Multiple orthogonal polynomials. Two stops: spectrality and hypergeometric series for type I

4º Workshop "Dos días de polinomios ortogonales"

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Universidad Complutense

December 15th, 2023

¡Felicidades a mis amigos Guillermo y Andrei!

Advances in Mathematics 434 (2023) 109313

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Spectral theory for bounded banded matrices with positive bidiagonal factorization and mixed multiple orthogonal polynomials

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ARTICLE INFO

Article history:
Received 2 January 2023
Received in revised form 3 September 2023
Accepted 16 September 2023
Available online xxxx
Communicated by C. Fefferman

ABSTRACT

Spectral and factorization properties of oscillatory matrices lead to a spectral Favard theorem for bounded banded matrices, that admit a positive bidiagonal factorization, in terms of sequences of mixed multiple orthogonal polynomials with respect to a set positive Lebesgue–Stieltjes measures. A mixed multiple Gauss quadrature formula with corresponding degrees of precision is given.

Banded semi-infinite matrices

Banded matrices

$$T^{[N]} = \begin{bmatrix} T_{0,0} & \cdots & T_{0,q} & & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & & \ddots & \vdots \\ T_{p,0} & & \ddots & & \ddots & & 0 \\ \vdots & \ddots & & \ddots & & \ddots & T_{N-q,N} \\ 0 & & \ddots & & \ddots & & \vdots \\ \vdots & \ddots & & \ddots & & \ddots & T_{N,N} \\ 0 & \cdots & 0 & & T_{N,N-p} & \cdots & T_{N,N} \end{bmatrix}$$

where the extreme diagonal entries are nonzero

with $T = \lim_{N\to\infty} T^{[N]}$ a banded semi-infinite matrix

Recursion polynomials

Type I recursion polynomials

$$A^{(a)}(x) = \begin{bmatrix} A_0^{(a)}(x) & A_1^{(a)}(x) \cdots \cdots \end{bmatrix}, \qquad a \in \{1, \ldots, p\}$$

- Left eigenvectors:

$$A^{(a)}(x)T = xA^{(a)}(x), \qquad a \in \{1, \ldots, p\}$$

Recursion polynomials

Type I recursion polynomials

- Initial conditions,

$$\begin{cases} A_0^{(1)} = 1, \\ A_1^{(1)} = \nu_1^{(1)}, \\ \quad \vdots \\ A_{p-1}^{(1)} = \nu_{p-1}^{(1)}, \end{cases} \quad \begin{cases} A_0^{(2)} = 0, \\ A_1^{(2)} = 1, \\ A_2^{(2)} = \nu_2^{(2)}, \\ \quad \vdots \\ A_{p-1}^{(2)} = \nu_{p-1}^{(2)}, \end{cases} \quad \cdots \quad \begin{cases} A_0^{(p)} = 0, \\ \quad \vdots \\ A_{p-2}^{(p)} = 0, \\ A_{p-1}^{(p)} = 1, \end{cases}$$

$\nu_j^{(i)}$ being arbitrary constants

Recursion polynomials

Type I recursion polynomials

- Initial condition matrix

$$\nu := \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \nu_1^{(1)} & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ \nu_{p-1}^{(1)} & \cdots & \cdots & \nu_{p-1}^{(p-1)} & \cdots \end{bmatrix}$$

- $(p+q+1)$-term recursion relation, $A_{-q}^{(a)} = \cdots = A_{-1}^{(a)} = 0$, $a \in \{1, \ldots, p\}$

$$A_{n-q}^{(a)} T_{n-q,n} + \cdots + A_{n+p}^{(a)} T_{n+p,n} = x A_n^{(a)}, \quad n \in \mathbb{N}_0$$

- $\deg A_n^{(a)} = \left\lceil \frac{n+2-a}{p} \right\rceil - 1$

Recursion polynomials

Type I recursion polynomials

- Initial condition matrix

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- $(p+q+1)$-term recursion relation, $A_{-q}^{(a)} = \cdots = A_{-1}^{(a)} = 0$, $a \in \{1, \ldots, p\}$

$$A_{n-q}^{(a)} T_{n-q,n} + \cdots + A_{n+p}^{(a)} T_{n+p,n} = x A_n^{(a)}, \quad n \in \mathbb{N}_0$$

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- $(p+q+1)$-term recursion relation, $A_{-q}^{(a)} = \cdots = A_{-1}^{(a)} = 0$, $a \in \{1, \ldots, p\}$

$$A_{n-q}^{(a)} T_{n-q,n} + \cdots + A_{n+p}^{(a)} T_{n+p,n} = x A_n^{(a)}, \quad n \in \mathbb{N}_0$$

- $\deg A_n^{(a)} = \left\lceil \frac{n+2-a}{p} \right\rceil - 1$

Recursion polynomials

Type II recursion polynomials

$$B^{(b)}(x) = \begin{bmatrix} B_0^{(b)}(x) & B_1^{(b)}(x) \cdots \cdots \end{bmatrix}^\top, \qquad b \in \{1, \ldots, q\}$$

- Right eigenvectors:

$$TB^{(b)}(x) = xB^{(b)}(x), \qquad b \in \{1, \ldots, q\}$$

Recursion polynomials

Type II recursion polynomials

- Initial conditions,

$$\begin{cases} B_0^{(1)} = 1, \\ B_1^{(1)} = \xi_1^{(1)}, \\ \quad \vdots \\ B_{q-1}^{(1)} = \xi_{q-1}^{(1)}, \end{cases} \quad \begin{cases} B_0^{(2)} = 0, \\ B_1^{(2)} = 1, \\ B_2^{(2)} = \xi_2^{(2)}, \\ \quad \vdots \\ B_{p-1}^{(2)} = \xi_{q-1}^{(2)}, \end{cases} \quad \cdots \quad \begin{cases} B_0^{(q)} = 0, \\ \quad \vdots \\ B_{q-2}^{(q)} = 0, \\ B_{q-1}^{(q)} = 1, \end{cases}$$

$\xi_j^{(i)}$ being arbitrary constants

Recursion polynomials

Type II recursion polynomials

- Initial condition matrix

$$\xi := \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \xi_1^{(1)} & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \xi_{q-1}^{(1)} & \cdots & \cdots & \xi_{q-1}^{(q-1)} & 1 \end{bmatrix}$$

- $(p+q+1)$-term recursion relation, $B_{-p}^{(b)} = \cdots = B_{-1}^{(b)} = 0$, $b \in \{1, \ldots, q\}$

$$T_{n,n-p} B_{n-p}^{(b)} + \cdots + T_{n,n+q} B_{n+q}^{(b)} = x B_n^{(b)}, \quad n \in \mathbb{N}_0$$

- $\deg B_n^{(b)} = \left\lceil \frac{n+2-b}{q} \right\rceil - 1$

Recursion polynomials

Type II recursion polynomials

- Initial condition matrix

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- $\deg B_n^{(b)} = \left\lceil \frac{n+2-b}{q} \right\rceil - 1$

Recursion polynomials

Characteristic polynomials

For the semi-infinite matrix T we consider the polynomials $P_N(x)$, $\deg P_N = N$, as the characteristic polynomials of $T^{[N-1]}$

$$P_N(x) := \begin{cases} 1, & N = 0 \\ \det\left(xI_N - T^{[N-1]}\right), & N \in \mathbb{N} \end{cases}$$

Left and right recursion polynomials determinants

$$A_N := \begin{bmatrix} A_N^{(1)} & \cdots\cdots & A_{N+p-1}^{(1)} \\ \vdots & & \vdots \\ A_N^{(p)} & \cdots\cdots & A_{N+p-1}^{(p)} \end{bmatrix}, \quad B_N := \begin{bmatrix} B_N^{(1)} & \cdots\cdots\cdots & B_N^{(q)} \\ \vdots & & \vdots \\ B_{N+q-1}^{(1)} & \cdots\cdots & B_{N+q-1}^{(q)} \end{bmatrix}$$

$\alpha_N := (-1)^{(p-1)N} T_{p,0} \cdots T_{N+p-1,N-1}$, $\beta_N := (-1)^{(q-1)N} T_{0,q} \cdots T_{N-1,N+q-1}$ for $N \in \mathbb{N}$ and $\alpha_0 = \beta_0 = 1$

Recursion polynomials

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Recursion and associated polynomials

Theorem

Determinantal expressions for the characteristic polynomials

$$P_N(x) = \alpha_N \det A_N(x) = \beta_N \det B_N(x)$$

Associated polynomials

$$Q_{n,N} := \begin{vmatrix} A_n^{(1)} & \cdots & A_n^{(p)} \\ A_{N+1}^{(1)} & \cdots & A_{N+1}^{(p)} \\ \vdots & & \vdots \\ A_{N+p-1}^{(1)} & \cdots & A_{N+p-1}^{(p)} \end{vmatrix}, \quad R_{n,N} := \begin{vmatrix} B_n^{(1)} & \cdots & B_n^{(q)} \\ B_{N+1}^{(1)} & \cdots & B_{N+1}^{(q)} \\ \vdots & & \vdots \\ B_{N+q-1}^{(1)} & \cdots & B_{N+q-1}^{(q)} \end{vmatrix}$$

$$Q_N := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots \end{bmatrix}, \; Q^{\langle N \rangle} := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots Q_{N,N} \end{bmatrix}$$
$$R_N := \begin{bmatrix} R_{0,N} & R_{1,N} \cdots \end{bmatrix}^\top, \; R^{\langle N \rangle} := \begin{bmatrix} R_{0,N} & R_{1,N} \cdots R_{N,N} \end{bmatrix}^\top$$

Spectral properties

Key spectral property of truncations

$$Q^{\langle N\rangle}T^{[N]} + \begin{bmatrix} 0 & \cdots & \cdots & 0 & T_{N+p,N}Q_{N+p,N} \end{bmatrix} = xQ^{\langle N\rangle}$$

$$T^{[N]}R^{\langle N\rangle} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ T_{N,N+q}R_{N+q,N} \end{bmatrix} = xR^{\langle N\rangle}$$

Christoffel–Darboux type formulas

$$\sum_{n=0}^{N} Q_{n,N}(x)R_{n,N}(y) = \frac{1}{\alpha_N\beta_N}\frac{P_{N+1}(x)P_N(y) - P_N(x)P_{N+1}(y)}{x-y}$$

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Spectral properties

Assume that P_{N+1} has **simple zeros** at the set $\{\lambda_k^{[N]}\}_{k=1}^{N+1}$

- Biorthogonal sets of left and right eigenvectors, $\{w_k^{\langle N\rangle}\}_{k=1}^{N+1}$, $\{u_k^{\langle N\rangle}\}_{k=1}^{N+1}$, are given by

$$w_k^{\langle N\rangle} = \frac{Q^{\langle N\rangle}(\lambda_k^{[N]})}{\beta_N \sum_{l=0}^{N} Q_{l,N}(\lambda_k^{[N]}) R_{l,N}(\lambda_k^{[N]})}, \quad u_k^{\langle N\rangle} = \beta_N R^{\langle N\rangle}(\lambda_k^{[N]})$$

- The following expression holds

$$w_{k,n}^{\langle N\rangle} = \frac{\alpha_N Q_{n-1,N}(\lambda_k^{[N]})}{P_N(\lambda_k^{[N]}) P'_{N+1}(\lambda_k^{[N]})}, \qquad u_{k,n}^{\langle N\rangle} = \beta_N R_{n-1,N}(\lambda_k^{[N]})$$

Spectral properties

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- Biorthogonal sets of left and right eigenvectors, $\{w_k^{\langle N\rangle}\}_{k=1}^{N+1}$, $\{u_k^{\langle N\rangle}\}_{k=1}^{N+1}$, are given by

$$w_k^{\langle N\rangle} = \frac{Q^{\langle N\rangle}\left(\lambda_k^{[N]}\right)}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda_k^{[N]}\right) R_{l,N}\left(\lambda_k^{[N]}\right)}, \quad u_k^{\langle N\rangle} = \beta_N R^{\langle N\rangle}\left(\lambda_k^{[N]}\right)$$

- The following expression holds

$$w_{k,n}^{\langle N\rangle} = \frac{\alpha_N Q_{n-1,N}\left(\lambda_k^{[N]}\right)}{P_N\left(\lambda_k^{[N]}\right) P'_{N+1}\left(\lambda_k^{[N]}\right)}, \qquad u_{k,n}^{\langle N\rangle} = \beta_N R_{n-1,N}\left(\lambda_k^{[N]}\right)$$

Spectral properties

- We can write $w_{k,n}^{\langle N\rangle} = A_{n-1}^{(1)}\left(\lambda_k^{[N]}\right)\mu_{k,1}^{[N]} + \cdots + A_{n-1}^{(p)}\left(\lambda_k^{[N]}\right)\mu_{k,p}^{[N]}$
  Christoffel numbers

$$\mu_{k,1}^{[N]} := \frac{\begin{vmatrix} A_{n+1}^{(2)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & A_{n+1}^{(p)}\left(\lambda_k^{[N]}\right) \\ \vdots & \ddots & \vdots \\ A_{n+p-1}^{(2)}\left(\lambda_k^{[N]}\right) & \cdots & A_{n+p-1}^{(p)}\left(\lambda_k^{[N]}\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda_k^{[N]}\right) R_{l,N}\left(\lambda_k^{[N]}\right)}$$

$$\mu_{k,2}^{[N]} := -\frac{\begin{vmatrix} A_{n+1}^{(1)}\left(\lambda_k^{[N]}\right) & A_{n+1}^{(3)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & A_{n+1}^{(p)}\left(\lambda_k^{[N]}\right) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n+p-1}^{(1)}\left(\lambda_k^{[N]}\right) & A_{n+p-1}^{(3)}\left(\lambda_k^{[N]}\right) & \cdots & A_{n+p-1}^{(p)}\left(\lambda_k^{[N]}\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda_k^{[N]}\right) R_{l,N}\left(\lambda_k^{[N]}\right)}$$

$$\vdots$$

$$\mu_{k,p}^{[N]} := (-1)^{p-1} \frac{\begin{vmatrix} A_{n+1}^{(1)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & A_{n+1}^{(p-1)}\left(\lambda_k^{[N]}\right) \\ \vdots & \ddots & \vdots \\ A_{n+p-1}^{(1)}\left(\lambda_k^{[N]}\right) & \cdots & A_{n+p-1}^{(p-1)}\left(\lambda_k^{[N]}\right) \end{vmatrix}}{\beta_N \sum_{l=0}^{N} Q_{l,N}\left(\lambda_k^{[N]}\right) R_{l,N}\left(\lambda_k^{[N]}\right)}$$

Spectral properties

- We can write $u_{k,n}^{\langle N\rangle} = B_{n-1}^{(1)}\left(\lambda_k^{[N]}\right)\rho_{k,1}^{[N]} + \cdots + B_{n-1}^{(q)}\left(\lambda_k^{[N]}\right)\rho_{k,q}^{[N]}$
  Christoffel numbers

$$\rho_{k,1}^{[N]} := \beta_N \begin{vmatrix} B_{N+1}^{(2)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & B_{N+1}^{(q)}\left(\lambda_k^{[N]}\right) \\ \vdots & & \vdots \\ B_{N+q-1}^{(2)}\left(\lambda_k^{[N]}\right) & \cdots & B_{N+q-1}^{(p)}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

$$\rho_{k,2}^{[N]} := -\beta_N \begin{vmatrix} B_{N+1}^{(1)}\left(\lambda_k^{[N]}\right) & B_{N+1}^{(3)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & B_{N+1}^{(q)}\left(\lambda_k^{[N]}\right) \\ \vdots & \vdots & & \vdots \\ B_{N+q-1}^{(1)}\left(\lambda_k^{[N]}\right) & B_{n+q-1}^{(3)}\left(\lambda_k^{[N]}\right) & \cdots & B_{N+q-1}^{(p)}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

$$\vdots$$

$$\rho_{k,q}^{[N]} := (-1)^{q-1}\beta_N \begin{vmatrix} B_{N+1}^{(1)}\left(\lambda_k^{[N]}\right) & \cdots\cdots & B_{N+1}^{(q-1)}\left(\lambda_k^{[N]}\right) \\ \vdots & & \vdots \\ B_{N+q-1}^{(1)}\left(\lambda_k^{[N]}\right) & \cdots & B_{N+q-1}^{(q-1)}\left(\lambda_k^{[N]}\right) \end{vmatrix}$$

Discrete orthogonality

Step functions

$$\psi_{b,a}^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{k,b}^{[N]}\mu_{k,a}^{[N]} & \lambda_{k+1}^{[N]} \leqslant x < \lambda_{k}^{[N]},\ k \in \{1,\ldots,N\} \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} & x \geqslant \lambda_{1}^{[N]} \end{cases}$$

Finite sums

For $a \in \{1,\ldots,p\}$ and $b \in \{1,\ldots,q\}$, we have

$$\rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} = (\xi^{-1} I_{q,p} \nu^{-\top})_{b,a}$$

$I_{p,q} \in \mathbb{R}^{p\times q}$, $(I_{p,q})_{i,j} = \delta_{i,j}$

Discrete orthogonality

Step functions

$$\psi_{b,a}^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{k,b}^{[N]}\mu_{k,a}^{[N]} & \lambda_{k+1}^{[N]} \leqslant x < \lambda_{k}^{[N]},\ k \in \{1,\ldots,N\} \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} & x \geqslant \lambda_{1}^{[N]} \end{cases}$$

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For $a \in \{1,\ldots,p\}$ and $b \in \{1,\ldots,q\}$, we have

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$I_{p,q} \in \mathbb{R}^{p\times q}$, $(I_{p,q})_{i,j} = \delta_{i,j}$

Discrete orthogonality

Matrix of discrete measures

$q \times p$ matrix of functions: $\Psi^{[N]} := \begin{bmatrix} \psi_{1,1}^{[N]} & \cdots & \psi_{1,p}^{[N]} \\ \vdots & & \vdots \\ \psi_{q,1}^{[N]} & \cdots & \psi_{q,p}^{[N]} \end{bmatrix}$ $q \times p$ matrix of discrete Lebesgue–Stieltjes measures supported at the zeros of P_{N+1}:

$$\mathrm{d}\Psi^{[N]} = \begin{bmatrix} \mathrm{d}\psi_{1,1}^{[N]} & \cdots & \mathrm{d}\psi_{1,p}^{[N]} \\ \vdots & & \vdots \\ \mathrm{d}\psi_{q,1}^{[N]} & \cdots & \mathrm{d}\psi_{q,p}^{[N]} \end{bmatrix} = \sum_{k=1}^{N+1} \begin{bmatrix} \rho_{k,1}^{[N]} \\ \vdots \\ \rho_{k,q}^{[N]} \end{bmatrix} \begin{bmatrix} \mu_{k,1}^{[N]} & \cdots & \mu_{k,p}^{[N]} \end{bmatrix} \delta\left(x - \lambda_k^{[N]}\right)$$

Discrete orthogonality

Theorem

The following **biorthogonal relations** *hold*

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int B_n^{(b)}(x)\mathrm{d}\psi_{b,a}^{[N]}(x)A_m^{(a)}(x)=\delta_{n,m},\qquad n,m\in\{0,\dots,N\}$$

Corollary

The following **discrete mixed multiple orthogonality** *for* $m\in\{1,\dots,N\}$ *are satisfied:*

$$\sum_{a=1}^{p}\int x^n\mathrm{d}\psi_{b,a}^{[N]}A_m^{(a)}=0,\quad n\in\{0,\dots,\deg B_{m-1}^{(b)}\},\quad b\in\{1,\dots,q\}$$

$$\sum_{b=1}^{q}\int B_m^{(b)}\mathrm{d}\psi_{b,a}^{[N]}x^n=0,\quad n\in\{0,\dots,\deg A_{m-1}^{(a)}\},\quad a\in\{1,\dots,p\}$$

Positive bidiagonal factorization

Positive bidiagonal factorization (PBF)

We say that a banded matrix T admits a PBF if

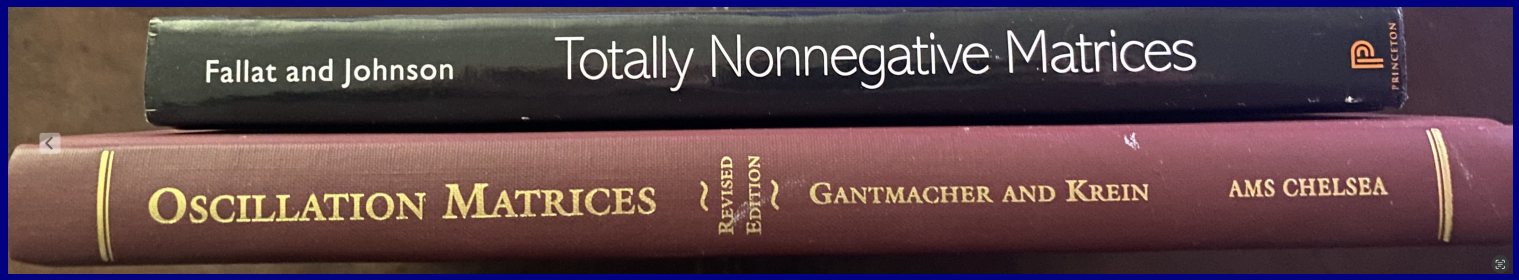
$$T = L_1 \cdots L_p \Delta U_q \cdots U_1$$

with $\Delta = \operatorname{diag}(\Delta_0, \Delta_1, \dots)$ and bidiagonal matrices given respectively by

$$L_k := \begin{bmatrix} 1 & 0 & & \\ L_{k|0} & 1 & \ddots & \\ 0 & L_{k|1} & 1 & \ddots \\ \vdots & \ddots & \ddots & \end{bmatrix}, \quad U_k := \begin{bmatrix} 1 & U_{k|0} & 0 & \\ 0 & 1 & U_{k|1} & \ddots \\ \vdots & \ddots & 1 & \ddots \\ & & \ddots & \end{bmatrix}$$

with $L_{k|i}, U_{k|i}, \Delta_i > 0$, for $i \in \mathbb{N}_0$

Oscillatory matrices



Totally nonnegative (TN)

All its minors are non-negative

Invertible totally nonnegative (InTN)

All its minors are non-negative and is nonsingular

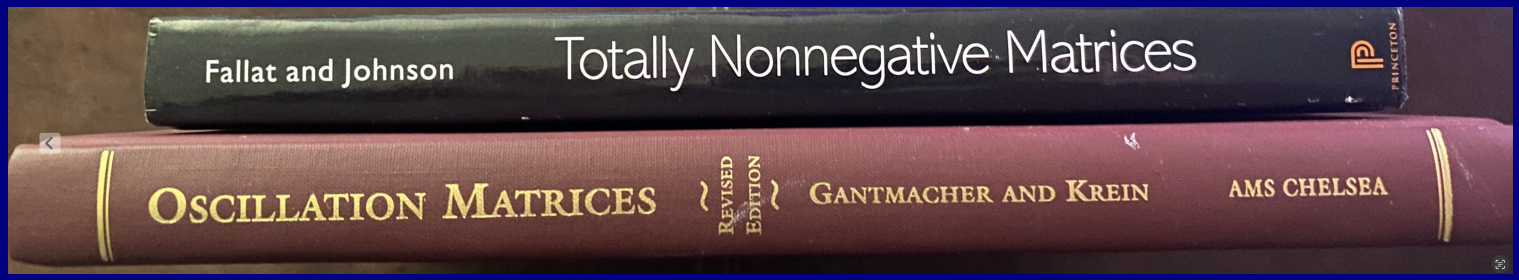
Totally positive (TP)

All its minors are positive

Oscillatory Matrix (IITN)

A totally non negative matrix A such that for some n, the matrix A^n is totally positive

Oscillatory matrices



Totally nonnegative (TN)

All its minors are non-negative

Invertible totally nonnegative (InTN)

All its minors are non-negative and is nonsingular

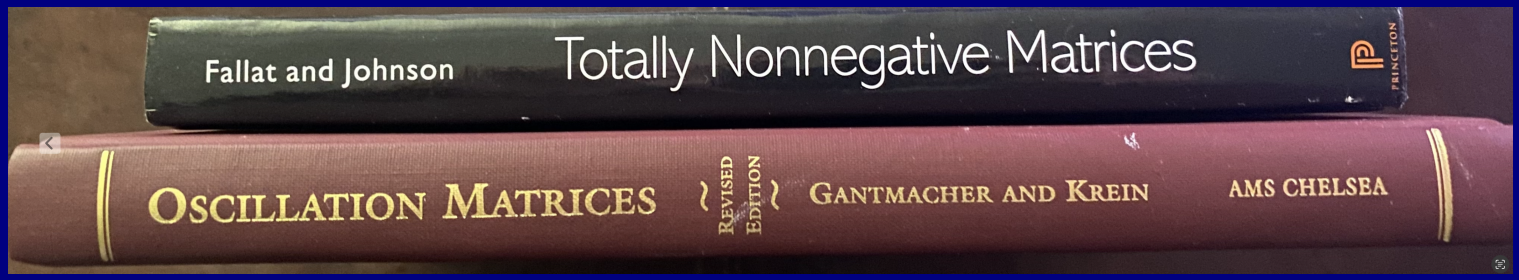
Totally positive (TP)

All its minors are positive

Oscillatory Matrix (IITN)

A totally non negative matrix A such that for some n, the matrix A^n is totally positive

Oscillatory matrices



Totally nonnegative (TN)

All its minors are non-negative

Invertible totally nonnegative (InTN)

All its minors are non-negative and is nonsingular

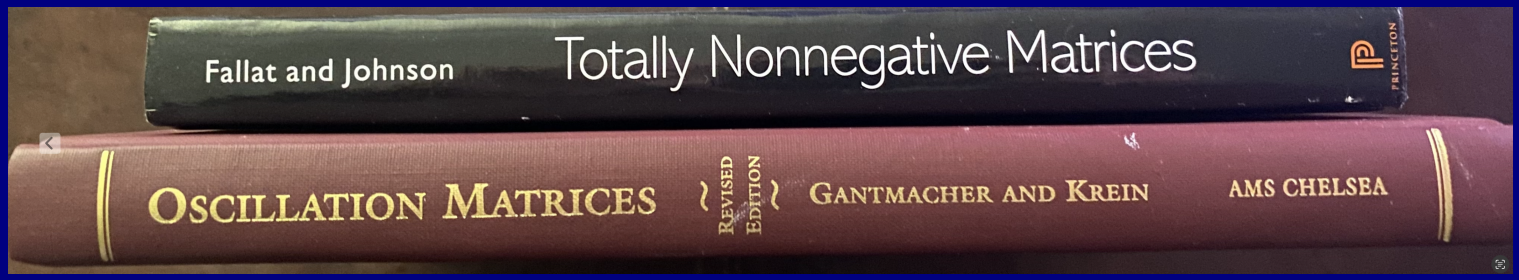
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From Cauchy–Binet Theorem one can deduce the invariance of these sets of matrices under the usual matrix product
PBF $\Rightarrow$ oscillatory

Eigenvalues

The eigenvalues are simple and positive

Interlacing property

The eigenvalues strictly interlace the eigenvalues of the principal submatrix (deleting first row and column) (also last column and row)

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Translations of bounded Jacobi matrices are oscillatory matrices

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Interlacing

Let us assume that T is oscillatory. Then:

1. The polynomial P_{N+1} interlaces P_N

$$(P'_{N+1}P_N)\big|_{x=\lambda_k^{[N]}} > 0, \qquad (P_{N+1}P'_N)\big|_{x=\lambda_k^{[N-1]}} < 0$$

2. The confluent kernel is a positive function; i.e., $\alpha_N\beta_N \sum_{n=0}^{N} Q_{n,N}(x)R_{n,N}(x) > 0$ for $x \in \mathbb{R}$

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Christoffel numbers are positive for PBF

$$\Lambda := \left[\Lambda^{(1)} \cdots\cdots \Lambda^{(p)}\right] \in \mathbb{R}^{p\times p}, \qquad \Upsilon := \begin{bmatrix} \Upsilon^{(1)} \\ \vdots \\ \vdots \\ \Upsilon^{(q)} \end{bmatrix} \in \mathbb{R}^{q\times q}$$

where

$$\Lambda^{(1)} := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \qquad \Lambda^{(k)} := \frac{1}{r_k} L_1^{[p-1]} \cdots L_{k-1}^{[p-1]} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$$\Upsilon^{(1)} := \begin{bmatrix} 1 & 0 \cdots\cdots 0 \end{bmatrix}, \qquad \Upsilon^{(k)} := \frac{1}{s_k} \begin{bmatrix} 1 & 0 \cdots\cdots 0 \end{bmatrix} U_1^{[q-1]} \cdots U_{k-1}^{[q-1]}$$

with

$$r_k := L_{k|0} L_{k-1|1} \cdots L_{1|k-1}, \qquad s_k := U_{k|0} U_{k-1|1} \cdots U_{1|k-1}$$

Christoffel numbers are positive for PBF

Lemma

The matrices Λ and Υ are positive upper and lower unitriangular matrices, respectively

Theorem (Christoffel coefficients positivity)

Let us assume that T has a PBF and choose the matrices of initial conditions as

$$\nu^{-\top} = \Lambda \mathcal{A}, \qquad \xi^{-1} = \mathcal{B}\Upsilon$$

for some upper and lower unitriangular nonnegative matrices $\mathcal{A} \in \mathbb{R}^{p\times p}$ and $\mathcal{B} \in \mathbb{R}^{q\times q}$, respectively. Then,

$$\rho_{k,b}^{[N]} > 0, \quad \mu_{k,a}^{[N]} > 0, \quad k \in \{1, \dots, N+1\}, \quad a \in \{1, \dots, p\}, \quad b \in \{1, \dots, q\}$$

Helly's tools

1. **Helly's Selection Principle**: for any uniformly bounded sequence $\{\psi^{[N]}\}_{N=0}^{\infty}$ of non-decreasing functions defined in $\mathbb{R}$, there exists a convergent subsequence converging to a non-decreasing function ψ defined in $\mathbb{R}$
2. **Helly's second theorem**: Let us assume a uniformly bounded sequence $\{\psi^{[N]}\}_{N=0}^{\infty}$ of non-decreasing functions on a compact interval $[a,b]$ with limit function ψ, then for any continuous function f in $[a,b]$ we have $\lim_{N\to\infty}\int_a^b f(x)\mathrm{d}\psi^{[N]}(x) = \int_a^b f(x)\mathrm{d}\psi(x)$

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Favard theorem

Theorem (Favard spectral representation)

Let us assume that

1. *The banded matrix T is bounded and there exist $s \geqslant 0$ such that $T + sI$ has a PBF*
2. *The sequences $\{A_n^{(1)}, \ldots, A_n^{(p)}\}_{n=0}^{\infty}, \{B_n^{(1)}, \ldots, B_n^{(q)}\}_{n=0}^{\infty}$ of recursion polynomials are determined by the initial condition matrices ν and ξ,respectively, such that $\nu^{-\top} = \Lambda\mathcal{A}$, $\xi^{-1} = \mathcal{B}\Upsilon$, and $\mathcal{A} \in \mathbb{R}^{p\times p}$ is a nonnegative upper unitriangular matrices and $\mathcal{B} \in \mathbb{R}^{q\times q}$ is a nonnegative lower unitriangular matrix*

Then, there exists pq non decreasing positive functions $\psi_{b,a}$, $a \in \{1, \ldots, p\}$ and $b \in \{1, \ldots, q\}$ and corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\psi_{b,a}$ with compact support $[\zeta, \eta]$ such that the following biorthogonality holds

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int_{\zeta}^{\eta} B_l^{(b)}(x)\mathrm{d}\psi_{b,a}(x)A_k^{(a)}(x) = \delta_{k,l}, \qquad k, l \in \mathbb{N}_0$$

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Favard theorem / Gaussian quadrature

Mixed multiple orthogonal relations

$$\sum_{a=1}^{p} \int_{\zeta}^{\eta} x^n \mathrm{d}\psi_{b,a}(x) A_m^{(a)}(x) = 0, \quad n \in \{0, \dots, \deg B_{m-1}^{(b)}\}, \quad b \in \{1, \dots, q\}$$

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Degrees of precision

$$d_{b,a}(N) = \deg A_N^{(a)} + \deg B_N^{(b)} + 1 = \left\lceil \frac{N+2-a}{p} \right\rceil + \left\lceil \frac{N+2-b}{q} \right\rceil - 1$$

Mixed multiple Gaussian quadrature formulas

$$\int_{\zeta}^{\eta} x^n \mathrm{d}\psi_{b,a}(x) = \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} \left(\lambda_k^{[N]}\right)^n, \qquad 0 \leqslant n \leqslant d_{b,a}(N)$$

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J. Math. Anal. Appl. 528 (2023) 127471

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Hahn multiple orthogonal polynomials of type I: Hypergeometric expressions

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ARTICLE INFO

Article history:
Received 25 February 2023
Available online 5 June 2023
Submitted by A.A.J. Durán

Keywords:
Multiple orthogonal polynomials
Askey scheme
Generalized hypergeometric functions
Kampé de Feriet hypergeometric functions

ABSTRACT

Explicit expressions for the Hahn multiple polynomials of type I, in terms of Kampé de Fériet hypergeometric series, are given. Orthogonal and biorthogonal relations are proven. Then, part of the Askey scheme for multiple orthogonal polynomials type I is completed. In particular, explicit expressions in terms of generalized hypergeometric series and Kampé de Fériet hypergeometric series, are given for the multiple orthogonal polynomials of type I for the Jacobi–Piñeiro, Meixner I, Meixner II, Kravchuk, Laguerre I, Laguerre II and Charlier families.

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On multiple orthogonal polynomials of type I

Relevance of type I MOPs

Type I and II polynomials play a crucial role in characterizing the spectral properties of bounded banded matrices. Consequently, their significance extends to practical applications involving orthogonalities, as seen in scenarios such as Markov chains with banded transition matrices. This is particularly evident in the context of constructing Markov chains through the combination of pure birth and pure death chains

Explicit expressions for type II MOPs

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- A. I. Aptekarev, A. Branquinho, & W. Van Assche, *Multiple orthogonal polynomials for classical weights*, Trans. Am. Math. Soc. **355** (2003), No. 10, 3887-3914
- W. Van Assche & E. Coussement, *Some classical multiple orthogonal polynomials*, J. Comput. Appl. Math. **127** (2001), 317-347
- J. Arvesu, J. Coussement, & W. Van Assche, *Some discrete multiple orthogonal polynomials*, J.Comput.Appl.Math. **153** (2003), 19-45

On multiple orthogonal polynomials of type I

Explicit expressions for type I are rare, very rare

1. Very few examples of type I MOPs are known
2. **Hahn discrete multiple orthogonal polynomials** lacked of such explicit expressions
3. From Hahn, the Askey scheme leads to: Jacobi–Piñeiro, Meixner I, Meixner II, Kravchuk, Laguerre I, Laguerre II and Charlier families

On the Askey scheme



B. Beckerman, J. Coussement & W. Van Assche, *Multiple Wilson and Jacobi–Piñeiro polynomials*, J.Approx.Theory **132** (2005), 155-181



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Discrete multiple orthogonal polynomials

AT system of two **weight functions** $w_1, w_2 : \Delta \subseteq \mathbb{Z} \to \mathbb{R}$ and a **multi-index sequence** $\{(n_1, n_2)\}_{(n_1,n_2)} \in \mathbb{N}_0^2$,

Type II

$B_{(n_1,n_2)}$ monic, $\deg B_{(n_1,n_2)} = n_1 + n_2$

$$\sum_{k\in\Delta} k^j B_{(n_1,n_2)}(k) w_i(k) = 0, \quad i \in \{1,2\}, \quad j \in \{0, \cdots, n_i - 1\}$$

Type I

$A^{(1)}_{(n_1,n_2)}, A^{(2)}_{(n_1,n_2)}$, $\deg A^{(i)}_{(n_1,n_2)} = n_i - 1$, for $j \in \{0, \cdots, n_1 + n_2 - 1\}$

$$\sum_{k\in\Delta} k^j \left(A^{(1)}_{(n_1,n_2)}(k) w_1(k) + A^{(2)}_{(n_1,n_2)}(k) w_2(k) \right) = \delta_{n_1+n_2-1,j}$$

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Hypergeomeric series

Pochhammer symbols: $(x)_n := \dfrac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)\cdots(x+n-1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$

Generalized hypergeometric series

$$ {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ \alpha_1, \cdots, \alpha_q \end{matrix}; x\right] := \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_p)_l}{(\alpha_1)_l \cdots (\alpha_q)_l} \frac{x^l}{l!} $$

Kampé de Fériet series

$$ F^{p:r;s}_{q:n;k}\left[\left.\begin{matrix} (a_1, \cdots, a_p) : (b_1, \cdots, b_r); (c_1, \cdots, c_s) \\ (\alpha_1, \cdots, \alpha_q) : (\beta_1, \cdots, \beta_n); (\gamma_1, \cdots, \gamma_k) \end{matrix}\right| x, y\right] $$

$$:= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a_1)_{l+m} \cdots (a_p)_{l+m}}{(\alpha_1)_{l+m} \cdots (\alpha_q)_{l+m}} \frac{(b_1)_l \cdots (b_r)_l}{(\beta_1)_l \cdots (\beta_n)_l} \frac{(c_1)_m \cdots (c_s)_m}{(\gamma_1)_m \cdots (\gamma_k)_m} \frac{x^l}{l!} \frac{y^m}{m!} $$

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Type II Hahn discrete MOPs for two weights

Weight functions, $i \in \{1,2\}$, $\Delta = \{0, \cdots, N\}$, $N \in \mathbb{N}_0$, $\alpha_1, \alpha_2, \beta > -1$ and $\alpha_1 - \alpha_2 \notin \mathbb{Z}$

$$w_i(x) = \frac{\Gamma(\alpha_i + x + 1)}{\Gamma(\alpha_i + 1)\Gamma(x+1)} \frac{\Gamma(\beta + N - x + 1)}{\Gamma(\beta+1)\Gamma(N-x+1)};$$

Type II Hahn MOPs $Q_{(n_1,n_2)}(x)$

$$\frac{(\alpha_1+1)_{n_1}(\alpha_2+1)_{n_2}(-N)_{n_1+n_2}}{(\alpha_1+\beta+n_1+n_2+1)_{n_1}(\alpha_2+\beta+n_1+n_2+1)_{n_2}} \times F^{2:3;1}_{2:2;0}\left[\begin{matrix}(-x, \alpha_1+\beta+n_1+1) : (-n_2, \alpha_1+n_1+1, \alpha_2+\beta+n_1+n_2+1); (-n_1) \\ (-N, \alpha_1+1) : (\alpha_2+1, \alpha_1+\beta+n_1+1); --\end{matrix}\middle| 1, 1\right]$$

J. Arvesu, J. Coussement, & W. Van Assche, *Some discrete multiple orthogonal polynomials*, J.Comput.Appl.Math. **153** (2003), 19-45

Our new finding for type I Hahn MOPs

Hahn multiple orthogonal polynomials of type I $Q^{(i)}_{(n_1,n_2)}(x)$

$$\frac{(-1)^{n_i-1}(n_1+n_2-2)!(N+1-n_1-n_2)!(\hat{\alpha}_i+\beta+\hat{n}_i+1)_{n_1+n_2-1}}{(n_1-1)!(n_2-1)!(\beta+1)_{n_1+n_2-1}(\alpha_i+\beta+n_1+n_2+n_i)_{N+1-n_1-n_2}(\alpha_i-\hat{\alpha}_i-\hat{n}_i+1)_{n_1+n_2-1}}$$

$$\times F^{2:3;1}_{2:2;0}\left[\begin{matrix}(-n_i+1,-N):(\alpha_i+\beta+n_1+n_2,\alpha_i-\hat{\alpha}_i-\hat{n}_i+1,-x);(\hat{\alpha}_i-\alpha_i-n_i+1)\\(-n_1-n_2+2,\hat{\alpha}_i+\beta+\hat{n}_i+1):(\alpha_i+1,-N);--\end{matrix}\middle|1,1\right]$$

with $\hat{\alpha}_i := \delta_{i,2}\alpha_1+\delta_{i,1}\alpha_2$ and $\hat{n}_i := \delta_{i,2}n_1+\delta_{i,1}n_2$

Tools for the proof

- Pfaff–Saalschütz summation formula

D. B. Karp & E. G. Prilepkina, *Extensions of Karlsson–Minton summation theorem and some consequences of the first Miller–Paris transformation*, Integral Transforms Spec. Funct. **29** (2018), 955-970

M. A. Rakha & A. K. Rathie, *Certain reduction and transformation formulas for the Kampé de Fériet function*, Commun. Korean Math. Soc. **37** (2022), No. 2, 473–496

Our new finding for type I Hahn MOPs

Hahn multiple orthogonal polynomials of type I $Q^{(i)}_{(n_1,n_2)}(x)$

$$\frac{(-1)^{n_i-1}(n_1+n_2-2)!(N+1-n_1-n_2)!(\hat{\alpha}_i+\beta+\hat{n}_i+1)_{n_1+n_2-1}}{(n_1-1)!(n_2-1)!(\beta+1)_{n_1+n_2-1}(\alpha_i+\beta+n_1+n_2+n_i)_{N+1-n_1-n_2}(\alpha_i-\hat{\alpha}_i-\hat{n}_i+1)_{n_1+n_2-1}}$$

$$\times F^{2:3;1}_{2:2;0}\left[\begin{matrix}(-n_i+1,-N):(\alpha_i+\beta+n_1+n_2,\alpha_i-\hat{\alpha}_i-\hat{n}_i+1,-x);(\hat{\alpha}_i-\alpha_i-n_i+1)\\(-n_1-n_2+2,\hat{\alpha}_i+\beta+\hat{n}_i+1):(\alpha_i+1,-N);--\end{matrix}\middle|\,1,1\right]$$

with $\hat{\alpha}_i := \delta_{i,2}\alpha_1+\delta_{i,1}\alpha_2$ and $\hat{n}_i := \delta_{i,2}n_1+\delta_{i,1}n_2$

Tools for the proof

- Pfaff–Saalschütz summation formula



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Descendants of Hahn in the multiple Askey scheme

