UNIVERSIDAD COMPLUTENSE MADRID

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The Inverse Problem in Neutron Stars: Listening to the Equation of State

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Mathematical Methods in Physics Meeting

Celebrating Luis Martínez Alonso's Career

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Relativistic Stars

A relativistic star is a neutron star whose behavior is well described in the framework of General Relativity.

A neutron star is the remnant of the explosion of a very massive star. ($\sim 8 - 30 M_\odot$).
Some properties: of neutron stars:

- They are the most compact stars known.
- Mass: between 1 and, perhaps, $3M_\odot$
- Radius: between $\sim 9 - 15\text{km}$.
- They could act as gravitational lenses
- Sources of the recently observed gravitational waves.

Why are neutron stars studied?

The study of neutron stars is useful for understanding how matter behaves under such extreme physical conditions..

The behavior of matter in the interior of neutron stars is modeled by an effective perfect fluid with a barotropic equation of state,

$$p = p(\epsilon)$$

Some recent interesting results on neutron stars:

- Discovery of I-Love-Q universal relations.
- Possibility of the existence of neutron stars with exotic matter (quarks, hyperons,...), in addition to the standar stellar nuclear matter.
- Discovery of gravitational waves produced by the coalescence of two compact objects (neutron stars) (GW170817).

Main targets:

- To develop an efficient numerical method to compute quasi-normal modes.
- To study the constraints imposed by the observation of the gravitational wave event GW170817.
- Develop a numerical method to solve the inverse stellar problem.

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Static and spherical stars

It is possible to choose coordinates in which the line element is parameterized as follows

$$ds^2 = -e^{\nu(r)}(cdt)^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

We consider the matter in the interior of the star as an effective perfect fluid,

$$T_{\mu\nu} = \left(\epsilon(x) + \frac{p(x)}{c^2}\right) u_\mu u_\nu + p(x) g_{\mu\nu}, \quad (2)$$

with a barotropic equation of state,

$$p = p(\epsilon). \quad (3)$$

We consider equilibrium and static configurations, i.e., the coordinates chosen can be considered as as comoving with the fluid This means that $u^\mu \propto \delta_t^{\ \mu}$. Imposing $u^2 = -c^2$, it follows that

$$u^\mu = (ce^{-\nu/2}, \vec{0}). \quad (4)$$

Once one have the metric and the energy-momentum tensor $\rightarrow$ Einstein's equations can be written

Equations for static and spherical stars

After writing down the Einstein equations and the conservation equations, we find that the system of ordinary differential equations to be solved for the interior of a spherically symmetric, static star is given by (TOV equations):

$$\frac{dm}{dr} = 4\pi r^2 \epsilon \tag{5a}$$

$$\frac{d\nu}{dr} = \frac{2G}{r^2 c^2} \frac{m + \frac{4\pi}{c^2} r^3 p}{1 - \frac{2Gm}{c^2 r}} \tag{5b}$$

$$\frac{dp}{dr} = -\left(\epsilon + \frac{p}{c^2}\right) \frac{G}{r^2} \frac{m + \frac{4\pi}{c^2} r^3 p}{1 - \frac{2Gm}{c^2 r}} \tag{5c}$$

Remarks:

- The equation of state, $p = p(\epsilon)$, closes the above system of equations
- Functions to be determined: ν, p, ϵ y m.
- Outside of the star, the equations are the same but with $p = 0$ and $\epsilon = 0$.

Exterior

Outside the star, the equations can be solved analytically (Schwarzschild solution),

$$\nu(r) = -\lambda(r) = \log\left(1 - \frac{2GM}{c^2 r}\right). \quad (6)$$

Interior

On the inside, they must be solved numerically. They are solved from the inside out and matched to the outside. The condition for each function in $r \to 0$ is obtained by making an asymptotic analysis:

$$\epsilon(r) = \epsilon_0 + \mathcal{O}(r^2), \quad (7a)$$

$$m(r) = \frac{4}{3}\pi r^3 \epsilon_0 + \mathcal{O}(r^5), \quad (7b)$$

$$p(r) = p_0 - \frac{2}{3}\pi G r^2 \left(\epsilon_0 + \frac{p_0}{c^2}\right)\left(\epsilon_0 + 3\frac{p_0}{c^2}\right) + \mathcal{O}(r^4), \quad (7c)$$

$$\nu(r) = \nu_0 + \frac{4}{3c^2}\pi G r^2 \left(\epsilon_0 + 3\frac{p_0}{c^2}\right) + \mathcal{O}(r^4). \quad (7d)$$

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Directly related to gravitational wave emission.

- There are different types of perturbations: radial or non-radial: axial and polar.
- Non-radial perturbations of compact objects (such as neutron stars) are accompanied by gravitational wave emission.
- The characteristic oscillations of neutron stars have a frequency and a damping time

$$\omega = 2\pi\nu + i\frac{1}{\tau}. \tag{8}$$

The perturbed metric has the expression

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(\varepsilon^2). \tag{9}$$

We will consider axial perturbations.

The axial disturbances are described by

$$h_{\mu\nu}^{(axial)} = \sum_l \begin{bmatrix} 0 & 0 & 0 & h_0^l \\ 0 & 0 & 0 & h_1^l \\ 0 & 0 & 0 & 0 \\ h_0^l & h_1^l & 0 & 0 \end{bmatrix} \sin\theta \frac{\partial P_l(\cos\theta)}{\partial\theta} \tag{10}$$

y

$$\delta u^{(axial)\mu} = \sum_l \frac{h_2^l}{\sin^2\theta} \left[0, 0, 0, -\frac{\partial P_l(\cos\theta)}{\partial\theta}\right]. \tag{11}$$

The Einstein equations are uncoupled for the different values of l. Let us consider the case of $l = 2$.

We define a new metric function Z as:

$$Z(t, r) = e^{\frac{\nu(r)-\lambda(r)}{2}} \frac{h_1(t, r)}{r} = e^{-i\omega t} Z(r). \tag{12}$$

From Einstein's equations, we get:

$$\frac{d^2 Z}{dr^2} + \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \left\{ -\left[-\frac{2Gm}{r^2 c^2} + \frac{4\pi G}{c^2} r \left(\epsilon - \frac{p}{c^2}\right)\right] \frac{dZ}{dr} \right.$$
$$\left. - \left[\frac{6}{r^2}\left(1 - \frac{Gm}{rc^2}\right) + \frac{4\pi G}{c^2}\left(\epsilon - \frac{p}{c^2}\right)\right] Z + \frac{\omega^2}{c^2} e^{-\nu} Z \right\} = 0. \tag{13}$$

Quasi-normal modes

Quasi-normal modes are solutions of the axial (and polar) equations that satisfy the following 3 boundary conditions:

1. All perturbations (in our case, Z) have a regular behavior in $r = 0$.

   In the interior, we expand Z(r) in taylor series about $r = 0$,

$$Z(r) = a_3 \left[r^3 + \frac{16\pi G \left(\epsilon_0 - p_0/c^2\right) - \omega^2 e^{-\nu_0}}{14c^2} r^5 + \mathcal{O}(r^7) \right]. \tag{14}$$

❷The perturbations behave as outgoing waves in the infinite.

On the outside, we integrate the equation for the phase g in a complex and compactified radial variable (exterior complex scaling).

$$g = \frac{d \log Z}{dr} \quad \text{y} \quad r = R + \frac{1-x}{x} e^{i\alpha}, \quad x \in [1, 0). \tag{15}$$

The asymptotic behavior of g for an outgoing wave is

$$g(x) = -i \frac{\omega}{c} \left[1 + \frac{2GM}{c^2} e^{-i\alpha} x + \mathcal{O}(x^2) \right]. \tag{16}$$

❸ The inner and outer solutions coincide in the surface of the star,

$$f(\omega) \equiv g_{in}^{(\omega)}(R) - g_{out}^{(\omega)}(R) = 0. \tag{17}$$

A numerical method is needed to search for the roots of the $f(\omega)$. For example, the Muller's method.

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The tidal effects are generated by even perturbations in the perturbation parameter χ. Así,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h^{(tidal)}_{\mu\nu} + \mathcal{O}(\chi^4), \tag{18}$$

with

$$h^{(tidal)}_{\mu\nu} = \sum_l \begin{bmatrix} -e^{\nu} H_0^l & 0 & 0 & 0 \\ 0 & e^{\lambda} H_2^l & 0 & 0 \\ 0 & 0 & r^2 K^l & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta K^l \end{bmatrix} P_l(\cos\theta), \tag{19}$$

We focus on the perturbations with $l = 2$. From Einstein's equations, we get

$$\frac{d^2 H_0}{dr^2} + \left\{ \frac{2}{r} + e^{\lambda} \left[\frac{2Gm}{r^2 c^2} - \frac{4\pi G}{c^2} \left(\rho - \frac{p}{c^2} \right) r \right] \right\} \frac{dH_0}{dr} + \left\{ -\frac{6e^{\lambda}}{r^2} \right.$$
$$\left. + \frac{4\pi G}{c^2} e^{\lambda} \left[5\rho + 9\frac{p}{c^2} + c^2 \left(\rho + \frac{p}{c^2} \right) \frac{d\rho}{dp} \right] - \nu'^2 \right\} H_0 = 0. \tag{20}$$

We solve the above equation numerically in the interior of the star. This allows us to calculate the tidal Love λ^{tid} , which is related to how easy or difficult it would be to deform our star.

Contents

Some realistic equations of state

To integrate the stellar structure equations one must specify an equation of state. We will consider the following ones:

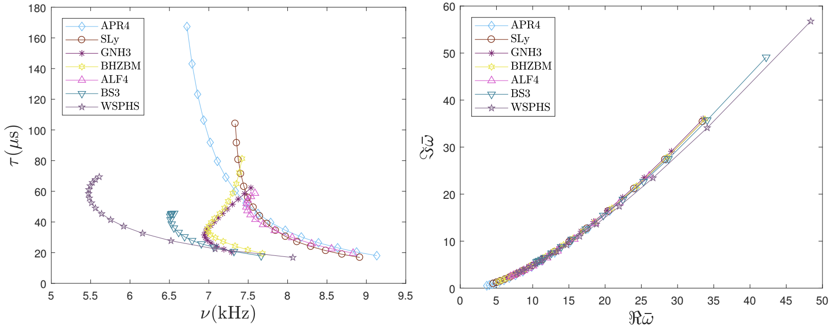
- Standard nuclear material (p, n, μ, e): SLy, APR4.
- A mixing of nuclear matter and hyperons: GNH3, BHZBM.
- Hybrid stars (nuclear matter and quarks): ALF4, BS3.
- Quark stars: WSPHS.

Numerical results: quasi-normal modes

Relations between the fundamental mode frequency *wI* and its damping time

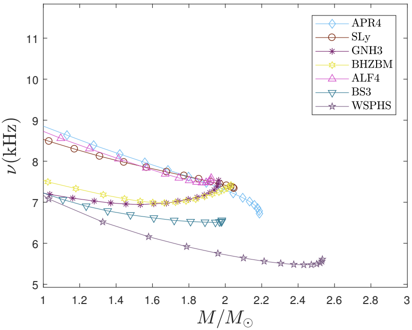
We find a universal relation if we do the following

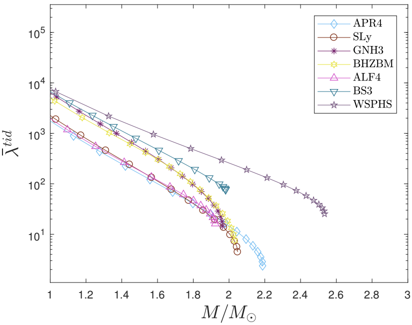
$$\omega = 2\pi\nu + i\frac{1}{\tau} \longrightarrow \bar{\omega} = \frac{c}{\sqrt{Gp_0}}\omega. \qquad (21)$$



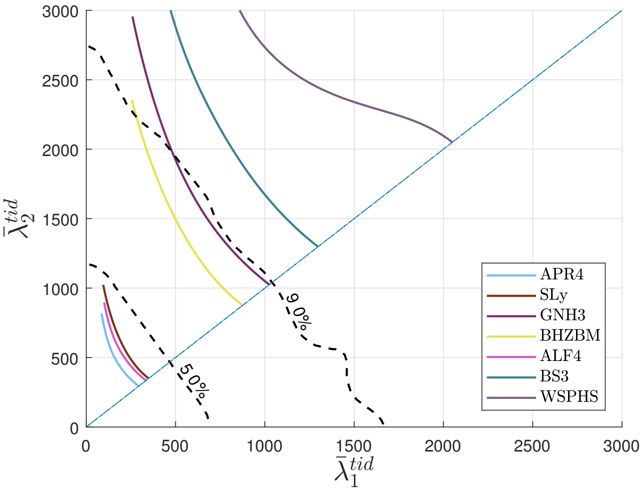
Numerical results: quasi-normal modes and tidal Love parameters

Relationship between the mass and the frequency of the fundamental mode *wl*, and between the mass and the tidal Love parameter





Numerical results: restrictions of GW170817



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Piecewise polytropic parametrization of the EOS

An equation of state is polytropic if

$$p(\rho)/c^2 = K\rho^\Gamma \quad \text{y} \quad \epsilon(\rho) = (1+a)\rho + \frac{K}{\Gamma - 1}\rho^\Gamma. \tag{22}$$

An equation of state is piecewise polytropic in$\rho \geq \rho_0$ if, for a set of densities $\rho_0 < \rho_1 < \rho_2 < \ldots$, the pressure and energy density are continuous and are given, in the interval $\rho_{i-1} \leq \rho \leq \rho_i$, by

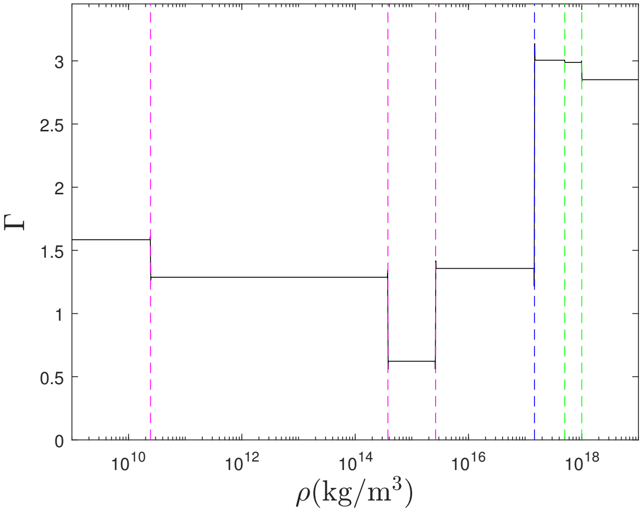
$$p(\rho)/c^2 = K_i\rho^{\Gamma_i} \quad \text{y} \quad \epsilon(\rho) = (1+a_i)\rho + \frac{K_i}{\Gamma_i - 1}\rho^{\Gamma_i}, \tag{23}$$

respectively, with

$$a_i = \frac{\epsilon(\rho_{i-1})}{\rho_{i-1}\Gamma_i - 1} - 1 - \frac{K_i}{}\rho_{i-1}^{\Gamma_i - 1}. \tag{24}$$

Specifying only 4 polytropic parameters, namely. $\{\log_{10} p_1, \Gamma_1, \Gamma_2, \Gamma_3\}$, one can parameterize with very good accuracy a wide variety of realistic equations of state.

Example: parameterization of the equation of state SLy



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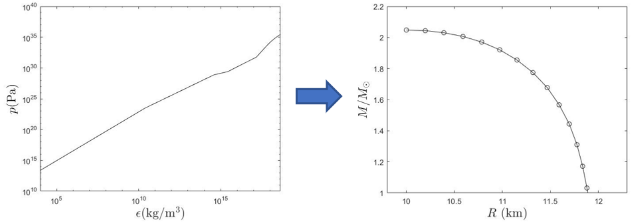
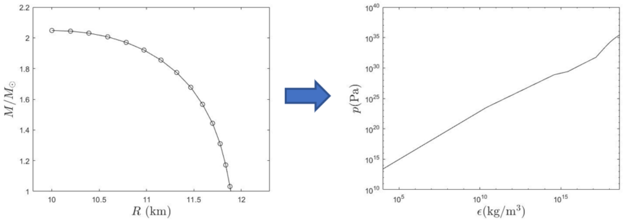
Direct Problem



Inverse Problem



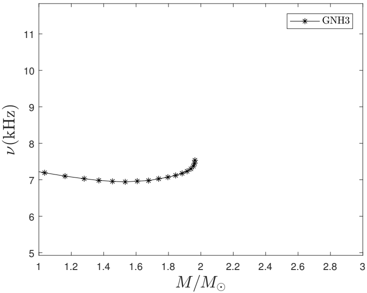
We want to reconstruct the equation of state from data of the mass (M), the frequency of the fundamental mode wI (ν) and/or the tidal LOVE parameter λ^{tid}

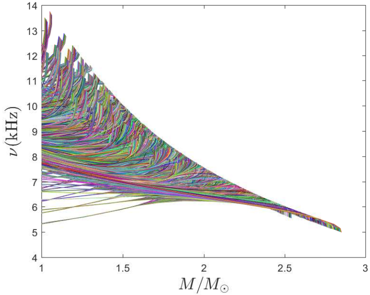
Due to the lack of experimental data, we use as input data the data obtained for the equation of state **GNH3.**

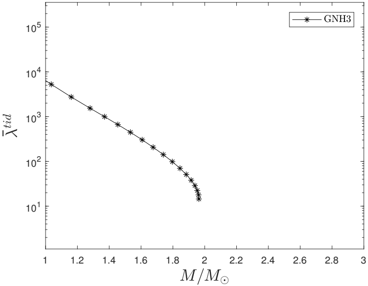
1. We generate a large number of equations of state in a 4-volume of polytropic parameters that includes most of the realistic equations of state used today. For example,

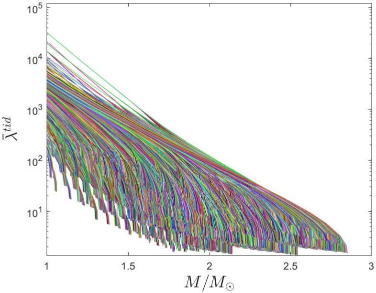
$$\log_{10} p_1 = [34.2, 34.7]_{10}, \quad \Gamma_1 = [2, 3.8]_{14}, \quad \Gamma_2 = [1.8, 3.8]_{14}, \quad \Gamma_3 = [1.8, 3.8]_{14}.$$

2. For each of these $10 \times 14^3 = \mathbf{27440}$ equations of state, we calculate 16 stellar configurations by solving the structure equations.


GNH3
ν(kHz)
$M/M_\odot$


ν(kHz)
$M/M_\odot$


GNH3
$\bar{\lambda}^{tid}$
$M/M_\odot$


$\bar{\lambda}^{tid}$
$M/M_\odot$

The method for quantifying the closeness between curves consists of calculating

$$e_i = a_i \cdot b_i \cdot c_i \cdot d_i, \tag{25}$$

with

$$a_i = \max_j \left\{ \frac{\left|\nu_i(M_j) - \nu_{\text{GNH3}}(M_j)\right|}{\frac{1}{2}\left[\nu_i(M_j) + \nu_{\text{GNH3}}(M_j)\right]} \right\}, \quad b_i = \max_j \left\{ \frac{\left|\overline{\lambda}_i^{tid}(M_j) - \overline{\lambda}_{\text{GNH3}}^{tid}(M_j)\right|}{\frac{1}{2}\left[\overline{\lambda}_i^{tid}(M_j) + \overline{\lambda}_{\text{GNH3}}^{tid}(M_j)\right]} \right\}, \tag{26}$$

$$c_i = \frac{1}{10^5}\sum_{j=1}^{10^5} \left\{ \frac{\left|\nu_i(M_j) - \nu_{\text{GNH3}}(M_j)\right|}{\frac{1}{2}\left[\nu_i(M_j) + \nu_{\text{GNH3}}(M_j)\right]} \right\}, \quad d_i = \frac{1}{10^5}\sum_{j=1}^{10^5} \left\{ \frac{\left|\overline{\lambda}_i^{tid}(M_j) - \overline{\lambda}_{\text{GNH3}}^{tid}(M_j)\right|}{\frac{1}{2}\left[\overline{\lambda}_i^{tid}(M_j) + \overline{\lambda}_{\text{GNH3}}^{tid}(M_j)\right]} \right\}, \tag{27}$$

$i = 1, \ldots, 27440$ y $M_j \in [M_\odot, M_{\text{GNH3}}^{max}]_{10^5}$.

First iteration of the method

$$\log_{10} p_1 = [34.2, 34.7]_{10}, \quad \Gamma_1 = [2, 3.8]_{14}, \quad \Gamma_2 = [1.8, 3.8]_{14}, \quad \Gamma_3 = [1.8, 3.8]_{14}.$$

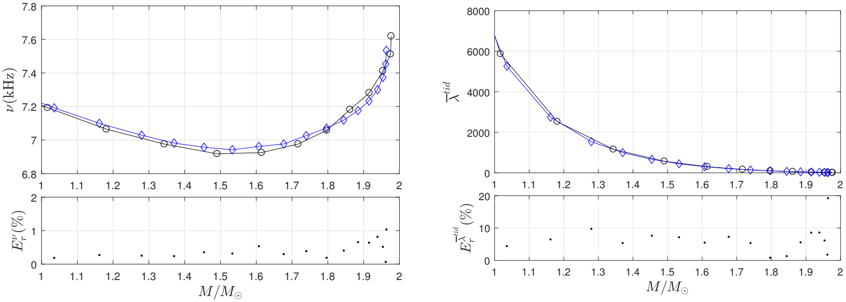
e_i	$\log_{10} p_1$	Γ_1	Γ_2	Γ_3
2.896e-06	34.644	2.6923	2.1077	2.7231
2.9258e-06	34.644	2.6923	2.1077	2.8769
3.959e-06	34.644	2.5538	2.1077	2.8769
4.1674e-06	34.644	2.6923	2.1077	3.0308
4.6182e-06	34.644	2.5538	2.1077	2.7231
4.7503e-06	34.7	2.9692	1.8	3.0308
4.9676e-06	34.7	2.9692	1.8	3.1846

Second iteration of the method

$$\log_{10} p_1 = [34.589, 34.756]_8, \quad \Gamma_1 = [2.4154, 3.1077]_8, \quad \Gamma_2 = [1.6462, 2.2615]_8, \quad \Gamma_3 = [2.5692, 3.3385]_8.$$

e_i	$\log_{10} p_1$	Γ_1	Γ_2	Γ_3
3.2693e-07	34.661	2.7121	2.0857	2.5692
4.0727e-07	34.661	2.811	2.0857	2.5692
1.0093e-06	34.661	2.811	2.0857	2.6791
1.0797e-06	34.661	2.7121	2.0857	2.6791
1.2461e-06	34.637	2.6132	2.1736	2.5692
1.5159e-06	34.661	2.811	2.0857	2.789
1.6026e-06	34.637	2.6132	2.1736	2.6791

Comparison of GNH3 and the reconstructed equation of state (in stellar parameters)



Comparison of GNH3 and the reconstructed equation of state (in polytropic parameters)

EOS	$\log_{10} p_1$	Γ_1	Γ_2	Γ_3
Best fit GNH3	34.648	2.664	2.194	2.304
First EOS	34.661	2.7121	2.0857	2.5692

Contents

Conclusions

- A series of symbolic calculation programs were developed to obtain the stellar structure equations.
- Numerical calculation programs were developed for the integration of the equations (the most complicated was that of the quasi-normal modes).
- A method has been developed to solve the inverse problem that works quite well.

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