Banded total positive matrices and multiple orthogonality: Favard, Gauss quadrature and normality

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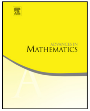

Spectral theory for bounded banded matrices with positive bidiagonal factorization and mixed multiple orthogonal polynomials

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A B S T R A C T

Spectral and factorization properties of oscillatory matrices lead to a spectral Favard theorem for bounded banded matrices, that admit a positive bidiagonal factorization, in terms of sequences of mixed multiple orthogonal polynomials with respect to a set positive Lebesgue–Stieltjes measures. A mixed multiple Gauss quadrature formula with corresponding degrees of precision is given.

Banded semi-infinite matrices

Banded matrices

$$T^{[N]} = \begin{bmatrix} T_{0,0} & \cdots & T_{0,q} & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ T_{p,0} & & \ddots & & \ddots & 0 \\ 0 & \ddots & & \ddots & & T_{N-q,N} \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & T_{N,N-p} & \cdots & T_{N,N} \end{bmatrix}$$

where the extreme diagonal entries are nonzero

with $T = \lim_{N\to\infty} T^{[N]}$ a banded semi-infinite matrix

Recursion polynomials

Type I recursion polynomials

$$A^{(a)}(x) = \begin{bmatrix} A_0^{(a)}(x) & A_1^{(a)}(x) \cdots \cdots \end{bmatrix}, \qquad a \in \{1, \ldots, p\}$$

- Left eigenvectors:

$$A^{(a)}(x)T = xA^{(a)}(x), \qquad a \in \{1, \ldots, p\}$$

Recursion polynomials

Type I recursion polynomials

- Initial conditions,

$$\begin{cases} A_0^{(1)} = 1, \\ A_1^{(1)} = \nu_1^{(1)}, \\ \vdots \\ A_{p-1}^{(1)} = \nu_{p-1}^{(1)}, \end{cases} \qquad \begin{cases} A_0^{(2)} = 0, \\ A_1^{(2)} = 1, \\ A_2^{(2)} = \nu_2^{(2)}, \\ \vdots \\ A_{p-1}^{(2)} = \nu_{p-1}^{(2)}, \end{cases} \qquad \dots \qquad \begin{cases} A_0^{(p)} = 0, \\ \vdots \\ A_{p-2}^{(p)} = 0, \\ A_{p-1}^{(p)} = 1, \end{cases}$$

$\nu_j^{(i)}$ being arbitrary constants

Recursion polynomials

Type I recursion polynomials

- Initial condition matrix

$$A_0 := \begin{bmatrix} 1 & \nu_1^{(1)} & \cdots & \nu_{p-1}^{(1)} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \nu_{p-1}^{(p-1)} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

- $(p+q+1)$-term recursion relation, $A_{-q}^{(a)} = \cdots = A_{-1}^{(a)} = 0$, $a \in \{1, \ldots, p\}$

$$A_{n-q}^{(a)} T_{n-q,n} + \cdots + A_{n+p}^{(a)} T_{n+p,n} = x A_n^{(a)}, \quad n \in \mathbb{N}_0$$

- $\deg A_n^{(a)} \leq \left\lceil \frac{n+2-a}{p} \right\rceil - 1$. **Normality** iff $\leq$ is replaced by $=$

Recursion polynomials

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$$A_{n-q}^{(a)} T_{n-q,n} + \cdots + A_{n+p}^{(a)} T_{n+p,n} = x A_n^{(a)}, \quad n \in \mathbb{N}_0$$

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Recursion polynomials

Type II recursion polynomials

$$B^{(b)}(x) = \begin{bmatrix} B_0^{(b)}(x) & B_1^{(b)}(x) \cdots \end{bmatrix}^\top, \qquad b \in \{1, \ldots, q\}$$

- Right eigenvectors:

$$TB^{(b)}(x) = xB^{(b)}(x), \qquad b \in \{1, \ldots, q\}$$

Recursion polynomials

Type II recursion polynomials

- Initial conditions,

$$\begin{cases} B_0^{(1)} = 1, \\ B_1^{(1)} = \xi_1^{(1)}, \\ \qquad \vdots \\ B_{q-1}^{(1)} = \xi_{q-1}^{(1)}, \end{cases} \qquad \begin{cases} B_0^{(2)} = 0, \\ B_1^{(2)} = 1, \\ B_2^{(2)} = \xi_2^{(2)}, \\ \qquad \vdots \\ B_{p-1}^{(2)} = \xi_{q-1}^{(2)}, \end{cases} \qquad \dots \qquad \begin{cases} B_0^{(q)} = 0, \\ \qquad \vdots \\ B_{q-2}^{(q)} = 0, \\ B_{q-1}^{(q)} = 1, \end{cases}$$

$\xi_j^{(i)}$ being arbitrary constants

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$$B_0 := \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \xi_1^{(1)} & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \xi_{q-1}^{(1)} & \cdots & \cdots & \xi_{q-1}^{(q-1)} & 1 \end{bmatrix}$$

- $(p+q+1)$-term recursion relation, $B_{-p}^{(b)} = \cdots = B_{-1}^{(b)} = 0$, $b \in \{1, \dots, q\}$

$$T_{n,n-p}B_{n-p}^{(b)} + \cdots + T_{n,n+q}B_{n+q}^{(b)} = xB_n^{(b)}, \quad n \in \mathbb{N}_0$$

- $\deg B_n^{(b)} \leq \left\lceil \frac{n+2-b}{q} \right\rceil - 1$. **Normality** iff $\leq$ is replaced by $=$

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Recursion polynomials

Characteristic polynomials

For the semi-infinite matrix T we consider the polynomials $P_N(x)$, $\deg P_N = N$, as the characteristic polynomials of $T^{[N-1]}$

$$P_N(x) \coloneqq \begin{cases} 1, & N = 0 \\ \det\left(xI_N - T^{[N-1]}\right), & N \in \mathbb{N} \end{cases}$$

Left and right recursion polynomials determinants

$$A_N \coloneqq \begin{bmatrix} A_N^{(1)} & \cdots\cdots & A_{N+p-1}^{(1)} \\ \vdots & & \vdots \\ A_N^{(p)} & \cdots\cdots & A_{N+p-1}^{(p)} \end{bmatrix}, \quad B_N \coloneqq \begin{bmatrix} B_N^{(1)} & \cdots\cdots & B_N^{(q)} \\ \vdots & & \vdots \\ B_{N+q-1}^{(1)} & \cdots\cdots & B_{N+q-1}^{(q)} \end{bmatrix}$$

$\alpha_N \coloneqq (-1)^{(p-1)N} T_{p,0} \cdots T_{N+p-1,N-1}$, $\beta_N \coloneqq (-1)^{(q-1)N} T_{0,q} \cdots T_{N-1,N+q-1}$ for $N \in \mathbb{N}$ and $\alpha_0 = \beta_0 = 1$

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Left and right recursion polynomials determinants

$$A_N := \begin{bmatrix} A^{(1)}_N & \cdots\cdots & A^{(1)}_{N+p-1} \\ \vdots & & \vdots \\ A^{(p)}_N & \cdots\cdots & A^{(p)}_{N+p-1} \end{bmatrix}, \quad B_N := \begin{bmatrix} B^{(1)}_N & \cdots\cdots & B^{(q)}_N \\ \vdots & & \vdots \\ B^{(1)}_{N+q-1} & \cdots\cdots & B^{(q)}_{N+q-1} \end{bmatrix}$$

$\alpha_N := (-1)^{(p-1)N} T_{p,0} \cdots T_{N+p-1,N-1}$, $\beta_N := (-1)^{(q-1)N} T_{0,q} \cdots T_{N-1,N+q-1}$ for $N \in \mathbb{N}$ and $\alpha_0 = \beta_0 = 1$

Recursion and associated polynomials

Theorem

Determinantal expressions for the characteristic polynomials

$$P_N(x) = \alpha_N \det A_N(x) = \beta_N \det B_N(x)$$

Associated polynomials

$$Q_{n,N} := \begin{vmatrix} A_n^{(1)} & \cdots\cdots & A_n^{(p)} \\ A_{N+1}^{(1)} & \cdots\cdots & A_{N+1}^{(p)} \\ \vdots & & \vdots \\ A_{N+p-1}^{(1)} & \cdots\cdots & A_{N+p-1}^{(p)} \end{vmatrix}, \quad R_{n,N} := \begin{vmatrix} B_n^{(1)} & \cdots\cdots & B_n^{(q)} \\ B_{N+1}^{(1)} & \cdots\cdots & B_{N+1}^{(q)} \\ \vdots & & \vdots \\ B_{N+q-1}^{(1)} & \cdots\cdots & B_{N+q-1}^{(q)} \end{vmatrix}$$

$Q_N := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots \end{bmatrix}$, $Q^{\langle N \rangle} := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots\cdots Q_{N,N} \end{bmatrix}$

$R_N := \begin{bmatrix} R_{0,N} & R_{1,N} \cdots \end{bmatrix}^\top$, $R^{\langle N \rangle} := \begin{bmatrix} R_{0,N} & R_{1,N} \cdots\cdots R_{N,N} \end{bmatrix}^\top$

Spectral properties

Key spectral property of truncations

$$Q^{\langle N\rangle}T^{[N]} + \begin{bmatrix} 0 \cdots\cdots 0 & T_{N+p,N}Q_{N+p,N}\end{bmatrix} = xQ^{\langle N\rangle}$$

$$T^{[N]}R^{\langle N\rangle} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ T_{N,N+q}R_{N+q,N}\end{bmatrix} = xR^{\langle N\rangle}$$

Christoffel–Darboux type formulas

$$\sum_{n=0}^{N} Q_{n,N}(x)R_{n,N}(y) = \frac{1}{\alpha_N\beta_N}\frac{P_{N+1}(x)P_N(y) - P_N(x)P_{N+1}(y)}{x-y}$$

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Spectral properties

Assume that P_{N+1} has **simple zeros** at the set $\left\{\lambda_k^{[N]}\right\}_{k=1}^{N+1}$

- Biorthogonal sets of left and right eigenvectors, $\left\{w_k^{\langle N\rangle}\right\}_{k=1}^{N+1}$, $\left\{u_k^{\langle N\rangle}\right\}_{k=1}^{N+1}$, are given by

$$w_k^{\langle N\rangle} = \frac{Q^{\langle N\rangle}\big(\lambda_k^{[N]}\big)}{\beta_N \sum_{l=0}^{N} Q_{l,N}\big(\lambda_k^{[N]}\big) R_{l,N}\big(\lambda_k^{[N]}\big)}, \quad u_k^{\langle N\rangle} = \beta_N R^{\langle N\rangle}\big(\lambda_k^{[N]}\big)$$

- The following expression holds

$$w_{k,n}^{\langle N\rangle} = \frac{\alpha_N Q_{n-1,N}\big(\lambda_k^{[N]}\big)}{P_N\big(\lambda_k^{[N]}\big) P'_{N+1}\big(\lambda_k^{[N]}\big)}, \qquad u_{k,n}^{\langle N\rangle} = \beta_N R_{n-1,N}\big(\lambda_k^{[N]}\big)$$

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$$w_k^{\langle N\rangle} = \frac{Q^{\langle N\rangle}(\lambda_k^{[N]})}{\beta_N \sum_{l=0}^{N} Q_{l,N}(\lambda_k^{[N]}) R_{l,N}(\lambda_k^{[N]})}, \quad u_k^{\langle N\rangle} = \beta_N R^{\langle N\rangle}(\lambda_k^{[N]})$$

- The following expression holds

$$w_{k,n}^{\langle N\rangle} = \frac{\alpha_N Q_{n-1,N}(\lambda_k^{[N]})}{P_N(\lambda_k^{[N]}) P'_{N+1}(\lambda_k^{[N]})}, \qquad u_{k,n}^{\langle N\rangle} = \beta_N R_{n-1,N}(\lambda_k^{[N]})$$

Spectral properties

- We can write $w^{\langle N\rangle}_{k,n} = A^{(1)}_{n-1}(\lambda^{[N]}_k)\mu^{[N]}_{k,1} + \cdots + A^{(p)}_{n-1}(\lambda^{[N]}_k)\mu^{[N]}_{k,p}$
  Christoffel numbers

$$\mu^{[N]}_{k,1} := \frac{\begin{vmatrix} A^{(2)}_{n+1}\left(\lambda^{[N]}_k\right) & \cdots\cdots\cdots & A^{(p)}_{n+1}\left(\lambda^{[N]}_k\right) \\ \vdots & \ddots & \vdots \\ A^{(2)}_{n+p-1}\left(\lambda^{[N]}_k\right) & \cdots\cdots & A^{(p)}_{n+p-1}\left(\lambda^{[N]}_k\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda^{[N]}_k\right) R_{l,N}\left(\lambda^{[N]}_k\right)}$$

$$\mu^{[N]}_{k,2} := -\frac{\begin{vmatrix} A^{(1)}_{n+1}\left(\lambda^{[N]}_k\right) & A^{(3)}_{n+1}\left(\lambda^{[N]}_k\right) & \cdots\cdots\cdots & A^{(p)}_{n+1}\left(\lambda^{[N]}_k\right) \\ \vdots & \vdots & \ddots & \vdots \\ A^{(1)}_{n+p-1}\left(\lambda^{[N]}_k\right) & A^{(3)}_{n+p-1}\left(\lambda^{[N]}_k\right) & \cdots\cdots & A^{(p)}_{n+p-1}\left(\lambda^{[N]}_k\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda^{[N]}_k\right) R_{l,N}\left(\lambda^{[N]}_k\right)}$$

$$\vdots$$

$$\mu^{[N]}_{k,p} := (-1)^{p-1}\frac{\begin{vmatrix} A^{(1)}_{n+1}\left(\lambda^{[N]}_k\right) & \cdots\cdots\cdots & A^{(p-1)}_{n+1}\left(\lambda^{[N]}_k\right) \\ \vdots & \ddots & \vdots \\ A^{(1)}_{n+p-1}\left(\lambda^{[N]}_k\right) & \cdots\cdots & A^{(p-1)}_{n+p-1}\left(\lambda^{[N]}_k\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda^{[N]}_k\right) R_{l,N}\left(\lambda^{[N]}_k\right)}$$

Spectral properties

- We can write $u_{k,n}^{\langle N\rangle} = B_{n-1}^{(1)}(\lambda_k^{[N]})\rho_{k,1}^{[N]} + \cdots + B_{n-1}^{(q)}(\lambda_k^{[N]})\rho_{k,q}^{[N]}$
  Christoffel numbers

$$\rho_{k,1}^{[N]} := \beta_N \begin{vmatrix} B_{N+1}^{(2)}\left(\lambda_k^{[N]}\right) \cdots\cdots\cdot B_{N+1}^{(q)}\left(\lambda_k^{[N]}\right) \\ \vdots \qquad\qquad \vdots \\ B_{N+q-1}^{(2)}\left(\lambda_k^{[N]}\right) \cdots\cdot B_{N+q-1}^{(p)}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

$$\rho_{k,2}^{[N]} := -\beta_N \begin{vmatrix} B_{N+1}^{(1)}\left(\lambda_k^{[N]}\right) & B_{N+1}^{(3)}\left(\lambda_k^{[N]}\right) \cdots\cdots\cdot B_{N+1}^{(q)}\left(\lambda_k^{[N]}\right) \\ \vdots & \vdots \qquad\qquad \vdots \\ B_{N+q-1}^{(1)}\left(\lambda_k^{[N]}\right) & B_{n+q-1}^{(3)}\left(\lambda_k^{[N]}\right) \cdots\cdot B_{N+q-1}^{(p)}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

$$\vdots$$

$$\rho_{k,q}^{[N]} := (-1)^{q-1}\beta_N \begin{vmatrix} B_{N+1}^{(1)}\left(\lambda_k^{[N]}\right) \cdots\cdots\cdot B_{N+1}^{(q-1)}\left(\lambda_k^{[N]}\right) \\ \vdots \qquad\qquad \vdots \\ B_{N+q-1}^{(1)}\left(\lambda_k^{[N]}\right) \cdots\cdot B_{N+q-1}^{(q-1)}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

Discrete orthogonality

Step functions

$$\psi_{b,a}^{[N]} \coloneqq \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{k,b}^{[N]}\mu_{k,a}^{[N]} & \lambda_{k+1}^{[N]} \leqslant x < \lambda_k^{[N]},\ k \in \{1,\dots,N\} \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} & x \geqslant \lambda_1^{[N]} \end{cases}$$

Finite sums

For $a \in \{1,\dots,p\}$ and $b \in \{1,\dots,q\}$, we have

$$\rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} = (B_0^{-1} I_{q,p} A_0^{-1})_{b,a}$$

$I_{p,q} \in \mathbb{R}^{p\times q}$, $(I_{p,q})_{i,j} = \delta_{i,j}$

Discrete orthogonality

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Discrete orthogonality

Matrix of discrete measures

$q \times p$ matrix of functions: $\Psi^{[N]} := \begin{bmatrix} \psi_{1,1}^{[N]} & \cdots & \psi_{1,p}^{[N]} \\ \vdots & & \vdots \\ \psi_{q,1}^{[N]} & \cdots & \psi_{q,p}^{[N]} \end{bmatrix}$ $q \times p$ matrix of discrete Lebesgue–Stieltjes measures supported at the zeros of P_{N+1}:

$$\mathrm{d}\Psi^{[N]} = \begin{bmatrix} \mathrm{d}\psi_{1,1}^{[N]} & \cdots\cdots & \mathrm{d}\psi_{1,p}^{[N]} \\ \vdots & & \vdots \\ \mathrm{d}\psi_{q,1}^{[N]} & \cdots\cdots & \mathrm{d}\psi_{q,p}^{[N]} \end{bmatrix} = \sum_{k=1}^{N+1} \begin{bmatrix} \rho_{k,1}^{[N]} \\ \vdots \\ \rho_{k,q}^{[N]} \end{bmatrix} \begin{bmatrix} \mu_{k,1}^{[N]} & \cdots\cdots & \mu_{k,p}^{[N]} \end{bmatrix} \delta\left(x - \lambda_k^{[N]}\right)$$

Discrete orthogonality

Theorem

The following **biorthogonal relations** *hold*

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int B_n^{(b)}(x)\mathrm{d}\psi_{b,a}^{[N]}(x)A_m^{(a)}(x)=\delta_{n,m}, \qquad n,m\in\{0,\dots,N\}$$

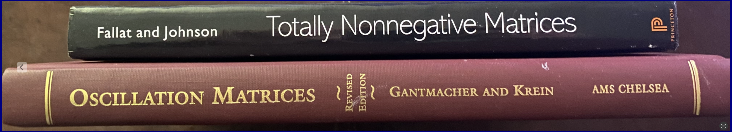
Corollary

The following **discrete mixed multiple orthogonality** *for* $m\in\{1,\dots,N\}$ *are satisfied:*

$$\sum_{a=1}^{p}\int x^n\mathrm{d}\psi_{b,a}^{[N]}A_m^{(a)}=0, \quad n\in\{0,\dots,\deg B_{m-1}^{(b)}\}, \quad b\in\{1,\dots,q\}$$

$$\sum_{b=1}^{q}\int B_m^{(b)}\mathrm{d}\psi_{b,a}^{[N]}x^n=0, \quad n\in\{0,\dots,\deg A_{m-1}^{(a)}\}, \quad a\in\{1,\dots,p\}$$

Totally nonnegative matrices



Totally nonnegative (TN)

All its minors are non-negative

Invertible totally nonnegative (InTN)

All its minors are non-negative and is nonsingular

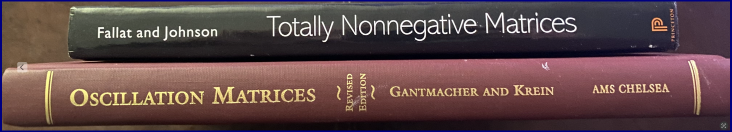
Totally positive (TP)

All its minors are positive

Oscillatory Matrix (IITN)

A totally non negative matrix A such that for some n, the matrix A^n is totally positive

Totally nonnegative matrices



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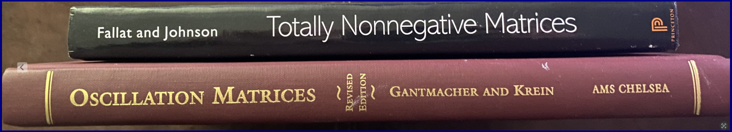

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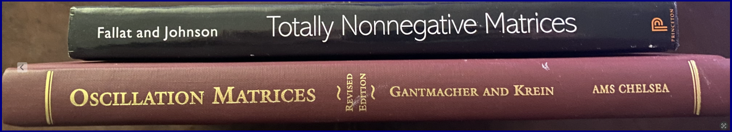
Totally positive (TP)

All its minors are positive

Oscillatory Matrix (IITN)

A totally non negative matrix A such that for some n, the matrix A^n is totally positive

Totally nonnegative matrices



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Factorization I

From Cauchy–Binet Theorem one can deduce the invariance of these sets of matrices under the usual matrix product
PBF $\Rightarrow$ oscillatory

Eigenvalues

The eigenvalues are simple and positive

Interlacing property

The eigenvalues strictly interlace the eigenvalues of the principal submatrix (deleting first row and column) (also last column and row)

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Translations of bounded Jacobi matrices are oscillatory matrices

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Banded total positivity

1. Banded matrices are characterized by having p subdiagonals and q superdiagonals. These matrices are denoted as $\mathrm{B}_{p,q}$.
2. We define $\mathrm{B}_{p,q}$TN as the subset of $\mathrm{B}_{p,q}$ that intersects with TN, $\mathrm{B}_{p,q}$InTN as the subset of $\mathrm{B}_{p,q}$ that intersects with InTN, and $\mathrm{B}_{p,q}$IITN as the subset of $\mathrm{B}_{p,q}$ that intersects with IITN.
3. A particular case of banded matrices arises when $p = n$ and $q = 0$, resulting in lower triangular matrices, and when $p = 0$ and $q = n$, resulting in upper triangular matrices. These specific cases are denoted by $\llcorner$TN or $\urcorner$TN for lower and upper totally nonnegative matrices, respectively

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Banded total positivity

Price, 1965

Let $T \in \mathsf{B}_{p,q}$ be a nonsingular banded matrix with contiguous minors

$$T\begin{pmatrix} i & i+1 \cdots\cdots i+r-1 \\ j & j+1 \cdots\cdots j+r-1 \end{pmatrix} > 0$$

for $i-p \leq j \leq i+q$, where $i, j \in \{1, \dots, n\}$. Then, T is a oscillatory matrix

Banded total positivity

Trivial submatrices $T\begin{bmatrix} i_1 & \cdots\cdots & i_r \\ j_1 & \cdots\cdots & j_r \end{bmatrix}$ with $T_{i_k,j_k} = 0$ above the q-th superdiagonal are of the form:

$$T\begin{bmatrix} i_1 & \cdots\cdots & i_r \\ j_1 & \cdots\cdots & j_r \end{bmatrix} = \begin{bmatrix} * & \cdots & * & 0 & & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & & \vdots \\ * & \cdots & * & 0 & & \cdots & 0 \\ * & \cdots & * & 0 & & \cdots & 0 \\ * & \cdots & * & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & * & \cdots & * \end{bmatrix}$$

Banded total positivity

and for $T_{i_k,j_k} = 0$ below the p-th subdiagonal

$$\begin{bmatrix} * & \cdots & * & * & \cdots & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & & & \vdots \\ * & \cdots & * & * & \cdots & \cdots & * \\ 0 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & * & \cdots & * \end{bmatrix}$$

Banded total positivity

Banded totally positive

A banded matrix is said to be banded totally positive if all nontrivial minors are positive. The set of such matrices will be denoted by BTP or by $\mathrm{B}_{p,q}\mathrm{TP}$

Nontrivial contiguous submatrices

For $T \in \mathbb{R}^{n\times n}$ such that $T \in \mathrm{B}_{(p,q)}\mathrm{TN}$, the corresponding nontrivial contiguous submatrices, associated with zero dispersion sequences of indexes,

$$T\begin{bmatrix} i & i+1\cdots\cdots\cdots i+r-1 \\ j & j+1\cdots\cdots\cdots j+r-1 \end{bmatrix}$$

are those satisfying $i-p \leq j \leq i+q$, with $i,j \in \{1,\ldots,n\}$

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Banded total positivity

Price in disguise

Banded matrices with all nontrivial contiguous minors being positive are oscillatory

Our result on BTP based on Pinkus

A matrix is banded totally positive if and only if all nontrivial contiguous minors are positive

Initial submatrices

For any matrix $T \in \mathbb{R}^{n\times n}$, we define the Δ initial submatrices as follows:

$$\Delta^i[T] := T\begin{bmatrix} 1 \cdots\cdots\cdots i \\ n-i+1 \cdots\cdots n \end{bmatrix}, \quad \Delta_i[T] := T\begin{bmatrix} n-i+1 \cdots\cdots n \\ 1 \cdots\cdots\cdots i \end{bmatrix},$$

and corresponding minors $\Delta^i(T)$ and $\Delta_i(T)$

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Our result on BTP based on Metelmann

Given a banded matrix T, the following equivalences hold:

1. $T \in$BTP
2. All nontrivial submatrices $\Delta_i[T], \Delta^i[T]$ are InTN
3. All nontrivial initial minors are positive
4. There exists a nonnegative bidiagonal factorization of T into $T = \hat{L}_1 \cdots \hat{L}_p \cdot D \cdot \hat{U}_q \cdots \hat{U}_1$ where $\hat{L}_i := \begin{bmatrix} I_{p-i} & 0 \\ 0 & L_i \end{bmatrix}$ is a nonnegative bidiagonal lower unitriangular matrix with L_i being a positive bidiagonal lower unitriangular matrix, and $\hat{U}_i := \begin{bmatrix} I_{q-i} & 0 \\ 0 & U_i \end{bmatrix}$ is a nonnegative bidiagonal upper unitriangular matrix with U_i being a positive bidiagonal upper unitriangular matrix, and D is a positive diagonal matrix

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Banded total positivity

Positive bidiagonal matrices are BTP

For $(L)_{i,i-1}, (U)_{i-1,n} > 0$, $i \in \{1, \dots, n\}$, the following bidiagonal matrices:

$$L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ (L)_{1,0} & 1 & \ddots & & \vdots \\ 0 & (L)_{2,1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (L)_{n-1,n-2} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & (U)_{1,0} & 0 & \cdots & 0 \\ 0 & 1 & (U)_{2,1} & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & (U)_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

are banded totally positive matrices

Banded total positivity

Positive of bidiagonal factorization

A matrix T is said to be PBF (positive of bidiagonal factorization) if it can be expressed as:

$$T = L_1 \cdots L_p D U_q \cdots U_1$$

where L_i and U_i are positive bidiagonal matrices and D is a positive diagonal matrix

PBF vs BTP

PBF matrices are banded totally positive matrices

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PBF vs BTP

PBF matrices are banded totally positive matrices

Christoffel numbers are positive for PBF

$$
\Lambda_p := \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
0 & (L_1)_{1,0} & (L_1L_2)_{1,0} & \cdots & (L_1\cdots L_{p-1})_{1,0} & (L_1\cdots L_p)_{1,0} \\
0 & 0 & (L_1L_2)_{2,0} & \cdots & (L_1\cdots L_{p-1})_{2,0} & (L_1\cdots L_p)_{2,0} \\
\vdots & & \ddots & \ddots & & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & (L_1\cdots L_p)_{p,0}
\end{bmatrix}
$$

$$
\Upsilon_q := \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & (U_1)_{0,1} & 0 & \ddots & \vdots \\
1 & (U_2U_1)_{0,1} & (U_2U_1)_{0,2} & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \\
1 & (U_{p-1}\cdots U_1)_{0,1} & (U_{p-1}\cdots U_1)_{0,2} & \ddots & 0 \\
1 & (U_p\cdots U_1)_{0,1} & (U_p\cdots U_1)_{0,2} & \cdots & (U_p\cdots U_1)_{0,p}
\end{bmatrix}
$$

Total positivity

The matrices Λ_p and Υ_q belong to ∇TP and ΔTP, respectively

Christoffel numbers are positive for PBF

$$\Lambda_p := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & (L_1)_{1,0} & (L_1L_2)_{1,0} & \cdots & (L_1\cdots L_{p-1})_{1,0} & (L_1\cdots L_p)_{1,0} \\ 0 & 0 & (L_1L_2)_{2,0} & \cdots & (L_1\cdots L_{p-1})_{2,0} & (L_1\cdots L_p)_{2,0} \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & (L_1\cdots L_p)_{p,0} \end{bmatrix}$$

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Christoffel numbers are positive for PBF

Theorem (Christoffel coefficients positivity)

Let us assume that T has a PBF and that $A_0^{-1} = \Lambda_p \mathcal{A}$, $B_0^{-1} = \mathcal{B}\Upsilon_q$, and $\mathcal{A} \in \mathbb{R}^{p\times p}$ is an upper unitriangular totally positive matrix and $\mathcal{B} \in \mathbb{R}^{q\times q}$ is a lower unitriangular totally positive matrix. Then,

$$\rho_{k,b}^{[N]} > 0, \quad \mu_{k,a}^{[N]} > 0, \quad k \in \{1,\dots,N+1\}, \quad a \in \{1,\dots,p\}, \quad b \in \{1,\dots,q\}$$

Favard theorem

Theorem (Favard spectral representation and normality)

Let us assume that

1. *The banded matrix T is bounded and there exist $s \geqslant 0$ such that $T + sI$ has a PBF*
2. *The sequences $\left\{A_n^{(1)}, \dots, A_n^{(p)}\right\}_{n=0}^{\infty}$, $\left\{B_n^{(1)}, \dots, B_n^{(q)}\right\}_{n=0}^{\infty}$ of recursion polynomials are determined by the initial condition matrices A_0 and B_0,respectively, such that $A_0^{-1} = \Lambda_p \mathcal{A}$, $B_0^{-1} = \mathcal{B}\Upsilon_q$, and $\mathcal{A} \in \mathbb{R}^{p\times p}$ is a upper unitriangular totally positive matrix and $\mathcal{B} \in \mathbb{R}^{q\times q}$ is a lower unitriangular totally positive matrix*

Then, there exists pq non decreasing positive functions $\psi_{b,a}$, $a \in \{1, \dots, p\}$ and $b \in \{1, \dots, q\}$ and corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\psi_{b,a}$ with compact support $[\zeta, \eta]$ such that the following biorthogonality holds

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int_{\zeta}^{\eta} B_l^{(b)}(x)\mathrm{d}\psi_{b,a}(x)A_k^{(a)}(x) = \delta_{k,l}, \qquad k,l \in \mathbb{N}_0$$

and we have normal degrees

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Favard theorem / Gaussian quadrature

Mixed multiple orthogonal relations

$$\sum_{a=1}^{p}\int_{\zeta}^{\eta} x^n \mathrm{d}\psi_{b,a}(x) A_m^{(a)}(x) = 0, \quad n \in \{0,\dots,\deg B_{m-1}^{(b)}\}, \quad b \in \{1,\dots,q\}$$

$$\sum_{b=1}^{q}\int_{\zeta}^{\eta} B_m^{(b)}(x) \mathrm{d}\psi_{b,a}(x) x^n = 0, \quad n \in \{0,\dots,\deg A_{m-1}^{(a)}\}, \quad a \in \{1,\dots,p\}$$

Degrees of precision

$$d_{b,a}(N) = \deg A_N^{(a)} + \deg B_N^{(b)} + 1 = \left\lceil \frac{N+2-a}{p} \right\rceil + \left\lceil \frac{N+2-b}{q} \right\rceil - 1$$

Mixed multiple Gaussian quadrature formulas

$$\int_{\zeta}^{\eta} x^n \mathrm{d}\psi_{b,a}(x) = \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} \big(\lambda_k^{[N]}\big)^n, \qquad 0 \leqslant n \leqslant d_{b,a}(N)$$

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