Positive bidiagonal factorization of oscillatory tetradiagonal Hessenberg matrices

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A B S T R A C T

Recently, a spectral Favard theorem was presented for bounded banded lower Hessenberg matrices that possess a positive bidiagonal factorization. The paper establishes conditions, expressed in terms of continued fractions, under which an oscillatory tetradiagonal Hessenberg matrix can have such a positive bidiagonal factorization. Oscillatory tetradiagonal Toeplitz matrices are examined as a case study of matrices that admit a positive bidiagonal factorization. Furthermore, the paper proves that oscillatory banded Hessenberg matrices are organized in rays, where the origin of the ray does not have a positive bidiagonal factorization, but all the interior points of the ray do have such a positive bidiagonal factorization.

Banded Hessenberg semi-infinite matrices

Tetradiagonal Hessenberg matrices

$$T = \begin{bmatrix} c_0 & 1 & 0 & \cdots & \cdots \\ b_1 & c_1 & 1 & \ddots & \\ a_2 & b_2 & c_2 & 1 & \ddots \\ 0 & a_3 & b_3 & c_3 & 1 \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Banded Hessenberg semi-infinite matrices

Positive Bidiagonal Factorization (PBF)

$$T = L_1 L_2 U$$

where L_1, L_2, and U are bidiagonal matrices.

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \alpha_2 & 1 & 0 & \cdots \\ 0 & \alpha_5 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \alpha_3 & 1 & 0 & \cdots \\ 0 & \alpha_6 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad U = \begin{bmatrix} \alpha_1 & 1 & 0 & \cdots \\ 0 & \alpha_4 & 1 & \ddots \\ 0 & 0 & \alpha_7 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

positivity requirement: $\alpha_j > 0$, $j \in \mathbb{N}$

Important issue for the spectral Favard theorem for multiple orthogonal polynomials with two weights

Jacobi matrices

Jacobi matrices are tridiagonal real matrices of the form

$$J := \begin{bmatrix} m_0 & 1 & 0 & \cdots & \cdots \\ \ell_1 & m_1 & 1 & \ddots & \\ 0 & \ell_2 & m_2 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with $\ell_j > 0$, $j \in \mathbb{N}$
Leading principal submatrix:

$$J^{[N]} := \begin{bmatrix} m_0 & 1 & 0 & \cdots & 0 \\ \ell_1 & m_1 & 1 & \ddots & \vdots \\ 0 & \ell_2 & m_2 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \ell_N & m_N \end{bmatrix}, \qquad \Delta_N := \left|J^{[N]}\right|$$

Jacobi matrices

Contiguous principal submatrices:

$$J^{[N,k]} := \begin{bmatrix} m_k & 1 & 0 & \cdots & \cdots & 0 \\ \ell_{k+1} & m_{k+1} & 1 & \ddots & & \vdots \\ 0 & \ell_{k+2} & m_{k+2} & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & & \ell_N & m_N \end{bmatrix}, \quad \Delta_{N,k} := |J^{[N,k]}|$$

Oscillatory Jacobi matrices (Gatmancher, Krein)

A tridiagonal matrix is oscillatory if and only if

1. The matrix entries of the first subdiagonal and first superdiagonal are positive
2. All leading principal minors are positive

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For $\|J\|_\infty < \infty$, all the possible eigenvalues of the submatrices $J^{[N]}$ belong to the disk $D(0, \|J\|_\infty)$. As all the eigenvalues are real, let b be the supremum of the absolute values of all negative eigenvalues. Notice that $b \leqslant \|J\|_\infty$

Jacobi and total positivity

For $s > b$, the matrix $J_s = J + sI$ is oscillatory

Proof.

Take $s > b$, then J_s has the eigenvalues of its leading principal submatrices $J_s^{[N]} = J^{[N]} + sI_{N+1}$ all positive. The corresponding characteristic polynomials are $P_{N+1}(x - s) = \det(xI_{N+1} - J_s^{[N]})$, so that $\det J_s^{[N]} = (-1)^{N+1} P_{N+1}(-s)$. As $-s$ is a lower bound for any possible zero of this monic polynomial, we have that $(-1)^{N+1} P_{N+1}(-s) > 0$. Hence, the leading principal minors of J_s are all positive, and the entries on the subdiagonal and superdiagonal are positive. Thus, we conclude that J_s is an oscillatory matrix □

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Positive bidiagonal factorization

A Jacobi matrix is oscillatory if and only if it admits a PBF

Proof.

Let us assume that the Jacobi matrix $J^{[N,1]}$ is oscillatory. Then, the Gauss–Borel factorization of

$$J^{[N,1]} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \beta_2 & \ddots & \ddots & & \vdots \\ 0 & \beta_4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{2N-2} & 1 \end{bmatrix} \begin{bmatrix} \beta_1 & 1 & 0 & \cdots & 0 \\ 0 & \beta_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \beta_5 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \beta_{2N-1} \end{bmatrix}$$

leads to $\ell_n = \beta_{2n-2}\beta_{2n-3}$, with $\beta_0 := 0$. Hence, as $\ell_n > 0$, $n \in \{2, 3, \dots\}$, we get that $\beta_n > 0$ for $n \in \mathbb{N}$.
If the Jacobi matrix $J^{[N,1]}$ admits a PBF, we deduce that it is oscillatory from the Gantmacher–Krein Criterion Theorem

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If the Jacobi matrix $J^{[N,1]}$ admits a PBF, we deduce that it is oscillatory from the Gantmacher–Krein Criterion Theorem □

Tetradiagonal matrices

$$T^{[N]} := \begin{bmatrix} c_0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ b_1 & c_1 & 1 & \ddots & & & \vdots \\ a_2 & b_2 & c_2 & 1 & \ddots & & \vdots \\ 0 & a_3 & b_3 & c_3 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_{N-1} & b_{N-1} & c_{N-1} & 1 \\ 0 & \cdots & \cdots & 0 & a_N & b_N & c_N \end{bmatrix}, \quad \delta^{[N]} := |T^{[N]}|$$

$$T^{[N,k]} := \begin{bmatrix} c_k & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ b_{k+1} & c_{k+1} & 1 & \ddots & & & \vdots \\ a_{k+2} & b_{k+2} & c_{k+2} & 1 & \ddots & & \vdots \\ 0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_{N-1} & b_{N-1} & c_{N-1} & 1 \\ 0 & \cdots & \cdots & 0 & a_N & b_N & c_N \end{bmatrix}$$

Tetradiagonal matrices. Gauss–Borel factorization

$$T^{[N]} = L^{[N]} U^{[N]}$$

$$L^{[N]} := \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ m_1 & 1 & \ddots & & & \vdots \\ \ell_2 & m_2 & 1 & \ddots & & \\ 0 & \ell_3 & m_3 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ell_N & m_N & 1 \end{bmatrix}$$

$$U^{[N]} := \begin{bmatrix} \alpha_1 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_7 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & & 0 & \alpha_{3N+1} \end{bmatrix}$$

Tetradiagonal matrices. Gauss–Borel factorization

Gauss–Borel factorization of tetradiagonal matrices

The Gauss–Borel factorization exists if and only if all leading principal minors $\delta^{[n]}$, $n \in \{0,1,\dots,N\}$, of $T^{[N]}$ are nonzero, i.e., $\delta^{[n]} \neq 0$ for $n \in \{0,1,\dots,N\}$. For $n \in \mathbb{N}$, the following expressions for the coefficients hold:

$$\ell_{n+1} = \frac{a_{n+1}\delta^{[n-2]}}{\delta^{[n-1]}} \qquad m_n = c_n - \frac{\delta^{[n]}}{\delta^{[n-1]}} \qquad \alpha_{3n-2} = \frac{\delta^{[n-1]}}{\delta^{[n-2]}}$$

where $\delta^{[-1]} = 1$ and $a_1 = 0$. The recurrence relation for the determinants is given by:

$$\delta^{[n]} = a_n\delta^{[n-3]} - b_n\delta^{[n-2]} + c_n\delta^{[n-1]}$$

and it is satisfied for $n \in \{0,1,\dots,N\}$

Auxiliary submatrices and continued fractions

1. Given the lower triangular factor $L^{[N]}$, determined by the Gauss–Borel factorization, we consider its complementary submatrix by deleting the first row and last column. This submatrix is called the auxiliary Jacobi matrix and is denoted as $J^{[N,1]} = L^{[N]}(\{1\},\{N+1\}) \in \mathbb{R}^{N\times N}$
2. $T_1^{[N]} := T(\{1\},\{N+1\}) \in \mathbb{R}^{N\times N}$ as the complementary submatrix obtained by removing the first row and last column of $T^{[N]}$, $\delta_1^{[N]} := |T_1^{[N]}|$
3. $T_1^{[N,k]}$ is obtained form $T_1^{[N]}$ by removing the first k columns and rows. It should be noted that $T_1^{[N,0]} = T_1^{[N]}$ and $\delta_1^{[N,0]} = \delta_1^{[N]}$
4. Finite continued fractions, $\mathscr{K}[k+1,k] := m_k$ and

$$\mathscr{K}[n,k] := m_k - \cfrac{\ell_{k+1}}{m_{k+1} - \cfrac{\ell_{k+2}}{m_{k+2} - \cfrac{\ell_{k+3}}{\ddots\, m_{n-1} - \cfrac{\ell_n}{m_n}}}}, \quad n \in \{k+1, k+2, \dots\}$$

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Determinants and continued fractions

Assume that $T^{[N]}$ is oscillatory

1. The triangular factors $L^{[N]}$ and $U^{[N]}$ in the Gauss–Borel factorization of $T^{[N]}$ are invertible totally nonnegative matrices

2. $\ell_2, \dots, \ell_N, m_1, \dots, m_N, \alpha_1, \alpha_4, \dots, \alpha_{3N-2} > 0$

3. For $k \in \mathbb{N}$, the recurrence relation

$$D(n+1) = m_{k+n} D(n) - \ell_{k+n} D(n-1), \qquad n \in \mathbb{N}$$

for the initial conditions $D(0) = 1, D(1) = m_k$ has a solution $D(n) = \Delta_{k+n-1,k}$, and for the initial conditions $D(0) = 0$ and $D(1) = 1$ has a solution $D(n) = \Delta_{k+n-1,k+1}$

4. The ratio of consecutive determinants is bounded:

$$\frac{a_{n+2}}{b_{n+2}} < \frac{\delta^{[n]}}{\delta^{[n-1]}} < c_n, \quad \frac{\ell_{n+1}}{m_{n+1}} < \frac{\Delta_{n,1}}{\Delta_{n-1,1}} < m_n$$

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Determinants and continued fractions

5. For $k \in \mathbb{N}$,

$$\mathscr{K}[n,k] = \frac{\Delta_{n,k}}{\Delta_{n,k+1}}$$

6. For $k = 1$, $\Delta_{n,1} > 0$

7. $J^{[N,1]}$ is oscillatory

8. The following factorizations are valid:

$$T_1^{[N]} = J^{[N,1]}U^{[N-1]}, \qquad T_1^{[N,k]} - m_{k+1}E_{1,1} = J^{[N,k+1]}U^{[N-1,k]}$$

Moreover, $\delta_1^{[N]} > 0$, and the following relation between determinants holds:

$$\Delta_{N,1} = \frac{\delta_1^{[N]}}{\delta^{[N-1]}}, \qquad \Delta_{N,k+1} = \alpha_1 \cdots \alpha_{3k-2} \frac{\delta_1^{[N,k]} - m_{k+1}\delta_1^{[N,k+1]}}{\delta^{[N-1]}}$$

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Determinants and continued fractions

9. The submatrix $T_1^{[N]}$ is oscillatory

10. The submatrices $J^{[N,k+1]}$ and $T_1^{[N,k]}$ are also oscillatory. In particular, $\Delta_{N,k+1}, \delta_1^{[N,k]} > 0$

11. The following relations are satisfied:

$$\Delta_{N,2}\delta^{[N-1]} = c_0\delta_1^{[N,1]} - a_2\delta_1^{[N,2]}$$
$$\frac{\Delta_{N,1}}{\Delta_{N,2}} = \frac{\delta_1^{[N]}}{c_0\delta_1^{[N,1]} - a_2\delta_1^{[N,2]}}$$

12. The recursion relation in k is satisfied:

$$\Delta_{N,k+1} = m_{k+1}\Delta_{N,k+2} - \ell_{k+2}\Delta_{N,k+3}$$

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$$\Delta_{N,2}\delta^{[N-1]} = c_0\delta_1^{[N,1]} - a_2\delta_1^{[N,2]}$$

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Continued fractions

Infinite continued fraction and tails

We introduce the following infinite continued fraction

$$\mathscr{K}[1] := m_1 - \cfrac{\ell_2}{m_2 - \cfrac{\ell_3}{m_3 - \ddots}}$$

and its tails

$$\mathscr{K}[k+1] := m_{k+1} - \cfrac{\ell_{k+2}}{m_{k+2} - \cfrac{\ell_{k+3}}{m_{k+3} - \ddots}}, \qquad k \in \mathbb{N}$$

Continued fractions

Continued fraction and quotient of determinants

$$\mathcal{K}[1] = \lim_{N\to\infty} \frac{\delta_1^{[N]}}{c_0\delta_1^{[N,1]} - a_2\delta_1^{[N,2]}}$$

Infinite continued fractions

1. For $k \in \mathbb{N}_0$, the sequences $\{\mathcal{K}[n,k]\}_{n=k+1}^{\infty}$ of the finite continued fractions are positive and strictly decreasing
2. The infinite continued fraction $\mathcal{K}[1]$ converges and is **nonnegative**
3. The corresponding tails converge and are **positive**, i.e. $\mathcal{K}[k+1] > 0$ for $k \in \mathbb{N}$

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Bidiagonal factorization

The factorization of any lower triangular matrix of the form

$$L^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ m_1 & 1 & \ddots & & & \vdots \\ \ell_2 & m_2 & 1 & & & \\ 0 & \ell_3 & m_3 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ell_N & m_N & 1 \end{bmatrix}$$

can be decomposed into bidiagonal factors, $L^{[N]} = L_1^{[N]} L_2^{[N]}$ with

$$L_1^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \alpha_2 & \ddots & \ddots & \vdots \\ 0 & \alpha_5 & \ddots & \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 \quad \alpha_{3N-1} & 1 \end{bmatrix}, \quad L_2^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \alpha_3 & \ddots & \ddots & \vdots \\ 0 & \alpha_6 & \ddots & \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 \quad \alpha_{3N} & 1 \end{bmatrix}$$

Bidiagonal factorization

$$\alpha_{3n} = m_n - \cfrac{\ell_n}{m_{n-1} - \cfrac{\ell_{n-1}}{m_{n-2} - \cfrac{\ell_{n-2}}{\ddots\, m_2 - \cfrac{\ell_2}{m_1 - \alpha_2}}}}$$

$$\alpha_{3n-1} = \cfrac{\ell_n}{m_{n-1} - \cfrac{\ell_{n-1}}{m_{n-2} - \cfrac{\ell_{n-2}}{\ddots\, m_2 - \cfrac{\ell_2}{m_1 - \alpha_2}}}}$$

The factorization exists if and only if $\alpha_{3n} \neq 0$ for $n \in \{1, \ldots, N-1\}$

Bidiagonal factorization

For each $\alpha_2 < \mathcal{K}[N,1]$, the factorization of $L^{[N]} = L_1^{[N]} L_2^{[N]}$ into bidiagonal factors given by

$$L_1^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \alpha_2 & \ddots & \ddots & & \vdots \\ 0 & \alpha_5 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{3N-1} & 1 \end{bmatrix}, \quad L_2^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \alpha_3 & \ddots & \ddots & & \vdots \\ 0 & \alpha_6 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{3N} & 1 \end{bmatrix}$$

where $\alpha_3, \alpha_5, \alpha_6, \alpha_8, \dots, \alpha_{3n-1}, \alpha_{3n} > 0$, exists and is unique. If $\alpha_2 \in [0, \mathcal{K}[N,1])$, then both $L_1^{[N]}$ and $L_2^{[N]}$ are invertible totally nonnegative matrices

Positive bidiagonal factorization

For $\alpha_2 < \mathcal{K}[N,1]$, we can determine a positive sequence α_1, α_3, α_4, α_5,..., α_{3N+1} that satisfies the factorization $T^{[N]} = L_1^{[N]} L_2^{[N]} U^{[N]}$ with:

$$L_1^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha_2 & \ddots & \ddots & & \vdots \\ 0 & \alpha_5 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{3N-1} & 1 \end{bmatrix}, L_2^{[N]} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha_3 & \ddots & \ddots & & \vdots \\ 0 & \alpha_6 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{3N} & 1 \end{bmatrix}$$

$$U^{[N]} = \begin{bmatrix} \alpha_1 & 1 & 0 & \cdots & 0 \\ 0 & \alpha_4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_7 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \alpha_{3N+1} \end{bmatrix}$$

Positive bidiagonal factorization

PBF in the semi-infinite case

Let us assume that the banded Hessenberg tetradiagonal matrix T is oscillatory. For $\alpha_2 < \mathscr{K}[1]$, there exists a unique positive sequence $\{\alpha_1, \alpha_3, \alpha_4, \dots\}$ such that the PBF holds. Additionally, if $\alpha_2 \in [0, \mathscr{K}[1])$, then L_1, L_2, and U belong to InTN.

Furthermore, the matrix entries satisfy the following relations:

$$\begin{cases} c_n = \alpha_{3n+1} + \alpha_{3n} + \alpha_{3n-1}, \\ b_n = \alpha_{3n}\alpha_{3n-2} + \alpha_{3n-1}\alpha_{3n-2} + \alpha_{3n-1}\alpha_{3n-3}, \\ a_n = \alpha_{3n-1}\alpha_{3n-3}\alpha_{3n-5}. \end{cases}$$

Example: oscillatory Toeplitz tetradiagonal matrices

We now consider the Hessenberg matrix T is a banded Toeplitz matrix given by:

$$T = \begin{bmatrix} c & 1 & 0 & \cdots & & \\ b & c & 1 & \ddots & & \\ a & b & c & 1 & & \ddots \\ 0 & a & b & c & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Edrei–Schoenberg

The Toeplitz matrix T is oscillatory if and only if there exist positive numbers $\gamma_1 \geqslant \gamma_2 \geqslant \gamma_3 > 0$ such that:

$$a = \gamma_1\gamma_2\gamma_3, \quad b = \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3, \quad c = \gamma_1 + \gamma_2 + \gamma_3$$

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Example: oscillatory Toeplitz tetradiagonal matrices

The determinants $\delta^{[n]}$ for $\gamma_1 > \gamma_2 > \gamma_3 > 0$

$$\delta^{[n]} = \frac{\gamma_1^{n+2}}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)} + \frac{\gamma_2^{n+2}}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)} + \frac{\gamma_3^{n+2}}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)}$$

Large N ratio asymptotics of the determinants $\delta^{[N]}$

$$\lim_{N\to\infty} \frac{\delta^{[N]}}{\delta^{[N-1]}} = \gamma_1$$

Infinite continued fractions and harmonic mean

$$\mathscr{K}[1] = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}$$

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Theorem

A tetradiagonal Toeplitz matrix is oscillatory if and only if it admits a PBF