacobi-like positive-factorization chamber with the requisite AT region produces analogous global positive coefficientwise solutions expressed by ${}_qF_q$ determinants.
- (3) The sparse matrix built from the prescribed factor order retains every original bidiagonal coefficient. Grouping consecutive heights makes it block tridiagonal and gives a factor-resolved matrix continued fraction. Its finite height-block truncations have explicit Padé-type contact, denominator factorization, backward evaluation, and residual error bounds. For bounded nonnegative factors these convergents are entrywise monotone, their denominators are nonsingular M -matrices, and their limit is the least nonnegative solution. Stochastic normalization gives convergence throughout the unit disk. A second grouping produces the q -state recursion, which for $q = 1$ is the classical p -branched Stieltjes recursion. A rectangular projection of the bounded resolvent is the Weyl matrix of the mixed-type system, so all its Stieltjes transforms are approximated simultaneously. For unbounded factorizations the coefficientwise approximation and finite algorithm remain valid. A specified closed realization satisfying the core and uniform finite-section stability hypotheses gives convergence to its projected resolvent; this limit is a measure-defined Weyl matrix only under the additional spectral identification.

Together these results extend the classical passage

$$\text{Jacobi matrix} \longrightarrow \text{Stieltjes fraction} \longrightarrow \text{scalar Weyl function}$$

to a two-sided mixed-type setting. The role of the Jacobi coefficients is played by the individual positive bidiagonal factors. Their prescribed order produces rational approximants to the rectangular Weyl matrix in the bounded or spectrally identified setting, and formal moment approximants in general; their minimum-height factorizations produce the two-diagonal recurrences and their moment systems. The Piñeiro example displays the full construction and its certified numerical error. In the Jacobi-like specialization, cyclic transforms may leave the original special-function family even though the PBF and its spectral consequences persist.

The remaining questions include uniform finite-section stability outside the domains controlled by the norm, spectral radius, and numerical range; ratio asymptotics and convergence rates of the matrix convergents; comparison with finite spectral approximants obtained from principal truncations in the unbounded case; and an extension of the star-lifting criterion beyond the compactly supported, determinate radial case, including intrinsic conditions that select noncompact representatives with no atom at the origin in nonzero modes and the required negative fractional moments. On the integrable side, the principal open problem is not the existence of scalar support-preserving flows for noncoprime p, q , but their PBF-relevant block realization when $d = \gcd(p, q) > 1$, including a matrix determinant or quasideterminant solution and its relation to the independent times of the full multi-component $2D$ Toda hierarchy.

The occurrence of Christoffel words suggests a further combinatorial and spectral direction. If coprime pairs (P_n, Q_n) are chosen along the continued-fraction convergents to an irrational slope, the corresponding Christoffel

patterns have a natural Sturmian limit. It remains to determine whether the associated sparse recurrences and matrix continued fractions admit a corresponding aperiodic limit, and whether the classical connection between Christoffel words, ordinary continued fractions, and Markoff theory has a spectral counterpart for positive bidiagonal factorizations [8].

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Data availability. No external data set is used. The Python and Sage scripts and their generated output are supplied as Supplementary Material with the manuscript. The numerical script implements the cited Piñeiro Gamma–Pochhammer formulas and reproduces Table 3, including the 12-digit reference resolvent block, the finite-section dimensions, and the a priori and residual bounds. The exact-arithmetic regression suite checks the cyclic powers and Darboux intertwining, the height multiplicities and block grouping, the finite Schur recursion and determinant factorization, the first omitted path, the q -state recursion, the rectangular moment projection, representative Jacobi-like Weyl coefficients, finite monotonicity, and the exact residual identity. Two additional exact-arithmetic Maple scripts reconstruct the determinant solution of the sparse Lax hierarchy. One checks the coefficient formula for several coprime pairs (p, q) ; the other verifies the first nonlinear flow coefficientwise for the rational Piñeiro $(3, 2)$ example. A further exact Sage check verifies, for the same set of coprime pairs, that restriction of the bilateral Lax identity to the semi-infinite zero boundary introduces no correction.

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