HYPERGEOMETRIC MIXED-TYPE MULTIPLE ORTHOGONAL POLYNOMIALS

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ABSTRACT. Two hypergeometric families of mixed-type multiple orthogonal forms are constructed on the positive real line for rank-one $q \times p$ matrices of weights, with arbitrary p and q: Jacobi-like and Laguerre-like systems. In the admissible near-diagonal range, both mixed forms are obtained explicitly. The components on the power vector are single terminating generalized hypergeometric polynomials. The components on the hypergeometric vector are finite sums of terminating generalized hypergeometric polynomials whose number grows with the total degree; in the Jacobi-like case these sums are reorganized as finite combinations of terminating Kampé de Fériet polynomials evaluated at $(x, 1)$. In the mixed beta–Euler Laguerre-like case, the corresponding regrouping is replaced by a finite triangular connection system coupling the finite-pole residues to the Euler-derivative components. Gamma-quotient Mellin formulas, Meijer G-representations, and Rodrigues-type formulas are obtained for the complete mixed forms.

On the step-line, the two biorthogonal sequences satisfy dual recurrences governed by a matrix with p subdiagonals and q superdiagonals. Every band coefficient is given by a finite Gamma–Pochhammer expression, after a finite confluent coefficient system in the Laguerre-like case. Under the stated normality and nonzero-pivot hypotheses, Christoffel transformations and Gauss–Borel factorization yield a bidiagonal factorization of each recurrence matrix. The lower factors have closed Pochhammer formulas, while the upper factors are represented by Christoffel tau-determinants and, equivalently, by cross-ratios of shifted moment minors; both chains close explicitly in the mixed Piñeiro specialization. Finally, moving-Christoffel/power scaling of the scalar corners recovers the full $p \times q$ Daems–Kuijlaars Gaussian mixed system, while the canonical Laguerre-like vector has a distinct gamma-convolution/Hermite-jet limit.

CONTENTS

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Date: August 20, 2026.

2020 *Mathematics Subject Classification.* Primary 33C45; Secondary 42C05, 33C20, 15A23, 47B36.

Key words and phrases. mixed-type multiple orthogonal polynomials; Jacobi-like and Laguerre-like weights; hypergeometric and Kampé de Fériet functions; Mellin transforms and Meijer G-functions; step-line recurrences; Christoffel transformations; bidiagonal factorization; Gaussian mixed systems; confluent limits.

1. Introduction

Multiple orthogonal polynomials and multiple orthogonal forms extend the classical theory to systems involving several weights or moment functionals. Besides their intrinsic approximation-theoretic interest, they appear in random matrices, nonintersecting paths, Hermite–Padé approximation, integrable systems, spectral theory of banded operators, and stochastic processes. Step-line recurrences and their associated banded matrices provide the analogue of the Jacobi matrix for multiple orthogonality.

In the scalar theory, orthogonal polynomials satisfy a three-term recurrence relation governed by a tridiagonal Jacobi matrix. For multiple orthogonality, the recurrence matrix is banded. Its structure depends on the geometry of the underlying multi-indices and on the interaction between the different weights. On the step-line, these recurrences give rise to banded matrices whose spectral theory is closely connected with matrix-valued moments, generalized Favard theorems, and block or rectangular Weyl functions.

In the mixed-type multiple orthogonality considered here, one side of the orthogonality structure is given by a vector of powers,

$$\mathbf{u}(x) = \left(x^{\theta_1}, \dots, x^{\theta_p}\right),$$

while the other side is given by a vector of weights

$$\mathbf{v}(x) = \left(v_1(x), \dots, v_q(x)\right).$$

The resulting matrix of measures has density

$$\mathrm{d}\boldsymbol{\mu}(x) = \mathbf{v}(x)^{\mathsf{T}}\mathbf{u}(x)\,\mathrm{d}x,$$

so it is a rectangular $q \times p$ matrix of pointwise rank one, with pq scalar weights. It leads to rectangular moment matrices and rectangular Weyl functions. The mixed setting contains, as particular cases, several well-known families of multiple orthogonal systems, including Jacobi–Piñeiro and multiple Laguerre systems.

Mixed-type multiple orthogonal polynomials for arbitrary numbers p and q of weights were introduced by Daems and Kuijlaars [6]. Their Gaussian system, arising from nonintersecting Brownian motions with several initial and final points, provides the basic genuinely mixed rank-one example and leads to a $(p + q) \times (p + q)$ Riemann–Hilbert problem and a mixed Christoffel–Darboux formula. That construction does not provide explicit component formulas for general p and q. A non-Gaussian explicit mixed construction associated with modified Bessel functions was later obtained by Zhang [19], in the restricted 2×2 setting.

The Jacobi-like and Laguerre-like families considered here are inspired by the hypergeometric moment constructions of Wolfs [18]. Their mixed extension and the study of the corresponding bidiagonal factorizations were also motivated by the preceding work on mixed Piñeiro orthogonality by Branquinho, Díaz, Foulquié-Moreno, and Mañas [3]. In that power-vector setting, the mixed forms, their hypergeometric representations, and the complete bidiagonal factorization of the step-line recurrence matrix can all be made explicit. In the mixed Jacobi-like system, the right-hand power vector is replaced by the Jacobi-like hypergeometric weights arising from Wolfs's construction, while the boundary degeneration $\boldsymbol{a}, \boldsymbol{b} \to \boldsymbol{\beta}$ recovers exactly the mixed Piñeiro matrix. The Piñeiro system [3] therefore provides the power-vector boundary specialization of the Jacobi-like mixed construction.

The present construction gives two explicit non-Gaussian hypergeometric families of mixed-type systems on the positive real line for arbitrary p and q: Jacobi-like systems and Laguerre-like systems. Their components are written in terms of terminating generalized hypergeometric functions together with Gamma-quotient moment formulas. In particular, the number of terminating hypergeometric blocks in the components on the hypergeometric vector grows with the total degree. In the Jacobi-like case, this finite reconstruction can also be grouped into a finite sum of terminating Kampé de Fériet polynomials evaluated at $(x, 1)$ [15]. The analogous two-variable regrouping is not available uniformly in the mixed beta–Euler Laguerre-like family: for deep confluent indices, the polynomial part at infinity changes the finite-pole residues through a triangular system involving the Euler-derivative block. At the formal boundary $s = 0$, the formulas reduce to the Jacobi-like Kampé de Fériet formula, whereas $r = 0$ admits terminating hypergeometric component formulas obtained by block Newton interpolation in the Euler basis. Rodrigues-type representations for the complete mixed forms by means of compositions of differential operators acting on suitable seed weights yield explicit formulas for the mixed forms and their moments.

On the step-line, the mixed forms constitute two biorthogonal sequences. Multiplication by the independent variable is represented by a (p, q)-banded matrix. Every band entry is evaluated directly as a finite pairing of the explicit components. In the Jacobi-like case this gives finite Gamma–Pochhammer sums. In the Laguerre-like case it gives finite Gamma–Pochhammer sums after the finite confluent system determining the components has been solved. Only the step-line sequence is used; no off-step-line recurrences between neighbouring multi-indices are required.

Bidiagonal factorizations of these step-line recurrence matrices are also constructed. Such factorizations are fundamental in the theory of totally positive and oscillatory matrices and play a central role in spectral theory [4]. Under the normality and nonzero-pivot conditions stated below, Christoffel transformations and the Gauss–Borel factorization of the matrix of measures give lower bidiagonal factorizations for both mixed families. In the mixed Piñeiro specialization [3], where both sides of the matrix of measures are power vectors, the factorization becomes completely explicit, including all upper bidiagonal factors. Outside that specialization, the upper factors are characterized by finite Christoffel determinants and, equivalently, by cross-ratios of shifted moment minors. The general construction is verified in an exact rational $p = q = 2$ example.

The one-weight reductions recover the known factorizations for the Jacobi–Piñeiro and multiple Laguerre systems of the first kind.

The explicit forms are considered in the admissible near-diagonal range, under the parameter and nonresonance hypotheses stated below. There the required finite Mellin cancellations hold, the cited AT results supply normality, and the hypergeometric expressions terminate. The recurrence part uses only the single sequence of indices on the step-line.

The paper is organized as follows. Section 2 introduces mixed orthogonality, rectangular moments, and step-line recurrences. Sections devoted to the Jacobi-like and Laguerre-like systems develop the explicit hypergeometric constructions and their Rodrigues formulas. The step-line recurrences and the bidiagonal factorizations of the resulting band matrices, including the shifted-minor formulas, the explicit mixed Piñeiro specialization [3], and a rational $p = q = 2$ example, are treated next. The final section establishes the Jacobi–Laguerre confluence, derives the full $p \times q$ Gaussian mixed system of

Daems–Kuijlaars [6] from a rectangular array assembled out of moving Christoffel orbits of the scalar corners, and treats separately the gamma-convolution/jet limit of the canonical Laguerre-like vector.

1.1. **Hypergeometric series, Mellin transform and Meijer G-function notation.** The Pochhammer symbol is denoted by

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}, \qquad k \in \mathbb{N}_0.$$

For a parameter vector

$$\boldsymbol{a} = (a_1, \dots, a_r),$$

the shorthand notation

$$\Gamma(z\mathbf{1}_r + \boldsymbol{a}) := \prod_{\rho=1}^{r} \Gamma(z + a_\rho), \qquad (z\mathbf{1}_r + \boldsymbol{a})_k := \prod_{\rho=1}^{r} (z + a_\rho)_k$$

is used. More generally, if

$$\boldsymbol{c} = (c_1, \dots, c_r), \qquad \boldsymbol{\nu} = (\nu_1, \dots, \nu_r) \in \mathbb{N}_0^r,$$

the notation

$$(\boldsymbol{c})_{\boldsymbol{\nu}} := \prod_{\rho=1}^{r} (c_\rho)_{\nu_\rho}$$

is used. Products over an empty parameter vector are understood to be equal to one.

For parameter vectors

$$\boldsymbol{a} = (a_1, \dots, a_p), \qquad \boldsymbol{b} = (b_1, \dots, b_q),$$

the generalized hypergeometric function is

$${}_pF_q\left(\begin{matrix}\boldsymbol{a}\\ \boldsymbol{b}\end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(\boldsymbol{a})_k}{(\boldsymbol{b})_k} \frac{z^k}{k!},$$

where the series is understood in its domain of convergence, or by analytic continuation when appropriate. The hypergeometric series occurring as polynomial components below are terminating series.

The classical Kampé de Fériet double hypergeometric series is also used; see [15, Chapter 1]. If the parameter strings $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$ have respective lengths p, q, k, ℓ, m, n, then

$$F^{p:q;k}_{\ell:m;n}\left(\begin{matrix}\boldsymbol{a} : \boldsymbol{b}; \boldsymbol{c}\\ \boldsymbol{d} : \boldsymbol{e}; \boldsymbol{f}\end{matrix}\middle| x, y\right) := \sum_{r,\lambda \ge 0} \frac{(\boldsymbol{a})_{r+\lambda} (\boldsymbol{b})_r (\boldsymbol{c})_\lambda}{(\boldsymbol{d})_{r+\lambda} (\boldsymbol{e})_r (\boldsymbol{f})_\lambda} \frac{x^r}{r!} \frac{y^\lambda}{\lambda!}.$$

Thus the first parameter pair is coupled to $r + \lambda$, while the second and third pairs depend only on r and λ, respectively. Empty parameter strings are allowed.

The Mellin transform of a function f on $\mathbb{R}_+$ is denoted by

$$\mathcal{M}[f](s) := \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x, \qquad s \in \mathcal{S}_f,$$

where $\mathcal{S}_f$ denotes the fundamental strip of f, that is, the vertical strip of values of s for which the integral converges. If f is supported on $(0, 1)$, the same notation is used with the integral restricted to $(0, 1)$. The elementary shift rule

$$\mathcal{M}[x^\ell f](s) = \mathcal{M}[f](s + \ell), \qquad \ell \in \mathbb{N}_0,$$

is used whenever both sides are defined.

For functions on $\mathbb{R}_+$, the Mellin convolution is

$$(f * g)(x) := \int_0^\infty f(t) g\left(\frac{x}{t}\right) \frac{\mathrm{d}t}{t}.$$

Whenever the integrals are convergent, the Mellin transform turns Mellin convolution into multiplication:

$$\mathcal{M}[f * g](s) = \mathcal{M}[f](s)\mathcal{M}[g](s). \tag{1}$$

These identities will also be used in the standard meromorphic-continuation sense when both sides have meromorphic continuations.

The Meijer G-function is defined by the Mellin–Barnes integral

$$G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right):=\frac{1}{2\pi\mathrm{i}}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)}x^{-s}\,\mathrm{d}s, \tag{2}$$

where the contour L separates the poles of the gamma factors $\Gamma(b_j+s)$, $j\in\{1,\dots,m\}$, from those of $\Gamma(1-a_j-s)$, $j\in\{1,\dots,n\}$. With this convention,

$$\int_0^\infty x^{z-1}G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)\mathrm{d}x=\frac{\prod_{j=1}^{m}\Gamma(b_j+z)\prod_{j=1}^{n}\Gamma(1-a_j-z)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-z)\prod_{j=n+1}^{p}\Gamma(a_j+z)}, \tag{3}$$

in the corresponding fundamental strip. Standard contour and convergence conditions are given in the NIST Digital Library of Mathematical Functions [13].

2. General mixed orthogonality and the step-line

Before introducing the two mixed systems, we fix the notation used throughout the paper. In this section, x denotes the variable of the mixed system and $\mathrm{d}\nu(x)$ denotes the corresponding scalar measure on an interval $\Delta\subset\mathbb{R}$.

We consider a p-component power vector

$$\mathbf{u}(x)=\left(x^{\theta_1},\dots,x^{\theta_p}\right)$$

and a q-component weight vector

$$\mathbf{v}(x)=\left(v_1(x),\dots,v_q(x)\right).$$

In the Jacobi-like and Laguerre-like settings, the parameters $\boldsymbol{\theta}=(\theta_1,\dots,\theta_p)\in\mathbb{R}^p$ are denoted by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

We adopt throughout the $q\times p$ matrix-of-measures convention

$$\mathrm{d}\boldsymbol{\mu}(x):=\left[v_j(x)x^{\theta_i}\right]_{\substack{j\in\{1,\dots,q\}\\ i\in\{1,\dots,p\}}}\mathrm{d}\nu(x)=\begin{bmatrix}v_1(x)x^{\theta_1}&\cdots&v_1(x)x^{\theta_p}\\ \vdots&\ddots&\vdots\\ v_q(x)x^{\theta_1}&\cdots&v_q(x)x^{\theta_p}\end{bmatrix}\mathrm{d}\nu(x). \tag{4}$$

Thus the rows correspond to the weight vector and the columns to the power vector. Whenever convenient, we use the bilinear pairing

$$\langle f,g\rangle_\nu:=\int_\Delta f(x)g(x)\,\mathrm{d}\nu(x).$$

The convention in (4) is fixed from the outset because it is the one used below for the moment matrix, the Gauss–Borel factorization, and the left and right Christoffel transformations.

2.1. Mixed forms and biorthogonality.

For $N\in\mathbb{Z}$, we denote by $\mathbb{P}_N$ the vector space of complex polynomials of degree at most N, with the convention $\mathbb{P}_N=\{0\}$ for $N<0$. In this paper $\mathbb{N}:=\{1,2,\dots\}$ and $\mathbb{N}_0:=\{0,1,2,\dots\}$.

Let

$$\boldsymbol{n}=(n_1,\dots,n_p)\in\mathbb{N}_0^p,\qquad \boldsymbol{m}=(m_1,\dots,m_q)\in\mathbb{N}_0^q.$$

When $|\boldsymbol{n}|=|\boldsymbol{m}|+1$, a mixed form expanded in the power vector is an expression

$$\mathscr{A}_{\boldsymbol{n},\boldsymbol{m}}(z):=\sum_{i=1}^{p}A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}(z)z^{\theta_i},\qquad A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}\in\mathbb{P}_{n_i-1}, \tag{5}$$

satisfying

$$\left\langle z^k\mathscr{A}_{\boldsymbol{n},\boldsymbol{m}},v_j\right\rangle_\nu=0,\qquad j\in\{1,\dots,q\},\quad k\in\{0,\dots,m_j-1\}. \tag{6}$$

When $|\boldsymbol{m}|=|\boldsymbol{n}|+1$, a mixed form expanded in the weight vector is an expression

$$\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(z):=\sum_{j=1}^{q}B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}(z)v_j(z),\qquad B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}\in\mathbb{P}_{m_j-1}, \tag{7}$$

satisfying

$$\left\langle z^k \mathcal{B}_{\boldsymbol{n},\boldsymbol{m}}, z^{\theta_i} \right\rangle_\nu = 0, \qquad i \in \{1,\dots,p\}, \quad k \in \{0,\dots,n_i - 1\}. \tag{8}$$

Definition 2.1 (Normality and perfectness)**.** Following the terminology for multiple orthogonality used by Van Assche in Chapter 23 of Ismail's monograph [8], an index pair $(\boldsymbol{n},\boldsymbol{m})$ satisfying $|\boldsymbol{n}| = |\boldsymbol{m}| + 1$ is called *normal for the $\mathcal{A}$-problem* if the conditions (5)–(6) determine $\mathcal{A}_{\boldsymbol{n},\boldsymbol{m}}$ uniquely up to multiplication by a nonzero constant. Likewise, an index pair satisfying $|\boldsymbol{m}| = |\boldsymbol{n}| + 1$ is called *normal for the $\mathcal{B}$-problem* if the conditions (7)–(8) determine $\mathcal{B}_{\boldsymbol{n},\boldsymbol{m}}$ uniquely up to multiplication by a nonzero constant. The mixed system is called *perfect* when all index pairs in both problems are normal in this sense.

Remark 2.2. The preceding terminology is not Mahler's stronger convention, in which normality also incorporates simultaneous attainment of all componentwise degree bounds. Whenever the nonvanishing of a particular leading coefficient is needed below, it is stated separately.

Definition 2.3 (Normalization convention for mixed forms)**.** Assume that the index pair under consideration is normal for the corresponding mixed problem. Then the space of solutions of the orthogonality conditions is one-dimensional. By a *normalized* $\mathcal{A}$-form or a *normalized* $\mathcal{B}$-form we mean the unique representative of this one-dimensional solution space selected by the scalar normalization condition explicitly stated in the corresponding setting.

This normalization is not an L^2-normalization. It fixes the remaining scalar freedom in a normal mixed form. Depending on the context, this scalar may be fixed by an adjacent mixed pairing, by the leading coefficient specified on the step-line, by a Mellin moment identity, or by a contour/residue identity. Thus the polynomial spaces in (5) and (7) determine the allowed components, while the displayed normalization formula fixes the representative. Once this condition is imposed, the notation $\mathcal{A}_{\boldsymbol{n},\boldsymbol{m}}$ or $\mathcal{B}_{\boldsymbol{n},\boldsymbol{m}}$ always refers to the normalized form.

Whenever the index pairs involved are normal, the two forms satisfy the fundamental vanishing relations

$$\left\langle \mathcal{A}_{\boldsymbol{n},\boldsymbol{m}}, \mathcal{B}_{\boldsymbol{\nu},\boldsymbol{\mu}} \right\rangle_\nu = 0 \quad \text{if} \quad \begin{cases} n_i \le \nu_i, & i \in \{1,\dots,p\}, \\ \text{or} & \\ m_j \ge \mu_j, & j \in \{1,\dots,q\}. \end{cases}$$

The adjacent nonzero pairings may be fixed by a choice of normalization. On the step-line below this choice produces a biorthonormal pair of sequences.

We restrict both families of mixed forms to the step-line. This gives two biorthogonal sequences indexed by a single integer N, and multiplication by the independent variable is represented by a banded matrix. No recurrence between arbitrary neighbouring multi-indices is introduced.

2.2. **Step-line recurrences and the banded matrix.** The step-line chooses one multi-index of each total degree. Along this ordered sequence, multiplication by the variable is represented on the B- and A-sides by a band matrix and its transpose.

For $d \in \mathbb{N}$, the step-line in $\mathbb{N}_0^d$ consists of the multi-indices whose components differ by at most one and are written in nonincreasing order. More explicitly, if

$$N = dm + j, \qquad m \in \mathbb{N}_0, \quad j \in \{0,\dots,d-1\},$$

then the step-line multi-index of norm N is

$$\boldsymbol{\sigma}_d(N) = \boldsymbol{\sigma}_d(dm+j) := (\underbrace{m+1,\dots,m+1}_{j \text{ entries}}, \underbrace{m,\dots,m}_{d-j \text{ entries}}).$$

Thus, for example,

$$\boldsymbol{\sigma}_d(0) = (0,\dots,0), \qquad \boldsymbol{\sigma}_d(1) = (1,0,\dots,0), \qquad \boldsymbol{\sigma}_d(d) = (1,\dots,1).$$

For later use, we denote the quotient and the remainder in the Euclidean division of N by d by

$$N = d\rho_d(N) + \ell_d(N), \qquad \rho_d(N) = m, \quad \ell_d(N) = j.$$

It follows directly that

$$\boldsymbol{\sigma}_d(N+1) - \boldsymbol{\sigma}_d(N) = \boldsymbol{e}^{(d)}_{\ell_d(N)+1}. \tag{9}$$

Exactly one coordinate increases at each step: from N to $N+1$ it is coordinate $\ell_d(N)+1$. These coordinates occur in the cyclic order

$$1, 2, \dots, d, 1, 2, \dots .$$

This is the convention used below when the general rectangular chains are restricted to the step-line.

The step-line mixed forms are

$$\mathscr{B}_N := \mathscr{B}_{\sigma_p(N), \sigma_q(N+1)}, \qquad \mathscr{A}_N := \mathscr{A}_{\sigma_p(N+1), \sigma_q(N)}, \qquad N \in \mathbb{N}_0. \tag{10}$$

Normality along this sequence is essential. It makes the relevant leading minors of the moment matrix nonzero, so that the two normalized forms and the Gauss–Borel factors defining T exist at every step. For the Jacobi-like and Laguerre-like families, this hypothesis is supplied in the parameter ranges established later by Corollary 3.9 and Proposition 4.4, respectively. Assuming that the index pairs along the step-line are normal, normalize them by

$$B_N^{(\ell_q(N)+1)}\left[\left(\boldsymbol{\sigma}_q(N+1)\right)_{\ell_q(N)+1} - 1\right] = 1, \qquad \left\langle z^{(\sigma_q(N))_{\ell_q(N)+1}} \mathscr{A}_N, v_{\ell_q(N)+1}\right\rangle_v = 1. \tag{11}$$

The orthogonality conditions then give

$$\langle \mathscr{B}_N, \mathscr{A}_M \rangle_v = \delta_{N,M}, \qquad N, M \in \mathbb{N}_0. \tag{12}$$

Proposition 2.4 (Step-line recurrence matrix). *Under the preceding normality assumptions, the step-line forms satisfy*

$$z\mathscr{B}_N(z) = \sum_{j=1}^{q} b_N^j \mathscr{B}_{N+j}(z) + b_N^0 \mathscr{B}_N(z) + \sum_{i=1}^{p} b_N^{-i} \mathscr{B}_{N-i}(z), \tag{13}$$

$$z\mathscr{A}_N(z) = \sum_{j=1}^{q} b_{N-j}^j \mathscr{A}_{N-j}(z) + b_N^0 \mathscr{A}_N(z) + \sum_{i=1}^{p} b_{N+i}^{-i} \mathscr{A}_{N+i}(z), \tag{14}$$

where forms with negative indices are understood to be zero. The coefficients are

$$b_N^k = \langle z\mathscr{B}_N, \mathscr{A}_{N+k} \rangle_v, \qquad k \in \{-p, \dots, -1, 0, 1, \dots, q\}. \tag{15}$$

The normalization (11) *gives*

$$b_N^q = 1, \qquad N \in \mathbb{N}_0.$$

Proof. The moment matrix satisfies the intertwining identity $\Lambda_{[q]}\mathscr{M} = \mathscr{M}\Lambda_{[p]}^{\mathsf{T}}$, recorded explicitly in (17) below. Dressing this identity by the two Gauss–Borel factors gives the multiplication matrix

$$T = \mathscr{L}\Lambda_{[q]}\mathscr{L}^{-1} = \mathscr{U}^{-1}\Lambda_{[p]}^{\mathsf{T}}\mathscr{U}.$$

The representation $T = \mathscr{L}\Lambda_{[q]}\mathscr{L}^{-1}$ shows that no entry lies above the q-th superdiagonal, while $T = \mathscr{U}^{-1}\Lambda_{[p]}^{\mathsf{T}}\mathscr{U}$ shows that no entry lies below the p-th subdiagonal. Hence the N-th component of $T\boldsymbol{\mathscr{B}} = z\boldsymbol{\mathscr{B}}$ contains only $\mathscr{B}_{N-p}, \dots, \mathscr{B}_{N+q}$, which gives (13).

Pairing that recurrence with $\mathscr{A}_{N+k}$ and using (12) isolates its k-th coefficient and gives (15). On the A-side the matrix is T^{T}. Since $T_{M,N} = b_M^{N-M}$, the N-th component of $T^{\mathsf{T}}\boldsymbol{\mathscr{A}} = z\boldsymbol{\mathscr{A}}$ is exactly (14).

It remains to identify the coefficient of $\mathscr{B}_{N+q}$. Put $j = \ell_q(N)+1$. The first condition in (11) makes the coefficient of $z^{(\sigma_q(N+1))_j - 1}$ in $B_N^{(j)}$ equal to one. Therefore the j-th component of $z\mathscr{B}_N$ has coefficient one at degree $(\boldsymbol{\sigma}_q(N+1))_j$. Every $\mathscr{B}_M$ with $M < N+q$ that occurs in the recurrence has smaller degree in its j-th component, whereas

$$\boldsymbol{\sigma}_q(N+q+1) = \boldsymbol{\sigma}_q(N+1) + \mathbf{1}_q$$

and the same normalization makes the coefficient of that degree in the j-th component of $\mathscr{B}_{N+q}$ equal to one. Comparison of these coefficients gives $b_N^q = 1$. □

If

$$\boldsymbol{\mathscr{B}}(z) := \begin{bmatrix} \mathscr{B}_0(z) \\ \mathscr{B}_1(z) \\ \mathscr{B}_2(z) \\ \vdots \end{bmatrix}, \qquad \boldsymbol{\mathscr{A}}(z) := \begin{bmatrix} \mathscr{A}_0(z) \\ \mathscr{A}_1(z) \\ \mathscr{A}_2(z) \\ \vdots \end{bmatrix},$$

then (13)–(14) can be written as

$$T\boldsymbol{\mathscr{B}}(z) = z\boldsymbol{\mathscr{B}}(z), \qquad T^{\mathsf{T}}\boldsymbol{\mathscr{A}}(z) = z\boldsymbol{\mathscr{A}}(z), \tag{16}$$

where T is the (p, q)-banded recurrence matrix defined by

$$T_{N,N+k} = b_N^k, \qquad N \in \mathbb{N}_0, \quad k \in \{-p, \dots, q\},$$

whenever $N + k \geq 0$, and with $b_N^q = 1$. Moreover,

$$\left(T^\ell\right)_{N,M} = \left\langle z^\ell \mathscr{B}_N, \mathscr{A}_M \right\rangle_\nu, \qquad \ell, N, M \in \mathbb{N}_0.$$

For the Jacobi-like and Laguerre-like systems below, the pairings in (15) are evaluated directly by finite Gamma–Pochhammer sums. No recurrence outside the step-line is needed.

2.3. Moment matrix realization and Gauss–Borel factorization.

With the matrix of measures fixed in (4), introduce the block monomial vector and block shift matrix

$$X_{[d]}(z) := \begin{bmatrix} I_d \\ zI_d \\ z^2 I_d \\ \vdots \end{bmatrix}, \qquad \Lambda_{[d]} := \begin{bmatrix} 0_d & I_d & 0_d & \cdots \\ 0_d & 0_d & I_d & \ddots \\ 0_d & 0_d & 0_d & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

so that

$$\Lambda_{[d]} X_{[d]}(z) = z X_{[d]}(z).$$

The step-line moment matrix is

$$\mathscr{M} := \int X_{[q]}(z)\, \mathrm{d}\boldsymbol{\mu}(z)\, X_{[p]}^{\mathsf{T}}(z),$$

and satisfies the Hankel-type identity

$$\Lambda_{[q]}\mathscr{M} = \mathscr{M}\Lambda_{[p]}^{\mathsf{T}}. \tag{17}$$

Assume that the leading principal truncations of $\mathscr{M}$ are nonsingular. Its Gauss–Borel factorization is written as

$$\mathscr{M} = \mathscr{L}^{-1}\mathscr{U}^{-1}, \tag{18}$$

where $\mathscr{L}$ is lower unitriangular and $\mathscr{U}$ is upper triangular. The matrix polynomial systems

$$B(z) := \mathscr{L} X_{[q]}(z), \qquad A(z) := X_{[p]}^{\mathsf{T}}(z)\mathscr{U}$$

provide the component vectors of the step-line forms introduced in (10), after the normalizations in (11). In this realization, (17) gives

$$T = \mathscr{L}\Lambda_{[q]}\mathscr{L}^{-1} = \mathscr{U}^{-1}\Lambda_{[p]}^{\mathsf{T}}\mathscr{U},$$

which is the moment-matrix realization of the banded recurrence matrix in (16).

2.4. **Christoffel perturbations and the bidiagonal factorization scheme.** For the bidiagonal factorization, we first state the Christoffel–Gauss–Borel construction and then evaluate the elementary factors inside each explicit hypergeometric family. The lower factors come from the column-side Christoffel chain and the upper factors from the row-side Christoffel chain. Whenever the corresponding chain remains inside the same parametric family, the factor is evaluated from the transformed mixed forms; otherwise it is kept in the finite Christoffel tau-determinant form. This distinction is essential for mixed systems.

We recall the factorization developed for mixed step-line systems in [1, 5], in the form needed below. Let

$$\mathfrak{X}_{[d]}(z) := \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ z & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \qquad \mathfrak{X}_{[d]}(z)^d = zI_d.$$

Thus the elementary cycle is

$$(\theta_1, \dots, \theta_d) \longmapsto (\theta_2, \dots, \theta_d, \theta_1 + 1),$$

which is the convention compatible with the step-line order (9). The elementary left and right Christoffel perturbations are

$$\mathrm{d}\boldsymbol{\mu}_{\mathrm{L}}^{(k)} := \mathrm{d}\boldsymbol{\mu}\left(\mathfrak{X}_{[p]}^{k}\right)^{\mathsf{T}}, \qquad \mathrm{d}\boldsymbol{\mu}_{\mathrm{R}}^{(k)} := \mathfrak{X}_{[q]}^{k}\mathrm{d}\boldsymbol{\mu}.$$

Their moment matrices satisfy

$$\mathscr{M}_{\mathrm{L}}^{(k)} = \mathscr{M}\left(\Lambda^{k}\right)^{\mathsf{T}}, \qquad \mathscr{M}_{\mathrm{R}}^{(k)} = \Lambda^{k}\mathscr{M}, \qquad \Lambda := \Lambda_{[1]}.$$

Assume that the Gauss–Borel factorizations of these perturbed moment matrices exist for all shifts which occur below.

Proposition 2.5 (Step-line bidiagonal factorization scheme)**.** *Assume that the step-line mixed system is normal along the Christoffel chains used below and that all pivots in the construction are nonzero. Then the step-line multiplication matrix T admits the bidiagonal factorization*

$$T = L_1 \cdots L_p\, U_q \cdots U_1, \tag{19}$$

where $L_1, \dots, L_p$ are lower bidiagonal matrices with unit main diagonal, and $U_1, \dots, U_q$ are upper bidiagonal matrices with unit first superdiagonal.

For $k \in \{0, \dots, p\}$, let

$$\mathscr{A}_N^{\langle k\rangle}(z) = \sum_{i=1}^{k} A_N^{\langle k\rangle,(i)}(z) z^{\theta_i+1} + \sum_{i=k+1}^{p} A_N^{\langle k\rangle,(i)}(z) z^{\theta_i}$$

be the step-line A-form for the system obtained by multiplying the first k left powers by z, in the step-line biorthonormal normalization. For a polynomial P, write $\operatorname{LC}(P)$ for its leading coefficient. Then

$$(L_k)_{N+1,N} = \frac{\operatorname{LC}\left(A_N^{\langle k\rangle,(\ell_p(N)+1)}\right)}{\operatorname{LC}\left(A_{N+1}^{\langle k-1\rangle,(\ell_p(N+1)+1)}\right)}, \qquad k \in \{1, \dots, p\}. \tag{20}$$

For $b \in \{0, \dots, q\}$, set

$$v_j^{[b]}(z) = \begin{cases} z v_j(z), & j \in \{1, \dots, b\}, \\ v_j(z), & j \in \{b+1, \dots, q\}. \end{cases}$$

In particular, $v_j^{[0]} = v_j$. Let

$$\mathscr{B}_N(z) = \sum_{j=1}^{q} B_N^{(j)}(z) v_j(z)$$

be the untransformed step-line B-form in the same biorthonormal normalization. Define

$$\tau^B_{0,N} := 1, \qquad \tau^B_{b,N} := \det\left[B^{(\nu)}_{N+\alpha-1}(0)\right]^b_{\alpha,\nu=1}, \qquad b \in \{1,\dots,q\}. \tag{21}$$

Whenever these Christoffel tau-functions do not vanish, the upper factors are

$$(U_b)_{N,N} = -\frac{\tau^B_{b-1,N}\tau^B_{b,N+1}}{\tau^B_{b-1,N+1}\tau^B_{b,N}}, \qquad b \in \{1,\dots,q\}. \tag{22}$$

If the Christoffel chain associated with the upper bidiagonal factors is closed inside the same parametric family, let

$$\mathscr{A}^{[b]}_N(z) = \sum_{i=1}^{p} A^{[b],(i)}_N(z) z^{\theta_i}$$

be the step-line A-form for the system with right weights

$$v^{[b]}_1, \dots, v^{[b]}_q,$$

in the corresponding biorthonormal normalization. Then the same quantities may equivalently be written as quotients of leading coefficients:

$$(U_b)_{N,N} = \frac{\mathrm{LC}\left(A^{[b-1],(\ell_p(N)+1)}_N\right)}{\mathrm{LC}\left(A^{[b],(\ell_p(N)+1)}_N\right)}, \qquad b \in \{1,\dots,q\}. \tag{23}$$

The identity (19) is an entrywise identity of semi-infinite matrices. No convergence of infinite matrix products is involved: there are only $p+q$ bidiagonal factors, and each entry of their product is a finite sum. Equivalently, for any fixed set of rows and all columns meeting their band, the identity is obtained from a sufficiently large finite Christoffel truncation. This is the form of the general factorization theorem from [1, 5] used here; below we evaluate its factors for the two hypergeometric families.

The distinction in Proposition 2.5 is essential. The lower factors are evaluated from the left Christoffel chain. In the two mixed settings below this chain is closed because the p-component side is a power vector. Thus the left-transformed A-forms are obtained by the same formulas after a cyclic affine shift of the power exponents. The upper factors are different: outside a closed right chain they must be computed from the Christoffel tau-functions (21).

Define the cyclic affine shift

$$\mathscr{C}_d(\theta_1,\dots,\theta_d) := (\theta_2,\dots,\theta_d,\theta_1+1).$$

In the Jacobi-like and Laguerre-like settings, respectively, the left Christoffel chain preserves the family by

$$\boldsymbol{\alpha} \longmapsto \mathscr{C}^k_p(\boldsymbol{\alpha}), \qquad \boldsymbol{\beta} \longmapsto \mathscr{C}^k_p(\boldsymbol{\beta}).$$

Bilateral preservation occurs in the mixed Piñeiro specialization of [3] in the Jacobi-like setting, and formally at the $s=0$ boundary of the Laguerre-like setting; in those cases one may use either the closed coefficient formula (23) or the tau-formula (22).

3. Jacobi-like mixed-type multiple orthogonal systems

We first consider a rectangular mixed extension of the Jacobi-like hypergeometric moment systems introduced by Wolfs [18]. The power vector is

$$\mathbf{u}(x) = (x^{\alpha_1},\dots,x^{\alpha_p}),$$

whereas the weight vector is formed by Jacobi-like Mellin weights. When $q=1$, the construction reduces to the ordinary Jacobi–Piñeiro system with p weights. When $p=1$ and $\alpha_1=0$, it reduces to the ordinary Jacobi-like system of Wolfs [18]. The boundary degeneration $\boldsymbol{a},\boldsymbol{b}\to\boldsymbol{\beta}$ gives the mixed Piñeiro matrix of [3]. Finally, the case $q=2$ gives a Gauss hypergeometric specialization. After matching normalizations, the relevant two-dimensional space of weights is related to the Lima–Loureiro Gauss pair [11] by an invertible constant change of basis.

Let

$$\boldsymbol{a} = (a_1, \dots, a_q), \qquad \boldsymbol{b} = (b_1, \dots, b_q), \qquad -1 < a_j < b_j, \quad j \in \{1, \dots, q\}.$$

For $-1 < a < b$, we use the Mellin-normalized beta weight

$$\mathscr{B}_{a,b}(x) := \frac{x^a (1-x)^{b-a-1}}{\Gamma(b-a)} \mathbf{1}_{(0,1)}(x), \qquad \mathscr{M}[\mathscr{B}_{a,b}](s) = \frac{\Gamma(s+a)}{\Gamma(s+b)}.$$

For vector-valued parameter strings, scalar shifts, addition, subtraction, and products are understood componentwise; in particular,

$$\Gamma(s\mathbf{1}_q + \boldsymbol{a}) := \prod_{j=1}^{q} \Gamma(s + a_j), \qquad (\boldsymbol{a})_k := \prod_{j=1}^{q} (a_j)_k.$$

The vector obtained by removing the j-th component of $\boldsymbol{a}$ is denoted by $\boldsymbol{a}^{*j}$.

The finite nonresonance assumptions needed for the residue formulas are stated explicitly in each result.

The basic Jacobi-like weight is

$$w_0(x; \boldsymbol{a}, \boldsymbol{b}) := \mathscr{B}_{a_1,b_1} * \cdots * \mathscr{B}_{a_q,b_q}(x).$$

Since each factor in the Mellin convolution is supported on $(0, 1)$, the iterated convolution is also supported on $(0, 1)$. Moreover, by the Mellin transform of the beta density and by (1), it satisfies

$$\mathscr{M}[w_0(\cdot; \boldsymbol{a}, \boldsymbol{b})](s) = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b})}. \tag{24}$$

With the Meijer G-function convention fixed in (2)–(3), the upper parameters of the Meijer G-function appear in the denominator of the Mellin transform, whereas the lower parameters appear in the numerator. Therefore

$$w_0(x; \boldsymbol{a}, \boldsymbol{b}) = G^{q,0}_{q,q}\left(x \,\middle|\, \begin{matrix} \boldsymbol{b} \\ \boldsymbol{a} \end{matrix}\right), \qquad 0 < x < 1.$$

Following Wolfs [18], define

$$w_j(x; \boldsymbol{a}, \boldsymbol{b}) := w_0(x; \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{e}_j), \qquad j \in \{1, \dots, q\}.$$

Then

$$\mathscr{M}[w_j(\cdot; \boldsymbol{a}, \boldsymbol{b})](s) = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b} + \boldsymbol{e}_j)} = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b})} \frac{1}{s + b_j}. \tag{25}$$

Assume that

$$a_\rho - a_\sigma \notin \mathbb{Z}, \qquad \rho \neq \sigma.$$

With the Meijer G-normalization fixed in (2)–(3), Slater's expansion of the Meijer G-function near $x = 0$, obtained by summing the residues at the poles $s = -a_\rho - k$, gives, for $0 < x < 1$ and every $j \in \{1, \dots, q\}$,

$$w_j(x; \boldsymbol{a}, \boldsymbol{b}) = \sum_{\rho=1}^{q} C_{j,\rho}\, x^{a_\rho}\, {}_qF_{q-1}\left(\begin{matrix} (1 + a_\rho)\mathbf{1}_q - \boldsymbol{b} - \boldsymbol{e}_j \\ (1 + a_\rho)\mathbf{1}_{q-1} - \boldsymbol{a}^{*\rho} \end{matrix}; x\right),$$

where

$$C_{j,\rho} := \frac{\Gamma(\boldsymbol{a}^{*\rho} - a_\rho \mathbf{1}_{q-1})}{\Gamma(\boldsymbol{b} + \boldsymbol{e}_j - a_\rho \mathbf{1}_q)}.$$

Thus each Jacobi-like weight is, generically, a linear combination of Frobenius-type terms $x^{a_\rho}{}_qF_{q-1}(x)$ at the origin.

Remark 3.1 (The Jacobi–Piñeiro specialization: $q = 1$). If $q = 1$, write $\boldsymbol{a} = (a)$ and $\boldsymbol{b} = (b)$. Then

$$w_1(x; a, b) = w_0(x; a, b + 1) = \frac{x^a (1-x)^{b-a}}{\Gamma(b - a + 1)}.$$

Consequently, multiplication by arbitrary powers gives

$$x^{\alpha_i} w_1(x;a,b) = \frac{x^{\alpha_i+a}(1-x)^{b-a}}{\Gamma(b-a+1)}, \qquad i \in \{1,\dots,p\},$$

which, up to a common nonzero constant, are precisely the Jacobi–Piñeiro weights after setting

$$\widetilde{\alpha}_i := \alpha_i + a, \qquad \gamma := b-a.$$

This agrees with the Mellin-transform normalization for Jacobi–Piñeiro weights used, for instance, by Smet and Van Assche [14]. Although the sufficient hypotheses used above to construct the full positive Mellin-convolution family impose $-1 < a < b$, the specialized Jacobi–Piñeiro weights are integrable under the usual conditions $\widetilde{\alpha}_i > -1$ and $\gamma > -1$.

Remark 3.2 (Relation with the Gauss system of Lima and Loureiro). For $q = 2$, set

$$\boldsymbol{a} = (a-1, b-1), \qquad \boldsymbol{b} = (c-1, d-1), \qquad \delta := c+d-a-b.$$

Then

$$w_0(x;(a-1,b-1),(c-1,d-1)) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)} W_{\mathrm{LL}}(x;a,b;c,d),$$

where W_{LL} denotes the Gauss hypergeometric weight used by Lima and Loureiro [11]. We write

$$\mathcal{W}(x;a,b;c,d) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)} W_{\mathrm{LL}}(x;a,b;c,d)$$

for its Mellin-normalized version. Their second weight is $W_{\mathrm{LL}}(x;a,b+1;c+1,d)$. The shifted Jacobi-like weight

$$w_0(x;(a-1,b-1),(c,d-1))$$

corresponds, in Mellin normalization, to $\mathcal{W}(x;a,b;c+1,d)$, and the constant contiguous relation

$$\mathcal{W}(x;a,b;c+1,d) = \frac{1}{c-b}\mathcal{W}(x;a,b;c,d) - \frac{1}{c-b}\mathcal{W}(x;a,b+1;c+1,d)$$

shows that the two pairs of weights span the same two-dimensional space, provided $c-b \neq 0$. Thus the $q = 2$ Jacobi-like construction recovers the Lima–Loureiro Gauss system up to normalization and an invertible constant change of basis, in the corresponding triangular range of multi-indices.

Remark 3.3 (Piñeiro degeneration). This is the mixed Piñeiro system studied in [3]. If one takes the boundary limit

$$\boldsymbol{a}, \boldsymbol{b} \longrightarrow \boldsymbol{\beta},$$

then (25) gives

$$\mathcal{M}[w_j(\cdot;\boldsymbol{\beta},\boldsymbol{\beta})](s) = \frac{1}{s+\beta_j},$$

and therefore

$$w_j(x;\boldsymbol{\beta},\boldsymbol{\beta}) = x^{\beta_j}, \qquad j \in \{1,\dots,q\}. \tag{26}$$

Let

$$\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_p),$$

and assume that

$$\alpha_i + a_\rho > -1, \qquad i \in \{1,\dots,p\}, \quad \rho \in \{1,\dots,q\}.$$

This condition guarantees the absolute convergence of all pairings appearing below. Equivalently, if

$$a_{\min} := \min_{\rho \in \{1,\dots,q\}} a_\rho,$$

it is enough to require $\alpha_i > -1 - a_{\min}$ for every $i \in \{1,\dots,p\}$. Consider the two vectors

$$\mathbf{u}(x) = \left(x^{\alpha_1},\dots,x^{\alpha_p}\right), \qquad \mathbf{w}(x) = \big(w_1(x;\boldsymbol{a},\boldsymbol{b}),\dots,w_q(x;\boldsymbol{a},\boldsymbol{b})\big).$$

Following the convention (4), we define the $q \times p$ matrix of measures

$$\mathrm{d}\boldsymbol{\mu}^{\mathrm{J}}(x) := \big[w_j(x;\boldsymbol{a},\boldsymbol{b})x^{\alpha_i}\big]_{\substack{j\in\{1,\dots,q\}\\ i\in\{1,\dots,p\}}}\,\mathrm{d}x. \tag{27}$$

The preceding specializations pass directly to this matrix of measures. In the case $q=1$, Remark 3.1 gives, up to a common nonzero constant, the ordinary Jacobi–Piñeiro vector with parameters $\widetilde{\alpha}_i = \alpha_i + a$ and $\gamma = b - a$. If $p = 1$ and $\alpha_1 = 0$, the matrix reduces to the ordinary Jacobi-like vector studied by Wolfs. More generally, multiplication by x^{α_1} corresponds, at the level of Mellin transforms, to the simultaneous shift

$$(\boldsymbol{a},\boldsymbol{b}) \longmapsto (\boldsymbol{a} + \alpha_1\mathbf{1}_q, \boldsymbol{b} + \alpha_1\mathbf{1}_q).$$

Finally, in the boundary degeneration $\boldsymbol{a},\boldsymbol{b} \to \boldsymbol{\beta}$, (26) gives the mixed Piñeiro matrix of [3],

$$\mathrm{d}\boldsymbol{\mu}^{\mathrm{J}}(x) = \big[x^{\beta_j+\alpha_i}\big]_{\substack{j\in\{1,\dots,q\}\\ i\in\{1,\dots,p\}}}\,\mathrm{d}x.$$

3.1. **Leading moment determinants.** The leading moment determinants can be evaluated at every step, including a final incomplete cycle. For $N \geq 1$, let

$$m_j(N) := \left\lfloor \frac{N+q-j}{q} \right\rfloor, \qquad n_i(N) := \left\lfloor \frac{N+p-i}{p} \right\rfloor. \tag{28}$$

Thus $m_j(N)$ and $n_i(N)$ count the occurrences of the j-th row component and the i-th column component among the first N step-line indices, and

$$\sum_{j=1}^{q} m_j(N) = \sum_{i=1}^{p} n_i(N) = N.$$

For $0 \leq v < n_i(N)$, set

$$z_{v,i} := v + \alpha_i + 1,$$

and order the resulting N nodes by the scalar step-line index $pv + i - 1$. Denote this ordered list by $\mathfrak{X}_N$, and write

$$\mathrm{VdM}(\mathfrak{X}_N) := \prod_{\lambda<\mu}(z_\mu - z_\lambda).$$

The leading $N \times N$ moment block is

$$\mathscr{M}_N^{\mathrm{J}} := \left[\int_0^1 x^{u+v+\alpha_i} w_j(x;\boldsymbol{a},\boldsymbol{b})\,\mathrm{d}x\right]_{\substack{0\leq u<m_j(N),\ 1\leq j\leq q\\ 0\leq v<n_i(N),\ 1\leq i\leq p}},$$

where the rows and columns follow their scalar step-line order. Define

$$D_N(z) := \prod_{j=1}^{q}(z+b_j)_{m_j(N)},$$

$$\Delta_N^{\mathrm{J}} := \prod_{j=1}^{q}\left[\prod_{h=1}^{j-1}(b_j - b_h)^{m_j(N)}\prod_{\rho=1}^{q}\prod_{u=0}^{m_j(N)-1}(b_j + 1 - a_\rho)_u\right].$$

Proposition 3.4 (All leading Jacobi-like moment determinants)**.** *Assume that the Gamma factors and denominators below are finite and nonzero. For every $N \geq 1$,*

$$\det \mathscr{M}_N^{\mathrm{J}} = \mathrm{VdM}(\mathfrak{X}_N)\,\Delta_N^{\mathrm{J}}\prod_{z\in\mathfrak{X}_N}\frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_q + \boldsymbol{b})D_N(z)}. \tag{29}$$

Proof. Put

$$G(z) := \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_q + \boldsymbol{b})}$$

and, for every row index (u,j), define

$$F_{u,j}(z) := \frac{\prod_{\rho=1}^{q}(z+a_\rho)_u}{(z+b_j)_{u+1}\prod_{\substack{1\leq h\leq q\\ h\neq j}}(z+b_h)_u}.$$

Formula (25), with $s = u + v + \alpha_i + 1 = u + z_{v,i}$, gives

$$\int_0^1 x^{u+v+\alpha_i} w_j(x; \boldsymbol{a}, \boldsymbol{b})\, \mathrm{d}x = G(z_{v,i}) F_{u,j}(z_{v,i}).$$

The component counts in (28) differ by at most one, and the components with the larger count occur first. If $u < m_j(N)$, then $u + 1 \le m_j(N)$ and $u \le m_h(N)$ for $h \ne j$. Hence the denominator of $F_{u,j}$ divides D_N, and

$$P_{u,j}^{(N)}(z) \coloneqq D_N(z) F_{u,j}(z)$$

is a polynomial of degree at most $N - 1$. Let $\mathscr{C}_N$ be the coefficient matrix of these N polynomials in the ascending monomial basis. Evaluation at $\mathfrak{X}_N$ factors the moment block into a coefficient matrix, a Vandermonde matrix, and a diagonal matrix. Therefore,

$$\det \mathscr{M}_N^{\mathrm{J}} = \det \mathscr{C}_N \ \mathrm{VdM}(\mathfrak{X}_N) \prod_{z \in \mathfrak{X}_N} \frac{G(z)}{D_N(z)}. \tag{30}$$

It remains to calculate $\det \mathscr{C}_N$. Write

$$N = qm + j, \qquad m \ge 0, \quad j \in \{1, \dots, q\}.$$

The row added when passing from order $N - 1$ to order N is (m, j), and

$$D_N(z) = D_{N-1}(z)(z + b_j + m).$$

Here and below, $D_0(z) = 1$. Thus every old polynomial acquires the factor $z + b_j + m$, whereas the new polynomial is

$$P_{m,j}^{(N)}(z) = \prod_{\rho=1}^{q} (z + a_\rho)_m \prod_{h=1}^{j-1} (z + b_h + m).$$

At the root $-b_j - m$ of the new factor,

$$P_{m,j}^{(N)}(-b_j - m) = (-1)^{N-1} \prod_{h=1}^{j-1} (b_j - b_h) \prod_{\rho=1}^{q} (b_j + 1 - a_\rho)_m. \tag{31}$$

Reducing the last polynomial modulo $z + b_j + m$ and expanding the coefficient determinant gives

$$\det \mathscr{C}_N = (-1)^{N-1} P_{m,j}^{(N)}(-b_j - m) \det \mathscr{C}_{N-1}.$$

The two signs cancel by (31), and hence

$$\frac{\det \mathscr{C}_N}{\det \mathscr{C}_{N-1}} = \prod_{h=1}^{j-1} (b_j - b_h) \prod_{\rho=1}^{q} (b_j + 1 - a_\rho)_m.$$

Starting with $\det \mathscr{C}_0 = 1$ and iterating in the scalar step-line order yields $\det \mathscr{C}_N = \Delta_N^{\mathrm{J}}$. Substitution in (30) proves (29). □

Corollary 3.5 (Step-line normality)**.** *Under the hypotheses of Proposition 3.4, the block $\mathscr{M}_N^{\mathrm{J}}$ is nonsingular if and only if the nodes in $\mathfrak{X}_N$ are distinct and*

$$b_j \ne b_h \quad \text{for } 1 \le h < j \le q \text{ with } m_j(N) > 0,$$

and

$$b_j + r \ne a_\rho \quad \text{for } 1 \le \rho, j \le q, \quad 1 \le r < m_j(N).$$

Consequently, if these conditions hold for every N, all leading principal truncations of the step-line moment matrix are nonsingular and the Gauss–Borel factorization (18) *exists. This criterion concerns the chosen step-line. The AT result below additionally provides normality for the stated class of multi-indices, as well as the zero and interlacing properties of the mixed forms.*

Proposition 3.6 (Normality and zero structure of the mixed Jacobi-like system in the AT range)**.** *Assume that*

$$b_j - a_j \in \mathbb{N}_0, \qquad j \in \{1, \dots, q\}, \qquad b_i - b_j \notin \mathbb{Z}, \quad i \neq j,$$

and that

$$\alpha_i - \alpha_h \notin \mathbb{Z}, \qquad i, h \in \{1, \dots, p\}, \quad i \neq h.$$

Let $\mathcal{J}$ be a class of multi-indices for which the Jacobi-like vector

$$\big(w_1(\cdot; \boldsymbol{a}, \boldsymbol{b}), \dots, w_q(\cdot; \boldsymbol{a}, \boldsymbol{b})\big)$$

is an AT-system on $(0, 1)$*. Then the mixed Jacobi-like system associated with the matrix of measures* (27) *is normal for every pair of multi-indices for which the multi-index attached to the Jacobi-like vector belongs to $\mathcal{J}$.*

In the balance $|\boldsymbol{n}| = |\boldsymbol{m}| + 1$, the linear form $\mathcal{A}_{\boldsymbol{n},\boldsymbol{m}}$ has exactly $|\boldsymbol{m}|$ simple zeros in $(0, 1)$. In the dual balance $|\boldsymbol{m}| = |\boldsymbol{n}| + 1$, the linear form $\mathcal{B}_{\boldsymbol{n},\boldsymbol{m}}$ has exactly $|\boldsymbol{n}|$ simple zeros in $(0, 1)$. Moreover, the corresponding neighboring mixed forms interlace their zeros whenever the two neighboring pairs remain in the same AT range.

Proof. The nonresonance condition on $\boldsymbol{\alpha}$ implies that the functions

$$x^{\alpha_i+\ell}, \qquad i \in \{1, \dots, p\}, \quad \ell \in \{0, \dots, n_i - 1\},$$

have pairwise distinct exponents for every $\boldsymbol{n}$. Hence they form a Chebyshev system on $(0, 1)$. Equivalently, the power vector

$$\mathbf{u}(x) = (x^{\alpha_1}, \dots, x^{\alpha_p})$$

is an AT-system on $(0, 1)$ for every multi-index $\boldsymbol{n}$.

Let

$$\mathbf{w}(x) = \big(w_1(x; \boldsymbol{a}, \boldsymbol{b}), \dots, w_q(x; \boldsymbol{a}, \boldsymbol{b})\big)\,.$$

By assumption, $\mathbf{w}$ is an AT-system on $(0, 1)$ for every multi-index in the class $\mathcal{J}$. Moreover, the matrix of measures (27) has the form

$$\mathrm{d}S(x) = \mathbf{w}(x)^{\mathsf{T}}\mathbf{u}(x)\,\mathrm{d}x.$$

Therefore, for every pair of multi-indices whose Jacobi-like component belongs to $\mathcal{J}$, this is an AT matrix measure in the sense of Fidalgo Prieto, Medina Peralta and Mínguez [7, Definition 5].

By Theorem 2 of Fidalgo Prieto, Medina Peralta and Mínguez [7, Theorem 2], for a mixed type multiple orthogonal polynomial with respect to an AT matrix measure, all polynomial components have maximal degree and the solution is unique up to multiplication by a nonzero constant. This gives normality in the stated range. Their zero theorem for AT matrix measures [7, Theorem 1] gives exactly $|\boldsymbol{m}|$ simple zeros for $\mathcal{A}_{\boldsymbol{n},\boldsymbol{m}}$ in the balance $|\boldsymbol{n}| = |\boldsymbol{m}| + 1$, and exactly $|\boldsymbol{n}|$ simple zeros for $\mathcal{B}_{\boldsymbol{n},\boldsymbol{m}}$ after applying the same assertion to the transposed matrix in the dual balance $|\boldsymbol{m}| = |\boldsymbol{n}| + 1$. The interlacing assertion is the corresponding neighboring-pair result of Fidalgo Prieto, Medina Peralta and Mínguez [7, Theorem 5], applied to the matrix or to its transpose. □

The preceding result reduces normality of the mixed Jacobi-like problems to the AT property of the Jacobi-like vector. Under the hypotheses stated in the next lemma, this AT property holds in the near-diagonal range treated by Wolfs.

Definition 3.7 (Near-diagonal multi-index)**.** A multi-index $\boldsymbol{m} = (m_1, \dots, m_q) \in \mathbb{N}_0^q$ is said to be near the diagonal if

$$|m_i - m_j| \leq 1, \qquad i, j \in \{1, \dots, q\}.$$

Lemma 3.8 (Wolfs [18, Corollary 2.4])**.** *Assume that*

$$b_j - a_j \in \mathbb{N}_0, \qquad j \in \{1, \dots, q\},$$

and

$$b_i - b_j \notin \mathbb{Z}, \qquad i, j \in \{1, \dots, q\}, \quad i \neq j.$$

Then the Jacobi-like vector

$$\mathbf{w}(x) := \big(w_1(x; \boldsymbol{a}, \boldsymbol{b}), \dots, w_q(x; \boldsymbol{a}, \boldsymbol{b})\big)$$

is an AT-system on $(0,1)$ *for every near-diagonal multi-index* $\boldsymbol{m} \in \mathbb{N}_0^q$.

Corollary 3.9 (Near-diagonal mixed normality)**.** *Assume the hypotheses of Lemma 3.8. Assume also that*

$$\alpha_i - \alpha_h \notin \mathbb{Z}, \qquad i, h \in \{1, \dots, p\}, \quad i \neq h.$$

Let $\boldsymbol{m} \in \mathbb{N}_0^q$ *be near the diagonal. Then the mixed Jacobi-like orthogonality problems associated with the matrix of measures*

$$\mathrm{d}\boldsymbol{\mu}^{\mathrm{J}}(x) = \left[w_j(x; \boldsymbol{a}, \boldsymbol{b}) x^{\alpha_i}\right]_{\substack{j \in \{1,\dots,q\} \\ i \in \{1,\dots,p\}}} \mathrm{d}x$$

are normal in the following two balances. The mixed type I problem is normal when

$$|\boldsymbol{n}| = |\boldsymbol{m}| + 1,$$

and the mixed type II problem is normal when

$$|\boldsymbol{m}| = |\boldsymbol{n}| + 1.$$

In the first balance, $\mathscr{A}_{\boldsymbol{n},\boldsymbol{m}}$ *has exactly* $|\boldsymbol{m}|$ *simple zeros in* $(0,1)$*. In the second balance,* $\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}$ *has exactly* $|\boldsymbol{n}|$ *simple zeros in* $(0,1)$*. Neighboring mixed forms interlace whenever both neighboring pairs remain in the near-diagonal AT range.*

Proof. By Lemma 3.8, the Jacobi-like vector $\mathbf{w}$ is an AT-system on $(0,1)$ for the near-diagonal multi-index $\boldsymbol{m}$. The nonresonance condition on $\boldsymbol{\alpha}$ implies that the power vector $\mathbf{u}$ is an AT-system on $(0,1)$ for every multi-index $\boldsymbol{n}$. Therefore the matrix of measures is an AT matrix measure, and the assertion follows from the mixed-type normality criterion of Fidalgo Prieto, Medina Peralta and Mínguez [7, Theorem 2]. □

We obtain the two mixed forms associated with the rectangular matrix (27). The first form is expanded in the power vector $\mathbf{u} = (x^{\alpha_i})_{i=1}^p$, whereas the second form is expanded in the Jacobi-like vector $\mathbf{w} = (w_j)_{j=1}^q$.

In the statements below, we use the notation $\boldsymbol{b}^{*j}$ for the vector obtained from $\boldsymbol{b}$ by deleting its j-th component. Proposition 3.6 explains the structural source of normality whenever the two underlying systems are AT-systems. In the explicit formulas below, the near-diagonal hypothesis provides the finite cancellations needed in the contour argument for the form on the power vector and in the rational Mellin reconstruction of the form on the Jacobi-like vector. All displayed identities are identities of meromorphic functions of the parameters wherever both sides are defined.

Theorem 3.10 (Explicit mixed Jacobi-like forms)**.** *Let*

$$\boldsymbol{n} = (n_1, \dots, n_p) \in \mathbb{N}_0^p, \qquad \boldsymbol{m} = (m_1, \dots, m_q) \in \mathbb{N}_0^q.$$

(i) The form on the power vector. *Assume that* $|\boldsymbol{n}| = |\boldsymbol{m}| + 1$ *and that* $\boldsymbol{m}$ *is near the diagonal. Assume also the finite nonresonance condition that the shifted nodes*

$$\alpha_i + \ell, \qquad i \in \{1, \dots, p\} \text{ with } n_i \geq 1, \quad \ell \in \{0, \dots, n_i - 1\},$$

are pairwise distinct. Then there exists a mixed type I form

$$\mathscr{A}_{\boldsymbol{n},\boldsymbol{m}}(x) = \sum_{i=1}^p A_{\boldsymbol{n},\boldsymbol{m}}^{(i)}(x) x^{\alpha_i},$$

where

$$A_{\boldsymbol{n},\boldsymbol{m}}^{(i)} \in \mathbb{P}_{n_i - 1} \quad \text{if } n_i \geq 1, \qquad A_{\boldsymbol{n},\boldsymbol{m}}^{(i)} \equiv 0 \quad \text{if } n_i = 0.$$

This form satisfies

$$\int_0^1 x^r w_j(x; \boldsymbol{a}, \boldsymbol{b}) \mathscr{A}_{\boldsymbol{n},\boldsymbol{m}}(x)\, \mathrm{d}x = 0, \qquad r \in \{0, \dots, m_j - 1\}, \quad j \in \{1, \dots, q\} \text{ with } m_j \geq 1. \tag{32}$$

Define

$$D_{\boldsymbol{\alpha},\boldsymbol{n}}(t) \coloneqq ((t+1)\mathbf{1}_p - \boldsymbol{\alpha} - \boldsymbol{n})_{\boldsymbol{n}}, \qquad P_{\boldsymbol{b},\boldsymbol{m}}(t) \coloneqq ((t+1)\mathbf{1}_q + \boldsymbol{b})_{\boldsymbol{m}}.$$

The following contour formula fixes a particular representative. Whenever the corresponding problem is normal, it is the normalized $\mathscr{A}$-form in the sense of Definition 2.3:

$$\mathscr{A}_{\boldsymbol{n},\boldsymbol{m}}(x) = \frac{(-1)^{|\boldsymbol{n}|}}{2\pi \mathrm{i}} \int_{\Sigma} \frac{P_{\boldsymbol{b},\boldsymbol{m}}(t)}{D_{\alpha,\boldsymbol{n}}(t)} \frac{\Gamma(t\mathbf{1}_q + \boldsymbol{b} + \mathbf{1}_q)}{\Gamma(t\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)} x^t \, \mathrm{d}t, \tag{33}$$

where Σ encloses all nodes $\alpha_i, \dots, \alpha_i + n_i - 1$, with $n_i \geq 1$, and no other pole of the integrand. Its polynomial components, for $i \in \{1, \dots, p\}$ with $n_i \geq 1$, are

$$A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}(x) = \kappa_i^{\mathrm{A}} \; {}_{p+q}F_{p+q-1}\left(\begin{matrix} -n_i + 1,\ (\alpha_i + 1)\mathbf{1}_q + \boldsymbol{b} + \boldsymbol{m},\ (\alpha_i + 1)\mathbf{1}_{p-1} - \boldsymbol{\alpha}^{*i} - \boldsymbol{n}^{*i} \\ (\alpha_i + 1)\mathbf{1}_q + \boldsymbol{a},\ (\alpha_i + 1)\mathbf{1}_{p-1} - \boldsymbol{\alpha}^{*i} \end{matrix}; x\right), \tag{34}$$

where

$$\kappa_i^{\mathrm{A}} := (-1)^{|\boldsymbol{n}^{*i}|+1} \frac{\Gamma((\alpha_i + 1)\mathbf{1}_q + \boldsymbol{b})}{\Gamma((\alpha_i + 1)\mathbf{1}_q + \boldsymbol{a})} \frac{((\alpha_i + 1)\mathbf{1}_q + \boldsymbol{b})_{\boldsymbol{m}}}{(n_i - 1)!((\alpha_i + 1)\mathbf{1}_{p-1} - \boldsymbol{\alpha}^{*i} - \boldsymbol{n}^{*i})_{\boldsymbol{n}^{*i}}}. \tag{35}$$

(ii) The form on the Jacobi-like vector. *Assume that $|\boldsymbol{m}| = |\boldsymbol{n}| + 1$ and that $\boldsymbol{m}$ is near the diagonal. Assume also the finite nonresonance conditions that the nodes*

$$-b_j - \ell, \qquad j \in \{1, \dots, q\} \text{ with } m_j \geq 1, \quad \ell \in \{0, \dots, m_j - 1\},$$

are pairwise distinct and that no denominator appearing in the finite reconstruction formula below vanishes. Then there exists a mixed type II form

$$\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(x) = \sum_{j=1}^{q} B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}(x) w_j(x; \boldsymbol{a}, \boldsymbol{b}),$$

where

$$B^{(j)}_{\boldsymbol{n},\boldsymbol{m}} \in \mathbb{P}_{m_j-1} \quad \text{if } m_j \geq 1, \qquad B^{(j)}_{\boldsymbol{n},\boldsymbol{m}} \equiv 0 \quad \text{if } m_j = 0.$$

This form satisfies

$$\int_0^1 \mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(x) x^{\alpha_i + r} \, \mathrm{d}x = 0, \qquad r \in \{0, \dots, n_i - 1\}, \quad i \in \{1, \dots, p\} \text{ with } n_i \geq 1. \tag{36}$$

The following Mellin identity fixes a particular representative. Whenever the corresponding problem is normal, it is the normalized $\mathscr{B}$-form in the sense of Definition 2.3:

$$\mathscr{M}[\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}](s) = -\frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b} + \boldsymbol{m})} (\boldsymbol{\alpha} + (1 - s)\mathbf{1}_p)_{\boldsymbol{n}}. \tag{37}$$

If, in addition,

$$a_\rho - a_\sigma \notin \mathbb{Z}, \qquad \rho \neq \sigma,$$

then the simple-pole expansion of the complete form is

$$\begin{aligned} \mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(x) = -\sum_{j=1}^{q} & \frac{\Gamma(\boldsymbol{a}^{*j} - a_j \mathbf{1}_{q-1})(\boldsymbol{\alpha} + (a_j + 1)\mathbf{1}_p)_{\boldsymbol{n}}}{\Gamma(\boldsymbol{b} + \boldsymbol{m} - a_j \mathbf{1}_q)} x^{a_j} \\ & \times \; {}_{p+q}F_{p+q-1}\left(\begin{matrix} \boldsymbol{\alpha} + \boldsymbol{n} + (a_j + 1)\mathbf{1}_p,\ (a_j + 1)\mathbf{1}_q - \boldsymbol{b} - \boldsymbol{m} \\ \boldsymbol{\alpha} + (a_j + 1)\mathbf{1}_p,\ (a_j + 1)\mathbf{1}_{q-1} - \boldsymbol{a}^{*j} \end{matrix}; x\right). \end{aligned} \tag{38}$$

For every parameter choice covered above, the same complete form admits the Meijer G-representation

$$\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(x) = -G^{q,p}_{p+q,p+q}\left(x \,\middle|\, \begin{matrix} -\boldsymbol{\alpha} - \boldsymbol{n},\ \boldsymbol{b} + \boldsymbol{m} \\ \boldsymbol{a},\ -\boldsymbol{\alpha} \end{matrix}\right), \qquad 0 < x < 1. \tag{39}$$

At coalescent values of the a_ρ, (38) *is understood by continuity, or equivalently by evaluating the higher-order residues of* (39).

Moreover, for $j \in \{1, \dots, q\}$ with $m_j \geq 1$, the polynomial components are given by the finite hypergeometric expansion

$$B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}(x) = \sum_{\substack{J \in \{1, \dots, q\} \\ K \in \{0, \dots, m_J - 1\} \\ K - 1 + \delta_{j,J} \geq 0}} \mathscr{C}^{\mathscr{B}}_{j;J,K} \; {}_{q+1}F_q\left(\begin{matrix} 1,\ \boldsymbol{b} + (1 - b_J - K)\mathbf{1}_q - \boldsymbol{e}_j \\ \boldsymbol{a} + (1 - b_J - K)\mathbf{1}_q \end{matrix}; x\right), \tag{40}$$

where, for the indices appearing in (40),

$$\begin{aligned}\mathscr{C}^{\mathscr{B}}_{j;J,K} &:= (-1)^{K+1}\frac{(\boldsymbol{\alpha}+(b_J+K+1)\mathbf{1}_p)_{\boldsymbol{n}}}{K!(m_J-K-1)!(\boldsymbol{b}^{*J}-(b_J+K)\mathbf{1}_{q-1})_{\boldsymbol{m}^{*J}}}\\ &\quad\times\frac{(\boldsymbol{a}-b_j\mathbf{1}_q)_1}{(\boldsymbol{b}^{*j}-b_j\mathbf{1}_{q-1})_1}\frac{(\boldsymbol{b}^{*j}-(b_J+K)\mathbf{1}_{q-1})_1}{(\boldsymbol{a}-(b_J+K)\mathbf{1}_q)_1}.\end{aligned}\tag{41}$$

Each hypergeometric series appearing in (40) *is terminating, of degree* $K-1+\delta_{j,J}$.

Proof. Step 1: Construction of the form on the power vector. The only poles of the integrand in (33) enclosed by Σ are the simple poles of $D_{\alpha,\boldsymbol{n}}(t)^{-1}$. They are located at

$$t=\alpha_i+\ell,\qquad i\in\{1,\dots,p\}\text{ with }n_i\geq 1,\quad \ell\in\{0,\dots,n_i-1\}.$$

Evaluating the contour integral by residues gives

$$\mathscr{A}_{\boldsymbol{n},\boldsymbol{m}}(x)=\sum_{\substack{i=1\\ n_i\geq 1}}^{p}\sum_{\ell=0}^{n_i-1}c_{i,\ell}x^{\alpha_i+\ell},$$

where

$$c_{i,\ell}=(-1)^{|\boldsymbol{n}|}\operatorname*{Res}_{t=\alpha_i+\ell}\left[\frac{P_{\boldsymbol{b},\boldsymbol{m}}(t)}{D_{\alpha,\boldsymbol{n}}(t)}\frac{\Gamma(t\mathbf{1}_q+\boldsymbol{b}+\mathbf{1}_q)}{\Gamma(t\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)}\right].$$

Equivalently,

$$\mathscr{A}_{\boldsymbol{n},\boldsymbol{m}}(x)=\sum_{\substack{i=1\\ n_i\geq 1}}^{p}x^{\alpha_i}\sum_{\ell=0}^{n_i-1}c_{i,\ell}x^{\ell}.$$

Thus the coefficient multiplying x^{α_i} lies in $\mathbb{P}_{n_i-1}$ for every $i\in\{1,\dots,p\}$, with the convention $\mathbb{P}_{-1}=\{0\}$. Hence $A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}\in\mathbb{P}_{n_i-1}$ when $n_i\geq 1$, and $A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}\equiv 0$ when $n_i=0$.

Let $j\in\{1,\dots,q\}$ with $m_j\geq 1$, and let $r\in\{0,\dots,m_j-1\}$. By (25),

$$\int_0^1 x^r w_j(x;\boldsymbol{a},\boldsymbol{b})\mathscr{A}_{\boldsymbol{n},\boldsymbol{m}}(x)\,\mathrm{d}x=\frac{(-1)^{|\boldsymbol{n}|}}{2\pi\mathrm{i}}\int_\Sigma\frac{P_{\boldsymbol{b},\boldsymbol{m}}(t)}{D_{\alpha,\boldsymbol{n}}(t)}\frac{(t\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)_r}{(t\mathbf{1}_q+\boldsymbol{b}+\mathbf{1}_q)_r(t+b_j+r+1)}\,\mathrm{d}t.$$

Since $\boldsymbol{m}$ is near the diagonal, $r\leq m_h$ for every $h\in\{1,\dots,q\}$. Therefore $P_{\boldsymbol{b},\boldsymbol{m}}(t)$ cancels the factor $(t\mathbf{1}_q+\boldsymbol{b}+\mathbf{1}_q)_r$. Moreover, since $r+1\leq m_j$, it also cancels the remaining factor $t+b_j+r+1$. The resulting integrand is rational and is $\mathrm{O}(t^{-2})$ at infinity, because $|\boldsymbol{n}|=|\boldsymbol{m}|+1$. Therefore its residue at infinity is zero. Equivalently, the sum of its finite residues vanishes. Since the contour Σ encloses precisely the poles $t=\alpha_i+\ell$, this sum of finite residues is exactly the contour integral above. Hence the integral is zero, and (32) follows.

Step 2: Hypergeometric components on the power vector. We evaluate the residues in the contour representation and show that, for each $i\in\{1,\dots,p\}$ with $n_i\geq 1$, the component $A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}$ is a terminating generalized hypergeometric polynomial. Fix such an index i. The coefficient of $x^{\alpha_i+\ell}$ in (33) is

$$c_{i,\ell}=(-1)^{|\boldsymbol{n}|}\frac{P_{\boldsymbol{b},\boldsymbol{m}}(\alpha_i+\ell)}{D'_{\alpha,\boldsymbol{n}}(\alpha_i+\ell)}\frac{\Gamma((\alpha_i+\ell)\mathbf{1}_q+\boldsymbol{b}+\mathbf{1}_q)}{\Gamma((\alpha_i+\ell)\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)}.$$

For $\ell=0$,

$$D'_{\alpha,\boldsymbol{n}}(\alpha_i)=(-1)^{n_i-1}(n_i-1)!(\alpha_i\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}-\boldsymbol{n}^{*i}+\mathbf{1}_{p-1})_{\boldsymbol{n}^{*i}},$$

and therefore $c_{i,0}=\kappa^{\mathrm{A}}_i$. For $\ell\geq 0$, direct use of Pochhammer identities gives

$$\frac{c_{i,\ell}}{c_{i,0}}=\frac{(-n_i+1)_\ell(\alpha_i\mathbf{1}_q+\boldsymbol{b}+\boldsymbol{m}+\mathbf{1}_q)_\ell(\alpha_i\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}-\boldsymbol{n}^{*i}+\mathbf{1}_{p-1})_\ell}{\ell!((\alpha_i)\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)_\ell(\alpha_i\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}+\mathbf{1}_{p-1})_\ell}.$$

Thus the residue expansion for the i-th component can be written as

$$A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}(x)=c_{i,0}\sum_{\ell=0}^{n_i-1}\frac{(-n_i+1)_\ell(\alpha_i\mathbf{1}_q+\boldsymbol{b}+\boldsymbol{m}+\mathbf{1}_q)_\ell(\alpha_i\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}-\boldsymbol{n}^{*i}+\mathbf{1}_{p-1})_\ell}{((\alpha_i)\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)_\ell(\alpha_i\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}+\mathbf{1}_{p-1})_\ell}\frac{x^\ell}{\ell!}.$$

This is exactly the terminating generalized hypergeometric series in (34), with $c_{i,0} = \kappa_i^{\mathrm{A}}$. Hence (34) and (35) follow. In particular, for every $i \in \{1, \dots, p\}$ with $n_i \geq 1$, the component $A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}$ is a single terminating polynomial of type ${}_{p+q}F_{p+q-1}$.

Step 3: A finite reconstruction formula on the Jacobi-like vector. We prove the elementary Mellin reconstruction formula used in Step 4. The goal is to realize each simple rational factor

$$\frac{1}{s + b_J + K}, \qquad J \in \{1, \dots, q\}, \quad K \in \{0, \dots, m_J - 1\},$$

as the Mellin transform of a finite linear combination of the Jacobi-like weights with polynomial coefficients. For fixed $J \in \{1, \dots, q\}$ with $m_J \geq 1$, and $K \in \{0, \dots, m_J - 1\}$, we construct polynomials

$$\mathscr{R}_{j;J,K}(x) \in \mathbb{P}_{m_j - 1}, \qquad j \in \{1, \dots, q\},$$

such that the following elementary Mellin reconstruction identity holds:

$$\mathscr{M}\left[\sum_{j=1}^{q} \mathscr{R}_{j;J,K}(\cdot) w_j(\cdot; \boldsymbol{a}, \boldsymbol{b})\right](s) = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b})} \frac{1}{s + b_J + K}. \tag{42}$$

By linearity, any finite partial-fraction expansion whose poles are simple and belong to the set of nodes $-b_J - K$, $J \in \{1, \dots, q\}$, $K \in \{0, \dots, m_J - 1\}$, can then be reconstructed by taking the corresponding linear combination of these elementary polynomials.

By (25), the Mellin transform of $x^\ell w_j(x; \boldsymbol{a}, \boldsymbol{b})$ is

$$\mathscr{M}[x^\ell w_j(\cdot; \boldsymbol{a}, \boldsymbol{b})](s) = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b})} \frac{(s\mathbf{1}_q + \boldsymbol{a})_\ell}{(s\mathbf{1}_q + \boldsymbol{b})_\ell (s + b_j + \ell)}.$$

Thus, after factoring out the common quotient $\Gamma(s\mathbf{1}_q + \boldsymbol{a})/\Gamma(s\mathbf{1}_q + \boldsymbol{b})$, it remains to prove a rational identity in the variable s.

For fixed $J \in \{1, \dots, q\}$ with $m_J \geq 1$, and $K \in \{0, \dots, m_J - 1\}$, define

$$\mathscr{R}_{j;J,K}(x) := \frac{(\boldsymbol{a} - b_j \mathbf{1}_q)_1}{(\boldsymbol{b}^{*j} - b_j \mathbf{1}_{q-1})_1} \sum_{\ell=0}^{K-1+\delta_{j,J}} \frac{(\boldsymbol{b}^{*j} - (b_J + K)\mathbf{1}_{q-1})_{\ell+1} (b_j - b_J - K)_\ell}{(\boldsymbol{a} - (b_J + K)\mathbf{1}_q)_{\ell+1}} x^\ell, \tag{43}$$

where the convention is that a sum with upper limit -1 is empty. Since $\boldsymbol{m}$ is near the diagonal, $0 \leq K \leq m_J - 1$ implies

$$K - 1 + \delta_{j,J} \leq m_j - 1.$$

Hence $\mathscr{R}_{j;J,K} \in \mathbb{P}_{m_j - 1}$.

We claim that these polynomials satisfy (42). Using the Mellin transform of $x^\ell w_j$, this claim is equivalent to the rational identity

$$\sum_{j=1}^{q} \sum_{\ell=0}^{K-1+\delta_{j,J}} \tau_{j;J,K,\ell} \frac{(s\mathbf{1}_q + \boldsymbol{a})_\ell}{(s\mathbf{1}_q + \boldsymbol{b})_\ell (s + b_j + \ell)} = \frac{1}{s + b_J + K}, \tag{44}$$

where

$$\tau_{j;J,K,\ell} = \frac{(\boldsymbol{a} - b_j \mathbf{1}_q)_1}{(\boldsymbol{b}^{*j} - b_j \mathbf{1}_{q-1})_1} \frac{(\boldsymbol{b}^{*j} - (b_J + K)\mathbf{1}_{q-1})_{\ell+1} (b_j - b_J - K)_\ell}{(\boldsymbol{a} - (b_J + K)\mathbf{1}_q)_{\ell+1}}.$$

We prove (44) by decomposing the single fraction $1/(s + B)$ in K successive steps. Put $B := b_J + K$. The nonresonance assumptions imply that $B \neq b_j + \ell$, for $j \in \{1, \dots, q\}$ and $\ell \in \{0, \dots, K - 1\}$, and that all poles appearing below are simple. If $K = 0$, then the left-hand side of (44) contains only the term $j = J$, $\ell = 0$, and $\tau_{J;J,0,0} = 1$. Hence the identity is immediate. We assume from now on that $K \geq 1$.

The successive steps are arranged as follows. At step ℓ, with $\ell \in \{0, \dots, K - 1\}$, the part of $1/(s + B)$ not yet decomposed is written as the sum of the fractions with poles

$$s = -b_j - \ell, \qquad j \in \{1, \dots, q\},$$

and a new part to be decomposed at the next step. For $\ell \geq 0$, set

$$E_\ell(s) := \frac{(\boldsymbol{b} - B\mathbf{1}_q)_\ell}{(\boldsymbol{a} - B\mathbf{1}_q)_\ell} \frac{(s\mathbf{1}_q + \boldsymbol{a})_\ell}{(s\mathbf{1}_q + \boldsymbol{b})_\ell}.$$

Then $E_0(s) = 1$, so the procedure starts with

$$\frac{E_0(s)}{s+B} = \frac{1}{s+B}.$$

At step ℓ, the part not yet decomposed is

$$\frac{E_\ell(s)}{s+B},$$

and the next part to be decomposed is

$$\frac{E_{\ell+1}(s)}{s+B}.$$

Thus it is enough to prove the transition from step ℓ to step $\ell+1$. After the K-th step, the part not yet decomposed is $E_K(s)/(s+B)$.

For $\ell \in \{0, \dots, K-1\}$, set

$$\tau_{j,\ell} := \frac{(\boldsymbol{a} - b_j \mathbf{1}_q)_1}{(\boldsymbol{b}^{*j} - b_j \mathbf{1}_{q-1})_1} \frac{(\boldsymbol{b}^{*j} - B\mathbf{1}_{q-1})_{\ell+1}(b_j - B)_\ell}{(\boldsymbol{a} - B\mathbf{1}_q)_{\ell+1}}, \qquad j \in \{1, \dots, q\}.$$

Since $B = b_J + K$, we have $\tau_{j,\ell} = \tau_{j;J,K,\ell}$. We claim that

$$\frac{E_\ell(s)}{s+B} = \sum_{j=1}^{q} \tau_{j,\ell} \frac{(s\mathbf{1}_q + \boldsymbol{a})_\ell}{(s\mathbf{1}_q + \boldsymbol{b})_\ell (s + b_j + \ell)} + \frac{E_{\ell+1}(s)}{s+B}. \tag{45}$$

This is the transition identity at step ℓ.

To prove (45), divide both sides by $(s\mathbf{1}_q + \boldsymbol{a})_\ell/(s\mathbf{1}_q + \boldsymbol{b})_\ell$. Since

$$\frac{E_\ell(s)}{(s\mathbf{1}_q + \boldsymbol{a})_\ell/(s\mathbf{1}_q + \boldsymbol{b})_\ell} = \frac{(\boldsymbol{b} - B\mathbf{1}_q)_\ell}{(\boldsymbol{a} - B\mathbf{1}_q)_\ell},$$

and

$$\frac{E_{\ell+1}(s)}{(s\mathbf{1}_q + \boldsymbol{a})_\ell/(s\mathbf{1}_q + \boldsymbol{b})_\ell} = \frac{(\boldsymbol{b} - B\mathbf{1}_q)_{\ell+1}}{(\boldsymbol{a} - B\mathbf{1}_q)_{\ell+1}} \frac{(s\mathbf{1}_q + \boldsymbol{a} + \ell\mathbf{1}_q)_1}{(s\mathbf{1}_q + \boldsymbol{b} + \ell\mathbf{1}_q)_1},$$

the required identity is equivalent to

$$\frac{(\boldsymbol{b} - B\mathbf{1}_q)_\ell}{(\boldsymbol{a} - B\mathbf{1}_q)_\ell} \frac{1}{s+B} = \sum_{j=1}^{q} \frac{\tau_{j,\ell}}{s + b_j + \ell} + \frac{(\boldsymbol{b} - B\mathbf{1}_q)_{\ell+1}}{(\boldsymbol{a} - B\mathbf{1}_q)_{\ell+1}} \frac{(s\mathbf{1}_q + \boldsymbol{a} + \ell\mathbf{1}_q)_1}{(s\mathbf{1}_q + \boldsymbol{b} + \ell\mathbf{1}_q)_1} \frac{1}{s+B}. \tag{46}$$

We verify this identity by comparing principal parts.

At $s = -B$, the terms in the sum over j have no pole, because $B \neq b_j + \ell$. The residue of the last term on the right-hand side of (46) is

$$\begin{aligned} &\frac{(\boldsymbol{b} - B\mathbf{1}_q)_{\ell+1}}{(\boldsymbol{a} - B\mathbf{1}_q)_{\ell+1}} \frac{((-B)\mathbf{1}_q + \boldsymbol{a} + \ell\mathbf{1}_q)_1}{((-B)\mathbf{1}_q + \boldsymbol{b} + \ell\mathbf{1}_q)_1} \\ &\quad = \frac{(\boldsymbol{b} - B\mathbf{1}_q)_\ell}{(\boldsymbol{a} - B\mathbf{1}_q)_\ell} \frac{(\boldsymbol{b} - B\mathbf{1}_q + \ell\mathbf{1}_q)_1}{(\boldsymbol{a} - B\mathbf{1}_q + \ell\mathbf{1}_q)_1} \frac{(\boldsymbol{a} - B\mathbf{1}_q + \ell\mathbf{1}_q)_1}{(\boldsymbol{b} - B\mathbf{1}_q + \ell\mathbf{1}_q)_1} \\ &\quad = \frac{(\boldsymbol{b} - B\mathbf{1}_q)_\ell}{(\boldsymbol{a} - B\mathbf{1}_q)_\ell}. \end{aligned}$$

which is the residue of the left-hand side.

Now fix $j \in \{1, \dots, q\}$. At $s = -b_j - \ell$, the left-hand side of (46) has no pole, because $B \neq b_j + \ell$. On the right-hand side, only two terms may have a pole: the j-th simple fraction in the sum and the last term. The residue of the latter is

$$\frac{(\boldsymbol{b} - B\mathbf{1}_q)_{\ell+1}}{(\boldsymbol{a} - B\mathbf{1}_q)_{\ell+1}} \frac{1}{B - b_j - \ell} \operatorname*{Res}_{s=-b_j-\ell} \frac{(s\mathbf{1}_q + \boldsymbol{a} + \ell\mathbf{1}_q)_1}{(s\mathbf{1}_q + \boldsymbol{b} + \ell\mathbf{1}_q)_1} = \frac{(\boldsymbol{b} - B\mathbf{1}_q)_{\ell+1}}{(\boldsymbol{a} - B\mathbf{1}_q)_{\ell+1}} \frac{1}{B - b_j - \ell} \frac{(\boldsymbol{a} - b_j\mathbf{1}_q)_1}{(\boldsymbol{b}^{*j} - b_j\mathbf{1}_{q-1})_1}.$$

Using

$$(\boldsymbol{b} - B\mathbf{1}_q)_{\ell+1} = (\boldsymbol{b}^{*j} - B\mathbf{1}_{q-1})_{\ell+1}(b_j - B)_{\ell+1}, \qquad \frac{(b_j - B)_{\ell+1}}{B - b_j - \ell} = -(b_j - B)_\ell,$$

this residue becomes

$$-\frac{(\boldsymbol{a}-b_j\mathbf{1}_q)_1}{(\boldsymbol{b}^{*j}-b_j\mathbf{1}_{q-1})_1}\frac{(\boldsymbol{b}^{*j}-B\mathbf{1}_{q-1})_{\ell+1}(b_j-B)_\ell}{(\boldsymbol{a}-B\mathbf{1}_q)_{\ell+1}}=-\tau_{j,\ell}.$$

This cancels the residue of $\tau_{j,\ell}/(s+b_j+\ell)$. Thus both sides of (46) have the same principal part at every finite pole. Their difference is therefore a polynomial. Since both sides are $\mathrm{O}(s^{-1})$ as $s\to\infty$, that polynomial is zero. This proves (46), and hence (45).

We apply (45) successively for $\ell\in\{0,\dots,K-1\}$. When these K identities are added, the terms

$$\frac{E_{\ell+1}(s)}{s+B},\qquad \ell\in\{0,\dots,K-2\},$$

cancel with the same terms appearing on the left-hand side of the next identity. Since $E_0(s)=1$, the sum of the identities gives

$$\frac{1}{s+B}=\sum_{j=1}^{q}\sum_{\ell=0}^{K-1}\tau_{j;J,K,\ell}\frac{(s\mathbf{1}_q+\boldsymbol{a})_\ell}{(s\mathbf{1}_q+\boldsymbol{b})_\ell(s+b_j+\ell)}+\frac{E_K(s)}{s+B}. \tag{47}$$

It remains to identify the last term. By definition,

$$\frac{E_K(s)}{s+B}=\frac{(\boldsymbol{b}-B\mathbf{1}_q)_K}{(\boldsymbol{a}-B\mathbf{1}_q)_K}\frac{(s\mathbf{1}_q+\boldsymbol{a})_K}{(s\mathbf{1}_q+\boldsymbol{b})_K(s+B)}.$$

Since $B=b_J+K$, we claim that

$$\tau_{J;J,K,K}=\frac{(\boldsymbol{b}-B\mathbf{1}_q)_K}{(\boldsymbol{a}-B\mathbf{1}_q)_K}.$$

Indeed,

$$\begin{aligned}\tau_{J;J,K,K}&=\frac{(\boldsymbol{a}-b_J\mathbf{1}_q)_1}{(\boldsymbol{b}^{*J}-b_J\mathbf{1}_{q-1})_1}\frac{(\boldsymbol{b}^{*J}-B\mathbf{1}_{q-1})_{K+1}(b_J-B)_K}{(\boldsymbol{a}-B\mathbf{1}_q)_{K+1}}\\&=\frac{(\boldsymbol{a}-b_J\mathbf{1}_q)_1}{(\boldsymbol{b}^{*J}-b_J\mathbf{1}_{q-1})_1}\frac{(\boldsymbol{b}^{*J}-B\mathbf{1}_{q-1})_K(\boldsymbol{b}^{*J}-b_J\mathbf{1}_{q-1})_1(b_J-B)_K}{(\boldsymbol{a}-B\mathbf{1}_q)_K(\boldsymbol{a}-b_J\mathbf{1}_q)_1}\\&=\frac{(\boldsymbol{b}^{*J}-B\mathbf{1}_{q-1})_K(b_J-B)_K}{(\boldsymbol{a}-B\mathbf{1}_q)_K}=\frac{(\boldsymbol{b}-B\mathbf{1}_q)_K}{(\boldsymbol{a}-B\mathbf{1}_q)_K}.\end{aligned}$$

Therefore,

$$\frac{E_K(s)}{s+B}=\tau_{J;J,K,K}\frac{(s\mathbf{1}_q+\boldsymbol{a})_K}{(s\mathbf{1}_q+\boldsymbol{b})_K(s+b_J+K)}.$$

Replacing the last term in (47) by this expression gives (44). This proves (44), and therefore (42).

Step 4: Construction and components of the form on the Jacobi-like vector. We apply the elementary reconstruction formula of Step 3 to the rational function required by the form on the Jacobi-like vector. The orthogonality conditions will be imposed by forcing zeros of the Mellin transform at the nodes $s=\alpha_i+r+1$. Consider

$$R^{\mathscr{B}}_{\boldsymbol{n},\boldsymbol{m}}(s):=-\frac{(\boldsymbol{\alpha}+(1-s)\mathbf{1}_p)_{\boldsymbol{n}}}{(s\mathbf{1}_q+\boldsymbol{b})_{\boldsymbol{m}}}.$$

The denominator has simple poles at the nodes $s=-b_J-K$, with $J\in\{1,\dots,q\}$, $m_J\geq 1$, and $K\in\{0,\dots,m_J-1\}$. Hence

$$R^{\mathscr{B}}_{\boldsymbol{n},\boldsymbol{m}}(s)=\sum_{\substack{J=1\\ m_J\geq 1}}^{q}\sum_{K=0}^{m_J-1}\frac{\pi^{\mathscr{B}}_{J,K}}{s+b_J+K},$$

where

$$\pi^{\mathscr{B}}_{J,K}=\lim_{s\to -b_J-K}(s+b_J+K)R^{\mathscr{B}}_{\boldsymbol{n},\boldsymbol{m}}(s).$$

A direct evaluation gives

$$\pi^{\mathscr{B}}_{J,K}=(-1)^{K+1}\frac{(\boldsymbol{\alpha}+(b_J+K+1)\mathbf{1}_p)_{\boldsymbol{n}}}{K!(m_J-K-1)!(\boldsymbol{b}^{*J}-(b_J+K)\mathbf{1}_{q-1})_{\boldsymbol{m}^{*J}}}.$$

For $j \in \{1, \dots, q\}$, define

$$B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}(x) := \sum_{\substack{J=1\\ m_J \geq 1}}^{q} \sum_{K=0}^{m_J-1} \pi^{\mathscr{B}}_{J,K}\, \mathscr{R}_{j;J,K}(x),$$

where the elementary reconstructed polynomials $\mathscr{R}_{j;J,K}$ are those defined in (43). Since $\mathscr{R}_{j;J,K} \in \mathbb{P}_{m_j-1}$, we have $B^{(j)}_{\boldsymbol{n},\boldsymbol{m}} \in \mathbb{P}_{m_j-1}$. Now set

$$\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(x) := \sum_{j=1}^{q} B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}(x) w_j(x; \boldsymbol{a}, \boldsymbol{b}).$$

By (42) and linearity,

$$\mathscr{M}[\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}](s) = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b})} R^{\mathscr{B}}_{\boldsymbol{n},\boldsymbol{m}}(s),$$

that is,

$$\mathscr{M}[\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}](s) = -\frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b} + \boldsymbol{m})}(\boldsymbol{\alpha} + (1-s)\mathbf{1}_p)_{\boldsymbol{n}},$$

which is (37).

We also obtain the Meijer G-representation of the complete form from this Mellin transform. Using

$$(\alpha_i + 1 - s)_{n_i} = \frac{\Gamma(\alpha_i + n_i + 1 - s)}{\Gamma(\alpha_i + 1 - s)},$$

the Mellin transform (37) can be written as

$$\mathscr{M}[\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}](s) = -\frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})\Gamma(\boldsymbol{\alpha} + \boldsymbol{n} + (1-s)\mathbf{1}_p)}{\Gamma(s\mathbf{1}_q + \boldsymbol{b} + \boldsymbol{m})\Gamma(\boldsymbol{\alpha} + (1-s)\mathbf{1}_p)}. \tag{48}$$

Since the form $\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}$ is represented on $(0,1)$, this Mellin transform may be read as a Mellin transform on $\mathbb{R}_+$. With the Meijer G-function convention fixed in (2)–(3), (48) is the Mellin transform of

$$-G^{q,p}_{p+q,p+q}\left(x \,\middle|\, \begin{matrix} -\boldsymbol{\alpha} - \boldsymbol{n},\ \boldsymbol{b} + \boldsymbol{m} \\ \boldsymbol{a},\ -\boldsymbol{\alpha} \end{matrix}\right).$$

This proves (39).

We derive the displayed formula for the polynomial components. Substituting (43) into the definition of $B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}$, we get

$$B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}(x) = \frac{(\boldsymbol{a} - b_j\mathbf{1}_q)_1}{(\boldsymbol{b}^{*j} - b_j\mathbf{1}_{q-1})_1} \sum_{\substack{J=1\\ m_J \geq 1}}^{q} \sum_{K=0}^{m_J-1} \pi^{\mathscr{B}}_{J,K} \sum_{\ell=0}^{K-1+\delta_{j,J}} \frac{(\boldsymbol{b}^{*j} - (b_J + K)\mathbf{1}_{q-1})_{\ell+1}(b_j - b_J - K)_\ell}{(\boldsymbol{a} - (b_J + K)\mathbf{1}_q)_{\ell+1}} x^\ell.$$

If $K = 0$ and $j \neq J$, the inner sum is empty. Otherwise, extracting the $\ell = 0$ factor from the inner sum gives

$$\frac{(\boldsymbol{b}^{*j} - (b_J + K)\mathbf{1}_{q-1})_1}{(\boldsymbol{a} - (b_J + K)\mathbf{1}_q)_1} \sum_{\ell=0}^{K-1+\delta_{j,J}} \frac{(\boldsymbol{b}^{*j} + (1 - b_J - K)\mathbf{1}_{q-1})_\ell (b_j - b_J - K)_\ell}{(\boldsymbol{a} + (1 - b_J - K)\mathbf{1}_q)_\ell} x^\ell.$$

Since

$$(\boldsymbol{b}^{*j} - (b_J + K)\mathbf{1}_{q-1} + \mathbf{1}_{q-1})_\ell (b_j - b_J - K)_\ell = (\boldsymbol{b} + (1 - b_J - K)\mathbf{1}_q - \boldsymbol{e}_j)_\ell$$

and $(1)_\ell/\ell! = 1$, this finite sum is

$${}_{q+1}F_q\left(\begin{matrix} 1,\ \boldsymbol{b} + (1 - b_J - K)\mathbf{1}_q - \boldsymbol{e}_j \\ \boldsymbol{a} + (1 - b_J - K)\mathbf{1}_q \end{matrix}; x\right).$$

Combining this expression with the formula for $\pi^{\mathscr{B}}_{J,K}$ gives exactly (40) and (41). Notice that the case $K = 0$, $j \neq J$, does not contribute, because the corresponding reconstructed polynomial $\mathscr{R}_{j;J,0}$ is identically zero.

If $s = \alpha_i + r + 1$, with $n_i \geq 1$ and $0 \leq r \leq n_i - 1$, then

$$(\alpha_i + 1 - s)_{n_i} = (-r)_{n_i} = 0.$$

Thus (36) follows from (37). Finally, Mellin inversion applied to (37), or equivalently to (48), followed by residue evaluation at the poles

$$s = -a_j - k, \qquad j \in \{1, \dots, q\}, \quad k \in \mathbb{N}_0,$$

gives (38). Under the additional nonresonance condition on the a_ρ, these poles are simple, and collecting the residues with fixed j gives exactly the j-th hypergeometric term in (38). Thus (38) is the residue expansion of the Meijer G-function in (39). When poles coalesce, the Meijer G-identity remains valid and the same expansion is obtained by continuity, equivalently from the corresponding higher-order residues. □

The finite expression in (40) is directly computable. Its terms can also be grouped according to the pole family J, and each group is then a terminating Kampé de Fériet polynomial. This reformulation is not needed for the orthogonality proof, but it gives a compact hypergeometric form of each component. The diagonal family $J = j$ and the families $J \neq j$ must be kept separate, since their first arguments are different.

Corollary 3.11 (Kampé de Fériet form of the B-components)**.** *Under the assumptions of Theorem 3.10*(ii)*, let $j \in \{1, \dots, q\}$. If $m_j = 0$, then $B^{(j)}_{\boldsymbol{n},\boldsymbol{m}} \equiv 0$. If $m_j \geq 1$, then*

$$B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}(x) = \mathcal{K}^{\mathcal{B}}_{j,j}\mathcal{F}_j(x) + \sum_{\substack{J=1 \\ J \neq j,\ m_J \geq 2}}^{q} \mathcal{K}^{\mathcal{B}}_{j,J}\mathcal{F}_{j,J}(x), \tag{49}$$

where

$$\mathcal{K}^{\mathcal{B}}_{j,j} := -\frac{(\boldsymbol{\alpha} + (b_j + 1)\mathbf{1}_p)_{\boldsymbol{n}}}{(m_j - 1)!(\boldsymbol{b}^{*j} - b_j\mathbf{1}_{q-1})_{\boldsymbol{m}^{*j}}},$$

and

$$\mathcal{F}_j(x) := F^{p+q:1;q}_{p+q:0;q-1}\left[\begin{matrix} 1 - m_j,\ \boldsymbol{\alpha} + (b_j + 1)\mathbf{1}_p + \boldsymbol{n},\ b_j\mathbf{1}_{q-1} - \boldsymbol{b}^{*j} + \mathbf{1}_{q-1} - \boldsymbol{m}^{*j} : 1;\ b_j\mathbf{1}_q - \boldsymbol{a} \\ \boldsymbol{\alpha} + (b_j + 1)\mathbf{1}_p,\ b_j\mathbf{1}_q - \boldsymbol{a} + \mathbf{1}_q : \text{—};\ b_j\mathbf{1}_{q-1} - \boldsymbol{b}^{*j} \end{matrix}\middle| x, 1\right].$$

For $J \neq j$ with $m_J \geq 2$,

$$\mathcal{K}^{\mathcal{B}}_{j,J} := \frac{1}{(m_J - 2)!}\frac{(\boldsymbol{\alpha} + (b_J + 2)\mathbf{1}_p)_{\boldsymbol{n}}}{(\boldsymbol{b}^{*J} - (b_J + 1)\mathbf{1}_{q-1})_{\boldsymbol{m}^{*J}}}\frac{(\boldsymbol{a} - b_j\mathbf{1}_q)_1}{(\boldsymbol{b}^{*j} - b_j\mathbf{1}_{q-1})_1}\frac{(\boldsymbol{b}^{*j} - (b_J + 1)\mathbf{1}_{q-1})_1}{(\boldsymbol{a} - (b_J + 1)\mathbf{1}_q)_1},$$

while

$$\mathcal{F}_{j,J}(x) := F^{p+q:1;q}_{p+q:0;q-1}\left[\begin{matrix} 2 - m_J,\ \boldsymbol{\alpha} + (b_J + 2)\mathbf{1}_p + \boldsymbol{n},\ b_J\mathbf{1}_{q-1} - \boldsymbol{b}^{*J} + 2\mathbf{1}_{q-1} - \boldsymbol{m}^{*J} : 1;\ b_J\mathbf{1}_q - \boldsymbol{a} + \mathbf{1}_q \\ \boldsymbol{\alpha} + (b_J + 2)\mathbf{1}_p,\ b_J\mathbf{1}_q - \boldsymbol{a} + 2\mathbf{1}_q : \text{—};\ b_J\mathbf{1}_{q-1} - \boldsymbol{b}^{*J} + \mathbf{1}_{q-1} + \boldsymbol{e}^{*J}_j \end{matrix}\middle| x, 1\right].$$

Here $\boldsymbol{e}^{*J}_j$ is obtained from the j-th coordinate vector $\boldsymbol{e}_j \in \mathbb{R}^q$ by deleting its J-th component. The parameter $1 - m_j$ restricts $\mathcal{F}_j$ to $u + \lambda \leq m_j - 1$, whereas $2 - m_J$ restricts $\mathcal{F}_{j,J}$ to $u + \lambda \leq m_J - 2$.

Proof. Fix j and expand every terminating ${}_{q+1}F_q$ in (40). We treat the pole family $J = j$ first. Write

$$K = u + \lambda, \qquad u, \lambda \in \mathbb{N}_0, \qquad u + \lambda \leq m_j - 1.$$

Here u is the summation index of the hypergeometric polynomial and $\lambda = K - u$. Thus this change of indices includes the term $K = 0$ at $u = \lambda = 0$. Put $s = u + \lambda$. From (41),

$$\frac{\mathcal{C}^{\mathcal{B}}_{j;j,s}}{\mathcal{C}^{\mathcal{B}}_{j;j,0}} = \frac{(1 - m_j)_s}{s!}\frac{(\boldsymbol{\alpha} + (b_j + 1)\mathbf{1}_p + \boldsymbol{n})_s}{(\boldsymbol{\alpha} + (b_j + 1)\mathbf{1}_p)_s}\frac{(b_j\mathbf{1}_{q-1} - \boldsymbol{b}^{*j} + \mathbf{1}_{q-1} - \boldsymbol{m}^{*j})_s}{(b_j\mathbf{1}_{q-1} - \boldsymbol{b}^{*j} + \mathbf{1}_{q-1})_s}. \tag{50}$$

The j-th upper parameter of the hypergeometric polynomial is $-s$, and therefore

$$(-s)_u = (-1)^u\frac{s!}{\lambda!}.$$

Using, componentwise,

$$(a + s)_n = (a)_n\frac{(a + n)_s}{(a)_s}, \qquad (a - s)_n = (a)_n\frac{(1 - a)_s}{(1 - a - n)_s},$$

in (50) and in the expanded hypergeometric term, the summand indexed by (u,λ) becomes

$$\mathcal{K}^{\mathcal{B}}_{j,j}\frac{(1-m_j)_{u+\lambda}(\boldsymbol{\alpha}+(b_j+1)\mathbf{1}_p+\boldsymbol{n})_{u+\lambda}}{(\boldsymbol{\alpha}+(b_j+1)\mathbf{1}_p)_{u+\lambda}} \times\frac{(b_j\mathbf{1}_{q-1}-\boldsymbol{b}^{*j}+\mathbf{1}_{q-1}-\boldsymbol{m}^{*j})_{u+\lambda}}{(b_j\mathbf{1}_q-\boldsymbol{a}+\mathbf{1}_q)_{u+\lambda}}\frac{(1)_u(b_j\mathbf{1}_q-\boldsymbol{a})_\lambda}{(b_j\mathbf{1}_{q-1}-\boldsymbol{b}^{*j})_\lambda}\frac{x^u}{u!}\frac{1}{\lambda!}. \tag{51}$$

To see the cancellations explicitly, the j-th entry of $b_j\mathbf{1}_q-\boldsymbol{b}+\boldsymbol{e}_j$ is 1, and its remaining entries form $b_j\mathbf{1}_{q-1}-\boldsymbol{b}^{*j}$. These entries cancel the corresponding scalar and $(q-1)$-strings produced by the two shift identities. The common string $b_j\mathbf{1}_q-\boldsymbol{a}$ cancels in the coupled block and remains in the λ-dependent numerator. The signs from the reflected factors cancel in this diagonal case. Consequently, (51) is exactly the (u,λ)-term of $\mathcal{K}^{\mathcal{B}}_{j,j}\mathcal{F}_j(x)$. Its first argument is x.

Now let $J\neq j$. Write

$$K=1+u+\lambda,\qquad u,\lambda\in\mathbb{N}_0,\qquad u+\lambda\le m_J-2.$$

This family is therefore absent when $m_J<2$. At $u=\lambda=0$, the coefficient is $\mathcal{K}^{\mathcal{B}}_{j,J}$. With $s=u+\lambda$, direct simplification of (41) gives

$$\frac{\mathcal{C}^{\mathcal{B}}_{j;J,1+s}}{\mathcal{C}^{\mathcal{B}}_{j;J,1}}=\frac{(2-m_J)_s}{(2)_s}\frac{(\boldsymbol{\alpha}+(b_J+2)\mathbf{1}_p+\boldsymbol{n})_s}{(\boldsymbol{\alpha}+(b_J+2)\mathbf{1}_p)_s}\frac{(b_J\mathbf{1}_{q-1}-\boldsymbol{b}^{*J}+2\mathbf{1}_{q-1}-\boldsymbol{m}^{*J})_s}{(b_J\mathbf{1}_{q-1}-\boldsymbol{b}^{*J}+2\mathbf{1}_{q-1})_s}.$$

The J-th upper parameter is now $1-K=-s$, so again

$$(-s)_u=(-1)^u\frac{s!}{\lambda!}.$$

Applying the same two shift identities as above, now about the base point $K=1$, transforms the summand into

$$\mathcal{K}^{\mathcal{B}}_{j,J}\frac{(2-m_J)_{u+\lambda}(\boldsymbol{\alpha}+(b_J+2)\mathbf{1}_p+\boldsymbol{n})_{u+\lambda}}{(\boldsymbol{\alpha}+(b_J+2)\mathbf{1}_p)_{u+\lambda}} \times\frac{(b_J\mathbf{1}_{q-1}-\boldsymbol{b}^{*J}+2\mathbf{1}_{q-1}-\boldsymbol{m}^{*J})_{u+\lambda}}{(b_J\mathbf{1}_q-\boldsymbol{a}+2\mathbf{1}_q)_{u+\lambda}}\frac{(1)_u(b_J\mathbf{1}_q-\boldsymbol{a}+\mathbf{1}_q)_\lambda}{(b_J\mathbf{1}_{q-1}-\boldsymbol{b}^{*J}+\mathbf{1}_{q-1}+\boldsymbol{e}^{*J}_j)_\lambda}\frac{x^u}{u!}\frac{1}{\lambda!}. \tag{52}$$

Indeed, the J-th entry of $b_J\mathbf{1}_q-\boldsymbol{b}+\mathbf{1}_q+\boldsymbol{e}_j$ is 1. Deleting it leaves the λ-dependent denominator string displayed above. The remaining common strings cancel between numerator and denominator, and the reflected signs cancel as well. Hence (52) is the (u,λ)-term of $\mathcal{K}^{\mathcal{B}}_{j,J}\mathcal{F}_{j,J}(x)$. Summing the diagonal family and all nonempty off-diagonal families gives (49). □

Remark 3.12 (Beyond the near-diagonal range). The near-diagonal assumption in Theorem 3.10 is a sufficient condition for the finite cancellations needed in the Mellin construction. Under the parameter hypotheses of Lemma 3.8, it also supplies the AT property of the underlying Jacobi-like vector. It is not, however, intrinsic to the residue argument itself. Outside the near-diagonal range, the same construction can still be carried out whenever suitable finite cancellation conditions hold for the extra Mellin factors.

That extension is not considered here. We use the near-diagonal formulation, for which cancellation is automatic and, under the hypotheses of Corollary 3.9, normality follows from the cited AT result; the formulas then have a terminating hypergeometric form. The off-step-line finite-cancellation theory beyond the near-diagonal range lies outside the scope of this paper.

Remark 3.13 (The ordinary Wolfs specialization: $p=1$). If

$$p=1,\qquad \alpha_1=0,$$

then the rectangular matrix reduces to the ordinary Jacobi-like vector

$$\big(w_1(x;\boldsymbol{a},\boldsymbol{b}),\dots,w_q(x;\boldsymbol{a},\boldsymbol{b})\big)\,\mathrm{d}x$$

studied by Wolfs [18]. In this specialization, the form $\mathcal{A}_{\boldsymbol{n},\boldsymbol{m}}$ in part (i) is the ordinary Jacobi-like type II polynomial. Since there is only one power component, formula (34) becomes a single terminating polynomial of type ${}_{q+1}F_q$. On the other hand, the form $\mathcal{B}_{\boldsymbol{n},\boldsymbol{m}}$ in part (ii) is the ordinary Jacobi-like type I form. Formula (38) writes the complete type I form as a sum of q generalized hypergeometric functions of type ${}_{q+1}F_q$, whereas each of its polynomial components remains, generically, a finite linear combination of terminating ${}_{q+1}F_q$ polynomials through (40). Thus the finite hypergeometric combination occurring for the components of $\mathscr{B}$ is already present in the ordinary Jacobi-like theory and is not an artefact of the rectangular mixed extension.

Remark 3.14 (Meijer G-representation of the complete form). The sum of generalized hypergeometric functions in (38) can be compressed at the level of the complete form. Indeed, using

$$(\alpha_i+1-s)_{n_i}=\frac{\Gamma(\alpha_i+n_i+1-s)}{\Gamma(\alpha_i+1-s)},$$

formula (37) becomes

$$\mathscr{M}[\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}](s)=-\frac{\Gamma(s\mathbf{1}_q+\boldsymbol{a})\Gamma(\boldsymbol{\alpha}+\boldsymbol{n}+(1-s)\mathbf{1}_p)}{\Gamma(s\mathbf{1}_q+\boldsymbol{b}+\boldsymbol{m})\Gamma(\boldsymbol{\alpha}+(1-s)\mathbf{1}_p)}.$$

Since the form $\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}$ is represented on $(0,1)$, this Mellin transform may be read as a Mellin transform on $\mathbb{R}_+$. Consequently, with the Meijer G-function convention fixed in (2)–(3),

$$\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(x)=-G^{q,p}_{p+q,p+q}\left(x\,\middle|\,\begin{matrix}-\boldsymbol{\alpha}-\boldsymbol{n},\ \boldsymbol{b}+\boldsymbol{m}\\ \boldsymbol{a},\ -\boldsymbol{\alpha}\end{matrix}\right),\qquad 0<x<1.$$

Under the corresponding nonresonance assumptions, the hypergeometric sum in (38) is the residue expansion of this single Meijer G-function. Formula (39) concerns the complete mixed form $\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}$; it does not, in general, replace each individual polynomial component $B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}$ by one Meijer G-function with comparably simple parameters. In the Piñeiro degeneration of [3], the simplification of the components arises instead from re-expansion in the degenerate power basis, as explained below.

Corollary 3.15 (Degeneration to the mixed Piñeiro family)**.** *Take the boundary degeneration $\boldsymbol{a},\boldsymbol{b}\to\boldsymbol{\beta}$ in the Mellin identities above. Assume that $\alpha_i+\beta_j>-1$ for all i,j, and that the denominators in the two limiting component formulas below do not vanish. Then* (27) *reduces to the mixed Piñeiro matrix of [3],*

$$\mathrm{d}\boldsymbol{\mu}(x)=\left[x^{\beta_j+\alpha_i}\right]_{\substack{j\in\{1,\dots,q\}\\ i\in\{1,\dots,p\}}}\mathrm{d}x. \tag{53}$$

With the normalizations fixed in Theorem 3.10, the nonzero polynomial components become

$$A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}(x)=-\frac{((\alpha_i+1)\mathbf{1}_q+\boldsymbol{\beta})_{\boldsymbol{m}}}{(n_i-1)!(\boldsymbol{\alpha}^{*i}-\alpha_i\mathbf{1}_{p-1})_{\boldsymbol{n}^{*i}}}\times{}_{p+q}F_{p+q-1}\left(\begin{matrix}-n_i+1,\ (\alpha_i+1)\mathbf{1}_q+\boldsymbol{\beta}+\boldsymbol{m},\ (\alpha_i+1)\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}-\boldsymbol{n}^{*i}\\ (\alpha_i+1)\mathbf{1}_q+\boldsymbol{\beta},\ (\alpha_i+1)\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}\end{matrix};x\right), \tag{54}$$

for $i\in\{1,\dots,p\}$ with $n_i\geq 1$, while $A^{(i)}_{\boldsymbol{n},\boldsymbol{m}}\equiv 0$ if $n_i=0$. Similarly,

$$B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}(x)=-\frac{(\boldsymbol{\alpha}+(\beta_j+1)\mathbf{1}_p)_{\boldsymbol{n}}}{(m_j-1)!(\boldsymbol{\beta}^{*j}-\beta_j\mathbf{1}_{q-1})_{\boldsymbol{m}^{*j}}}\times{}_{p+q}F_{p+q-1}\left(\begin{matrix}-m_j+1,\ \boldsymbol{\alpha}+\boldsymbol{n}+(\beta_j+1)\mathbf{1}_p,\ (\beta_j+1)\mathbf{1}_{q-1}-\boldsymbol{\beta}^{*j}-\boldsymbol{m}^{*j}\\ \boldsymbol{\alpha}+(\beta_j+1)\mathbf{1}_p,\ (\beta_j+1)\mathbf{1}_{q-1}-\boldsymbol{\beta}^{*j}\end{matrix};x\right), \tag{55}$$

for $j\in\{1,\dots,q\}$ with $m_j\geq 1$, while $B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}\equiv 0$ if $m_j=0$.

Proof. Equation (26) gives the degeneration of the weights to the mixed Piñeiro system studied in [3]. In part (i) of Theorem 3.10, the factor $(-1)^{|\boldsymbol{n}|}$ in (33) compensates the residue sign $(-1)^{|\boldsymbol{n}|-1}$, and hence the formula reduces exactly to (54).

For the type II components, the specialization must be taken at the level of the Mellin transform. Before degeneration, formula (40) expresses each component $B^{(j)}_{\boldsymbol{n},\boldsymbol{m}}$ as a finite linear combination of terminating polynomials of type ${}_{q+1}F_q$. This is a representation in the Jacobi-like basis and it is not the convenient formula to specialize term by term in the boundary limit $\boldsymbol{a},\boldsymbol{b}\to\boldsymbol{\beta}$: in this limit the Jacobi-like weights degenerate to pure powers and the relevant basis changes.

Instead, specialize the Mellin transform (37) itself. Since $\boldsymbol{a}, \boldsymbol{b} \to \boldsymbol{\beta}$, it becomes

$$\int_0^1 x^{s-1} \mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(x)\,\mathrm{d}x = -\frac{(\boldsymbol{\alpha} + (1-s)\mathbf{1}_p)_{\boldsymbol{n}}}{(s\mathbf{1}_q + \boldsymbol{\beta})_{\boldsymbol{m}}}. \tag{56}$$

Moreover, $w_j(x; \boldsymbol{\beta}, \boldsymbol{\beta}) = x^{\beta_j}$, so we may write

$$\mathscr{B}_{\boldsymbol{n},\boldsymbol{m}}(x) = \sum_{\substack{j=1\\ m_j \geq 1}}^{q} \left(\sum_{\ell=0}^{m_j-1} c_{j,\ell} x^{\ell} \right) x^{\beta_j}.$$

The coefficients $c_{j,\ell}$ are precisely the coefficients obtained by expanding the rational function in (56) at its poles $s = -\beta_j - \ell$. For fixed j, their quotient with the constant coefficient is

$$\frac{c_{j,\ell}}{c_{j,0}} = \frac{(-m_j+1)_\ell}{\ell!} \prod_{i=1}^{p} \frac{(\alpha_i + \beta_j + n_i + 1)_\ell}{(\alpha_i + \beta_j + 1)_\ell} \prod_{\substack{k=1\\ k \neq j}}^{q} \frac{(\beta_j - \beta_k - m_k + 1)_\ell}{(\beta_j - \beta_k + 1)_\ell}.$$

Thus all terms belonging to the degenerate power weight x^{β_j} collect into one terminating hypergeometric series of type ${}_{p+q}F_{p+q-1}$. With the normalization fixed in part (ii) of Theorem 3.10, this gives exactly (55). Hence the apparent simplification from a finite combination of ${}_{q+1}F_q$ polynomials to a single ${}_{p+q}F_{p+q-1}$ polynomial is a consequence of degeneration and re-expansion in the power basis, rather than a termwise collapse of (40). □

Remark 3.16 (Bilateral Christoffel preservation in the Piñeiro specialization). The degeneration in Corollary 3.15, which underlies the mixed Piñeiro system studied in [3], differs from the nonspecialized Jacobi-like case. In fact, both sides of (53) are power vectors. Hence the left and right Christoffel chains preserve the same Piñeiro family through the cyclic affine shifts

$$\boldsymbol{\alpha} \longmapsto \mathscr{C}_p^k(\boldsymbol{\alpha}), \qquad \boldsymbol{\beta} \longmapsto \mathscr{C}_q^k(\boldsymbol{\beta}),$$

respectively. Thus both the lower factors $L_1, \dots, L_p$ and the upper factors $U_1, \dots, U_q$ in (19) are obtained from leading coefficients of forms that remain in the Piñeiro family. Away from this degeneration, only the left chain is automatically preserved inside the Jacobi-like parametric class.

4. Laguerre-like mixed-type multiple orthogonal systems

In this section the power vector has length p, whereas the Laguerre-like vector has length

$$q = r + s.$$

The first r Laguerre-like weights are beta-shifted weights and the last s are Mellin-Euler derivative weights. Thus the mixed matrix below is a $q \times p$ matrix of measures. In parallel with the Jacobi-like notation, the form expanded in the Laguerre-like vector is denoted by $\mathscr{B}$, whereas the dual form expanded in the power vector is denoted by $\mathscr{A}$.

The notation in this section is adapted to the Laguerre-like construction. The multi-index on the Laguerre-like vector will be denoted by $\boldsymbol{\lambda} = (\boldsymbol{n}, \boldsymbol{\eta}) \in \mathbb{N}_0^q$, whereas $\boldsymbol{m} \in \mathbb{N}_0^p$ denotes the multi-index on the power vector. Thus the order of the two multi-indices is not meant to duplicate the Jacobi-like notation, but to keep the Laguerre-like side visible in the formulas below.

Let

$$q = r + s, \qquad r \geq 0, \qquad s \geq 1,$$

and let

$$\boldsymbol{a} = (a_1, \dots, a_q) \in (-1, \infty)^q, \qquad \boldsymbol{b} = (b_1, \dots, b_r) \in (-1, \infty)^r.$$

When $r > 0$, assume

$$a_h < b_h, \qquad h \in \{1, \dots, r\}.$$

We use the beta density $\mathscr{B}_{a,b}$ introduced in the Jacobi-like section. For $a > -1$, define the gamma weight

$$\mathscr{G}_a(x) := x^a \mathrm{e}^{-x} \mathbf{1}_{(0,\infty)}(x), \qquad \mathscr{M}[\mathscr{G}_a](z) = \Gamma(z + a).$$

The Laguerre-like base weight is the Mellin convolution

$$w_0(x;\boldsymbol{a},\boldsymbol{b}) := \mathscr{B}_{a_1,b_1} * \cdots * \mathscr{B}_{a_r,b_r} * \mathscr{G}_{a_{r+1}} * \cdots * \mathscr{G}_{a_q}(x), \tag{57}$$

with the convention that the beta block is absent when $r = 0$. Hence

$$\mathscr{M}[w_0(\cdot;\boldsymbol{a},\boldsymbol{b})](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b})}. \tag{58}$$

Equivalently, with the Mellin–Barnes convention fixed in (2)–(3),

$$w_0(x;\boldsymbol{a},\boldsymbol{b}) = G^{q,0}_{r,q}\left(x \,\middle|\, \begin{matrix}\boldsymbol{b}\\ \boldsymbol{a}\end{matrix}\right), \qquad x > 0.$$

Following Wolfs's Laguerre-like construction, define the shifted beta weights

$$w_h(x;\boldsymbol{a},\boldsymbol{b}) := w_0(x;\boldsymbol{a},\boldsymbol{b}+\boldsymbol{e}_h), \qquad h \in \{1,\dots,r\}. \tag{59}$$

For the derivative block we use the Mellin-Euler operator

$$\mathscr{D} := -x\frac{\mathrm{d}}{\mathrm{d}x},$$

so that, after integration by parts,

$$\mathscr{M}[\mathscr{D}f](z) = z\mathscr{M}[f](z).$$

Set

$$v_\ell(x;\boldsymbol{a},\boldsymbol{b}) := \mathscr{D}^{\ell-1}w_0(x;\boldsymbol{a},\boldsymbol{b}), \qquad \ell \in \{1,\dots,s\}. \tag{60}$$

Then

$$\mathscr{M}[w_h(\cdot;\boldsymbol{a},\boldsymbol{b})](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{e}_h)}, \qquad h \in \{1,\dots,r\}, \tag{61}$$

$$\mathscr{M}[v_\ell(\cdot;\boldsymbol{a},\boldsymbol{b})](z) = z^{\ell-1}\frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b})}, \qquad \ell \in \{1,\dots,s\}. \tag{62}$$

We collect these $q = r + s$ functions in the Laguerre-like vector

$$\mathbf{U}^{\mathrm{L}}(x) := (w_1,\dots,w_r,v_1,\dots,v_s) = \left(U_1^{\mathrm{L}},\dots,U_q^{\mathrm{L}}\right).$$

Let the independent power vector be

$$\mathbf{V}(x) := \left(x^{\beta_1},\dots,x^{\beta_p}\right).$$

Assume that

$$\beta_i + a_\rho > -1, \qquad i \in \{1,\dots,p\}, \quad \rho \in \{1,\dots,q\}.$$

Under this condition all pairings appearing in the mixed orthogonality relations below are absolutely convergent. Equivalently, if

$$a_{\min} := \min_{\rho\in\{1,\dots,q\}} a_\rho,$$

it is enough to require $\beta_i > -1 - a_{\min}$ for every $i \in \{1,\dots,p\}$. The rectangular mixed matrix of measures is

$$\mathrm{d}\boldsymbol{\mu}^{\mathrm{L}}(x) := \left[U_\rho^{\mathrm{L}}(x)x^{\beta_i}\right]_{\substack{\rho\in\{1,\dots,q\}\\ i\in\{1,\dots,p\}}} \mathrm{d}x. \tag{63}$$

Definition 4.1 (Laguerre admissibility)**.** Let

$$\boldsymbol{\lambda} = (\boldsymbol{n},\boldsymbol{\eta}) = (n_1,\dots,n_r,\eta_1,\dots,\eta_s) \in \mathbb{N}_0^r \times \mathbb{N}_0^s.$$

Following the index range used by Wolfs for Laguerre-like weights, we say that $\boldsymbol{\lambda}$ is *Laguerre admissible* if the full multi-index $\boldsymbol{\lambda}$ is near the diagonal and the derivative multi-index $\boldsymbol{\eta}$ lies on the step-line in $\mathbb{N}_0^s$.

Proposition 4.2 (Combinatorial criterion for Laguerre admissibility)**.** *Let*

$$\boldsymbol{\lambda} = (\boldsymbol{n}, \boldsymbol{\eta}) = (n_1, \dots, n_r, \eta_1, \dots, \eta_s) \in \mathbb{N}_0^r \times \mathbb{N}_0^s.$$

Then $\boldsymbol{\lambda}$ is Laguerre admissible if and only if the following four explicit inequalities hold:

$$n_h - 1 \le n_u, \qquad h \in \{1, \dots, r\} \text{ with } n_h \ge 1, \quad u \in \{1, \dots, r\}, \tag{64a}$$

$$\eta_\ell - 1 \le n_u, \qquad \ell \in \{1, \dots, s\} \text{ with } \eta_\ell \ge 1, \quad u \in \{1, \dots, r\}, \tag{64b}$$

$$s(n_h - 1) \le |\boldsymbol{\eta}|, \qquad h \in \{1, \dots, r\} \text{ with } n_h \ge 1, \tag{64c}$$

$$s(\eta_\ell - 1) + \ell \le |\boldsymbol{\eta}|, \qquad \ell \in \{1, \dots, s\} \text{ with } \eta_\ell \ge 1. \tag{64d}$$

When $r = 0$, the conditions involving $\boldsymbol{n}$ are absent. The criterion is purely combinatorial: it rewrites the condition that $\boldsymbol{\lambda}$ is near the diagonal and that $\boldsymbol{\eta}$ is on the step-line without referring to those two definitions separately.

Proof. If $r = 0$, the assertion reduces to the standard characterization of the step-line condition for $\boldsymbol{\eta}$, namely (64d). Thus we assume $r > 0$ in the rest of the proof.

Write

$$|\boldsymbol{\eta}| = sQ + R, \qquad Q \in \mathbb{N}_0, \quad R \in \{0, \dots, s-1\}.$$

With our step-line convention,

$$\eta_\ell = \begin{cases} Q + 1, & \ell \in \{1, \dots, R\}, \\ Q, & \ell \in \{R + 1, \dots, s\}. \end{cases}$$

This is equivalent to

$$s(\eta_\ell - 1) + \ell \le |\boldsymbol{\eta}|, \qquad \ell \in \{1, \dots, s\} \text{ with } \eta_\ell \ge 1.$$

Indeed, these inequalities imply

$$\eta_\ell \le Q + 1 \quad \text{for } \ell \in \{1, \dots, R\}, \qquad \eta_\ell \le Q \quad \text{for } \ell \in \{R + 1, \dots, s\},$$

and, since the sum of the η_ℓ's is $sQ + R$, all these upper bounds must be attained.

Assume first that $\boldsymbol{\lambda}$ is Laguerre admissible. Then $\boldsymbol{\lambda}$ is near the diagonal, so the first two inequalities in (64) follow immediately. The fourth inequality is the step-line condition just described. Since the minimum component of $\boldsymbol{\eta}$ is Q, near diagonality gives $n_h \le Q + 1$. Hence

$$s(n_h - 1) \le sQ \le |\boldsymbol{\eta}|,$$

which is the third inequality.

Conversely, assume (64). The first inequality says that the components of $\boldsymbol{n}$ differ from each other by at most one. The second says that no component of $\boldsymbol{\eta}$ exceeds any component of $\boldsymbol{n}$ by more than one. The fourth says that $\boldsymbol{\eta}$ lies on the step-line, and therefore $\min_\ell \eta_\ell = Q$. The third inequality gives $n_h \le Q + 1$, so no component of $\boldsymbol{n}$ exceeds any component of $\boldsymbol{\eta}$ by more than one. Hence all components of $\boldsymbol{\lambda} = (\boldsymbol{n}, \boldsymbol{\eta})$ differ by at most one, and $\boldsymbol{\lambda}$ is near the diagonal. Thus $\boldsymbol{\lambda}$ is Laguerre admissible. □

Remark 4.3. The preceding proposition is an admissibility criterion, not a normality statement. The role of the four inequalities will be made explicit in Theorem 4.5. Namely, (64a) and (64b) ensure that the relevant gamma quotients are polynomials, whereas (64c) and (64d) give the required membership in the polynomial spaces $\mathbb{P}_{m_i - 1}$, $\mathbb{P}_{n_h - 1}$, and $\mathbb{P}_{\eta_\ell - 1}$ in the contour argument. Thus the admissible range is used here only to keep the residue formulas inside the prescribed mixed polynomial spaces; normality itself is supplied separately by Proposition 4.4.

Proposition 4.4 (Normality in the Laguerre admissible range)**.** *Let*

$$\boldsymbol{\lambda} = (\boldsymbol{n}, \boldsymbol{\eta}) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$$

be Laguerre admissible. Assume that

$$b_h - a_h \in \mathbb{N}_0 \cup (|\boldsymbol{n}| + |\boldsymbol{\eta}| - 1, \infty), \qquad h \in \{1, \dots, r\},$$

and that

$$\beta_i - \beta_j \notin \mathbb{Z}, \qquad i, j \in \{1, \dots, p\}, \quad i \ne j.$$

Then the mixed Laguerre-like system associated with (63) *is normal in the two balances*

$$|\boldsymbol{m}| = |\boldsymbol{n}| + |\boldsymbol{\eta}| + 1 \qquad \text{and} \qquad |\boldsymbol{m}| = |\boldsymbol{n}| + |\boldsymbol{\eta}| - 1.$$

Equivalently, the A- and B-form problems considered below have maximal degrees and are unique up to a nonzero multiplicative constant after the corresponding normalization is fixed.

Proof. The derivative block is written here with $\mathscr{D} = -x\mathrm{d}/\mathrm{d}x$. This differs from the normalization used by Wolfs only by a constant triangular change of basis in the derivative block, and therefore it preserves the AT property for the same step-line ranges.

With our notation, Proposition 3.3 of Wolfs [18], in the range specified before Lemma 3.2, states that the Laguerre-like vector

$$\left(U_1^{\mathrm{L}}, \dots, U_q^{\mathrm{L}}\right) = (w_1, \dots, w_r, v_1, \dots, v_s)$$

is an AT-system on $(0, \infty)$ for every Laguerre admissible multi-index $\boldsymbol{\lambda} = (\boldsymbol{n}, \boldsymbol{\eta})$, under the stated index-dependent hypotheses on the beta shifts $b_h - a_h$.

The nonresonance condition on $\boldsymbol{\beta}$ implies that the functions

$$x^{\beta_i+\ell}, \qquad i \in \{1, \dots, p\}, \quad \ell \in \{0, \dots, m_i - 1\},$$

have pairwise distinct exponents for every $\boldsymbol{m}$. Hence the power vector

$$\mathbf{V}(x) = \left(x^{\beta_1}, \dots, x^{\beta_p}\right)$$

is an AT-system on $(0, \infty)$ for every multi-index $\boldsymbol{m}$.

Thus, in each of the two balances above involving the fixed Laguerre-like multi-index $\boldsymbol{\lambda}$, both the Laguerre-like vector and the power vector are AT-systems. Hence the matrix of measures (63) is an AT matrix measure in the sense of Fidalgo Prieto, Medina Peralta and Mínguez [7, Definition 5]. The normality statement follows from their mixed-type normality theorem for AT matrix measures [7, Theorem 2]. ◻

Let

$$\boldsymbol{n} = (n_1, \dots, n_r) \in \mathbb{N}_0^r, \qquad \boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \mathbb{N}_0^s,$$

and put

$$\boldsymbol{\lambda} := (\boldsymbol{n}, \boldsymbol{\eta}) = (n_1, \dots, n_r, \eta_1, \dots, \eta_s), \qquad N := |\boldsymbol{n}| + |\boldsymbol{\eta}|.$$

Empty sums and empty products are understood in the usual sense. In the Laguerre admissible range considered below, the normality input is supplied by Proposition 4.4. Once this normality is known, the word normalized is used in the sense of Definition 2.3: the formulas below choose a specific representative of the one-dimensional space of solutions.

Theorem 4.5 (Explicit mixed Laguerre-like forms)**.** *Assume that*

$$\boldsymbol{\lambda} = (\boldsymbol{n}, \boldsymbol{\eta}) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$$

is Laguerre admissible in the sense of Definition 4.1, and that the hypotheses of Proposition 4.4 hold.

(i) The A-form. *Let*

$$\boldsymbol{m} = (m_1, \dots, m_p) \in \mathbb{N}_0^p, \qquad |\boldsymbol{m}| = N + 1,$$

and assume that the nodes

$$\beta_i + k, \qquad i \in \{1, \dots, p\} \text{ with } m_i \geq 1, \quad k \in \{0, \dots, m_i - 1\},$$

are pairwise distinct. Set

$$D_{\boldsymbol{\beta},\boldsymbol{m}}(t) := \prod_{i=1}^{p} (\beta_i - t)_{m_i}.$$

The contour-normalized representative of the mixed form on the power vector is

$$\mathscr{A}_{\boldsymbol{\lambda},\boldsymbol{m}}^{\mathrm{L}}(x) := \frac{1}{2\pi\mathrm{i}} \int_{\Sigma} \frac{\Gamma(t\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n} + \mathbf{1}_r)}{\Gamma(t\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)} \frac{x^t}{D_{\boldsymbol{\beta},\boldsymbol{m}}(t)} \, \mathrm{d}t, \tag{65}$$

where Σ encloses precisely the nodes $\beta_i, \dots, \beta_i + m_i - 1$, $i \in \{1, \dots, p\}$ with $m_i \geq 1$, and no other pole of the integrand. Its residue expansion is

$$\mathscr{A}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) = \sum_{i=1}^{p} A^{(i)}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) x^{\beta_i},$$

where $A^{(i)}_{\boldsymbol{\lambda},\boldsymbol{m}} \equiv 0$ if $m_i = 0$, and $A^{(i)}_{\boldsymbol{\lambda},\boldsymbol{m}} \in \mathbb{P}_{m_i-1}$ if $m_i \geq 1$. More explicitly, for $m_i \geq 1$,

(66)
$$A^{(i)}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) = \kappa^{\mathrm{L},\mathscr{A}}_i(\boldsymbol{\lambda}, \boldsymbol{m}) \; {}_{p+r}F_{p+q-1}\left(\begin{matrix} -m_i + 1,\ (\beta_i + 1)\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n},\ (\beta_i + 1)\mathbf{1}_{p-1} - \boldsymbol{\beta}^{*i} - \boldsymbol{m}^{*i} \\ (\beta_i + 1)\mathbf{1}_q + \boldsymbol{a},\ (\beta_i + 1)\mathbf{1}_{p-1} - \boldsymbol{\beta}^{*i} \end{matrix} ; x \right),$$

where

$$\kappa^{\mathrm{L},\mathscr{A}}_i(\boldsymbol{\lambda}, \boldsymbol{m}) := -\frac{\Gamma((\beta_i + 1)\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}{(m_i - 1)!\, \Gamma((\beta_i + 1)\mathbf{1}_q + \boldsymbol{a})\, (\boldsymbol{\beta}^{*i} - \beta_i \mathbf{1}_{p-1})_{\boldsymbol{m}^{*i}}}. \tag{67}$$

If $m_i = 0$, then $A^{(i)}_{\boldsymbol{\lambda},\boldsymbol{m}} \equiv 0$. The A-form satisfies

$$\int_0^\infty \mathscr{A}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) x^k w_h(x; \boldsymbol{a}, \boldsymbol{b})\, \mathrm{d}x = 0, \qquad k \in \{0, \dots, n_h - 1\}, \quad h \in \{1, \dots, r\} \text{ with } n_h \geq 1, \tag{68}$$

$$\int_0^\infty \mathscr{A}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) x^k v_\ell(x; \boldsymbol{a}, \boldsymbol{b})\, \mathrm{d}x = 0, \qquad k \in \{0, \dots, \eta_\ell - 1\}, \quad \ell \in \{1, \dots, s\} \text{ with } \eta_\ell \geq 1. \tag{69}$$

(ii) The B-form. *Let*

$$\boldsymbol{m} = (m_1, \dots, m_p) \in \mathbb{N}_0^p, \qquad |\boldsymbol{m}| = N - 1.$$

Assume also, for the residue expansion below, that

$$a_\rho - a_\sigma \notin \mathbb{Z}, \qquad \rho, \sigma \in \{1, \dots, q\}, \quad \rho \neq \sigma.$$

There exists a unique representative of the normal $\mathscr{B}$-form, normalized by the Mellin identity below,

$$\mathscr{B}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) = \sum_{h=1}^{r} B^{(h)}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) w_h(x; \boldsymbol{a}, \boldsymbol{b}) + \sum_{\ell=1}^{s} D^{(\ell)}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) v_\ell(x; \boldsymbol{a}, \boldsymbol{b}), \tag{70}$$

where

$$B^{(h)}_{\boldsymbol{\lambda},\boldsymbol{m}} \in \mathbb{P}_{n_h-1} \quad \text{if } n_h \geq 1, \qquad B^{(h)}_{\boldsymbol{\lambda},\boldsymbol{m}} \equiv 0 \quad \text{if } n_h = 0,$$

and

$$D^{(\ell)}_{\boldsymbol{\lambda},\boldsymbol{m}} \in \mathbb{P}_{\eta_\ell-1} \quad \text{if } \eta_\ell \geq 1, \qquad D^{(\ell)}_{\boldsymbol{\lambda},\boldsymbol{m}} \equiv 0 \quad \text{if } \eta_\ell = 0.$$

It satisfies

$$\int_0^\infty \mathscr{B}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) x^{\beta_i + k}\, \mathrm{d}x = 0, \qquad k \in \{0, \dots, m_i - 1\}, \quad i \in \{1, \dots, p\}, \tag{71}$$

where this condition is absent when $m_i = 0$. The normalization is prescribed by

$$\mathscr{M}[\mathscr{B}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})} (\boldsymbol{\beta} + (1 - z)\mathbf{1}_p)_{\boldsymbol{m}}. \tag{72}$$

The complete form also has the hypergeometric residue expansion

$$\begin{aligned} \mathscr{B}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) = \sum_{\rho=1}^{q} & \frac{\Gamma(\boldsymbol{a}^{*\rho} - a_\rho \mathbf{1}_{q-1})(\boldsymbol{\beta} + (a_\rho + 1)\mathbf{1}_p)_{\boldsymbol{m}}}{\Gamma(\boldsymbol{b} + \boldsymbol{n} - a_\rho \mathbf{1}_r)} x^{a_\rho} \\ & \times {}_{r+p}F_{q+p-1}\left(\begin{matrix} (a_\rho + 1)\mathbf{1}_r - \boldsymbol{b} - \boldsymbol{n},\ \boldsymbol{\beta} + \boldsymbol{m} + (a_\rho + 1)\mathbf{1}_p \\ (a_\rho + 1)\mathbf{1}_{q-1} - \boldsymbol{a}^{*\rho},\ \boldsymbol{\beta} + (a_\rho + 1)\mathbf{1}_p \end{matrix} ; (-1)^s x \right). \end{aligned} \tag{73}$$

Equivalently, the same complete form can be written compactly as the Meijer G-function

$$\mathscr{B}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) = G^{q,p}_{p+r,p+q}\left(x \,\middle|\, \begin{matrix} -\boldsymbol{\beta} - \boldsymbol{m},\ \boldsymbol{b} + \boldsymbol{n} \\ \boldsymbol{a},\ -\boldsymbol{\beta} \end{matrix} \right). \tag{74}$$

The shifted-beta components also admit an explicit closed hypergeometric form. For the formulas below, assume additionally that the nodes

$$-b_H - K, \qquad H \in \{1,\dots,r\}, \quad K \in \{0,\dots,n_H-1\},$$

are pairwise distinct and that every displayed denominator is nonzero. These extra finite nonresonance conditions are not needed for the Mellin or Meijer G-representations of the complete form. Let

$$\Pi^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}(z) := \left[\frac{(\boldsymbol{\beta}+(1-z)\mathbf{1}_p)_{\boldsymbol{m}}}{(z\mathbf{1}_r+\boldsymbol{b})_{\boldsymbol{n}}}\right]_+$$

be the polynomial part at $z=\infty$. For $\ell \in \{1,\dots,s\}$ and $k \in \{0,\dots,\eta_\ell - 1\}$, set

$$E^{\mathrm{L}}_{\ell,k}(z) := \left[\frac{(z\mathbf{1}_q+\boldsymbol{a})_k}{(z\mathbf{1}_r+\boldsymbol{b})_k}(z+k)_{\ell-1}\right]_+ .$$

The coefficients $\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}$ are determined by the finite expansion

$$\Pi^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}(z) = \sum_{\ell=1}^{s}\sum_{k=0}^{\eta_\ell-1}\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}E^{\mathrm{L}}_{\ell,k}(z),$$

with the convention that the sum is absent when $|\boldsymbol{\eta}|=0$. For $H \in \{1,\dots,r\}$ and $K \in \{0,\dots,n_H-1\}$, define the effective residues directly by

$$\widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K} := (-1)^K \frac{(\boldsymbol{\beta}+(b_H+K+1)\mathbf{1}_p)_{\boldsymbol{m}}}{K!(n_H-K-1)!(\boldsymbol{b}^{*H}-(b_H+K)\mathbf{1}_{r-1})_{\boldsymbol{n}^{*H}}} - \sum_{\ell=1}^{s}\sum_{k=K+1}^{\eta_\ell-1}\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}\frac{(k-b_H-K)_{\ell-1}(\boldsymbol{a}-(b_H+K)\mathbf{1}_q)_k}{(-1)^K K!(k-1-K)!(\boldsymbol{b}^{*H}-(b_H+K)\mathbf{1}_{r-1})_k},$$

where empty sums are absent. Then, for each $h \in \{1,\dots,r\}$ with $n_h \geq 1$,

$$B^{(h)}_{\lambda,\boldsymbol{m}}(x) = \sum_{\substack{H=1\\ n_H\geq 1}}^{r}\sum_{K=0}^{n_H-1}\widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K}\mathscr{R}^{\mathrm{L}}_{h;H,K}(x), \tag{75}$$

where

$$\mathscr{R}^{\mathrm{L}}_{h;H,K}(x) = 0 \quad \text{if } h \neq H \text{ and } K = 0,$$

and, in all other cases,

(76)

$$\mathscr{R}^{\mathrm{L}}_{h;H,K}(x) := \frac{(\boldsymbol{a}-b_h\mathbf{1}_q)_1}{(\boldsymbol{b}^{*h}-b_h\mathbf{1}_{r-1})_1}\frac{(\boldsymbol{b}^{*h}-(b_H+K)\mathbf{1}_{r-1})_1}{(\boldsymbol{a}-(b_H+K)\mathbf{1}_q)_1}\,{}_{r+1}F_q\left(\begin{matrix}1,\ \boldsymbol{b}+(1-b_H-K)\mathbf{1}_r-\boldsymbol{e}_h\\ \boldsymbol{a}+(1-b_H-K)\mathbf{1}_q\end{matrix};x\right).$$

Each hypergeometric block is terminating; its degree is K if $h=H$, and $K-1$ if $h \neq H$.

Proof. Step 1: Residue formula and hypergeometric components for the form on the power vector. We first turn to the A-form. The contour expression (65) has possible poles inside Σ only at the zeros of $D_{\boldsymbol{\beta},\boldsymbol{m}}$. These nodes are assumed to be pairwise distinct, and they are $t=\beta_i+k$, with $i \in \{1,\dots,p\}$, $m_i \geq 1$, and $k \in \{0,\dots,m_i-1\}$. Hence all enclosed poles are simple.

Evaluating the contour integral by residues gives

$$\mathscr{A}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}(x) = \sum_{\substack{i=1\\ m_i\geq 1}}^{p}\sum_{k=0}^{m_i-1}c_{i,k}x^{\beta_i+k},$$

where

$$c_{i,k} = \operatorname*{Res}_{t=\beta_i+k}\left[\frac{\Gamma(t\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n}+\mathbf{1}_r)}{\Gamma(t\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)}\frac{1}{D_{\boldsymbol{\beta},\boldsymbol{m}}(t)}\right].$$

Equivalently,

$$\mathscr{A}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}(x) = \sum_{i=1}^{p}A^{(i)}_{\lambda,\boldsymbol{m}}(x)x^{\beta_i},$$

where $A^{(i)}_{\lambda,\boldsymbol{m}} \equiv 0$ if $m_i = 0$, and, for $m_i \geq 1$,

$$A^{(i)}_{\lambda,\boldsymbol{m}}(x) = \sum_{k=0}^{m_i-1} c_{i,k} x^k \in \mathbb{P}_{m_i-1}.$$

This already gives the required expansion in the power vector and the degree bounds for its components.

We compute the coefficients explicitly. Fix $i \in \{1, \dots, p\}$ with $m_i \geq 1$. At the node $t = \beta_i + k$, direct evaluation of the simple residue gives

$$c_{i,k} = \frac{(-1)^{k+1}}{k!(m_i - k - 1)!(\boldsymbol{\beta}^{*i} - \beta_i \mathbf{1}_{p-1} - k\mathbf{1}_{p-1})_{\boldsymbol{m}^{*i}}} \frac{\Gamma((\beta_i + k)\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n} + \mathbf{1}_r)}{\Gamma((\beta_i + k)\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)},$$

for $k \in \{0, \dots, m_i - 1\}$. In particular,

$$c_{i,0} = -\frac{\Gamma((\beta_i)\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n} + \mathbf{1}_r)}{(m_i - 1)!\,\Gamma((\beta_i)\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)\,(\boldsymbol{\beta}^{*i} - \beta_i \mathbf{1}_{p-1})_{\boldsymbol{m}^{*i}}} = \kappa_i^{\mathrm{L},\mathscr{A}}(\boldsymbol{\lambda}, \boldsymbol{m}).$$

For $k \geq 0$, the quotient $c_{i,k}/c_{i,0}$ is obtained by the usual Pochhammer identities. One gets

$$\frac{c_{i,k}}{c_{i,0}} = \frac{(-m_i + 1)_k}{k!} \frac{((\beta_i)\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n} + \mathbf{1}_r)_k (\beta_i \mathbf{1}_{p-1} - \boldsymbol{\beta}^{*i} - \boldsymbol{m}^{*i} + \mathbf{1}_{p-1})_k}{((\beta_i)\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)_k (\beta_i \mathbf{1}_{p-1} - \boldsymbol{\beta}^{*i} + \mathbf{1}_{p-1})_k}.$$

Therefore

$$A^{(i)}_{\lambda,\boldsymbol{m}}(x) = \kappa_i^{\mathrm{L},\mathscr{A}}(\boldsymbol{\lambda}, \boldsymbol{m}) \sum_{k=0}^{m_i-1} \frac{(-m_i + 1)_k}{k!} \frac{((\beta_i)\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n} + \mathbf{1}_r)_k (\beta_i \mathbf{1}_{p-1} - \boldsymbol{\beta}^{*i} - \boldsymbol{m}^{*i} + \mathbf{1}_{p-1})_k}{((\beta_i)\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)_k (\beta_i \mathbf{1}_{p-1} - \boldsymbol{\beta}^{*i} + \mathbf{1}_{p-1})_k} x^k.$$

This is exactly the terminating hypergeometric polynomial (66), with the normalization (67).

Step 2: Orthogonality of the A-form against the shifted beta block. Let $h \in \{1, \dots, r\}$ with $n_h \geq 1$, and let $k \in \{0, \dots, n_h - 1\}$. We pair the contour expression (65) with $x^k w_h(x; \boldsymbol{a}, \boldsymbol{b})$. The exchange of the contour integral with the x-integral is legitimate. Indeed, Σ is a finite union of compact contours around the nodes $\beta_i + j$, and it can be chosen so that, for every $t \in \Sigma$, the Mellin parameter $t + k + 1$ lies in the common fundamental strip of the weights involved. Equivalently, the functions $x^{t+k} w_h(x; \boldsymbol{a}, \boldsymbol{b})$ are absolutely integrable uniformly for $t \in \Sigma$. Therefore Fubini's theorem gives

$$\int_0^\infty \mathscr{A}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}(x) x^k w_h(x; \boldsymbol{a}, \boldsymbol{b})\,\mathrm{d}x = \frac{1}{2\pi\mathrm{i}} \int_\Sigma \frac{\Gamma(t\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n} + \mathbf{1}_r)}{\Gamma(t\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q) D_{\boldsymbol{\beta},\boldsymbol{m}}(t)} \left(\int_0^\infty x^{t+k} w_h(x; \boldsymbol{a}, \boldsymbol{b})\,\mathrm{d}x \right) \mathrm{d}t.$$

Using (61),

$$\mathscr{M}[w_h](t + k + 1) = \frac{\Gamma((t + k)\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)}{\Gamma((t + k)\mathbf{1}_r + \boldsymbol{b} + \mathbf{1}_r + \boldsymbol{e}_h)}.$$

Thus the required pairing is

$$\left\langle x^k \mathscr{A}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}, w_h \right\rangle = \frac{1}{2\pi\mathrm{i}} \int_\Sigma \frac{R_{h,k}(t)}{D_{\boldsymbol{\beta},\boldsymbol{m}}(t)}\,\mathrm{d}t,$$

where

$$R_{h,k}(t) = \frac{\Gamma(t\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n} + \mathbf{1}_r)}{\Gamma(t\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)} \frac{\Gamma((t + k)\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)}{\Gamma((t + k)\mathbf{1}_r + \boldsymbol{b} + \mathbf{1}_r + \boldsymbol{e}_h)}.$$

We check that $R_{h,k}$ is a polynomial of sufficiently small degree. By (64a), one has $k \leq n_h - 1 \leq n_u$ for every $u \in \{1, \dots, r\}$. Hence $\boldsymbol{n} - k\mathbf{1}_r - \boldsymbol{e}_h \in \mathbb{N}_0^r$, and

$$R_{h,k}(t) = (t\mathbf{1}_q + \boldsymbol{a} + \mathbf{1}_q)_k (t\mathbf{1}_r + \boldsymbol{b} + (k + 1)\mathbf{1}_r + \boldsymbol{e}_h)_{\boldsymbol{n} - k\mathbf{1}_r - \boldsymbol{e}_h}.$$

This is a polynomial of degree $qk + |\boldsymbol{n}| - rk - 1 = |\boldsymbol{n}| + sk - 1$. Since (64c) gives $sk \leq s(n_h - 1) \leq |\boldsymbol{\eta}|$, we have $R_{h,k} \in \mathbb{P}_{N-1}$.

Now $\deg D_{\boldsymbol{\beta},\boldsymbol{m}} = |\boldsymbol{m}| = N + 1$, so $R_{h,k}(t)/D_{\boldsymbol{\beta},\boldsymbol{m}}(t) = \mathrm{O}(t^{-2})$ as $t \to \infty$. Hence its residue at infinity is zero. By the residue theorem, the sum of all finite residues is zero. Since Σ encloses precisely the poles at the nodes $\beta_i + j$, with $i \in \{1, \dots, p\}$, $m_i \geq 1$, and $j \in \{0, \dots, m_i - 1\}$, the contour integral vanishes. This proves (68).

Step 3: Orthogonality of the A-form against the Euler-derivative block. Let $\ell \in \{1, \dots, s\}$ with $\eta_\ell \geq 1$, and let $k \in \{0, \dots, \eta_\ell - 1\}$. We pair (65) with $x^k v_\ell(x; \boldsymbol{a}, \boldsymbol{b})$. The same justification of the exchange of integrations applies to the weights v_ℓ: after the integrations by parts used in (60), the Mellin transform (62) is valid in the corresponding fundamental strip, and the chosen compact contour Σ lies inside it for the Mellin parameters $t+k+1$. Since Σ is a finite union of compact contours and the integrand is absolutely integrable uniformly for $t\in\Sigma$, Fubini's theorem allows us to interchange the order of integration. Hence

$$\begin{aligned}\left\langle x^k \mathscr{A}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}, v_\ell\right\rangle &= \int_0^\infty \mathscr{A}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}(x) x^k v_\ell(x;\boldsymbol{a},\boldsymbol{b})\,\mathrm{d}x\\ &= \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\Gamma(t\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n}+\mathbf{1}_r)}{\Gamma(t\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)D_{\boldsymbol{\beta},\boldsymbol{m}}(t)}\left(\int_0^\infty x^{t+k}v_\ell(x;\boldsymbol{a},\boldsymbol{b})\,\mathrm{d}x\right)\mathrm{d}t.\end{aligned}$$

Using (62),

$$\mathscr{M}[v_\ell](t+k+1) = (t+k+1)^{\ell-1}\frac{\Gamma((t+k)\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)}{\Gamma((t+k)\mathbf{1}_r+\boldsymbol{b}+\mathbf{1}_r)}.$$

Therefore the required pairing is

$$\left\langle x^k \mathscr{A}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}, v_\ell\right\rangle = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{S_{\ell,k}(t)}{D_{\boldsymbol{\beta},\boldsymbol{m}}(t)}\,\mathrm{d}t,$$

with

$$S_{\ell,k}(t) = \frac{\Gamma(t\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n}+\mathbf{1}_r)}{\Gamma(t\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)}\frac{\Gamma((t+k)\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)}{\Gamma((t+k)\mathbf{1}_r+\boldsymbol{b}+\mathbf{1}_r)}(t+k+1)^{\ell-1}.$$

By (64b), one has $k\le\eta_\ell-1\le n_u$ for every $u\in\{1,\dots,r\}$. Thus $\boldsymbol{n}-k\mathbf{1}_r\in\mathbb{N}_0^r$, and

$$S_{\ell,k}(t) = (t\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)_k(t\mathbf{1}_r+\boldsymbol{b}+(k+1)\mathbf{1}_r)_{\boldsymbol{n}-k\mathbf{1}_r}(t+k+1)^{\ell-1}.$$

This is a polynomial of degree $qk+|\boldsymbol{n}|-rk+\ell-1 = |\boldsymbol{n}|+sk+\ell-1$. By (64d), one has $sk+\ell\le s(\eta_\ell-1)+\ell\le|\boldsymbol{\eta}|$. Hence $S_{\ell,k}\in\mathbb{P}_{N-1}$.

Again, $\deg D_{\boldsymbol{\beta},\boldsymbol{m}} = |\boldsymbol{m}| = N+1$, and therefore $S_{\ell,k}(t)/D_{\boldsymbol{\beta},\boldsymbol{m}}(t) = \mathrm{O}(t^{-2})$ as $t\to\infty$. Its residue at infinity is zero, so the sum of the finite residues enclosed by Σ vanishes. The contour integral is zero, and (69) follows.

Step 4: Mellin transforms of the elementary building blocks for the B-form. We compute the Mellin transforms of the terms that appear in the B-form. The purpose of this step is to show that, after taking out the common factor

$$\frac{\Gamma(z\mathbf{1}_q+\boldsymbol{a})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n})},$$

the remaining factor should be a polynomial in z, of degree at most $N-1$. This is what allows us to reconstruct the B-form by solving a finite polynomial identity.

For the shifted beta weights and the Euler-derivative weights, we have

$$\mathscr{M}[w_h](z) = \frac{\Gamma(z\mathbf{1}_q+\boldsymbol{a})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{e}_h)},\qquad h\in\{1,\dots,r\},$$

and

$$\mathscr{M}[v_\ell](z) = z^{\ell-1}\frac{\Gamma(z\mathbf{1}_q+\boldsymbol{a})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b})},\qquad \ell\in\{1,\dots,s\}.$$

Multiplication by x^k shifts the Mellin variable. Thus $\mathscr{M}[x^k w_h](z) = \mathscr{M}[w_h](z+k)$, and we get

$$\mathscr{M}[x^k w_h](z) = \frac{\Gamma(z\mathbf{1}_q+\boldsymbol{a})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n})}(z\mathbf{1}_q+\boldsymbol{a})_k\frac{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+k\mathbf{1}_r+\boldsymbol{e}_h)}. \tag{77}$$

Similarly, since $\mathscr{M}[x^k v_\ell](z) = \mathscr{M}[v_\ell](z+k)$, we obtain

$$\mathscr{M}[x^k v_\ell](z) = \frac{\Gamma(z\mathbf{1}_q+\boldsymbol{a})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n})}(z\mathbf{1}_q+\boldsymbol{a})_k\frac{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+k\mathbf{1}_r)}(z+k)^{\ell-1}. \tag{78}$$

We use Laguerre admissibility. Let $h\in\{1,\dots,r\}$ with $n_h\ge1$, and let $k\in\{0,\dots,n_h-1\}$. By (64a), we have $k\le n_h-1\le n_u$ for every $u\in\{1,\dots,r\}$. Hence $\boldsymbol{n}-k\mathbf{1}_r-\boldsymbol{e}_h\in\mathbb{N}_0^r$. Therefore the quotient of gamma functions in (77) is a finite Pochhammer product:

$$\frac{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+k\mathbf{1}_r+\boldsymbol{e}_h)} = (z\mathbf{1}_r+\boldsymbol{b}+k\mathbf{1}_r+\boldsymbol{e}_h)_{\boldsymbol{n}-k\mathbf{1}_r-\boldsymbol{e}_h}.$$

Thus

$$\mathscr{M}[x^k w_h](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}(z\mathbf{1}_q + \boldsymbol{a})_k (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r + \boldsymbol{e}_h)_{\boldsymbol{n}-k\mathbf{1}_r-\boldsymbol{e}_h}. \tag{79}$$

Now let $\ell \in \{1, \ldots, s\}$ with $\eta_\ell \geq 1$, and let $k \in \{0, \ldots, \eta_\ell - 1\}$. By (64b), we have $k \leq \eta_\ell - 1 \leq n_u$ for every $u \in \{1, \ldots, r\}$. Hence $\boldsymbol{n} - k\mathbf{1}_r \in \mathbb{N}_0^r$. Therefore

$$\frac{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r)} = (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r)_{\boldsymbol{n}-k\mathbf{1}_r}.$$

Substituting this into (78) gives

$$\mathscr{M}[x^k v_\ell](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}(z\mathbf{1}_q + \boldsymbol{a})_k (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r)_{\boldsymbol{n}-k\mathbf{1}_r}(z+k)^{\ell-1}. \tag{80}$$

The polynomial factors in (79) and (80) are the two finite polynomial families that enter the coefficient system below.

It remains to check the degree bounds. In (79), the polynomial factor is

$$(z\mathbf{1}_q + \boldsymbol{a})_k (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r + \boldsymbol{e}_h)_{\boldsymbol{n}-k\mathbf{1}_r-\boldsymbol{e}_h}.$$

The first factor has degree qk, and the second one has degree $|\boldsymbol{n}| - rk - 1$. Hence the total degree is

$$qk + |\boldsymbol{n}| - rk - 1 = |\boldsymbol{n}| + sk - 1.$$

Since $k \leq n_h - 1$, condition (64c) gives $sk \leq s(n_h - 1) \leq |\boldsymbol{\eta}|$. Therefore

$$|\boldsymbol{n}| + sk - 1 \leq |\boldsymbol{n}| + |\boldsymbol{\eta}| - 1 = N - 1.$$

So the polynomial factor in (79) belongs to $\mathbb{P}_{N-1}$.

In (80), the polynomial factor is

$$(z\mathbf{1}_q + \boldsymbol{a})_k (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r)_{\boldsymbol{n}-k\mathbf{1}_r}(z+k)^{\ell-1}.$$

Its degree is

$$qk + |\boldsymbol{n}| - rk + \ell - 1 = |\boldsymbol{n}| + sk + \ell - 1.$$

Since $k \leq \eta_\ell - 1$, condition (64d) gives

$$sk + \ell \leq s(\eta_\ell - 1) + \ell \leq |\boldsymbol{\eta}|.$$

Hence

$$|\boldsymbol{n}| + sk + \ell - 1 \leq |\boldsymbol{n}| + |\boldsymbol{\eta}| - 1 = N - 1.$$

Thus the polynomial factor in (80) also belongs to $\mathbb{P}_{N-1}$.

We have therefore proved that every polynomial combination in (70) has a Mellin transform of the form

$$\mathscr{M}[\mathscr{B}](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}F(z), \qquad F \in \mathbb{P}_{N-1}. \tag{81}$$

Step 5: Determination of the form on the Laguerre-like vector. We choose the polynomial F in (81). The choice is dictated by the orthogonality conditions against the power vector. Thus the problem is reduced to expanding a prescribed polynomial in a finite family of polynomials.

Let $\mathscr{V}_\lambda$ be the space of component vectors

$$\left(\{B^{(h)}\}_{h=1}^r, \{D^{(\ell)}\}_{\ell=1}^s\right)$$

with $B^{(h)} \in \mathbb{P}_{n_h-1}$ if $n_h \geq 1$, and $B^{(h)} \equiv 0$ if $n_h = 0$, together with $D^{(\ell)} \in \mathbb{P}_{\eta_\ell-1}$ if $\eta_\ell \geq 1$, and $D^{(\ell)} \equiv 0$ if $\eta_\ell = 0$. This space has dimension

$$\dim \mathscr{V}_\lambda = |\boldsymbol{n}| + |\boldsymbol{\eta}| = N.$$

By Step 4, the Mellin transform defines a linear map

$$\Phi_\lambda : \mathscr{V}_\lambda \longrightarrow \mathbb{P}_{N-1}$$

through the identity

$$\mathscr{M}[\mathscr{B}](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})} \Phi_\lambda \left(\{B^{(h)}\}_{h=1}^r, \{D^{(\ell)}\}_{\ell=1}^s \right)(z). \tag{82}$$

If

$$B^{(h)}(x) = \sum_{k=0}^{n_h-1} b_{h,k} x^k, \qquad D^{(\ell)}(x) = \sum_{k=0}^{\eta_\ell-1} d_{\ell,k} x^k,$$

then this map is explicitly

$$\begin{aligned}\Phi_\lambda \left(\{B^{(h)}\}_{h=1}^r, \{D^{(\ell)}\}_{\ell=1}^s \right)(z) &= \sum_{\substack{h=1\\ n_h \geq 1}}^{r} \sum_{k=0}^{n_h-1} b_{h,k} (z\mathbf{1}_q + \boldsymbol{a})_k \frac{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r + \boldsymbol{e}_h)} \\ &\quad + \sum_{\substack{\ell=1\\ \eta_\ell \geq 1}}^{s} \sum_{k=0}^{\eta_\ell-1} d_{\ell,k} (z\mathbf{1}_q + \boldsymbol{a})_k \frac{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r)} (z+k)^{\ell-1}.\end{aligned}$$

Using the Pochhammer rewritings from Step 4, this becomes

$$\begin{aligned}\Phi_\lambda \left(\{B^{(h)}\}_{h=1}^r, \{D^{(\ell)}\}_{\ell=1}^s \right)(z) &= \sum_{\substack{h=1\\ n_h \geq 1}}^{r} \sum_{k=0}^{n_h-1} b_{h,k} (z\mathbf{1}_q + \boldsymbol{a})_k (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r + \boldsymbol{e}_h)_{\boldsymbol{n} - k\mathbf{1}_r - \boldsymbol{e}_h} \\ &\quad + \sum_{\substack{\ell=1\\ \eta_\ell \geq 1}}^{s} \sum_{k=0}^{\eta_\ell-1} d_{\ell,k} (z\mathbf{1}_q + \boldsymbol{a})_k (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r)_{\boldsymbol{n} - k\mathbf{1}_r} (z+k)^{\ell-1}.\end{aligned} \tag{83}$$

So the coefficient problem for $\mathscr{B}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}$ is exactly the problem of expanding a prescribed polynomial in $\mathbb{P}_{N-1}$ in the finite family appearing in (83). Equivalently, the coefficients $b_{h,k}$ and $d_{\ell,k}$ satisfy the finite identity

$$\begin{aligned}&\sum_{\substack{h=1\\ n_h \geq 1}}^{r} \sum_{k=0}^{n_h-1} b_{h,k} (z\mathbf{1}_q + \boldsymbol{a})_k (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r + \boldsymbol{e}_h)_{\boldsymbol{n} - k\mathbf{1}_r - \boldsymbol{e}_h} \\ &\quad + \sum_{\substack{\ell=1\\ \eta_\ell \geq 1}}^{s} \sum_{k=0}^{\eta_\ell-1} d_{\ell,k} (z\mathbf{1}_q + \boldsymbol{a})_k (z\mathbf{1}_r + \boldsymbol{b} + k\mathbf{1}_r)_{\boldsymbol{n} - k\mathbf{1}_r} (z+k)^{\ell-1} = (\boldsymbol{\beta} + (1-z)\mathbf{1}_p)_{\boldsymbol{m}}.\end{aligned} \tag{84}$$

We prove that this expansion is unique. Suppose that a component vector lies in the kernel of Φ_λ. Then the associated mixed form has identically vanishing Mellin transform in its strip of definition. By uniqueness of the Mellin transform, this implies

$$\mathscr{B}(x) \equiv 0$$

on $(0, \infty)$. By Proposition 4.4, the Laguerre-like vector is an AT-system for the multi-index λ. Hence the functions

$$x^k w_h(x; \boldsymbol{a}, \boldsymbol{b}), \qquad h \in \{1, \dots, r\}, \quad k \in \{0, \dots, n_h - 1\},$$

together with

$$x^k v_\ell(x; \boldsymbol{a}, \boldsymbol{b}), \qquad \ell \in \{1, \dots, s\}, \quad k \in \{0, \dots, \eta_\ell - 1\},$$

are linearly independent on $(0, \infty)$. Therefore all coefficients of the component vector vanish. Thus Φ_λ is injective. Since $\dim \mathscr{V}_\lambda = N = \dim \mathbb{P}_{N-1}$, it follows that Φ_λ is an isomorphism.

We choose

$$F(z) = (\boldsymbol{\beta} + (1-z)\mathbf{1}_p)_{\boldsymbol{m}}.$$

Because $|\boldsymbol{m}| = N - 1$, this polynomial belongs to $\mathbb{P}_{N-1}$. Since Φ_λ is an isomorphism, there is a unique component vector such that

$$\Phi_\lambda \left(\{B^{(h)}\}_{h=1}^r, \{D^{(\ell)}\}_{\ell=1}^s \right)(z) = (\boldsymbol{\beta} + (1-z)\mathbf{1}_p)_{\boldsymbol{m}}.$$

Substitution in (82) gives

$$\mathscr{M}[\mathscr{B}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}(\boldsymbol{\beta} + (1-z)\mathbf{1}_p)_{\boldsymbol{m}}, \tag{85}$$

which is (72).

The orthogonality now follows directly from the zeros of this Mellin transform. Let $i \in \{1, \dots, p\}$ with $m_i \geq 1$, and let $k \in \{0, \dots, m_i - 1\}$. At $z = \beta_i + k + 1$, the factor $(\beta_i + 1 - z)_{m_i} = (-k)_{m_i}$ vanishes. Therefore

$$\mathscr{M}[\mathscr{B}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}](\beta_i + k + 1) = 0,$$

which means exactly that

$$\int_0^\infty \mathscr{B}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}(x) x^{\beta_i + k}\,\mathrm{d}x = 0.$$

This proves (71).

Step 6: Effective finite-pole residues and the shifted-beta components. We prove the explicit formula (75). Set

$$R^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}(z) := \frac{(\boldsymbol{\beta} + (1-z)\mathbf{1}_p)_{\boldsymbol{m}}}{(z\mathbf{1}_r + \boldsymbol{b})_{\boldsymbol{n}}}.$$

Dividing the identity (84) by $(z\mathbf{1}_r + \boldsymbol{b})_{\boldsymbol{n}}$ is equivalent to decomposing R into partial fractions at the finite beta poles,

$$R^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}(z) = \Pi^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}(z) + \sum_{\substack{H=1\\ n_H \geq 1}}^{r} \sum_{K=0}^{n_H - 1} \frac{\rho^{\mathscr{B},\mathrm{L}}_{H,K}}{z + b_H + K},$$

where $\Pi^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}$ is the polynomial part at infinity. Evaluating the simple residue of R at $z = -b_H - K$ gives

$$\rho^{\mathscr{B},\mathrm{L}}_{H,K} := (-1)^K \frac{(\boldsymbol{\beta} + (b_H + K + 1)\mathbf{1}_p)_{\boldsymbol{m}}}{K!(n_H - K - 1)!(\boldsymbol{b}^{*H} - (b_H + K)\mathbf{1}_{r-1})_{\boldsymbol{n}^{*H}}}.$$

The polynomial part is expanded in the polynomial parts of the confluent extraction symbols. For $\ell \in \{1, \dots, s\}$ and $k \in \{0, \dots, \eta_\ell - 1\}$, define

$$\widetilde{Q}^{\mathrm{L}}_{\ell,k}(z) := \frac{(z\mathbf{1}_q + \boldsymbol{a})_k}{(z\mathbf{1}_r + \boldsymbol{b})_k}(z + k)_{\ell - 1}, \qquad E^{\mathrm{L}}_{\ell,k}(z) := \left[\widetilde{Q}^{\mathrm{L}}_{\ell,k}(z)\right]_+, \tag{86}$$

where $[\cdot]_+$ denotes the polynomial part at $z = \infty$, and write

$$\Pi^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}(z) = \sum_{\nu=0}^{|\boldsymbol{\eta}|-1} \pi_\nu z^\nu, \qquad E^{\mathrm{L}}_{\ell,k}(z) = \sum_{\nu=0}^{|\boldsymbol{\eta}|-1} e^{\mathrm{L}}_{\nu;\ell,k} z^\nu.$$

Since $\boldsymbol{\eta}$ is on the step-line, the degrees $sk + \ell - 1$, with $\ell \in \{1, \dots, s\}$ and $k \in \{0, \dots, \eta_\ell - 1\}$, are precisely $0, \dots, |\boldsymbol{\eta}| - 1$, and $E^{\mathrm{L}}_{\ell,k}$ is monic of degree $sk + \ell - 1$. If $|\boldsymbol{\eta}| = 0$ the construction is empty; otherwise let

$$\mathscr{E}^{\mathrm{L}}_{\boldsymbol{\eta}} := \left(e^{\mathrm{L}}_{\nu;\ell,k}\right)_{\substack{\nu \in \{0,\dots,|\boldsymbol{\eta}|-1\}\\ \ell \in \{1,\dots,s\},\ k \in \{0,\dots,\eta_\ell - 1\}}},$$

with columns ordered by the degree $sk + \ell - 1$. By the previous remark $\mathscr{E}^{\mathrm{L}}_{\boldsymbol{\eta}}$ is unitriangular in this ordering, so the connection coefficients defined by

$$\left(\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}\right)_{\ell,k} = \left(\mathscr{E}^{\mathrm{L}}_{\boldsymbol{\eta}}\right)^{-1} (\pi_0, \dots, \pi_{|\boldsymbol{\eta}|-1})^T, \tag{87}$$

equivalently, by Cramer's rule,

$$\delta^{\mathscr{B},\mathrm{L}}_{\ell,k} = \frac{\det \mathscr{E}^{\mathrm{L}}_{\boldsymbol{\eta}}(\ell, k)}{\det \mathscr{E}^{\mathrm{L}}_{\boldsymbol{\eta}}},$$

where $\mathscr{E}^{\mathrm{L}}_{\boldsymbol{\eta}}(\ell,k)$ is obtained from $\mathscr{E}^{\mathrm{L}}_{\boldsymbol{\eta}}$ by replacing the column indexed by (ℓ,k) with $(\pi_0,\dots,\pi_{|\boldsymbol{\eta}|-1})^T$, give the unique expansion

$$\Pi^{\mathscr{B},\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(z)=\sum_{\ell=1}^{s}\sum_{k=0}^{\eta_\ell-1}\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}E^{\mathrm{L}}_{\ell,k}(z). \tag{88}$$

The entries of $\mathscr{E}^{\mathrm{L}}_{\boldsymbol{\eta}}$ are explicit: with $u=z^{-1}$, put

$$A_k(u):=\prod_{\rho=1}^{q}\prod_{j=0}^{k-1}\big(1+(a_\rho+j)u\big),\qquad B_k(u):=\prod_{h=1}^{r}\prod_{j=0}^{k-1}\big(1+(b_h+j)u\big);$$

and, with the empty product understood as one,

$$C_{\ell,k}(u):=\prod_{j=1}^{\ell-1}\big(1+(k+j-1)u\big).$$

Then, from $\widetilde{Q}^{\mathrm{L}}_{\ell,k}(z)=z^{sk+\ell-1}C_{\ell,k}(u)A_k(u)/B_k(u)$,

$$e^{\mathrm{L}}_{\nu;\ell,k}=\big[u^{sk+\ell-1-\nu}\big]\,C_{\ell,k}(u)\frac{A_k(u)}{B_k(u)}.$$

For the same symbol $\widetilde{Q}^{\mathrm{L}}_{\ell,k}$, the finite beta-pole contribution is obtained by ordinary residues. For $H\in\{1,\dots,r\}$, $K\in\{0,\dots,n_H-1\}$, and $k\ge K+1$, define

$$\sigma^{(\ell,k)}_{H,K}:=\operatorname*{Res}_{z=-b_H-K}\widetilde{Q}^{\mathrm{L}}_{\ell,k}(z)=\frac{(k-b_H-K)_{\ell-1}(\boldsymbol{a}-(b_H+K)\mathbf{1}_q)_k}{(-1)^K K!(k-1-K)!(\boldsymbol{b}^{*H}-(b_H+K)\mathbf{1}_{r-1})_k}.$$

The same confluent symbol is therefore used both at infinity and at the finite beta nodes. Since the polynomial part of $\sum_{\ell,k}\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}\widetilde{Q}^{\mathrm{L}}_{\ell,k}$ is $\Pi^{\mathscr{B},\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}$ by (88), the rational function

$$R^{\mathscr{B},\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(z)-\sum_{\ell=1}^{s}\sum_{k=0}^{\eta_\ell-1}\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}\widetilde{Q}^{\mathrm{L}}_{\ell,k}(z)$$

has no polynomial part at $z=\infty$. Its beta-pole residues are therefore exactly

$$\widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K}:=\rho^{\mathscr{B},\mathrm{L}}_{H,K}-\sum_{\ell=1}^{s}\sum_{k=K+1}^{\eta_\ell-1}\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}\sigma^{(\ell,k)}_{H,K}, \tag{89}$$

where an inner sum with upper limit smaller than its lower limit is absent. Equivalently,

$$R^{\mathscr{B},\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(z)-\sum_{\ell=1}^{s}\sum_{k=0}^{\eta_\ell-1}\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}\widetilde{Q}^{\mathrm{L}}_{\ell,k}(z)=\sum_{\substack{H=1\\ n_H\ge1}}^{r}\sum_{K=0}^{n_H-1}\frac{\widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K}}{z+b_H+K}. \tag{90}$$

The inequalities in Proposition 4.2 imply that every finite beta pole arising from these connection coefficients satisfies $K\le n_H-1$, so all poles lie in the displayed range of (75).

We show that the shifted-beta blocks $\mathscr{R}^{\mathrm{L}}_{h;H,K}$ realize the effective finite-pole residues $\widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K}$. The pole-by-pole realization available in the Jacobi-like case $s=0$ is *not* available here: when $s\ge1$ the reduced rational symbol of a single shifted-beta block is not a pure simple fraction. Each finite pole is instead reconstructed by its shifted-beta block *modulo the Euler-symbol space*.

Fix a finite beta node and set $B = b_H + K$. Let

$$\begin{aligned}\mathscr{W}^{\mathrm{L}}_{H,K}(z) &:= \sum_{h=1}^{r}\sum_{j=0}^{K-1+\delta_{h,H}} \tau^{H,K}_{h,j}\\ &\qquad\times \frac{(z\mathbf{1}_q+\boldsymbol{a})_j}{(z\mathbf{1}_r+\boldsymbol{b})_j\,(z+b_h+j)},\\ \tau^{H,K}_{h,j} &= \frac{(\boldsymbol{a}-b_h\mathbf{1}_q)_1}{(\boldsymbol{b}^{*h}-b_h\mathbf{1}_{r-1})_1}\\ &\qquad\times\frac{(\boldsymbol{b}^{*h}-B\mathbf{1}_{r-1})_{j+1}(b_h-B)_j}{(\boldsymbol{a}-B\mathbf{1}_q)_{j+1}},\end{aligned}\tag{91}$$

be the reduced rational symbol of the block vector $(\mathscr{R}^{\mathrm{L}}_{1;H,K},\dots,\mathscr{R}^{\mathrm{L}}_{r;H,K})$; the coefficients $\tau^{H,K}_{h,j}$ are the Jacobi-like reconstruction coefficients of (44), evaluated at the present parameters, and the expansion of the hypergeometric block (76) agrees coefficientwise with (91). We prove

$$\frac{1}{z+b_H+K}-\mathscr{W}^{\mathrm{L}}_{H,K}(z)\in\operatorname{span}\big\{\widetilde{Q}^{\mathrm{L}}_{\ell,j}\,:\,\ell\in\{1,\dots,s\},\ j\in\{0,\dots,K-1\}\big\}.\tag{92}$$

The modified telescoping. For $j\geq 0$ put

$$\mathscr{E}_j(z;B):=\frac{(\boldsymbol{b}-B\mathbf{1}_r)_j}{(\boldsymbol{a}-B\mathbf{1}_q)_j}\,\frac{(z\mathbf{1}_q+\boldsymbol{a})_j}{(z\mathbf{1}_r+\boldsymbol{b})_j},\qquad \mathscr{E}_0(z;B)=1.$$

A direct computation gives, for each j, the transition identity

$$\frac{\mathscr{E}_j(z;B)}{z+B}=\sum_{h=1}^{r}\tau^{H,K}_{h,j}\frac{(z\mathbf{1}_q+\boldsymbol{a})_j}{(z\mathbf{1}_r+\boldsymbol{b})_j\,(z+b_h+j)}+\frac{\mathscr{E}_{j+1}(z;B)}{z+B}+\frac{(z\mathbf{1}_q+\boldsymbol{a})_j}{(z\mathbf{1}_r+\boldsymbol{b})_j}P_j(z;B),\tag{93}$$

where $P_j(\,\cdot\,;B)$ is a polynomial. For $s=0$ the remainder P_j is absent, whereas for $s\geq 1$ it is a nonzero element of $\mathbb{P}_{s-1}$.

To see this, divide (93) by $(z\mathbf{1}_q+\boldsymbol{a})_j/(z\mathbf{1}_r+\boldsymbol{b})_j$, which gives

$$P_j(z;B)=\frac{(\boldsymbol{b}-B\mathbf{1}_r)_j}{(\boldsymbol{a}-B\mathbf{1}_q)_j}\frac{1}{z+B}-\sum_{h=1}^{r}\frac{\tau^{H,K}_{h,j}}{z+b_h+j}-\frac{(\boldsymbol{b}-B\mathbf{1}_r)_{j+1}}{(\boldsymbol{a}-B\mathbf{1}_q)_{j+1}}\frac{(z\mathbf{1}_q+\boldsymbol{a}+j\mathbf{1}_q)_1}{(z\mathbf{1}_r+\boldsymbol{b}+j\mathbf{1}_r)_1}\frac{1}{z+B}.\tag{94}$$

At $z=-B$ the residue of the first term is $(\boldsymbol{b}-B\mathbf{1}_r)_j/(\boldsymbol{a}-B\mathbf{1}_q)_j$, while that of the third term is

$$-\frac{(\boldsymbol{b}-B\mathbf{1}_r)_{j+1}}{(\boldsymbol{a}-B\mathbf{1}_q)_{j+1}}\frac{(\boldsymbol{a}-B\mathbf{1}_q+j\mathbf{1}_q)_1}{(\boldsymbol{b}-B\mathbf{1}_r+j\mathbf{1}_r)_1}=-\frac{(\boldsymbol{b}-B\mathbf{1}_r)_j}{(\boldsymbol{a}-B\mathbf{1}_q)_j},$$

so the two cancel. At each node $z=-b_h-j$ the residue of the third term equals $-\tau^{H,K}_{h,j}$, which cancels the residue of the corresponding summand in the middle term. Hence the right-hand side of (94) has no poles and is a polynomial. Its degree is controlled by the only term that grows at infinity, $(z\mathbf{1}_q+\boldsymbol{a}+j\mathbf{1}_q)_1(z+B)^{-1}/(z\mathbf{1}_r+\boldsymbol{b}+j\mathbf{1}_r)_1$: the numerator has degree $q=r+s$ and the denominator degree $r+1$, so the growth is $(r+s)-(r+1)=s-1$. Thus $P_j(\,\cdot\,;B)\in\mathbb{P}_{s-1}$.

Since $\{(z+j)_{\ell-1}\,:\,\ell=1,\dots,s\}$ is a basis of $\mathbb{P}_{s-1}$, there are scalars $\gamma^{H,K}_{\ell,j}$ with $P_j(z;B)=\sum_{\ell=1}^{s}\gamma^{H,K}_{\ell,j}(z+j)_{\ell-1}$, whence, by (86),

$$\frac{(z\mathbf{1}_q+\boldsymbol{a})_j}{(z\mathbf{1}_r+\boldsymbol{b})_j}P_j(z;B)=\sum_{\ell=1}^{s}\gamma^{H,K}_{\ell,j}\,\widetilde{Q}^{\mathrm{L}}_{\ell,j}(z).$$

The remainder is therefore an Euler symbol and belongs to the Euler-derivative block.

Summing (93) over $j=0,\dots,K-1$, the terms $\mathscr{E}_{j+1}(z;B)/(z+B)$ telescope. Using $\mathscr{E}_0(z;B)=1$ and $B=b_H+K$, the terminal term is

$$\frac{\mathscr{E}_K(z;B)}{z+B}=\tau^{H,K}_{H,K}\frac{(z\mathbf{1}_q+\boldsymbol{a})_K}{(z\mathbf{1}_r+\boldsymbol{b})_K\,(z+b_H+K)},$$

which is precisely the $h = H,\ j = K$ summand of (91). Collecting all shifted-beta summands into $\mathscr{W}^{\mathrm{L}}_{H,K}$ yields

$$\frac{1}{z + b_H + K} = \mathscr{W}^{\mathrm{L}}_{H,K}(z) + \sum_{j=0}^{K-1} \sum_{\ell=1}^{s} \gamma^{H,K}_{\ell,j}\, \widetilde{Q}^{\mathrm{L}}_{\ell,j}(z), \tag{95}$$

which is (92). The Euler symbols used here lie within the finite system: (64c) gives $sK \le s(n_H - 1) \le |\boldsymbol{\eta}|$, and since $\boldsymbol{\eta}$ is on the step-line this forces $K \le \min_\ell \eta_\ell$; hence $j \le K - 1 \le \eta_\ell - 1$ for every ℓ.

Conclusion. Insert (95) into the effective decomposition (90). Multiplying by $\widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K}$ and summing over the finite nodes gives

$$R^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}(z) = \sum_{\substack{H=1\\ n_H \ge 1}}^{r} \sum_{K=0}^{n_H-1} \widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K}\, \mathscr{W}^{\mathrm{L}}_{H,K}(z) + \sum_{\ell=1}^{s} \sum_{k=0}^{\eta_\ell-1} \widetilde{\delta}_{\ell,k}\, \widetilde{Q}^{\mathrm{L}}_{\ell,k}(z),$$

where $\widetilde{\delta}_{\ell,k}$ are the coefficients $\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}$ corrected by the Euler contributions (95). By (91) the first sum is exactly the reduced symbol of $\sum_h B^{(h)} w_h$ with $B^{(h)} = \sum_{H,K} \widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K} \mathscr{R}^{\mathrm{L}}_{h;H,K}$, and the second is the reduced symbol of an Euler-derivative block. This exhibits $R^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}$ as the reduced symbol of a mixed form of the type (70). By the isomorphism of Step 5 its component vector is unique; hence the shifted-beta components are (75), and the remaining coordinates are the components multiplying the weights v_ℓ. This proves (75).

Step 7: Meijer and hypergeometric representations of the complete B-form.

We derive the Meijer G-representation of the complete B-form. Using

$$(\beta_i + 1 - z)_{m_i} = \frac{\Gamma(\beta_i + m_i + 1 - z)}{\Gamma(\beta_i + 1 - z)},$$

the Mellin transform (85) can be written as

$$\mathscr{M}[\mathscr{B}^{\mathrm{L}}_{\lambda,\boldsymbol{m}}](z) = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})\Gamma(\boldsymbol{\beta} + \boldsymbol{m} + (1-z)\mathbf{1}_p)}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})\Gamma(\boldsymbol{\beta} + (1-z)\mathbf{1}_p)}.$$

With the Mellin–Barnes convention fixed in (2)–(3), this is the Mellin transform of

$$G^{q,p}_{p+r,p+q}\left(x \,\middle|\, \begin{matrix} -\boldsymbol{\beta} - \boldsymbol{m},\ \boldsymbol{b} + \boldsymbol{n} \\ \boldsymbol{a},\ -\boldsymbol{\beta} \end{matrix}\right).$$

This proves (74).

We next give the complete hypergeometric expression of the linear form and account for the resonant parameters excluded in part (ii). The Meijer G-representation (74) is defined, and equals the complete B-form, for *every* admissible parameter choice considered here: it has a finite analytic continuation in the parameters $\boldsymbol{a}$ across the coalescence hyperplanes relevant here, and the residue hypothesis $a_\rho - a_\sigma \notin \mathbb{Z}$ of part (ii) serves only to keep the poles of the Mellin transform simple, so that the inversion integral can be summed termwise.

Apply Mellin inversion to (85), closing the contour to the left and collecting the residues of

$$\frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})\,\Gamma(\boldsymbol{\beta} + \boldsymbol{m} + (1-z)\mathbf{1}_p)}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})\,\Gamma(\boldsymbol{\beta} + (1-z)\mathbf{1}_p)}\, x^{-z}$$

at the poles $z = -a_\rho - k$, $\rho \in \{1, \dots, q\}$, $k \in \mathbb{N}_0$, contributed by $\Gamma(z\mathbf{1}_q + \boldsymbol{a})$. When $a_\rho - a_\sigma \notin \mathbb{Z}$ every such pole is simple, with $\operatorname{Res}_{z=-a_\rho-k} \Gamma(z + a_\rho) = (-1)^k/k!$, and summing the resulting family over k yields the ${}_{r+p}F_{q+p-1}$ series (73). The argument $(-1)^s x$, and its sign, come from the excess $q - r$, namely s, of numerator gamma factors over the beta-shifted denominator factors.

Coalescent parameters. If $a_\sigma - a_\rho = \nu \in \mathbb{Z}$ for some pair, the towers $\{-a_\rho - k\}_k$ and $\{-a_\sigma - k\}_k$ overlap and the corresponding poles merge. The right-hand side of (73) is then to be read as the limit $a_\sigma \to a_\rho + \nu$ of its values at nearby non-resonant parameters; this limit exists and is finite, because by the first paragraph the complete form is the Meijer G-function (74), with a finite analytic continuation in

$\boldsymbol{a}$ across the relevant coalescence hyperplanes. Equivalently, one inverts the Mellin transform directly. Write its Mellin–Barnes integrand as

$$\mathcal{I}(z;x) := \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})\Gamma(\boldsymbol{\beta} + \boldsymbol{m} + (1-z)\mathbf{1}_p)}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})\Gamma(\boldsymbol{\beta} + (1-z)\mathbf{1}_p)} x^{-z}.$$

A point at which j of the poles $-a_\rho - k$ coincide is a pole of order at most j, equal to j unless a reciprocal gamma factor cancels part of the singularity, and its contribution is the exact residue

$$\operatorname*{Res}_{z=z_0} \mathcal{I}(z;x) = \frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\left[(z-z_0)^j \mathcal{I}(z;x)\right]_{z=z_0}. \tag{96}$$

Since $\dfrac{\mathrm{d}}{\mathrm{d}z}x^{-z} = -x^{-z}\log x$ and the derivatives of the surviving gamma factors produce digamma values, (96) contributes a term of the form $x^{a_\rho+k}P_{j-1}(\log x)$, with P_{j-1} a polynomial of degree at most $j-1$ in $\log x$. In particular, an uncancelled two-fold coincidence ($j=2$) replaces the two coalescing ${}_{r+p}F_{q+p-1}$ series by a logarithmic reduction-of-order solution carrying one power of $\log x$. This determines (73) for all admissible parameters. □

Remark 4.6 (Kampé de Fériet reduction and the confluent correction). The formula (75) gives the shifted-beta components as finite sums of terminating ${}_{r+1}F_q$ polynomials. There is a structural reason why the Jacobi-like Kampé de Fériet reduction does not extend to these formulas in a comparably compact uniform form. For each Jacobi pole family, expansion of the terminating hypergeometric block followed by the substitution $K = u + \lambda$ produces coefficients made from fixed Pochhammer strings in u, λ, and $u+\lambda$; this is exactly a classical Kampé de Fériet polynomial.

In the Laguerre-like case the coefficient of the block indexed by (H,K) is instead

$$\widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K} = \rho^{\mathscr{B},\mathrm{L}}_{H,K} - \sum_{\ell=1}^{s}\sum_{k=K+1}^{\eta_\ell-1}\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}\sigma^{(\ell,k)}_{H,K}.$$

Thus the substitution $K = u+\lambda$ leaves an additional confluent index $k > u+\lambda$, whose coefficients $\delta^{\mathscr{B},\mathrm{L}}_{\ell,k}$ are obtained from the inverse unitriangular connection system (87). The result is a triangular nested sum, rather than one double hypergeometric series with fixed parameter strings. Reordering the triangular correction does not remove the connection coefficients and gives no compression of the formula. The finite sum in (75), together with the connection system, is therefore the natural uniform closed form.

The correction from $\rho^{\mathscr{B},\mathrm{L}}_{H,K}$ to $\widehat{\rho}^{\mathscr{B},\mathrm{L}}_{H,K}$ is the contribution of the polynomial part at infinity of $R^{\mathscr{B},\mathrm{L}}_{\lambda,\boldsymbol{m}}$. If $|\boldsymbol{\eta}| \le s$, this correction is absent, and the finite-pole residues alone give the shifted-beta components; in this shallow case the shifted-beta part may be regrouped by the same double-index argument. This does not provide a formula for the full Laguerre-like component vector, because the Euler-derivative components remain to be determined. For deep indices, $|\boldsymbol{\eta}| > s$, the correction is necessary; it is encoded explicitly by (87)– (89) through the confluent extraction symbol $\widetilde{Q}^{\mathrm{L}}_{\ell,k}$. The remaining components $D^{(\ell)}_{\lambda,\boldsymbol{m}}$, which multiply v_ℓ, are then determined, together with the displayed B-components, by the finite identity (84).

Remark 4.7 (The two pure boundaries for $q > 1$). The preceding obstruction is specific to the mixed beta–Euler case $r, s > 0$. The two pure boundaries have different structures.

If the formal boundary $s = 0$, $r = q$, is allowed, the gamma block is absent and the weight vector is exactly the Jacobi-like vector. Under the index identification

$$\boldsymbol{n}^{\mathrm{J}} = \boldsymbol{m}^{\mathrm{L}}, \qquad \boldsymbol{m}^{\mathrm{J}} = \boldsymbol{n}^{\mathrm{L}}, \qquad \boldsymbol{\alpha} = \boldsymbol{\beta},$$

the component formula is therefore the Jacobi-like formula, and Corollary 3.11 gives its terminating Kampé de Fériet representation.

If $r = 0$ and $s = q$, all components belong to the pure Euler block. In this case the polynomial system is a block Newton interpolation problem. The next proposition gives the resulting terminating hypergeometric expression for every component.

Proposition 4.8 (Terminating hypergeometric components at the pure Euler boundary)**.** *Assume $r = 0$, $s = q > 1$, and put*

$$P(z) := (\boldsymbol{\beta} + (1-z)\mathbf{1}_p)_{\boldsymbol{m}}, \qquad A_k(z) := (z\mathbf{1}_q + \boldsymbol{a})_k.$$

For $0 \le k \le \eta_\ell - 1$, set

$$\boldsymbol{K}^{k,\ell} := \left(K_1^{k,\ell}, \dots, K_q^{k,\ell}\right), \qquad K_\sigma^{k,\ell} := k + \begin{cases} 1, & \sigma \le \ell, \\ 0, & \sigma > \ell, \end{cases} \qquad n_\rho^{k,\ell} := K_\rho^{k,\ell} - 1,$$

and abbreviate

$$\boldsymbol{B}_\rho := (B_{1\rho}, \dots, B_{p\rho}), \qquad B_{i\rho} := \beta_i + a_\rho + 1, \qquad \boldsymbol{C}_\rho := (C_{1\rho}, \dots, C_{q\rho}), \qquad C_{\sigma\rho} := a_\sigma - a_\rho.$$

Suppose first that $a_\rho - a_\sigma \notin \mathbb{Z}$ for $\rho \ne \sigma$. Define

$$c_{k,\ell} := \sum_{\substack{\rho=1 \\ K_\rho^{k,\ell} \ge 1}}^{q} \frac{(\boldsymbol{B}_\rho)_{\boldsymbol{m}}}{n_\rho^{k,\ell}!\,(\boldsymbol{C}_\rho^{*\rho})_{(\boldsymbol{K}^{k,\ell})^{*\rho}}} \times {}_{p+q}F_{p+q-1}\left(\begin{matrix} -n_\rho^{k,\ell},\ \boldsymbol{B}_\rho + \boldsymbol{m},\ \mathbf{1}_{q-1} - \boldsymbol{C}_\rho^{*\rho} - (\boldsymbol{K}^{k,\ell})^{*\rho} \\ \boldsymbol{B}_\rho,\ \mathbf{1}_{q-1} - \boldsymbol{C}_\rho^{*\rho} \end{matrix}; 1\right). \tag{97}$$

The series terminates at $n_\rho^{k,\ell}$. Extend the definition by $c_{k,\ell} = 0$ when $k \ge \eta_\ell$, and let

$$d_{h,k} := \sum_{\ell=h}^{q} e_{\ell-h}(a_1, \dots, a_{\ell-1}) c_{k,\ell}, \qquad h \in \{1, \dots, q\},$$

where e_j denotes the elementary symmetric polynomial and $e_0 = 1$. Then the pure Euler components are

$$D_{\boldsymbol{\eta},\boldsymbol{m}}^{(h)}(x) = \sum_{k=0}^{\eta_h - 1} d_{h,k} x^k, \qquad h \in \{1, \dots, q\}. \tag{98}$$

If two lattice nodes coalesce, the same formula is understood by continuity, equivalently by replacing ordinary divided differences with confluent ones.

Proof. Order the lattice nodes as

$$-a_1, \dots, -a_q,\ -a_1 - 1, \dots, -a_q - 1, \dots.$$

Because $\boldsymbol{\eta}$ is on the step-line, the pairs (k,ℓ) with $0 \le k \le \eta_\ell - 1$ form the first $N = |\boldsymbol{\eta}|$ positions in this order. Let $c_{k,\ell}$ be the divided difference of P at all nodes up to and including $-a_\ell - k$. Newton interpolation gives

$$P(z) = \sum_{\ell=1}^{q} \sum_{k=0}^{\eta_\ell - 1} c_{k,\ell} A_k(z) \prod_{\sigma=1}^{\ell-1} (z + a_\sigma + k). \tag{99}$$

The node polynomial used in this divided difference is

$$\Omega_{k,\ell}(z) = A_k(z) \prod_{\sigma=1}^{\ell} (z + a_\sigma + k) = \prod_{\sigma=1}^{q} (z + a_\sigma)_{K_\sigma^{k,\ell}}.$$

Its simple roots are $-a_\rho - j$, with $0 \le j \le K_\rho^{k,\ell} - 1$, and the barycentric formula yields

$$c_{k,\ell} = \sum_{\substack{\rho=1 \\ K_\rho^{k,\ell} \ge 1}}^{q} \sum_{j=0}^{K_\rho^{k,\ell} - 1} \frac{(\boldsymbol{\beta} + (a_\rho + j + 1)\mathbf{1}_p)_{\boldsymbol{m}}}{(-1)^j j!\,(K_\rho^{k,\ell} - 1 - j)! \prod\limits_{\substack{\sigma=1 \\ \sigma \ne \rho}}^{q} (a_\sigma - a_\rho - j)_{K_\sigma^{k,\ell}}}. \tag{100}$$

Use

$$(B + j)_m = (B)_m \frac{(B + m)_j}{(B)_j}, \qquad (C - j)_K = (C)_K \frac{(1 - C)_j}{(1 - C - K)_j},$$

together with

$$\frac{1}{(-1)^j j!(n-j)!} = \frac{(-n)_j}{n!j!}.$$

Substitution in (100) gives exactly (97).

Finally, with $t = z + k$,

$$\prod_{\sigma=1}^{\ell-1}(t + a_\sigma) = \sum_{h=1}^{\ell} e_{\ell-h}(a_1, \dots, a_{\ell-1}) t^{h-1}.$$

Insert this expansion in (99) and collect the powers $(z+k)^{h-1}$. The result is

$$P(z) = \sum_{h=1}^{q} \sum_{k=0}^{\eta_h - 1} d_{h,k}(z\mathbf{1}_q + \boldsymbol{a})_k (z+k)^{h-1},$$

which is the specialization of (84) to $r = 0$. Uniqueness of that expansion proves (98). Multiplication by the common Mellin factor $\Gamma(z\mathbf{1}_q + \boldsymbol{a})$ gives

$$\mathscr{M}[\mathscr{B}^{\mathrm{L}}_{\boldsymbol{\eta},\boldsymbol{m}}](z) = \Gamma(z\mathbf{1}_q + \boldsymbol{a})(\boldsymbol{\beta} + (1-z)\mathbf{1}_p)_{\boldsymbol{m}}.$$

At $z = \beta_i + j + 1$, with $0 \le j \le m_i - 1$, the factor $(\beta_i + 1 - z)_{m_i} = (-j)_{m_i}$ vanishes. Hence the components in (98) satisfy all the required orthogonality conditions. □

Remark 4.9 (Operator interpretation of the pure Euler formula). At $r = 0$, Mellin transformation gives

$$\mathscr{B}^{\mathrm{L}}_{\boldsymbol{\eta},\boldsymbol{m}} = P(\mathscr{D})w_0, \qquad xw_0 = \prod_{\rho=1}^{q}(\mathscr{D} + a_\rho)w_0.$$

Thus (98) is the normal remainder of $P(\mathscr{D})$ in the ordered basis $x^k \mathscr{D}^{h-1} w_0$. It is a finite sum with terminating ${}_{p+q}F_{p+q-1}(1)$ coefficients, rather than a single classical Kampé de Fériet polynomial in x.

Remark 4.10 (Beyond the near-diagonal range). We use the near-diagonal range for the Laguerre-like mixed system because the Mellin cancellations are then automatic and the resulting $\mathscr{A}$- and $\mathscr{B}$-forms admit terminating hypergeometric representations. Outside this range, the same contour and finite reconstruction formulas may still be applied under additional finite cancellation conditions. The corresponding formulas are less involved, especially for the $\mathscr{B}$-form, where auxiliary polynomial reconstructions may involve extra components.

We therefore restrict the Laguerre-like results to the near-diagonal range. The finite-cancellation extension beyond this range is not used below.

Remark 4.11 (Components versus the complete form). The individual polynomials $B^{(h)}$ and $D^{(\ell)}$ are obtained from the finite identity (84). In general they are not single hypergeometric polynomials in one component. The hypergeometric formula applies to the complete linear form $\mathscr{B}$, rather than to each component separately. The pure Euler boundary is the explicit exception described in Proposition 4.8: its components are finite sums with terminating hypergeometric coefficients.

Remark 4.12 (Ordinary Laguerre-like situation). Take $p = 1$. If, for simplicity, $\beta_1 = 0$, then the power vector is the scalar vector (1), and the mixed matrix (63) reduces to the Laguerre-like vector itself. In part (ii) of Theorem 4.5, the condition $|\boldsymbol{m}| = N - 1$ means $m_1 = N - 1$, and the normalization becomes

$$\int_0^\infty x^{z-1}\mathscr{B}^{\mathrm{L}}_{\lambda,N-1}(x)\,\mathrm{d}x = \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}(1-z)_{N-1}.$$

Consequently,

$$\int_0^\infty x^k \mathscr{B}^{\mathrm{L}}_{\lambda,N-1}(x)\,\mathrm{d}x = 0, \qquad k \in \{0, \dots, N-2\},$$

which is the ordinary Laguerre-like type I normalization. The dual formula in part (i) has a single power component. With $m_1 = N+1$, one obtains

$$A^{(1)}_{\lambda,N+1}(x) = -\frac{\Gamma(\boldsymbol{b}+\boldsymbol{n}+\mathbf{1}_r)}{N!\,\Gamma(\boldsymbol{a}+\mathbf{1}_q)}\,{}_{r+1}F_q\left(\begin{matrix}-N,\ \boldsymbol{b}+\boldsymbol{n}+\mathbf{1}_r\\ \boldsymbol{a}+\mathbf{1}_q\end{matrix};x\right).$$

Thus the mixed theorem collapses to Wolfs's ordinary Laguerre-like pair of type I and type II formulae, up to the harmless global normalizations fixed here.

Example 4.13 (The smallest confluent mixed form)**.** Let $q=2$, $r=1$ and $s=1$, so that

$$\mathbf{U}^{\mathrm{L}} = (w_1, v_1), \qquad v_1 = w_0.$$

Take $\boldsymbol{n} = (1)$, $\boldsymbol{\eta} = (1)$, hence $N = 2$, and let the power vector consist of the single function x^β, with $p = 1$ and $\boldsymbol{m} = (1)$. Since

$$B^{(1)} \in \mathbb{P}_{n_1-1} = \mathbb{P}_0, \qquad D^{(1)} \in \mathbb{P}_{\eta_1-1} = \mathbb{P}_0,$$

the mixed form on the Laguerre-like vector has the form

$$\mathscr{B}^{\mathrm{L}}_{\beta}(x) = b_0 w_1(x) + d_0 v_1(x).$$

The finite identity (84) reduces to

$$b_0 + d_0(z+b) = \beta + 1 - z.$$

Therefore

$$d_0 = -1, \qquad b_0 = \beta + b + 1,$$

and

$$\mathscr{B}^{\mathrm{L}}_{\beta}(x) = (\beta+b+1)w_1(x) - v_1(x). \tag{101}$$

Its Mellin transform is

$$\int_0^\infty x^{z-1}\mathscr{B}^{\mathrm{L}}_{\beta}(x)\,\mathrm{d}x = \frac{\Gamma(z+a_1)\Gamma(z+a_2)}{\Gamma(z+b+1)}(\beta+1-z),$$

and hence

$$\int_0^\infty x^{\beta}\mathscr{B}^{\mathrm{L}}_{\beta}(x)\,\mathrm{d}x = 0.$$

Thus, already in the first confluent case, the form expanded in the Laguerre-like vector is a nontrivial linear combination of the beta-shifted and Euler-generated weights.

Example 4.14 (The exponential-integral specialization)**.** In Example 4.13, choose

$$a_1 = 0, \qquad a_2 = b, \qquad b > 0.$$

Then

$$\mathscr{M}[w_0](z) = \frac{\Gamma(z)\Gamma(z+b)}{\Gamma(z+b)} = \Gamma(z),$$

so

$$w_0(x) = \mathrm{e}^{-x}.$$

Moreover,

$$\mathscr{M}[w_1](z) = \frac{\Gamma(z)\Gamma(z+b)}{\Gamma(z+b+1)} = \frac{\Gamma(z)}{z+b}.$$

With the standard notation

$$E_{\nu+1}(x) = \int_1^\infty \mathrm{e}^{-xt}t^{-\nu-1}\,\mathrm{d}t,$$

one has

$$\int_0^\infty x^{z-1}E_{\nu+1}(x)\,\mathrm{d}x = \frac{\Gamma(z)}{z+\nu},$$

and therefore

$$w_1(x) = E_{b+1}(x), \qquad v_1(x) = \mathrm{e}^{-x}.$$

Formula (101) becomes the explicit mixed exponential-integral identity

$$\mathscr{B}^{\mathrm{L}}_{\beta}(x) = (\beta + b + 1)E_{b+1}(x) - \mathrm{e}^{-x},$$

with

$$\int_0^{\infty} x^{\beta}\big((\beta + b + 1)E_{b+1}(x) - \mathrm{e}^{-x}\big)\,\mathrm{d}x = 0.$$

This gives the first-order member of the exponential-integral specialization of the Laguerre-like construction.

Example 4.15 (A rectangular calculation)**.** Keep $q = 2$, $r = 1$, $s = 1$, but now take two powers

$$\mathbf{V} = (x^{\beta_1}, x^{\beta_2}).$$

Choose

$$\boldsymbol{n} = (2), \qquad \boldsymbol{\eta} = (1), \qquad N = 3, \qquad \boldsymbol{m} = (1,1).$$

Then the form on the Laguerre-like vector has the shape

$$\mathscr{B}^{\mathrm{L}}(x) = (b_0 + b_1 x)w_1(x) + d_0 v_1(x). \tag{102}$$

The identity (84) becomes

$$b_0(z + b + 1) + b_1(z + a_1)(z + a_2) + d_0(z + b)(z + b + 1) = (\beta_1 + 1 - z)(\beta_2 + 1 - z).$$

Assume

$$\Delta := (a_1 - b - 1)(a_2 - b - 1) \neq 0.$$

Comparison of the coefficients of z^2, z, and the constant term gives

$$\begin{aligned} b_1 &= \frac{(b + \beta_1 + 2)(b + \beta_2 + 2)}{\Delta}, & d_0 &= 1 - b_1,\\ b_0 &= \frac{(\beta_1 + 1)(\beta_2 + 1) - b_1 a_1 a_2 - d_0 b(b + 1)}{b + 1}. \end{aligned}$$

Thus (102) is completely explicit. Its Mellin transform is

$$\frac{\Gamma(z + a_1)\Gamma(z + a_2)}{\Gamma(z + b + 2)}(\beta_1 + 1 - z)(\beta_2 + 1 - z),$$

and therefore

$$\int_0^{\infty} x^{\beta_j}\mathscr{B}^{\mathrm{L}}(x)\,\mathrm{d}x = 0, \qquad j \in \{1, 2\}.$$

This is the first case where the rectangular nature of the construction is visible: one combines two Laguerre-like functions against two independent powers.

Remark 4.16 (A formal degeneration to the Piñeiro system). The target of this formal degeneration is the mixed Piñeiro system of [3]. The Laguerre-like setting considered above assumes $s \geq 1$, so that a gamma block remains in the Mellin transform and the corresponding weights are supported on $(0, \infty)$. If one formally allows $s = 0$, then $q = r$ and the gamma block disappears. If, in addition,

$$\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{\gamma},$$

then

$$\frac{\Gamma(z\mathbf{1}_q + \boldsymbol{\gamma})}{\Gamma(z\mathbf{1}_q + \boldsymbol{\gamma} + \boldsymbol{e}_j)} = \frac{1}{z + \gamma_j} = \int_0^1 x^{z-1}x^{\gamma_j}\,\mathrm{d}x, \qquad j \in \{1, \dots, q\}.$$

Thus the resulting weights are $x^{\gamma_j}\mathbf{1}_{(0,1)}(x)$, and the matrix of measures becomes, equivalently,

$$\big[x^{\beta_i + \gamma_j}\big]_{\substack{j \in \{1,\dots,q\}\\ i \in \{1,\dots,p\}}}\,\mathrm{d}x, \qquad x \in (0, 1),$$

which is the mixed Piñeiro matrix studied in [3]. Hence this formal specialization does not remain in the Laguerre-like setting with $s \geq 1$; it degenerates to the Jacobi–Piñeiro setting.

Remark 4.17 (Confluent and exponential-integral cases). For $(q, r, s) = (2, 1, 1)$, the Laguerre-like vector has one shifted beta weight and one Mellin-Euler derivative weight,

$$\mathbf{U}^{\mathrm{L}} = (w_1, v_1), \qquad v_1 = w_0.$$

The base weight

$$w_0 = \mathcal{B}_{a_1,b_1} * \mathcal{G}_{a_2}$$

has Mellin transform

$$\mathcal{M}[w_0](z) = \frac{\Gamma(z + a_1)\Gamma(z + a_2)}{\Gamma(z + b_1)}.$$

Equivalently, up to normalization,

$$w_0(x) = x^{a_2} \mathrm{e}^{-x} U(b_1 - a_1, a_2 - a_1 + 1, x),$$

so this case contains the confluent hypergeometric weights appearing in the work of Lima and Loureiro [10], after matching parameters and normalizations.

The non-mixed exponential-integral weight system of Van Assche and Wolfs [17] is obtained as a boundary degeneration of the same $(q, r, s) = (2, 1, 1)$ block. Indeed, taking

$$a_2 = \alpha, \qquad a_1 = \alpha + \nu, \qquad b_1 = a_1$$

gives

$$\mathcal{M}[w_0](z) = \Gamma(z + \alpha), \qquad \mathcal{M}[w_1](z) = \frac{\Gamma(z + \alpha)}{z + \alpha + \nu}.$$

These are the Mellin transforms of $x^\alpha \mathrm{e}^{-x}$ and $x^\alpha E_{\nu+1}(x)$, respectively. Thus the non-mixed Van Assche–Wolfs pair appears as a boundary degeneration of the beta block, not as an interior case under the strict assumption $a_1 < b_1$.

The mixed exponential-integral system of Van Assche and Wolfs is also recovered from the present mixed construction, again as a boundary degeneration. Namely, take

$$(q, r, s) = (2, 1, 1), \qquad p = 2, \qquad (a_1, a_2, b_1) = (\nu, 0, \nu), \qquad \boldsymbol{\beta} = (\alpha, \beta).$$

Then

$$v_1(x) = \mathrm{e}^{-x}, \qquad w_1(x) = E_{\nu+1}(x),$$

up to the normalization of $E_{\nu+1}$. Hence the mixed matrix

$$\mathrm{d}\boldsymbol{\mu}^{\mathrm{L}}(x) = \begin{bmatrix} w_1(x)x^\alpha & w_1(x)x^\beta \\ v_1(x)x^\alpha & v_1(x)x^\beta \end{bmatrix} \mathrm{d}x$$

is, after interchanging the two rows,

$$\begin{bmatrix} \mathrm{e}^{-x}x^\alpha & \mathrm{e}^{-x}x^\beta \\ E_{\nu+1}(x)x^\alpha & E_{\nu+1}(x)x^\beta \end{bmatrix} \mathrm{d}x.$$

This is the mixed exponential-integral matrix built from the weight vector $(\mathrm{e}^{-x}, E_{\nu+1})$ and the power vector (x^α, x^β). In the present construction the Laguerre-like vector is ordered as $(w_1, v_1) = (E_{\nu+1}, \mathrm{e}^{-x})$, whereas the usual exponential-integral order is $(\mathrm{e}^{-x}, E_{\nu+1})$. Thus the underlying matrix of weights is identified after a row permutation. At the level of the mixed orthogonality problems, one must also permute the corresponding components of the multi-index.

Remark 4.18 (Zhang's Bessel I/K system and the 2×2 Laguerre-like case). Zhang's mixed system associated with modified Bessel functions [19] is also related to the Laguerre-like level, but in a different way from the exponential-integral system of Van Assche and Wolfs. The relevant Laguerre-like specialization has

$$r = 0, \qquad s = 2, \qquad q = 2, \qquad p = 2.$$

Choose the Laguerre-like seed so that

$$\mathcal{M}[w_0](z) = \Gamma(z)\Gamma(z + \nu).$$

Thus, up to normalization and a dilation of the variable,

$$w_0(x) = x^{\nu/2} K_\nu(2b\sqrt{x}) =: \rho_{\nu,b}(x).$$

The second Laguerre-like component is obtained by applying the Euler operator. With $\mathscr{D} = -x\mathrm{d}/\mathrm{d}x$, the elementary identity for modified Bessel functions gives

$$\mathscr{D}\rho_{\nu,b} = b\rho_{\nu+1,b} - \nu\rho_{\nu,b}.$$

Consequently

$$\operatorname{span}\{w_0, \mathscr{D}w_0\} = \operatorname{span}\{\rho_{\nu,b}, \rho_{\nu+1,b}\},$$

again up to the harmless normalization and dilation. Hence the K-Bessel vector used by Zhang is the Laguerre-like vector of this $q = 2$ specialization, written in an equivalent two-dimensional basis.

The other side of Zhang's mixed system is not the power vector itself but its Bessel ${}_0F_1$ analogue. Namely,

$$\omega_{\mu,a}(x) := x^{\mu/2} I_\mu(2a\sqrt{x}) = \frac{a^\mu x^\mu}{\Gamma(\mu+1)} {}_0F_1\left(\left.\begin{matrix} - \\ \mu+1 \end{matrix}\right| a^2 x\right),$$

and similarly for $\omega_{\mu+1,a}$. Thus $(\omega_{\mu,a}, \omega_{\mu+1,a})$ is a Bessel-hypergeometric deformation of the power vector $(x^\mu, x^{\mu+1})$. After multiplying $\omega_{\mu+i,a}$ by $\Gamma(\mu+i+1)a^{-\mu-i}$, the limit $a \to 0$ is $x^{\mu+i}$, for $i \in \{0,1\}$. Thus the two functions reduce to the two powers after an explicit diagonal normalization.

The precise link is at the level of bimoments. For $0 < a < b$, Zhang uses

$$\int_0^\infty \omega_{\mu+i,a}(x)\rho_{\nu+j,b}(x)\,\mathrm{d}x = \frac{a^{\mu+i} b^{\nu+j}}{2(b^2-a^2)^{\mu+\nu+1+i+j}} \Gamma(\mu+\nu+1+i+j), \qquad i,j \in \mathbb{N}_0.$$

Set

$$c = b^2 - a^2, \qquad \lambda = \mu + \nu + 1,$$

and normalize the shifted Bessel families by

$$\Phi_i(x) = c^i a^{-\mu-i} \omega_{\mu+i,a}(x), \qquad \Psi_j(x) = c^j b^{-\nu-j} \rho_{\nu+j,b}(x).$$

Then

$$\int_0^\infty \Phi_i(x)\Psi_j(x)\,\mathrm{d}x = \frac{1}{2c^\lambda}\Gamma(\lambda+i+j) = \frac{1}{2c^\lambda}\int_0^\infty t^i t^j t^{\lambda-1} \mathrm{e}^{-t}\,\mathrm{d}t.$$

Therefore Zhang's shifted Bessel bases have, up to a constant factor, the same bimoment matrix as the gamma moments associated with the Laguerre-like specialization above.

This does not mean that the component polynomials are literally the same functions of the same variable. Rather, the Laguerre coefficients are written in Bessel bases. If

$$\widehat{L}_n^{(\lambda-1)}(t) = \sum_{i=0}^n \ell_{n,i} t^i$$

is the corresponding Laguerre polynomial, then the form before reducing Bessel orders is

$$\sum_{i=0}^n \ell_{n,i}\Phi_i(x),$$

which, after undoing the normalization, is Zhang's finite expansion in the shifted functions $\omega_{\mu+i,a}$. The final two-component form is obtained by reducing the shifted orders to the basis $(\omega_{\mu,a}, \omega_{\mu+1,a})$. The same argument applies on the dual side, where shifted functions $\rho_{\nu+j,b}$ are reduced to $(\rho_{\nu,b}, \rho_{\nu+1,b})$. The coefficients of these order reductions are the Lommel polynomials, equivalently terminating hypergeometric polynomials. Thus Zhang's 2×2 mixed polynomials are the Laguerre-like moment problem written in the Bessel bases $(\omega_{\mu,a}, \omega_{\mu+1,a})$ and $(\rho_{\nu,b}, \rho_{\nu+1,b})$.

5. Rodrigues formulas

The Mellin-transform identities obtained above for the mixed forms expanded in the weight vector admit a direct differential-operator interpretation. In the ordinary Jacobi-like and Laguerre-like systems of Wolfs, the Mellin transform of the complete type I function contains a single descending Pochhammer factor. In the rectangular mixed extension, the power vector has p components, and the Mellin transform contains one factor

$$(\gamma_i + 1 - s)_{n_i}$$

for each of them. The Rodrigues formula is therefore obtained by composing one differential operator for each entry of the power vector.

The identity concerns the complete mixed form expanded in the weight vector. Only when that vector has a single component does it reduce to a Rodrigues formula for an individual polynomial. In particular, in the Jacobi-like setting the specialization $q = 1$ gives a Rodrigues formula for the Jacobi–Piñeiro type II polynomial, whereas the specialization $p = 1$, with the corresponding power exponent equal to zero, recovers the Rodrigues formula for the ordinary Jacobi-like type I function of Wolfs.

For $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}_0$, define

$$\mathscr{R}_{\gamma,n}[f](x) \coloneqq x^{-\gamma}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{\gamma+n}f(x)\right).$$

Set

$$\vartheta \coloneqq x\frac{\mathrm{d}}{\mathrm{d}x}.$$

Then

$$\mathscr{R}_{\gamma,n} = (\vartheta+\gamma+1)_n. \tag{103}$$

Indeed, both sides act on a monomial x^λ as multiplication by

$$(\lambda+\gamma+1)_n.$$

Consequently, the operators $\mathscr{R}_{\gamma_i,n_i}$ commute. For

$$\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_p), \qquad \boldsymbol{n} = (n_1,\dots,n_p) \in \mathbb{N}_0^p,$$

we may therefore define

$$\mathscr{R}_{\boldsymbol{\gamma},\boldsymbol{n}} \coloneqq \mathscr{R}_{\gamma_1,n_1}\circ\cdots\circ\mathscr{R}_{\gamma_p,n_p}, \tag{104}$$

without regard to the order of the factors.

We use the following elementary Mellin-transform identity. Here and below, $\mathscr{M}$ denotes the Mellin transform on the corresponding real-line support Δ, namely $(0,1)$ in the Jacobi-like setting and $(0,\infty)$ in the Laguerre-like setting.

Proposition 5.1 (Mellin action of the Rodrigues operators)**.** *Let f be a function on Δ such that the Mellin transforms below exist in a common fundamental strip and the endpoint terms produced by the integrations by parts vanish in that strip. Then*

$$\mathscr{M}[\mathscr{R}_{\gamma,n}[f]](s) = (\gamma+1-s)_n\mathscr{M}[f](s). \tag{105}$$

Consequently, if the same condition holds for the intermediate functions appearing in a partial composition of the operators

$$\mathscr{R}_{\gamma_1,n_1},\dots,\mathscr{R}_{\gamma_p,n_p},$$

then

$$\mathscr{M}[\mathscr{R}_{\boldsymbol{\gamma},\boldsymbol{n}}[f]](s) = (\boldsymbol{\gamma}+(1-s)\mathbf{1}_p)_{\boldsymbol{n}}\mathscr{M}[f](s). \tag{106}$$

Proof. By (103), it is enough to determine the Mellin action of ϑ. In the fundamental strip under consideration, and with no endpoint contribution in the integration by parts,

$$\begin{aligned}\mathscr{M}[\vartheta f](s) &= \int_\Delta x^{s-1}x\frac{\mathrm{d}}{\mathrm{d}x}f(x)\,\mathrm{d}x = \int_\Delta x^s\frac{\mathrm{d}}{\mathrm{d}x}f(x)\,\mathrm{d}x = -s\int_\Delta x^{s-1}f(x)\,\mathrm{d}x\\ &= -s\mathscr{M}[f](s).\end{aligned}$$

Hence

$$\begin{aligned}\mathscr{M}[\mathscr{R}_{\gamma,n}[f]](s) &= \mathscr{M}[(\vartheta+\gamma+1)_nf](s)\\ &= (\gamma+1-s)_n\mathscr{M}[f](s),\end{aligned}$$

which proves (105). Since the operators in (104) commute, successive application of this identity gives (106). □

Lemma 5.2 (Endpoint behavior for the Rodrigues formulas)**.** *In the Jacobi-like setting, assume*

$$a_j > -1, \qquad b_j > a_j, \qquad j \in \{1, \dots, q\}.$$

Let $|\boldsymbol{m}| = |\boldsymbol{n}| + 1$*. Then the shifted seed*

$$w_0(x; \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{m})$$

and all functions obtained from it by partial compositions of the Rodrigues operators

$$\mathscr{R}_{\alpha_1,n_1}, \dots, \mathscr{R}_{\alpha_p,n_p}$$

have no endpoint contribution in the integrations by parts used in Proposition 5.1, in their common fundamental Mellin strip.

In the Laguerre-like setting, assume

$$a_\rho > -1, \qquad \rho \in \{1, \dots, q\},$$

and, for the beta-shifted block,

$$b_h > a_h, \qquad h \in \{1, \dots, r\}.$$

Let

$$|\boldsymbol{m}| = |\boldsymbol{n}| + |\boldsymbol{\eta}| - 1.$$

Then the shifted seed

$$w_0(x; \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{n})$$

and all functions obtained from it by partial compositions of the Rodrigues operators

$$\mathscr{R}_{\beta_1,m_1}, \dots, \mathscr{R}_{\beta_p,m_p}$$

have no endpoint contribution in the integrations by parts used in Proposition 5.1, in their common fundamental Mellin strip.

Proof. In the Jacobi-like case the support is $(0,1)$. Near 0, the seed and its partial Rodrigues transforms have power-type behavior, so the Mellin transform is defined in a nonempty vertical strip. Near 1, the shifted seed $w_0(x; \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{m})$ has a beta-type endpoint behavior with total endpoint exponent

$$\sum_{j=1}^{q} (b_j + m_j - a_j) - 1 = \sum_{j=1}^{q} (b_j - a_j) + |\boldsymbol{m}| - 1.$$

Since $|\boldsymbol{m}| = |\boldsymbol{n}| + 1$, this exponent is

$$\sum_{j=1}^{q} (b_j - a_j) + |\boldsymbol{n}|.$$

A partial composition of the Rodrigues operators has order at most $|\boldsymbol{n}|$. Therefore, in every integration by parts occurring before or during the composition, the endpoint exponent at 1 remains positive, because $b_j > a_j$ for every j. Hence the boundary term at 1 vanishes. The boundary term at 0 vanishes in the common fundamental Mellin strip.

In the Laguerre-like case the support is $(0, \infty)$. Near 0, the seed and its partial Rodrigues transforms again have power-type behavior, so there is a nonempty Mellin strip. Near ∞, the gamma block in the Laguerre-like seed gives rapid, generally stretched-exponential decay. Applying finitely many Rodrigues operators only produces finite linear combinations of derivatives multiplied by powers of x, and therefore preserves sufficiently rapid decay at infinity. Hence the boundary term at ∞ vanishes. The boundary term at 0 vanishes in the common fundamental Mellin strip. □

5.1. **Jacobi-like mixed Rodrigues formula.** The Jacobi-like Mellin identity for the mixed form expanded in the Jacobi-like vector contains one factor

$$(\alpha_i + 1 - s)_{n_i}$$

for each component x^{α_i} of the power vector. By Proposition 5.1, these factors are generated by applying the composed operator $\mathcal{R}_{\boldsymbol{\alpha},\boldsymbol{n}}$ to the seed weight whose parameter vector $\boldsymbol{b}$ has been shifted by $\boldsymbol{m}$.

In this subsection, we write $\mathcal{B}^{\mathrm{J}}_{\boldsymbol{n},\boldsymbol{m}}$ for the form $\mathcal{B}_{\boldsymbol{n},\boldsymbol{m}}$ introduced in Theorem 3.10, in order to distinguish it from its Laguerre-like analogue below.

Theorem 5.3 (Rodrigues formula for the Jacobi-like mixed form)**.** *Assume the hypotheses of Theorem 3.10*(ii)*, so that*

$$|\boldsymbol{m}| = |\boldsymbol{n}| + 1.$$

Assume also

$$a_j > -1, \qquad b_j > a_j, \qquad j \in \{1, \dots, q\}.$$

Then the normalized mixed form expanded in the Jacobi-like vector is given by

$$\mathcal{B}^{\mathrm{J}}_{\boldsymbol{n},\boldsymbol{m}}(x) = -\mathcal{R}_{\boldsymbol{\alpha},\boldsymbol{n}}\left[w_0(x; \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{m})\right]. \tag{107}$$

Equivalently,

$$\mathcal{B}^{\mathrm{J}}_{\boldsymbol{n},\boldsymbol{m}}(x) = -\left(\mathcal{R}_{\alpha_1,n_1} \circ \cdots \circ \mathcal{R}_{\alpha_p,n_p}\right)\left[w_0(x; \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{m})\right].$$

Proof. By (24), with $\boldsymbol{b}$ replaced by $\boldsymbol{b} + \boldsymbol{m}$, we have

$$\mathcal{M}[w_0(\,\cdot\,; \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{m})](s) = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b} + \boldsymbol{m})}.$$

By Lemma 5.2, the integrations by parts required in Proposition 5.1 have no endpoint contributions for this seed and its partial Rodrigues transforms. Therefore, Proposition 5.1 yields

$$\begin{aligned}&\mathcal{M}\left[-\mathcal{R}_{\boldsymbol{\alpha},\boldsymbol{n}}\left[w_0(\,\cdot\,; \boldsymbol{a}, \boldsymbol{b} + \boldsymbol{m})\right]\right](s)\\ &\quad= -\frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_q + \boldsymbol{b} + \boldsymbol{m})}(\boldsymbol{\alpha} + (1-s)\mathbf{1}_p)_{\boldsymbol{n}}.\end{aligned}$$

The right-hand side is precisely the Mellin transform prescribed in (37) for $\mathcal{B}^{\mathrm{J}}_{\boldsymbol{n},\boldsymbol{m}}$. Since both sides have the same Mellin transform in the present class of functions, uniqueness of the Mellin transform gives (107). □

The preceding identity is a Rodrigues formula for the complete mixed form. Its two boundary specializations recover familiar polynomial or type I formulas. When $q = 1$, the Jacobi-like vector has one component, so the complete form is one polynomial times one weight; the corresponding polynomial is of Jacobi–Piñeiro type. When $p = 1$ and the unique power exponent is zero, the composed operator reduces to a single ordinary derivative operator and the formula recovers the ordinary Jacobi-like type I Rodrigues identity of Wolfs.

Corollary 5.4 (Jacobi–Piñeiro polynomial Rodrigues formula)**.** *In the specialization $q = 1$, the mixed form expanded in the Jacobi-like vector has a single component:*

$$\mathcal{B}^{\mathrm{J}}_{\boldsymbol{n},m}(x) = B^{\mathrm{J},(1)}_{\boldsymbol{n},m}(x) w_1(x; a, b).$$

Consequently, for $m = |\boldsymbol{n}| + 1$,

$$B^{\mathrm{J},(1)}_{\boldsymbol{n},m}(x) = -\frac{\mathcal{R}_{\boldsymbol{\alpha},\boldsymbol{n}}\left[w_0(x; a, b + m)\right]}{w_1(x; a, b)}. \tag{108}$$

In particular, the quotient on the right-hand side belongs to $\mathbb{P}_{m-1}$.

Proof. When $q = 1$, the mixed form $\mathcal{B}^{\mathrm{J}}_{\boldsymbol{n},m}$ has exactly one polynomial component in the Jacobi-like vector. Substituting its expression into (107) and dividing by the nonzero weight $w_1(x; a, b)$ gives (108). The asserted belonging to $\mathbb{P}_{m-1}$ is the polynomial-space condition prescribed by the mixed orthogonality problem. □

Remark 5.5 (Ordinary Jacobi-like specialization). When $p = 1$ and $\alpha_1 = 0$, formula (107) becomes, up to the normalization used here,

$$\mathscr{B}^{\mathrm{J}}_{(n),\boldsymbol{m}}(x) = -\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^n w_0(x;\boldsymbol{a},\boldsymbol{b}+\boldsymbol{m})\right], \qquad n = |\boldsymbol{m}| - 1.$$

This is the Rodrigues formula for the ordinary Jacobi-like type I function obtained by Wolfs [18].

5.2. **Laguerre-like mixed Rodrigues formula.** The Laguerre-like case is obtained in the same way, with the roles of the two multi-indices adapted to the mixed organization. The power vector is indexed by $\boldsymbol{m}$, while the shift of the seed weight is determined by the beta-shifted part $\boldsymbol{n}$ of $\boldsymbol{\lambda} = (\boldsymbol{n}, \boldsymbol{\eta})$. Consequently, the factors appearing in the Mellin transform are reproduced by $\mathscr{R}_{\boldsymbol{\beta},\boldsymbol{m}}$.

Theorem 5.6 (Rodrigues formula for the Laguerre-like mixed form)**.** *Assume the hypotheses of Theorem 4.5*(ii)*. Let*

$$\boldsymbol{\lambda} = (\boldsymbol{n}, \boldsymbol{\eta}), \qquad |\boldsymbol{m}| = |\boldsymbol{n}| + |\boldsymbol{\eta}| - 1.$$

Assume also

$$a_\rho > -1, \qquad \rho \in \{1, \dots, q\},$$

and

$$b_h > a_h, \qquad h \in \{1, \dots, r\}.$$

Then the normalized form on the Laguerre-like vector is given by

$$\mathscr{B}^{\mathrm{L}}_{\boldsymbol{\lambda},\boldsymbol{m}}(x) = \mathscr{R}_{\boldsymbol{\beta},\boldsymbol{m}}\left[w_0(x;\boldsymbol{a},\boldsymbol{b}+\boldsymbol{n})\right] = \left(\mathscr{R}_{\beta_1,m_1} \circ \cdots \circ \mathscr{R}_{\beta_p,m_p}\right)\left[w_0(x;\boldsymbol{a},\boldsymbol{b}+\boldsymbol{n})\right]. \tag{109}$$

Proof. From (58), with $\boldsymbol{b}$ replaced by $\boldsymbol{b}+\boldsymbol{n}$, one has

$$\mathscr{M}[w_0(\,\cdot\,;\boldsymbol{a},\boldsymbol{b}+\boldsymbol{n})](s) = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}.$$

By Lemma 5.2, the integrations by parts required in Proposition 5.1 have no endpoint contributions for this seed and its partial Rodrigues transforms. Therefore Proposition 5.1 gives

$$\mathscr{M}\left[\mathscr{R}_{\boldsymbol{\beta},\boldsymbol{m}}\left[w_0(\,\cdot\,;\boldsymbol{a},\boldsymbol{b}+\boldsymbol{n})\right]\right](s) = \frac{\Gamma(s\mathbf{1}_q + \boldsymbol{a})}{\Gamma(s\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{n})}(\boldsymbol{\beta} + (1-s)\mathbf{1}_p)_{\boldsymbol{m}}.$$

This is exactly the Mellin transform in (72). Mellin inversion proves (109). □

The ordinary Laguerre-like family is recovered when the power vector contains a single component. In this case the composed operator reduces to one ordinary derivative and the formula agrees with the type I Rodrigues identity obtained by Wolfs.

Remark 5.7 (Ordinary Laguerre-like specialization). For $p = 1$ and $\beta_1 = 0$, let

$$N = |\boldsymbol{n}| + |\boldsymbol{\eta}|, \qquad m_1 = N - 1.$$

Then (109) reduces to

$$\mathscr{B}^{\mathrm{L}}_{\boldsymbol{\lambda},N-1}(x) = \frac{\mathrm{d}^{N-1}}{\mathrm{d}x^{N-1}}\left[x^{N-1} w_0(x;\boldsymbol{a},\boldsymbol{b}+\boldsymbol{n})\right],$$

which is the Rodrigues-type formula for the ordinary Laguerre-like type I function obtained by Wolfs [18].

6. The hypergeometric settings in the step-line

The general step-line construction of Section 2 can now be specialized to the two systems obtained above. In this section, we write the resulting step-line forms explicitly, including their polynomial components and normalization constants. The recurrence coefficients are then evaluated directly from the step-line pairing $\langle x\mathscr{B}_N, \mathscr{A}_{N+k}\rangle$. This avoids introducing any recurrence between off-step-line neighbouring multi-indices.

For $d \in \mathbb{N}$, let

$$\boldsymbol{\sigma}_d(N) \in \mathbb{N}_0^d, \qquad N \in \mathbb{N}_0,$$

denote the step-line multi-index introduced in Section 2. We also set

$$\jmath_N := \ell_q(N) + 1, \qquad N \in \mathbb{N}_0,$$

so that

$$\boldsymbol{\sigma}_q(N+1)-\boldsymbol{\sigma}_q(N)=\boldsymbol{e}^{(q)}_{J_N}.$$

6.1. **Jacobi-like step-line forms.** In the Jacobi-like setting, the first multi-index refers to the p-component power vector, whereas the second refers to the q-component Jacobi-like vector. Define

$$\begin{aligned}\boldsymbol{\nu}_N &:= \boldsymbol{\sigma}_p(N), & \boldsymbol{\mu}_N &:= \boldsymbol{\sigma}_q(N+1),\\ \boldsymbol{\nu}^*_N &:= \boldsymbol{\sigma}_p(N+1), & \boldsymbol{\mu}^*_N &:= \boldsymbol{\sigma}_q(N),\end{aligned}\qquad N\in\mathbb{N}_0.$$

Then

$$|\boldsymbol{\mu}_N|=|\boldsymbol{\nu}_N|+1,\qquad |\boldsymbol{\nu}^*_N|=|\boldsymbol{\mu}^*_N|+1,$$

and

$$\boldsymbol{\mu}_N-\boldsymbol{\mu}^*_N=\boldsymbol{e}^{(q)}_{J_N}.$$

Thus, the unnormalized Jacobi-like step-line forms are obtained from the index pairs

$$\widehat{\mathscr{A}}^{\mathrm{J}}_N := \mathscr{A}^{\mathrm{J}}_{\boldsymbol{\nu}^*_N,\boldsymbol{\mu}^*_N},\qquad \widehat{\mathscr{B}}^{\mathrm{J}}_N := \mathscr{B}^{\mathrm{J}}_{\boldsymbol{\nu}_N,\boldsymbol{\mu}_N}.$$

6.1.1. *The form on the power vector.* The normalized form on the power vector is

$$\mathscr{A}^{\mathrm{J}}_N(x)=\frac{1}{g^{\mathrm{J}}_N}\widehat{\mathscr{A}}^{\mathrm{J}}_N(x)=\frac{1}{g^{\mathrm{J}}_N}\sum_{i=1}^{p}\mathscr{A}^{\mathrm{J},(i)}_N(x)x^{\alpha_i}.$$

For $i\in\{1,\dots,p\}$ such that $(\boldsymbol{\nu}^*_N)_i\geq 1$, define

$$\begin{aligned}\mathscr{A}^{\mathrm{J},(i)}_N(x)&=C^{\mathrm{J},\mathscr{A}}_{N,i}\\ &\times {}_{p+q}F_{p+q-1}\left(\begin{matrix}-(\boldsymbol{\nu}^*_N)_i+1,\ (\alpha_i+1)\mathbf{1}_q+\boldsymbol{b}+\boldsymbol{\mu}^*_N,\ (\alpha_i+1)\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}-(\boldsymbol{\nu}^*_N)^{*i}\\ (\alpha_i+1)\mathbf{1}_q+\boldsymbol{a},\ (\alpha_i+1)\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}\end{matrix};x\right),\end{aligned}\tag{110}$$

where

$$C^{\mathrm{J},\mathscr{A}}_{N,i}:=(-1)^{|(\boldsymbol{\nu}^*_N)^{*i}|+1}\frac{\Gamma((\alpha_i+1)\mathbf{1}_q+\boldsymbol{b})}{\Gamma((\alpha_i+1)\mathbf{1}_q+\boldsymbol{a})}\frac{((\alpha_i+1)\mathbf{1}_q+\boldsymbol{b})_{\boldsymbol{\mu}^*_N}}{((\boldsymbol{\nu}^*_N)_i-1)!((\alpha_i+1)\mathbf{1}_{p-1}-\boldsymbol{\alpha}^{*i}-(\boldsymbol{\nu}^*_N)^{*i})_{(\boldsymbol{\nu}^*_N)^{*i}}}.$$

If $(\boldsymbol{\nu}^*_N)_i=0$, we set $\mathscr{A}^{\mathrm{J},(i)}_N\equiv 0$.

6.1.2. *The form on the Jacobi-like vector.* The normalized form on the Jacobi-like vector is

$$\mathscr{B}^{\mathrm{J}}_N(x)=\frac{1}{h^{\mathrm{J}}_N}\widehat{\mathscr{B}}^{\mathrm{J}}_N(x)=\frac{1}{h^{\mathrm{J}}_N}\sum_{j=1}^{q}\mathscr{B}^{\mathrm{J},(j)}_N(x)w_j(x;\boldsymbol{a},\boldsymbol{b}).$$

For $J\in\{1,\dots,q\}$ such that $(\boldsymbol{\mu}_N)_J\geq 1$, and $K\in\{0,\dots,(\boldsymbol{\mu}_N)_J-1\}$, define

$$\Pi^{\mathrm{J}}_{N;J,K}:=-\frac{(\boldsymbol{\alpha}+(b_J+K+1)\mathbf{1}_p)_{\boldsymbol{\nu}_N}}{(-1)^K K!((\boldsymbol{\mu}_N)_J-K-1)!(\boldsymbol{b}^{*J}-(b_J+K)\mathbf{1}_{q-1})_{\boldsymbol{\mu}^{*J}_N}}.$$

For $j\in\{1,\dots,q\}$ and $K-1+\delta_{j,J}\geq 0$, set

$$\Psi^{\mathrm{J}}_{N;j;J,K}(x):=\frac{(\boldsymbol{b}^{*j}-(b_J+K)\mathbf{1}_{q-1})_1}{(\boldsymbol{a}-(b_J+K)\mathbf{1}_q)_1}\,{}_{q+1}F_q\left(\begin{matrix}1,\ \boldsymbol{b}+(1-b_J-K)\mathbf{1}_q-\boldsymbol{e}_j\\ \boldsymbol{a}+(1-b_J-K)\mathbf{1}_q\end{matrix};x\right),\tag{111}$$

and put $\Psi^{\mathrm{J}}_{N;j;J,0}:=0$, $j\neq J$. Indeed, the J-th upper parameter in (111) is $-K$ when $j=J$, and $1-K$ when $j\neq J$. Consequently, every nonzero $\Psi^{\mathrm{J}}_{N;j;J,K}$ is a terminating polynomial of degree $K-1+\delta_{j,J}$.

For $j\in\{1,\dots,q\}$ such that $(\boldsymbol{\mu}_N)_j\geq 1$, the step-line component polynomial is

$$\mathscr{B}^{\mathrm{J},(j)}_N(x)=\frac{(\boldsymbol{a}-b_j\mathbf{1}_q)_1}{(\boldsymbol{b}^{*j}-b_j\mathbf{1}_{q-1})_1}\sum_{\substack{J=1\\(\boldsymbol{\mu}_N)_J\geq 1}}^{q}\sum_{K=0}^{(\boldsymbol{\mu}_N)_J-1}\Pi^{\mathrm{J}}_{N;J,K}\Psi^{\mathrm{J}}_{N;j;J,K}(x).\tag{112}$$

If $(\boldsymbol{\mu}_N)_j=0$, we set

$$\mathscr{B}^{\mathrm{J},(j)}_N\equiv 0.$$

6.1.3. *Normalization and Mellin identity.* The normalization constants fixing the biorthonormal Jacobi-like step-line system are

$$h_N^{\mathrm{J}} := [x^{(\mu_N)_{J_N}-1}]\mathscr{B}_N^{\mathrm{J},(J_N)}(x), \qquad g_N^{\mathrm{J}} := \int_0^1 x^{(\mu_N^*)_{J_N}}\widehat{\mathscr{A}}_N^{\mathrm{J}}(x)w_{J_N}(x;\boldsymbol{a},\boldsymbol{b})\,\mathrm{d}x. \tag{113}$$

Here $[x^d]P(x)$ denotes the coefficient of x^d in P. The terminating expression (112) evaluates h_N^{J}, while expansion of (110) followed by (25) evaluates g_N^{J}. Thus both constants are finite Gamma–Pochhammer expressions. The Mellin identity of the normalized form on the Jacobi-like vector is

$$\int_0^1 x^{s-1}\mathscr{B}_N^{\mathrm{J}}(x)\,\mathrm{d}x = -\frac{1}{h_N^{\mathrm{J}}}\frac{\Gamma(s\mathbf{1}_q+\boldsymbol{a})}{\Gamma(s\mathbf{1}_q+\boldsymbol{b}+\boldsymbol{\mu}_N)}(\boldsymbol{\alpha}+(1-s)\mathbf{1}_p)_{\boldsymbol{\nu}_N}.$$

Under the boundary degeneration

$$\boldsymbol{a},\boldsymbol{b}\longrightarrow\boldsymbol{\beta},$$

these formulas specialize to the mixed Piñeiro step-line forms. These are precisely the step-line forms developed in [3].

6.2. **Laguerre-like step-line forms.** In the Laguerre-like setting, the notation of the explicit forms places first the multi-index of the q-component Laguerre-like vector and second the multi-index of the p-component power vector. Define

$$\boldsymbol{\lambda}_N^{\mathrm{L}} := \boldsymbol{\sigma}_q(N) = (\boldsymbol{n}_N^{\mathrm{L}},\boldsymbol{\eta}_N^{\mathrm{L}}), \qquad \boldsymbol{\mu}_N^{\mathrm{L}} := \boldsymbol{\sigma}_p(N), \qquad N\in\mathbb{N}_0,$$

where

$$\boldsymbol{n}_N^{\mathrm{L}}\in\mathbb{N}_0^r, \qquad \boldsymbol{\eta}_N^{\mathrm{L}}\in\mathbb{N}_0^s.$$

Then

$$|\boldsymbol{\mu}_{N+1}^{\mathrm{L}}| = |\boldsymbol{\lambda}_N^{\mathrm{L}}|+1, \qquad |\boldsymbol{\lambda}_{N+1}^{\mathrm{L}}| = |\boldsymbol{\mu}_N^{\mathrm{L}}|+1,$$

and

$$\boldsymbol{\lambda}_{N+1}^{\mathrm{L}}-\boldsymbol{\lambda}_N^{\mathrm{L}} = \boldsymbol{e}_{J_N}^{(q)}.$$

Accordingly, the unnormalized Laguerre-like step-line forms are

$$\widehat{\mathscr{A}}_N^{\mathrm{L}} := \mathscr{A}^{\mathrm{L}}_{\boldsymbol{\lambda}_N^{\mathrm{L}},\boldsymbol{\mu}_{N+1}^{\mathrm{L}}}, \qquad \widehat{\mathscr{B}}_N^{\mathrm{L}} := \mathscr{B}^{\mathrm{L}}_{\boldsymbol{\lambda}_{N+1}^{\mathrm{L}},\boldsymbol{\mu}_N^{\mathrm{L}}}.$$

6.2.1. *The form on the power vector.* The normalized form on the power vector is

$$\mathscr{A}_N^{\mathrm{L}}(x) = \frac{1}{g_N^{\mathrm{L}}}\widehat{\mathscr{A}}_N^{\mathrm{L}}(x) = \frac{1}{g_N^{\mathrm{L}}}\sum_{i=1}^p \mathscr{A}_N^{\mathrm{L},(i)}(x)x^{\beta_i}.$$

For $i\in\{1,\dots,p\}$ such that $(\boldsymbol{\mu}_{N+1}^{\mathrm{L}})_i\geq 1$, define

$$\begin{aligned}\mathscr{A}_N^{\mathrm{L},(i)}(x) &= C_{N,i}^{\mathrm{L},\mathscr{A}}\\ &\times {}_{r+p}F_{q+p-1}\left(\begin{matrix}-(\boldsymbol{\mu}_{N+1}^{\mathrm{L}})_i+1,\ (\beta_i+1)\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n}_N^{\mathrm{L}},\ (\beta_i+1)\mathbf{1}_{p-1}-\boldsymbol{\beta}^{*i}-(\boldsymbol{\mu}_{N+1}^{\mathrm{L}})^{*i}\\ (\beta_i+1)\mathbf{1}_q+\boldsymbol{a},\ (\beta_i+1)\mathbf{1}_{p-1}-\boldsymbol{\beta}^{*i}\end{matrix};x\right),\end{aligned} \tag{114}$$

where

$$C_{N,i}^{\mathrm{L},\mathscr{A}} := -\frac{\Gamma((\beta_i+1)\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n}_N^{\mathrm{L}})}{\Gamma((\beta_i+1)\mathbf{1}_q+\boldsymbol{a})}\frac{1}{((\boldsymbol{\mu}_{N+1}^{\mathrm{L}})_i-1)!(\boldsymbol{\beta}^{*i}-\beta_i\mathbf{1}_{p-1})_{(\boldsymbol{\mu}_{N+1}^{\mathrm{L}})^{*i}}}.$$

If $(\boldsymbol{\mu}_{N+1}^{\mathrm{L}})_i=0$, we set $\mathscr{A}_N^{\mathrm{L},(i)}\equiv 0$.

6.2.2. *The form on the Laguerre-like vector.* The normalized form on the Laguerre-like vector is

$$\mathscr{B}^{\mathrm{L}}_N(x) = \frac{1}{h^{\mathrm{L}}_N}\widehat{\mathscr{B}}^{\mathrm{L}}_N(x) = \frac{1}{h^{\mathrm{L}}_N}\left(\sum_{h=1}^{r}\mathscr{B}^{\mathrm{L},(h)}_N(x)w_h(x;\boldsymbol{a},\boldsymbol{b}) + \sum_{\ell=1}^{s}\mathscr{D}^{\mathrm{L},(\ell)}_N(x)v_\ell(x;\boldsymbol{a},\boldsymbol{b})\right). \tag{115}$$

For $h \in \{1,\dots,r\}$ such that $(\boldsymbol{n}^{\mathrm{L}}_{N+1})_h \geq 1$, write

$$\mathscr{B}^{\mathrm{L},(h)}_N(x) = \sum_{k=0}^{(\boldsymbol{n}^{\mathrm{L}}_{N+1})_h-1} b^{\mathrm{L}}_{N;h,k}x^k,$$

whereas, if $(\boldsymbol{n}^{\mathrm{L}}_{N+1})_h = 0$, we set $\mathscr{B}^{\mathrm{L},(h)}_N \equiv 0$. For $\ell \in \{1,\dots,s\}$ such that $(\boldsymbol{\eta}^{\mathrm{L}}_{N+1})_\ell \geq 1$, write

$$\mathscr{D}^{\mathrm{L},(\ell)}_N(x) = \sum_{k=0}^{(\boldsymbol{\eta}^{\mathrm{L}}_{N+1})_\ell-1} d^{\mathrm{L}}_{N;\ell,k}x^k,$$

whereas, if $(\boldsymbol{\eta}^{\mathrm{L}}_{N+1})_\ell = 0$, we set $\mathscr{D}^{\mathrm{L},(\ell)}_N \equiv 0$. Introduce, for the step-line data, the compact basis polynomials

$$\begin{aligned}
P^{\mathrm{L}}_{N;h,k}(z) &:= (z\mathbf{1}_q+\boldsymbol{a})_k(z\mathbf{1}_r+\boldsymbol{b}+k\mathbf{1}_r+\boldsymbol{e}_h)_{\boldsymbol{n}^{\mathrm{L}}_{N+1}-k\mathbf{1}_r-\boldsymbol{e}_h},\\
D^{\mathrm{L}}_{N;\ell,k}(z) &:= (z\mathbf{1}_q+\boldsymbol{a})_k(z\mathbf{1}_r+\boldsymbol{b}+k\mathbf{1}_r)_{\boldsymbol{n}^{\mathrm{L}}_{N+1}-k\mathbf{1}_r}(z+k)^{\ell-1}.
\end{aligned}$$

The polynomial components in (115) are uniquely determined by the finite confluent identity

(116)
$$\sum_{\substack{h=1\\(\boldsymbol{n}^{\mathrm{L}}_{N+1})_h\geq 1}}^{r}\ \sum_{k=0}^{(\boldsymbol{n}^{\mathrm{L}}_{N+1})_h-1} b^{\mathrm{L}}_{N;h,k}P^{\mathrm{L}}_{N;h,k}(z) + \sum_{\substack{\ell=1\\(\boldsymbol{\eta}^{\mathrm{L}}_{N+1})_\ell\geq 1}}^{s}\ \sum_{k=0}^{(\boldsymbol{\eta}^{\mathrm{L}}_{N+1})_\ell-1} d^{\mathrm{L}}_{N;\ell,k}D^{\mathrm{L}}_{N;\ell,k}(z) = (\boldsymbol{\beta}+(1-z)\mathbf{1}_p)_{\boldsymbol{\mu}^{\mathrm{L}}_N}.$$

If $a_\rho - a_\sigma \notin \mathbb{Z}$, $\rho \neq \sigma$, then the complete unnormalized form $\widehat{\mathscr{B}}^{\mathrm{L}}_N$ admits the residue expansion

$$\begin{aligned}
\widehat{\mathscr{B}}^{\mathrm{L}}_N(x) = \sum_{\rho=1}^{q}&\frac{\Gamma(\boldsymbol{a}^{*\rho}-a_\rho\mathbf{1}_{q-1})(\boldsymbol{\beta}+(a_\rho+1)\mathbf{1}_p)_{\boldsymbol{\mu}^{\mathrm{L}}_N}}{\Gamma(\boldsymbol{b}+\boldsymbol{n}^{\mathrm{L}}_{N+1}-a_\rho\mathbf{1}_r)}x^{a_\rho}\\
&\times {}_{r+p}F_{q+p-1}\left(\begin{matrix}(a_\rho+1)\mathbf{1}_r-\boldsymbol{b}-\boldsymbol{n}^{\mathrm{L}}_{N+1},\ \boldsymbol{\beta}+\boldsymbol{\mu}^{\mathrm{L}}_N+(a_\rho+1)\mathbf{1}_p\\ (a_\rho+1)\mathbf{1}_{q-1}-\boldsymbol{a}^{*\rho},\ \boldsymbol{\beta}+(a_\rho+1)\mathbf{1}_p\end{matrix};(-1)^s x\right).
\end{aligned}\tag{117}$$

Consequently,

$$h^{\mathrm{L}}_N\mathscr{B}^{\mathrm{L}}_N(x) = \widehat{\mathscr{B}}^{\mathrm{L}}_N(x)$$

is given by the right-hand side of (117).

6.2.3. *Normalization and Mellin identity.* For $\rho \in \{1,\dots,q\}$, set

$$\mathscr{U}^{\mathrm{L},(\rho)}_N := \begin{cases}\mathscr{B}^{\mathrm{L},(\rho)}_N, & \rho\in\{1,\dots,r\},\\ \mathscr{D}^{\mathrm{L},(\rho-r)}_N, & \rho\in\{r+1,\dots,q\}.\end{cases}$$

The normalization constants fixing the biorthonormal Laguerre-like step-line system are

$$h^{\mathrm{L}}_N := [x^{(\lambda^{\mathrm{L}}_{N+1})_{J_N}-1}]\mathscr{U}^{\mathrm{L},(J_N)}_N(x), \qquad g^{\mathrm{L}}_N := \int_0^\infty x^{(\lambda^{\mathrm{L}}_N)_{J_N}}\widehat{\mathscr{A}}^{\mathrm{L}}_N(x)U^{\mathrm{L}}_{J_N}(x)\,\mathrm{d}x. \tag{118}$$

The finite component system (116) evaluates h^{L}_N; expanding (114) and applying the Mellin transform of $U^{\mathrm{L}}_{J_N}$ evaluates g^{L}_N. The Mellin identity for the normalized form on the Laguerre-like vector is

$$\int_0^\infty x^{z-1}\mathscr{B}^{\mathrm{L}}_N(x)\,\mathrm{d}x = \frac{1}{h^{\mathrm{L}}_N}\frac{\Gamma(z\mathbf{1}_q+\boldsymbol{a})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{n}^{\mathrm{L}}_{N+1})}(\boldsymbol{\beta}+(1-z)\mathbf{1}_p)_{\boldsymbol{\mu}^{\mathrm{L}}_N}.$$

Proposition 6.1 (Biorthonormal step-line systems)**.** *Assume that the relevant step-line index pairs are normal and that the normalization constants*

$$h_N^{\mathrm{J}}, \qquad g_N^{\mathrm{J}}, \qquad h_N^{\mathrm{L}}, \qquad g_N^{\mathrm{L}}$$

do not vanish. Then

$$\left\langle \mathscr{B}_N^{\mathrm{J}}, \mathscr{A}_M^{\mathrm{J}} \right\rangle = \left\langle \mathscr{B}_N^{\mathrm{L}}, \mathscr{A}_M^{\mathrm{L}} \right\rangle = \delta_{N,M}, \qquad N, M \in \mathbb{N}_0. \tag{119}$$

Proof. The displayed Jacobi-like and Laguerre-like forms are precisely the specializations of the general step-line forms associated with the index pairs defined above. For $\star \in \{\mathrm{J}, \mathrm{L}\}$, their orthogonality conditions give zero when $N \neq M$, and the constants $h_N^\star$ and $g_N^\star$ impose $\langle \mathscr{B}_N^\star, \mathscr{A}_N^\star \rangle = 1$. Hence the general step-line biorthogonality relation gives (119). □

6.3. **Explicit step-line recurrence coefficients.** We introduce coefficient notation for the terminating polynomials displayed above:

$$\begin{aligned}
&\mathscr{A}_M^{\mathrm{J},(i)}(x) = \sum_{u \geq 0} \mathsf{a}_{M;i,u}^{\mathrm{J}} x^u, \qquad \mathscr{B}_N^{\mathrm{J},(j)}(x) = \sum_{v \geq 0} \mathsf{b}_{N;j,v}^{\mathrm{J}} x^v, \\
&\mathscr{A}_M^{\mathrm{L},(i)}(x) = \sum_{u \geq 0} \mathsf{a}_{M;i,u}^{\mathrm{L}} x^u, \qquad \mathscr{B}_N^{\mathrm{L},(h)}(x) = \sum_{v \geq 0} \mathsf{b}_{N;h,v}^{\mathrm{L}} x^v, \qquad \mathscr{D}_N^{\mathrm{L},(\ell)}(x) = \sum_{v \geq 0} \mathsf{d}_{N;\ell,v}^{\mathrm{L}} x^v.
\end{aligned} \tag{120}$$

All these sums are finite; the appropriate component is identically zero when its degree bound is negative.

For $N, M \in \mathbb{N}_0$, the Jacobi-like pairing $\langle x\mathscr{B}_N^{\mathrm{J}}, \mathscr{A}_M^{\mathrm{J}} \rangle$ is the finite sum

$$\mathscr{P}_{N,M}^{\mathrm{J}} := \frac{1}{h_N^{\mathrm{J}} g_M^{\mathrm{J}}} \sum_{\substack{1 \leq i \leq p,\ 1 \leq j \leq q \\ u, v \geq 0}} \mathsf{a}_{M;i,u}^{\mathrm{J}} \mathsf{b}_{N;j,v}^{\mathrm{J}} \frac{\Gamma((\alpha_i + u + v + 2)\mathbf{1}_q + \boldsymbol{a})}{\Gamma((\alpha_i + u + v + 2)\mathbf{1}_q + \boldsymbol{b} + \boldsymbol{e}_j)}. \tag{121}$$

For the Laguerre-like pairing, put $\zeta_{i,u,v} := \beta_i + u + v + 2$ and define

$$\begin{aligned}
\mathscr{P}_{N,M}^{\mathrm{L}} := \frac{1}{h_N^{\mathrm{L}} g_M^{\mathrm{L}}} \sum_{\substack{1 \leq i \leq p \\ u \geq 0}} \mathsf{a}_{M;i,u}^{\mathrm{L}} \Bigg[&\sum_{\substack{1 \leq h \leq r \\ v \geq 0}} \mathsf{b}_{N;h,v}^{\mathrm{L}} \frac{\Gamma(\zeta_{i,u,v}\mathbf{1}_q + \boldsymbol{a})}{\Gamma(\zeta_{i,u,v}\mathbf{1}_r + \boldsymbol{b} + \boldsymbol{e}_h)} \\
&+ \sum_{\substack{1 \leq \ell \leq s \\ v \geq 0}} \mathsf{d}_{N;\ell,v}^{\mathrm{L}} \zeta_{i,u,v}^{\ell-1} \frac{\Gamma(\zeta_{i,u,v}\mathbf{1}_q + \boldsymbol{a})}{\Gamma(\zeta_{i,u,v}\mathbf{1}_r + \boldsymbol{b})} \Bigg].
\end{aligned} \tag{122}$$

Thus the Jacobi-like coefficients are finite Gamma–Pochhammer sums, while the Laguerre-like coefficients are finite Gamma–Pochhammer sums after the finite confluent system (116) has been solved.

Theorem 6.2 (Step-line recurrences for the two hypergeometric systems)**.** *Assume the hypotheses of Proposition 6.1. For $\star \in \{\mathrm{J}, \mathrm{L}\}$, $k \in \{-p, \dots, q\}$, and $N + k \geq 0$, set*

$$b_N^{\star,k} := \mathscr{P}_{N,N+k}^\star. \tag{123}$$

Then

$$x\mathscr{B}_N^\star(x) = \sum_{\substack{k=-p \\ N+k \geq 0}}^{q} b_N^{\star,k} \mathscr{B}_{N+k}^\star(x), \tag{124}$$

and the corresponding recurrence for the $\mathscr{A}$-forms is

$$x\mathscr{A}_N^\star(x) = \sum_{\substack{k=-q \\ N+k \geq 0}}^{p} b_{N+k}^{\star,-k} \mathscr{A}_{N+k}^\star(x). \tag{125}$$

Moreover,

$$b_N^{\mathrm{J},q} = b_N^{\mathrm{L},q} = 1.$$

Proof. Proposition 2.4 already proves that only the indices $N-p,\dots,N+q$ occur and identifies the coefficient of $\mathscr{B}^\star_{N+k}$ as

$$\left\langle x\mathscr{B}^\star_N, \mathscr{A}^\star_{N+k}\right\rangle.$$

Insert first the two Jacobi-like component expansions from (120). A term with polynomial degrees u and v contains the moment

$$\int_0^1 x^{\alpha_i+u+v+1} w_j(x;\boldsymbol{a},\boldsymbol{b})\,\mathrm{d}x = \frac{\Gamma((\alpha_i+u+v+2)\mathbf{1}_q+\boldsymbol{a})}{\Gamma((\alpha_i+u+v+2)\mathbf{1}_q+\boldsymbol{b}+\boldsymbol{e}_j)},$$

by (25). Including the two normalization constants gives exactly (121).

For the Laguerre-like system, a term containing the weight w_h gives

$$\int_0^\infty x^{\zeta_{i,u,v}-1} w_h(x)\,\mathrm{d}x = \frac{\Gamma(\zeta_{i,u,v}\mathbf{1}_q+\boldsymbol{a})}{\Gamma(\zeta_{i,u,v}\mathbf{1}_r+\boldsymbol{b}+\boldsymbol{e}_h)}$$

by (61). A term containing the weight v_ℓ gives

$$\int_0^\infty x^{z-1} v_\ell(x)\,\mathrm{d}x = z^{\ell-1}\frac{\Gamma(z\mathbf{1}_q+\boldsymbol{a})}{\Gamma(z\mathbf{1}_r+\boldsymbol{b})}$$

by (62); here $z=\zeta_{i,u,v}$. Summing the finite component expansions gives (122). This proves (123) and (124).

Let $T^\star$ be the recurrence matrix, so that $(T^\star)_{N,N+k}=b^{\star,k}_N$. The N-th component of $(T^\star)^\top\boldsymbol{\mathscr{A}}^\star = x\boldsymbol{\mathscr{A}}^\star$ is

$$x\mathscr{A}^\star_N = \sum_{\substack{k=-q\\ N+k\geq 0}}^{p} (T^\star)_{N+k,N}\mathscr{A}^\star_{N+k} = \sum_{\substack{k=-q\\ N+k\geq 0}}^{p} b^{\star,-k}_{N+k}\mathscr{A}^\star_{N+k},$$

which is (125). Finally, (113) and (118) are the specializations of (11); hence the coefficient of $\mathscr{B}^\star_{N+q}$ is one by the last part of Proposition 2.4. □

Consequently, each system defines a (p,q)-banded step-line recurrence matrix by

$$(T^\star)_{N,N+k} = b^{\star,k}_N, \qquad \star\in\{\mathrm{J},\mathrm{L}\},\quad k\in\{-p,\dots,q\},\quad N+k\geq 0.$$

7. Bidiagonal factorization of the step-line recurrence matrix

This section concerns the ordered step-line matrix T. The finite pairings of the previous section evaluate its band entries, while the Christoffel–Gauss–Borel factorization explains how the same matrix decomposes into elementary bidiagonal factors. Closed formulas are available precisely for the Christoffel chains which remain inside the explicit family; otherwise the factors are kept in tau-determinant form.

The left Christoffel chain remains within each of the two explicit families: it acts on the power vector by the cyclic affine shift of its exponent parameters. Thus

$$\mathscr{A}^{\mathrm{J},[k]}_{\mathrm{L},N} = \mathscr{A}^{\mathrm{J}}_N\big|_{\boldsymbol{\alpha}\mapsto\boldsymbol{\alpha}^{[k]}}, \qquad \mathscr{A}^{\mathrm{L},[k]}_{\mathrm{L},N} = \mathscr{A}^{\mathrm{L}}_N\big|_{\boldsymbol{\beta}\mapsto\boldsymbol{\beta}^{[k]}},$$

where

$$\boldsymbol{\alpha}^{[k]} := \mathscr{C}^k_p(\boldsymbol{\alpha}), \qquad \boldsymbol{\beta}^{[k]} := \mathscr{C}^k_p(\boldsymbol{\beta}).$$

To display the cyclic shifts in the factor formulas, extend the exponent parameters by

$$\alpha_{sp+i} := \alpha_i + s, \qquad \beta_{sp+i} := \beta_i + s, \qquad s\in\mathbb{N}_0,\quad i\in\{1,\dots,p\}.$$

For each $N\in\mathbb{N}_0$, write

$$N = pu_N + \ell_N = qv_N + j_N, \qquad \ell_N\in\{0,\dots,p-1\},\quad j_N\in\{0,\dots,q-1\},$$

and set

$$\iota_N := \ell_N + 1, \qquad \chi_N := j_N + 1, \qquad \boldsymbol{d}_N := \boldsymbol{\sigma}_p(N+1), \qquad d_{N,i} := (\boldsymbol{d}_N)_i.$$

Throughout this section, assume the normality and nonzero-pivot hypotheses of Proposition 2.5 along every Christoffel chain used. All denominators in the formulas below are assumed to be nonzero.

7.1. **Closed evaluation of the left normalizing factors.** The normalizing constants in the left Christoffel chain can be computed directly from the contour representations of the forms on the power vector; no expansion of the hypergeometric components is needed.

Proposition 7.1 (Closed normalizing factors)**.** *For $k \in \{0,\dots,p\}$, define*

$$g_N^{\mathrm{J},[k]} := g_N^{\mathrm{J}}\big|_{\boldsymbol{\alpha}\mapsto\boldsymbol{\alpha}^{[k]}}, \qquad g_N^{\mathrm{L},[k]} := g_N^{\mathrm{L}}\big|_{\boldsymbol{\beta}\mapsto\boldsymbol{\beta}^{[k]}}.$$

Then

$$g_N^{\mathrm{J},[k]} = -\frac{(\boldsymbol{a}-(b_{\chi_N}+v_N)\mathbf{1}_q)_{v_N}\prod_{h=1}^{\chi_N-1}(b_h-b_{\chi_N})}{(\boldsymbol{\alpha}^{[k]}+(b_{\chi_N}+v_N+1)\mathbf{1}_p)_{\boldsymbol{d}_N}}. \tag{126}$$

For the Laguerre-like system one has

$$g_N^{\mathrm{L},[k]} = \begin{cases} -\frac{(\boldsymbol{a}-(b_{\chi_N}+v_N)\mathbf{1}_q)_{v_N}\prod_{h=1}^{\chi_N-1}(b_h-b_{\chi_N})}{(\boldsymbol{\beta}^{[k]}+(b_{\chi_N}+v_N+1)\mathbf{1}_p)_{\boldsymbol{d}_N}}, & \chi_N \in \{1,\dots,r\},\\[2ex] (-1)^{N+1}, & \chi_N \in \{r+1,\dots,q\}. \end{cases} \tag{127}$$

Proof. Consider first the Jacobi-like system. Inserting the contour representation (33) in the normalizing pairing in (11), with

$$\boldsymbol{n} = \boldsymbol{\sigma}_p(N+1), \qquad \boldsymbol{m} = \boldsymbol{\sigma}_q(N), \qquad \boldsymbol{\alpha}\mapsto\boldsymbol{\alpha}^{[k]},$$

and using

$$(\boldsymbol{\sigma}_q(N))_{\chi_N} = v_N,$$

reduces the integral to

$$\frac{(-1)^{N+1}}{2\pi\mathrm{i}}\int_\Sigma \frac{(t\mathbf{1}_q+\boldsymbol{a}+\mathbf{1}_q)_{v_N}\prod_{h=1}^{\chi_N-1}(t+b_h+v_N+1)}{(t+b_{\chi_N}+v_N+1)\prod_{i=1}^{p}\prod_{\ell=0}^{d_{N,i}-1}(t-\alpha_i^{[k]}-\ell)}\,\mathrm{d}t.$$

The integrand is $\mathrm{O}(t^{-2})$ at infinity. Moving the contour to the exterior leaves the simple pole at $t=-b_{\chi_N}-v_N-1$, whose residue gives (126).

In the Laguerre-like setting, use (65). If $\chi_N \in \{1,\dots,r\}$, coordinate χ_N corresponds to the weight w_{χ_N}, and the same exterior-pole calculation gives the first line of (127), with $\boldsymbol{\alpha}^{[k]}$ replaced by $\boldsymbol{\beta}^{[k]}$. If $\chi_N \in \{r+1,\dots,q\}$, it corresponds to v_{χ_N-r}. The corresponding rational integrand has leading behavior

$$\frac{(-1)^{N+1}}{t}+\mathrm{O}(t^{-2}),$$

and its finite residues therefore sum to $(-1)^{N+1}$. This proves the second line. □

7.2. **Simplification of the leading-coefficient quotients.** The leading coefficients in (20) also simplify before they are combined with the normalizing factors.

Proposition 7.2 (Leading-coefficient quotient reduction)**.** *For $k \in \{1,\dots,p\}$, the quotient of the leading coefficients of the unnormalized left Christoffel forms is*

$$-\frac{\alpha_{k+\ell_N+1}-\alpha_k+u_N}{\alpha_{k+\ell_N+1}+u_N+1+b_{\chi_N}+v_N} \tag{128}$$

in the Jacobi-like setting. In the Laguerre-like setting it is

$$\begin{cases} -\frac{\beta_{k+\ell_N+1}-\beta_k+u_N}{\beta_{k+\ell_N+1}+u_N+1+b_{\chi_N}+v_N}, & \chi_N \in \{1,\dots,r\},\\[2ex] -\left(\beta_{k+\ell_N+1}-\beta_k+u_N\right), & \chi_N \in \{r+1,\dots,q\}. \end{cases} \tag{129}$$

Proof. Insert the highest-degree coefficients obtained from (110) and (114), after the corresponding cyclic parameter shift. Since

$$\alpha_{\iota_N}^{[k]}+d_{N,\iota_N} = \alpha_{\iota_{N+1}}^{[k-1]}+d_{N+1,\iota_{N+1}} = \alpha_{k+\ell_N+1}+u_N+1,$$

the Gamma quotients involving $\boldsymbol{a}$ cancel in the Jacobi-like calculation. The change

$$\boldsymbol{\sigma}_q(N+1)=\boldsymbol{\sigma}_q(N)+\boldsymbol{e}^{(q)}_{\chi_N}$$

produces the denominator in (128); the remaining Pochhammer and factorial factors reduce to its numerator.

For Laguerre-like forms, the same cancellation occurs on the power side. The vector $\boldsymbol{n}^{\mathrm{L}}_N$ changes from N to $N+1$ only when $\chi_N\leq r$, giving the denominator in the first line of (129); if $\chi_N>r$, the beta-shifted block does not change and no such denominator remains. □

7.3. **Explicit lower bidiagonal factors.**

Theorem 7.3 (Jacobi-like lower bidiagonal factors)**.** *For $k\in\{1,\dots,p\}$,*

$$(L^{\mathrm{J}}_k)_{N+1,N}=-\frac{g^{\mathrm{J},[k-1]}_{N+1}}{g^{\mathrm{J},[k]}_N}\frac{\alpha_{k+\ell_N+1}-\alpha_k+u_N}{\alpha_{k+\ell_N+1}+u_N+1+b_{\chi_N}+v_N},\tag{130}$$

where the two normalizing factors are the Pochhammer products in (126).

Proof. The factorization formula (20) uses the leading coefficients after the left normalization. Hence it is the product of the quotient of normalizing constants and the quotient of unnormalized leading coefficients. Apply Proposition 7.1 and Proposition 7.2. □

Theorem 7.4 (Laguerre-like lower bidiagonal factors)**.** *Assume first that $r=0$. Then, for $k\in\{1,\dots,p\}$,*

$$(L^{\mathrm{L}}_k)_{N+1,N}=\beta_{k+\ell_N+1}-\beta_k+u_N.\tag{131}$$

Assume now that $r>0$. Then

$$\begin{aligned}
(L^{\mathrm{L}}_k)_{N+1,N}&=-\frac{g^{\mathrm{L},[k-1]}_{N+1}}{g^{\mathrm{L},[k]}_N}\frac{\beta_{k+\ell_N+1}-\beta_k+u_N}{\beta_{k+\ell_N+1}+u_N+1+b_{\chi_N}+v_N}, && \chi_N\in\{1,\dots,r-1\},\\
(L^{\mathrm{L}}_k)_{N+1,N}&=-\frac{(-1)^{N+2}}{g^{\mathrm{L},[k]}_N}\frac{\beta_{k+\ell_N+1}-\beta_k+u_N}{\beta_{k+\ell_N+1}+u_N+1+b_r+v_N}, && \chi_N=r,\\
(L^{\mathrm{L}}_k)_{N+1,N}&=\beta_{k+\ell_N+1}-\beta_k+u_N, && \chi_N\in\{r+1,\dots,q-1\},\\
(L^{\mathrm{L}}_k)_{N+1,N}&=-\frac{g^{\mathrm{L},[k-1]}_{N+1}}{(-1)^{N+1}}\left(\beta_{k+\ell_N+1}-\beta_k+u_N\right), && \chi_N=q.
\end{aligned}$$

Here every $g^{\mathrm{L},[a]}_M$ appearing in a beta-block position is the explicit Pochhammer product in the first line of (127)*; empty ranges of conditions are omitted.*

Proof. If $r=0$, every coordinate of the Laguerre-like vector corresponds to one of the weights v_ℓ. Therefore

$$\frac{g^{\mathrm{L},[k-1]}_{N+1}}{g^{\mathrm{L},[k]}_N}=-1,$$

while the second line of (129) contributes another minus sign, proving (131).

Let $r>0$. From step N to step $N+1$, coordinate χ_N is incremented. At the next step the increment occurs in coordinate χ_N+1, unless $\chi_N=q$, in which case it returns to coordinate 1. The four displayed equations correspond, respectively, to $\chi_N<r$, $\chi_N=r$, $r<\chi_N<q$, and $\chi_N=q$. Combining (127) with (129) proves the formulas. □

7.4. **Consistency with the classical one-weight reductions.** The cases $q = 1$ in the two real-line settings provide a direct check of the normalizations and of the Christoffel indexing. In the Jacobi-like setting, write

$$\widetilde{a}_i := \alpha_i + a_1 + 1, \qquad \widetilde{b} := b_1 - a_1,$$

so that

$$x^{\alpha_i} w_1(x; a_1, b_1) \propto x^{\widetilde{a}_i - 1}(1-x)^{\widetilde{b}}.$$

Extend these parameters cyclically by

$$\widetilde{a}_{sp+i} := \widetilde{a}_i + s, \qquad s \in \mathbb{N}_0, \quad i \in \{1, \dots, p\}.$$

Writing $N = pm + k$, with $k \in \{0, \dots, p-1\}$, the specialization of the displayed Jacobi-like formulas gives

$$(U_1^{\mathrm{J}})_{N,N} = \frac{(\widetilde{a}_{k+1} + m) \prod_{j=1}^{p} (\widetilde{a}_j + \widetilde{b} + pm + k)}{\prod_{j=1}^{p+1} (\widetilde{a}_{k+j} + \widetilde{b} + (p+1)m + k)},$$

and

$$(L_i^{\mathrm{J}})_{N+1,N} = \frac{(\widetilde{a}_{k+i+1} - \widetilde{a}_i + m)(\widetilde{b} + pm + k + 1) \prod_{j=1}^{p-1} (\widetilde{a}_{i+j} + \widetilde{b} + pm + k)}{\prod_{j=1}^{p+1} (\widetilde{a}_{k+i+j} + \widetilde{b} + (p+1)m + k)}, \qquad i \in \{1, \dots, p\}.$$

These are equations (5.20)–(5.21) of [2], after the identification $r = p$, $a_i = \widetilde{a}_i$, and $b = \widetilde{b}$.

In the Laguerre-like setting, $q = 1$ implies $r = 0$, $s = 1$. With

$$\widetilde{a}_i := \beta_i + a_1 + 1,$$

one has

$$x^{\beta_i} w_0(x; a_1) \propto x^{\widetilde{a}_i - 1} \mathrm{e}^{-x}.$$

The one-weight specialization gives

$$(U_1^{\mathrm{L}})_{N,N} = \widetilde{a}_{k+1} + m,$$

and

$$(L_i^{\mathrm{L}})_{N+1,N} = \widetilde{a}_{k+i+1} - \widetilde{a}_i + m, \qquad i \in \{1, \dots, p\},$$

where the cyclic extension $\widetilde{a}_{sp+i} = \widetilde{a}_i + s$ is used. Equivalently, when this is unfolded into the fundamental range, the second branch acquires the extra term $+1$. These formulas reproduce Theorem 5.4, equivalently equations (5.26)–(5.27), of [2].

7.5. **The mixed Piñeiro specialization.** In the boundary degeneration $\boldsymbol{a}, \boldsymbol{b} \to \boldsymbol{\sigma}$ of the Jacobi-like matrix, both vectors in the matrix of measures are power vectors, giving the mixed Piñeiro system of [3]. Put

$$\boldsymbol{\rho} = (\rho_1, \dots, \rho_p), \qquad \boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_q), \qquad \boldsymbol{\rho}^{[k]} := \mathscr{C}_p^k(\boldsymbol{\rho}), \qquad \boldsymbol{\sigma}^{[k]} := \mathscr{C}_q^k(\boldsymbol{\sigma}).$$

With

$$\boldsymbol{d}_N := \boldsymbol{\nu}_N^*, \qquad \boldsymbol{c}_N := \boldsymbol{\mu}_N^*,$$

define the explicitly normalized leading coefficients

(132)
$$\Lambda_{\mathrm{L},N,i}^{\mathrm{P},[k]} := \frac{(\boldsymbol{\rho}^{[k]} + (\sigma_{\chi_N} + (\boldsymbol{c}_N)_{\chi_N} + 1)\mathbf{1}_p)_{\boldsymbol{d}_N}}{(\boldsymbol{\sigma} - (\sigma_{\chi_N} + (\boldsymbol{c}_N)_{\chi_N})\mathbf{1}_q)_{\boldsymbol{c}_N}} \frac{(-1)^{(\boldsymbol{d}_N)_i - 1}((\rho_i^{[k]} + (\boldsymbol{d}_N)_i)\mathbf{1}_q + \boldsymbol{\sigma})_{\boldsymbol{c}_N}}{((\boldsymbol{d}_N)_i - 1)!(\boldsymbol{\rho}^{[k],*i} + (1 - \rho_i^{[k]} - (\boldsymbol{d}_N)_i)\mathbf{1}_{p-1})_{\boldsymbol{d}_N^{*i}}},$$

and

$$\Lambda_{\mathrm{R},N,i}^{\mathrm{P},[k]} := \frac{(\boldsymbol{\rho} + (\sigma_{\chi_N}^{[k]} + (\boldsymbol{c}_N)_{\chi_N} + 1)\mathbf{1}_p)_{\boldsymbol{d}_N}}{(\boldsymbol{\sigma}^{[k]} - (\sigma_{\chi_N}^{[k]} + (\boldsymbol{c}_N)_{\chi_N})\mathbf{1}_q)_{\boldsymbol{c}_N}} \frac{(-1)^{(\boldsymbol{d}_N)_i - 1}((\rho_i + (\boldsymbol{d}_N)_i)\mathbf{1}_q + \boldsymbol{\sigma}^{[k]})_{\boldsymbol{c}_N}}{((\boldsymbol{d}_N)_i - 1)!(\boldsymbol{\rho}^{*i} + (1 - \rho_i - (\boldsymbol{d}_N)_i)\mathbf{1}_{p-1})_{\boldsymbol{d}_N^{*i}}}. \tag{133}$$

These are finite Pochhammer products obtained from the displayed mixed Piñeiro step-line forms of [3] after the left and right cyclic Christoffel shifts, respectively. Moreover, after setting

$$\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{\sigma}, \qquad \alpha = \boldsymbol{\rho},$$

the product in (126), combined with the Jacobi-like leading coefficient before normalization, gives precisely (132). Therefore the Jacobi-like lower factor formula (130) specializes exactly to (134).

Corollary 7.5 (Complete mixed Piñeiro bidiagonal factorization)**.** *The mixed Piñeiro step-line recurrence matrix of [3] admits*

$$T^{\mathrm{P}} = L^{\mathrm{P}}_1 \cdots L^{\mathrm{P}}_p U^{\mathrm{P}}_q \cdots U^{\mathrm{P}}_1,$$

where

$$(L^{\mathrm{P}}_k)_{N+1,N} = \frac{\Lambda^{\mathrm{P},[k]}_{\mathrm{L},N,\iota_N}}{\Lambda^{\mathrm{P},[k-1]}_{\mathrm{L},N+1,\iota_{N+1}}}, \qquad k \in \{1,\dots,p\}, \tag{134}$$

$$(U^{\mathrm{P}}_k)_{N,N} = \frac{\Lambda^{\mathrm{P},[k-1]}_{\mathrm{R},N,\iota_N}}{\Lambda^{\mathrm{P},[k]}_{\mathrm{R},N,\iota_N}}, \qquad k \in \{1,\dots,q\}. \tag{135}$$

Proof. The lower and upper entries are the quotients of the normalized leading coefficients of the left and right Christoffel transforms, respectively. Equations (132) and (133) evaluate each of those coefficients as a finite Pochhammer product, so that the displayed factorization contains no unevaluated integral or determinant. □

An independent Cauchy-minor derivation of the same factor entries, together with the positivity regions, stochastic normalization, and Markov-chain applications of this specialization, is given in [12].

7.6. **Upper factors when the right Christoffel chain is not closed.** Outside the two one-weight reductions and the mixed Piñeiro specialization of [3], the right Christoffel transforms do not in general remain inside the same explicit parametric family. The upper factors must then be written in the Christoffel tau-form. The formulas for the two families follow.

For the Jacobi-like system define the Gauss–Borel normalized values

$$\mathfrak{b}^{\mathrm{J},[0]}_{N,j} := \left(\mathscr{B}^{\mathrm{J}}_N\right)^{(j)}(0) = \frac{\mathscr{B}^{\mathrm{J},(j)}_N(0)}{h^{\mathrm{J}}_N}, \qquad j \in \{1,\dots,q\}.$$

For the Laguerre-like system set

$$\mathfrak{b}^{\mathrm{L},[0]}_{N,\rho} := \left(\mathscr{B}^{\mathrm{L}}_N\right)^{(\rho)}(0) = \frac{\mathscr{U}^{\mathrm{L},(\rho)}_N(0)}{h^{\mathrm{L}}_N}, \qquad \rho \in \{1,\dots,q\},$$

where $\mathscr{U}^{\mathrm{L},(\rho)}_N$ denotes the unified Laguerre-like component introduced before (118). These are values of the normalized B-forms, not values of the unnormalized hypergeometric representatives.

Define

$$\tau^{\mathrm{J},B}_{0,N} := 1, \qquad \tau^{\mathrm{J},B}_{b,N} := \det\left[\mathfrak{b}^{\mathrm{J},[0]}_{N+\alpha-1,\nu}\right]^{b}_{\alpha,\nu=1}, \qquad b \in \{1,\dots,q\},$$

and

$$\tau^{\mathrm{L},B}_{0,N} := 1, \qquad \tau^{\mathrm{L},B}_{b,N} := \det\left[\mathfrak{b}^{\mathrm{L},[0]}_{N+\alpha-1,\nu}\right]^{b}_{\alpha,\nu=1}, \qquad b \in \{1,\dots,q\}.$$

When the displayed determinants do not vanish, the upper factors are

$$(U^{\mathrm{J}}_b)_{N,N} = -\frac{\tau^{\mathrm{J},B}_{b-1,N}\tau^{\mathrm{J},B}_{b,N+1}}{\tau^{\mathrm{J},B}_{b-1,N+1}\tau^{\mathrm{J},B}_{b,N}}, \qquad b \in \{1,\dots,q\}, \tag{136}$$

$$(U^{\mathrm{L}}_b)_{N,N} = -\frac{\tau^{\mathrm{L},B}_{b-1,N}\tau^{\mathrm{L},B}_{b,N+1}}{\tau^{\mathrm{L},B}_{b-1,N+1}\tau^{\mathrm{L},B}_{b,N}}, \qquad b \in \{1,\dots,q\}. \tag{137}$$

Thus the nonclosed upper factors are finite Christoffel determinants built from the original Gauss–Borel normalized B-forms. In the closed Piñeiro case of [3], these determinants reduce to the leading-coefficient quotients in (135); outside the closed case the tau-determinants above are the correct finite formulas.

The tau-form also admits an equivalent expression in terms of moment minors, which does not require the right Christoffel chain to close. For either family $\star \in \{\mathrm{J}, \mathrm{L}\}$, let $\mathscr{M}^\star$ be its scalar step-line moment matrix and define the shifted leading minors

$$\Delta^\star_{b,0} := 1, \qquad \Delta^\star_{b,N} := \det \left[\mathscr{M}^\star_{b+\alpha,\beta}\right]_{\alpha,\beta=0}^{N-1}, \qquad b \in \{0,\dots,q\}, \quad N \geq 1. \tag{138}$$

Thus $\Delta^\star_{0,N}$ is the ordinary leading moment minor and $\Delta^\star_{b,N}$ is obtained by starting the row sequence after b cyclic row steps.

Proposition 7.6 (Shifted-minor formula for the upper factors)**.** *Assume that the minors in* (138) *which occur below are nonzero. Then*

$$\tau^{\star,B}_{b,N} = (-1)^{bN} \frac{\Delta^\star_{b,N}}{\Delta^\star_{0,N}}, \qquad b \in \{0,\dots,q\}, \quad N \in \mathbb{N}_0. \tag{139}$$

Consequently,

$$(U^\star_b)_{N,N} = \frac{\Delta^\star_{b-1,N}\Delta^\star_{b,N+1}}{\Delta^\star_{b-1,N+1}\Delta^\star_{b,N}}. \tag{140}$$

Proof. In the Gauss–Borel realization (18), the value at the origin of the ν-th component of the N-th normalized B-form is the entry $\mathscr{L}_{N,\nu-1}$. Therefore $\tau^{\star,B}_{b,N}$ is the minor of $\mathscr{L}$ with rows $N,\dots,N+b-1$ and columns $0,\dots,b-1$. Apply the complementary-minor identity to the leading $(N+b)$-square Gauss–Borel factorization. The upper-triangular factor cancels, and moving the first b rows past the following N rows contributes $(-1)^{bN}$. The remaining quotient is precisely $\Delta^\star_{b,N}/\Delta^\star_{0,N}$, which proves (139). Substitution in (136) or (137) cancels the four ordinary leading minors and the four signs, giving (140). □

7.7. **The first mixed upper chain:** $q = 2$**.** For $q = 2$, the upper Christoffel chain has two factors. Let p be arbitrary and write

$$N = pu_N + \ell_N = 2v_N + \varepsilon_N, \qquad \ell_N \in \{0,\dots,p-1\}, \qquad \varepsilon_N \in \{0,1\}.$$

Then

$$\iota_N = \ell_N + 1, \qquad \chi_N = \varepsilon_N + 1.$$

For either the Jacobi-like or the Laguerre-like family, let $\mathfrak{b}^{[0]}_{N,j}$ denote the Gauss–Borel normalized value at the origin of the j-th component of the step-line B-form. Thus one reads $\mathfrak{b}^{[0]}_{N,j} = \mathfrak{b}^{\mathrm{J},[0]}_{N,j}$ in the Jacobi-like case and $\mathfrak{b}^{[0]}_{N,j} = \mathfrak{b}^{\mathrm{L},[0]}_{N,j}$ in the Laguerre-like case. The two upper Christoffel tau-functions are

$$\tau^B_{1,N} = \mathfrak{b}^{[0]}_{N,1}, \tag{141}$$

$$\tau^B_{2,N} = \det \begin{bmatrix} \mathfrak{b}^{[0]}_{N,1} & \mathfrak{b}^{[0]}_{N,2} \\ \mathfrak{b}^{[0]}_{N+1,1} & \mathfrak{b}^{[0]}_{N+1,2} \end{bmatrix}. \tag{142}$$

To make these determinants directly computable, introduce the constant terms and the normalizing leading coefficients

$$\mathscr{C}^{\mathrm{J}}_{N,j} := \mathscr{B}^{\mathrm{J},(j)}_N(0), \quad \mathscr{H}^{\mathrm{J}}_N := h^{\mathrm{J}}_N,$$
$$\mathscr{C}^{\mathrm{L}}_{N,j} := \mathscr{U}^{\mathrm{L},(j)}_N(0), \quad \mathscr{H}^{\mathrm{L}}_N := h^{\mathrm{L}}_N.$$

Thus, in either family,

$$\mathfrak{b}^{[0]}_{N,j} = \frac{\mathscr{C}_{N,j}}{\mathscr{H}_N}.$$

The quantities $\mathscr{C}^{\mathrm{J}}_{N,j}$ are obtained by setting $x = 0$ in the finite sum (112); the Laguerre-like quantities $\mathscr{C}^{\mathrm{L}}_{N,j}$ are obtained in the same way from the finite component system (116). No limiting operation or unevaluated integral is involved.

Put

$$\Delta_N^{(2)} := \mathscr{C}_{N,1}\mathscr{C}_{N+1,2} - \mathscr{C}_{N,2}\mathscr{C}_{N+1,1}.$$

Substitution in (141) and (142) gives

$$\tau_{1,N}^B = \frac{\mathscr{C}_{N,1}}{\mathscr{H}_N}, \qquad \tau_{2,N}^B = \frac{\Delta_N^{(2)}}{\mathscr{H}_N\mathscr{H}_{N+1}}.$$

Consequently, whenever the displayed denominators do not vanish, the two upper bidiagonal factors are the finite expressions

$$(U_1)_{N,N} = -\frac{\mathscr{C}_{N+1,1}\mathscr{H}_N}{\mathscr{C}_{N,1}\mathscr{H}_{N+1}},$$

$$(U_2)_{N,N} = -\frac{\mathscr{C}_{N,1}\Delta_{N+1}^{(2)}\mathscr{H}_{N+1}}{\mathscr{C}_{N+1,1}\Delta_N^{(2)}\mathscr{H}_{N+2}}.$$

Thus the step-line recurrence matrix factors as

$$T = L_1 \cdots L_p U_2 U_1. \tag{143}$$

The lower factors in (143) are still obtained from the closed column-Christoffel chain. In the Jacobi-like case this gives

$$(L_k^{\mathrm{J}})_{N+1,N} = -\frac{g_{N+1}^{\mathrm{J},[k-1]}}{g_N^{\mathrm{J},[k]}} \frac{\alpha_{k+\ell_N+1} - \alpha_k + u_N}{\alpha_{k+\ell_N+1} + u_N + 1 + b_{\varepsilon_N+1} + v_N}, \qquad k \in \{1, \dots, p\},$$

where

$$g_N^{\mathrm{J},[k]} = \begin{cases} -\dfrac{(\boldsymbol{a}-(b_1+v_N)\mathbf{1}_q)_{v_N}}{(\alpha^{[k]}+(b_1+v_N+1)\mathbf{1}_p)_{\boldsymbol{d}_N}}, & \varepsilon_N = 0, \\[2ex] -\dfrac{(\boldsymbol{a}-(b_2+v_N)\mathbf{1}_q)_{v_N}(b_1-b_2)}{(\alpha^{[k]}+(b_2+v_N+1)\mathbf{1}_p)_{\boldsymbol{d}_N}}, & \varepsilon_N = 1. \end{cases}$$

For the Laguerre-like family the same formula holds with $\boldsymbol{\alpha}$ replaced by $\boldsymbol{\beta}$, and with the Laguerre normalizers $g_N^{\mathrm{L},[k]}$. Since $q = 2$, there are only two Laguerre regimes: when $r = 0$ one has the pure Euler formula

$$(L_k^{\mathrm{L}})_{N+1,N} = \beta_{k+\ell_N+1} - \beta_k + u_N,$$

and when $r = 1$ the two transitions are the beta–Euler and Euler–beta cases of Theorem 7.4. This example shows the general pattern: the closed column side gives the lower factors by leading-coefficient quotients, while the nonclosed two-step row side gives the upper factors through the rank-one and rank-two Christoffel tau-functions (141)–(142).

7.8. **An exact mixed Jacobi-like factorization.** Consider a mixed example in which both Christoffel chains have length two. Take

$$p = q = 2, \qquad \boldsymbol{\alpha} = [\tfrac{1}{5}\ \tfrac{2}{5}], \qquad \boldsymbol{a} = [0\ \tfrac{1}{3}], \qquad \boldsymbol{b} = [1\ \tfrac{4}{3}].$$

If

$$s_{u,v,i} := u + v + \alpha_i + 1,$$

then the bimoments are rational:

$$\mathscr{M}_{(u,j),(v,i)} = \frac{1}{(s_{u,v,i})_{1+\delta_{j,1}}(s_{u,v,i}+\frac{1}{3})_{1+\delta_{j,2}}}, \qquad u, v \in \mathbb{N}_0. \tag{144}$$

Gauss–Borel elimination may therefore be performed without floating-point arithmetic. The leading 5×5 block of the resulting $(2,2)$-banded recurrence matrix is

$$T^{[5]} \doteq \begin{bmatrix} 0.226974 & 3.54657 & 1 & 0 & 0 \\ 0.00845235 & 0.269142 & 0.201390 & 1 & 0 \\ 0.00374871 & 0.217645 & 0.346208 & 4.47968 & 1 \\ 0 & 0.00217769 & 0.0114150 & 0.332614 & 0.196135 \\ 0 & 0 & 0.00416138 & 0.278947 & 0.365588 \end{bmatrix}.$$

The subdiagonal entries of the two lower factors are

$$\operatorname{subdiag} L_1^{[5]} \doteq (0.00962773, 0.598156, 0.0282521, 0.853697),$$
$$\operatorname{subdiag} L_2^{[5]} \doteq (0.0276116, 0.562346, 0.0350197, 0.830354),$$

and the diagonal entries of the two upper factors are

$$\operatorname{diag} U_1^{[5]} \doteq (0.868421, 3.28520, 0.122428, 2.18223, 0.0839606),$$
$$\operatorname{diag} U_2^{[5]} \doteq (0.261364, 0.0417234, 1.13695, 0.0489025, 1.33328).$$

The displayed decimals are only for readability. Using the exact rational values obtained from (144), one has

$$T^{[5]} = L_1^{[5]} L_2^{[5]} U_2^{[5]} U_1^{[5]}$$

entry by entry. More generally, a direct comparison of the first eight rows and all ten columns containing their band entries gives the zero 8×10 difference matrix. This checks simultaneously the closed formulas for the lower factors, the τ-determinant formulas for the upper factors, and the cyclic ordering of the four Christoffel steps.

8. Confluent limits and the Askey scheme

The two real-line families of this paper occupy the continuous branch of the Askey scheme [9]: the Jacobi-like system sits at the apex and the Laguerre-like system is its confluent limit. We first assemble a rectangular rank-one array from moving Christoffel transforms of their scalar corners and moving power parameters on the opposite side. Its central limit recovers the full $p \times q$ Gaussian mixed system of Daems–Kuijlaars [6], with distinct starting and ending points. The one-sided Jacobi–Piñeiro and Laguerre-I limits behind this construction are classical [16, Equations (3.35), (3.39), and (3.40)]; here they are assembled on both sides of a rectangular mixed matrix. We then study a different simultaneous scaling of the canonical Laguerre-like vector. That fixed vector yields a Hermite jet with one-sided gamma convolutions, rather than the generic Gaussian system. All limits are formulated at the level of moments and weight vectors, with finite-index consequences whenever the relevant limiting determinant is nonzero.

8.1. **From Jacobi-like to Laguerre-like.**

We begin with the elementary one-factor limit and then perform the simultaneous confluence of the full weight vector. Writing $b = a + 1 + \kappa$, the beta density of the Jacobi-like construction satisfies, for $x > 0$,

$$\kappa^a \Gamma(\kappa+1) \mathscr{B}_{a,b}\left(\frac{x}{\kappa}\right) = x^a \left(1 - \frac{x}{\kappa}\right)^{\kappa} \xrightarrow[\kappa\to+\infty]{} x^a \mathrm{e}^{-x} = \mathscr{G}_a(x),$$

uniformly on compact subsets of $(0,\infty)$, with relative error $\mathrm{O}(\kappa^{-1})$.

Let $q = r + s$. Fix pairwise distinct positive numbers $\omega_1, \dots, \omega_s$ and introduce the Jacobi-like lower parameters

$$b_{r+\ell}(t) := t\omega_\ell, \qquad \kappa_\ell(t) := t\omega_\ell - a_{r+\ell} - 1, \qquad \ell \in \{1, \dots, s\}.$$

They are admissible for all sufficiently large t. Set

$$c_t := \prod_{\ell=1}^{s} \kappa_\ell(t), \qquad d_t := \prod_{\ell=1}^{s} \kappa_\ell(t)^{a_{r+\ell}} \Gamma(\kappa_\ell(t) + 1),$$

and write

$$\boldsymbol{b}^{\mathrm{J}}(t) := \big(b_1, \dots, b_r, b_{r+1}(t), \dots, b_q(t)\big).$$

Unless stated otherwise, every limit below is taken as $t \to +\infty$, with $x > 0$, z in a compact subset of a common fundamental strip, $\boldsymbol{a}, \boldsymbol{b}, \omega_1, \dots, \omega_s$, the power exponents, and the multi-indices fixed. Only $b_{r+\ell}(t)$, $\kappa_\ell(t)$, c_t, and d_t vary with t. For the scaled Jacobi-like base weight

$$W_0^{(t)}(x) := d_t\, w_0\left(\frac{x}{c_t}; \boldsymbol{a}, \boldsymbol{b}^{\mathrm{J}}(t)\right)$$

the Mellin transform is

$$\mathcal{M}[W_0^{(t)}](z) = d_t c_t^z \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b}) \prod_{\ell=1}^s \Gamma(z + t\omega_\ell)} \xrightarrow[t\to+\infty]{} \frac{\Gamma(z\mathbf{1}_q + \boldsymbol{a})}{\Gamma(z\mathbf{1}_r + \boldsymbol{b})}.$$

Indeed,

$$\frac{\kappa_\ell(t)^{z+a_{r+\ell}}\Gamma(\kappa_\ell(t)+1)}{\Gamma(\kappa_\ell(t)+z+a_{r+\ell}+1)} \xrightarrow[t\to+\infty]{} 1$$

locally uniformly in z. Thus $\mathcal{M}[W_0^{(t)}](z) \to \mathcal{M}[w_0](z)$ locally uniformly on compact subsets of every common fundamental strip, where w_0 is the Laguerre-like base weight (57). In particular, every fixed Mellin moment converges.

For $h \in \{1, \dots, r\}$, define

$$W_h^{(t)}(x) := d_t\, w_h\left(\frac{x}{c_t}; \boldsymbol{a}, \boldsymbol{b}^{\mathrm{J}}(t)\right).$$

Its Mellin transform is

$$\mathcal{M}[W_h^{(t)}](z) = \frac{\mathcal{M}[W_0^{(t)}](z)}{z + b_h},$$

so every fixed Mellin moment of $W_h^{(t)}$ converges to the corresponding moment of w_h in (59).

The last s shifted Jacobi-like weights collapse to one direction under the common dilation. To recover the Euler directions while preserving the step-line flags, put

$$\varepsilon_\ell(t) := (t\omega_\ell)^{-1}, \qquad Z_\ell^{(t)}(x) := d_t\, w_{r+\ell}\left(\frac{x}{c_t}; \boldsymbol{a}, \boldsymbol{b}^{\mathrm{J}}(t)\right).$$

Lemma 8.1 (Prefix-compatible Newton confluence)**.** *For $1 \le \ell \le k \le s$, set*

$$R_{k\ell}^{(t)} := \frac{(-1)^{k-1}}{\varepsilon_\ell(t) \prod\limits_{\substack{1\le h\le k\\ h\neq \ell}} \big(\varepsilon_\ell(t) - \varepsilon_h(t)\big)}, \qquad R_{k\ell}^{(t)} := 0 \quad (\ell > k),$$

and define

$$\widehat{V}_k^{(t)}(x) := \sum_{\ell=1}^k R_{k\ell}^{(t)} Z_\ell^{(t)}(x).$$

Then, identically in z,

$$\sum_{\ell=1}^k \frac{R_{k\ell}^{(t)}}{z + t\omega_\ell} = \frac{z^{k-1}}{\prod\limits_{h=1}^k \big(1 + \varepsilon_h(t) z\big)}. \tag{145}$$

Consequently,

$$\mathcal{M}[\widehat{V}_k^{(t)}](z) = \mathcal{M}[W_0^{(t)}](z) \frac{z^{k-1}}{\prod_{h=1}^k (1 + \varepsilon_h(t) z)} \longrightarrow z^{k-1}\mathcal{M}[w_0](z) = \mathcal{M}[v_k](z) \tag{146}$$

locally uniformly on every compact subset of a common fundamental strip.

Proof. For $f_z(\varepsilon) = \varepsilon/(1+z\varepsilon)$, the Newton divided difference satisfies

$$[0, \varepsilon_1, \dots, \varepsilon_k] f_z = \frac{(-z)^{k-1}}{\prod_{h=1}^k (1 + z\varepsilon_h)}.$$

Its barycentric expansion is exactly (145). Since $\mathcal{M}[Z_\ell^{(t)}](z) = \mathcal{M}[W_0^{(t)}](z)/(z + t\omega_\ell)$, this proves (146).

The matrix $R_t = [R_{k\ell}^{(t)}]_{k,\ell=1}^s$ is lower triangular, with

$$R_{kk}^{(t)} = \frac{(-1)^{k-1}}{\varepsilon_k(t) \prod_{h<k} (\varepsilon_k(t) - \varepsilon_h(t))} \neq 0.$$

This triangularity is the prefix property needed below. Indeed, if $m_1 \geq \cdots \geq m_s$, $\widehat{\boldsymbol{v}} = R_t \boldsymbol{v}$, and $Q_k \in \mathbb{P}_{m_k-1}$, then the coefficient of v_ℓ in $\sum_k Q_k \widehat{v}_k$ is $\sum_{k\geq\ell} R^{(t)}_{k\ell} Q_k \in \mathbb{P}_{m_\ell-1}$. The inverse matrix has the same triangularity, so the two polynomial flags coincide. The corresponding orthogonality conditions also coincide, because, at each degree, their active components form a prefix on which the leading principal block of R_t is invertible. □

Thus the transformed vector

$$\widehat{\mathbf{U}}^{(t)} := \big(W_1^{(t)}, \dots, W_r^{(t)}, \widehat{V}_1^{(t)}, \dots, \widehat{V}_s^{(t)}\big) \tag{147}$$

converges to the Laguerre-like vector $\mathbf{U}^{\mathrm{L}} = (w_1, \dots, w_r, v_1, \dots, v_s)$ at every fixed Mellin moment. The triangular transformation preserves every incomplete step-line cycle. More generally, it preserves the component spaces occurring in the Laguerre admissible range because it only mixes the Euler block and $\boldsymbol{\eta}$ is a nonincreasing step-line multi-index.

Identify $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $\boldsymbol{n}^{\mathrm{J}} = \boldsymbol{m}$, and $\boldsymbol{m}^{\mathrm{J}} = (\boldsymbol{n}, \boldsymbol{\eta})$. Whenever the limiting normalizing determinant is nonzero, the finite orthogonality systems and Stirling's quotient give

$$\frac{\mathscr{A}^{\mathrm{J}}_{\boldsymbol{m},(\boldsymbol{n},\boldsymbol{\eta})}(x/c_t)}{\displaystyle\prod_{\ell=1}^{s} \Gamma(t\omega_\ell + \eta_\ell + 1)} \longrightarrow \mathscr{A}^{\mathrm{L}}_{(\boldsymbol{n},\boldsymbol{\eta}),\boldsymbol{m}}(x),$$

$$-d_t \prod_{\ell=1}^{s} \kappa_\ell(t)^{\eta_\ell} \mathscr{B}^{\mathrm{J}}_{\boldsymbol{m},(\boldsymbol{n},\boldsymbol{\eta})}(x/c_t) \longrightarrow \mathscr{B}^{\mathrm{L}}_{(\boldsymbol{n},\boldsymbol{\eta}),\boldsymbol{m}}(x).$$

For the i-th power component the only large-parameter quotient is

$$\prod_{\ell=1}^{s} \frac{\Gamma(\beta_i + t\omega_\ell + \eta_\ell + 1)}{\Gamma(t\omega_\ell + \eta_\ell + 1)\, \kappa_\ell(t)^{\beta_i}} \longrightarrow 1,$$

which proves the first limit coefficientwise. The second limit follows directly at the Mellin-transform level, without an interchange of nonterminating hypergeometric series. Indeed, the transform of its left-hand side is

$$\mathscr{M}[W_0^{(t)}](z) \frac{(\boldsymbol{\beta} + (1-z)\mathbf{1}_p)_{\boldsymbol{m}}}{\displaystyle\prod_{h=1}^{r} (z + b_h)_{n_h}} \prod_{\ell=1}^{s} \frac{\kappa_\ell(t)^{\eta_\ell}}{(z + t\omega_\ell)_{\eta_\ell}},$$

which converges locally uniformly to (72). The Laguerre-like hypergeometric expression (73) is then the already established residue expansion of this limiting Mellin transform.

Proposition 8.2 (Finite-index stability of the confluence)**.** *Let $\widehat{\mathscr{M}}^{(t)}$ be the moment matrix of* (147)*, and let $\mathscr{M}^{\mathrm{L}}$ be the Laguerre-like moment matrix. Fix K. Assume that the leading minors, through the orders needed at level K, of*

$$\mathscr{M}^{\mathrm{L}}, \qquad \Lambda^a \mathscr{M}^{\mathrm{L}} \quad (0 \leq a \leq q), \qquad \mathscr{M}^{\mathrm{L}} (\Lambda^b)^{\top} \quad (0 \leq b \leq p),$$

which occur in the Christoffel chains are nonzero. Then, for all sufficiently large t, the corresponding Gauss–Borel factorizations exist. At every fixed index not exceeding K, the normalized forms, the recurrence coefficients, and the entries of all canonical lower and upper bidiagonal factors converge to their Laguerre-like counterparts.

Proof. Equation (146) and the analogous limits for $W_h^{(t)}$ give $\widehat{\mathscr{M}}^{(t)} \to \mathscr{M}^{\mathrm{L}}$ entry by entry; the same holds after every fixed left or right shift. Each determinant in the statement is a polynomial in finitely many moments, so its nonzero limit remains nonzero for large t. Finite Gauss elimination, Cramer's rule, the recurrence pairings, and the Christoffel tau-quotients are rational functions of those same moments, with precisely these determinants as denominators. Their convergence follows. □

Finally, this transport is related exactly to the original Jacobi-like step-line matrix. Put

$$\mathsf{R}_t := \operatorname{diag}(I_r, R_t), \qquad j_N := \ell_q(N) + 1, \qquad g_{t,N} := (\mathsf{R}_t)_{j_N j_N} c_t^{\rho_q(N)}, \qquad G_t := \operatorname{diag}(g_{t,0}, g_{t,1}, \dots).$$

Because R_t is lower triangular, its repeated block action on the scalar row order restricts to every leading prefix. Hence the normalized step-line forms satisfy

$$\widehat{\mathscr{B}}_N^{(t)}(x) = d_t g_{t,N} \mathscr{B}_N^{\mathrm{J}}(x/c_t), \qquad \widehat{\mathscr{A}}_N^{(t)}(x) = \frac{\mathscr{A}_N^{\mathrm{J}}(x/c_t)}{c_t d_t g_{t,N}}.$$

Therefore their recurrence matrix is

$$\boxed{\widehat{T}_t = G_t\big(c_t T_{\mathrm{J}}(t)\big)G_t^{-1}} \xrightarrow[t\to+\infty]{\text{entrywise at fixed indices}} T_{\mathrm{L}}. \tag{148}$$

Since $g_{t,N+q} = c_t g_{t,N}$, the q-th superdiagonal remains normalized to one. Proposition 8.2 applies to the canonical bidiagonal factors of $\widehat{T}_t$; the diagonal gauge in (148) is the normalization intrinsic to the Jacobi–Laguerre confluence.

8.2. **Moving-Christoffel/power confluence to the full Daems–Kuijlaars system.** The generic Gaussian mixed system requires Christoffel orders separated on the central-limit scale. It is therefore obtained from an associated rectangular array assembled out of q members of a moving orbit of the scalar corner of our families, rather than from a single canonical q-component vector. The claims below concern its weights, moments, and fixed-index forms; they do not transfer the factorization formulas of the canonical q-component family to this assembled array. Put

$$\phi_0(t) := \pi^{-1/2}\mathrm{e}^{-t^2}, \qquad \varepsilon_A := \sqrt{2/A}, \qquad \mathrm{d}\nu_A(x) := \frac{x^{A-1}\mathrm{e}^{-x}}{\Gamma(A)}\,\mathrm{d}x.$$

The measure $\mathrm{d}\nu_A$ is the normalized $r = 0, s = 1$ Laguerre-like member with parameter $a = A - 1$.

Theorem 8.3 (Moving-Christoffel/power limit to the full Gaussian mixed system)**.** *Fix real vectors $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ and $\boldsymbol{\eta} = (\eta_1, \dots, \eta_q)$. Choose $H > -\min_j \eta_j$, integers $k_{j,A} \geq 0$, and real numbers $\beta_{i,A}$ such that*

$$\varepsilon_A k_{j,A} \longrightarrow \eta_j + H, \qquad \varepsilon_A \beta_{i,A} \longrightarrow \xi_i - H.$$

The auxiliary H is a common scalar rank-one gauge: it contributes e^{Ht} on the row side and e^{-Ht} on the column side and therefore cancels from every matrix entry. Define the $q \times p$ rank-one matrix of measures

$$\mathrm{d}\Psi_{ji}^{(A)}(x) := \left(\frac{x}{A}\right)^{k_{j,A}} \left(\frac{x}{A}\right)^{\beta_{i,A}} \mathrm{d}\nu_A(x). \tag{149}$$

For $t > -\varepsilon_A^{-1}$, define its pushed-forward density by

$$\widehat{\psi}_{ji}^{(A)}(t) := A\varepsilon_A \frac{\mathrm{d}\Psi_{ji}^{(A)}}{\mathrm{d}x}\big(A(1+\varepsilon_A t)\big), \qquad \mathrm{d}\widehat{\Psi}_{ji}^{(A)}(t) := \widehat{\psi}_{ji}^{(A)}(t)\,\mathrm{d}t,$$

and extend $\widehat{\psi}_{ji}^{(A)}$ by zero to the remaining real line. Then, locally uniformly in t and with convergence of every polynomial moment,

$$\widehat{\psi}_{ji}^{(A)}(t) \longrightarrow \phi_0(t)\mathrm{e}^{\eta_j t}\mathrm{e}^{\xi_i t}. \tag{150}$$

Thus, up to an affine change of variable, transposition, and nonzero diagonal gauges, the limit is the $q \times p$ weight matrix with p starting and q ending points in the Gaussian mixed system of Daems–Kuijlaars. If the entries of $\boldsymbol{\xi}$ are pairwise distinct and, separately, the entries of $\boldsymbol{\eta}$ are pairwise distinct, these points are distinct and the limiting system is perfect. Consequently, at every fixed mixed order, the moment determinants formed from the pushed-forward measures converge, and so do the normalized forms written in the variable t, after the triangular affine change of polynomial basis; the corresponding prelimit determinants are nonzero for all sufficiently large A.

Proof. For one entry put $d_A = k_{j,A} + \beta_{i,A}$ and $c = \eta_j + \xi_i$, so that $\varepsilon_A d_A \to c$. Its total mass is

$$A^{-d_A}\frac{\Gamma(A+d_A)}{\Gamma(A)} \longrightarrow \mathrm{e}^{c^2/4}.$$

Indeed, uniformly for $d_A = \mathrm{O}(\sqrt{A})$,

$$\log\left(A^{-d_A}\frac{\Gamma(A+d_A)}{\Gamma(A)}\right) = \frac{d_A^2 - d_A}{2A} + \mathrm{O}\left(\frac{1+|d_A|^3}{A^2}\right).$$

After division by this mass, the variable x has the gamma law with shape $A + d_A$. Hence

$$\frac{x-A}{A\varepsilon_A} \Longrightarrow N(c/2, 1/2).$$

The same Gamma-quotient calculation, or equivalently the convergence of all centered gamma cumulants, gives convergence of every polynomial moment. Since

$$\mathrm{e}^{c^2/4}\pi^{-1/2}\mathrm{e}^{-(t-c/2)^2} = \phi_0(t)\mathrm{e}^{ct},$$

this proves (150). Local uniformity follows from Stirling's formula on compact t-sets.

To identify the Brownian system, write

$$P_\tau(a,y) := \frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{(y-a)^2}{2\tau}\right), \qquad 0<\tau<1.$$

With $y = \sqrt{2\tau(1-\tau)}\,t$, put

$$\xi_i = a_i\sqrt{\frac{2(1-\tau)}{\tau}}, \qquad \eta_j = b_j\sqrt{\frac{2\tau}{1-\tau}}.$$

Then $P_\tau(a_i,y)P_{1-\tau}(b_j,y)\,\mathrm{d}y$ is a common nonzero constant, times separate row and column constants, times $\phi_0(t)\mathrm{e}^{\eta_j t}\mathrm{e}^{\xi_i t}\,\mathrm{d}t$. This is precisely the matrix in the Brownian formulation of [6, Equation (6.2)]. Perfectness for distinct points follows from the nondegeneracy argument in [6, Lemma 6.1 and Equation (6.5)]. Finite determinant continuity gives the final assertion. □

Each row in (149) has, up to a nonzero constant, the scalar Laguerre-like base weight $\mathscr{G}_{A-1+k_{j,A}}$, multiplied by the column power vector. Moreover,

$$\varepsilon_A(k_{j,A} - k_{\ell,A}) \longrightarrow \eta_j - \eta_\ell,$$

so distinct η_j's retain distinct ending points on the central scale. The same orbit is already present in the Jacobi-like scalar corner, since

$$w_1(z; A-1+k, A-1+k+\kappa) = \frac{z^{A-1+k}(1-z)^\kappa}{\Gamma(\kappa+1)}. \tag{151}$$

After $x = \kappa z$, the common gauge $\Gamma(\kappa+1)\kappa^A$, the row gauge κ^k, and the corresponding column power gauges turn (151) into $x^{A-1+k}(1-x/\kappa)^\kappa\,\mathrm{d}x$. Together with the row and column normalizations $\Gamma(A)^{-1}A^{-k}$ and $A^{-\beta_{i,A}}$ already used in (149), these are exactly the normalized Laguerre arrays above in the limit $\kappa\to\infty$. A simultaneous choice $\kappa_A/A^2 \to \infty$ gives the same Gaussian limit directly, including all fixed polynomial moments. Indeed, under each tilted gamma law $X = \mathrm{O}_{\mathbb{P}}(A)$ and $X^2/\kappa_A \to 0$, while gamma exponential-moment bounds give uniform integrability. In this precise sense, the complete Daems–Kuijlaars system is a confluence of rectangular arrays assembled from the moving Christoffel orbits of both scalar reductions of the present families.

8.3. **A different central limit of the canonical Laguerre-like vector.** Here the Euler orders $0,\dots,s-1$ remain fixed, in contrast with the moving orders of Theorem 8.3. The central scaling must be applied to the complete Mellin convolution. For $\rho>0$ and $d>0$, define the one-sided gamma-convolution operator

$$(K_{\rho,d}f)(t) := \frac{\rho^d}{\Gamma(d)}\int_0^\infty u^{d-1}\mathrm{e}^{-\rho u}f(t+u)\,\mathrm{d}u, \qquad K_{\rho,0} := I. \tag{152}$$

Thus $K_{\rho,d}K_{\rho,e} = K_{\rho,d+e}$ and $K_{\rho,1} = \rho J_\rho$, where $(J_\rho f)(t) := \int_0^\infty \mathrm{e}^{-\rho u}f(t+u)\,\mathrm{d}u$.

Theorem 8.4 (Canonical gamma-convolution and Hermite-jet limit)**.** *Let $A \to +\infty$. For $j \le s$ and $h \le r$, assume*

$$a_{r+j}(A) + 1 = A + \sqrt{2A}\,\widehat{a}_j + \mathrm{O}(1),$$

$$\varepsilon_A := \sqrt{2s/A}, \quad c_A := A^s, \quad \widehat{a}_* := s^{-1/2}\sum_{j=1}^{s}\widehat{a}_j, \quad x = c_A(1+\varepsilon_A t),$$

$$\varepsilon_A(a_h(A)+1) \longrightarrow \rho_h > 0, \qquad b_h(A) - a_h(A) \longrightarrow d_h \ge 0.$$

Write $w_{0,A}$, $w_{h,A}$, and $v_{\ell,A} = \mathscr{D}^{\ell-1}w_{0,A}$ for the corresponding weights and put

$$Z_A := \mathscr{M}[w_{0,A}](1) = \frac{\prod_{\nu=1}^{r+s}\Gamma(a_\nu(A)+1)}{\prod_{h=1}^{r}\Gamma(b_h(A)+1)}, \qquad Z_{h,A} := \mathscr{M}[w_{h,A}](1) = \frac{Z_A}{b_h(A)+1},$$

and define

$$\phi_{\widehat{a}_*}(t) := \pi^{-1/2}\mathrm{e}^{-(t-\widehat{a}_*)^2}, \qquad F := K_{\rho_1,d_1}\cdots K_{\rho_r,d_r}\phi_{\widehat{a}_*}.$$

Then, weakly for the beta rows and distributionally for the Euler rows, with convergence of every fixed polynomial moment,

$$\frac{c_A\varepsilon_A}{Z_{h,A}}w_{h,A}(c_A(1+\varepsilon_A t)) \longrightarrow K_{\rho_h,1}F(t), \qquad h \in \{1,\dots,r\},$$

$$\frac{c_A\varepsilon_A}{Z_A}(-\varepsilon_A)^{\ell-1}v_{\ell,A}(c_A(1+\varepsilon_A t)) \longrightarrow F^{(\ell-1)}(t), \qquad \ell \in \{1,\dots,s\}.$$

Hence, up to nonzero row constants, the limiting vector is

$$\left(J_{\rho_1}F,\dots,J_{\rho_r}F, F, F',\dots,F^{(s-1)}\right).$$

Proof. After division by Z_A, the Mellin convolution is the density of the product $X_A = \prod_h Y_{h,A}\prod_j G_{j,A}$ of independent variables, with $G_{j,A} \sim \mathrm{Gamma}(a_{r+j}(A)+1,1)$ and $Y_{h,A} \sim \mathrm{Beta}(a_h(A)+1, b_h(A)-a_h(A))$. The multivariate central limit theorem and the delta method give $(c_A^{-1}\prod_j G_{j,A} - 1)/\varepsilon_A \Longrightarrow N(\widehat{a}_*, 1/2)$. If $Y_A \sim \mathrm{Beta}(P_A, D_A)$, $P_A \to +\infty$, and $D_A \to d \ge 0$, then $P_A(1-Y_A) \Longrightarrow \mathrm{Gamma}(d,1)$, with $d = 0$ interpreted as the point mass at the origin. Consequently, $(1-Y_{h,A})/\varepsilon_A$ tends to a gamma variable of shape d_h and rate ρ_h. Expanding the finite product shows that the central variable tends to a $N(\widehat{a}_*, 1/2)$ variable minus those independent gamma variables, whose density is F. In $w_{h,A}$, the h-th beta gap is increased by one, so the semigroup property in (152) gives the first limit.

Finally, set $F_A(t) := c_A\varepsilon_A Z_A^{-1}w_{0,A}(c_A(1+\varepsilon_A t))$. The exact identity

$$\frac{c_A\varepsilon_A}{Z_A}(-\varepsilon_A)^k\mathscr{D}^k w_{0,A}(c_A(1+\varepsilon_A t)) = ((1+\varepsilon_A t)\partial_t)^k F_A(t)$$

proves the Euler limit after testing against a smooth compactly supported function and integrating by parts. Exact product moments give the asserted moment convergence. □

This boundary is different from the full Gaussian limit in Theorem 8.3. When $r = 0$, one has $F = \phi_{\widehat{a}_*}$, and the fixed Euler block gives the finite Hermite jet $F, F', \dots, F^{(s-1)}$, not distinct Gaussian centers. For $s > 1$ these separated jet modules do not define a globally normal step line; for example,

$$t\phi_{\widehat{a}_*} = \widehat{a}_*\phi_{\widehat{a}_*} - \tfrac{1}{2}\phi'_{\widehat{a}_*}.$$

Thus sufficiently large leading minors vanish, and normalized forms converge only at finite orders having an explicitly nonzero limiting determinant. When $r > 0$, the rows $J_{\rho_h}F$ are one-sided gamma convolutions and lie outside the Gaussian family. This is precisely why the moving orders $k_{j,A} = \mathrm{O}(\sqrt{A})$ are needed in the preceding subsection.

The scale is already forced by $r = 0$, $s = 2$. If $a_1 = a_2 = A - 1$, then $w_{0,A}(x) = 2x^{A-1}K_0(2\sqrt{x})$, $Z_A = \Gamma(A)^2$, $\mathbb{E}X_A = A^2$, and $\mathrm{Var}(X_A) = 2A^3 + A^2$. Thus the correct central variable is $(x - A^2)/(2A^{3/2})$, whose limiting variance is $1/2$; the one-factor scale $(x-A)/\sqrt{2A}$ cannot give this limit.

If $\varepsilon_A\beta_i(A) \to \xi_i$, with $\xi_i + \rho_h > 0$ for all i, h when $r > 0$, then $c_A^{-\beta_i(A)}x^{\beta_i(A)} \to \mathrm{e}^{\xi_i t}$. After the affine polynomial change of basis, the fixed mixed moment matrices converge to the rank-one gamma-convolution/jet moments described above. At any order for which the relevant limiting determinant is nonzero, the corresponding normalized forms converge as well.

9. Conclusions and outlook

Two non-Gaussian hypergeometric families of rank-one mixed-type systems have been constructed for arbitrary p and q. In the admissible near-diagonal range, their individual components are given by terminating generalized hypergeometric polynomials. The components on the Jacobi-like vector are finite reconstructions whose number of blocks grows with the total degree, and the same reconstructions admit a compact description in terms of terminating Kampé de Fériet polynomials. In the mixed beta–Euler Laguerre-like case, the polynomial part at infinity introduces a triangular coupling with the Euler-derivative block; this is why the natural uniform description remains a finite hypergeometric reconstruction plus a finite connection system rather than a compact Kampé de Fériet regrouping. The formal boundary $s = 0$ recovers the Jacobi-like Kampé de Fériet representation, whereas the pure Euler boundary $r = 0$ admits explicit terminating hypergeometric component formulas through block Newton interpolation. Gamma-quotient Mellin formulas, Meijer G-representations, and Rodrigues formulas complete the explicit description of the two mixed forms.

The associated step-line sequences satisfy dual recurrences governed by (p, q)-banded matrices, with every band coefficient reduced to a finite Gamma–Pochhammer expression. Under the stated normality and nonzero-pivot hypotheses, the recurrence matrices admit bidiagonal factorizations obtained from the two Christoffel chains. The lower factors are explicit Pochhammer products; the upper factors are finite tau-determinants and, equivalently, cross-ratios of shifted moment minors. The shifted-minor formula remains finite when the corresponding Christoffel chain does not close inside the original parametric family. In the mixed Piñeiro specialization, both chains close and every factor becomes an explicit product. The rational $p = q = 2$ example verifies the general construction exactly.

The one-weight reductions agree with the Jacobi–Piñeiro and multiple Laguerre polynomials of the first kind. Rectangular arrays assembled from moving Christoffel orbits of these scalar reductions converge to the full $p \times q$ Gaussian mixed system of Daems–Kuijlaars, including distinct points on both sides. This is the two-sided rectangular form of the classical Jacobi–Piñeiro/Laguerre-I to Hermite confluence. Separately, the fixed canonical Laguerre-like vector has a gamma-convolution/Hermite-jet limit. Its jet components may be polynomially dependent, so no global step-line normality is claimed for that different boundary without the stated nonvanishing determinants.

Use of artificial intelligence

OpenAI Codex was used for adversarial review of arguments, exact and symbolic checks, and improvements to clarity. All suggested changes and computations were independently checked and validated by the author, who assumes full responsibility for the content of the manuscript. It did not originate the mathematical results or determine the conclusions.

Acknowledgements

The author was supported by the research project PID2024-155133NB-I00, *Ortogonalidad, aproximación e integrabilidad: aplicaciones en procesos estocásticos clásicos y cuánticos*.

Conflict of interest

The author declares no conflict of interest.

Data availability

No datasets were generated or analyzed in this study.

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