Brenke polynomials with real zeros and the Riemann Hypothesis

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If $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$ are two formal power series, with $a_n, b_n \in \mathbb{R}$ and $a_0 = b_0 = 1$, the polynomials $(p_n)_n$ defined by the generating function

$$A(z)B(xz) = \sum_{n=0}^{\infty} p_n(x) z^n$$

are called the Brenke polynomials generated by $A$ and associated to $B$. We say that $A \in \mathcal{R}_B$ if the Brenke polynomials $(p_n)_n$ have only real zeros.

In this talk we show necessary and sufficient conditions on $B$ such that $\mathcal{R}_B = \mathcal{L}\text{-}\mathcal{P}_0$, where $\mathcal{L}\text{-}\mathcal{P}_0$ denotes the Laguerre-Pólya class of entire functions (normalized so that if $A \in \mathcal{L}\text{-}\mathcal{P}_0$ then $A(0) = 1$). These results can be considered an extension to Brenke polynomials of the Jensen, and Pólya and Schur characterization $\mathcal{R}_{e^z} = \mathcal{L}\text{-}\mathcal{P}_0$, for Appell polynomials. When applying our results to a relative of the Riemann zeta function, we find new equivalencies for the Riemann Hypothesis in terms of real-rootedness of some sequences of Brenke polynomials.