Multiple orthogonality, spectral Favard theorem and Markov chains

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Simple case: Jacobi matrix

Spectral Favard theorem for bounded Jacobi matrices

Recurrence Relation

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \qquad b_n \in \mathbb{R}, \qquad a_n > 0$$

Favard's Theorem

$\exists\psi(x)$, non-decreasing piece-wise continuous function, such that

$$\int P_n(x)P_m(x)\mathrm{d}\psi(x) = k_n\delta_{n,m}$$

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Jacobi matrix and recursion relation

$$J := \begin{bmatrix} b_0 & 1 & 0 & \cdots & \cdots \\ a_1 & b_1 & 1 & \ddots & \\ 0 & a_2 & b_2 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \qquad J \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{bmatrix}$$

Characteristic Polynomial

$$P_{n+1} = \begin{vmatrix} x-b_0 & -1 & 0 & \cdots & \cdots & 0 \\ -a_1 & x-b_1 & -1 & \ddots & & \vdots \\ 0 & -a_2 & x-b_2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & & -a_n & x-b_n \end{vmatrix}$$

Proof

This determinantal polynomial satisfies the same three recurrence relation with the same initial conditions $P_{-1} = 0, P_0 = 1$

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Polynomials of the second kind

$P_{-1}^{(1)} = 1, P_0^{(1)} = 0$

Determinantal form of polynomials of the second kind

$$P_{n+1}^{(1)} = \begin{vmatrix} x-b_1 & -1 & 0 & \cdots & \cdots & 0 \\ -a_2 & x-b_2 & -1 & \ddots & & \vdots \\ 0 & -a_3 & x-b_3 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -a_n & x-b_n \end{vmatrix}$$

Adjugates and second kind

$P_{n+1}^{(1)} = e_1^{\top} \operatorname{adj}\left(xI_{n+1} - J^{[n]}\right) e_1$

Eigenvalues and zeros

1. The zeros $\left\{\lambda_1^{[N]},\dots,\lambda_{N+1}^{[N]}\right\}$ of the orthogonal polynomials are the eigenvalues of the truncated Jacobi matrix

$$J^{[N]}=\begin{bmatrix} b_0 & 1 & 0 & \cdots & \cdots & 0\\ a_1 & b_1 & 1 & \ddots & & \vdots\\ 0 & a_2 & b_2 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & 0 & a_N & b_N \end{bmatrix}$$

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Proof.

$$J^{[N]} \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_N(x) \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_N(x) \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_{N+1}(x) \end{bmatrix}$$

□

Right eigenvector matrix

$$U := \begin{bmatrix} P_0(\lambda_1^{[N]}) & P_0(\lambda_2^{[N]}) & \cdots\cdots\cdots & P_0(\lambda_{N+1}^{[N]}) \\ P_1(\lambda_1^{[N]}) & P_1(\lambda_2^{[N]}) & \cdots\cdots\cdots & P_1(\lambda_{N+1}^{[N]}) \\ \vdots & \vdots & & \vdots \\ P_N(\lambda_1^{[N]}) & P_N(\lambda_2^{[N]}) & \cdots\cdots\cdots & P_N(\lambda_{N+1}^{[N]}) \end{bmatrix}$$

Diagonalization

$$J^{[N]}U = UD, \qquad D := \operatorname{diag}(\lambda_1^{[N]}, \dots, \lambda_{N+1}^{[N]})$$

Left eigenvector matrix condition

$$WJ^{[N]} = DW$$

Proof.

Let us consider $H = \operatorname{diag}\left(1, a_1, \dots, a_1 \dots a_N\right)$, with $H_0 = 1$ and $H_k = a_1 \cdots a_k > 0$ for $k \in \mathbb{N}$ such that

$$H^{-1}J^{[N]} = {J^{[N]}}^\top H^{-1}$$

so that it holds

$$U^\top H^{-1}J^{[N]} = U^\top {J^{[N]}}^\top H^{-1} = DU^\top H^{-1}$$

and we get that $\tilde{W} = U^\top H^{-1}$ is a left eigenvector matrix □

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$\tilde{W}$ Left Eigenvector Matrix

$$\tilde{W} = \begin{bmatrix} Q_0(\lambda_1^{[N]}) & Q_1(\lambda_1^{[N]}) \cdots\cdots & Q_N(\lambda_1^{[N]}) \\ Q_0(\lambda_2^{[N]}) & Q_1(\lambda_2^{[N]}) \cdots\cdots & Q_N(\lambda_2^{[N]}) \\ \vdots & \vdots & \vdots \\ Q_0(\lambda_{N+1}^{[N]}) & Q_1(\lambda_{N+1}^{[N]}) \cdots\cdots & Q_N(\lambda_{N+1}^{[N]}) \end{bmatrix}, \quad Q_n(x) := \frac{P_n(x)}{H_n}$$

Orthogonality condition

$$D\tilde{W}U = \tilde{W}J^{[N]}U = \tilde{W}UD \qquad \Rightarrow \qquad \tilde{W}U = \tilde{D}$$

so that

$$\tilde{D} = \operatorname{diag}\Big(\sum_{k=0}^{N} \frac{(P_k(\lambda_1^{[N]}))^2}{H_k}, \dots, \sum_{k=0}^{N} \frac{(P_k(\lambda_{N+1}^{[N]}))^2}{H_k}\Big)$$

$\tilde{W}$ Left Eigenvector Matrix

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Bi-orthogonality

In terms of the Christoffel numbers (they are clearly positive!)

$$\mu_k^{[n]} := \frac{1}{\sum_{l=0}^{N} \frac{(P_l(\lambda_k^{[N]}))^2}{H_l}} > 0$$

we have

$$\tilde{D}^{-1}\tilde{W}U = U\tilde{D}^{-1}\tilde{W} = I, \qquad \tilde{D}^{-1} = \operatorname{diag}(\mu_1^{[N]}, \dots, \mu_{N+1}^{[N]})$$

Normalised left eigenvectors

$$W = \tilde{D}^{-1}\tilde{W}$$

Discrete orthogonality relations

$$\sum_{k=1}^{N+1} \frac{P_r(\lambda_k^{[N]})P_s(\lambda_k^{[N]})}{H_s}\mu_k^{[N]} = \delta_{r,s}$$

Discrete weight measure

$$\mu^{[N]} = \sum_{k=1}^{N+1} \mu_k^{[N]}\delta(z-\lambda_k^{[N]})$$

Lebesgue–Stieltjes representation

In terms of the non-decreasing piecewise continuous function

$$\psi^{[N]} := \begin{cases} 0, & x < \lambda_{n+1}^{[N]}, \\ \mu_1^{[N]} + \cdots + \mu_k^{[N]}, & \lambda_{k+1}^{[N]} \leqslant x < \lambda_k^{[N]}, \quad k \in \{1,\dots,N\} \\ \mu_1^{[N]} + \cdots + \mu_{N+1}^{[N]} = 1, & x \geqslant \lambda_1^{[N]} \end{cases}$$

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Moments of the discrete measure

$$UD^mW = \left(J^{[N]}\right)^m$$

$$\sum_{k=1}^{N+1} (\lambda_k^{[N]})^m \mu_k^{[N]} = e_1^\top \left(J^{[N]}\right)^m e_1$$

Resolvent matrix function

$$R_z^{[N]} := \left(zI_{N+1} - J^{[N]}\right)^{-1} = U\left(zI_{N+1} - D\right)^{-1} W$$

Weyl function

$$S^{[N]}(z) := e_1^\top \left(zI_{N+1} - J^{[N]}\right)^{-1} e_1 = \sum_{k=1}^{N+1} \frac{\mu_k^{[N]}}{z - \lambda_k^{[N]}} = \frac{P_{N+1}^{(1)}(z)}{P_{N+1}(z)}$$

Truncated polynomials

$$P_{n+1}^{[0]} = P_{n+1} = \begin{vmatrix} x-b_0 & -1 & 0 & \cdots & 0 \\ -a_1 & x-b_1 & -1 & \ddots & \vdots \\ 0 & -a_2 & x-b_2 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -a_n & x-b_n \end{vmatrix}$$

$$P_{n+1}^{[1]} = P_{n+1}^{(1)} = \begin{vmatrix} x-b_1 & -1 & 0 & \cdots & 0 \\ -a_2 & x-b_2 & -1 & \ddots & \vdots \\ 0 & -a_3 & x-b_3 & -1 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -a_n & x-b_n \end{vmatrix}$$

For $k \in \{0, 1, \dots, n\}$

$$P^{[k]}_{n+1} = \begin{vmatrix} x - b_k & 1 & 0 & \cdots & \cdots & 0 \\ -a_{k+1} & x - b_{k+1} & 1 & \ddots & & \vdots \\ 0 & -a_{k+2} & x - b_{k+2} & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & -a_n & x - b_n \end{vmatrix}$$

Recursion relation

$$a_{k+1} P^{[k+2]}_{n+1} + b_k P^{[k+1]}_{N+1} + P^{[k]}_{N+1} = x P^{[k+1]}_{N+1}, \qquad k \in \{0, 1, \dots, n\}$$

with $P^{[n+1]}_{n+1} = 1, P^{[n+2]}_{n+1} = 0$

Left eigenvectors truncated version

$$\left[P^{[1]}_{N+1}(\lambda^{[N]}_k)\cdots\cdot P^{[N+1]}_{N+1}(\lambda^{[N]}_k)\right]J^{[N]}=\lambda^{[N]}_k\left[P^{[1]}_{N+1}(\lambda^{[N]}_k)\cdots\cdot P^{[N+1]}_{N+1}(\lambda^{[N]}_k)\right]$$

Comparing truncated and original

Using CD formulas

$$H_N\frac{Q_{r-1}\big(\lambda^{[N]}_k\big)}{P_N\big(\lambda^{[N]}_k\big)P'_{N+1}\big(\lambda^{[N]}_k\big)}=\frac{P^{[r]}_{N+1}\big(\lambda^{[N]}_k\big)}{\sum_{l=0}^{N}P^{[l+1]}_{N+1}\big(\lambda^{[N]}_k\big)P_l\big(\lambda^{[N]}_k\big)}=\frac{P^{[r]}_{N+1}\big(\lambda^{[N]}_k\big)}{P'_{N+1}\big(\lambda^{[N]}_k\big)}$$

In particular, for $r=1$, $\mu^{[N]}_k=\frac{P^{(1)}_{N+1}\left(\lambda^{[N]}_k\right)}{P'_{N+1}\left(\lambda^{[N]}_k\right)}$. As the Christoffel coefficients are positive we conclude that recursion polynomials P_{n+1} strictly interlace its second kind polynomials $P^{(1)}_{n+1}$

Helly's tools

1. **Helly's Selection Principle**: for any uniformly bounded sequence $\{\psi^{[n]}\}_{n=0}^{\infty}$ of non-decreasing functions defined in $\mathbb{R}$, there exists a convergent subsequence converging to a non-decreasing function ψ defined in $\mathbb{R}$
2. **Helly's second theorem**: Let us assume a uniformly bounded sequence $\{\psi^{[n]}\}_{n=0}^{\infty}$ of non-decreasing functions on a compact interval $[a, b]$ with limit function ψ, then for any continuous function f in $[a, b]$ we have $\lim_{n\to\infty} \int_a^b f(x)\mathrm{d}\psi^{[n]}(x) = \int_a^b f(x)\mathrm{d}\psi(x)$

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Spectral Favard's theorem

Helly's results lead to the existence of a nondecreasing functions ψ and corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\psi$ with compact support Δ such that the orthogonal relations of "type II"

$$\int_\Delta x^k P_n(x)\mathrm{d}\psi(x) = 0, \qquad k = 0,\dots,n-1$$

and of "type I"

$$\int_\Delta Q_{k-1}(x)\mathrm{d}\psi(x)x^l = 0, \quad l \in \{0,1,\dots,k-1\}, \quad k \in \{1,\dots,n\}$$

hold. These polynomial sequences of types II and I are biorthogonal, i.e.,

$$\int_\Delta Q_k(x)\mathrm{d}\psi(x)P_l(x) = \delta_{k,l}$$

for $k,l \in \mathbb{N}_0$. Recall that $H_kQ_k = P_k$, and the biorthogonality reads

$$\int_\Delta P_k(x)\mathrm{d}\psi(x)P_l(x) = \delta_{k,l}H_k$$

Spectral Favard's theorem

Helly's second theorem leads to the spectral representation in terms of the spectral function ψ

$$s_k = e_1^\top J^k e_1 = \int_\Delta t^k \mathrm{d}\psi(t), \qquad \hat{\psi} = e_1^\top (zI - J)^{-1} e_1 = \int_\Delta \frac{\mathrm{d}\psi(t)}{z-t},$$

of the moments s_k and of the Stieltjes–Markov function $\hat{\psi}$. The Markov theorem ensures that the Weyl function converges uniformly in $\bar{\mathbb{C}} \setminus \Delta$ to the Stieltjes–Markov function

$$S^{[n]} \rightrightarrows \hat{\psi}, \qquad n \to \infty$$

Multiple case: banded Hessenberg

Banded Hessenberg semi-infinite matrices and Favard theorem

Banded monic lower Hessenberg semi-infinite matrix

$$T = \begin{bmatrix} a_{0,0} & 1 & 0 & \cdots & & \\ a_{1,0} & a_{1,1} & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ a_{p,0} & & & \ddots & \ddots & \ddots \\ 0 & \ddots & & & \ddots & \ddots \\ \vdots & \ddots & \ddots & & & \ddots \end{bmatrix}$$

Recursion polynomials

Type II recursion polynomials

$$B(x) = \begin{bmatrix} B_0(x) & B_1(x) \cdots \cdots \end{bmatrix}^\top, \qquad \deg B_n = n$$

- Right eigenvectors:

$$TB(x) = xB(x)$$

- Initial condition: $B_0 = 1$
- Type II recursion polynomials: $B_n(x)$
- $(p+2)$-term recurrence relation

$$B_{n+1} = (x - a_{n,n})B_n - a_{n,n-1}B_{n-1} - \cdots - a_{n,n-p}B_{n-p}, \quad n \in \mathbb{N}_0$$

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Recursion polynomials

Type I recursion polynomials

$$A^{(a)}(x) = \left[A_0^{(a)}(x) \quad A_1^{(a)}(x) \cdots \right], \qquad a = 1, \ldots, p,$$

- Left eigenvectors:

$$A^{(a)}(x)T = xA^{(a)}(x), \qquad a = 1, \ldots, p.$$

Recursion polynomials

Type I recursion polynomials

- Initial conditions,

$$\begin{cases} A_0^{(1)} = 1, \\ A_1^{(1)} = \nu_1^{(1)}, \\ \quad\vdots \\ A_{p-1}^{(1)} = \nu_{p-1}^{(1)}, \end{cases} \qquad \begin{cases} A_0^{(2)} = 0, \\ A_1^{(2)} = 1, \\ A_2^{(2)} = \nu_2^{(2)}, \\ \quad\vdots \\ A_{p-1}^{(2)} = \nu_{p-1}^{(2)}, \end{cases} \qquad \dots \qquad \begin{cases} A_0^{(p)} = 0, \\ \quad\vdots \\ A_{p-2}^{(p)} = 0, \\ A_{p-1}^{(p)} = 1, \end{cases}$$

$\nu_j^{(i)}$ being arbitrary constants

Recursion polynomials

Type I recursion polynomials

- Initial condition matrix

$$\nu := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \nu_1^{(1)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \nu_{p-1}^{(1)} & \cdots & \nu_{p-1}^{(p-1)} & 1 \end{bmatrix}$$

- $(p+2)$-term recursion relation, $a \in \{1, \ldots, p\}$, $A_{-1}^{(a)} = 0$

$$A_{n-1}^{(a)} + a_{n,n} A_n^{(a)} + \cdots + a_{n+p,n} A_{n+p}^{(a)} = x A_n^{(a)}, \quad n \in \{0, 1, \ldots\}$$

- $\deg A_{kp+j}^{(r)} = k$, for $r = 1, \ldots, j+1$, $\deg A_{kp+j}^{(r)} = k-1$ for $r = j+2, \ldots, p$

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Recursion polynomials

Recursion polynomials and characteristic polynomials

Jonathan Coussement and Walter Van Assche, *Gaussian quadrature for multiple orthogonal polynomials*, Journal of Computational and Applied Mathematics **178** (2005) 131–145.

$$B_n = h_n \begin{vmatrix} A_n^{(1)} & \cdots\cdots\cdots & A_n^{(p)} \\ \vdots & \ddots & \vdots \\ A_{n+p-1}^{(1)} & \cdots\cdots & A_{n+p-1}^{(p)} \end{vmatrix}, \qquad n \in \mathbb{N}_0$$

where

$$h_n := (-1)^{(p-1)n} H_n, \qquad H_n := a_{p,0}a_{p+1,1}\cdots a_{n-1+p,n-1}$$

Associated polynomials

$$Q_{n,N} := \begin{vmatrix} A_n^{(1)} & \cdots\cdots & A_n^{(p)} \\ A_{N+1}^{(1)} & \cdots\cdots & A_{N+1}^{(p)} \\ \vdots & \ddots & \vdots \\ A_{N+p-1}^{(1)} & \cdots\cdots & A_{N+p-1}^{(p)} \end{vmatrix}$$

$Q_N := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots \end{bmatrix}$, $Q^{\langle N\rangle} := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots\cdots Q_{N,N} \end{bmatrix}$

1. $Q_{N+1,N} = \cdots = Q_{N+p-1,N} = 0$
2. $Q_{N,N} = h_N^{-1} B_N$ and $Q_{N+p,N} = (-1)^{p-1} h_{N+1}^{-1} B_{N+1}$
3. $Q_N T = x Q_N$
4. $Q^{\langle N\rangle} T^{[N]} + \begin{bmatrix} 0 \cdots\cdots 0 & a_{N+p,N} Q_{N+p,N} \end{bmatrix} = x Q^{\langle N\rangle}$

Associated polynomials

$$Q_{n,N} := \begin{vmatrix} A_n^{(1)} & \cdots\cdots & A_n^{(p)} \\ A_{N+1}^{(1)} & \cdots\cdots & A_{N+1}^{(p)} \\ \vdots & \ddots & \vdots \\ A_{N+p-1}^{(1)} & \cdots\cdots & A_{N+p-1}^{(p)} \end{vmatrix}$$

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Spectral properties

Assume that B_{N+1} has simple zeros at the set $\{\lambda_k^{[N]}\}_{k=1}^{N+1}$, so that the vectors $u_k^{\langle N\rangle} := B^{\langle N\rangle}(\lambda_k^{[N]})$ and $\tilde{w}_k^{\langle N\rangle} := Q^{\langle N\rangle}(\lambda_k^{[N]})$ are right and left eigenvectors of $T^{[N]}$, respectively, $k = 1, \dots, N+1$. Then

- The normalized basis of left eigenvectors $\{w_k^{\langle N\rangle}\}_{k=1}^{N+1}$, which is biorthogonalto the basis of right eigenvectors $\{u_k^{\langle N\rangle}\}_{k=1}^{N+1}$, is given by

$$w_k^{\langle N\rangle} = \frac{Q^{\langle N\rangle}(\lambda_k^{[N]})}{\sum_{l=0}^{N} Q_{l,N}(\lambda_k^{[N]}) B_l(\lambda_k^{[N]})}$$

- The following expression holds

$$w_{k,n}^{\langle N\rangle} = \frac{Q_{n-1,N}(\lambda_k^{[N]})}{\sum_{l=0}^{N} Q_{l,N}(\lambda_k^{[N]}) B_l(\lambda_k^{[N]})} = \frac{Q_{n-1,N}(\lambda_k^{[N]})}{Q_{N,N}(\lambda_k^{[N]}) B'_{N+1}(\lambda_k^{[N]})}$$

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Spectral properties

- We can write $w_{k,n}^{\langle N\rangle}=A^{(1)}_{n-1}\big(\lambda_k^{[N]}\big)\mu_{k,1}^{[N]}+\cdots+A^{(p)}_{n-1}\big(\lambda_k^{[N]}\big)\mu_{k,p}^{[N]}$
  Christoffel numbers

$$\mu_{k,1}^{[N]}:=\frac{\begin{vmatrix} A^{(2)}_{n+1}\big(\lambda_k^{[N]}\big) & \cdots\cdots & A^{(p)}_{n+1}\big(\lambda_k^{[N]}\big)\\ \vdots & \ddots & \vdots\\ A^{(2)}_{n+p-1}\big(\lambda_k^{[N]}\big) & \cdots & A^{(p)}_{n+p-1}\big(\lambda_k^{[N]}\big)\end{vmatrix}}{\sum_{l=0}^{N} Q_{l,N}\big(\lambda_k^{[N]}\big)B_l\big(\lambda_k^{[N]}\big)}$$

$$\mu_{k,2}^{[N]}:=-\frac{\begin{vmatrix} A^{(1)}_{n+1}\big(\lambda_k^{[N]}\big) & A^{(3)}_{n+1}\big(\lambda_k^{[N]}\big) & \cdots\cdots & A^{(p)}_{n+1}\big(\lambda_k^{[N]}\big)\\ \vdots & \vdots & \ddots & \vdots\\ A^{(1)}_{n+p-1}\big(\lambda_k^{[N]}\big) & A^{(3)}_{n+p-1}\big(\lambda_k^{[N]}\big) & \cdots & A^{(p)}_{n+p-1}\big(\lambda_k^{[N]}\big)\end{vmatrix}}{\sum_{l=0}^{N} Q_{l,N}\big(\lambda_k^{[N]}\big)B_l\big(\lambda_k^{[N]}\big)}$$

$$\vdots$$

$$\mu_{k,p}^{[N]}:=(-1)^{p-1}\frac{\begin{vmatrix} A^{(1)}_{n+1}\big(\lambda_k^{[N]}\big) & \cdots\cdots & A^{(p-1)}_{n+1}\big(\lambda_k^{[N]}\big)\\ \vdots & \ddots & \vdots\\ A^{(1)}_{n+p-1}\big(\lambda_k^{[N]}\big) & \cdots & A^{(p-1)}_{n+p-1}\big(\lambda_k^{[N]}\big)\end{vmatrix}}{\sum_{l=0}^{N} Q_{l,N}\big(\lambda_k^{[N]}\big)B_l\big(\lambda_k^{[N]}\big)}$$

Spectral properties

- It holds that

$$\begin{bmatrix} \mu^{[N]}_{k,1} \\ \mu^{[N]}_{k,2} \\ \vdots \\ \mu^{[N]}_{k,p} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \nu^{(1)}_1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \nu^{(1)}_{p-1} & \cdots & \cdots & \nu^{(p-1)}_{p-1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} w^{\langle N\rangle}_{k,1} \\ w^{\langle N\rangle}_{k,2} \\ \vdots \\ w^{\langle N\rangle}_{k,p} \end{bmatrix}$$

- Matrices U (with columns the right eigenvectors u_k arranged in the standard order) and W and (with rows the left eigenvectors w_k arranged in the standard order) satisfy

$$UW = WU = I_{N+1}$$

- In terms of the diagonal matrix $D = \operatorname{diag}\left(\lambda^{[N]}_1, \dots, \lambda^{[N]}_{N+1}\right)$ we have

$$UD^nW = \left(T^{[N]}\right)^n$$

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Orthogonality

Discrete measures: $\mu_a^{[N]} := \sum_{j=1}^{N+1} \mu_{j,a}^{[N]} \delta\big(x - \lambda_j^{[N]}\big)$, $a \in \{1, \dots, p\}$

Multiple discrete biorthogonalities

Assume that the recursion polynomials B_{N+1} have simple zeros $\{\lambda_k\}_{k=1}^{N+1}$. For $k, l \in \{0, \dots, N\}$, the following biorthogonal relations hold

$$\left\langle A_k^{(1)} \mu_1^{[N]} + \dots + A_k^{(p)} \mu_p^{[N]}, B_l \right\rangle = \delta_{k,l}$$

Proof.

$$U = \begin{bmatrix} B_0\big(\lambda_1^{[N]}\big) & \cdots & B_0\big(\lambda_{N+1}^{[N]}\big) \\ \vdots & \ddots & \vdots \\ B_N\big(\lambda_1^{[N]}\big) & \cdots & B_N\big(\lambda_{N+1}^{[N]}\big) \end{bmatrix}, \quad W = \begin{bmatrix} w_{1,1}^{[N]} & \cdots & w_{1,N+1}^{[N]} \\ \vdots & \ddots & \vdots \\ w_{N+1,1}^{[N]} & \cdots & w_{N+1,N+1}^{[N]} \end{bmatrix}$$

satisfy $UW = I$, and biorthogonality follows immediately □

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Orthogonality

Multiple discrete orthogonalities

Assume that the recursion polynomials B_{N+1} have simple zeros $\{\lambda_k^{[N]}\}_{k=1}^{N+1}$. For k such that $kp+j \leqslant N$, the following type II multiple orthogonal conditions are satisfied

$$\left\langle \mu_r^{[N]}, x^m B_{kp+j} \right\rangle = 0, \qquad m = 0, \dots, k, \qquad r = 1 \dots, j$$
$$\left\langle \mu_r^{[N]}, x^m B_{kp+j} \right\rangle = 0, \qquad m = 0, \dots, k-1, \qquad r = j+1 \dots, p$$

For the recursion polynomials of type I we have the following discrete type I multiple orthogonality

$$\left\langle A^{(1)}_{kp+j} \mu_1^{[N]} + \dots + A^{(p)}_{kp+j} \mu_2^{[N]}, x^n \right\rangle = 0$$

for $n \in \{0, 1, \dots, kp+j-1\}$, $k \in \{1, \dots, N\}$

Orthogonality

Lebesgue–Stieltjes representation of the measures

In terms of the following piecewise continuous functions

$$\psi_a^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]} \\ \mu_{1,a}^{[N]} + \cdots + \mu_{k,a}^{[N]}, & \lambda_{k+1}^{[N]} \leqslant x \leqslant \lambda_k^{[N]} \\ \mu_{1,a}^{[N]} + \cdots + \mu_{N+1,a}^{[N]} = -(\nu^{-\top})_{1,a} & x > \lambda_1^{[N]} \end{cases}$$

we can write $\mu_a^{[N]} = \mathrm{d}\psi_a^{[N]}$ for $a \in \{1, \ldots, p\}$

Positivity of Christoffel numbers

For simple orthogonality the Christoffel number was clearly positive from definition. (And from this positivity interlacing was proven)

Now, for multiple orthogonality, is not clear at all.

Can we find conditions that ensure that Christoffel numbers are positive???

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Positive bidiagonal factorization

1. We now introduce the very important idea of **positive bidiagonal factorization (PBF)**
2. This factorization is very natural for banded matrices as all the subdiagonals may be constructed in terms of simpler bidiagonal matrices

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Positive bidiagonal factorization

PBF: $T = L_1 L_2 \cdots L_p U$

$$L_k := \begin{bmatrix} 1 & 0 & \cdots & & \\ (L_k)_{1,0} & 1 & \ddots & & \\ 0 & (L_k)_{2,1} & 1 & \ddots & \\ \vdots & \ddots & (L_k)_{3,2} & 1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \qquad (L_k)_{j+1,j} = \alpha_{k+1+j(p+1)}$$

$$U := \begin{bmatrix} U_{0,0} & 1 & 0 & \cdots \\ 0 & U_{1,1} & 1 & \ddots \\ \vdots & \ddots & U_{2,2} & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix} \qquad U_{j,j} = \alpha_{1+j(p+1)}$$

with $\alpha_i > 0$, for $i \in \mathbb{N}$

Finite PBF: $T^{[N]} = L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]} U^{[N]}$

$$L_k^{[N]} = \begin{bmatrix} 1 & & 0 & \cdots & \cdots & 0 \\ \alpha_{k+1} & \ddots & & \ddots & & \vdots \\ 0 & \alpha_{k+1+(p+1)} & \ddots & & \ddots & \\ \vdots & \ddots & & \ddots & & 0 \\ 0 & \cdots & \cdots & 0 & \alpha_{k+1+(p+1)N} & 1 \end{bmatrix}$$

$$U^{[N]} = \begin{bmatrix} \alpha_1 & 1 & 0 & \cdots & 0 & \\ 0 & \alpha_{1+(p+1)} & \ddots & \ddots & \vdots & \\ \vdots & \ddots & \alpha_{1+2(p+1)} & \ddots & 0 & \\ \vdots & & \ddots & \ddots & 1 & \\ 0 & \cdots & \cdots & 0 & \alpha_{1+(p+1)N} & \end{bmatrix}$$

Oscillatory Matrices

Totally nonnegative (TN)

All its minors are non-negative

Invertible totally nonnegative (InTN)

All its minors are non-negative and is nonsingular

Totally positive (TP)

All its minors are positive

Oscillatory Matrix (IITN)

A totally non negative matrix A such that for some n, the matrix A^n is totally positive

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Oscillatory matrices

Gantmacher-Krein Criterion

A totally non negative matrix is oscillatory if and only if it is nonsingular and the elements of the first subdiagonal and superdiagonal are positive.

Oscillatory Jacobi Matrix

If and only if the elements of the first subdiagonal and superdiagonal are positive, and the leading principal minors are positive

Factorization I

From Cauchy–Binet Theorem one can deduce the invariance of these sets of matrices under the usual matrix product

The product of matrices in InTN is again InTN (similar statements hold for TN or oscillatory matrices)

Factorization II

PBF $\Rightarrow$ oscillatory

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Eigenvalues

The eigenvalues are simple and positive

Interlacing property

The eigenvalues strictly interlace the eigenvalues of the principal submatrix (deleting first row and column) (also last column and row)

Left and right eigenvectors $w^{(k)}, u^{(k)}$ to the k-th largest eigenvalue

$$U = \left[u^{(1)} \cdots\cdots u^{(n)}\right], \qquad W = \begin{bmatrix} w^{(1)} \\ \vdots \\ \vdots \\ w^{(n)} \end{bmatrix}, \qquad UW = I$$

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Oscillatory matrices

Sign-variation

Number of variations in the eigenvectors will lead us to interlacing properties of polynomials

Translations

Translations of bounded Jacobi matrices are oscillatory matrices

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Darboux transformations

Darboux transformations

Let us assume that T admits a bidiagonal factorization (not necessarily positive). For each of its truncations $T^{[N]}$ we consider a chain of new auxiliary matrices, called Darboux transformations, given by the consecutive permutation of the triangular matrices in the factorization

$$\begin{cases} \hat{T}^{[N,1]} = L_2^{[N]} \cdots L_p^{[N]} U^{[N]} L_1^{[N]} \\ \hat{T}^{[N,2]} = L_3^{[N]} \cdots L_p^{[N]} U^{[N]} L_1^{[N]} L_2^{[N]} \\ \qquad \vdots \\ \hat{T}^{[N,p-1]} = L_p^{[N]} U^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_{p-1}^{[N]} \\ \hat{T}^{[N,p]} = U^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]} \end{cases}$$

Darboux transformations

Darboux transformations are banded Hessenberg matrices with only its first p subdiagonals different from zero

PBF and Darboux transformations

Let us assume that the PBF holds. Then, for $k \in \{1, \ldots, p\}$, we find

1. The Darboux transformation $\hat{T}^{[N,k]}$ is oscillatory
2. The characteristic polynomial of the Darboux transformation $\hat{T}^{[N,k]}$ is B_{N+1}
3. If w is a left eigenvector of $T^{[N]}$, then $\hat{w} = wL_1^{[N]} \cdots L_k^{[N]}$ is a left eigenvector of $\hat{T}^{[N,k]}$

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Darboux transformations

Proof.

1. Each bidiagonal factor belongs to InTN. Then, the Darboux transformation $\hat{T}^{[N,k]}$ is a product of matrices in InTN and, consequently, belongs to InTN. Moreover, the entries in the first superdiagonal are all 1, while the entries in the first subdiagonal are sum of products of α's. Thus, all entries in these two diagonals are positive. According to Gantmacher–Krein Criterion is an oscillatory matrix.
2. As $\hat{T}^{[N,k]} = (L_1^{[N]} \cdots L_k^{[N]})^{-1} T^{[N]} L_1^{[N]} \cdots L_k^{[N]}$ its characteristic polynomial is B_{N+1}
3. We see that

$$\begin{aligned}\lambda \hat{w} &= \lambda w L_1^{[N]} \cdots L_k^{[N]} = w L_1^{[N]} \cdots L_k^{[N]} L_{k+1}^{[N]} \cdots L_p^{[N]} U^{[N]} L_1^{[N]} \cdots L_k^{[N]} \\ &= \hat{w} \hat{T}^{[N]}\end{aligned}$$

Darboux transformations

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Christoffel numbers are positive for PBF

Gathering α's

$$\mathscr{L} := \begin{bmatrix} \mathscr{L}_1 & \mathscr{L}_2 \cdots\cdots \mathscr{L}_p \end{bmatrix} \in \mathbb{R}^{p\times p}$$

with columns

$$\mathscr{L}_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathscr{L}_k :- \frac{1}{d_k} L_1^{[p-1]} \cdots L_{k-1}^{[p-1]} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \qquad k \in \{2, \dots, p\}$$

with $d_k := \alpha_k \alpha_{k+p} \alpha_{k+2p} \alpha_{k+3p} \cdots \alpha_{k+(k-2)p}$

$\mathscr{L}$ is a non-negative upper unitriangular matrix

Christoffel numbers are positive for PBF

Christoffel coefficients positivity

Let us assume that $T^{[N]}$ has a PBF and that the initial conditions is such that

$$\nu^{-\top} = \mathscr{L}\mathscr{C}$$

for some nonnegative upper unitriangular matrix $\mathscr{C}$. Then, the Christoffel numbers of the discrete measures given for for $T^{[N]}$ are positive, i.e.,

$$\mu_{n,a}^{[N]} > 0, \qquad n \in \{1, \dots, N+1\}, \qquad a \in \{1, \dots, p\}$$

Idea of the proof

For $\mu_{n,1}^{[N]} > 0$, $n \in \{1, \dots, N+1\}$, we consider the left eigenvector

$$Q^{\langle N \rangle}(\lambda_k^{[N]}) = \left[Q_{0,N}(\lambda_k^{[N]}) \quad Q_{1,N}(\lambda_k^{[N]}) \cdots\cdots Q_{N,N}(\lambda_k^{[N]}) \right]$$

that has $Q_{N,N}(\lambda_k^{[N]}) \neq 0$ (oscillatory property), we can normalize the last entry to 1

$$\omega_k^{\langle N \rangle} := \left[\frac{Q_{0,N}\left(\lambda_k^{[N]}\right)}{Q_{N,N}\left(\lambda_k^{[N]}\right)} \quad \frac{Q_{1,N}\left(\lambda_k^{[N]}\right)}{Q_{N,N}\left(\lambda_k^{[N]}\right)} \cdots\cdots 1 \right]$$

The change sign properties (oscillatory) leads to

$$\frac{Q_{0,N}(\lambda_1^{[N]})}{Q_{N,N}(\lambda_1^{[N]})} > 0, \qquad \frac{Q_{0,N}(\lambda_2^{[N]})}{Q_{N,N}(\lambda_2^{[N]})} < 0, \qquad \frac{Q_{0,N}(\lambda_3^{[N]})}{Q_{N,N}(\lambda_3^{[N]})} > 0,$$

and so on, alternating the sign

Idea of the proof

As the polynomial B_{N+1} is monic for the derivative B'_{N+1} evaluated at the zeros $\lambda_k^{[N]}$ we have

$$B'_{N+1}(\lambda_1^{[N]}) > 0, \qquad B'_{N+1}(\lambda_2^{[N]}) < 0, \qquad B'_{N+1}(\lambda_3^{[N]}) > 0$$

and so on, alternating the sign. Then, it holds that

$$\mu_{k,1}^{[N]} = \frac{Q_{0,N}\big(\lambda_k^{[N]}\big)}{Q_{N,N}\big(\lambda_k^{[N]}\big)B'_{N+1}\big(\lambda_k^{[N]}\big)} > 0, \qquad k \in \{1, \dots, N+1\}.$$

Idea of the proof

For the Christoffel coefficient $\mu_{k,2}^{[N]}$, for $k \in \{1, \dots, N+1\}$ recall that $\hat{T}^{[N,1]}$ is an oscillatory matrix with characteristic polynomial B_{N+1}. Then, a left eigenvector for the eigenvalue $\lambda_k^{[N]}$ can be chosen as

$$\begin{aligned}\omega_k^{\langle N\rangle} L_1^{[N]} &= \begin{bmatrix} \frac{Q_{0,N}(\lambda_k^{[N]})}{Q_{N,N}(\lambda_k^{[N]})} & \frac{Q_{1,N}(\lambda_k^{[N]})}{Q_{N,N}(\lambda_k^{[N]})} & \cdots \cdots 1 \end{bmatrix} L_1^{[N]} \\ &= \begin{bmatrix} \frac{(Q_{0,N}+\alpha_2 Q_{1,N})(\lambda_k^{[N]})}{Q_{N,N}(\lambda_k^{[N]})} & \cdots \cdots 1 \end{bmatrix}\end{aligned}$$

Using the sign properties of the left eigenvector (oscillatory)

$$\frac{(Q_{0,N}+\alpha_2 Q_{1,N})(\lambda_1^{[N]})}{Q_{N,N}(\lambda_1^{[N]})} > 0, \quad \frac{(Q_{0,N}+\alpha_2 Q_{1,N})(\lambda_2^{[N]})}{Q_{N,N}(\lambda_2^{[N]})} < 0, \quad \frac{(Q_{0,N}+\alpha_2 Q_{1,N})(\lambda_3^{[N]})}{Q_{N,N}(\lambda_3^{[N]})} > 0$$

and so on, alternating the sign. As before, using the interlacing properties of the polynomial B'_{N+1}:

$$\tilde{\mu}_{k,2}^{[N]} := \frac{(\frac{1}{\alpha_2} Q_{0,N} + Q_{1,N})(\lambda_k^{[N]})}{Q_{N,N}(\lambda_k^{[N]}) B'_{N+1}(\lambda_k^{[N]})} > 0, \qquad k \in \{1, \dots, N+1\}$$

Idea of the proof

We repeat this argument for each $a \in \{1, \ldots, p\}$ getting some positive numbers $\tilde{\mu}_{k,a}^{[N]} > 0$, for example

$$\tilde{\mu}_{k,3}^{[N]} := \frac{\left(\frac{1}{\alpha_3 \alpha_{3+p}} Q_{0,N} + \frac{\alpha_2 + \alpha_3}{\alpha_3 \alpha_{3+p}} Q_{1,N} + Q_{2,N}\right)\left(\lambda_k^{[N]}\right)}{Q_{N,N}\left(\lambda_k^{[N]}\right) B'_{N+1}\left(\lambda_k^{[N]}\right)} > 0$$

In general, these positive numbers are

$$\tilde{\mu}_{k,1}^{[N]} := \begin{bmatrix} w_{k,1}^{\langle N \rangle} & \cdots\cdots & w_{k,p}^{\langle N \rangle} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \qquad \tilde{\mu}_{k,j}^{[N]} := \begin{bmatrix} w_{k,1}^{\langle N \rangle} & \cdots\cdots & w_{k,p}^{\langle N \rangle} \end{bmatrix} \mathscr{L}_j$$

Idea of the proof

Entrywise positive row vector

$$\begin{bmatrix}\tilde{\mu}_{k,1}^{[N]} & \tilde{\mu}_{k,2}^{[N]} & \cdots\cdots & \tilde{\mu}_{k,p}^{[N]}\end{bmatrix} = \begin{bmatrix}w_{k,1}^{\langle N\rangle} & \cdots & w_{k,p}^{\langle N\rangle}\end{bmatrix}\mathscr{L}$$

Then, from

$$\begin{bmatrix}\mu_{k,1}^{[N]} & \mu_{k,2}^{[N]} & \cdots\cdots & \mu_{k,p}^{[N]}\end{bmatrix} = \begin{bmatrix}w_{k,1}^{\langle N\rangle} & w_{k,2}^{\langle N\rangle} & \cdots\cdots & w_{k,p}^{\langle N\rangle}\end{bmatrix}\nu^{-\top}$$

we get the result

$$\begin{aligned}\begin{bmatrix}\mu_{k,1}^{[N]} & \mu_{k,2}^{[N]} & \cdots\cdots & \mu_{k,p}^{[N]}\end{bmatrix} &= \begin{bmatrix}w_{k,1}^{\langle N\rangle} & w_{k,2}^{\langle N\rangle} & \cdots\cdots & w_{k,p}^{\langle N\rangle}\end{bmatrix}\mathscr{L}\mathscr{C} \\ &= \begin{bmatrix}\tilde{\mu}_{k,1}^{[N]} & \tilde{\mu}_{k,2}^{[N]} & \cdots\cdots & \tilde{\mu}_{k,p}^{[N]}\end{bmatrix}\mathscr{C}\end{aligned}$$

which is a entrywise positive row vector

Second kind polynomials

We consider

1.

$$T^{[N,k]} := \begin{bmatrix} a_{k,k} & 1 & 0 & \cdots & \cdots & 0 \\ a_{k+1,k} & a_{k+1,k+1} & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ a_{p,k} & & & \ddots & \ddots & 0 \\ 0 & \ddots & & & \ddots & 1 \\ \vdots & & \ddots & & & \\ 0 & \cdots & 0 & a_{N,N-p} & \cdots & a_{N,N} \end{bmatrix}$$

2. Truncated polynomials

$$B^{[k]}_{N+1}(x) :- \det\left(xI_{N+1} - T^{[N,k]}\right)$$

Second kind polynomials

$(p+2)$ terms linear homogeneous recurrence

For $k \in \{0, 1, \dots, N\}$

$$a_{p,k} B_{N+1}^{[k+p+1]} + \cdots + a_{k+1,k} B_{N+1}^{[k+2]} + a_{k,k} B_{N+1}^{[k+1]} + B_{N+1}^{[k]} = x B_{N+1}^{[k+1]}$$

Normalized left eigenvectors

$$\omega_n^{\langle N \rangle} := \begin{bmatrix} B_{N+1}^{[1]} & \cdots & \cdots & B_{N+1}^{[N+1]} \end{bmatrix}\Big|_{x=\lambda_n^{[N]}}, \qquad n \in \{1, \dots, N+1\},$$

are the left eigenvectors of $T^{[N]}$ with last entry normalized to 1

Second kind polynomials

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are the left eigenvectors of $T^{[N]}$ with last entry normalized to 1

Christoffel–Darboux

Christoffel–Darboux type formulas for truncated polynomials

1. Christoffel–Darboux type relation holds

$$\sum_{n=0}^{N} B_{N+1}^{[n+1]}(x) B_n(y) = \frac{B_{N+1}(x) - B_{N+1}(y)}{x - y}$$

2. Confluent Christoffel–Darboux type formula is satisfied

$$\sum_{n=0}^{N} B_{N+1}^{[n+1]} B_n = B'_{N+1}$$

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Second kind polynomials

For $k \in \{0, 1, \dots, N\}$

$$\frac{Q_{n,N}(\lambda_k^{[N]})}{Q_{N,N}(\lambda_k^{[N]})} = B_{N+1}^{[n+1]}(\lambda_k^{[N]}), \qquad w_{k,n}^{\langle N \rangle} = \frac{B_{N+1}^{[n]}(\lambda_k^{[N]})}{B'_{N+1}(\lambda_k^{[N]})}$$

Second kind polynomials

The second kind polynomials $B_{N+1}^{(k)}$, $k \in \{1, \dots, p\}$, are the entries of the following row vector

$$\begin{bmatrix} B_{N+1}^{(1)} & \cdots & \cdots & B_{N+1}^{(p)} \end{bmatrix} = \begin{bmatrix} B_{N+1}^{[1]} & \cdots & \cdots & B_{N+1}^{[p]} \end{bmatrix} \nu^{-\top}$$

Second kind polynomials

If $\{e_1, \dots, e_{N+1}\}$ is the canonical basis of $\mathbb{R}^{N+1}$ we have the modified basis $e_k^\nu := \nu^{-\top} e_k$. For example, $e_1^\nu = e_1$, $e_2^\nu = e_2 - \nu_1^{(1)} e_1$ and $e_3^\nu = e_3 - \nu_2^{(2)} e_2 + \left(\nu_1^{(1)} \nu_2^{(2)} - \nu_2^{(1)}\right) e_1$

Second kind polynomials and adjugate matrix

The second kind polynomials are given as

$$B_{N+1}^{(k)}(x) = e_1^\top \operatorname{adj}(xI_{N+1} - T^{[N]}) e_k^\nu$$

Resolvent and Weyl functions

Resolvent matrix $R_z^{[N]}$ of the leading principal submatrix $T^{[N]}$

$$\begin{aligned} R_z^{[N]} &:= \left(zI_{N+1} - T^{[N]}\right)^{-1} = \frac{\operatorname{adj}\left(zI_{N+1} - T^{[N]}\right)}{\det(zI_{N+1} - T^{[N]})} \\ &= U(zI_{N+1} - D)^{-1} W \end{aligned}$$

Weyl's functions

$$S_a^{[N]}(z) := e_1^\top \left(zI_{N+1} - T^{[N]}\right)^{-1} e_a^\nu = \frac{B_{N+1}^{(a)}(z)}{B_{N+1}(z)} = \sum_{n=1}^{N+1} \frac{\mu_{n,a}^{[N]}}{z - \lambda_n^{[N]}}$$

The Christoffel coefficients are residues at the simple poles of the Weyl functions

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Resolvent matrix $R_z^{[N]}$ of the leading principal submatrix $T^{[N]}$

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Spectral Favard Theorem and Multiple Orthogonal Polynomials

Favard spectral theorem

At this point we are ready to give one of the main results of the talk, that establish the existence of multiple orthogonal polynomials and corresponding positive Lebesgue–Stieltjes measures for a given bounded banded Hessenberg matrix that admit a positive bidiagonal factorization. The result is based in the positivity of the Christoffel coefficients

Favard spectral theorem

Favard spectral representation

Let us assume that

1. The banded Hessenberg matrix T is bounded and admit a PBF
2. The sequences $\{A_n^{(1)}, \dots, A_n^{(p)}\}_{n=0}^{\infty}$ of recursion polynomials of type I, are determined by the initial condition matrix ν such that $\nu^{-\top} = \mathscr{L}\mathscr{C}$

Then, there exists p non decreasing functions $\{\psi_k\}_{k=1}^{p}$, and corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\psi_k$ with compact support Δ such that the following biorthogonality holds

$$\int_{\Delta} \big(A_k^{(1)}(x)\mathrm{d}\psi_1(x) + \cdots + A_k^{(p)}(x)\mathrm{d}\psi_p(x)\big) B_l(x) = \delta_{k,l}, \quad k, l \in \mathbb{N}_0$$

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Favard spectral theorem

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Favard spectral theorem

Proof.

The sequences $\{\psi_k^{[N]}\}_{N=0}^{\infty}$, $k \in \{1, \dots, p\}$ are uniformly bounded and nondecreasing. Consequently, following Helly's results, there exist subsequences that converge when $N \to \infty$ to nondecreasing functions $\psi_1, \dots, \psi_p$. Being T bounded its eigenvalues lay in a bounded set Δ, and we deduce that these measures have compact support. □

Favard spectral theorem

Multiple orthogonal relations

Multiple orthogonal relations of type II

$$\begin{aligned} &\int_\Delta x^m B_{kp+j} \mathrm{d}\psi_r(x) = 0, && m = 0, \dots, k, && r = 1, \dots, j \\ &\int_\Delta x^m B_{kp+j} \mathrm{d}\psi_r(x) = 0, && m = 0, \dots, k-1, && r = j+1, \dots, p \end{aligned}$$

and of type I, for $n \in \{0, 1, \dots, kp+j-1\}$,

$$\int_\Delta \big(A^{(1)}_{kp+j}(x)\mathrm{d}\psi_1(x) + \cdots + A^{(p)}_{kp+j}(x)\mathrm{d}\psi_p(x)\big) x^n = 0$$

Favard spectral theorem

Spectral representation of moments and Stieltjes–Markov functions

Helly's second theorem leads to the spectral representation for the moments and Stieltjes–Markov functions $\hat{\psi}_k$ of the full banded Hessenberg matrix in terms of the spectral functions $\psi_1, \dots, \psi_p$:

$$e_1^\top T^k e_k^\nu = \int_\Delta t^k \mathrm{d}\psi_k(t), \qquad \hat{\psi}_k := e_1^\top (zI - T)^{-1} e_1^\nu = \int_\Delta \frac{\mathrm{d}\psi_k(t)}{z-t}$$

For the Weyl functions we have in $\bar{\mathbb{C}} \setminus \Delta$ uniform convergence to the Stieltjes–Markov functions

$$S_k^{[N]} \rightrightarrows \hat{\psi}_k, \qquad N \to \infty$$

Markov chains beyond birth and death

Banded stochastic matrices

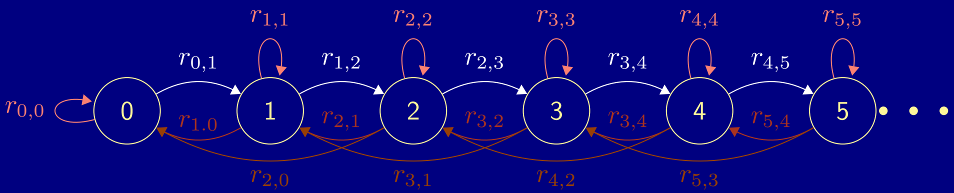
Stochastic matrices: all the entries in each row are nonnegative and satisfy that its sum is 1

Banded stochastic matrices

$$
P_{II} = \begin{bmatrix}
r_{0,0} & r_{0,1} & 0 & \cdots & \cdots & \cdots & \cdots \\
r_{1,0} & r_{1,1} & r_{1,2} & \ddots & & & \\
\vdots & & \ddots & \ddots & \ddots & & \\
r_{p,0} & \cdots & \cdots & r_{p,p} & r_{p,p+1} & \ddots & \\
0 & r_{p+1,1} & \cdots & \cdots & r_{p+1,p+1} & r_{p+1,p+2} & \ddots \\
\vdots & \ddots & \ddots & & & \ddots & \ddots
\end{bmatrix}
$$

Banded stochastic matrices

$p = 2$

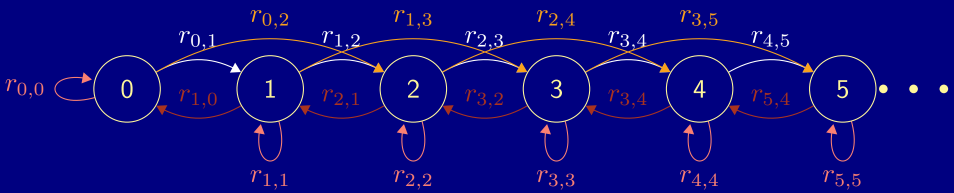


Banded stochastic matrices

$$P_I = \begin{bmatrix}
r_{0,0} & r_{0,1} & \cdots & r_{0,p} & 0 & \cdots & \\
r_{1,0} & r_{1,1} & & \vdots & r_{1,p+1} & \ddots & \\
0 & r_{2,1} & \ddots & \vdots & \vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & r_{p,p} & \vdots & & \ddots \\
\vdots & & \ddots & r_{p+1,p} & r_{p+1,p+1} & \ddots & \\
\vdots & & & \ddots & r_{p+1,p+2} & \ddots & \ddots \\
\vdots & & & & \ddots & & \ddots
\end{bmatrix}$$

Banded stochastic matrices

$p = 2$



Banded stochastic matrices

These stochastic matrices of Hessenberg type are connected to the monic banded Hessenberg matrix T assuming that $r_{n,n+1} > 0$, we have

$$P_{II} = H_{II} T H_{II}^{-1}$$

$$H_{II} = \operatorname{diag}(1, H_{II,1}, H_{II,2}, \dots), \qquad H_{II,n} = \frac{1}{r_{0,1} r_{1,2} \cdots r_{n,n-1}}$$

Assuming that $r_{n+1,n} > 0$

$$P_I = H_I^{-1} T^{\top} H_I$$

$$H_I = \operatorname{diag}(1, H_{I,1}, H_{I,2}, \dots), \qquad H_{I,n} = \frac{1}{r_{1,0} r_{2,1} \cdots r_{n-1,n}}$$

Markov chains can be described by the spectral methods we have constructed for monic Hessenberg semi-infinite matrices with positive bidiagonal factorization

Banded stochastic matrices

Positive stochastic bidiagonal factorization (PSBF)

$P_{II} = \Pi_1 \cdots \Pi_p \Upsilon$ with stochastic bidiagonal factors ($a \in \{1, \dots, p\}$)

$$\Pi_a := \begin{bmatrix} (\Pi_a)_{0,0} & 0 & \cdots & & \\ (\Pi_a)_{1,0} & (\Pi_k)_{1,1} & \ddots & & \\ 0 & (\Pi_a)_{2,1} & (\Pi_a)_{2,2} & \ddots & \\ \vdots & \ddots & (\Pi_a)_{3,2} & (\Pi_a)_{3,3} & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\Upsilon := \begin{bmatrix} \Upsilon_{0,0} & \Upsilon_{0,1} & 0 & \cdots \\ 0 & \Upsilon_{1,1} & \Upsilon_{1,2} & \ddots \\ \vdots & \ddots & \Upsilon_{2,2} & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

Finite banded stochastic matrices

A finite matrix with a positive stochastic bidiagonal factorization is oscillatory

PSBF vs PBF

Let us assume a banded stochastic matrix, P_{II}. Then, P_{II} has a positive stochastic bidiagonal factorization if and only if it is similar, via a positive diagonal matrix, to a monic Hessenberg matrix T with positive bidiagonal factorization

$$P_{II} = \Pi_1 \cdots \Pi_p \Upsilon \qquad \Longleftrightarrow \qquad T = L_1 \cdots L_p U$$

with $L_1, \ldots, L_p$ positive lower bidiagonal matrices and U an upper positive bidiagonal matrix

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with $L_1, \dots, L_p$ positive lower bidiagonal matrices and U an upper positive bidiagonal matrix

Karlin-McGregor

$$\Theta_{II,k,l} := \frac{H_{II,k}}{H_{II,l}} = \begin{cases} \dfrac{1}{r_{l,l+1}\cdots r_{k-1,k}}, & l < k \\ 1, & l = k \\ r_{k,k+1}\cdots r_{l-1,l}, & l > k \end{cases}$$

$$\Theta_{I,l,k} := \frac{H_{I,l}}{H_{I,k}} = \begin{cases} r_{l,l+1}\cdots r_{k,k+1}, & l < k \\ 1, & l = k, \\ \dfrac{1}{r_{k+1,k}\cdots r_{l,l-1}}, & l > k \end{cases}$$

Karlin-McGregor

Let us consider a Markov chain with transition matrix a PSBF $(p+2)$-diagonal matrix such that admits a positive stochastic bidiagonal factorization. Then, there is sequence of multiple orthogonal polynomials of type II, $\{B_n\}_{n=0}^{\infty}$, and of type I, $\{A_n^{(1)},\dots,A_n^{(p)}\}_{n=0}^{\infty}$, associated with positive Lebesgue–Stieltjes measures $\mathrm{d}\psi_1,\dots,\mathrm{d}\psi_p$ such that:

Karlin–McGregor spectral representation formula

The iterated probabilities have the following spectral representation

$$\big((P_{II})^n\big)_{k,l} = \Theta_{II,k,l}\int_0^1 (A_l^{(1)}\mathrm{d}\psi_1(x)+\cdots+A_l^{(p)}\mathrm{d}\psi_p(x))x^nB_k(x)$$

$$\big((P_{I})^n\big)_{k,l} = \Theta_{I,l,k}\int_0^1 (A_k^{(1)}\mathrm{d}\psi_1(x)+\cdots+A_k^{(p)}\mathrm{d}\psi_p(x))x^nB_l(x)$$

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Spectral representation of generating functions

Spectral representation of generating functions. For $|s| < 1$, the corresponding transition probability generating functions are

$$(P_{II}(s))_{k,l} = \Theta_{II,k,l} \int_0^1 \left(A_l^{(1)} \mathrm{d}\psi_1(x) + \cdots + A_l^{(p)} \mathrm{d}\psi_p(x)\right) \frac{B_k(x)}{1 - sx}$$

$$(P_I(s))_{k,l} = \Theta_{I,l,k} \int_0^1 \left(A_k^{(1)} \mathrm{d}\psi_1(x) + \cdots + A_k^{(p)} \mathrm{d}\psi_p(x)\right) \frac{B_l(x)}{1 - sx}$$

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Spectral representation of generating functions

For $k \neq l$, the first passage generating functions are

$$(F_{II}(s))_{k,l} = \Theta_{II,k,l} \frac{\int_0^1 \left(A_l^{(1)} \mathrm{d}\psi_1(x) + \cdots + A_l^{(p)} \mathrm{d}\psi_p(x)\right) \frac{B_k(x)}{1-sx}}{\int_0^1 \left(A_l^{(1)} \mathrm{d}\psi_1(x) + \cdots + A_l^{(p)} \mathrm{d}\psi_p(x)\right) \frac{B_l(x)}{1-sx}}$$

$$(F_I(s))_{k,l} = \Theta_{I,l,k} \frac{\int_0^1 \left(A_k^{(1)} \mathrm{d}\psi_1(x) + \cdots + A_k^{(p)} \mathrm{d}\psi_p(x)\right) \frac{B_l(x)}{1-sx}}{\int_0^1 \left(A_l^{(1)} \mathrm{d}\psi_1(x) + \cdots + A_l^{(p)} \mathrm{d}\psi_p(x)\right) \frac{B_l(x)}{1-sx}}$$

For $k = l$ the first passage generating functions are the same for type I and II, namely

$$F_{ll}^{[N]}(s) = 1 - \frac{1}{\int_0^1 \left(A_l^{(1)} \mathrm{d}\psi_1(x) + \cdots + A_l^{(p)} \mathrm{d}\psi_p(x)\right) \frac{B_l(x)}{1-sx}}$$

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Recurrent vs transient

The Markov chain is recurrent if and only if the integral

$$\int_0^1 \frac{\mathrm{d}\psi_1(x)}{1-x}$$

diverges. Otherwise is transient

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Ergodic Markov chains

The Markov chain is ergodic (or positive recurrent) if and only 1 is a mass point of $\mathrm{d}\psi_1, \mathrm{d}\psi_2, \ldots, \mathrm{d}\psi_p$ with masses $m_1 > 0$ and $m_2, \ldots, m_p \geqslant 0$, respectively. In that case, the corresponding stationary distribution is

$$\pi = \begin{bmatrix} \pi_1 & \pi_2 \cdots \cdots \end{bmatrix}, \qquad \pi_{n+1} = (A_n^{(1)}(1) m_1 + \cdots + A_n^{(p)}(1) m_p) B_n(1)$$