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**Sobolev orthogonal polynomials and boundary value problems
for Schrödinger equations with polynomial potentials**

**NonLinear Phenomena in Mathematical Physics and
Applications**

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The role of standard orthogonal polynomials

Let μ be a positive Borel measure supported in a subset E of the real line. Determine a polynomial Q_{n-1} of degree $\leq n - 1$ that minimizes

$$\int_E |x^n - \pi_{n-1}(x)|^2 d\mu(x).$$

among all the polynomials π_{n-1} of degree $\leq n - 1$. This is equivalent to find the n -th monic polynomial orthogonal $P_n(x) = x^n - Q_{n-1}(x)$ with respect to the inner product

$$\langle f, g \rangle = \int_E f(x)g(x)d\mu(x). \quad (1)$$

The role of standard orthogonal polynomials

These polynomials satisfy a three term recurrence relation (TTRR)

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x), n \geq 0, \quad (2)$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$.

The connection with Jacobi matrices and discretization of Schrödinger operators

$$(Ju)_n = a_{n+1}u_{n+1} + b_nu_n + a_nu_{n-1}, n \geq 0, \quad (3)$$

with real entries, $a_n > 0$, for $n \geq 1$, and $a_0 = 0$.

The role of standard orthogonal polynomials

Inverse problem (Favard's Theorem):

Given a sequence of monic polynomials satisfying a TTRR

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x), n \geq 0, \quad (4)$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, they are orthogonal with respect to a positive Borel measure (not unique in general).

Jacobi polynomial sequence $\{P_n^{(\alpha, \beta)}\}_{n \geq 0}$ is orthogonal with respect to the measure $d\mu(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, supported on $[-1, 1]$. But notice that you can consider orthogonality with respect to linear functionals (quasi-definite case) when $\alpha \neq -N, \beta \neq -N, \alpha + \beta \neq -N - 1$, where N is a positive integer number.

On the other hand, let remind that $P_n^{(\alpha, \beta)}$ is, up to a constant factor, the hypergeometric function ${}_2F_1(-n, \alpha + \beta + n + 1; \alpha + 1 | \frac{1-x}{2})$.

The role of standard orthogonal polynomials

The three term relation for Legendre polynomials $\{P_n^{(0,0)}\}_{n \geq 0}$ reads as

$$x P_n(x) = P_{n+1} + \gamma_n P_{n-1}(x), \quad n \geq 0, \quad (5)$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$, and

$$\gamma_n = \frac{n^2}{4n^2 - 1} = \frac{1}{4} + \frac{1}{8} \left(\frac{1}{2n - 1} - \frac{1}{2n + 1} \right).$$

Iterating the above recurrence relation we get

$$x^2 P_n(x) = P_{n+2} + (\gamma_{n+1} + \gamma_n) P_n(x) + \gamma_n \gamma_{n-1} P_{n-2}(x), \quad n \geq 1. \quad (6)$$

The role of standard orthogonal polynomials

Zeros of orthogonal polynomials are real, simple and they are located in the interior of the convex hull of the support E of the measure $d\mu$.

Let us denote $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ the zeros of $P_n(x)$.

Theorem

The zeros of P_n are the eigenvalues of the leading principal submatrix of size $n \times n$ of the Jacobi matrix.

Theorem

(Interlacing property).

Between two consecutive zeros of P_{n+1} there is exactly one zero of P_n .

Zeros of orthogonal polynomials as nodes in Gaussian quadrature and positivity of the Christoffel-Cotes numbers as a consequence of the interlacing property.

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An inner product is said to be a Sobolev inner product if

$$\langle f, g \rangle_S := \int_{E_0} f(x)g(x)d\mu_0 + \sum_{k=1}^m \int_{E_k} f^{(k)}(x)g^{(k)}(x)d\mu_k,$$

where $(d\mu_0, \dots, d\mu_m)$ is a vector of positive Borel measures and $E_k = \text{supp } d\mu_k$, $k = 0, 1, \dots, m$.

Using Gram-Schmidt method for the canonical basis $(x^n)_{n \geq 0}$ you get a sequence of monic orthogonal polynomials. Thus, the n -th orthogonal polynomial is an extremal polynomial in terms of the Sobolev norm among all monic polynomials of degree exactly n .

Taking into account $\langle xf, g \rangle_S \neq \langle f, xg \rangle_S$ these polynomials do not satisfy a TTRR. Thus, a basic property of standard orthogonal polynomials is lost. A natural question is to compare analytic properties of these polynomials and the standard ones.

A natural framework is the implementation of spectral methods for Boundary Value Problems for elliptic differential operators. Instead of the use of standard orthogonal polynomials when you deal with the Galerkin approximation in the variational problem it seems to be natural to use Sobolev orthogonal polynomials taking into account the matrix problem reduces to a diagonal problem. Unfortunately, there is not a general theory about these families of orthogonal polynomials like in the standard case.

A second interesting problem is related to quadrature formulas involving derivatives. Again, very few results are known about the zeros of Sobolev orthogonal polynomials as nodes in such quadrature rules.

Third, the analysis of Fourier expansions in terms of Sobolev orthogonal polynomials seems to be more accurate than the standard ones in order to increase the speed of convergence as well as to analyze the behavior of the approximation at the end points of the support of the measures, assuming it is a bounded interval.

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The birth time of Sobolev OP

In 1947 D. C. Lewis stated the following problem in the framework of polynomial least square approximations: Let w_0, \dots, w_p be monotonic, non-decreasing functions defined on $[a, b]$ and let f be a function on $[a, b]$ that satisfies certain regularity conditions. Determine a polynomial P_n of degree $\leq n$ that minimizes

$$\sum_{k=0}^p \int_a^b |f^{(k)}(x) - P_n^{(k)}(x)|^2 dw_k(x).$$

Lewis did not use Sobolev orthogonal polynomials and gave a formula for the remainder term of the approximation as an integral of the Peano kernel.

Notice that the polynomial P_n is the n -th partial Fourier sum in terms of orthogonal polynomials with respect to the Sobolev inner product associated with the vector of weights w_0, \dots, w_p .

On the other hand, if you deal with the above problem when $f(x) = x^{n+1}$, then the monic polynomial $S_{n+1}(x) = x^{n+1} - P_n(x)$ has a minimum Sobolev norm i.e. is a extremal polynomial and, as a consequence, is orthogonal to every polynomial of degree at most n with respect to the above inner product.

The birth time of Sobolev OP

The first paper on Sobolev orthogonal polynomials was published by Althammer in 1962, who attributed his motivation to Lewis's paper. These Sobolev OP are orthogonal with respect to the inner product

$$\langle f, g \rangle_S = \int_{-1}^1 f(x)g(x)dx + \lambda \int_{-1}^1 f'(x)g'(x)dx, \quad \lambda > 0. \quad (7)$$

The birth time of Sobolev OP

Let $S_n(\cdot; \lambda)$ denote the orthogonal polynomial of degree n with respect to the inner product $\langle \cdot, \cdot \rangle_S$, normalized so that $S_n(1; \lambda) = 1$, and let P_n denote the n -th Legendre polynomial. The following properties hold for $S_n(\cdot; \lambda)$:

- ① $\{S_n(\cdot; \lambda)\}_{n \geq 0}$ satisfies a differential equation

$$\lambda S_n''(x; \lambda) - S_n(x; \lambda) = A_n P_{n+1}'(x) + B_n P_{n-1}'(x),$$

where A_n and B_n are constants that can be given by explicit formulas.

- ② $\{S_n(\cdot; \lambda)\}_{n \geq 0}$ satisfies a recursive relation

$$S_n(x; \lambda) - S_{n-2}(x; \lambda) = a_n(P_n(x) - P_{n-2}(x)), \quad n = 1, 2, \dots$$

- ③ $S_n(\cdot; \lambda)$ has n real simple zeros in $(-1, 1)$,

The Sobolev-Legendre polynomials were also studied by Gröbner (1967), who established a version of the Rodrigues formula which states that, up to a constant factor c_n ,

$$S_n(x; \lambda) = c_n \frac{\partial^n}{1 - \lambda \partial^2} ((x^2 - x)^n - \alpha_n (x^2 - x)^{n-1})$$

where α_n are real numbers explicitly given in terms of λ and n .

Althammer also gave an example in which he replaced dx in the second integral in $\langle \cdot, \cdot \rangle_S$ by $w(x)dx$ with $w(x) = 10$ for $-1 \leq x < 0$ and $w(x) = 1$ for $0 \leq x \leq 1$, and made the observation that $S_2(x; \lambda)$ for this new inner product has one real zero outside of $(-1, 1)$.

Brenner considered the inner product

$$\langle f, g \rangle := \int_0^\infty f(x)g(x)e^{-x} dx + \lambda \int_0^\infty f'(x)g'(x)e^{-x} dx, \quad \lambda > 0,$$

and obtained results similar to those of Althammer.

Sobolev orthogonal polynomials: general aspects

Essentially, there are three types of **Sobolev inner products (SIP)** on the real line:

- **Continuous Sobolev inner products.** All the measures μ_i with $k \in \{0, 1, \dots, m\}$ have continuous support.
- **Sobolev-type inner products.** μ_0 has continuous support and $\mu_i, i = 1, \dots, m,$ are supported on finite subsets.
- **Discrete-Continuous Sobolev inner products.** μ_m has continuous support and $\mu_i, i = 0, \dots, m - 1,$ are supported on finite subsets.

The methods to deal with these three kind of SIP are quite different.

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The study of Sobolev orthogonal polynomials unexpectedly became largely dormant for nearly two decades, from which it reemerged only when a new ingredient, *coherent pairs*, was introduced by Iserles, Koch, Nørsett and Sanz-Serna in 1991 in the framework of Fourier expansions in terms of Legendre-Sobolev orthogonal polynomials.

The comparison with standard Legendre-Fourier expansions reveals the interest of these new expansions despite the fact you have not explicit expressions neither for the polynomials nor the corresponding kernels.

Coherent pairs of measures and Sobolev OP

The coherent pair introduced in 1991 is defined for the inner product

$$\langle f, g \rangle_\lambda = \int_a^b f(x)g(x)d\mu_0(x) + \lambda \int_a^b f'(x)g'(x)d\mu_1(x), \quad (8)$$

where $-\infty \leq a < b \leq \infty$, μ_0 and μ_1 are positive Borel measures on the real line with finite moments of all orders. Let $P_n(\cdot; d\mu_i)$ denote the monic orthogonal polynomial of degree n with respect to $d\mu_i$, $i=0,1$.

Definition

The pair $\{d\mu_0, d\mu_1\}$ is called coherent if there exists a sequence of nonzero real numbers $\{a_n\}_{n \geq 1}$ such that

$$P_n(\cdot; d\mu_1) = \frac{P'_{n+1}(\cdot; d\mu_0)}{n+1} + a_n \frac{P'_n(\cdot; d\mu_0)}{n}, \quad n \geq 1. \quad (9)$$

Definition (cont.)

If $[a, b] = [-c, c]$ and $d\mu_0$ and $d\mu_1$ are both even, then $\{d\mu_0, d\mu_1\}$ is called a symmetrically coherent pair if

$$P_n(\cdot; d\mu_1) = \frac{P'_{n+1}(\cdot; d\mu_0)}{n+1} + a_n \frac{P'_{n-1}(\cdot; d\mu_0)}{n-1}, \quad n \geq 2. \quad (10)$$

In the case of $d\mu_1 = d\mu_0$, we call $d\mu_0$ self-coherent.

Theorem

If $\{d\mu_0, d\mu_1\}$ is a coherent pair, then

$$S_n(x; \lambda) + b_{n-1}(\lambda)S_{n-1}(x; \lambda) = P_n(x; d\mu_0) + \hat{a}_{n-1}P_{n-1}(x; d\mu_0), \quad (11)$$

where $\hat{a}_{n-1} = na_n/(n-1)$ and $b_{n-1}(\lambda) = \hat{a}_{n-1} \|P_{n-1}(\cdot; d\mu_0)\|_{d\mu_0}^2 / \|S_{n-1}(\cdot; \lambda)\|_{\lambda}^2$.

Theorem

If $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a coherent pair, then at least one of them has to be classical and the other one is a rational perturbation of it.

In 1997 H. G. Meijer proved that in the Jacobi case

Type I

- 1 $d\mu_0(x) = |x - \xi|(1 - x)^{\alpha-1}(1 + x)^{\beta-1}dx$ and $d\mu_1(x) = (1 - x)^\alpha(1 + x)^\beta dx$, where if $|\xi| > 1$ then $\alpha > 0$ and $\beta > 0$, if $\xi = 1$ then $\alpha > -1$ and $\beta > 0$, and if $\xi = -1$ then $\alpha > 0$ and $\beta > -1$.
- 2 $d\mu_0(x) = (1 + x)^{\beta-1}dx + M\delta_1$ and $d\mu_1(x) = (1 + x)^\beta dx$, where $\beta > 0$ and $M \geq 0$.
- 3 $d\mu_0(x) = (1 - x)^{\alpha-1}dx + M\delta_{-1}$ and $d\mu_1(x) = (1 - x)^\alpha dx$, where $\alpha > 0$ and $M \geq 0$.

Type II

- 1 $d\mu_0(x) = (1 - x)^\alpha(1 + x)^\beta dx$ and $d\mu_1(x) = \frac{1}{|x - \xi|}(1 - x)^{\alpha+1}(1 + x)^{\beta+1}dx + M\delta_\xi$, where $|\xi| > 1$, $\alpha > -1$ and $\beta > -1$, and $M \geq 0$.

Symmetrically coherent pairs

The similar analysis was also carried out for symmetrically coherent pairs. It leads to the following list of symmetrically coherent pairs in the Gegenbauer case.

- 1 $d\mu_0(x) = (1 - x^2)^{\alpha-1}dx$ and $d\mu_1(x) = \frac{1}{x^2 + \xi^2}(1 - x^2)^\alpha dx$, where $\xi \neq 0$ and $\alpha > 0$.
- 2 $d\mu_0(x) = (1 - x^2)^{\alpha-1}dx$ and $d\mu_1(x) = \frac{1}{\xi^2 - x^2}(1 - x^2)^\alpha dx + M\delta_\xi + M\delta_{-\xi}$, where $|\xi| \geq 1$, $\alpha > 0$ and $M \geq 0$.
- 3 $d\mu_0(x) = (x^2 + \xi^2)(1 - x^2)^{\alpha-1}dx$ and $d\mu_1(x) = (1 - x^2)^\alpha dx$, where $\alpha > 0$.
- 4 $d\mu_0(x) = (\xi^2 - x^2)(1 - x^2)^{\alpha-1}dx$ and $d\mu_1(x) = (1 - x^2)^\alpha dx$, where $|\xi| \geq 1$ and $\alpha > 0$.
- 5 $d\mu_0(x) = dx + M\delta_1 + M\delta_{-1}$ and $d\mu_1(x) = dx$, where $M \geq 0$.

Fourier series and coherent pairs

For a coherent pair (μ_0, μ_1) of measures of Jacobi type, let introduce the space $W_{0,1}^p = \{f : [-1, 1] \rightarrow \mathbb{R}, \|f\|_{0,1}^p = \|f\|_{\mu_0}^p + \lambda \|f'\|_{\mu_1}^p < \infty\}$.

Let consider the n th Fourier partial sum $S_n f$ of a function $f \in W_{0,1}^p$ in terms of the sequence of Sobolev orthonormal polynomials associated with the Jacobi coherent pair.

We are interested to analyze the convergence of the Fourier partial sums with the above Sobolev norm.

Theorem (Ciaurri, Mínguez 2021)

Let $\alpha > -1, \beta > 0$ and $1 < p < \infty$.

- If (μ_0, μ_1) is a Jacobi coherent pair of type I, then $\|S_n f\|_{W_{0,1}^p} \leq C \|f\|_{W_{0,1}^p}$ with a constant C independent of n and f if and only if the following conditions hold

$$|(\alpha + 1)\left(\frac{1}{p} - \frac{1}{2}\right)| < \frac{1}{4}, |(\beta + 1)\left(\frac{1}{p} - \frac{1}{2}\right)| < \frac{1}{4}. \quad (12)$$

- If (μ_0, μ_1) is a Jacobi coherent pair of type II, with $M = 0$, then $\|S_n f\|_{W_{0,1}^p} \leq C \|f\|_{W_{0,1}^p}$ with a constant C independent of n and f if and only if the following conditions hold

$$|(\alpha + 2)\left(\frac{1}{p} - \frac{1}{2}\right)| < \frac{1}{4}, |(\beta + 1)\left(\frac{1}{p} - \frac{1}{2}\right)| < \frac{1}{4}. \quad (13)$$

If the Sobolev space is complete and the polynomials constitute a dense subset then the uniform boundedness of S_n would be equivalent to the convergence in $W_{0,1}^p$. Indeed,

Theorem (Ciaurri, Mínguez 2021)

Let $\alpha > -1, \beta > 0$ and $1 < p < \infty$ and (μ_0, μ_1) be a coherent pair of Jacobi measures where $M = 0$ for type II. Then

- *The set of polynomials is dense in the space $W_{0,1}^p$.*
- *$W_{0,1}^p$ is a complete space.*

Theorem (Ciaurri, Mínguez 2021)

Let $\alpha > -1, \beta > 0$ and $1 < p < \infty$ and (μ_0, μ_1) be a coherent pair of Jacobi measures where $M = 0$ for type II. Then

$$\lim_{n \rightarrow \infty} \|S_n f - f\|_{W_{0,1}^p} = 0$$

if and only if

- For type I, (12) holds
- For type II, (13) holds

Open problems

- 1 What happens when you have $M \neq 0$ for type II Jacobi coherent pair of measures?
- 2 To analyze the convergence of the Fourier series expansion in the Sobolev space for Laguerre coherent pairs of measures.
- 3 To analyze the convergence of the Fourier series expansion in the Sobolev space for Hermite coherent pairs of measures.

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BVP for second order elliptic problems and SOP

X. Yu, Z. Wang and H. LI (2019) used basis of SOP to analyze spectral methods for second-order elliptic BVP problems in the interval $I = [-1, 1]$. For example, they deal with a second order problem with a harmonic potential,

$$\begin{cases} -u''(x) + \lambda x^2 u(x) = f(x), & \lambda \geq 0, x \in I, \\ u(-1) = u(1) = 0. \end{cases}$$

Let consider the space $H_0^1(I) = \{g : I \rightarrow \mathbb{R}, g, g' \in L^2(I), g(\pm 1) = 0\}$.

The basic idea is to have an weak (variational) formulation of the above problem, i.e. to find a function $u \in H_0^1$ such that $B(u, v) = \lambda(x^2 u, v) + (u', v') = (f, v)$ for every $v \in H_0^1$.

The generalized Jacobi spectral method for the above formulation yields to find a polynomial u_N of degree N , with $u_N(\pm 1) = 0$, such that $B(u_N, q) = (f, q)$, for every polynomial of degree N satisfying $q(\pm 1) = 0$.

BVP for second order elliptic problems and SOP

Notice that the polynomial $u_N(x)$ belongs to the span $\{P_n^{(-1,-1)}(x), 2 \leq n \leq N\}$.

Theorem

The sequence of polynomials $\{P_n^{(-1,-1)}(x)\}_{n \geq 2}$ is orthogonal with respect to the discrete-continuous Sobolev inner product

$$A(p, q) = ap(1)q(1) + bp(-1)q(-1) + \int_{-1}^1 p'(x)q'(x)dx,$$

assuming $a + b > 0, a + b + 2ab > 0,$

Let us remind that $P_n^{(-1,-1)}(x) = (x^2 - 1)P_{n-2}^{(1,1)}(x), n \geq 2$. On the other hand, by using the hypergeometric character of such polynomials,

$P_n^{(-1,1)}(x) = P_n(x) - a_n P_{n-2}(x), n \geq 2$, where $\{P_n\}_{n \geq 0}$ is the sequence of Legendre polynomials.

BVP for second order elliptic problems and SOP

Let consider the linear space of the polynomials $(x^2 - 1)\mathbb{P}$. Let $\{\Phi_n\}_{n \geq 2}$ be the set of monic polynomials in the above linear space with $\deg \Phi_n = n$ such that $B(\Phi_n, \Phi_m) = \rho_n \delta_{m,n}$, $m, n \geq 2$. Notice that $\Phi_k(x) = P_k^{(-1, -1)}(x)$, $k = 2, 3$, and $\Phi_n(x) - P_n^{(-1, -1)}(x)$, $n \geq 4$, is polynomial of degree at most $n - 2$ vanishing at ± 1 . This means that $\{\Phi_n\}_{n \geq 2}$ is an orthogonal basis in the above linear space with respect to the inner product given by the bilinear form B . Then,

Theorem

$P_n^{(-1, -1)}(x) = \Phi_n(x) + d_{n,1}\Phi_{n-2}(x) + d_{n,2}\Phi_{n-4}(x)$, $n \geq 4$. with the convention $\Phi_k(x) = 0$, $k = 0, 1$.

As a consequence,

Theorem

If u and u_N are the solutions of the above two variational problems, then

$$u_N(x) = \sum_{n=2}^N \frac{(f, \Phi_n)}{\rho_n} \Phi_n(x) \quad \text{and} \quad u(x) = \sum_{n=2}^{\infty} \frac{(f, \Phi_n)}{\rho_n} \Phi_n(x).$$

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An example of non coherent pair of measures

Let consider the Sobolev inner product

$B(u, v) = \lambda \int_{-1}^1 x^2 u(x)v(x)dx + \int_{-1}^1 u'(x)v'(x)dx$ and denote by $\{\phi_n\}_{n \geq 0}$ the sequence of monic orthogonal polynomials with respect to this inner product. It is easy to check that $\phi_0(x) = 1$, $\phi_1(x) = x$, and $\phi_2(x) = x^2 - \frac{3}{5}$.

Theorem

[Marcellán et al 202]

For $n \geq 2$,

$$P_n^{(-1, -1)}(x) = \phi_n(x) + a_n \phi_{n-2}(x) + b_n \phi_{n-4}(x), \quad b_n \neq 0, \quad (14)$$

where

$$a_n = \lambda (\gamma_n \gamma_{n-1} + d_n (\gamma_{n-1} + \gamma_{n-2}) + d_{n-2} - a_{n-2}) \frac{\|P_{n-2}\|_S^2}{\|\phi_{n-2}\|_S^2}, \quad n \geq 2, \quad (15)$$

$$b_n = \lambda d_n \frac{\|P_{n-2}\|_S^2}{\|\phi_{n-4}\|_S^2}, \quad n \geq 4. \quad (16)$$

An example of non coherent pair of measures

Theorem (Marcellán et al 2022))

The norms of the Sobolev orthogonal polynomials for $n \geq 4$ satisfy

$$\begin{aligned} \|\phi_n\|_S^2 &= \lambda(\gamma_n + \gamma_{n+1} + 2d_n - a_n)\|P_n\|^2 \\ &+ \lambda d_n((\gamma_{n-1} + \gamma_{n-2})(d_n - a_n) - a_n(d_{n-2} - a_{n-2} - b_n))\|P_{n-2}\|^2 \\ &+ n^2\|P_{n-1}\|^2. \end{aligned} \quad (17)$$

Moreover,

$$\begin{aligned} \|\phi_0\|_S^2 &= \frac{2\lambda}{3}, \quad \|\phi_1\|_S^2 = 2 + \frac{2\lambda}{5}, \quad \|\phi_2\|_S^2 = \frac{8}{3} + \frac{8\lambda}{175} \\ \|\phi_3\|_S^2 &= \lambda(\gamma_4 + \gamma_3 + 2d_3 - a_3)\|P_3\|^2 + 9\|P_2\|^2 \\ &+ \lambda d_3(\gamma_2 + \gamma_1)(d_3 - a_3)\|P_1\|^2. \end{aligned}$$

An example of non coherent pair of measures

An algorithm is implemented to find recursively the polynomials $\phi_n(x)$ from $\phi_{n-2}(x)$ and $\phi_{n-4}(x)$.

- 1 STEP 1. From a_{n-2} and $\|\phi_{n-2}\|_S^2$ you get a_n .
- 2 STEP 2. From $\|\phi_{n-4}\|_S^2$ you get b_n .
- 3 STEP 3. From connection formula (15) you get $\phi_n(x)$.

An example of non coherent pair of measures

Now we study some asymptotic relations of the norms of the Sobolev orthogonal polynomials.

Theorem (Marcellán et al 2022)

The following asymptotic relations hold.

- (i) $\lim_{n \rightarrow \infty} \frac{\|\phi_n\|_S^2}{n^2 \|P_{n-1}\|^2} = 1$, and the result is independent of λ .
- (ii) $\lim_{n \rightarrow \infty} \frac{\|\phi_n\|_S^2}{\|\phi_{n-1}\|_S^2} = \frac{1}{4}$.

An example of non coherent pair of measures

From (14)

$$P_n(x) + d_n P_{n-2}(x) = \phi_n(x) + a_n \phi_{n-2}(x) + b_n \phi_{n-4}(x). \quad (18)$$

Taking into account the outer ratio asymptotics for Legendre polynomials

$$\lim_{n \rightarrow \infty} \frac{P_n(x)}{P_{n-1}(x)} = \phi(x), \quad x \in \mathbb{C} \setminus [-1, 1],$$

where $\phi(x) = \frac{x + \sqrt{x^2 - 1}}{2}$, then

Theorem (Marcellán et al 2022)

$$\lim_{n \rightarrow \infty} \frac{\phi_n(x)}{P_n(x)} = \frac{1}{\phi'(x)}, \quad x \in \mathbb{C} \setminus [-1, 1]. \quad (19)$$

Some final comments

The above approach is also valid for potentials $V(x) = x^{2N}$ but the complexity of the connection relations increases from a computational point of view.

Conjecture 1.

In such a case, the outer relative asymptotics

$$\lim_{n \rightarrow \infty} \frac{\phi_n(x)}{P_n(x)} = \frac{1}{\phi'(x)}, \quad x \in \mathbb{C} \setminus [-1, 1]. \quad (20)$$

also holds.

Conjecture 2.

Zeros of the polynomials $\phi_n(x)$ are real, simple and interlace with the zeros of $P_{n+1}(x)$.

Some useful references

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