General Geronimus Perturbations for Mixed Multiple Orthogonal Polynomials

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Mixed Multiple Orthogonal Polynomials

Given a $q \times p$ matrix of measures, $\mathrm{d}\,\mu$, where each $\mu_{b,a}$ are supported on the interval $\Delta \subseteq \mathbb{R}$, and two matrix polynomials $A(x)$ and $B(x)$:

$$\mathrm{d}\,\mu = \begin{bmatrix} \mathrm{d}\,\mu_{1,1} & \cdots & \mathrm{d}\,\mu_{1,p} \\ \vdots & & \vdots \\ \mathrm{d}\,\mu_{q,1} & \cdots & \mathrm{d}\,\mu_{q,p} \end{bmatrix} \qquad B = \begin{bmatrix} B_0^{(1)} & \cdots & B_0^{(q)} \\ B_1^{(1)} & \cdots & B_1^{(q)} \\ B_2^{(1)} & \cdots & B_2^{(q)} \\ \vdots & & \vdots \end{bmatrix} \qquad A = \begin{bmatrix} A_0^{(1)} & A_1^{(1)} & A_2^{(1)} & \cdots \\ \vdots & \vdots & \vdots & \\ A_0^{(p)} & A_1^{(p)} & A_2^{(p)} & \cdots \end{bmatrix}$$

Associated to this matrix of measures, we introduce the following moment matrix: $\mathscr{M} := \int_\Delta X_{[q]}(x)\,\mathrm{d}\mu(x)\,X_{[p]}^\top(x)$ where $X_{[r]}^\top(x) = \begin{bmatrix} I_r & xI_r & x^2 I_r & \cdots \end{bmatrix}$

If all the leading principal submatrices $\mathscr{M}^{[k]}$ are nonsingular, then the Gauss–Borel factorization exists:

$$\mathscr{M} = \mathscr{L}^{-1}\mathscr{U}^{-1} \quad \Rightarrow \quad \begin{cases} \int_\Delta B(x)\,\mathrm{d}\mu(x) X_{[p]}^\top(x) = \mathscr{U}^{-1} \\ \int_\Delta X_{[p]}(x)\,\mathrm{d}\mu(x) A(x) = \mathscr{L}^{-1} \end{cases}$$

The following relation also holds: $\int_\Delta B(x)\,\mathrm{d}\mu(x)\,A(x) = I$, whose entries are the biorthogonality relations: $\int_\Delta \sum_{b=1}^q \sum_{a=1}^p B_n^{(b)}(x)\,\mathrm{d}\mu_{b,a}(x)\,A_m^{(a)}(x) = \delta_{n,m}$

Another crucial set of objects in this construction are the Cauchy transforms of the matrix polynomials:

$$C(z) = \int_\Delta \frac{\mathrm{d}\,\mu(x)}{z-x} A(x) \qquad D(z) = \int_\Delta B(x) \frac{\mathrm{d}\,\mu(x)}{z-x}$$

Let us introduce the Christoffel–Darboux (CD) kernel matrix polynomial:

$$K^{[n]}(x,y) = A^{[n]}(x)B^{[n]}(y) = \begin{bmatrix} A_0^{(1)}(x) & \cdots & A_{n-1}^{(1)}(x) \\ \vdots & & \vdots \\ A_0^{(p)}(x) & \cdots & A_{n-1}^{(p)}(x) \end{bmatrix} \begin{bmatrix} B_0^{(1)}(y) & \cdots & B_0^{(q)}(y) \\ \vdots & & \vdots \\ B_{n-1}^{(1)}(y) & \cdots & B_{n-1}^{(q)}(y) \end{bmatrix} \qquad K_{a,b}^{[n]}(x,y) = \sum_{i=0}^{n-1} A_i^{(a)}(x) B_i^{(b)}(y)$$

We can also establish additional families of CD kernels, asociated with $D(x)$ and $C(x)$:

$$K_C^{[n]}(x,y) = C^{[n]}(x)B^{[n]}(y) = \int_\Delta \frac{\mathrm{d}\,\mu(t)}{x-t} K^{[n]}(t,y) \qquad K_D^{[n]}(x,y) = A^{[n]}(x)D^{[n]}(y) = \int_\Delta K^{[n]}(x,t) \frac{\mathrm{d}\,\mu(t)}{y-t}$$

Geronimus Perturbation

A Geronimus perturbation for the original mixed multiple orthogonal polynomials is of the form:

$$\boxed{\mathrm{d}\,\check{\mu}(x) R(x) = \mathrm{d}\,\mu(x)}$$

The perturbed measure can be rewritten as:

$$\mathrm{d}\,\check{\mu}(x) = \mathrm{d}\,\mu(x) R^{-1}(x) + \sum_{i=1}^{M} \boldsymbol{\xi}_i(x) \boldsymbol{v}_i^{\boldsymbol{L}} \delta(x - x_i)$$

We can establish the following connection formulas:

$$\check{A}(x)\Omega = R(x)A(x) \qquad \Omega B(x) = \check{B}(x)$$

in terms of a connection matrix, Ω, which is lower unitriangular with M nonzero subdiagonals.

We aim to express the entries of the connection matrix in terms of the original orthogonal polynomials. Cauchy transforms will be highly useful in this context,

$$\check{C}(x)\Omega = C(x) \qquad \check{D}(x)R(x) = \Omega D(x) + \int_\Delta \check{B}(y)\,\mathrm{d}\,\check{\mu}(y) \frac{R(x) - R(y)}{x-y}$$

The connection formulas for $\check{D}(x)$ establish a linear system of M equations (M roots of $\det R(x)$) for M unkowns (Ω components).

Notation

Let us introduce the following notation:

$$\mathbb{D}_n^{(i)} = \sum_{a=1}^{p} D_n^{(a)}(x_i) v_{i;a}^R, \qquad \mathbb{W}_n^{(i)} = \sum_{b=1}^{q} B_n^{(b)}(x_i) \xi_{i;b}(x_i) \boldsymbol{v}_i^L R'(x_i) \boldsymbol{v}_i^R$$

$$\mathbb{K}^{[n-1],(i)}(x) = K_D^{[n-1]}(x, x_i)\boldsymbol{v}_i^R - K^{[n-1]}(x, x_i) \boldsymbol{\xi}_i \boldsymbol{v}_i^L R'(x_i) \boldsymbol{v}_i^R$$

As well as some τ-determinats:

$$\tau_n = \begin{vmatrix} \mathbb{D}_{n-M}^{(1)} - \mathbb{W}_{n-M}^{(1)} & \cdots & \mathbb{D}_{n-M}^{(M)} - \mathbb{W}_{n-M}^{(M)} \\ \vdots & & \vdots \\ \mathbb{D}_{n}^{(1)} - \mathbb{W}_{n}^{(1)} & \cdots & \mathbb{D}_{n}^{(M)} - \mathbb{W}_{n}^{(M)} \end{vmatrix}$$

These determinants represent the compatibility condition of the linear system.

Explicit Formulas for the Perturbed Polynomials

In terms of these τ determinants we find the following Christoffel formulas, for $n \geq M$:

$$\check{A}_n^{(a)}(x) = \sum_{\tilde{a}=1}^{p} \frac{R_{a,\tilde{a}}(x)}{\tau_n} \begin{vmatrix} \mathbb{D}_{n-M+1}^{(1)} - \mathbb{W}_{n-M+1}^{(1)} & \cdots & \mathbb{D}_{n-M+1}^{(M)} - \mathbb{W}_{n-M+1}^{(M)} \\ \vdots & & \vdots \\ \mathbb{D}_{n-1}^{(1)} - \mathbb{W}_{n-1}^{(1)} & \cdots & \mathbb{D}_{n-1}^{(M)} - \mathbb{W}_{n-1}^{(M)} \\ \mathbb{K}_{\tilde{a}}^{[n-M],(1)}(x) + \frac{v_{1;\tilde{a}}^R}{x-x_1} & \cdots & \mathbb{K}_{\tilde{a}}^{[n-M],(M)}(x) + \frac{v_{M;\tilde{a}}^R}{x-x_M} \end{vmatrix}$$

$$\check{B}_n^{(b)}(x) = \frac{1}{\tau_{n-1}} \begin{vmatrix} \mathbb{D}_{n-M}^{(1)} - \mathbb{W}_{n-M}^{(1)} & \cdots & \mathbb{D}_{n-M}^{(M)} - \mathbb{W}_{n-M}^{(M)} & B_{n-M}^{(b)}(x) \\ \vdots & & \vdots & \vdots \\ \mathbb{D}_{n-1}^{(1)} - \mathbb{W}_{n-1}^{(1)} & \cdots & \mathbb{D}_{n-1}^{(M)} - \mathbb{W}_{n-1}^{(M)} & B_{n-1}^{(b)}(x) \\ \mathbb{D}_{n}^{(1)} - \mathbb{W}_{n}^{(1)} & \cdots & \mathbb{D}_{n}^{(M)} - \mathbb{W}_{n}^{(M)} & B_{n}^{(b)}(x) \end{vmatrix}$$

Matrix Polinomial Perturbation. Simple Eigenvalues

For the matrix polynomial perturbation, we will consider the following matrices:

$$R(x) = \sum_{i=0}^{N} R_i x^i \quad R_l \in \mathbb{C}^{p\times p}$$

$$R_N(x) = \begin{bmatrix} 0_{(p-r)\times r} & I_{(p-r)} \\ 0_r & 0_{r\times(p-r)} \end{bmatrix} \quad R_{N-1} = \begin{bmatrix} [R_{N-1}^1]_{(p-r)\times r} & [R_{N-1}^2]_{(p-r)\times(p-r)} \\ I_r & [R_{N-1}^4]_{r\times(p-r)} \end{bmatrix}$$

$\det \boldsymbol{R(x)}$: the determinant of these matrix polynomials is a polynomial of degree $Np - r$.

Canonical Set of Jordan Chains

For simplicity, assume that the zeros of $\det R(x)$ are simple, i.e., there are $M = Np - r$ different roots of $\det R(x)$. For any given root x_i, there exists 2 vectors such that:

$$R(x_i)\boldsymbol{v}_i^R = 0 \qquad \boldsymbol{v}_i^L R(x_i) = 0$$

Generalizations

The results presented here can be extended to more general scenarios.

1. The assumption on the spectrum of the polynomial matrix can be relaxed to include zeros with arbitrary multiplicity. In this case, the general theory of canonical sets of Jordan chains is required.
2. Left perturbations of the measure matrix can also be studied, with similar results:

$$\boxed{\mathrm{d}\,\hat{\mu}(x) = L(x)\,\mathrm{d}\,\mu(x)}$$

On the Existence of Christoffel Perturbed Orthogonality

The existence of the perturbed orthogonality is related to the formulas for $\check{A}(x)$ and $\check{B}(x)$ in the case where $n < M$. A non spectral method yields a connection between the formulas for $\check{B}(x)$ and new τ_n determinats when $n < M$. Therefore, we find:

Theorem: *The perturbed mixed multiple orthogonality exists if and only if $\tau_n \neq 0$ for $n \in \mathbb{N}_0$.*

References

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M. Mañas, M. Rojas *General Geronimus Perturbations For Mixed Multiple Orthogonal Polynomials*, 2024. https://arxiv.org/abs/2411.16022

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