New Contributions to Hardy-type Inequalities and Boundary-Domain Integral Equations

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College of Natural and Computational Sciences

Mathematics

Doctoral Thesis | Addis Ababa University | 2025

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Certificate

I hereby certify that I have read this thesis prepared by **Bizuneh Minda Demissie** under my supervision and recommend that, it should by accepted as fulfilling the PhD thesis requirement.

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Examining Committee

This is to certify that this thesis written by Bizuneh Minda Demissie entitled: *New Contributions to Hardy-type Inequalities and Boundary-Domain Integral Equations* submitted in fulfillment of the requirements for the degree of doctor of philosophy (in mathematics) complies with the regulation of the university and meets the accepted standards with respect to originality and quality.

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Dedication

To my beloved father, **Minda Demissie**, whose love, wisdom, and guidance continue to inspire me every day. Though you are no longer with me, your presence is always felt in my heart. Your unwavering belief in me and the values you instilled in me have shaped the person I am today. This work is dedicated to you, as a tribute to the sacrifices you made and the strength you showed. I carry your memory with me, and I will always honor the legacy of love and resilience you left behind.

Abstract

For a large class of operators acting between weighted ℓ^∞ spaces, exact formulas are obtained for their norms and the norms of their restrictions to the cones of nonnegative sequences and nonnegative monotone sequences. The weights involved are arbitrary nonnegative sequences and may differ in the domain and codomain spaces. The results are applied to the Cesàro and Copson operators, giving their norms and their distances to the identity operator on the whole space and on the cones. Simplifications of these formulas are derived in the case of these operators acting on power-weighted ℓ^∞. As an application, best constants are given for inequalities relating the weighted ℓ^∞ norms of the Cesàro and Copson operators both for general weights and for power weights. Moreover, we characterize the optimal non-absolute domain for the Hardy operator (and its dual) minus the identity, in the Lebesgue space $L^p(0,\infty)$, $1 \leq p \leq \infty$.

For variable coefficient Helmholtz equation, using appropriate parametrix, we formulate boundary-domain integral equations (BDIEs) for the Dirichlet and mixed (Dirichlet-Neumann) boundary value problems (BVPs) in a two-dimensional bounded domain. The Dirichlet BVP is reduced to two different BDIE systems, depending on whether the trace or co-normal derivative of the third Green identity is employed on the boundary. On the other hand, the mixed BVP is reduced to four different BDIE systems, depending on whether the trace or co-normal derivative of the third Green identity is employed on the Dirichlet and Neumann boundaries. It is not clear in advance which of them will be more suitable for particular applications and for numerical implementation, and hence we analyzed all the BDIE systems. The equivalence between the BVPs and the formulated BDIE systems are shown. Fredholm properties, invertibility and unique solvability of BDIE systems are investigated in appropriate Sobolev spaces.

List of Papers

The main body of the thesis consists of the following papers:

I. S. Barza, B. M. Demissie, and G. Sinnamon, *End-point norm estimates for Cesàro and Copson operators*, Ann. Mat. Pura Appl. (4)**203**, (2024), no. 2, 989–1013.
https://doi.org/10.1007/s10231-023-01390-3

II. S. Barza, B. M. Demissie, and J. Soria, *Optimal non-absolute domains for the Hardy operator minus identity*, J. Math. Anal. and Appl., **538**, (2024), no. 1, 128324, 22 pp.
https://doi.org/10.1016/j.jmaa.2024.128324.

III. T. G. Ayele, B. M. Demissie, and S. E. Mikhailov, *Boundary-Domain Integral Equations for Variable-Coefficient Helmholtz BVPs In 2D*, J. Math. Sci. (N.Y.), (2024).
https://doi.org/10.1007/s10958-024-06993-6

Author's Contributions

I, Bizuneh Minda Demissie, have made the following contributions to the three papers which are part of this thesis.

I. In paper I, I was responsible for the proof of some of the main theorems and finalizing the first draft of the manuscript. Moreover, I calculated the norms of concrete operators like, Cesàro, Copson, Cesàro minus identity, and Copson minus identity, with power weights.

II. In paper II, I was involved in proving the main results and updated the manuscript based on our discussions.

III. In paper III, I was responsible to prove most of the theorems and wrote the first draft of the manuscript.

Acknowledgement

First and foremost, I would like to express my deepest gratitude to my supervisors prof. Tsegaye G. Ayele, Sergey E. Mikhailov, Sorina Barza, Javier Soria and Lars Erik Persson, for their invaluable guidance, continuous support, and encouragement throughout my PhD journey. Their expertise, insight, and patience have been instrumental in the successful completion of this work.

A heartfelt thanks goes to my colleagues and friends at department of mathematics, Addis Ababa university, for providing a collaborative and inspiring environment. Their intellectual exchanges have made this journey both productive and enjoyable.

On a personal note, I would like to thank my wife S/r Fana Tessema and my children Edna, Nathan and Henon for their unconditional love, patience, and sacrifices. To my father who instilled in me the value of education and perseverance, and to my brothers and sisters for their constant encouragement and support.

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Contents

1 General Introduction

Functional analysis, particularly the study of linear or quasi-linear operators, plays a crucial role in various fields of mathematics and has applications in quantum mechanics, differential and integral equations and signal processing, to mention just a few of them, see e.g., [7]. Among the most studied operators in my field of research are the Hardy operator and its dual or their discrete analogue Cesàro and Copson operators.

The Hardy operator has significant importance on its own, primarily within the context of analysis and functional spaces. The operator and the associated inequality are historically important in the development of functional analysis and remains a central topic in the theory of inequalities. Hardy's inequality provides a way to estimate integrals and sums involving functions and their derivatives.

The connections between the Hardy operator and partial differential equations are profound and illustrate the deep interplay between pure analysis and applied mathematical problems.

The multidimensional Hardy operator, in particular, has numerous applications in the spectral theory of operators, in the theory of partial differential equations, in the theory of integral equations, in the theory of function spaces, see [8] and the references therein. For the one dimensional case we refer to [49]. For instance, in [49] the connection between operator $I - H$ and the Euler differential equation

$$y'(x) - \frac{1}{x}y(x) = g(x), \quad y(0) = 0, \ \ x > 0 \tag{1}$$

is discussed. It has been shown that the solution of (1) is $y(x) = \int_0^x f(t)dt$ where $(I - H)f = g$ or $f = (I - H)^{-1}g$. Moreover, from the remarkable mapping property

$$\|(I - H)f\|_{L^2} = \|f\|_{L^2} \quad \text{for all} \ \ f \in L^2,$$

the relation $\|y'\|_{L^2} = \|g\|_{L^2}$ is found.

The study of embedding theorems, which play an important role in the study of elliptic equations, in arbitrary open sets in $\mathbb{R}^n$ requires investigation of the multi-dimensional Hardy operator, and there is a need for estimates of this operator in various function spaces, among which the weighted Lebesgue spaces are very important, see [8] and the references therein.

In the last thirty years, considerable attention has been devoted to the behavior of averaging operators on function spaces, especially the study of some functional properties as boundedness between weighted Lebesgue spaces or the determination of their optimal domains.

In this thesis, two distinct, yet related topics are discussed: the end-point norm estimates for Cesàro and Copson operators and the determination of optimal non-absolute domains for the Hardy operator (or dual Hardy operator) minus identity. Moreover, boundary domain integral equations are derived and analyzed for variable coefficient Helmholtz boundary value problems (BVPs) in 2D.

The results presented in this thesis contribute to the growing body of literature on operator theory, with potential applications in various areas of mathematics, including partial differential equations. In addition, they help us find the numerical solutions of the variable coefficient Helmholtz BVPs in 2D. Moreover, the techniques developed in the thesis are useful in the analysis of other operators and BVPs.

1.1 Preliminaries

1.1.1 The space of distributions $\mathcal{D}'$

The reader can refer to [41, 73] for the details of this section. Let Ω be a domain in $\mathbb{R}^n$. The support of a continuous function $f(x)$ in Ω, denoted by supp f is the closure of the set of all points $x \in \Omega$ such that $f(x) \neq 0$;

$$\text{supp } f = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

f is said to have compact support, if its support is a bounded set. Then the set of infinitely differentiable functions in Ω with compact support in Ω is called *test functions*;

$$\mathcal{D}(\Omega) := \{\varphi \in C^\infty(\Omega) \ : \text{supp}(\varphi) \text{ is compact in } \Omega\}$$

This set is also denoted by $C_0^\infty(\Omega)$ or $C_{\text{comp}}^\infty(\Omega)$. For example, the function

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1; \\ 0, & \text{otherwise}, \end{cases}$$

is infinitely differentiable for all $x \in \mathbb{R}^2$, with the compact support,

$$\text{supp}\ \varphi = \{x \in \mathbb{R}^2 : \ |x| \leq 1\}.$$

So, φ is a test function in $\mathbb{R}^2$. A sequence $\{\varphi_n\}_{n=1}^\infty$ in $\mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ as $n \to \infty$, written $\varphi_n \to \varphi$ as $n \to \infty$ if:

i) there is a compact set $K \subset \Omega$ with supp $\varphi_n \subset K$, $\forall n$ and

ii) for each multi-index $\alpha \in \mathbb{N}_0^n$,

$$D^\alpha \varphi_n \to D^\alpha \varphi \ \text{ uniformly in } K \text{ as } n \to \infty.$$

A distribution is a generalization of the classical concept of a function. The theory of distributions is far advanced and has numerous applications in physics and mathematics.

A linear functional f on the space $\mathcal{D}$ is an operation (or a rule) by which we assign to every test function $\varphi(x)$ a complex number denoted $\langle f, \varphi\rangle$, such that

$$\langle f, c_1\varphi_1 + c_2\varphi_2\rangle = c_1\langle f, \varphi_1\rangle + c_2\langle f, \varphi_2\rangle$$

for arbitrary test functions φ_1 and φ_2 and complex numbers c_1 and c_2.

A linear functional on $\mathcal{D}$ is continuous if and only if the sequence of numbers $\langle f, \varphi_n\rangle$ converges to $\langle f, \varphi\rangle$, when the sequence of test functions $\{\varphi_n\}$ converges to the test function φ. That is, a linear functional on $\mathcal{D}$ is continuous if

$$\varphi_n \to \varphi \ \text{ as } \ n \to \infty \ \text{ in } \ \mathcal{D} \quad \Rightarrow \quad \langle f, \varphi_n\rangle \to \langle f, \varphi\rangle \ \text{ as } \ n \to \infty \text{ in } \mathbb{C}.$$

A continuous linear functional on the space $\mathcal{D}$ of test functions is called a *distribution*. The set of all distributions in $\mathbb{R}^n$ is denoted by $\mathcal{D}'(\mathbb{R}^n)$. For example, let Ω be a domain in $\mathbb{R}^2$ and $a \in \Omega$. Then it follows immediately that the Dirac delta function $\delta_a := \delta(x-a)$, given by

$$\langle \delta_a, \varphi \rangle = \varphi(a), \quad \varphi \in \mathcal{D}(\Omega),$$

is a distribution, $\delta_a \in \mathcal{D}'(\Omega)$.

A sequence of distributions $\{f_n\}$ in $\mathcal{D}'$ converges to a distribution $f \in \mathcal{D}'$ if, for any $\varphi \in \mathcal{D}$

$$\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle \quad \text{as} \quad n \to \infty \ \text{ in } \mathbb{C}.$$

In this case we shall write $f_n \to f$ as $n \to \infty$ in $\mathcal{D}'$. This convergence is called weak convergence.

For all pairs of multi-indices α and β, a function $\varphi \in C^\infty(\mathbb{R}^n)$ is called rapidly decreasing if

$$\|\varphi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta \varphi(x) \right| < \infty.$$

The space of all such functions in $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$. This space does not coincide with $\mathcal{D}(\mathbb{R}^n)$. For instance, the function $e^{-|x|^2}$ is in $\mathcal{S}\left(\mathbb{R}^2\right)$ but not in $\mathcal{D}\left(\mathbb{R}^2\right)$ because it does not have compact support. Evidently, $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.

A sequence $\{\varphi_n\}$ in $\mathcal{S}(\mathbb{R}^n)$ converges to the function $\varphi \in \mathcal{S}(\mathbb{R}^n)$, written $\varphi_n \to \varphi$ as $n \to \infty$ in $\mathcal{S}(\mathbb{R}^n)$, if

$$x^\beta D^\alpha \varphi_n(x) \longrightarrow x^\beta D^\alpha \varphi(x), \;\; n \to \infty, \;\; x \in \mathbb{R}^n,$$

for all α and β. From the inclusion $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, we deduce that the convergence in $\mathcal{D}$ implies the convergence in $\mathcal{S}$. Evidently, $\mathcal{D}$ is dense in $\mathcal{S}$; that is, for any $\varphi \in \mathcal{S}$ there is a sequence $\varphi_k \in \mathcal{D}$, $k = 1, 2, 3, \cdots$, such that $\varphi_k \to \varphi$ as $k \to \infty$ in $\mathcal{S}$.

Each linear functional over the space of test functions $\mathcal{S}$ is called *generalized function of slow growth* (or *tempered distribution*). The set of tempered distributions is denoted by $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$.

1.1.2 Banach function spaces

Banach function spaces extend classical spaces like Lebesgue, Lorentz, and Orlicz spaces. Paper I and paper II are at the core of these spaces, especially the weighted Lebesgue spaces. The Banach function spaces are built on a measure space $(\Omega, \mathcal{R}, \mu)$ and defined by a function norm ρ that satisfies specific criteria. These spaces are defined next to Lebesgue spaces. For further details we refer the reader to e.g. [13, 29, 33].

Let $(\Omega, \mathcal{R}, \mu)$ be a measure space, $\mathcal{M}_\mu(\Omega)$ denotes the space of μ-measurable real-valued functions on $(\Omega, \mathcal{R}, \mu)$ and

$$\mathcal{M}_\mu^+(\Omega) := \{f \in \mathcal{M}_\mu(\Omega) : \ f \geq 0 \ \ \mu\text{-a.e.}\}.$$

We write simply $\mathcal{M}(\Omega)$ and $\mathcal{M}^+(\Omega)$ if μ is the Lebesgue measure and $\mathcal{R}$ is the σ-algebra of Lebesgue measurable sets of $\Omega = [0, \infty]$.

For $f \in \mathcal{M}(\Omega)$, the *Lebesgue space* $L^p(\Omega, \mathcal{R}, \mu)$, $p \in (0, \infty]$ or simply $L^p(\Omega)$ is defined as

$$L^p(\Omega) = \{f : (\Omega, \mathcal{R}, \mu) \to \mathbb{R}, f \text{ is } \mu\text{- measurable and } \|f\|_{L^p(\Omega)} < \infty\}$$

where

$$\|f\|_{L^p(\Omega)} := \Big(\int_\Omega |f(x)|^p d\mu(x)\Big)^{1/p}, \qquad \text{for } \ 0 < p < \infty,$$

$$\text{and} \qquad \|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| \qquad \text{for } \ p = \infty,$$

with

$$\operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\{a > 0 : |f| \leq a \ \mu\text{-a.e. on } \ \Omega\}.$$

In this case we apply the convention that $\inf \emptyset = \infty$.

Note that $\|\cdot\|_p$ is a norm for $p \in [1, \infty)$, but it is a quasi-norm for $p \in (0, 1)$. We write simply L^p instead of $L^p(\Omega)$ if there is no risk of confusion.

Weighted Lebesgue spaces, $L^p(w, \Omega)$ generalize the classical one $L^p(\Omega)$ by incorporating a nonnegative weight function w. Let $f \in \mathcal{M}(\Omega)$ and w be a nonnegative measurable function on Ω, called a weight function. The weighted

Lebesgue space on Ω, denoted by $L^p(w,\Omega)$, is the set of all $f \in \mathcal{M}(\Omega)$ for which $\|f\|_{L^p(w,\Omega)} < \infty$, where

$$\|f\|_{L^p(w,\Omega)} = \Big(\int_\Omega |f(x)|^p w(x)dx \Big)^{1/p}, \qquad \text{for } 0 < p < \infty$$
$$\|f\|_{L^\infty(w,\Omega)} = \operatorname*{ess\,sup}_{x>0} |f(x)|w(x), \qquad \text{for } p = \infty.$$

If there is no risk of confusion, we write simply $L^p(w)$ instead of $L^p(w,\Omega)$.

For a counting measure μ, $\Omega = \mathbb{N}$ and a weight sequence of non-negative terms $w = (w_n)$, the *weighted Lebesgue spaces* of sequences with weight w, denoted by $\ell^p(w)$, are defined as

$$\ell^p(w) = \{x = (x_n) : \|x\|_{\ell^p(w)} < \infty\},$$

where $x = (x_n)$ is a real (or complex) sequence and

$$\|x\|_{\ell^p(w)} = \Big(\sum_{n=1}^{\infty} |x_n|^p w_n \Big)^{1/p}, \qquad \text{for } 0 < p < \infty,$$
$$\|x\|_{\ell^\infty(w)} = \sup_{n\in\mathbb{N}} |x_n| w_n, \qquad \text{for } p = \infty.$$

If $w(x) \equiv 1$ for all x and $w_n = 1$ for all n, we get the classical Lebesgue spaces L^p and ℓ^p.

Whenever $1/p + 1/p' = 1$, the numbers $p, p' \in [1,\infty]$ are called *conjugate* or *dual*. Using the convention that $1/\infty = 0$, $p = \infty$ corresponds to $p' = 1$ and vice versa.

We repeatedly use the Lebesgue differentiation theorem to characterize the optimal domain of the Hardy operator minus identity on $L^p(0,\infty)$, for $p \in [1,\infty]$. Let $B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ denote the open ball with radius $r > 0$ and $|B(x,r)|$ denotes its Lebesgue measure. For a locally integrable function f on $\mathbb{R}^n$ the Lebesgue differentiation theorem states that

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x),$$

for almost every $x \in \mathbb{R}^n$.

W. A. J. Luxemburg defined Banach function spaces inspired by the properties of L^p spaces, see e.g. [13, 29].

For a measure space $(\Omega, \mathcal{R}, \mu)$, a mapping $\rho : \mathcal{M}_\mu^+(\Omega) \to [0, \infty]$ is called a *Banach function norm* (or simply a function norm) if, for all functions f, g, f_n, $(n = 1, 2, 3, \cdots)$ in $\mathcal{M}_\mu^+(\Omega)$, for all constants $a \geq 0$ and for all $\mu-$measurable subsets E of Ω the following properties hold:

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\quad \rho(af) = a\rho(f)$;
$\rho(f + g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f$, μ-a.e. $\Rightarrow \rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \uparrow f$, μ-a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

(P4) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$;

(P5) $\mu(E) < \infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$

for some constants C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.

The *Banach function space* $X = X(\rho)$ determined by a function norm ρ is defined as follows.

$$X(\rho) = \{f \in \mathcal{M}_\mu(\Omega) : \rho(|f|) < \infty\}.$$

For each $f \in X$ we denote $\|f\|_X = \rho(|f|)$.

For a nonnegative weight w, the Lebesgue spaces $L^p(w)$, $p \in [1, \infty]$ are Banach function spaces.

One of the properties of a Banach function space that we check for the domain of Hardy minus identity on Lebesgue spaces $L^p(0, \infty)$, $1 \leq p \leq \infty$ is the *lattice property*, (P2), namely if $|g| \leq |f|$ μ-a.e. and $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$.

The concept of associate norms ρ' leads to dual Banach function spaces X'. The *associate norm* ρ' of a function norm ρ is defined on $\mathcal{M}^+$ by

$$\rho'(g) = \sup\left\{\int_\Omega fg d\mu : f \in \mathcal{M}^+, \rho(f) \leq 1\right\}, \quad (g \in \mathcal{M}^+).$$

Let ρ be a function norm and let $X = X(\rho)$ be the BFS determined by ρ. The BFS $X(\rho')$ determined by the associate norm ρ' of ρ is called the *associate space* of X and is denoted by X'.

It follows from the norm of a function in a BFS X and the definition of $\rho'(g)$ that the norm of a function g in the associate space X' is given by

$$\|g\|_{X'} = \sup\Big\{ \int_\Omega |fg| d\mu : f \in X, \|f\|_X \leq 1 \Big\}. \qquad (2)$$

The general version of Hölder's inequality which connects the norms of functions in BFS X and its associate X' is defined as follows. Let X be a BFS with associate space X'. If $f \in X$ and $g \in X'$, then fg is integrable and

$$\int_\Omega |fg| d\mu \leq \|f\|_X \|g\|_{X'}. \qquad (3)$$

Since $L^{p'}$ is the associate space of L^p for $1 \leq p \leq \infty$, the Hölder's inequality (3) takes the form

$$\int_\Omega |fg| d\mu \leq \|f\|_{L^p} \|g\|_{L^{p'}},$$

for all $f \in L^p$ and $g \in L^{p'}$.

The particular class of Banach function space called *rearrangement-invariant spaces* (or simply r.i. space) is defined as follows. Let $f, g \in \mathcal{M}_\mu(\Omega)$. A Banach function norm ρ is called a *rearrangement invariant norm* if it satisfies

$$f^* = g^* \text{ on } (0, \mu(\Omega)) \Rightarrow \rho(f) = \rho(g),$$

where

$$f^*(t) := \inf\{s \geq 0 : \mu(\{x \in \mathbb{R}^n, |f(x)| > s\}) \leq t\}$$

is the *nonincreasing rearrangement* of f, $t \in (0, \mu(\Omega))$.

The weighted Lebesgue spaces $L^p(w)$, $1 \leq p \leq \infty$ are not rearrangement-invariant unless the weight function w is constant a.e. That is, the classical Lebesgue spaces L^p, $1 \leq p \leq \infty$, are rearrangement-invariant. The two spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$ over a resonant measure space are rearrangement-invariant spaces [13, Theorem 6.4], and are respectively the largest and the smallest of all r.i. spaces (that is, $L^1 \cap L^\infty \hookrightarrow X \hookrightarrow L^1 + L^\infty$, for an arbitrary r.i. space X over a resonant measure space) [13, Theorem 6.6].

1.1.3 Sobolev spaces

For the details of this section the reader can refer to e.g., [17, 73]. For two Banach spaces V and W with $V \subset W$, the space V is said to be *continuously embedded* in W if

$$\|v\|_W \leq c\|v\|_V \quad \forall v \in V. \tag{4}$$

We say the space V is *compactly embedded* in W if (4) holds and each bounded sequence in V has a convergent subsequence in W.

The weak partial derivative is defined as follows, see e.g., [73]. Let $\Omega \subset \mathbb{R}^n$ be open. Suppose $u, v \in L^1_{\text{loc}}(\Omega)$, and α is a multi-index. We say that v is the α^{th}- *weak partial derivative* of u, written $D^\alpha u = v$, provided

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v \varphi dx \quad \text{for all } \varphi \in D(\Omega).$$

If there does not exist such a function v, then u does not possess a weak α^{th}-partial derivative.

For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W_p^k(\Omega)$ is defined as

$$W_p^k(\Omega) = \Big\{ f \in L^p(\Omega) \ : \ D^\alpha f \text{ exists weakly and } D^\alpha f \in L^p(\Omega),\ \forall \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq k \Big\}$$

If $u \in W_p^k(\Omega)$, we define its norm to be

$$||u||_{W_p^k(\Omega)} := \begin{cases} \Big(\sum\limits_{|\alpha| \leq k} ||D^\alpha u||^p_{L^p(\Omega)} \Big)^{1/p}, & 1 \leq p < \infty; \\ \sum\limits_{|\alpha| \leq k} \operatorname*{ess\,sup}_\Omega |D^\alpha u|, & p = \infty. \end{cases}$$

For $\{u_m\}_{m=1}^\infty$, $u \in W_p^k(\Omega)$, we say u_m converges to u in $W_p^k(\Omega)$, written $u_m \to u$ in $W_p^k(\Omega)$, provided that $\lim\limits_{m \to \infty} ||u_m - u||_{W_p^k(\Omega)} = 0$.

The Sobolev space $W_p^k(\Omega)$, $k = 1, 2, 3, \cdots$ and $1 \leq p \leq \infty$, is a Banach space.

From the definition of Sobolev space we see that $L^p(\Omega) = W_p^0(\Omega$, and for $p = 2$, we usually write $H^k(\Omega) = W_2^k(\Omega),\ k = 0, 1, 2, \cdots$. Thus, $H^0(\Omega) = L^2(\Omega)$. Consequently,

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) : \sum_{|\alpha| \leq k} ||D^\alpha f||_{L^2(\Omega)} < \infty \right\}$$

is the Sobolev space of order k with the norm

$$||f||_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} ||D^\alpha f||^2_{L^2(\Omega)} \right)^{\frac{1}{2}} .$$

In particular, the norm in $H^1(\Omega)$ is given by

$$||u||^2_{H^1(\Omega)} = ||u||^2_{L^2(\Omega)} + ||\nabla u||^2_{L^2(\Omega)}.$$

The Bessel potential space $H^s(\mathbb{R}^n)$ extends the classical Sobolev spaces $H^k(\mathbb{R}^n)$ from $k \in \mathbb{N}$ to $s \in \mathbb{R}$. It is defined as

$$H^s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : w_s(\xi)\mathcal{F}f(\xi) \in L^2(\mathbb{R}^n) \},$$

where $w_s(\xi) = \left(1 + |\xi|^2\right)^{s/2}$, $\mathcal{F}f$ is the *Fourier transform* of f and is defined by

$$\langle \mathcal{F}[f], \varphi \rangle = \langle f, \mathcal{F}[\varphi] \rangle, \quad \forall \varphi \in S(\mathbb{R}^n).$$

Here, $\mathcal{F}[\varphi]$ is the classical Fourier trnasform of the Schwartz function φ, given by

$$\mathcal{F}\varphi(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) dx, \qquad \xi \in \mathbb{R}^n,$$

and provide it with the norm

$$||f||_{H^s(\mathbb{R}^n)} := \|w_s \mathcal{F}f\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\mathcal{F}f(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

which makes it a Hilbert space.

The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$, $\forall s$, see e.g., [55, pp 31]. For any non-empty open set $\Omega \subseteq \mathbb{R}^n$, according to [56, pp 77] we define

$$H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = U\,|_\Omega \ \text{ for some } \ U \in H^s(\mathbb{R}^n)\},$$

equipped with norm

$$\|u\|_{H^s(\Omega)} := \inf_{U \in H^s(\mathbb{R}^n), u = U|_\Omega} \|U\|_{H^s(\mathbb{R}^n)}.$$

We also define

$$\widetilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : \text{ supp } u \subset \overline{\Omega}\}.$$

For $s > 0$, we have the inclusions:

$$\widetilde{H}^s(\Omega) \subset H^s(\Omega) \subset L^2(\Omega) \subset \widetilde{H}^{-s}(\Omega)$$

with continuous injections. For $0 \leq s < \frac{1}{2}$, one has

$$\widetilde{H}^s(\Omega) = H^s(\Omega) \quad \text{and} \quad \widetilde{H}^{-s}(\Omega) = H^{-s}(\Omega),$$

which is not true any more for $s \geq \frac{1}{2}$.

For an open subset Ω of $\mathbb{R}^n$ with a continuous boundary, $\mathcal{D}(\overline{\Omega})$ and $\mathcal{D}(\Omega)$ are dense in $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$ respectively, for all $s > 0$, see e.g., [34, pp 24].

Now, the Rellich compact embedding theorem which is used repeatedly to prove the mapping properties of the surface and volume potential operators is stated as follows. For a bounded open subset Ω of $\mathbb{R}^n$ and $-\infty < s < t < \infty$, the inclusion $H^t(\Omega) \subseteq H^s(\Omega)$ is compact, see e.g., [56, Theorem 3.27].

1.1.4 Linear operators and equations

According to [56, pp 18], we define the kernel and image of a linear map. Suppose that X and Y are vector spaces, and let $A : X \to Y$ be a linear map. The *kernel* (or null space) of A is the subspace of X defined by

$$\text{ker } A = \{u \in X : Au = 0\},$$

and the *image* of A (or range of A) is the subspace of Y defined by

$$\text{im } A = \{f \in Y : \text{there exists } u \in X \text{ such that } f = Au\}.$$

A^{-1} exists if and only if ker $A = \{0\}$ and im $A = Y$.

Linear operator T from a vector space V to vector space W over the same field, written $T : V \to W$, is a mapping which satisfies

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

for $u, v \in V$ and scalars α, β.

For normed spaces X and Y, a linear operator $A: X \to Y$ is *bounded* if there exists a number $c > 0$ such that

$$\|Au\|_Y \leq c\|u\|_X, \quad \forall u \in X.$$

A linear operator is continuous if and only if it is bounded.

For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, a *linear operator* $T : \mathcal{H}_1 \to \mathcal{H}_2$ to be *symmetric* $\mathcal{H}_1$ has to be dense in $\mathcal{H}_2$, and

$$\langle Lh, g\rangle = \langle h, Lg\rangle, \quad \text{for all } h, g \in \mathcal{H}_1.$$

The meaning of semi-boundedness and compactness of a linear operator is important in the analysis of the Fredholm properties of the operators corresponding to the boundary domain integral equations. They are defined as follows.

A linear symmetric operator A in a Hilbert space $\mathcal{H}$ is called *semi-bounded* (or bounded from below) if there is a constant $c \in \mathbb{R}$ such that

$$\langle Ah, h\rangle \geq c\|h\|^2_{\mathcal{H}}, \quad \text{for } h \in \text{dom}(A).$$

If $c = 0$, then A is called *positive*, if $c > 0$, then A is called positive-definite. Again, a linear operator $K : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be *compact* if and only if every bounded sequence $\{x_n\}$ in $\mathcal{H}_1$ has a subsequence $\{x_{n_j}\}$ such that $\{Kx_{n_j}\}$ converges in $\mathcal{H}_2$. The second-order partial differential operator

$$L = \sum_{i,j=1}^{n} \partial_{x_j}\left[a_{ij}(x)\partial_{x_i}\right] + \sum_{i=1}^{n} b_i(x)\partial_{x_i} + c(x)$$

is called *elliptic* if there exists a constant $\beta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \beta|\xi|^2$$

for a.e. x in an open and bounded subset Ω of $\mathbb{R}^n$, and all $\xi \in \mathbb{R}^n$.

The continuity and coercivity of the bilinear form which help us analyse the variational formulation of the boundary value problems are defined as follows. Given two linear spaces V_1, V_2, a *bilinear form* in $V_1 \times V_2$ is a function

$$a: \ V_1 \times V_2 \longrightarrow \mathbb{R}$$

satisfying the following properties:

i) For every $y \in V_2$, the function $x \longrightarrow a(x, y)$ is linear in V_1.

ii) For every $x \in V_1$, the function $x \longrightarrow a(x, y)$ is linear in V_2.

When $V_1 = V_2 = v$, we simply say that a is a bilinear form in V. Moreover, a bilinear form $a: \ V \times V \to \mathbb{R}$, for a normed space V is said to be

i) *continuous* if there exists M such that

$$|a(u, v)| \leq M||u|| \, ||v||, \qquad \forall \, u, v \in V.$$

ii) *coercive* on V (or V- elliptic) if there exists a real number $\beta > 0$ such that

$$a(u, u) \geq \beta||u||^2, \qquad \forall u \in V.$$

For a linear operator L, the equation

$$Lu = f \tag{5}$$

is called a linear (inhomogeneous) equation. In (5) the element f is called the inhomogeneous term (free term, or right-hand side), and the unknown element u belonging to $\mathrm{Dom}(L)$ is called the solution of this equation. If in (5) $f \equiv 0$, the equation obtained,

$$Lu = 0 \tag{6}$$

is called the linear homogeneous equation corresponding to (5).

According to [73], the solution of (5) to be unique in $\mathrm{Dom}(L)$, it is necessary and sufficient that the corresponding homogeneous equation (6) have only a zero solution in $\mathrm{Dom}(L)$.

The distribution $E \in \mathcal{D}'$ is said to be the fundamental solution of the differential operator

$$L(D) = \sum_{|a|=0}^{m} a_\alpha D^\alpha.$$

with constant coefficients a_α, if $L(D)E = \delta(x)$ in $\mathbb{R}^n$.

Generally speaking, the fundamental solution $E(x)$ of the operator $L(D)$ is not unique; it is defined accurately as far as the term $E_0(x)$, which is a solution of the homogeneous equation $L(D)E_0 = 0$. In fact, the distribution $E(x) + E_0(x)$ is also a fundamental solution of the operator $L(D)$,

$$L(D)(E + E_0) = L(D)E + L(D)E_0 = \delta(x).$$

A fundamental solution for the Laplace operator Δ in 2D is given by

$$G(x) = \frac{1}{2\pi} \log \frac{|x|}{r_0} \, ,$$

for any constant $r_0 > 0$, see e.g. [56, Theorem 8.1]. If we shift the origin to a new point y, the PDE $\Delta u = 0$ is unchanged because of the translation invariance of the Laplace operator. That is, if $u(x)$ is harmonic then $u(x - y)$ is also harmonic for x different from y. Therefore,

$$P_\Delta(x, y) := G(x, y) = \frac{1}{2\pi} \log \left(\frac{|x - y|}{r_0} \right), \quad r_0 > 0,$$

is a fundamental solution for Laplace equation.

For general elliptic linear differential operators with variable coefficients, it may not always be possible to find a fundamental solution. For such operators if the fundamental solution does not exist, one can always construct a parametrix [38, pp 333], which we use instead of the fundamental solution.

1.1.5 Trace and trace theorems

Let Ω denote a bounded, open subset of $\mathbb{R}^n$, and the boundary $\partial\Omega$ be sufficiently regular for the outward unit normal v to be well defined. Sometimes we shall work with both the interior and the exterior domains

$$\Omega^+ = \Omega \qquad \text{and} \qquad \Omega^- = \mathbb{R}^n \setminus \big(\Omega^+ \cup \partial\Omega\big),$$

in which case, if the function u is defined on $\Omega^\pm$, we write

$$\gamma^\pm u(x) = \lim_{\Omega^\pm \ni y \to x \in \partial\Omega} u(y)$$

$$\text{and} \qquad \left[\frac{\partial u}{\partial v}\right]^\pm = \partial_v^\pm u(x) = \lim_{\Omega^\pm \ni y \to x \in \partial\Omega} v(x) \cdot \nabla u(y) \quad \text{for } x \in \partial\Omega$$

whenever these limits exist, see e.g., [56, pp 1],[60]. $\gamma^+ u(x)$ (or $\left[\frac{\partial u}{\partial v}\right]^+$) and $\gamma^- u(x)$ (or $\left[\frac{\partial u}{\partial v}\right]^-$) are respectively called the interior boundary trace (or interior normal derivative) and exterior boundary traces (or exterior normal derivative) of a given function $u(x),\ x \in \partial\Omega$.

The partial derivatives of order exactly r, denoted by $D^\beta f$ (where β is a multi-index with $|\beta| = r$), is said to satisfy the *Hölder condition* with exponent $\alpha \in (0, 1]$ if there exists a constant $c > 0$ such that for all $x, y \in \Omega$:

$$\left|D^\beta f(x) - D^\beta f(y)\right| \leq c|x - y|^\alpha.$$

The space $C^{r,\alpha}(\Omega)$, known as the Hölder space, consists of functions that are r-times continuously differentiable (i.e., C^r functions) and whose rth order derivatives satisfy a Hölder condition with exponent α. That is, let Ω be an open subset of $\mathbb{R}^n$. The *Hölder space* $C^{r,\alpha}(\Omega)$ is defined as:

$$C^{r,\alpha}(\Omega) = \left\{f \in C^r(\Omega) : D^\beta f \text{ is locally } \alpha\text{-Hölder continuous for all } |\beta| = r\right\}.$$

In particular, $C^{0,\alpha}(\Omega)$ is the space of functions that are just α-Hölder continuous, $C^{r,1}(\Omega)$ is the space of C^r functions whore rth derivatives are Lipschitz continuous and $C^{r,0}(\Omega) = C^r(\Omega)$ is the space of r-times continuously differentiable functions.

When we say that a domain $\Omega \subset \mathbb{R}^n$ is a $C^{k-1,1}$-domain, we mean that its boundary $\partial\Omega$ can be locally represented as the graph of a $C^{k-1,1}$ functions. This provides a level of smoothness to the boundary, ensuring that it is well-behaved up to its $(k-1)$th derivatives and that its $(k-1)$th derivatives are Lipschitz continuous.

Now the trace theorem, see e.g., [56, Theorem 3.37] is stated as follows. If we define the trace operator $\gamma : \mathcal{D}(\overline{\Omega}) \to \mathcal{D}(\partial\Omega)$ by $\gamma u := u|_{\partial\Omega}$ and Ω is a $C^{k-1,1}$- domain with $\frac{1}{2} < s \leq k$, then γ has a unique extension to a bounded linear operator

$$\gamma : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$$

and this extension has a continuous right inverse.

For a bounded C^∞ domain Ω in $\mathbb{R}^2$ with its boundary $\partial\Omega$, the following assertions are proved in [36, Theorem 4.24].

i) For $s > \frac{1}{2}$, the trace operator γ is a linear and bounded map of $H^s(\Omega)$ onto $H^{s-\frac{1}{2}}(\partial\Omega)$, $\gamma\left(H^s(\Omega)\right) = H^{s-\frac{1}{2}}(\partial\Omega)$.

ii) For $s > \frac{3}{2}$, the normal derivative operator $\gamma_1 = \gamma\left(\frac{\partial}{\partial n}\right)$ is a linear and bounded map of $H^s(\Omega)$ onto $H^{s-\frac{3}{2}}(\partial\Omega)$, $\gamma_1\left(H^s(\Omega)\right) = H^{s-\frac{3}{2}}(\partial\Omega)$.

The Gauss-Ostrogradski theorem is one of the important theorems which is used in the third paper to obtain the first Green identity for the variable coefficient Helmholtz equation in 2D. This theorem is stated as follows. Let Ω be a bounded, open subset of $\mathbb{R}^2$, and $\partial\Omega$ be C^1. If $h \in C_0^1(\overline{\Omega})$, then

$$\int_\Omega \frac{\partial}{\partial x_i} h(x)dx = \int_{\partial\Omega} \gamma^+ h(x) n_i(x) dS_x, \qquad i = 1, 2, \tag{7}$$

where dx denotes the infinitesimal element in area, $n(x) = (n_1(x), n_2(x))$ is the outward unit normal vector to $\partial\Omega$ at x, $n_i(x)$ is the ith component of the normal $n(x)$ in the x_i-direction and dS_x refers to the infinitesimal arc length element along $\partial\Omega$.

Using the results for density and trace theorems, the integral relation (7) holds for any $h \in H^1(\Omega)$. Indeed, since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ we have a sequence $h_n \in C^1(\overline{\Omega})$ such that $\|h_n - h\|_{H^1(\Omega)} \to 0$ as $n \to \infty$. Since $h_n \in C^1(\overline{\Omega})$, the integral relation (7) implies

$$\int_\Omega \frac{\partial}{\partial x_i} h_n dx = \int_{\partial\Omega} \gamma^+ h_n n_i dS_x, \quad i = 1, 2. \tag{8}$$

Taking the limit as $n \to \infty$ in the equation (8), we obtain

$$\begin{aligned} \left| \int_\Omega \frac{\partial h_n}{\partial x_i} dx - \int_\Omega \frac{\partial h}{\partial x_i} dx \right|^2 &\le c \int_\Omega \left| \frac{\partial h_n}{\partial x_i} - \frac{\partial h}{\partial x_i} \right|^2 dx \\ &\le c \|h_n - h\|_{H^1(\Omega)} \longrightarrow 0. \end{aligned}$$

Hence

$$\int_\Omega \frac{\partial h_n}{\partial x_i} dx \longrightarrow \int_\Omega \frac{\partial h}{\partial x_i} dx \quad \text{as } n \to \infty.$$

For the right side of equation (8), we get

$$\begin{aligned} \left| \int_{\partial\Omega} \gamma^+ h_n n_i dS_x - \int_{\partial\Omega} \gamma^+ h n_i dS_x \right| &\le \int_{\partial\Omega} \left| \gamma^+ (h_n - h) n_i \right| dS_x \\ &\le \|\gamma^+ (h_n - h)\|_{L^2(\partial\Omega)}, \end{aligned}$$

the continuity of trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ and hence to $L^2(\partial\Omega)$ implies,

$$\|\gamma^+ (h_n - h)\|_{L^2(\partial\Omega)} \le c \|h_n - h\|_{H^1(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Hence

$$\int_{\partial\Omega} \gamma^+ h_n n_i dS_x \longrightarrow \int_{\partial\Omega} \gamma^+ h n_i dS_x \quad \text{as } n \to \infty.$$

1.1.6 Variational formulations of boundary value problems

Let $\partial\Omega = \overline{\partial\Omega}_D \cup \overline{\partial\Omega}_N$ be a disjoint decomposition of the boundary $\partial\Omega$. The boundary value problem (BVP) is to find a scalar function satisfying the partial differential equation

$$(Lu)(x) = f(x) \quad \text{for } x \in \Omega, \tag{9}$$

the Dirichlet boundary condition

$$\gamma^+ u(x) = g_{_D}(x) \quad \text{for } \; x \in \partial\Omega_D, \tag{10}$$

and the Neumann boundary condition

$$T^+ u(x) = g_{_N}(x) \quad \text{for } \; x \in \partial\Omega_N, \tag{11}$$

where $f,\ g_{_D},$ and $g_{_N}$ are some given functions, and T^+ denotes the conormal derivative operator.

The BVP (9) and (10) with $\partial\Omega = \partial\Omega_D$ is called a *Dirichlet BVP*, the BVP (9) and (11) with $\partial\Omega = \partial\Omega_N$ is called a *Neumann BVP*, while the BVP (9)–(11) is called *mixed BVP*.

Variational methods proved to be very successful for a class of elliptic BVPs which admit an equivalent variational formulation in terms of coercive bilinear form. All the problems seen can be cast in the following abstract variational formulation: Find $u \in V$ such that

$$a(u, v) = b(v), \quad \forall v \in V,$$

where:

- V is a Hilbert space (with norm $||.||_{_V}$);
- $a: \ V \times V \to \mathbb{R}$ is a bilinear form, that is for $\alpha, \beta \in \mathbb{R},\ u, v, w \in V$,

$$\begin{aligned} a(\alpha u + \beta v, w) &= \alpha\ a(u, w) \ + \ \beta\ a(v, w), \\ a(w, \alpha u + \beta v) &= \alpha\ a(w, u) \ + \ \beta\ a(w, v). \end{aligned}$$

- $b : V \to \mathbb{R}$ is a linear functional: $b(\alpha u + \beta v) = \alpha\ b(u) \ + \ \beta\ b(v)$.

The existence and uniqueness of the solution of the above formulation are guaranteed by the *Lax-Milgram lemma* which is stated as follows. Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ be a bilinear form, continuous and coercive. Let b be a continuous linear form on V. Then the problem:

$$\text{Find } u \in V \text{ such that } \; a(u, v) = b(v), \text{ for all } v \in V$$

has a unique solution.

1.1.7 Helmholtz equation with constant coefficients

The Helmholtz equation, or reduced wave equation, has the form

$$\Delta u + k^2 u = 0. \tag{12}$$

It takes its name from the German physicist Hermann von Helmholtz (1821-1894), a pioneer in acoustics, electromagnetism and physiology. The equation arises naturally when one is looking for mono-frequency or time-harmonic solutions to the wave equation. If $u(x,t) = v(x)e^{-i\omega t}$ satisfies $u_{tt} = c^2 \Delta u$, then v satisfies (12) with $k = \omega/c$, hence the name reduced wave equation. Time-harmonic waves are of fundamental importance in applications as diverse as noise scattering, radar and sonar technology and seismology.

The quantity k is the wave number. It is often real and constant, but it can be complex if the medium of propagation is energy absorbing, or a function of space if the medium is inhomogeneous. When $k = 0$, (12) reduces to Laplace equation. When $k^2 < 0$ (i.e., for imaginary k), the equation becomes the space part of the diffusion equation. At low wave numbers, (12) behaves very much like the Laplace equation. However, solutions at large wave number are highly oscillatory, and this causes a great increase in complexity of analytical and numerical methods, see e.g. [23].

The inhomogeneous Helmholtz equation is an equation of the form

$$\Delta u + k^2 u = f, \tag{13}$$

where $k^2 \neq 0$, u is the unknown solution and f is a given function. u can either be a scalar function or a vector function. Equation (13) is the generalization of the following two cases:

- Laplace's equation, with $k = 0$ and $f = 0$.
- Poisson's equation, with $k = 0$ and $f \neq 0$.

1.1.8 Fredholm operator and alternative theorem

According to [56, pp 33], for Banach spaces X and Y, a bounded linear operator $A : X \to Y$ is said to be *Fredholm* if

- the subspace im A is closed in Y, and
- the subspaces ker A and $Y/\text{im}A$ are finite-dimensional.

The index of A, in this case is the integer defined by index$(A) = \dim(\ker A) - \dim(Y/\text{im}A)$.

The Fredholm alternative is stated as follows, see e.g., [56, Theorem 2.27]. Assume that $A : X \to Y$ is Fredholm with index$(A) = 0$. There are two, mutually exclusive possibilities:

(i) The homogeneous equation $Au = 0$ has only the trivial solution $u = 0$. In this case, for each $y \in Y$, the inhomogeneous equation $Au = f$ has a unique solution $u \in X$.

(ii) The homogeneous equation $Au = 0$ has exactly p linearly independent solutions $u_1, \cdots, u_p$ for some finite $p \geq 1$.

From the Fredholm alternative, we can draw the following assertion. If $A : X \to Y$ is Fredholm operator with zero index and injective, then for each $f \in Y$, the inhomogeneous equation $Au = f$ has a unique solution $u \in X$.

The assertion which states "every bounded sequence in a Hilbert space has a weakly convergent subsequence" is fundamental in functional analysis and has important applications in the study of variational methods and partial differential equations. The following two results are proved in [56, Lemma 2.32 and Theorem 2.33]. Let $\mathcal{H}^*$ be the dual of a Hilbert space $\mathcal{H}$.

(i) If the bounded linear operator $A : \mathcal{H} \to \mathcal{H}^*$ is positive and bounded below, then it has a bounded inverse $A^{-1} : \mathcal{H}^* \to \mathcal{H}$.

(ii) If $A = A_0 + K$, where $A_0 : \mathcal{H} \to \mathcal{H}^*$ is positive and bounded below, and $K : \mathcal{H} \to \mathcal{H}^*$ is compact, then $A : \mathcal{H} \to \mathcal{H}^*$ is Fredholm with zero index, and hence the Fredholm alternative holds for the equation $Au = f$.

From the above two facts we obtain the following assertion. If $A = A_0 + K$, where $A_0 : \mathcal{H} \to \mathcal{H}^*$ is invertible operator and $K : \mathcal{H} \to \mathcal{H}^*$ is compact, then $A : \mathcal{H} \to \mathcal{H}^*$ is Fredholm operator with zero index.

1.2 Concise literature review

We now provide selected information related to our topic based on some books and related papers.

1.2.1 Hardy-type operators and inequalities

G. H. Hardy introduced the so called Hardy operator and proved some inequalities in 1915, while he was trying to give a simpler proof of Hilbert's inequality, for details, see [50, 51].

The Hardy operator is defined as follows. For a locally integrable function f on $(0, \infty)$,

$$Hf(x) = \frac{1}{x} \int_0^x f(t)dt$$

is called the *Hardy averaging operator* (or simply Hardy operator). Its discrete counterpart is

$$Cx = \frac{1}{n} \sum_{k=1}^{n} x_k,$$

for a sequence $x = (x_n)$, is called the *discrete Hardy operator* or *Cesàro operator*.

As described in [52, pp 17], the duality on the weighted Lebesgue space $L^p(w)$ with $1 < p < \infty$ is defined by the inner product

$$\langle g, f \rangle = \int_a^b g(x)f(x)dx, \quad f \in L^p(w),$$

where $a, b \in \mathbb{R}$ satisfying $-\infty \leq a < b \leq \infty$. From this one can easily observe that the dual space to $L^p(w)$ can be identified with the space $L^{p'}(\widehat{w})$ where $p' = p/(p-1), \widehat{w} = w^{1-p'}$.

The dual operators to H, denoted by H^*, and to C, denoted by C^* are given by

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t}dt \qquad \text{and} \qquad (C^*x)_n = \sum_{k=n}^\infty \frac{x_k}{k}.$$

They are called the *dual Hardy operator* and the *Copson operator*, respectively.

G. H. Hardy [35] stated and proved the original form of the integral inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right) \int_0^\infty f^p(x)dx, \tag{14}$$

which has been extensively studied. Later, the Hardy's power-weighted version of the inequality (14) and its dual were respectively extended to the form

$$\left(\int_a^b \left(\int_a^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leq C\left(\int_a^b f^p(x)v(x)dx\right)^{1/p} \tag{15}$$

and

$$\left(\int_a^b \left(\int_x^b f(t)dt\right)^q u(x)dx\right)^{1/q} \leq C\left(\int_a^b f^p(x)v(x)dx\right)^{1/p} \tag{16}$$

with $a, b \in \mathbb{R}$ satisfying $-\infty \leq a < b \leq \infty$, u, v weight functions (which means measurable functions positive a.e. in the interval (a, b)), $p, q \in \mathbb{R}$ satisfying $0 < q \leq \infty, 1 \leq p \leq \infty$.

It is known, see e.g., [35, 52], that (15) and (16) respectively hold for all measurable functions $f \geq 0$ and $1 < p \leq q < \infty$, if and only if

$$\sup_{a<x<b} \left(\int_x^b u(t)dt\right)^{1/q} \left(\int_a^x v^{1-p'}(t)dt\right)^{1/p'} < \infty,$$

and

$$\sup_{a<x<b} \left(\int_a^x u(t)dt\right)^{1/q} \left(\int_x^b v^{1-p'}(t)dt\right)^{1/p'} < \infty.$$

For $p > 1$, the discrete analogue of the classical Hardy inequality (14) is given as

$$\sum_{n=1}^\infty \left(\frac{1}{n}\sum_{k=1}^n x_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty x_n^p, \tag{17}$$

unless all the $x_i's$ are zero. The constant is the best possible. Moreover, for $p > 1$ it has been proved that

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} x_k\right)^p < p^p \sum_{n=1}^{\infty}(nx_n)^p,$$

unless x_n is null. The constant is the best possible. The inequality (17) is also generalized for a non-negative sequence x_n and weight sequence w_n. For $p > 1$,

$$\sum_{n=1}^{\infty}\left(\frac{\sum_{k=1}^{n} x_k w_k}{\sum_{k=1}^{n} w_k}\right)^p w_n \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} x_n^p w_n.$$

Again the discrete analogue of (15) or the general weighted extension of (17) with weight sequences u_n and v_n has the form

$$\left[\sum_{n=1}^{\infty} u_n \left(\sum_{k=1}^{n} x_k\right)^q\right]^{1/q} \leq C \left(\sum_{n=1}^{\infty} v_n x_n^p\right)^{1/p} \tag{18}$$

for $0 < q < \infty, 1 < p < \infty$. This inequality holds for every non-negative sequence x_n and $1 < p \leq q < \infty$ if and only if

$$\sup_{k>0}\left(\sum_{n=k}^{\infty} u_n\right)^{1/q}\left(\sum_{n=0}^{k} v_n^{1-p'}\right)^{1/p'} < \infty,$$

which is the discrete analogue of the condition for (15).

For a sequence $x = x_n$ and $p > 1$, Cesàro space $ces(p)$ and Copson space $cop(p)$ are defined as follows.

$$ces(p) = \left\{x : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty\right\},$$

with the norm

$$\|x\|_{ces(p)} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{1/p}$$

and

$$cop(p) = \left\{x : \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{|x_k|}{k}\right)^p < \infty\right\}$$

with the norm

$$\|x\|_{cop(p)} = \left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{|x_k|}{k}\right)^p\right)^{1/p}.$$

G. Hardy, see e.g., [10] proved the embedding between these spaces. He was able to show that

$$\|x\|_{ces(p)} \leq \left(\sum_{k=1}^{\infty} k^{-p}\right)^{1/p} \|x\|_{cop(p)}, \tag{19}$$

for $p \geq 2$, and

$$\|x\|_{cop(p)} \leq (p-1)^{1/p}\|x\|_{ces(p)}, \tag{20}$$

for $1 < p \leq 2$. The constants are both best possible. Moreover, there is strict inequality (19) except when $x_2 = x_3 = \cdots = 0$, and in (20) except when $x = 0$.

Hardy's inequalities laid the foundation for extensive research into what today is referred to as Hardy-type inequalities, see e.g., [2, 43, 51, 52].

Consider an operator

$$Tf(x) = \int_a^x k(x,s)f(s)ds. \tag{21}$$

An inequality of the form

$$\left(\int_a^b (Tf(x))^q\, u(x)dx\right)^{1/q} \leq C\left(\int_a^b f^p(x)v(x)dx\right)^{1/p} \tag{22}$$

is the natural generalization of the Hardy inequality, where $k(x,s)$ is a non-negative measurable function on the set $\{(x,s) : a < s \leq x < b\}$. The inequality (22) can also be represented as a norm inequality

$$\|Tf\|_{L^q(u)} \leq C\|f\|_{L^p(v)}.$$

The kernel operator (3.1) is usually called the Hardy-Volterra operator or Volterra integral operator. We call it also Hardy-type operator. Consequently, the inequality (22) is called a Hardy-type inequality, see e.g., [2, 43, 51, 52].

The upper Boyd index $\overline{\alpha}$ and the lower Boyd index $\underline{\alpha}$ of an r.i. space X are given as follows, see e.g., [29]. The norm of the dilation operator D_t, where $D_t f(s) = f(\frac{s}{t})$ is given by

$$\|D_t\|_X = \sup_{\|f\|_X \leq 1} \|D_t f\|_X,$$

and

$$\overline{\alpha} = \lim_{t\to\infty} \frac{\log \|D_t\|_X}{\log t} \quad \text{and} \quad \overline{\alpha} = \lim_{t\to 0} \frac{\log \|D_t\|_X}{\log t}.$$

The lower Boyd index $\underline{\alpha}'$ and the upper Boyd index $\overline{\alpha}'$ of the associate space X' are given by $\underline{\alpha}' = 1 - \overline{\alpha}$ and $\overline{\alpha}' = 1 - \underline{\alpha}$. Since the Hardy operator H and its dual H^* are bounded on an r.i. space X for $\overline{\alpha} < 1$ and $\underline{\alpha} > 0$ respectively, see e.g., [13], $H : L^p(0,\infty) \to L^p(0,\infty)$ for $1 < p \leq \infty$ and $H^* : L^p(0,\infty) \to L^p(0,\infty)$ for $1 \leq p < \infty$ are bounded, cf. also [35, Theorems 327 and 328], [29].

The discrete Hardy inequality

$$\|Cx\|_{\ell^p} < p'\|x\|_{\ell^p}, \tag{23}$$

holds for $p > 1$ and nonnegative sequence of real numbers $x = (x_n)$, unless all the $x_n' s$ are zero. Moreover,

$$\|C^* x\|_{\ell^p} < p\|x\|_{\ell^p}, \tag{24}$$

unless $x = (x_n)$ is the null sequence. The constants p' and p in the above inequalities are best possible, see e.g., [29, 35, 50, 51]. These inequalities show that C and C^* are bounded operators with $\|C\|_{\ell^p} = p'$ and $\|C^*\|_{\ell^p} = p$ for $p > 1$.

The inequalities (23) and (24) can be extended to other operators with general weights, see e.g., [29, 35, 50, 51],

$$\sum_{n=1}^{\infty} \left(\frac{\sum_{k=1}^{n} x_k w_k}{\sum_{k=1}^{n} w_k} \right)^p w_n \leq (p')^p \sum_{n=1}^{\infty} x_n^p w_n, \tag{25}$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{x_k w_k}{\sum_{m=1}^{k} w_m}\right)^p w_n \leq p^p \sum_{n=1}^{\infty} x_n^p w_n, \tag{26}$$

for $p > 1$, $x_n \geq 0$ and $w_n > 0$. In each case, the constants $(p')^p$ and p^p are sharp.

For some values of the parameter p, G. Bennett [10] has found out the best constants α_p and β_p for the inequalities relating the Cesàro and Copson operators.

$$\|Cx\|_{\ell^p} \leq \alpha_p \|C^* x\|_{\ell^p}, \tag{27}$$

and

$$\|C^* x\|_{\ell^p} \leq \beta_p \|Cx\|_{\ell^p}. \tag{28}$$

Bennett has showed that $\alpha_p = \left(\sum_{k=1}^{\infty}\frac{1}{k^p}\right)^{1/p}$ for $p \geq 2$, and $\beta_p = (p-1)^{1/p}$ for $1 < p \leq 2$. These constants are sharp. There is strict inequality in (27) except when $x_2 = x_3 = \cdots = 0$, and in (28) except when $x \equiv 0$. Following Bennett's question of finding the best constants in (27) for $1 < p < 2$ and in (28) for $p > 2$, V. Kolyada [46] has found out that the best possible constant in (28) for $p > 2$ is $\beta_p = p - 1$. The problem of finding the optimal constant in inequality (27) for $1 < p < 2$ remains open. However, V. Kolyada has found some lower and upper bounds for α_p, namely

$$\frac{1}{p-1} \leq \alpha_p \leq \frac{\zeta(p)^{1/p}}{(p-1)^{1/p'}},$$

He proved also one optimal weighted type estimate for $1 < p \leq 2$ and nonnegative sequence (x_n):

$$\left[\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^p\right]^{1/p} \leq (p-1)^{-1/p'}\left[\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{x_k}{k} w_p(k)\right)^p\right]^{1/p} \tag{29}$$

where $w_p(n) = \left(n^{p-1}\sum_{k=n}^{\infty}\frac{1}{k^p}\right)^{1/p}$.

For $\alpha, \beta \in \mathbb{R}$, $1 < p \leq \infty$, the optimal constants in the integral inequalities

$$\|Hf\|_{L^p(t^\alpha)} \leq K_{\alpha,p}\|H^*f\|_{L^p(t^\alpha)} \quad \text{and} \quad \|H^*f\|_{L^p(t^\beta)} \leq K_{\beta,p}\|Hf\|_{L^p(t^\beta)}$$

are known for different cones of functions (see e.g. [15, 16, 29, 43, 45, 47, 51, 52, 72]. For $p = \infty$ and positive functions, see e.g.,[16] it has been shown that

$$K_{\alpha,\infty} = \begin{cases} 0, & \alpha < 0, \\ \dfrac{1}{1-\alpha}, & 0 \leq \alpha < 1, \\ \infty, & \alpha \geq 1, \end{cases} \quad \text{and} \quad K_{\beta,\infty} = \begin{cases} \infty, & \beta \leq 0, \\ \dfrac{1}{\beta}, & 0 < \beta \leq 1, \\ 0, & \beta > 1, \end{cases}$$

and these estimates are sharp. These results were the first motivation of paper I even though the techniques developed in [16] could not be transferred to the discrete case. Moreover, we were interested to treat for the general non-negative weight.

Let $\alpha, \beta \in \mathbb{R}$. The best possible constants in the inequalities

$$\|Hf\|_{L^p(t^\alpha)} \leq K_{\alpha,p}\|H^*f\|_{L^p(t^\alpha)} \text{ and } \|H^*f\|_{L^p(t^\beta)} \leq K_{\beta,p}\|Hf\|_{L^p(t^\beta)} \quad (30)$$

for positive or monotone function f are known for $1 < p \leq \infty$, see e.g. [15, 16, 43, 45, 47, 51, 52, 72]. For $p = \infty$ and positive functions, it has been found out that

$$K_{\alpha,\infty} = \begin{cases} 0, & \alpha < 0, \\ \dfrac{1}{1-\alpha}, & 0 \leq \alpha < 1, \\ \infty, & \alpha \geq 1, \end{cases} \text{ and } K_{\beta,\infty} = \begin{cases} \infty, & \beta \leq 0, \\ \dfrac{1}{\beta}, & 0 < \beta \leq 1, \\ 0, & \beta > 1, \end{cases} \quad (31)$$

and these estimates are sharp [15, 16].

The characterization of optimal domains has been considered for many different kinds of operators and function spaces, see e.g. [9, 28, 32, 65]. Let $T : X \to Y$ be a bounded linear operator, for function spaces X and Y. Then the optimal domain is the 'largest' space M (usually within a class of spaces with a priori conditions and with $X \subseteq M$) for which $T : M \to Y$ is still bounded. The space M is the largest in the sense that if $T : F \to Y$ is bounded (with $X \subseteq F$), then F is continuously embedded in M. Generally, these spaces are not well-defined. For instance, for an r.i. space X for which $H : X \to X$ is bounded, the class of functions for which $H(|f|) \in X$ is larger than X. It is not even a subspace of $(L^1 + L^\infty)(\mathbb{R}^+)$, (see, e.g., [28, Theorem 2.6]). This shows that the class of functions for which $H(|f|) \in X$ is not an r.i. space, see e.g., [13, Theorem 6.6].

The above considerations inspired us to investigate for the identity minus Hardy operator, $I - H$, defined as follows:

$$(I - H)f(x) = f(x) - \frac{1}{x}\int_0^x f(t)dt, \quad x > 0.$$

The special interest of this operator its mapping properties, see e.g., [37, 43, 48, 72].

One of such a result which has close connection to fractional order Hardy inequalities is [52, Proposition 5.38]: for $g \in L^p(x^{-\alpha p-1})$ with $p \geq 1$ and $\alpha > -1$, $\alpha \neq 0$,

$$\left(\int_0^\infty \left|\frac{g(x) - \frac{1}{x}\int_0^x g(y)dy}{x^\alpha}\right|^p \frac{dx}{x}\right)^{1/p} \approx \left(\int_0^\infty \left|\frac{g(x)}{x^\alpha}\right|^p \frac{dx}{x}\right)^{1/p}$$

with the equivalence constants $1 + 1/|\alpha|$ and $(\alpha + 1)/(\alpha + 2)$.

In particular, for $p = 2$ and $\alpha = -1/2$ we obtain the property of the L^2-spaces:

$$\left(\int_0^\infty ((I - H)g(x))^2 \, dx\right)^{1/2} \approx \left(\int_0^\infty g^2(x)dx\right)^{1/2} \tag{32}$$

with equivalence constants $\frac{1}{3}$ and 3. But even more surprising is that we have even equality in (32). More generally, it was proved in the paper [40] that this isometry holds even in a general weighted arithmetic mean operator H_w and its dual version $\widetilde{H}_w$ involved. They are defined as follows:

$$(H_w g)(x) := \frac{1}{W(x)}\int_0^x g(y)w(y)dy,$$

where w is a weight function on $(0,\infty)$ and

$$W(x) = \int_0^x w(y)dy < \infty \quad \text{for every } \ x > 0.$$

Similarly,

$$(\widetilde{H}_w g)(x) := \frac{1}{\widetilde{W}(x)} \int_x^\infty g(y)w(y)dy,$$

with

$$\widetilde{W}(x) = \int_x^\infty w(y)dy < \infty \quad \text{for every } \ x > 0.$$

This remarkable result reads as follows, (see [52, Theorem 5.45]: for $g \in L^2(w)$ and $W(\infty) = \widetilde{W}(0) = \infty$,

$$\|g\|_{L^2(w)} = \|g - H_w g\|_{L^2(w)} \qquad \text{and} \qquad \|g\|_{L^2(w)} = \|g - \widetilde{H}_w g\|_{L^2(w)}.$$

These few examples of results show the great interest of the remarkable operator 'Hardy minus identity' and a lot of problems around appear such as to generalize to more general result, e.g., to other Hilbert spaces, optimal non-absolute domains for this operator, etc.

The codomain of the Hardy operator H for a Banach function space X is denoted by

$$[H, X] = \{f : \mathbb{R}^+ \to \mathbb{R} \ \text{measurable}, \ \ H|f| \in X\}.$$

It is well known that for a rearrangement invariant space X and a bounded Hardy operator $H : X \to X$, $[H, X]$ is larger than X [28, Theorem 2.6]. In particular, $L^p(\mathbb{R}^+) \subsetneq [H, L^p(\mathbb{R}^+)]$ for $1 < p \leq \infty$, [28, Proposition 2.3]. It is known also that $[H, L^1(0,\infty)] = \{0\}$. The same result holds for $[C, \ell^1(\mathbb{N})]$. Some results for the sequence spaces can be also found, for C and C^*, see e.g., [10, 12, 27]. Lee in [54] dropped the positivity assumption on a sequence x for which $Cx \in \ell^p(\mathbb{N})$;

$$\mathrm{Dom}[C, \ell^p(\mathbb{N})] = \{x = (x_i)_{i\in\mathbb{N}} : Cx \in \ell^p(\mathbb{N})\},$$

and called it the non-absolute domain of C on ℓ^p-space.

Recently, Barza and Soria in [9] has studied conditions for a general sequence x such that $(C-I)x$ and $(C^*-I)x$ belong to an r.i. space X. It is known that the class of real sequences $x=(x_n)$ such that $(C-I)x \in X$ is larger than X, whenever $C : X \to X$ and the $C^* : X \to X$ are bounded [9, Lemma 2.2]. That is, for an r.i. space X if $C : X \to X$ and $C^* : X \to X$ are bounded,

$$\mathrm{Dom}[C-I, X] = X + \mathbb{R} \qquad \text{and} \qquad \mathrm{Dom}[C^*-I, X] = X,$$

Since $C, C^* : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ are bounded for $1 < p < \infty$, they deduce from the above general result that $[C-I, \ell^p(\mathbb{N})] = \ell^p(\mathbb{N})+\mathbb{R}$, $[C^*-I, \ell^p(\mathbb{N})] = \ell^p(\mathbb{N})$ for $1 < p < \infty$, cf. [9, Theorem 2.6 and 4.3]. They showed also that $[C-I, \ell^p(\mathbb{N})], 1 \le p \le \infty$, is not a space satisfying the lattice property [9, Remark 2.7].

1.2.2 Boundary value problems and integral equations

The boundary-integral equation (BIE) method also known as boundary-element method or elastic potential method has been intensively developed over recent decades both in theory and in engineering applications. Its popularity was due to reducing a boundary-value problem (BVP) for a partial differential equation (PDE) in a domain to an integral equation on the domain boundary, that is, to diminishing the problem dimensionality by one. The main ingredient necessary for the reduction of a BVP to a BIE is a fundamental solution to the original PDE.

Partial differential equations with variable coefficients often arise in mathematical modeling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogenity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows through porous media, and other areas of physics and engineering. Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of BVPs for such PDEs to explicit BIEs, which could be effectively solved numerically. However, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs) for further analysis and numerical solution of the latter, see e.g. [19, 21, 57, 64, 58, 59] and references therein.

The theory of BDIEs for the Dirichlet, Neumann and mixed BVPs for a second order elliptic PDE with variable coefficient

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[a(x) \frac{\partial u(x)}{\partial x_i} \right] = f(x), \tag{DE}$$

is well developed in 3D, see, e.g.,[57, 59, 19, 18, 20, 21, 22, 62, 63]. Recently, in [1] only the BDIEs for the mixed BVP for Helmholtz equation

$$Au(x) + k^2(x)u(x) = f(x) \tag{HE}$$

are formulated in 3D for numerical approximation of its solution.

The BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs.

Boundary-domain integral equations for variable-coefficient BVPs associated with (DE) in two-dimensional bounded domain are investigated in [5, 6, 31]. But this is not the case for BDIEs for variable-coefficient BVPs for Helmholtz equation (HE) in 2D.

1.3 Statement of the problem

In this thesis, two distinct, yet related problems in integral equations and inequalities are investigated.

The authors in [15, 16] have found out the optimal constants in the inequalities

$$\|Hf\|_{L^\infty(t^\alpha)} \leq K_\alpha \|H^* f\|_{L^\infty(t^\alpha)} \text{ and } \|H^* f\|_{L^\infty(t^\beta)} \leq K_\beta \|Hf\|_{L^\infty(t^\beta)} \tag{33}$$

for $\alpha, \beta \in \mathbb{R}$ and positive function f. It has been found out that

$$K_\alpha = \begin{cases} 0, & \alpha < 0, \\ \dfrac{1}{1-\alpha}, & 0 \leq \alpha < 1, \\ \infty, & \alpha \geq 1, \end{cases} \quad \text{and} \quad K_\beta = \begin{cases} \infty, & \beta \leq 0, \\ \dfrac{1}{\beta}, & 0 < \beta \leq 1, \\ 0, & \beta > 1, \end{cases} \tag{34}$$

and these estimates are sharp. In this thesis, we study the problem of finding the best possible constants for discrete counterparts of (33) and also extended it for general weighted Lebesgue spaces.

Moreover, inspired by the results obtained in [9] about the conditions for a general sequence x such that $(C - I)x$ and $(C^* - I)x$ belong to an r.i. space X we find the analogous results for the continuous case; the optimal non-absolute domains for the Hardy operator minus identity on a BFS X, and on the Lebesgue spaces $L^p, 1 \leq p \leq \infty$.

Furthermore, the boundary-domain integral equations (BDIEs) for variable-coefficient BVPs associated with (DE) in two-dimensional bounded domain are investigated in [5, 6, 31]. But this is not the case for BDIEs for variable-coefficient BVPs for Helmholtz equation (HE) in 2D. Therefore, in this thesis we formulate and analyze the BDIEs for variable-coefficient Helmholtz BVPs associated with (HE) in 2D bounded domain.

1.4 Objectives of the thesis

1.4.1 General objective

The general objective of this thesis is to study and investigate some Hardy-type inequalities and integral equations of some variable-coefficient BVPs.

1.4.2 Specific objectives

The specific objectives of this thesis are to:

- determine the best possible constant for the inequalities involving Cesàro and Copson operators.

- find the optimal non-absolute domain of the Hardy operator minus identity.
- derive and investigate the boundary-domain integral equations for the variable-coefficient Helmholtz BVPs in two-dimensional bounded domain.

1.5 Structure of the dissertation

The remaining part of the dissertation is organized as follows: Subsection **??** describes the main results of the three papers. Chapter 2 contains Paper I which deals with the discrete analogue of the continuous results obtained in [15, 16] and extended them to general weights. Conversely, the continuous analogue of the discrete results found in [9] is studied in Chapter 3, that is, paper II. In Chapter 4 which consists of paper III, we have derived and analyzed the BDIEs for the variable coefficient Helmholtz BVPs in two dimensional bounded domain which is the extension of the analysis made for the operator A in (DE), see e.g., [6, 31]. Below we provide short description of the main results of the papers.

1.5.1 Paper I

In paper I, we studied the problem of finding the best possible constants for discrete counterparts of (33) for $p = \infty$ and also extended it for general weighted Lebesgue spaces. That is, finding the smallest constant $A \in [0, \infty]$ such that

$$\|Cx\|_{\ell^\infty(v)} \leq A\|C^*x\|_{\ell^\infty(w)} \quad \text{and} \quad \|C^*x\|_{\ell^\infty(v)} \leq A\|Cx\|_{\ell^\infty(w)} \tag{35}$$

for non-negative weights v and w, for any real sequence x or non-negative real sequence x or monotone sequence x. This is just to find the sharp constants α_∞ and β_∞ of (27) and (28) on ℓ^∞-space for all real sequences or non-negative real sequences and extend it for general weights.

The techniques developed to achieve our goal are as follows. First we proved the reduction theorem which states the equivalence of the inequalities relating the two operators C and C^* (57) with a corresponding single operator inequalities for appropriate sequence of real numbers. Having obtained the exact operator norm of some matrix operators between weighted ℓ^{∞}-spaces, we have found the smallest constants $A \in [0, \infty]$ for (57), for general weights and power weights. We have also analyzed the sharp constants obtained for power weights for non-negative sequences with the result obtained for (33) for positive functions shown in (34).

As a by product, we have found the operator norm of C, C^*, $C - I$ and $C^* - I$, where I is the identity operator, for different cones of sequences in ℓ^{∞}-spaces for general weights as well as power weights. For a real sequence x, the least possible constant $A = 2$ obtained for $C - I$ for non-weighted case confirms the result obtained in [39, Proposition 1]. Moreover, the operator norm of $C - I$ for non-negative sequences, for non-increasing sequences and for real sequences coincides respectively with the results obtained for $H - I$ for positive functions, for decreasing functions and for general functions [15].

1.5.2 Paper II

The authors in [9] have studied conditions for a general sequence x such that $(C - I)x$ and $(C^* - I)x$ belong to an r.i. space X;

$$\mathrm{Dom}[C - I, X] = \{x : (Cx - x) \in X\},$$
$$\mathrm{Dom}[C^* - I, X] = \{x : (C^*x - x) \in X\}.$$

They have found out that, for an r.i. space X

$$\mathrm{Dom}[C - I, X] = X + \mathbb{R} \qquad \text{and} \qquad \mathrm{Dom}[C^* - I, X] = X,$$

Whenever $C : X \to X$ and $C^* : X \to X$ are bounded [9, Lemma 2.2]. In particular, they deduced from the above general result that $[C - I, \ell^p(\mathbb{N})] = \ell^p(\mathbb{N}) + \mathbb{R}$, $[C^* - I, \ell^p(\mathbb{N})] = \ell^p(\mathbb{N})$ for $1 < p < \infty$, cf. [9, Theorem 2.6 and 4.3]. Barza and Soria showed also that $[C - I, \ell^p(\mathbb{N})], 1 \leq p \leq \infty$, is not a space satisfying the lattice property [9, Remark 2.7].

The findings in paper II essentially contributes with new information concerning optimal non-absolute domains for the Hardy operator minus identity on a BFS X, and on the Lebesgue spaces $L^p, 1 \leq p \leq \infty$.

We were inspired by the results found in [9] and obtained analogous results for the continuous case; the optimal non-absolute domain of $H - I$;

$$\mathrm{Dom}[H - I, X] = \{f : (Hf - f) \in X\},$$

where X is a Banach function space, and analogously for $\mathrm{Dom}[H^* - I, X]$. The non-absolute domain emphasizes that the analysis does not rely on the absolute values of a function f. Instead, the focus is on conditions that allow f to belong to the domain. For a Banach function space X, it is shown that

$$\begin{aligned} \mathrm{Dom}[H - I, X] &= X + \ker(H - I), \\ \text{and} \qquad \mathrm{Dom}[H^* - I, X] &= X + \ker(H^* - I) \end{aligned}$$

whenever $H : X \to X$ and $H^* : X \to X$ are bounded. We deduced that

$$\begin{aligned} \mathrm{Dom}[H - I, L^p(0,\infty)] &= L^p(0,\infty) + \ker(H - I), \\ \text{and} \qquad \mathrm{Dom}[H^* - I, L^p(0,\infty)] &= L^p(0,\infty) + \ker(H^* - I) \end{aligned}$$

for $1 < p < \infty$. We have found also that

$$\mathrm{Dom}[H - I, L^1(0,\infty)] = \Big(\mathrm{Dom}[H, L^1(0,\infty)] \cap L^1_0(0,\infty)\Big) + \ker(H - I),$$

where $L^1_0 = \{f \in L^1(0,\infty) : \int_0^\infty f(x)dx = 0\}$, and the following inclusions:

$$\begin{aligned} L^\infty(0,\infty) &\subsetneq \mathrm{Dom}(H - I, L^\infty(0,\infty)], \\ L^1(0,\infty) &\subsetneq \mathrm{Dom}[H^* - I, L^1(0,\infty)]. \end{aligned}$$

Finally, we have shown that the spaces $\mathrm{Dom}[H - I, L^p(0,\infty)]$ and $\mathrm{Dom}[H^* - I, L^p(0,\infty)]$ for $1 \leq p \leq \infty$ do not satisfy the lattice property.

1.5.3 Paper III

The boundary-integral equation (BIE) method also known as boundary-element method or elastic potential method has been intensively developed over recent decades both in theory and in engineering applications. Its popularity was due to reducing a boundary-value problem (BVP) for a partial differential equation (PDE) in a domain to an integral equation on the domain boundary, that is, to diminishing the problem dimensionality by one. The main ingredient necessary for the reduction of a BVP to a BIE is a fundamental solution to the original PDE [57].

PDEs with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogenity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows through porous media, and other areas of physics and engineering. For such PDEs a fundamental solution is generally not available in explicit form, preventing reduction of BVPs for such PDEs to explicit BIEs. However, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit 'parametrix' associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs), for further analysis and numerical solution of the latter, see e.g. [19, 21, 57, 59, 64] and the references therein. These equations are well studied for Dirichlet, Neumann and Mixed (Dirichlet-Neumann) BVPs for variable-coefficient second order scalar "steady-state heat transfer" equation (DE) in 3D, see e.g., [19, 20, 21, 22, 59, 62, 63]. BDIEs in 2D need a special consideration due to their different equivalence properties. Therefore, we need to set conditions on the associated Sobolev spaces or choose appropriate scaling parameter in the parametrix form to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs.

Considerable investigations of BDIEs for variable-coefficient BVPs associated with a second order scalar "steady-state heat transfer" equation (DE) in two-dimensional domain have been made in [5, 6, 31]. However, this is not the case for the parametrix-based system of BDIEs for variable-coefficient Helmholtz equation. In this paper III, the Helmholtz equation

$$Au(x) + k^2(x)u(x) = f(x), \quad x \in \Omega \subseteq \mathbb{R}^2 \tag{HE}$$

is considered, where

$$A = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big[a(x)\frac{\partial}{\partial x_i}\Big], \tag{DO}$$

$k(x)$ is a real function of x, $a(x)$ is a known variable coefficient, u is unknown function and $f \in L^2(\Omega)$ is a given function in Ω. We assume that $a, k \in C^\infty(\overline{\Omega})$ and $0 < a_0 < a(x) < a_1 < \infty$ for some constants a_0 and a_1, for all $x \in \Omega$.

Using appropriate parametrix (Levi function) and applying previously developed techniques for (DE), we shall formulate and investigate BDIE systems for the following BVPs associated with PDE (HE) in appropriate function spaces in 2D.

Dirichlet BVP: Find a function $u \in H^1(\Omega)$ satisfying the conditions

$$\begin{aligned} A_k u &= f && \text{in } \Omega, \\ \gamma^+ u &= \varphi_0 && \text{on } \partial\Omega, \end{aligned} \tag{DP}$$

where $f \in L^2(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ are given functions. The first equation is understood in the distribution sense.

Mixed BVP: Let $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$, where $\partial\Omega_D$ and $\partial\Omega_N$ are non-empty and non-intersecting parts of $\partial\Omega$. We find a function $u \in H^1(\Omega)$ such that

$$\begin{aligned} A_k u &= f && \text{in } \Omega, \\ \gamma^+ u &= \varphi_0 && \text{on } \partial\Omega_D, \\ T^+ u &= \psi_0 && \text{on } \partial\Omega_N, \end{aligned} \tag{MP}$$

where $f \in L^2(\Omega)$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ are given functions. In both BVPs, the first equation is understood in the distribution sense.

The Dirichlet BVP can be naturally reduced to two different BDIE systems, depending on whether the trace or co-normal derivative of the third Green identity is employed on the boundary. On the other hand, the mixed BVP can be naturally reduced to four different BDIE systems, depending on whether the trace or co-normal derivative of the third Green identity is employed on the Dirichlet and Neumann boundaries.

In this paper, the equivalence of the BVPs with the formulated system of BDIEs are shown. Moreover, the Fredeholm properties, unique solvability, and invertibility of the BDIE systems are investigated in appropriate Sobolev spaces.

2 End-point norm estimates for Cesàro and Copson operators

2.1 Introduction

The Cesàro matrix, C, and its transpose the Copson matrix, C^*, are

$$C = \begin{pmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^* = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & & \frac{1}{3} & \frac{1}{4} & \cdots \\ & & & \frac{1}{4} & \cdots \\ & & & & \ddots \end{pmatrix}.$$

The same names denote the operators associated with these infinite matrices, defined by

$$(Cx)_n = \frac{1}{n}\sum_{k=1}^{n} x_k \quad \text{and} \quad (C^*x)_n = \sum_{k=n}^{\infty} \frac{x_k}{k}$$

for x an appropriate real sequence. The motivation for this work is to determine best constants in the weighted two-operator inequalities

$$\|Cx\|_{\ell^\infty(v)} \leq A\|C^*x\|_{\ell^\infty(1/u)} \quad \text{and} \quad \|C^*x\|_{\ell^\infty(v)} \leq A\|Cx\|_{\ell^\infty(1/u)} \tag{36}$$

for all x and also for all nonnegative x.

The Cesàro and Copson operators, together with their integral analogues,

$$Hf(x) = \int_0^x f(t)\,dt \quad \text{and} \quad H^*f(x) = \int_x^\infty f(t)\frac{dt}{t},$$

appear throughout classical and modern analysis. They were already standard tools in Fourier analysis when Hardy used them to give a simple proof of Hilbert's double series theorem from complex analysis. They serve as base cases and motivating examples for summability, positive operators, convolution inequalities, interpolation of operators, maximal functions and more.

Most relevant to our study, is their appearance in the theory of weighted norm inequalities. A remarkable array of techniques have been tried out on these operators for the first time and often the results set the standard for subsequent progress.

Recently, techniques for determination of exact operator norms, exact distances between operators, and best constants in two-operator inequalities have been worked out using H, H^*, C and C^* as motivating examples.

For the operators H and H^* a great deal of progress has been made in recent years. We refer to [71] and [72], which, besides establishing the current best results for exact operator norms, include in their introductions detailed accounts of recent work. The contributions of Boza and Soria deserve special mention, recently from [15] and [16], but going back to [14]. In the first, they make a clear case for the independent study of restrictions of operators to cones of monotone functions. In the second, they point out the significance of understanding the action of operators in endpoint cases, i.e., the $p = 1$, $p = \infty$, and weak type cases among ℓ^p spaces.

For the operators C and C^*, exact norms, distances and constants had already found an important place in Bennett's 1996 memoir [10]. Some were proved and others were left as open problems. A few of the open problems have settled quite recently, see [39, 46, 68].

Our focus on weighted ℓ^∞ spaces puts us firmly in the endpoint case, and greatly simplifies norm estimates. On the other hand our results apply for general weight sequences, something which is beyond the current reach when seeking exact operator norms in the ℓ^p spaces for $1 < p < \infty$. We also consider the restrictions of operators to cones of monotone sequences, something of proven value.

Our approach is in two steps. First, we reduce the best constant problems for the two-operator inequalities (36) for general x or for nonnegative x to the determination of the operator norm of a related matrix operator on a related cone of sequences. See Theorems 2.5 and 2.6. Second, we prove and apply a result on matrix operator norms between cones in weighted ℓ^∞ spaces that is general enough to include the ones we need to solve the best constant problems. See Theorem 2.9. This result is of independent interest and we apply it to give the operator norms of a number of related matrix operators that have appeared in recent literature. Here the operators $C - I$ and $C^* - I$ figure prominently. The results of our analysis of (36) are in Theorem 2.20.

The most commonly studied and applied weight sequences are the power weights. We illustrate our results throughout by giving concrete expressions for the best constants in the case of power weighted ℓ^∞. See Theorems 2.11 and 2.13 for exact operator norms for C and C^* on all four cones. See 2.15 for exact distances from C to the identity on all four cones. The exact distance from C^* to the identity is given in 2.17 on two of the cones. (The case of nondecreasing sequences is trivial and the case of nonincreasing sequences remains open.) The best constants in the two-operator inequalities are given in 2.21 and 2.22 on the cone of all sequences and on the cone of all nonnegative sequences.

2.1.1 Notation and Definitions

For an infinite matrix to represent an operator on sequences, we have to decide in what sense the sums involved in matrix multiplication should converge.

Definition 2.1. Let $B = (b_{n,k})$ be a real matrix. The domain, denoted $\mathcal{D}(B)$, of the associated matrix operator is the set of all real sequences x such that for each n, the sum $\sum_{k=1}^{\infty} b_{n,k}x_k$ converges to a real number. For $x \in \mathcal{D}(B)$, we define the sequence Bx by setting $(Bx)_n = \sum_{k=1}^{\infty} b_{n,k}x_k$.

If all entries of B and x are nonnegative, we extend this definition to permit $(Bx)_n$ to take the value ∞.

This definition gives us larger domains than if we insisted on absolute convergence in all matrix sums. It means that our matrix operators do not correspond to standard integral operators as well as they correspond to operators defined by principal value integrals.

Besides C and C^* we will encounter the matrices I, S, S^*, D and E. The first three are standard, the identity matrix, the right shift (with ones on the subdiagonal) and the left shift (with ones on the superdiagonal.) The other two are defined by

$$D = \begin{pmatrix} \frac{1}{2} & & & & \\ & \frac{1}{3} & & & \\ & & \frac{1}{4} & & \\ & & & \frac{1}{5} & \\ & & & & \ddots \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

so that $(Dx)_n = x_n/(n+1)$ and $(Ex)_n = \sum_{k=1}^n x_k$. The domain of each these matrix operators, with the exception of C^*, consists of all real sequences. Evidently, $\mathcal{D}(C^*)$ consists of all real sequences x for which $\sum_{k=1}^\infty \frac{x_k}{k}$ converges in $\mathbb{R}$.

Let ℓ, ℓ^+, $\ell^\downarrow$ and $\ell^\uparrow$ denote, respectively, the set of all sequences of real numbers, the set of all sequences of nonnegative real numbers, the set of all nonincreasing sequences of nonnegative real numbers, and the set of all nondecreasing sequences of nonnegative real numbers.

For weights $u, v \in \ell^+$ and for any $x, y \in \ell$ we define

$$\|y\|_{\ell^\infty(v)} = \sup_n |y_n| v_n \quad \text{and} \quad \|x\|_{\ell^\infty(1/u)} = \sup_k |x_k|/u_k.$$

The two definitions agree, except that if $u_k = 0$ for some k, the sequence $(1/u_1, 1/u_2, \dots)$ is not in ℓ^+. In this case we apply the convention $0/0 = 0$: If $u_k = 0$ for some k then $|x_k|/u_k = \infty$ when $x_k \neq 0$ and $|x_k|/u_k = 0$ when $x_k = 0$. Note that we permit these weighted "norms" to take the value ∞.

For a real number x, let $x^+ = (|x| + x)/2 \geq 0$ and $x^- = (|x| - x)/2 \geq 0$. Note that $x = x^+ - x^-$. This notation extends termwise to sequences and entrywise to matrices: If $x = (x_n)$ is a real sequence, then $x^+ = ((x_n)^+)$, $x^- = ((x_n)^-)$, and $|x| = (|x_n|)$. If $B = (b_{n,k})$ then $B^+ = ((b_{n,k})^+)$, $B^- = ((b_{n,k})^-)$ and $|B| = (|b_{n,k}|)$.

For $u \in \ell^+$ we define the greatest nonincreasing minorant $u^\downarrow$ of u and the greatest nondecreasing minorant $u^\uparrow$ of u by

$$u_k^\downarrow = (u^\downarrow)_k = \min_{j \leq k} u_j \quad \text{and} \quad u_k^\uparrow = (u^\uparrow)_k = \inf_{j \geq k} u_j.$$

Their relevance emerges from the following simple observation.

Lemma 2.2. *Let* $u \in \ell^+$.

(i) If $x \in \ell^\downarrow$, *then* $\|x\|_{\ell^\infty(1/u)} = \|x\|_{\ell^\infty(1/u^\downarrow)}$;

(ii) If $x \in \ell^\uparrow$, *then* $\|x\|_{\ell^\infty(1/u)} = \|x\|_{\ell^\infty(1/u^\uparrow)}$;

(iii) If $x \in \ell^\downarrow$ *and* $x \le u$ *then* $x \le u^\downarrow$.

(iv) If $x \in \ell^\uparrow$ *and* $x \le u$ *then* $x \le u^\uparrow$.

Proof. First observe that, since $u^\downarrow \le u$ and $u^\uparrow \le u$,

$$\|x\|_{\ell^\infty(1/u)} \le \|x\|_{\ell^\infty(1/u^\downarrow)} \quad \text{and} \quad \|x\|_{\ell^\infty(1/u)} \le \|x\|_{\ell^\infty(1/u^\uparrow)}.$$

Let $x \in \ell^\downarrow$. For each k,

$$\frac{x_k}{u_k^\downarrow} = \max_{j \le k} \frac{x_k}{u_j} \le \max_{j \le k} \frac{x_j}{u_j} \le \|x\|_{\ell^\infty(1/u)}.$$

Take the supremum over all k to get $\|x\|_{\ell^\infty(1/u^\downarrow)} \le \|x\|_{\ell^\infty(1/u)}$.

Let $x \in \ell^\uparrow$. For each k,

$$\frac{x_k}{u_k^\uparrow} = \sup_{j \ge k} \frac{x_k}{u_j} \le \sup_{j \ge k} \frac{x_j}{u_j} \le \|x\|_{\ell^\infty(1/u)}.$$

Take the supremum over all k to get $\|x\|_{\ell^\infty(1/u^\uparrow)} \le \|x\|_{\ell^\infty(1/u)}$.

If $x \in \ell^\downarrow$ and $x \le u$, then $\|x\|_{\ell^\infty(1/u)} \le 1$ so $\|x\|_{\ell^\infty(1/u^\downarrow)} \le 1$ and therefore $x \le u^\downarrow$. If $x \in \ell^\uparrow$ and $x \le u$, then $\|x\|_{\ell^\infty(1/u)} \le 1$ so $\|x\|_{\ell^\infty(1/u^\uparrow)} \le 1$ and therefore $x \le u^\uparrow$. □

2.2 Two Identities

In this section we use two matrix identities to connect the inequalities (36) to norm inequalities for related operators. The identities are

$$C = (C - S^*)C^* \quad \text{and} \quad C^* = (C^* - S)DE.$$

In Section 10 of [10], Bennett uses the first identity and one closely related to the second, namely, $C^* = (C^* - I)SC$, to explore two-operator inequalities involving C and C^*. Either of the two second identities would suffice in this analysis; our aim was to simplify intermediate results.

In matrix form, the identity $(C - S^*)C^* = C$ may be written as

$$\begin{pmatrix} 1 & -1 & & & & \\ \frac{1}{2} & \frac{1}{2} & -1 & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & & \frac{1}{3} & \frac{1}{4} & \cdots \\ & & & \frac{1}{4} & \cdots \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Viewed as an operator identity we can prove that it is valid on $\mathcal{D}(C^*)$, the domain of the operator C^*.

Lemma 2.3. *If $x \in \mathcal{D}(C^*)$ then $(C - S^*)C^*x = Cx$.*

Proof. Let $x \in \mathcal{D}(C^*)$ and set $y = C^*x$ to see that $(Cx)_n$ is equal to

$$\frac{1}{n}\sum_{j=1}^{n}\frac{x_j}{j}\sum_{k=1}^{j}1 = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=k}^{n}\frac{x_j}{j} = \frac{1}{n}\sum_{k=1}^{n}(y_k - y_{n+1}) = (Cy)_n - y_{n+1}.$$

Therefore $(Cx)_n = ((C - S^*)y)_n = ((C - S^*)C^*x)_n$ for all n. □

The second identity is a bit more complicated because the matrix multiplication involves infinite sums and extra care has to be taken with the domain of the matrix operators.

In matrix form, the identity $(C^* - S)DE = C^*$ may be written as

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ -1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & -1 & \frac{1}{3} & \frac{1}{4} & \cdots \\ & & -1 & \frac{1}{4} & \cdots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \frac{1}{2} & & & & \\ \frac{1}{3} & \frac{1}{3} & & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & & \frac{1}{3} & \frac{1}{4} & \cdots \\ & & & \frac{1}{4} & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Next we show that it is also an operator identity on $\mathcal{D}(C^*)$.

Lemma 2.4. *For $x \in \ell$, $x \in \mathcal{D}(C^*)$ if and only if*

$$Ex \in \mathcal{D}(C^*D) \quad \textit{and} \quad (DEx)_N \to 0 \textit{ as } N \to \infty.$$

In this case $(C^* - S)DEx = C^*x$.

Proof. Fix a real sequence x and a positive integer n. If $N < n$, then

$$\sum_{j=1}^{N} x_j \Big(\frac{1}{N+1} + \sum_{k=\max(n,j)}^{N} \frac{1}{k(k+1)} \Big) = (DEx)_N + \sum_{k=n}^{N} \frac{1}{k(k+1)} \sum_{j=1}^{k} x_j,$$

which telescopes to

$$\sum_{j=n}^{N} \frac{x_j}{j} + \frac{1}{n} \sum_{j=1}^{n-1} x_j = \sum_{j=1}^{N} \frac{x_j}{\max(n,j)} = (DEx)_N + \sum_{k=n}^{N} \frac{(DEx)_k}{k}. \tag{37}$$

Suppose $Ex \in \mathcal{D}(C^*D)$ and $(DEx)_N \to 0$ as $N \to \infty$. Then the right-hand side of (37) converges as $N \to \infty$. So does the left-hand side, so $x \in \mathcal{D}(C^*)$.

Conversely, suppose $x \in \mathcal{D}(C^*)$ and let $N \to \infty$ in (37). Since the left-hand side converges, so does the right-hand side. Setting $y = C^*x$ we get $y_N \to 0$. It follows that the averages $(Cy)_N \to 0$ and the shifts $(S^*y)_N \to 0$. Now Lemma 2.3 shows that $(Cx)_N = (Cy)_N - (S^*y)_N \to 0$. But $(DEx)_N = \frac{N}{N+1}(Cx)_N$ so $(DEx)_N \to 0$. Since the first term of the right-hand side of (37) converges, so does the second term. It follows that $Ex \in \mathcal{D}(C^*D)$, which completes the equivalence.

Letting $N \to \infty$, (37) becomes $(C^*x)_n + (SDEx)_n = (C^*DEx)_n$. Since n was arbitrary, $(C^* - S)DEx = C^*x$. □

These two identities are the keys to proving the following two theorems that reduce inequalities relating C and C^* to inequalities involving a single operator.

Theorem 2.5. *Let $u, v \in \ell^+$ and $A \in [0, \infty)$. Then (38) if only if (39) and (40) if only if (41), where*

$$\|Cx\|_{\ell^\infty(v)} \le A\|C^*x\|_{\ell^\infty(1/u)} \quad \textit{for } x \in \mathcal{D}(C^*); \tag{38}$$

$$\|(C - S^*)y\|_{\ell^\infty(v)} \le A\|y\|_{\ell^\infty(1/u)} \quad \textit{for } y \in \ell, y_n \to 0; \tag{39}$$

$$\|Cx\|_{\ell^\infty(v)} \le A\|C^*x\|_{\ell^\infty(1/u)} \quad \textit{for } x \in \ell^+ \cap \mathcal{D}(C^*); \tag{40}$$

$$\|(C - S^*)y\|_{\ell^\infty(v)} \le A\|y\|_{\ell^\infty(1/u)} \quad \textit{for } y \in \ell^\downarrow, y_n \to 0. \tag{41}$$

Proof. Let $x \in \mathcal{D}(C^*)$, set $y = C^*x$, and note that $y_n \to 0$ as $n \to \infty$. If (39) holds, then by Lemma 2.3,

$$\|Cx\|_{\ell^\infty(v)} = \|(C - S^*)y\|_{\ell^\infty(v)} \leq A\|y\|_{\ell^\infty(1/u)} = A\|C^*x\|_{\ell^\infty(1/u)},$$

so (38) holds. If (41) holds and $x \in \ell^+ \cap \mathcal{D}(C^*)$, then $y \in \ell^\downarrow$ so the same estimate gives (40).

Now let $y \in \ell$ with $y_n \to 0$ and set $x_k = k(y_k - y_{k+1})$. We have

$$\sum_{k=n}^{N} \frac{x_k}{k} = \sum_{k=n}^{N} (y_k - y_{k+1}) = y_n - y_N \to y_n$$

as $N \to \infty$. So $x \in \mathcal{D}(C^*)$ and $C^*x = y$. If (38) holds, then Lemma 2.3 shows

$$\|(C - S^*)y\|_{\ell^\infty(v)} = \|Cx\|_{\ell^\infty(v)} \leq A\|C^*x\|_{\ell^\infty(1/u)} = A\|y\|_{\ell^\infty(1/u)},$$

so (39) holds. If (40) holds, and $y \in \ell^\downarrow$ with $y_n \to 0$, then $x \in \ell^+$ and the same estimate gives (41). □

Theorem 2.6. *Let* $u, v \in \ell^+$ *and* $A \in [0, \infty)$. *Set* $w_k = ku_k$ *for each* k. *Then* (42) *if only if* (43) *and* (44) *if only if* (45), *where*

$$\|C^*x\|_{\ell^\infty(v)} \leq A\|Cx\|_{\ell^\infty(1/u)}, \quad x \in \mathcal{D}(C^*); \tag{42}$$

$$\|(C^* - S)Dz\|_{\ell^\infty(v)} \leq A\|z\|_{\ell^\infty(1/w)}, \quad z \in \mathcal{D}(C^*D), (Dz)_n \to 0; \tag{43}$$

$$\|C^*x\|_{\ell^\infty(v)} \leq A\|Cx\|_{\ell^\infty(1/u)}, \quad x \in \ell^+ \cap \mathcal{D}(C^*); \tag{44}$$

$$\|(C^* - S)Dz\|_{\ell^\infty(v)} \leq A\|z\|_{\ell^\infty(1/w)}, \quad z \in \ell^\uparrow \cap \mathcal{D}(C^*D), (Dz)_n \to 0. \tag{45}$$

Proof. Let $x \in \mathcal{D}(C^*)$ and set $z = Ex$. The definition of E shows $\|z\|_{\ell^\infty(1/w)} = \|Cx\|_{\ell^\infty(1/u)}$. From Lemma 2.4 we get $(Dz)_n \to 0$, $z \in \mathcal{D}(C^*D)$ and $C^*x = (C^* - S)Dz$. If (43) holds, then

$$\|C^*x\|_{\ell^\infty(v)} = \|(C^* - S)Dz\|_{\ell^\infty(v)} \leq A\|z\|_{\ell^\infty(1/w)} = A\|Cx\|_{\ell^\infty(1/u)},$$

that is, (42) holds. If $x \in \ell^+$ then $z \in \ell^\uparrow$ so the same estimate shows that if (45) holds, so does (44).

Now let $z \in \mathcal{D}(C^*D)$ with $(Dz)_N \to 0$ as $N \to \infty$, set $z_0 = 0$ and $x_k = z_k - z_{k-1}$ for all k. Then $z = Ex$ so Lemma 2.4 shows that $x \in \mathcal{D}(C^*)$ and $(C^* - S)Dz = C^*x$. If (42) holds, then

$$\|(C^* - S)Dz\|_{\ell^\infty(v)} = \|C^*x\|_{\ell^\infty(v)} \leq A\|Cx\|_{\ell^\infty(1/u)} = A\|z\|_{\ell^\infty(1/w)},$$

so (43) holds. If (44) holds and $z \in \ell^\uparrow$, then $x \in \ell^+$ so the same estimate gives (45). This completes the proof.

2.3 Operator norms for some matrix operators on cones

The simple form of weighted ℓ^∞ norms permits direct computation of the norms of matrix operators from one weighted space to another and from the positive cone of one weighted space to another. For the cones of decreasing sequences and increasing sequences, the situation is more delicate but for each of these cones we identify a class of matrix operators for which it simplifies nicely. The operators involved in our analysis of the inequalities in (36) are in those classes.

Definition 2.7. Let $b \in \ell$. We say that b has *positives before negatives* provided that for all $j, k \in \mathbb{Z}^+$, $b_j > 0 > b_k$ only if $j < k$. We say that b has *negatives before positives* if $-b$ has positives before negatives.

Note that if b has positives before negatives or has negatives before positives then $\sum_{k=1}^N b_k$ is a monotone function of N for sufficiently large N so the sum $\sum_{k=1}^\infty b_k$, exists in $[-\infty, \infty]$. We call it the *sum of* b.

Let $u, v \in \ell^+$. For a matrix B, let $A(B)$, $A^+(B)$, $A^\downarrow(B)$ and $A^\uparrow(B)$ denote the smallest constant $A \in [0, \infty]$ such that inequality

$$\|Bx\|_{\ell^\infty(v)} \leq A\|x\|_{\ell^\infty(1/u)} \tag{46}$$

holds for all $x \in \mathcal{D}(B)$, $x \in \ell^+ \cap \mathcal{D}(B)$, $x \in \ell^\downarrow \cap \mathcal{D}(B)$, and $x \in \ell^\uparrow \cap \mathcal{D}(B)$, respectively. In the next theorem we may need to apply the convention $\infty \cdot 0 = 0$.

Remark 2.8. Multiplying the matrix B on the left by a complex diagonal matrix has no effect on the left-hand side of (46), provided the weight sequence v

is adjusted appropriately. This simple observation substantially extends the applicability of the next theorem. Rather than unduly complicate its statement, we trust that, in applications, suitable row-by-row "preprocessing" will have been carried out to ensure that the hypotheses of the theorem are satisfied. One simple form of this preprocessing allows some subset of the rows of a real matrix B to be multiplied by -1 to permit the use of parts (iii) or (iv) of the theorem.

Some expressions in what follows need to be understood according to the convention $\infty \cdot 0 = 0$.

Theorem 2.9. *Let B be a matrix with real entries.*

(i) The least $A \in [0, \infty]$ such that (46) holds for all $x \in \mathcal{D}(B)$ is

$$A(B) = \| |B| u \|_{\ell^\infty(v)}.$$

(ii) The least $A \in [0, \infty]$ such that (46) holds for all $x \in \ell^+ \cap \mathcal{D}(B)$ is

$$A^+(B) = \max(\|B^+ u\|_{\ell^\infty(v)}, \|B^- u\|_{\ell^\infty(v)}).$$

(iii) Suppose each row of B has positives before negatives and a nonnegative sum. The least $A \in [0, \infty]$ such that (46) holds for all $x \in \ell^\downarrow \cap \mathcal{D}(B)$ is

$$A^\downarrow(B) = \|B^+(u^\downarrow)\|_{\ell^\infty(v)}.$$

(iv) Suppose each row of B has negatives before positives and a nonnegative sum. If each row of B has a finite sum then the least $A \in [0, \infty]$ such that (46) holds for all $x \in \ell^\uparrow \cap \mathcal{D}(B)$ is

$$A^\uparrow(B) = \|B^+(u^\uparrow)\|_{\ell^\infty(v)}.$$

If some row of B has an infinite sum, then $\ell^\uparrow \cap \mathcal{D}(B) = \{0\}$ so (46) holds trivially with $A^\uparrow(B) = 0$.

Proof. Let ρ^+ be the result of replacing all nonzero entries of B^+ by 1 and let ρ^- be the result of replacing all nonzero entries of B^- by 1. Then $b_{n,k}\rho^+_{n,k} = b^+_{n,k}$ and $b_{n,k}\rho^-_{n,k} = -b^-_{n,k}$ for all n and k.

If $x \in \mathcal{D}(B)$ and $|x| \leq u$, then $\|x\|_{\ell^\infty(1/u)} \leq 1$ and, for each n, $|(Bx)_n| \leq (|B||x|)_n \leq (|B|u)_n$ so we get $\|Bx\|_{\ell^\infty(v)} \leq \||B|u\|_{\ell^\infty(v)}$. By positive homogeneity of the norm, $A(B) \leq \||B|u\|_{\ell^\infty(v)}$.

Fix n and K. Define a sequence x by setting $x_k = (\rho^+ - \rho^-)_{n,k}u_k$ if $k \leq K$ and $x_k = 0$ if $k > K$. Then $x \in \mathcal{D}(B)$ and $\|x\|_{\ell^\infty(1/u)} \leq 1$. Since $|b_{n,k}| = b_{n,k}(\rho^+ - \rho^-)_{n,k}$ for all k, we have

$$v_n \sum_{k=1}^{K} |b_{n,k}|u_k = v_n(Bx)_n \leq \|Bx\|_{\ell^\infty(v)} \leq A(B).$$

Letting $K \to \infty$ and taking the supremum over all n we get

$$\||B|u\|_{\ell^\infty(v)} = \sup_n v_n(|B|u)_n \leq A(B).$$

This proves (i).

If $x \in \ell^+ \cap \mathcal{D}(B)$ and $x \leq u$, then $\|x\|_{\ell^\infty(1/u)} \leq 1$ and, for each n, $0 \leq (B^+x)_n \leq (B^+u)_n$ and $0 \leq (B^-x)_n \leq (B^-u)_n$. If $(B^+u)_n$ and $(B^-u)_n$ are both finite, then

$$|(Bx)_n| = |(B^+x)_n - (B^-x)_n| \leq \max((B^+u)_n, (B^-u)_n),$$

an inequality that also holds if $(B^+u)_n = \infty$ or $(B^-u)_n = \infty$. Multiplying both sides by v_n, taking the supremum over all n, and interchanging the supremum and the maximum, yields

$$\|Bx\|_{\ell^\infty(v)} \leq \max(\|B^+u\|_{\ell^\infty(v)}, \|B^-u\|_{\ell^\infty(v)}).$$

By positive homogeneity of the norm,

$$A^+(B) \leq \max(\|B^+u\|_{\ell^\infty(v)}, \|B^-u\|_{\ell^\infty(v)}).$$

Fix n and K. First, define a sequence x by setting $x_k = \rho^+_{n,k} u_k$ if $k \le K$ and $x_k = 0$ if $k > K$. Then $x \in \mathcal{D}(B)$ and $\|x\|_{\ell^\infty(1/u)} \le 1$. Since $b^+_{n,k} = b_{n,k}\rho^+_{n,k}$,

$$v_n \Big| \sum_{k=1}^{K} b^+_{n,k} u_k \Big| = v_n |(Bx)_n| \le \|Bx\|_{\ell^\infty(v)} \le A^+(B).$$

Next, define a sequence x by setting $x_k = \rho^-_{n,k} u_k$ if $k \le K$ and $x_k = 0$ if $k > K$. Then $x \in \mathcal{D}(B)$ and $\|x\|_{\ell^\infty(1/u)} \le 1$. Since $b^-_{n,k} = -b_{n,k}\rho^-_{n,k}$,

$$v_n \Big| \sum_{k=1}^{K} b^-_{n,k} u_k \Big| = v_n |(Bx)_n| \le \|Bx\|_{\ell^\infty(v)} \le A^+(B).$$

Letting $K \to \infty$ and taking the supremum over all n in the two estimates above, we get

$$\max(\|B^+u\|_{\ell^\infty(v)}, \|B^-u\|_{\ell^\infty(v)}) \le A^+(B).$$

This proves (ii).

To prove (iii), suppose each row of B has positives before negatives and a nonnegative sum. Fix n and set $m = \sup\{k : b_{n,k} > 0\}$, taking $\sup \emptyset = 0$ if necessary. If $m = 0$, then $b_{n,k} \le 0$ for all k, but the nth row of B has a nonnegative sum so $b_{n,k} = 0$ for all k. If $m = \infty$, then $b_{n,k} \ge 0$ for all k because the nth row of B has positives before negatives. In the remaining case, $m \in \mathbb{Z}^+$, $b_{n,k} \ge 0$ for $k \le m$ and $b_{n,k} \le 0$ for $k > m$. This implies that for all $x \in \ell^\downarrow \cap \mathcal{D}(B)$, $b_{n,k} x_k \ge b_{n,k} x_m$ for all k and so $(Bx)_n \ge x_m \sum_{k=1}^\infty b_{n,k} \ge 0$, because the nth row of B has a nonnegative sum. In all three cases we get $(Bx)_n \ge 0$.

If $x \in \ell^\downarrow \cap \mathcal{D}(B)$ and $x \le u$, then $x \le u^\downarrow$ by Lemma 2.2. Therefore,

$$v_n |(Bx)_n| = v_n (Bx)_n \le v_n (B^+ x)_n \le v_n (B^+(u^\downarrow))_n.$$

Taking the supremum over all n, we get $\|Bx\|_{\ell^\infty(v)} \le \|B^+(u^\downarrow)\|_{\ell^\infty(v)}$. Positive homogeneity of the norm shows that $A^\downarrow(B) \le \|B^+(u^\downarrow)\|_{\ell^\infty(v)}$.

Fix n and K, and define m as above. Define x by setting $x_k = u^\downarrow_k$ if $k \le \min(m, K)$ and $x_k = 0$ otherwise. Then $x \in \ell^\downarrow \cap \mathcal{D}(B)$ and, by Lemma

2.2, $\|x\|_{\ell^\infty(1/u)} = \|x\|_{\ell^\infty(1/u^\downarrow)} \leq 1$. We have seen that $b^+_{n,k} = b_{n,k}$ for $k \leq m$ and $b^+_{n,k} = 0$ for $k > m$. Therefore,

$$v_n \sum_{k=1}^{K} b^+_{n,k} u^\downarrow_k = v_n (Bx)_n \leq \|Bx\|_{\ell^\infty(v)} \leq A^\downarrow(B).$$

Letting $K \to \infty$ we get

$$\|B^+(u^\downarrow)\|_{\ell^\infty(v)} \leq A^\downarrow(B).$$

To prove (iv), suppose each row of B has negatives before positives and a nonnegative sum. Fix n and set $m = \sup\{k : b_{n,k} < 0\}$, taking $\sup \emptyset = 0$ if necessary. If $m = \infty$, then $b_{n,k} \leq 0$ for all k because the nth row of B has negatives before positives, but the nth row of B has a nonnegative sum so $b_{n,k} = 0$ for all k. If $m = 0$, then $b_{n,k} \geq 0$ for all k. In the remaining case, $m \in \mathbb{Z}^+$, $b_{n,k} \leq 0$ for $k \leq m$ and $b_{n,k} \geq 0$ for $k > m$. This implies that for all $x \in \ell^\uparrow \cap \mathcal{D}(B)$, $b_{n,k} x_k \geq b_{n,k} x_m$ for all k and so $(Bx)_n \geq x_m \sum_{k=1}^\infty b_{n,k} \geq 0$, because the nth row of B has a nonnegative sum. In all three cases we get $(Bx)_n \geq 0$.

If $x \in \ell^\uparrow \cap \mathcal{D}(B)$ and $x \leq u$, then $x \leq u^\uparrow$ by Lemma 2.2. Therefore,

$$v_n |(Bx)_n| = v_n (Bx)_n \leq v_n (B^+ x)_n \leq v_n (B^+(u^\uparrow))_n.$$

Taking the supremum over all n, we get $\|Bx\|_{\ell^\infty(v)} \leq \|B^+(u^\uparrow)\|_{\ell^\infty(v)}$. Positive homogeneity of the norm shows that $A^\uparrow(B) \leq \|B^+(u^\uparrow)\|_{\ell^\infty(v)}$.

Case 1. Every row of B has a finite (nonnegative) sum. First we show that $\mathcal{D}(B)$ contains every nonnegative, bounded sequence. Suppose x is such a sequence and choose P so that $0 \leq x_k \leq P$ for all k. Fix n and let $m = \sup\{k : b_{n,k} < 0\}$ again. As we have seen, if $m = \infty$, then $b_{n,k} = 0$ for all k, so $\sum_{k=1}^\infty b_{n,k} x_k$ is trivially convergent. Otherwise, $\sum_{k=1}^K b_{n,k} x_k$ is a nondecreasing for $K > m$ and is bounded above by

$$\sum_{k=1}^{m} b_{n,k} x_k + P \sum_{k=m+1}^{\infty} b_{n,k} < \infty.$$

Again, $\sum_{k=1}^{\infty} b_{n,k}x_k$ is convergent. Since n was arbitrary, $x \in \mathcal{D}(B)$.

Now fix n and a real number P. Let $m = \sup\{k : b_{n,k} < 0\}$. Define x by setting $x_k = 0$ if $k \leq m$ and $x_k = \min(u_k^{\uparrow}, P)$ if $k > m$. Since x is bounded above by P, $x \in \ell^{\uparrow} \cap \mathcal{D}(B)$. Moreover, $|x| \leq u^{\uparrow}$ so $\|x\|_{\ell^\infty(1/u)} = \|x\|_{\ell^\infty(1/u^{\uparrow})} \leq 1$ by Lemma 2.2. Therefore,

$$v_n \sum_{k=1}^{\infty} b_{n,k}^{+} \min(u_k^{\uparrow}, P) = v_n \sum_{k=1}^{\infty} b_{n,k}x_k \leq \|Bx\|_{\ell^\infty(v)} \leq A^{\uparrow}(B).$$

Taking the limit as $P \to \infty$, we get

$$v_n \sum_{k=1}^{\infty} b_{n,k}^{+} u_k^{\uparrow} \leq A^{\uparrow}(B).$$

Taking the supremum over all n, we get $\|B^{+}(u^{\uparrow})\|_{\ell^\infty(v)} \leq A^{\uparrow}(B)$.

Case 2. For some n, the nth row of B has an infinite sum. Since its sum is nonnegative by hypothesis, the sum is ∞. Since this row has negatives before positives, there exists an m such that $b_{n,k} \geq 0$ when $k \geq m$. If $x \in \ell^{\uparrow}$ is not the zero sequence, then there exists a $K \geq m$ such that $x_K > 0$. Therefore,

$$\sum_{k=K}^{\infty} b_{n,k}x_k \geq x_K \sum_{k=K}^{\infty} b_{n,k} = \infty.$$

Thus, $x \notin \mathcal{D}(B)$. We conclude that $\ell^{\uparrow} \cap \mathcal{D}(B)$ contains only the zero sequence, so $A^{\uparrow}(B) = 0$. $\square$

2.3.1 The Cesàro and Copson operators

The Cesàro matrix C is nonnegative so all four parts of Theorem 2.9 apply.

Corollary 2.10. *Let $u, v \in \ell^{+}$. The inequality*

$$\sup_n \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right| v_n \leq A \sup_k \frac{|x_k|}{u_k}$$

holds for all real sequences x with $A = A(C)$; for all nonnegative sequences x with $A = A^{+}(C)$; for all nonnegative, nonincreasing sequences x with $A =$

$A^{\downarrow}(C)$*; and for all nonnegative, nondecreasing sequences* x *with* $A = A^{\uparrow}(C)$. *In each case the constant* A *is best possible. Here*

$$A(C) = A^{+}(C) = \|Cu\|_{\ell^\infty(v)} = \sup_n \frac{v_n}{n} \sum_{k=1}^{n} u_k;$$
$$A^{\downarrow}(C) = \|C(u^{\downarrow})\|_{\ell^\infty(v)} = \sup_n \frac{v_n}{n} \sum_{k=1}^{n} \min_{j \le k} u_j;$$
$$A^{\uparrow}(C) = \|C(u^{\uparrow})\|_{\ell^\infty(v)} = \sup_n \frac{v_n}{n} \sum_{k=1}^{n} \inf_{j \ge k} u_j.$$

Even using the formulas from this corollary, the operator norms of C as a map on cones in a power weighted ℓ^∞ space requires some work to simplify. This is done in the next theorem.

Theorem 2.11. *Let* $\alpha \in \mathbb{R}$. *The inequality*

$$\sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| n^\alpha \le A \sup_k |x_k| k^\alpha$$

holds for all real sequences x *if and only if it holds for all nonnegative sequences* x *if and only if it holds for all nonnegative, nonincreasing sequences* x. *In this case the best constant* A *is*

$$A = \begin{cases} 1, & \alpha < 0; \\ \dfrac{1}{1-\alpha}, & 0 \le \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

The inequality holds for all nonnegative, nondecreasing sequences x *with best constant*

$$A = \begin{cases} 1, & \alpha \le 0; \\ 0, & \alpha > 0. \end{cases}$$

Proof. Take $u_k = k^{-\alpha}$ and $v_n = n^\alpha$ in Corollary 2.10 to get

$$A(C) = A^+(C) = \sup_n n^{\alpha-1} \sum_{k=1}^n k^{-\alpha};$$
$$A^\downarrow(C) = \sup_n n^{\alpha-1} \sum_{k=1}^n \min_{j\le k} j^{-\alpha};$$
$$A^\uparrow(C) = \sup_n n^{\alpha-1} \sum_{k=1}^n \inf_{j\ge k} j^{-\alpha}.$$

We will make use of Proposition 3 of [11], which shows that

$$n^{\alpha-1} \sum_{k=1}^n k^{-\alpha}$$

increases with n when $\alpha \ge 0$ and decreases with n when $\alpha \le 0$. If $\alpha \ge 0$, then $\min_{j\le k} j^{-\alpha} = k^{-\alpha}$ so $A(C) = A^+(C) = A^\downarrow(C)$ and their common value is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{-\alpha} = \int_0^1 x^{-\alpha}\, dx = \begin{cases} \dfrac{1}{1-\alpha}, & 0 \le \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

If $\alpha < 0$, then $\min_{j\le k} j^{-\alpha} = 1$ so

$$A(C) = A^+(C) = \sup_n n^{\alpha-1} \sum_{k=1}^n k^{-\alpha} = 1 \text{ and } A^\downarrow(C) = \sup_n n^\alpha = 1.$$

If $\alpha > 0$, then $\inf_{j\ge k} j^{-\alpha} = 0$ and if $\alpha \le 0$, then $\inf_{j\ge k} j^{-\alpha} = k^{-\alpha}$. Therefore, $A^\uparrow(C) = 0$ when $\alpha > 0$ and

$$A^\uparrow(C) = \sup_n n^{\alpha-1} \sum_{k=1}^n k^{-\alpha} = 1$$

when $\alpha \le 0$. □

The Copson matrix C^* is nonnegative so all four parts of Theorem 2.9 apply, although the fourth part applies trivially. Recall that $\mathcal{D}(C^*)$ consists of all real sequences x for which $\sum_{k=1}^\infty \frac{x_k}{k}$ converges in $\mathbb{R}$.

Corollary 2.12. *Let $u, v \in \ell^+$. The inequality*

$$\sup_n \Big| \sum_{k=n}^{\infty} \frac{x_k}{k} \Big| v_n \le A \sup_k \frac{|x_k|}{u_k}$$

holds for all real sequences $x \in \mathcal{D}(C^*)$ with $A = A(C^*)$; for all nonnegative sequences $x \in \mathcal{D}(C^*)$ with $A = A^+(C^*)$; for all nonnegative, nonincreasing sequences $x \in \mathcal{D}(C^*)$ with $A = A^{\downarrow}(C^*)$; and for all nonnegative, nondecreasing sequences $x \in \mathcal{D}(C^*)$ with $A = A^{\uparrow}(C^*)$. In each case the constant A is best possible. Here

$$A(C^*) = A^+(C^*) = \|C^* u\|_{\ell^\infty(v)} = \sup_n v_n \sum_{k=n}^{\infty} \frac{u_k}{k};$$

$$A^{\downarrow}(C^*) = \|C^*(u^{\downarrow})\|_{\ell^\infty(v)} = \sup_n v_n \sum_{k=n}^{\infty} \frac{1}{k} \min_{j \le k} u_j;$$

$$A^{\uparrow}(C^*) = 0. \quad (\ell^{\uparrow} \cap \mathcal{D}(C^*) = \{0\}.)$$

The operator norms of C^* as a map on cones in a power weighted ℓ^∞ space are given in the next theorem. We use ζ to denote the Riemann zeta function.

Theorem 2.13. *Let $\alpha \in \mathbb{R}$. The inequality*

$$\sup_n \Big| \sum_{k=n}^{\infty} \frac{x_k}{k} \Big| n^\alpha \le A \sup_k |x_k| k^\alpha$$

holds for all real sequences $x \in \mathcal{D}(C^*)$ if and only if it holds for all nonnegative sequences $x \in \mathcal{D}(C^*)$ if and only if it holds for all nonnegative, nonincreasing sequences $x \in \mathcal{D}(C^*)$. In this case the best constant A is

$$A = \begin{cases} \infty, & \alpha \le 0; \\ \zeta(\alpha + 1), & \alpha > 0. \end{cases}$$

Except for the zero sequence, there are no nonnegative, nondecreasing sequences in $\mathcal{D}(C^*)$. The inequality holds for the zero sequence x with best constant $A = 0$.

Proof. Take $u_k = k^{-\alpha}$ and $v_n = n^\alpha$ in Corollary 2.12 to get

$$A(C^*) = A^+(C^*) = \sup_n n^\alpha \sum_{k=n}^\infty k^{-\alpha-1};$$
$$A^\downarrow(C^*) = \sup_n n^\alpha \sum_{k=n}^\infty \frac{1}{k} \min_{j\le k} j^{-\alpha}.$$

If $\alpha > 0$, then $\min_{j\le k} j^{-\alpha} = k^{-\alpha}$ so $A(C^*) = A^+(C^*) = A^\downarrow(C^*)$; their common value is

$$\sup_n n^\alpha \sum_{k=n}^\infty k^{-\alpha-1} \ge \sum_{k=1}^\infty k^{-\alpha-1} = \zeta(\alpha+1).$$

We show this is actually equality by supplying a proof of the first inequality from Remark 4.10 of [10]: $n^\alpha \sum_{k=n}^\infty k^{-\alpha-1}$ decreases with n. The derivative of $\log(x^{\alpha+1}(x^{-\alpha} - (x+1)^{-\alpha}))$ is

$$\frac{\left(1+\frac{1}{x}\right)^{\alpha+1} - \left(1+\frac{\alpha+1}{x}\right)}{(1+x)^{\alpha+1}(x^{-\alpha} - (x+1)^{-\alpha})},$$

which is positive for $x > 0$ by Bernoulli's inequality. Thus,

$$a_k = \frac{1}{k^{\alpha+1}(k^{-\alpha} - (k+1)^{-\alpha})}$$

is a decreasing sequence, and so is its moving average

$$\frac{\sum_{k=n}^\infty a_k(k^{-\alpha} - (k+1)^{-\alpha})}{\sum_{k=n}^\infty (k^{-\alpha} - (k+1)^{-\alpha})} = n^\alpha \sum_{k=n}^\infty k^{-\alpha-1}.$$

If $\alpha \le 0$, then $\min_{j\le k} j^{-\alpha} = 1$ and we have

$$A(C^*) = A^+(C^*) \ge A^\downarrow(C^*) = \sup_n n^\alpha \sum_{k=n}^\infty \frac{1}{k} = \infty.$$

The final statement of the theorem is evident. □

2.3.2 The Cesàro and Copson operators minus identity

The matrices we consider here are,

$$C - I = \begin{pmatrix} 0 & & & & \\ \frac{1}{2} & -\frac{1}{2} & & & \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^* - I = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ & & -\frac{2}{3} & \frac{1}{4} & \cdots \\ & & & -\frac{3}{4} & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Parts (i), (ii), and (iii) of Theorem 2.9 apply to $C - I$ and part (iv) applies to $I - C$. (See Remark 2.8.)

Corollary 2.14. *Let $u, v \in \ell^+$. The inequality*

$$\sup_n \left| \left(\frac{1}{n} \sum_{k=1}^n x_k \right) - x_n \right| v_n \le A \sup_k \frac{|x_k|}{u_k}$$

holds for all real sequences x with $A = A(C-I)$; for all nonnegative sequences x with $A = A^+(C - I)$; for all nonnegative, nonincreasing sequences x with $A = A^{\downarrow}(C - I)$ and for all nonnegative, nondecreasing sequences x with $A = A^{\downarrow}(I - C)$. In each case the constant A is best possible. Here

$$A(C - I) = \| |C - I| u \|_{\ell^\infty(v)} = \sup_n \frac{v_n}{n} \left((n-1) u_n + \sum_{k=1}^{n-1} u_k \right);$$

$$A^+(C - I) = \max(\|(C - I)^+ u\|_{\ell^\infty(v)}, \|(C - I)^- u\|_{\ell^\infty(v)})$$

$$= \sup_n \frac{v_n}{n} \max\left((n-1) u_n, \sum_{k=1}^{n-1} u_k \right);$$

$$A^{\downarrow}(C - I) = \|(C - I)^+ (u^{\downarrow})\|_{\ell^\infty(v)} = \sup_n \frac{v_n}{n} \sum_{k=1}^{n-1} \min_{j \le k} u_j;$$

$$A^{\uparrow}(I - C) = \|(I - C)^+ (u^{\uparrow})\|_{\ell^\infty(v)} = \sup_n \frac{v_n}{n} (n-1) \inf_{j \ge n} u_j.$$

Proposition 3.5 of [16] may be compared with the case $0 \le \alpha < 1$ of the next theorem: The norm of $C - I$ restricted to the cone of nonnegative, nonincreasing sequences coincides with the norm of $H - I$ restricted to the cone of nonnegative, nonincreasing functions.

On power-weighted ℓ^∞, $C-I$ exhibits different behavior on all four different cones, the cone of real sequences, the cone of nonnegative sequences, the cone of nonnegative, nonincreasing sequences and the cone of nonnegative, nonincreasing sequences. The dependence of the operator norm on the power gets particularly interesting for the third cone.

Theorem 2.15. *Let $\alpha \in \mathbb{R}$, set $s_1 = -\infty$ and set $s_m = 1 + \frac{\log(1-1/m)}{\log(1+1/m)}$ for $m = 2, 3, \ldots$. The inequality*

$$\sup_n \left| \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right) - x_n \right| n^\alpha \le A \sup_k |x_k| k^\alpha$$

holds for all real sequences x with best constant

$$A = \begin{cases} \dfrac{2-\alpha}{1-\alpha}, & \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

It holds for all nonnegative sequences x with best constant

$$A = \begin{cases} 1, & \alpha < 0; \\ \dfrac{1}{1-\alpha}, & 0 \le \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

It holds for all nonnegative, nonincreasing sequences x with best constant

$$A = \begin{cases} (m+1)^{\alpha-1} m, & s_m < \alpha \le s_{m+1},\ m = 1, 2, 3, \ldots; \\ \dfrac{1}{1-\alpha}, & 0 \le \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

It holds for all nonnegative, nondecreasing sequences x with best constant

$$A = \begin{cases} 1, & \alpha \le 0; \\ 0, & \alpha > 0. \end{cases}$$

Proof. Take $u_k = k^{-\alpha}$ and $v_n = n^\alpha$ in Corollary 2.14 to get

$$A(C-I) = \sup_n \Big(1 - \frac{1}{n} + n^{\alpha-1}\sum_{k=1}^{n-1} k^{-\alpha}\Big);$$
$$A^+(C-I) = \sup_n \max\Big(1 - \frac{1}{n}, n^{\alpha-1}\sum_{k=1}^{n-1} k^{-\alpha}\Big);$$
$$A^\downarrow(C-I) = \sup_n n^{\alpha-1}\sum_{k=1}^{n-1} \min_{j\le k} j^{-\alpha};$$
$$A^\uparrow(I-C) = \sup_n n^{\alpha-1}(n-1)\inf_{j\ge n} j^{-\alpha}.$$

Proposition 4 in [11] shows that $n^{\alpha-1}\sum_{k=1}^{n-1} k^{-\alpha}$ increases with n for all $\alpha \in \mathbb{R}$. It tends to

$$\int_0^1 x^{-\alpha}\,dx = \begin{cases} \dfrac{1}{1-\alpha}, & \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

The first two statements of the theorem follow.

If $\alpha \ge 0$, then $\min_{j\le k} j^{-\alpha} = k^{-\alpha}$ so $A^\downarrow(C-I) = A^+(C-I)$. If $\alpha < 0$, then $\min_{j\le k} j^{-\alpha} = 1$ so $A^\downarrow(C-I) = \sup_n n^{\alpha-1}(n-1)$. Consider the function $g(x) = x^{\alpha-1}(x-1)$ for $x \ge 1$. Looking at $g'(x)$ we find that g is strictly increasing on $(0, 1-1/\alpha)$ and strictly decreasing on $(1-1/\alpha, \infty)$. It follows that a positive integer m satisfies $\sup_n g(n) = g(m+1)$ if and only if $g(m+1) \ge g(m)$ and $g(m+1) \ge g(m+2)$. These two conditions may be expressed as $s_m \le \alpha \le s_{m+1}$.

If $\alpha > 0$, then $\inf_{j\ge n} j^{-\alpha} = 0$ so $A^\uparrow(I-C) = 0$. If $\alpha \le 0$, then $\inf_{j\ge n} j^{-\alpha} = n^{-\alpha}$ so $A^\uparrow(I-C) = \sup_n (n-1)/n = 1$. □

Parts (i), (ii), and (iv) of Theorem 2.9 apply to $C^* - I$, although part (iv) gives a trivial result. Note that $\mathcal{D}(C^* - I) = \mathcal{D}(C^*)$.

Corollary 2.16. *Let* $u, v \in \ell^+$. *The inequality*

$$\sup_n \Big|\Big(\sum_{k=n}^{\infty} \frac{x_k}{k}\Big) - x_n\Big| v_n \le A \sup_k \frac{|x_k|}{u_k}$$

holds for all real sequences $x \in \mathcal{D}(C^*)$ with $A = A(C^* - I)$; for all nonnegative sequences $x \in \mathcal{D}(C^*)$ with $A = A^+(C^* - I)$; and for all nonnegative, nondecreasing sequences $x \in \mathcal{D}(C^*)$ with $A = A^\uparrow(C^* - I)$. In each case the constant A is best possible. Here

$$A(C^* - I) = \||C^* - I|u\|_{\ell^\infty(v)} = \sup_n v_n\Big(\frac{n-1}{n}u_n + \sum_{k=n+1}^{\infty} \frac{u_k}{k}\Big);$$

$$A^+(C^* - I) = \max(\|(C^* - I)^+ u\|_{\ell^\infty(v)}, \|(C^* - I)^- u\|_{\ell^\infty(v)})$$

$$= \sup_n v_n \max\Big(\frac{n-1}{n}u_n, \sum_{k=n+1}^{\infty} \frac{u_k}{k}\Big);$$

$A^\uparrow(C^* - I) = 0$. *(The first row of $C^* - I$ has an infinite sum.)*

We omit the trivial case when considering the power-weighted inequalities.

Theorem 2.17. *Let $\alpha \in \mathbb{R}$. The inequality*

$$\sup_n \Big|\Big(\sum_{k=n}^{\infty} \frac{x_k}{k}\Big) - x_n\Big| n^\alpha \le A \sup_k |x_k| k^\alpha$$

holds for all real sequences $x \in \mathcal{D}(C^*)$ with

$$A = \begin{cases} \infty, & \alpha \le 0; \\ 1 + \dfrac{1}{\alpha}, & \alpha > 0. \end{cases}$$

It holds for all nonnegative sequences $x \in \mathcal{D}(C^*)$ with

$$A = \begin{cases} \infty, & \alpha \le 0; \\ \dfrac{1}{\alpha}, & 0 < \alpha < 1; \\ 1, & \alpha \ge 1. \end{cases}$$

In each case the value of A is best possible.

Proof. Take $u_k = k^{-\alpha}$ and $v_n = n^\alpha$ in Corollary 2.16 to get

$$A(C^* - I) = \sup_n \left(1 - \frac{1}{n} + n^\alpha \sum_{k=n+1}^{\infty} k^{-\alpha-1}\right);$$

$$A^+(C^* - I) = \sup_n \max\left(1 - \frac{1}{n}, n^\alpha \sum_{k=n+1}^{\infty} k^{-\alpha-1}\right).$$

If $\alpha \leq 0$ the sum diverges so both of these are infinite. If $\alpha > 0$ we need the second inequality from Remark 4.10 of [10]: $n^\alpha \sum_{k=n+1}^{\infty} k^{-\alpha-1}$ increases with n. The derivative of $\log(x^{\alpha+1}((x-1)^{-\alpha} - x^{-\alpha}))$ is

$$\frac{\left(1 - \frac{\alpha+1}{x}\right) - \left(1 - \frac{1}{x}\right)^{\alpha+1}}{(x-1)^{\alpha+1}((x-1)^{-\alpha} - x^{-\alpha})},$$

which is negative for $x > 1$ by Bernoulli's inequality. Thus,

$$a_k = \frac{1}{k^{\alpha+1}((k-1)^{-\alpha} - k^{-\alpha})}$$

is an increasing sequence, and so is its moving average

$$\frac{\sum_{k=n+1}^{\infty} a_k((k-1)^{-\alpha} - k^{-\alpha})}{\sum_{k=n+1}^{\infty}((k-1)^{-\alpha} - k^{-\alpha})} = n^\alpha \sum_{k=n+1}^{\infty} k^{-\alpha-1}.$$

We recognize these as (improper) Riemann sums, and get

$$n^\alpha \sum_{k=n+1}^{\infty} k^{-\alpha-1} = \frac{1}{n} \sum_{k=n+1}^{\infty} \left(\frac{k}{n}\right)^{-\alpha-1} \to \int_1^\infty x^{-\alpha-1}\,dx = \frac{1}{\alpha}$$

as $n \to \infty$. Therefore $A(C^* - I) = 1 + 1/\alpha$ and $A^+(C^* - I) = \max(1, 1/\alpha)$. This completes the proof. $\square$

2.3.3 Two required operators

The next two operators appear in Theorems 2.5 and 2.6. Their operators norms are needed to complete the work on (36). In matrix form, they are

$$C - S^* = \begin{pmatrix} 1 & -1 & & & & \\ \frac{1}{2} & \frac{1}{2} & -1 & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad (C^* - S)D = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \cdots \\ -\frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \cdots \\ & -\frac{1}{3} & \frac{1}{12} & \frac{1}{20} & \cdots \\ & & -\frac{1}{4} & \frac{1}{20} & \cdots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Parts (i), (ii), and (iii) of Theorem 2.9 apply to $C - S^*$ and part (iv) applies to $S^* - C$.

Corollary 2.18. *Let $u, v \in \ell^+$. The inequality*

$$\sup_n \Big| \Big(\frac{1}{n} \sum_{k=1}^{n} x_k \Big) - x_{n+1} \Big| v_n \le A \sup_k \frac{|x_n|}{u_n}$$

holds for all real sequences x with $A = A(C - S^*)$; for all nonnegative sequences x with $A = A^+(C - S^*)$; for all nonnegative, nonincreasing sequences x with $A = A^{\downarrow}(C - S^*)$ and for all nonnegative, nondecreasing sequences x with $A = A^{\uparrow}(S^* - C)$. In each case the constant A is best possible. Here

$$A(C - S^*) = \| |C - S^*| u \|_{\ell^\infty(v)} = \sup_n v_n \Big(u_{n+1} + \frac{1}{n} \sum_{k=1}^{n} u_k \Big);$$

$$A^+(C - S^*) = \max(\|(C - S^*)^+ u\|_{\ell^\infty(v)}, \|(C - S^*)^- u\|_{\ell^\infty(v)})$$

$$= \sup_n v_n \max \Big(\frac{1}{n} \sum_{k=1}^{n} u_k, u_{n+1} \Big);$$

$$A^{\downarrow}(C - S^*) = \|(C - S^*)^+ (u^{\downarrow})\|_{\ell^\infty(v)} = \sup_n \frac{v_n}{n} \sum_{k=1}^{n} \min_{j \le k} u_j;$$

$$A^{\uparrow}(S^* - C) = \|(S^* - C)^+ (u^{\uparrow})\|_{\ell^\infty(v)} = \sup_n v_n \inf_{j \le n+1} u_j.$$

Parts (i), (ii), and (iv) of Theorem 2.9 apply to $(C^* - S)D$ and part (iii) applies to $(S - C^*)D$.

Corollary 2.19. *Let $u, v \in \ell^+$ and for convenience let $u_0 = x_0 = 0$. The inequality*

$$\sup_n \Big| \Big(\sum_{k=n}^{\infty} \frac{x_k}{k(k+1)} \Big) - \frac{x_{n-1}}{n} \Big| v_n \le A \sup_k \frac{|x_n|}{u_n}$$

holds for all real sequences $x \in \mathcal{D}(C^*D)$ with $A = A((C^* - S)D)$; for all nonnegative sequences $x \in \mathcal{D}(C^*D)$ with $A = A^+((C^* - S)D)$; for all nonnegative, nonincreasing sequences $x \in \mathcal{D}(C^*D)$ with $A = A^{\downarrow}((S - C^*)D)$;

and for all nonnegative, nondecreasing sequences $x \in \mathcal{D}(C^*D)$ *with* $A = A^{\uparrow}((C^*-S)D)$. *In each case the constant* A *is best possible. Here*

$$\begin{aligned}
A((C^*-S)D) &= \| |(C^*-S)D| u \|_{\ell^\infty(v)} \\
&= \sup_n v_n \Big(\frac{u_{n-1}}{n} + \sum_{k=n}^{\infty} \frac{u_k}{k(k+1)} \Big); \\
A^{+}((C^*-S)D) &= \max(\| ((C^*-S)D)^{+} u \|_{\ell^\infty(v)}, \| ((C^*-S)D)^{-} u \|_{\ell^\infty(v)}) \\
&= \sup_n v_n \max\Big(\frac{u_{n-1}}{n}, \sum_{k=n}^{\infty} \frac{u_k}{k(k+1)} \Big); \\
A^{\downarrow}((S-C^*)D) &= \| ((S-C^*)D)^{+} (u^{\downarrow}) \|_{\ell^\infty(v)} = \sup_n \frac{v_n}{n} \min_{j \le n-1} u_j; \\
A^{\uparrow}((C^*-S)D) &= \| ((C^*-S)D)^{+} (u^{\uparrow}) \|_{\ell^\infty(v)} \\
&= \sup_n v_n \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \inf_{j \ge k} u_j.
\end{aligned}$$

We forgo an investigation of the power-weighted case for these operators. Their principal interest is their use in the proof of Theorem 2.20 and the special case is not required there. Theorem 2.21 and 2.22 deduce the power-weighted case of Theorem 2.20 directly.

2.4 Best constants in the two-operator inequalities

Combining Theorems 2.5 and 2.6 with the formulas given in the previous subsection for

$$A(C-S^*),\ A^{\downarrow}(C-S^*),\ A((C^*-S)D), \quad \text{and} \quad A^{\uparrow}((C^*-S)D)$$

gives us answers to our original questions, the best constants in the inequalities of (36).

Fix $u, v \in \ell^+$. Let $A(C, C^*)$ and $A^+(C, C^*)$ denote the smallest $A \ge 0$ such that inequality

$$\|Cx\|_{\ell^\infty(v)} \le A \|C^*x\|_{\ell^\infty(1/u)}$$

holds for all $x \in \mathcal{D}(C^*)$ and for all $x \in \ell^+ \cap \mathcal{D}(C^*)$, respectively. Similarly, let $A(C^*, C)$ and $A^+(C^*, C)$ denote the smallest $A \geq 0$ such that inequality

$$\|C^* x\|_{\ell^\infty(v)} \leq A \|Cx\|_{\ell^\infty(1/u)}$$

holds for all $x \in \mathcal{D}(C^*)$ and for all $x \in \ell^+ \cap \mathcal{D}(C^*)$, respectively.

Theorem 2.20. *Let $u, v \in \ell^+$. Then, taking $u_0 = 0$, we have*

$$A(C, C^*) = \sup_n v_n \Big(u_{n+1} + \frac{1}{n} \sum_{k=1}^n u_k \Big);$$

$$A^+(C, C^*) = \sup_n \frac{v_n}{n} \sum_{k=1}^n \min_{j \leq k} u_j;$$

$$A(C^*, C) = \sup_n v_n \Big(\frac{n-1}{n} u_{n-1} + \sum_{k=n}^\infty \frac{u_k}{k+1} \Big);$$

$$A^+(C^*, C) = \sup_n v_n \sum_{k=n}^\infty \frac{1}{k(k+1)} \inf_{j \geq k} j u_j.$$

Proof. Since (39) holds with $A = A(C - S^*)$, Theorem 2.5 shows that (38) does as well. Thus $A(C, C^*) \leq A(C - S^*)$. On the other hand, by definition, (38) holds with $A = A(C, C^*)$ and by Theorem 2.5, so does (39). Fix n and let

$$y = (u_1, u_2, \dots, u_n, -u_{n+1}, 0, 0, \dots).$$

Since $y_k \to 0$ as $k \to \infty$ and $\|y\|_{\ell^\infty(1/u)} \leq 1$, we get

$$v_n \Big(u_{n+1} + \frac{1}{n} \sum_{k=1}^n u_k \Big) = v_n((C - S^*)y)_n \leq \|(C - S^*)y\|_{\ell^\infty(v)} \leq A(C, C^*).$$

Using this in the formula for $A(C - S^*)$ from Corollary 2.18 yields

$$A(C - S^*) = \sup_n v_n \Big(u_{n+1} + \frac{1}{n} \sum_{k=1}^n u_k \Big) \leq A(C, C^*).$$

Therefore,

$$A(C, C^*) = \sup_n v_n \Big(u_{n+1} + \frac{1}{n} \sum_{k=1}^n u_k \Big).$$

Clearly, (41) holds with $A = A^{\downarrow}(C - S^*)$ and, by Theorem 2.5, so does (40). Thus $A^+(C, C^*) \le A^{\downarrow}(C - S^*)$. By definition, (40) holds with $A = A^+(C, C^*)$ and Theorem 2.5 shows that (41) does also. Fix n and let

$$y = (u_1^{\downarrow}, u_2^{\downarrow}, \dots, u_n^{\downarrow}, 0, 0, \dots).$$

Then $y \in \ell^{\downarrow}$, $y_k \to 0$ as $k \to \infty$ and, by Lemma 2.2, $\|y\|_{\ell^\infty(1/u)} = \|y\|_{\ell^\infty(1/u^{\downarrow})} \le 1$. Therefore

$$v_n\Big(\frac{1}{n}\sum_{k=1}^n u_k^{\downarrow}\Big) = v_n((C - S^*)y)_n \le \|(C - S^*)y\|_{\ell^\infty(v)} \le A^+(C, C^*).$$

The formula for $A^{\downarrow}(C - S^*)$ from Corollary 2.18 implies

$$A^{\downarrow}(C - S^*) = \sup_n \frac{v_n}{n}\sum_{k=1}^n \min_{j \le k} u_j \le A^+(C, C^*).$$

Therefore,

$$A^+(C, C^*) = \sup_n \frac{v_n}{n}\sum_{k=1}^n \min_{j \le k} u_j.$$

Let $w_k = ku_k$ for all k and note that $\mathcal{D}((C^* - S)D) = \mathcal{D}(C^*D)$.

Replacing u by w in the formula for $A((C^* - S)D)$ from Corollary 2.19 shows that (43) holds with A replaced by

$$\tilde{A} = \sup_n v_n\Big(\frac{w_{n-1}}{n} + \sum_{k=n}^\infty \frac{w_k}{k(k+1)}\Big) = \sup_n v_n\Big(\frac{n-1}{n}u_{n-1} + \sum_{k=n}^\infty \frac{u_k}{k+1}\Big).$$

By Theorem 2.6, (42) also holds with $A = \tilde{A}$. Thus $A(C^*, C) \le \tilde{A}$. Theorem 2.6 also shows that (42) and hence (43) holds with $A = A(C^*, C)$. Fix n and $K > n$, and let

$$z = (0, 0, \dots, 0, -w_{n-1}, w_n, w_{n+1}, \dots w_K, 0, 0, \dots).$$

Since $z \in \mathcal{D}(C^*D)$, $(Dz)_k \to 0$ as $k \to \infty$, and $\|z\|_{\ell^\infty(1/w)} \le 1$, we have $\|(C^* - S)Dz\|_{\ell^\infty(v)} \le A(C^*, C)$. It follows that

$$v_n\Big(\frac{w_{n-1}}{n} + \sum_{k=n}^K \frac{w_k}{k(k+1)}\Big) = v_n((C^* - S)Dz)_n \le A(C^*, C).$$

Letting $K \to \infty$ and taking the supremum over n, gives $\tilde{A} \le A(C^*, C)$, and we conclude that $A(C^*, C) = \tilde{A}$.

Replacing u by w in the formula for $A^{\uparrow}((C^* - S)D)$ from Corollary 2.19 shows that (45) holds with A replaced by

$$\tilde{A}^{\uparrow} = \sup_n v_n \sum_{k=n}^{\infty} \frac{w_k^{\uparrow}}{k(k+1)} = \sup_n v_n \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \inf_{j \ge k} j u_j.$$

By Theorem 2.6, (44) also holds with $A = \tilde{A}^{\uparrow}$. Thus $A^+(C^*, C) \le \tilde{A}^{\uparrow}$. Theorem 2.6 also shows that (44) and hence (45) holds with $A = A^+(C^*, C)$. Fix n and $K > n$, and let

$$z = (0, 0, \dots, 0, w_n^{\uparrow}, w_{n+1}^{\uparrow}, \dots w_K^{\uparrow}, w_K^{\uparrow}, w_K^{\uparrow}, \dots).$$

Then $(C^*Dz)_K = \sum_{k=K}^{\infty} \frac{w_K^{\uparrow}}{k(k+1)} < \infty$ so $z \in \ell^{\uparrow} \cap \mathcal{D}(C^*D)$, $(Dz)_k \to 0$ as $k \to \infty$, and, by Lemma 2.2, $\|z\|_{\ell^\infty(1/w)} = \|z\|_{\ell^\infty(1/w^{\uparrow})} \le 1$. Therefore,

$$v_n \sum_{k=n}^{K} \frac{w_k^{\uparrow}}{k(k+1)} \le v_n((C^* - S)Dz)_n \le \|(C^* - S)Dz\|_{\ell^\infty(v)} \le A^+(C^*, C).$$

Letting $K \to \infty$ and taking the supremum over n, we get $\tilde{A}^{\uparrow} \le A^+(C^*, C)$, and conclude that $A^+(C^*, C) = \tilde{A}^{\uparrow}$. □

We split the power-weighted inequalities for the two-operator inequalities into two theorems because the techniques of simplification differ.

Theorem 2.21. *Let $\alpha \in \mathbb{R}$. The inequality*

$$\sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| n^{\alpha} \le A \sup_n \left| \sum_{k=n}^{\infty} \frac{x_k}{k} \right| n^{\alpha}$$

holds for all $x \in \mathcal{D}(C^*)$ with

$$A = \begin{cases} 1 + 2^{-\alpha}, & \alpha \le 0; \\ \dfrac{2-\alpha}{1-\alpha}, & 0 < \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

It holds for all nonnegative $x \in \mathcal{D}(C^*)$ with

$$A = \begin{cases} 1, & \alpha \le 0; \\ \dfrac{1}{1-\alpha}, & 0 < \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

In each case the constant A is best possible.

Proof. Take $u_k = k^{-\alpha}$ and $v_n = n^\alpha$ in Theorem 2.20 to see that the best constant A, taken over all $x \in \mathcal{D}(C^*)$, is

$$A(C, C^*) = \sup_n \left(\left(\frac{n}{n+1} \right)^\alpha + n^{\alpha-1} \sum_{k=1}^n k^{-\alpha} \right)$$

and the best constant A, taken over all nonnegative $x \in \mathcal{D}(C^*)$, is

$$A^+(C, C^*) = \sup_n n^{\alpha-1} \sum_{k=1}^n \min_{j \le k} j^{-\alpha}.$$

By Proposition 3 of [11], $n^{\alpha-1} \sum_{k=1}^n k^{-\alpha}$ decreases with n when $\alpha \le 0$ and increases with n when $\alpha > 0$. The same is true of $\left(\frac{n}{n+1}\right)^\alpha$. So if $\alpha \le 0$, $A(C, C^*) = 1 + 2^{-\alpha}$ and if $\alpha > 0$, $\left(\frac{n}{n+1}\right)^\alpha \to 1$ and

$$n^{\alpha-1} \sum_{k=1}^n k^{-\alpha} = \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n} \right)^{-\alpha} \to \int_0^1 x^{-\alpha}\, dx = \begin{cases} \dfrac{1}{1-\alpha}, & 0 < \alpha < 1; \\ \infty, & \alpha \ge 1, \end{cases}$$

as $n \to \infty$ so

$$A(C, C^*) = \begin{cases} \dfrac{2-\alpha}{1-\alpha}, & 0 < \alpha < 1; \\ \infty, & \alpha \ge 1, \end{cases}$$

If $\alpha \le 0$, then $\min_{j \le k} j^{-\alpha} = 1$ so $A^+(C, C^*) = 1$. If $\alpha > 0$, then $\min_{j \le k} j^{-\alpha} = k^{-\alpha}$ so as above we have

$$A^+(C, C^*) = \sup_n n^{\alpha-1} \sum_{k=1}^n k^{-\alpha} = \begin{cases} \dfrac{1}{1-\alpha}, & 0 < \alpha < 1; \\ \infty, & \alpha \ge 1. \end{cases}$$

□

In the case of nonnegative sequences, the best constants given above agree with those that appear in Theorem 3.7 of [16] for the operators H and H^* on nonnegative functions, except when $\alpha < 0$. The best constants given below, again for nonnegative sequences, agree with the corresponding results from Theorem 3.7 of [16] for all values of α.

Theorem 2.22. *Let $\alpha \in \mathbb{R}$ and set $M_\alpha = \sum_{k=1}^{\infty} \frac{k^{-\alpha}}{k+1}$. The inequality*

$$\sup_n \Big| \sum_{k=n}^{\infty} \frac{x_k}{k} \Big| n^\alpha \le A \sup_n \Big| \frac{1}{n} \sum_{k=1}^{n} x_k \Big| n^\alpha$$

holds for all $x \in \mathcal{D}(C^*)$ with

$$A = \begin{cases} \infty, & \alpha \le 0; \\ 1 + \dfrac{1}{\alpha}, & 0 < \alpha \le 1; \\ 2^\alpha M_\alpha, & \alpha > 1. \end{cases}$$

It holds for all nonnegative $x \in \mathcal{D}(C^*)$ with

$$A = \begin{cases} \infty, & \alpha \le 0; \\ \dfrac{1}{\alpha}, & 0 < \alpha \le 1; \\ 0, & \alpha > 1. \end{cases}$$

In each case the constant A is best possible.

Proof. Take $u_k = k^{-\alpha}$ and $v_n = n^\alpha$ in Theorem 2.20 to see that the best constant A, taken over all $x \in \mathcal{D}(C^*)$, is

$$A(C^*, C) = \max\Big(M_\alpha, \sup_{n \ge 2} \Big(\Big(\frac{n-1}{n}\Big)^{1-\alpha} + n^\alpha \sum_{k=n}^{\infty} \frac{k^{-\alpha}}{k+1} \Big) \Big)$$

and the best constant A, taken over all nonnegative $x \in \mathcal{D}(C^*)$, is

$$A^+(C^*, C) = \sup_n n^\alpha \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \inf_{j \ge k} j^{1-\alpha}.$$

We will need another monotonicity result in the spirit of Remark 4.10 of [**?**]: If $0 < \alpha \le 1$, then

$$n^\alpha \sum_{k=n}^{\infty} \frac{k^{-\alpha}}{k+1}$$

increases with n to $1/\alpha$ and if $\alpha > 1$, it decreases with n. For all $x > 0$, the derivative of $\log((x+1)x^\alpha(x^{-\alpha} - (x+1)^{-\alpha}))$ is

$$\frac{\left(1+\frac{1}{x}\right)^\alpha - \left(1+\frac{\alpha}{x}\right)}{(x+1)^{\alpha+1}(x^{-\alpha} - (x+1)^{-\alpha})}$$

which is nonpositive when $\alpha \le 1$ and nonnegative when $\alpha \ge 1$ by Bernoulli's inequality. Thus,

$$a_k = \frac{1}{(k+1)k^\alpha(k^{-\alpha} - (k+1)^{-\alpha})}$$

is a nondecreasing sequence when $\alpha \le 1$ and is a decreasing sequence when $\alpha \ge 1$. Its moving average,

$$\frac{\sum_{k=n}^{\infty} a_k(k^{-\alpha} - (k+1)^{-\alpha})}{\sum_{k=n}^{\infty}(k^{-\alpha} - (k+1)^{-\alpha})} = n^\alpha \sum_{k=n}^{\infty} \frac{k^{-\alpha}}{k+1}.$$

shares its monotonicity. If $0 < \alpha \le 1$, then the last expression goes to $1/\alpha$, as can be seen from the estimates

$$n^\alpha \int_{n+1}^{\infty} (x+1)^{-\alpha-1}\, dx \le n^\alpha \sum_{k=n}^{\infty} \frac{k^{-\alpha}}{k+1} \le n^\alpha \int_{n-1}^{\infty} x^{-\alpha-1}\, dx.$$

If $\alpha \le 0$, $A(C^*, C) = \infty$ since the sums diverge. If $0 < \alpha \le 1$, then $A(C^*, C) = 1 + 1/\alpha$. If $\alpha > 1$, then

$$A(C^*, C) = \max(M_\alpha, 2^{\alpha-1} + 2^\alpha(M_\alpha - 1/2)) = 2^\alpha M_\alpha.$$

If $\alpha > 1$, then $\inf_{j \ge k} j^{1-\alpha} = 0$ so $A^+(C^*, C) = 0$. If $\alpha \le 1$, then $\inf_{j \ge k} j^{1-\alpha} = k^{1-\alpha}$ so $A^+(C, C^*) = \sup_n n^\alpha \sum_{k=1}^{\infty} \frac{k^{-\alpha}}{k+1}$, which is infinite when $\alpha \le 0$ and equals $1/\alpha$ when $0 < \alpha \le 1$. □

3 Optimal non-absolute domains for the Hardy operator minus identity

3.1 Introduction

We consider the Hardy averaging operator

$$Hf(x) = \frac{1}{x}\int_0^x f(s)ds$$

and its dual

$$H^*f(x) = \int_x^\infty \frac{f(s)}{s}ds.$$

It is well known in [13] that the Hardy operator H and its dual H^* are bounded on a rearrangement invariant Banach function space (BFS, for short) X if $\overline{\alpha} < 1$ and $\underline{\alpha} > 0$ respectively, where $\underline{\alpha}$ is the lower and $\overline{\alpha}$ is the upper Boyd index of X. Consequently, $H : L^p(0,\infty) \to L^p(0,\infty)$ for $1 < p \leq \infty$ and $H^* : L^p(0,\infty) \to L^p(0,\infty)$ for $1 \leq p < \infty$ are bounded.

As shown in Proposition 3.6*(i)*, the only non-negative function $f \in L^1(0,\infty)$ such that $Hf \in L^1(0,\infty)$ is the zero function. The motivation of this work is the great interest in this topic for the Hardy operators, but for non-negative functions, as shown in [26, 28, 65, 66, 69]. Moreover, if X is a BFS for which $H : X \to X$ is bounded, then the class of functions for which $H(|f|) \in X$ is known to be much larger than X and, in fact, not even a subspace of $(L^1 + L^\infty)(0,\infty)$ [28, Theorem 2.6].

It is well-known that subtracting the identity from an averaging operator provides some additional regularity and smoothness [44]. The authors in [9] determine the optimal non-absolute domain of $C - \mathrm{I}$ and $C^* - \mathrm{I}$ on a BFS X, where I is the identity operator, C and C^* are the Cesáro and Copson operators respectively. Our main goal is to characterize the optimal domains for $H - \mathrm{I}$, on Lebesgue spaces $L^p(0,\infty)$, $1 \leq p \leq \infty$. Moreover, we study conditions for a function f such that $(Hf - f) \in X$, where X is a BFS. That is, we determine the conditions to describe the optimal domain of $H - \mathrm{I}$;

$$\mathrm{Dom}[H - \mathrm{I}, X] = \{f : (Hf - f) \in X\}$$

on a BFS X. A similar definition holds for $\mathrm{Dom}[H^* - \mathrm{I}, X]$.

We recall some standard notations. The subspace $L_0^1(0,\infty)$ of the space $L^1(0,\infty)$ is defined as

$$L_0^1(0,\infty) = \left\{ f \in L^1(0,\infty) : \int_0^\infty f(x)dx = 0 \right\}.$$

For $1 \leq p < \infty$ and non-negative weight w, we recall that

$$L^p(w) := L^p(w,(0,\infty)) = \left\{ f : \int_0^\infty |f(x)|^p w(x)dx < \infty \right\}$$

with $\|f\|_{L^p(w)} = \left(\int_0^\infty |f(x)|^p w(x)dx \right)^{1/p}$, and analogously if $p = \infty$. Moreover, we define

$$L_0^p(w) := L_0^p(w,(0,\infty)) = L^p(w) \cap L_0^1(0,\infty),$$

with $\|f\|_{L_0^p(w)} = \|f\|_{L^p(w)} + \|f\|_{L_0^1(0,\infty)}$. If $w \equiv 1$, we denote $L^p(1,(0,\infty))$ and $L_0^p(1,(0,\infty))$ by $L^p(0,\infty)$ and $L_0^p(0,\infty)$ respectively.

Throughout, we use the standard notation $A \lesssim B$ to denote the existence of a positive constant $\alpha > 0$ (independent of the main parameters defining A and B) such that $A \leq \alpha B$ (analogously for the notation $A \gtrsim B$). If both $A \lesssim B$ and $A \gtrsim B$ hold true, we will write $A \approx B$.

The paper is structured as follows: in Section 3.2 we start by showing some general results for a BFS X. We then establish a useful equivalent expression for a function f such that $Hf - f \in X$ or $H^*f - f \in X$ as shown in Lemma 3.2, which enable us obtain the optimal domain of $H - \mathrm{I}$ and $H^* - \mathrm{I}$ on X. In section 3.3 we present our main result for $H - I$ as shown in Theorem 3.9, where we fully characterize $\mathrm{Dom}[H - \mathrm{I}, L^p(0,\infty)]$, $1 \leq p \leq \infty$. As a consequence of these results, we can see that $\mathrm{Dom}[H - \mathrm{I}, L^1(0,\infty)] \subsetneq L_0^1(0,\infty) + \ker(H - \mathrm{I})$. Finally, in Section 3.4 we find the optimal domains for the dual Hardy operator minus the identity, and show the analogous results in Theorem 3.13. In this case, we show that $\mathrm{Dom}[H^* - \mathrm{I}, L^\infty(0,\infty)] \subsetneq L^\infty(0,\infty) + \ker(H^* - \mathrm{I})$ and $\mathrm{Dom}[H^* - \mathrm{I}, L^1(0,\infty)] \subsetneq L^{1,\infty}(0,\infty)$.

3.2 Preliminaries

As a start we find the kernels of $H - \mathrm{I}$ and $H^* - \mathrm{I}$ on a BFS X. We find also an equivalent expression for a function f such that $Hf - f \in X$ or $H^*f - f \in X$. These expressions help us show that $H - \mathrm{I}$ and $H^* - \mathrm{I}$ are inverses of each other on $L^p(0,\infty)$ spaces, $1 < p < \infty$ (cf. Remark 3.3). As a general fact, when we write Hf or H^*f, we will be assuming that these expressions are well-defined.

Lemma 3.1. *Let X be a BFS on $(0,\infty)$, with the Lebesgue measure and $I : X \to X$ be the identity operator. Then*

(i) $\ker(H - \mathrm{I}) = \{f \in X : f(x) = c, c \in \mathbb{R}\}$

(ii) $\ker(H^* - \mathrm{I}) = \left\{g \in X : g(x) = \frac{c}{x} : \ c \in \mathbb{R}, x > 0\right\}$.

Proof. We prove first *(i)*. It is obvious that $\{f \in X : f(x) = c, c \in \mathbb{R}\} \subset \ker(H - \mathrm{I})$. To prove the reverse inclusion, let $f \in \ker(H - \mathrm{I})$. Then $\int_0^x (f(s) - f(x))ds = 0$, for all $x > 0$. From which we can deduce that f is C^∞ and, by the fundamental theorem of calculus, we obtain that $f'(x) = 0$ for all $x > 0$. Hence $f(x) = c$ for all $x > 0$, where c is a constant. The proof of *(ii)* follows similarly as in *(i)*. □

Lemma 3.2. *Let X be a BFS on $(0,\infty)$ with the Lebesgue measure.*

(i) Let $H^* : X \to X$ is bounded. If $g(x) = Hf(x) - f(x) \in X$, then f can be expressed as

$$f(x) = H^*g(x) - g(x) + \alpha,$$

where $\alpha = \lim_{s\to\infty} Hf(s) \in \mathbb{R}$, and $f \in X + \ker[H - I]$.

(ii) If $g(x) = H^*f(x) - f(x) \in X$*, then* f *can be expressed as*

$$f(x) = Hg(x) - g(x) + \frac{\beta}{x},$$

where $\beta = \lim_{s \to 0} (sH^*f(s)) \in \mathbb{R}$*. Moreover, if* $H : X \to X$ *is bounded, then* $f \in X + \ker[H^* - I]$.

Proof. To prove *(i)* we follow similar techniques as in [48, Theorem 1]). Let $g(t) = Hf(t) - f(t) \in X$ and let $0 < x < y < \infty$. Then dividing $g(s)$ by s and integrating from x to y we obtain

$$\int_x^y \frac{g(s)}{s} ds = \int_x^y \left(s^{-2} \int_0^s f(t)dt \right) ds - \int_x^y \frac{f(s)}{s} ds$$

Using Fubini's theorem on the right-hand side we get

$$\begin{aligned} \int_x^y \frac{g(s)}{s} ds &= \frac{1}{x} \int_0^x f(s)ds - \frac{1}{y} \int_0^y f(s)ds \\ &= Hf(x) - Hf(y) \\ &= f(x) + g(x) - Hf(y) \end{aligned}$$

Letting $y \to \infty$ we obtain

$$f(x) = H^*g(x) - g(x) + \lim_{y \to \infty} Hf(y).$$

Since $H^* : X \to X$ is bounded, $H^*g \in X$ and hence $\lim_{y \to \infty} Hf(y)$ exists. Consequently, $f \in X + \ker(H - \mathrm{I})$.

To prove *(ii)*, we let $g(t) = H^*f(t) - f(t) \in X$ and let $0 < y < x < \infty$. Integrating $g(t)$ from y to x and applying integration by parts we get

$$\begin{aligned} \int_y^x g(t)dt &= \int_y^x \int_t^\infty \frac{f(s)}{s} dsdt - \int_y^x f(t)dt \\ &= \left[t \int_t^\infty \frac{f(s)}{s} ds \right]_y^x + \int_y^x f(t)dt - \int_y^x f(t)dt \\ &= x \int_x^\infty \frac{f(s)}{s} ds - y \int_y^\infty \frac{f(s)}{s} ds \end{aligned}$$

Dividing by x and letting $y \to 0$ gives

$$f(x) = Hg(x) - g(x) + \frac{1}{x} \lim_{y \to 0} (yH^* f(y)),$$

where $\lim_{y \to 0} (yH^* f(y))$ exists. If $H : X \to X$ is bounded, $Hg \in X$, and hence $f \in X + \ker(H^* - \mathrm{I})$. □

Remark 3.3.

(i) The boundedness of $H^* : X \to X$ in Lemma 3.2-*(i)* is a necessary condition for the existence of the limit $\alpha = \lim_{s \to \infty} Hf(s)$. For example, if we consider $f(x) = \log x$, $g(x) = Hf(x) - f(x) = -1 \in L^\infty(0, \infty)$ but $H^* g(x) \notin L^\infty(0, \infty)$ for all $x > 0$, and α does not exist. Moreover, if $H^* : X \to X$ is bounded and $f \in \mathrm{Dom}[H - \mathrm{I}, X]$ then we get

$$f(x) - \alpha = (H^* - \mathrm{I})(H - \mathrm{I})(f(x) - \alpha).$$

(ii) Let $H : X \to X$ is bounded and $f \in \mathrm{Dom}[H^* - I, X]$. Then

$$f(x) - \frac{\beta}{x} = (H - \mathrm{I})(H^* - \mathrm{I})\Big(f(x) - \frac{\beta}{x}\Big),$$

where $\beta = \lim_{t \to 0} (tH^* f(t)) \in \mathbb{R}$.

(iii) As a special case of (i) and (ii), when $X = L^p(0, \infty)$, $1 < p < \infty$, we obtain that $\alpha = \beta = 0$, and hence $(H^* - \mathrm{I})(H - \mathrm{I}) = \mathrm{I} = (H - \mathrm{I})(H^* - \mathrm{I})$. In fact, for $f \in L^p(0, \infty), 1 < p < \infty$ and $1/p + 1/p' = 1$, the Hölder's inequality gives

$$\begin{aligned} |Hf(x)| &= \frac{1}{x} \left| \int_0^x f(t) dt \right| \le \frac{1}{x} \int_0^\infty |f(t)| \chi_{(0,x)}(t) dt \\ &\le \frac{1}{x} \Big(\int_0^\infty |f(t)|^p dt \Big)^{1/p} \Big(\int_0^\infty \chi_{(0,x)}^{p'}(t) dt \Big)^{1/p'} \\ &= \frac{1}{x} \|f\|_{L^p(0,\infty)} x^{1/p'} \\ &= \|f\|_{L^p(0,\infty)} x^{-1/p} \xrightarrow{x \to \infty} 0, \end{aligned}$$

and hence we get $\alpha = \lim_{x\to\infty} Hf(x) = 0$.

Again from the Hölder's inequality we get

$$\begin{aligned}
|tH^*f(t)| &= t\left|\int_t^\infty \frac{f(x)}{x}dx\right| \le t\int_t^\infty \frac{|f(x)|}{x}dx \\
&\le t\|f\|_{L^p(0,\infty)}\Big(\int_t^\infty \frac{1}{x^{p'}}dx\Big)^{1/p'} \\
&= t\|f\|_{L^p(0,\infty)}\Big(\frac{t^{1-p'}}{p'-1}\Big)^{1/p'} \\
&= \Big(\frac{1}{p'-1}\Big)^{1/p'}\|f\|_{L^p(0,\infty)}t^{1/p'} \xrightarrow{t\to 0} 0,
\end{aligned}$$

because $1 < p' < \infty$. Hence $\beta = \lim_{t\to 0}(tH^*f(t)) = 0$.

Lemma 3.4. *Let X be a BFS on $(0,\infty)$ with the Lebesgue measure, and X' be the associate space of X. Then the following are equivalent;*

(i) $H : X \to X$ *and* $H^* : X \to X$ *are bounded;*

(ii) $\mathrm{Dom}[H - I, X] = X + \ker(H - I)$ *and*
$\mathrm{Dom}[H - I, X'] = X' + \ker(H - I)$;

(iii) $\mathrm{Dom}[H^* - I, X] = X + \ker(H^* - I)$ *and*
$\mathrm{Dom}[H^* - I, X'] = X' + \ker(H^* - I)$.

Proof. We prove first that *(i)* implies *(ii)*. Since $H : X \to X$ is bounded, we have that $X + \ker(H - \mathrm{I}) \subset \mathrm{Dom}[H - I, X]$. Since $H^* : X \to X$ is equivalent to $H : X' \to X'$ we also get that $X' + \ker(H - \mathrm{I}) \subset \mathrm{Dom}[H - \mathrm{I}, X']$. The reverse inclusion of the first follows directly from Lemma 3.2-*(i)*. Similarly, since $H^* : X' \to X'$ is equivalent to $H : X \to X$, reversing the role of X and X' in the previous argument we also get that $\mathrm{Dom}[H-\mathrm{I}, X'] \subset X'+\ker(H-\mathrm{I})$.

Now let us prove *(ii)* implies *(i)*. Let $\mathrm{Dom}[H - \mathrm{I}, X] = X + \ker(H - \mathrm{I})$. If $f \in X$ and $\alpha \in \ker[H - \mathrm{I}]$, then $H(f + \alpha) - (f + \alpha) = Hf - f \in X$ and hence $Hf \in X$. Thus, by [13, Theorem 1.8] we conclude that $H : X \to X$ is bounded. By a similar and dual argument we can also get that $H^* : X \to X$ is bounded.

Next we prove *(i)* implies *(iii)*. Since $H^* : X \to X$ is bounded, we get that $X + \ker(H^* - \mathrm{I}) \subset \mathrm{Dom}[H^* - \mathrm{I}, X]$. The fact that $H^* : X' \to X'$ is bounded gives also that $X' + \ker(H^* - \mathrm{I}) \subset \mathrm{Dom}[H^* - \mathrm{I}, X']$. The reverse inclusion of the first follows from Lemma 3.2-*(ii)*. Consequently, $\mathrm{Dom}[H^* - \mathrm{I}, X] = X + \ker(H^* - \mathrm{I})$. Similarly, since $H : X' \to X'$ is bounded we get also that $\mathrm{Dom}[H^* - \mathrm{I}, X'] = X' + \ker(H^* - \mathrm{I})$.

To prove that *(iii)* implies *(i)*, pick $f \in X$. Then $H^* f - f \in X$ and hence $H^* f \in X$. By [13, Theorem 1.8] we conclude that $H^* : X \to X$ is bounded. By a similar and dual argument we also get that $H : X \to X$ is bounded. □

Remark 3.5. Since $\ker(H^* - \mathrm{I}) \not\subset L^1_{\mathrm{loc}}(0, \infty)$, we deduce that neither $\mathrm{Dom}[H^* - \mathrm{I}, X]$ nor $\mathrm{Dom}[H^* - \mathrm{I}, X']$ are r.i. spaces on $(0, \infty)$ [13].

3.3 Optimal domain for hardy minus identity

In this section, we characterize the optimal non-absolute domain for $H - \mathrm{I}$, in the Lebesgue space $L^p(0, \infty), 1 \le p \le \infty$. We start with some properties of H.

Proposition 3.6.

(i) If f is non-negative function such that $Hf \in L^1(0, \infty)$, then $f \equiv 0$. That is,

$$L^1_+(0, \infty) \cap \mathrm{Dom}[H, L^1(0, \infty)] = \{0\}.$$

(ii) There exists $0 \not\equiv f \in L^1(0, \infty)$ such that $Hf \in L^1(0, \infty)$. Moreover, any such function satisfies $\int_0^\infty f(x)dx = 0$. That is,

$$L^1(0, \infty) \cap \mathrm{Dom}[H, L^1(0, \infty)] = L^1_0(0, \infty) \cap \mathrm{Dom}[H, L^1(0, \infty)] \neq \emptyset.$$

(iii) There exists $f \notin L^1(0, \infty)$ such that $Hf \in L^1(0, \infty)$ and there exists $g \in L^1_0(0, \infty)$ such that $Hg \notin L^1(0, \infty)$. That is,

$$\mathrm{Dom}[H, L^1(0, \infty)] \not\subset L^1(0, \infty) \quad \textit{and} \quad L^1_0(0, \infty) \not\subset \mathrm{Dom}[H, L^1(0, \infty)].$$

Proof. If f is a positive, locally integrable function we have that $Hf \in L^1(0,\infty)$ if and only if $f = 0$, a.e. This proves *(i)*.

To prove the first part of *(ii)*, i.e. to show the existence of $0 \not\equiv f \in L^1(0,\infty)$ such that $Hf \in L^1(0,\infty)$, take $f(x) = (1-x)e^{-x}$.

Now let us prove the second part of *(ii)*. Let $f \in L^1(0,\infty)$ and $Hf \in L^1(0,\infty)$. Then $g(x) := Hf(x) - f(x) \in L^1(0,\infty)$ and $\lim_{s\to\infty} Hf(s) = 0$, and from Lemma 3.2-*(i)* we obtain that

$$f(x) = H^* g(x) - g(x). \tag{47}$$

Integrating both sides of (47) on $(0,\infty)$ and then applying Fubini's theorem we get

$$\begin{aligned}\int_0^\infty f(x)dx &= \int_0^\infty \left(\int_x^\infty \frac{g(t)}{t}dt\right) dx - \int_0^\infty g(x)dx \\ &= \int_0^\infty \frac{g(t)}{t}\left(\int_0^t dx\right) dt - \int_0^\infty g(x)dx \\ &= \int_0^\infty g(t)dt - \int_0^\infty g(x)dx = 0.\end{aligned}$$

Hence $f \in L^1_0(0,\infty)$, which completes the proof of *(ii)*.

For the first part of *(iii)*, we take $f_t(x) = t\chi_{(2^t,2^t+1)}(x) - t\chi_{(2^t+1,2^t+2)}(x)$, for a fixed $t > 0$. In fact, $\|f_t\|_{L^1(0,\infty)} = 2t \to \infty$, as $t \to \infty$. But,

$$Hf_t(x) = t\left(1 - \frac{2^t}{x}\right)\chi_{(2^t,2^t+1)}(x) + t\left(\frac{2+2^t}{x} - 1\right)\chi_{(2^t+1,2^t+2)}(x),$$

and

$$\|Hf_t\|_{L^1(0,\infty)} = \|Hf_t\|_{L^1(0,\infty)} = \ln\left(\frac{2^t(2^t+2)}{(2^t+1)^2}\right)^{t2^t}\left(\frac{2^t+2}{2^t+1}\right)^{2t} \xrightarrow{t\to\infty} 0.$$

For the second part let us consider the function

$$g(x) = \frac{1}{\log 2}\chi_{(0,1]}(x) - \frac{1}{(x+1)\log^2(x+1)}\chi_{(1,\infty)}(x),$$

which is in $L^1_0(0,\infty)$ but $Hg(x) \notin L^1(0,\infty)$, where

$$Hg(x) = \frac{1}{\log 2}\chi_{(0,1]}(x) + \frac{1}{x\log(x+1)}\chi_{(1,\infty)}(x).$$

□

Remark 3.7. From Proposition 3.6-*(iii)*, we observe that

$$L^1(0,\infty) \cap L^\infty(0,\infty) \not\subset \mathrm{Dom}[H, L^1(0,\infty)],$$
$$\mathit{and} \qquad \mathrm{Dom}[H, L^1(0,\infty)] \not\subset L^1(0,\infty) + L^\infty(0,\infty),$$

and hence $\mathrm{Dom}[H, L^1(0,\infty)]$ is not an r.i. space.

Proposition 3.8. *Let w be a weight. Then*

(i) $L^1_0(w) \subset \mathrm{Dom}[H - \mathrm{I}, L^1(0,\infty)] \Longleftrightarrow$
$$\|Hf - f\|_{L^1(0,\infty)} \lesssim \|f\|_{L^1(w)} + \|f\|_{L^1_0(0,\infty)}$$

(ii) $(\mathrm{Dom}[H, L^1(0,\infty)] \cap L^1_0(0,\infty)) \subset L^1_0(w) \Longleftrightarrow$
$$\|f\|_{L^1_0(w)} \lesssim \|Hf\|_{L^1(0,\infty)} + \|f\|_{L^1_0(0,\infty)}$$

(iii) $L^\infty(w) \subset \mathrm{Dom}[H - \mathrm{I}, L^\infty(0,\infty)] \Longleftrightarrow \|Hf - f\|_{L^\infty(0,\infty)} \lesssim \|f\|_{L^\infty(w)}$

Proof. In each case, we use the Closed Graph Theorem to prove the continuity of the embeddings. To prove *(i)*, we need to show that

$$\left.\begin{aligned} f_n(x) &\xrightarrow{L^1_0(w)} 0, \\ Hf_n(x) - f_n(x) &\xrightarrow{L^1(0,\infty)} g(x) \end{aligned}\right\} \Longrightarrow g(x) \equiv 0.$$

From the definition of the norm in $L^1_0(w)$, we get $f_n(x) \xrightarrow{L^1_0(0,\infty)} 0$. Then we obtain $|Hf_n(x)| \to 0$, for a fixed $x > 0$ because

$$\begin{aligned} |Hf_n(x)| &= \Big|\frac{1}{x}\int_0^\infty f_n(t)dt - \frac{1}{x}\int_x^\infty f_n(t)dt\Big| \\ &\le \frac{1}{x}\int_x^\infty |f_n(t)|dt \\ &\le \frac{1}{x}\|f_n\|_{L^1(0,\infty)} \to 0. \end{aligned}$$

Again from the inequality $|Hf_n - g| \le |Hf_n - f_n - g| + |f_n|$, we obtain that $Hf_n \xrightarrow{L^1(0,\infty)} g$. So, there is a subsequence of $Hf_n(x)$ which converges to $g(x)$ for a.e. $x > 0$. Consequently, we get $g(x) \equiv 0$ for a.e. $x > 0$.

To prove the continuity of the embedding in *(ii)*

$$\|f\|_{L_0^1(w)} \lesssim \|Hf\|_{L^1(0,\infty)} + \|f\|_{L_0^1(0,\infty)},$$

it suffices to show that

$$\left.\begin{aligned} f_n(x) &\xrightarrow{L_0^1(0,\infty)} 0, \\ Hf_n(x) &\xrightarrow{L^1(0,\infty)} 0, \\ f_n(x) &\xrightarrow{L_0^1(w)} g(x) \end{aligned}\right\} \Longrightarrow g(x) \equiv 0.$$

From $f_n(x) \xrightarrow{L_0^1(w)} g(x)$, we obtain that $f_n(x) \xrightarrow{L_0^1(0,\infty)} g(x)$. Hence we get the required result.

To prove *(iii)*, we need to show that

$$\left.\begin{aligned} f_n(x) &\xrightarrow{L^\infty(w)} 0, \\ Hf_n(x) - f_n(x) &\xrightarrow{L^\infty(0,\infty)} g(x) \end{aligned}\right\} \Longrightarrow g(x) \equiv 0.$$

Let $\epsilon > 0$ be given. Let $w > 0$ on $[0,\infty)$ and $w \in C[0,\infty)$. Fix $a > 0$ and let $m_a = \min\{w(x) : 0 \le x \le a\}$. Then for $f_n(x) \in L^\infty(w)$, there exists a number $N_1 > 0$ such that, if $n > N_1$,

$$\begin{aligned} |f_n(x)| = 1/w(x)|f_n(x)|w(x) &\le 1/m_a|f_n(x)|w(x) \\ &\le 1/m_a\|f_n\|_{L^\infty(w)} < \epsilon/3 \end{aligned} \tag{48}$$

for almost everywhere $0 \le x \le a$. Again for $n > N_1$, from (48) we get

$$|Hf_n(x)| \le \frac{1}{x}\int_0^x |f_n(t)|dt < \varepsilon/3$$

Similarly, there exists a number $N_2 > 0$ such that, if $n > N_2$,

$$|g(x) - (Hf_n(x) - f_n(x))| < \varepsilon/3$$

a.e. on $(0, \infty)$. Thus, for a number $N \geq N_0$ where $N_0 = \max\{N_1, N_2\}$, if $n > N_0$, for a.e. $0 \leq x \leq a$ we obtain

$$|g(x)| \leq |g(x) - (Hf_n(x) - f_n(x))| + |Hf_n(x)| + |f_n(x)| < \varepsilon,$$

and hence $g(x) \equiv 0$ a.e. on $[0, a]$. Hence $g(x) = 0$ a.e. on $(0, \infty)$. $\square$

Theorem 3.9. *(Optimal domain for $H - \mathrm{I}$ on $L^p(0, \infty)$, $1 \leq p \leq \infty$)*

(i) For $p = 1$*;*

$$\mathrm{Dom}[H - \mathrm{I}, L^1(0, \infty)] = \big(\mathrm{Dom}[H, L^1(0, \infty)] \cap L^1_0(0, \infty)\big) + \ker(H - I).$$

(a) $L^1_0(w) \subset \mathrm{Dom}[H - \mathrm{I}, L^1(0, \infty)] \iff w(x) \gtrsim |\log x|$. *Hence, the logarithmic space* $L^1_0(|\log x|)$ *is the largest* $L^1_0(w)$ *space contained in* $\mathrm{Dom}[H - \mathrm{I}, L^1(0, \infty)]$. *Moreover,*

$$L^1_0(|\log x|) \subsetneq \mathrm{Dom}[H - \mathrm{I}, L^1(0, \infty)].$$

(b) $(\mathrm{Dom}[H, L^1(0, \infty)] \cap L^1_0(0, \infty)) \subset L^1_0(w) \iff w \lesssim 1$. *Hence,* $L^1_0(0, \infty)$ *is the smallest* $L^1_0(w)$ *space containing* $\mathrm{Dom}[H, L^1(0, \infty)] \cap L^1_0(0, \infty)$. *Moreover,*

$$\mathrm{Dom}[H, L^1(0, \infty)] \cap L^1_0(0, \infty) \subsetneq L^1_0((0, \infty)).$$

(ii) For $1 < p < \infty$, $\mathrm{Dom}[H - \mathrm{I}, L^p(0, \infty)] = L^p(0, \infty) + \ker(H - \mathrm{I})$.

(iii) For $p = \infty$*;*

(a) $L^\infty(w) \subset \mathrm{Dom}[H - \mathrm{I}, L^\infty(0, \infty)] \iff w \gtrsim 1$. *Hence* $L^\infty(0, \infty)$ *is the largest* $L^\infty(w)$ *space contained in* $\mathrm{Dom}[H - \mathrm{I}, L^\infty(0, \infty)]$. *Moreover*

$$L^\infty(0, \infty) \subsetneq \mathrm{Dom}[H - \mathrm{I}, L^\infty(0, \infty)].$$

(b) $\|f\|_{L^\infty(w)} \lesssim \|Hf - f\|_{L^\infty(0,\infty)} + |Hf(1)| \iff w(x) \lesssim 1/(1 + |\log x|)$. *Hence* $L^\infty(1/(1+|\log x|))$ *is the smallest* $L^\infty(w)$ *space containing* $\mathrm{Dom}[H-I, L^\infty(0,\infty)]$. *Moreover,*

$$\mathrm{Dom}[H - \mathrm{I}, L^\infty(0,\infty)] \subsetneq L^\infty(1/(1+|\log x|)).$$

Proof. To prove the equality of the domains in *(i)*, suppose $f \in \mathrm{Dom}[H - \mathrm{I}, L^1(0,\infty)]$. Since $H^* : L^1(0,\infty) \to L^1(0,\infty)$ is bounded, from Lemma 3.2-*(i)* we get $f + \beta \in L^1(0,\infty)$, where $\beta \in \ker(H - \mathrm{I})$.

Now we show that $(f+\beta) \in \mathrm{Dom}[H, L^1(0,\infty)]$. Since

$$H(f+\beta) - (f+\beta) = Hf - f \in L^1(0,\infty)$$

and $(f+\beta) \in L^1(0,\infty)$ we have that $H(f+\beta) \in L^1(0,\infty)$, which means that $(f+\beta) \in \mathrm{Dom}[H, L^1(0,\infty)]$. Due to Proposition 3.6-*(ii)*, $f + \beta \in L^1_0(0,\infty)$. Thus

$$\mathrm{Dom}[H - \mathrm{I}, L^1(0,\infty)] \subset (\mathrm{Dom}[H, L^1(0,\infty)] \cap L^1_0((0,\infty)) + \ker(H - I)$$

Conversely, if $f + \beta \in \mathrm{Dom}[H, L^1(0,\infty)] \cap L^1_0(0,\infty) + \ker(H - I)$, where $\beta \in \ker(H - \mathrm{I})$ we obtain that

$$H(f+\beta) - (f+\beta) = Hf - f \in L^1(0,\infty).$$

That is, $f + \beta \in \mathrm{Dom}[H - \mathrm{I}, L^1(0,\infty)]$. This completes the proof of *(i)*.

To prove *(i)-(a)*, we let first that $w(x) \gtrsim |\log x|$. Let $f \in L^1_0(w)$. Then by definition $f \in L^1(w) \cap L^1_0(0,\infty)$. Since $L^1_0(w) \subset L^1_0(|\log x|)$, we get $f \in L^1_0(|\log x|)$. Now we remain to show that $f \in \mathrm{Dom}[H, L^1(0,\infty)]$. Due to the cancellation property of f (i.e., $f \in L^1_0(0,\infty)$) we see that

$$\int_0^x f(t)dt = -\int_x^\infty f(t)dt$$

and

$$\int_0^\infty |Hf(t)|dt = \int_0^1 \frac{1}{x}\Big|\int_0^x f(t)dt\Big|dx + \int_1^\infty \frac{1}{x}\Big|\int_x^\infty f(t)dt\Big|dx$$
$$\leq \int_0^1 \int_t^1 |f(t)|\frac{1}{x}dx\,dt + \int_1^\infty \int_1^t |f(t)|\frac{1}{x}dx\,dt$$
$$= \int_0^1 |f(t)|(-\log t)dt + \int_1^\infty |f(t)|\log t\,dt$$
$$= \int_0^\infty |f(t)||\log t|dt < \infty.$$

Hence $f \in \mathrm{Dom}[H, L^1(0,\infty)]$. To prove that the inclusion is strict we consider the function $f(x) = -x\chi_{(0,1]}(x) + \frac{1}{x(1+\log x)^3}\chi_{(1,\infty)}(x)$. It is not difficult to show that $f \in L_0^1(0,\infty)$ but $f \notin L_0^1(|\log x|)$. Again $Hf(x) = -\frac{1}{2}x\chi_{(0,1]}(x) - \frac{1}{2x(1+\log x)^2}\chi_{(1,\infty)}(x)$ which belongs to $L^1(0,\infty)$.

To prove the necessity, we assume that $L_0^1(w) \subset \mathrm{Dom}[H - \mathrm{I}, L^1(0,\infty)]$. Let $\varepsilon > 0$. Fix $\varepsilon < x_0 < 1$, and let $f(x) = \chi_{(x_0-\varepsilon,x_0)}(x) - \chi_{(1,1+\varepsilon)}(x)$. Then

$$Hf(x) = \begin{cases} 0, & x \leq x_0 - \varepsilon; \\ \dfrac{x - x_0 + \varepsilon}{x}, & x_0 - \varepsilon < x \leq x_0; \\ \dfrac{\varepsilon}{x}, & x_0 < x \leq 1; \\ \dfrac{\varepsilon - x + 1}{x}, & 1 < x \leq 1 + \varepsilon; \\ 0, & x \geq 1 + \varepsilon, \end{cases}$$

and

$$Hf(x) - f(x) = \begin{cases} 0, & x \leq x_0 - \varepsilon; \\ \dfrac{-x_0 + \varepsilon}{x}, & x_0 - \varepsilon < x \leq x_0; \\ \dfrac{\varepsilon}{x}, & x_0 < x \leq 1; \\ \dfrac{1 + \varepsilon}{x}, & 1 < x \leq 1 + \varepsilon; \\ 0, & x \geq 1 + \varepsilon. \end{cases}$$

From these inequalities we get

$$\|Hf - f\|_{L^1(0,\infty)} = (x_0 - \varepsilon)\log\left(\frac{x_0}{x_0-\varepsilon}\right) - \varepsilon\log x_0 + (1+\varepsilon)\log(1+\varepsilon).$$

From Proposition 3.8-*(i)* we obtain

$$(x_0 - \varepsilon)\log\left(\frac{x_0}{x_0-\varepsilon}\right) - \varepsilon\log x_0 + (1+\varepsilon)\log(1+\epsilon)$$
$$\lesssim \int_{x_0-\varepsilon}^{x_0} w(t)dt + \int_1^{1+\varepsilon} w(t)dt + 2\varepsilon.$$

Dividing by ϵ gives

$$\log\left(\frac{x_0}{x_0-\varepsilon}\right)^{\frac{x_0-\varepsilon}{\varepsilon}} - \log x_0 + \log(1+\varepsilon)^{\frac{1+\varepsilon}{\varepsilon}} \lesssim \frac{1}{\varepsilon}\int_{x_0-\varepsilon}^{x_0} w(t)dt + \frac{1}{\varepsilon}\int_1^{1+\varepsilon} w(t)dt + 2.$$

Letting $\varepsilon \to 0$ we obtain

$$2 - \log x_0 \lesssim w(x_0) + w(1) + 2.$$

If $f \in L_0^1(w)$, then $f \in L_0^1(0,\infty)$ and hence w is bounded below. Hence $w(x_0) \gtrsim -\log x_0$ for $0 < x_0 < 1$. Similarly, for $1+\varepsilon < x_0 < \infty$ consider $f(x) = \chi_{(1,1+\varepsilon)}(x) - \chi_{(x_0,x_0+\varepsilon)}(x)$. Then

$$Hf(x) = \begin{cases} 0, & x \leq 1; \\ \dfrac{x-1}{x}, & 1 < x \leq 1+\varepsilon; \\ \dfrac{\varepsilon}{x}, & 1+\varepsilon < x \leq x_0; \\ \dfrac{\varepsilon - x + x_0}{x}, & x_0 < x \leq x_0 + \varepsilon; \\ 0, & x \geq x_0 + \varepsilon, \end{cases}$$

and

$$Hf(x)-f(x)=\begin{cases}0, & x\le 1;\\ -\dfrac{1}{x}, & 1<x\le 1+\varepsilon;\\ \dfrac{\varepsilon}{x}, & 1+\varepsilon<x\le x_0;\\ \dfrac{\varepsilon+x_0}{x}, & x_0<x\le x_0+\varepsilon;\\ 0, & x\ge x_0+\varepsilon,\end{cases}$$

From the embedding and then letting $\varepsilon\to 0$ we obtain

$$\|Hf-f\|_{L^1(0,\infty)}=2+\log x_0\lesssim w(1)+w(x_0)+2=\|f\|_{L^1(w)}+\|f\|_{L^1_0(0,\infty)}.$$

Thus, $w(x_0)\gtrsim \log x_0$ for $1\le x_0<\infty$. Therefore, $w(x)\gtrsim |\log x|$ for $x>0$.

To prove *(i)-(b)*, suppose that $\mathrm{Dom}[H,L^1(0,\infty)]\cap L^1_0(0,\infty)\subset L^1_0(w)$; that is,

$$\|f\|_{L^1_0(w)}\lesssim \|Hf\|_{L^1(0,\infty)}+\|f\|_{L^1_0(0,\infty)}. \tag{49}$$

Consider $f(x)=\chi_{[x_0-\frac{1}{n},x_0]}(x)-\chi_{[x_0,x_0+\frac{1}{n}]}(x)$, for $\frac{1}{n}<x_0<\infty$. Then

$$Hf(x)=\Big(1-\frac{x_0+\frac{1}{n}}{x}\Big)\chi_{[x_0-\frac{1}{n},x_0]}(x)+\Big(\frac{x_0+\frac{1}{n}}{x}-1\Big)\chi_{(x_0,x_0+\frac{1}{n}]}(x)$$

Hence from (49) we obtain

$$\begin{aligned}\int_{x_0-\frac{1}{n}}^{x_0+\frac{1}{n}} w(x)dx &\lesssim \int_{x_0-\frac{1}{n}}^{x_0}\left(\frac{x_0+\frac{1}{n}}{x}-1\right)dx+\int_{x_0}^{x_0+\frac{1}{n}}\left(\frac{x_0+\frac{1}{n}}{x}-1\right)dx+\frac{2}{n}\\ &\lesssim -\Big(x_0+\frac{1}{n}\Big)\ln\Big(1-\frac{1}{nx_0}\Big)+\Big(x_0+\frac{1}{n}\Big)\ln\Big(1+\frac{1}{nx_0}\Big)+\frac{2}{n}\end{aligned}$$

Multiplying by $n/2$ we get

$$\frac{n}{2}\int_{x_0-\frac{1}{n}}^{x_0+\frac{1}{n}} w(x)dx\lesssim \frac{1}{2}\Big[\ln\Big(1-\frac{1}{nx_0}\Big)^{nx_0+1}+\ln\Big(1+\frac{1}{nx_0}\Big)^{nx_0+1}+2\Big].$$

Letting $n\to\infty$ and applying the Lebesgue differentiation theorem we get $w(x_0)\lesssim 1$, for arbitrary $x_0>0$. Hence $w(x)\lesssim 1$, a.e.

Conversely, if $w(x) \lesssim 1$, then trivially

$$\mathrm{Dom}[H, L^1(0,\infty)] \cap L^1_0(0,\infty) \subset L^1_0(0,\infty) \subset L^1_0(w).$$

Moreover, the inclusion is strict by the second part of Proposition 3.6-*(iii)*.

Since the operators $H, H^* : L^p(0,\infty) \to L^p(0,\infty)$ are bounded for $1 < p < \infty$ [35, Theorem 327 and 328], *(ii)* follows directly from Lemma 3.4.

Now let us prove *(iii)-(a)*. It is clear that if $w \gtrsim 1$, then $L^\infty(w) \subset L^\infty(0,\infty)$. Since $H : L^\infty(0,\infty) \to L^\infty(0,\infty)$ is bounded, we get the required result.

To prove the reverse implication, fix $x_0 > 0$ and let $N \in \mathbb{N}$ such that $N \geq 1/x_0$. Fix $n > N$. For $f_n(x) = \chi_{(x_0-1/n, x_0+1/n)}(x)$ we have

$$(Hf_n - f_n)(x) = \frac{\frac{1}{n} - x_0}{x}\chi_{(x_0-1/n,x_0+1/n)} + \frac{2}{nx}\chi_{[x_0+1/n,\infty)}.$$

Then, we get

$$\sup_{x_0-1/n<x<x_0+1/n} w(x) = \|f_n\|_{L^\infty(w)} \gtrsim \|Hf_n - f_n\|_{L^\infty(0,\infty)} = 1.$$

Letting $n \to \infty$ we get $w(x_0) \gtrsim 1$, for any $x_0 > 0$.

To show that the inclusion is strict, we take $f(x) = \log x$. Clearly, $f \notin L^\infty(0,\infty)$ but $Hf(x) - f(x) = -1 \in L^\infty(0,\infty)$.

To prove the necessity of *(iii)-(b)*, we take $f(x) = 1 + |\log x|$. Then

$$Hf(x) = \frac{1}{x}\int_0^x (1+|\log t|)dt = (2-\log x)\chi_{(0,1]}(x) + \Big(\frac{2}{x} + \log x\Big)\chi_{(1,\infty)}(x)$$

and

$$Hf(x) - f(x) = \chi_{(0,1]}(x) + \Big(\frac{2}{x} - 1\Big)\chi_{(1,\infty)}(x).$$

From this we obtain that $\|Hf - f\|_{L^\infty(0,\infty)} = 1$, and

$$(1+|\log x|)w(x) \leq \|f\|_{L^\infty(w)} \lesssim \|Hf - f\|_{L^\infty(0,\infty)} + |Hf(1)| = 3.$$

Hence $w(x) \lesssim 1/(1+|\log x|)$, for all $x > 0$.

To prove the converse, we let $w(x) \lesssim 1/(1+|\log x|)$, for all $x > 0$. It is clear that $L^\infty(1/(1+|\log x|)) \subset L^\infty(w)$. Then it suffices to show that

$$\|f\|_{L^\infty(1/(1+|\log x|))} \lesssim \|Hf - f\|_{L^\infty(0,\infty)} + |Hf(1)|.$$

Let $f \in \mathrm{Dom}[H - \mathrm{I}, L^\infty(0,\infty)]$, and set $g(x) = Hf(x) - f(x)$. Then for $0 < x < y < \infty$, we see that [cf. the proof of Lemma 3.2-*(i)*]

$$f(x) = \int_x^y \frac{g(s)}{s} ds - g(x) + Hf(y).$$

For $0 < x < 1$ (i.e., when $y = 1$), we get

$$\begin{aligned}\frac{|f(x)|}{1 + |\log x|} = \frac{|f(x)|}{1 - \log x} &\le \|g\|_{L^\infty(0,\infty)} + |Hf(1)| \frac{1}{1 - \log x} \\ &\le \|g\|_{L^\infty(0,\infty)} + |Hf(1)|\end{aligned}$$

Again for $0 < y < x < \infty$, we get

$$f(x) = -\int_y^x \frac{g(s)}{s} ds - g(x) + Hf(y).$$

Then for $1 < x < \infty$ (i.e., when $y = 1$) we obtain

$$\begin{aligned}\frac{|f(x)|}{1 + |\log x|} = \frac{|f(x)|}{1 + \log x} &\le \|g\|_{L^\infty(0,\infty)} + |Hf(1)| \frac{1}{1 + \log x} \\ &\le \|g\|_{L^\infty(0,\infty)} + |Hf(1)|\end{aligned}$$

Hence $f(x) \in L^\infty(1/(1 + |\log x|)$, and the result follows. To prove that the embedding is strict, we take $f(x) = \sin x \log x \chi_{[1,\infty)}(x)$. Since

$$\frac{|f(x)|}{1 + |\log x|} \le \frac{\log x}{1 + \log x} \chi_{(1,\infty)}(x) \le \chi_{[1,\infty)}(x),$$

$f(x) \in L^\infty(1/(1 + |\log x|))$. But

$$Hf(x) - f(x) = \Big(-\frac{\cos x \log x}{x} + \frac{1}{x} \int_1^x \frac{\cos t}{t} dt - \sin x \log x \Big) \chi_{[1,\infty)}(x), \quad (50)$$

does not belong to $L^\infty(0,\infty)$. In fact, since

$$\sup_{x>1} \Big| \frac{\cos x \log x}{x} \Big| < \infty, \qquad \sup_{x>1} \big| \sin x \log x \big| = \infty,$$

and

$$\sup_{x>1} \frac{1}{x} \Big| \int_1^x \frac{\cos t}{t} dt \Big| \le \sup_{x>1} \frac{\log x}{x} < \infty,$$

we conclude that $\sup_{x>1} |Hf(x) - f(x)| = \infty$. □

Remark 3.10. $\mathrm{Dom}[H-\mathrm{I}, L^p(0,\infty)]$, $1 \le p \le \infty$, does not satisfy the lattice property (i.e., if $|g| \le |f|$ and f belongs to the space, so does g). In fact,

$$f(x) = \chi_{[0,1]}(x) - \chi_{[1,2]}(x) \in \mathrm{Dom}[H-\mathrm{I}, L^1(0,\infty)],$$

but $g(x) = \chi_{[0,2]}(x) \notin \mathrm{Dom}[H-\mathrm{I}, L^1(0,\infty)]$. Observe that $g = |f|$ and

$$Hf(x) - f(x) = \frac{2}{x}\chi_{(1,2]}(x) \in L^1(0,\infty),$$

but $Hg(x) - g(x) = \dfrac{2}{x}\chi_{(2,\infty)} \notin L^1(0,\infty)$.

For $1 < p < \infty$, from Theorem 3.9-*(ii)* it suffices to consider

$$f(x) = 1 + \frac{1}{x}\chi_{[1/2,1]}(x) \in L^p(0,\infty) + \ker(H-\mathrm{I}),$$

but $g(x) = \sin x \notin L^p(0,\infty) + \ker(H-\mathrm{I})$.

For $p = \infty$, $f(x) = \log x \chi_{[1,\infty)}(x) \in \mathrm{Dom}[H-\mathrm{I}, L^\infty(0,\infty)]$, but as shown in (50),

$$g(x) = \sin x \log x \chi_{[1,\infty)}(x) \notin \mathrm{Dom}[H-\mathrm{I}, L^\infty(0,\infty)].$$

Indeed,

$$Hf(x) - f(x) = \Big(\frac{1}{x} - 1\Big)\chi_{[1,\infty)}(x) \in L^\infty(0,\infty).$$

3.4 Optimal domain for dual hardy minus identity

We are now going to study the analogous results of section 3.3 for the dual Hardy operator, H^*.

Proposition 3.11.

(i) There exists $f \in L^\infty(0,\infty)$ *such that* $H^*f \in L^\infty(0,\infty)$*; that is,*

$$L^\infty(0,\infty) \cap \mathrm{Dom}[H^*, L^\infty(0,\infty)] \neq \emptyset.$$

(ii) There exists $g \notin L^\infty(0,\infty)$ *such that* $H^*g \in L^\infty(0,\infty)$*; that is,*

$$\mathrm{Dom}[H^*, L^\infty(0,\infty)] \not\subset L^\infty(0,\infty).$$

Proof. To prove *(i)*, we take the function $f(x) = xe^{-x}$. For *(ii)*, define $\{I_k\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned}
&I_1 = [1, 3/2] = [a_1, a_2]\\
&I_2 = [3/2, 7/4] = [a_2, a_3]\\
&\ \vdots\\
&I_k = [a_k, a_{k+1}], \qquad I_{k+1} = [a_{k+1}, a_{k+1} + 1/2^{k+1}]
\end{aligned} \tag{51}$$

Hence, $|I_k| = 1/2^k$, $\cup_{k\in\mathbb{N}} I_k = [1, 2)$. Fix $1 < r < 2$ and define

$$w(t) = \begin{cases} \dfrac{1}{\sqrt{t}}, & t \in (0, 1);\\ \dfrac{1}{r^k}, & t \in I_k;\\ t, & t \in [2, \infty). \end{cases} \tag{52}$$

Then $g(x) = 1/w(x)$ proves *(ii)*. In fact, $g(x) \notin L^\infty(0, \infty)$ and

$$\begin{aligned}
\|H^* g\|_{L^\infty(0,\infty)} = \sup_{x>0} |H^* g(x)| = \int_0^\infty \frac{g(t)}{t} dt = \int_0^\infty \frac{1}{tw(t)} dt\\
= \int_0^1 \frac{dt}{\sqrt{t}} + \int_2^\infty \frac{dt}{t^2} + \sum_{k=1}^\infty r^k \int_{I_k} \frac{dt}{t}.
\end{aligned}$$

Observe that

$$a_1 = 1,\ a_2 = 1 + \frac{1}{2} = \frac{3}{2},\ a_3 = \frac{3}{2} + \frac{1}{2^2} = \frac{7}{4},\ a_4 = \frac{7}{4} + \frac{1}{2^3} = \frac{15}{8}, \cdots .$$

In general, $a_k = \dfrac{2^k - 1}{2^{k-1}}$ for $k \in \mathbb{N}$. Hence

$$\begin{aligned}
\|H^* g\|_{L^\infty(0,\infty)} &= 2 + \frac{1}{2} + \sum_{k=1}^\infty r^k \log \left(\frac{\frac{2^{k+1} - 1}{2^k}}{\frac{2^k - 1}{2^{k-1}}} \right)\\
&\approx \frac{5}{2} + \sum_{k=1}^\infty r^k \frac{1}{2^k} < \infty,
\end{aligned} \tag{53}$$

because

$$\frac{\frac{2^{k+1}-1}{2^k}}{\frac{2^k-1}{2^{k-1}}} = \frac{2^{k+1}-1}{2^{k+1}-2} = 1 + \frac{1}{2^{k+1}-2} \quad \text{and} \quad \frac{\log(x+1)}{x} \xrightarrow{x\to 0} 1.$$

□

Proposition 3.12. *For $x > 0$, let $w(x) > 0$ be a weight. Then,*

$$H^* : L^\infty(w) \longrightarrow L^\infty(0,\infty)$$

is bounded if and only if $A = \int_0^\infty \frac{1}{tw(t)}dt < \infty$. Moreover, $A = \|H^*\|$ is the best constant in the inequality $\|H^* f\|_{L^\infty(0,\infty)} \le A\|f\|_{L^\infty(w)}$.

Proof. To prove the necessity, we let $f_N(x) = (1/w(x))\chi_{(0,N]}(x),\ N > 0$. Since H^* is bounded, we get

$$\int_0^N \frac{1}{tw(t)}dt = \sup_{x>0} |H^* f_N(x)| = \|H^* f_N\|_{L^\infty(0,\infty)} \le \|H^*\|,$$

for all $N > 0$. Letting $N \longrightarrow \infty$, we obtain

$$A := \int_0^\infty \frac{1}{tw(t)}dt \le \|H^*\| < \infty.$$

We prove the sufficiency by using the homogeneity of norms. It is sufficient to see that $\|H^* f\|_{L^\infty(0,\infty)} \le A$, for all $f \in L^\infty(w)$, such that $|f(x)|w(x) \le 1$. Now,

$$\|H^* f\|_{L^\infty(0,\infty)} = \sup_{x>0}\left|\int_x^\infty \frac{f(t)}{t}dt\right| \le \sup_{x>0}\int_x^\infty \frac{1}{tw(t)}dt = \int_0^\infty \frac{1}{tw(t)}dt = A.$$

which shows that $\|H^*\| \le A < \infty$. Hence H^* is bounded and $A = \|H^*\|$. □

Theorem 3.13. *(Optimal domain for $H^* - \mathrm{I}$ on $L^p(0,\infty)$, $1 \le p \le \infty$)*

(i) For $p = 1$,

(a) $\|H^*f - f\|_{L^1(0,\infty)} \lesssim \|f\|_{L^1(w)} \iff w(x) \gtrsim 1$. *Hence,* $L^1(0,\infty)$ *is the largest weighted space* $L^1(w)$ *contained in* $\mathrm{Dom}[H^*{-}\mathrm{I}, L^1(0,\infty)]$. *Moreover,*

$$L^1(0,\infty) \subsetneq \mathrm{Dom}[H^* - \mathrm{I}, L^1(0,\infty)].$$

(b) $\mathrm{Dom}[H^* - \mathrm{I}, L^1(0,\infty)] \subsetneq L^{1,\infty}(0,\infty)$.

(ii) For $1 < p < \infty$, $\mathrm{Dom}[H^* - \mathrm{I}, L^p(0,\infty)] = L^p(0,\infty) + \ker(H^* - \mathrm{I})$

(iii) For $p = \infty$,

$$\mathrm{Dom}[H^*{-}\mathrm{I}, L^\infty(0,\infty)] = (\mathrm{Dom}[H^*, L^\infty(0,\infty)] \cap L^\infty(0,\infty)) {+} \ker(H^*{-}\mathrm{I}).$$

Moreover,

(a) if $W = \left\{ w(x) : w(x) \gtrsim 1, \text{ for } x > 0 \text{ and } \int_0^\infty \frac{1}{tw(t)} dt < \infty \right\}$*, then*

$$L^\infty(w) \subset (\mathrm{Dom}[H^*, L^\infty(0,\infty)] \cap L^\infty(0,\infty)) \iff w(x) \in W.$$

(b) $\|f\|_{L^\infty(w)} \lesssim \|H^*f\|_{L^\infty(0,\infty)} + \|f\|_{L^\infty(0,\infty)} \iff w(x) \lesssim 1$. *Hence,* $L^\infty(0,\infty)$ *is the smallest* $L^\infty(w)$ *space containing* $\mathrm{Dom}[H^*, L^\infty(0,\infty)] \cap L^\infty(0,\infty)$. *Moreover,*

$$(\mathrm{Dom}[H^*, L^\infty(0,\infty)] \cap L^\infty(0,\infty)) \subsetneq L^\infty(0,\infty).$$

Proof. To prove the necessity of *(i)-(a)*, take $f(x) = \chi_E(x)$, where $E = [x_0 - 1/n, x_0 + 1/n]$ for $x_0 > \frac{e+1}{e-1}\frac{1}{n}$. Then

$$H^*f(x) = \ln\left(\frac{x_0 + 1/n}{x_0 - 1/n}\right)\chi_{(0,x_0-1/n]}(x) + \ln\left(\frac{x_0 + 1/n}{x}\right)\chi_E(x)$$

and

$$H^*f(x) - f(x) = \log\left(\frac{x_0 + 1/n}{x_0 - 1/n}\right)\chi_{(0,x_0-1/n]}(x) + \left(-1 + \log\left(\frac{x_0 + 1/n}{x}\right)\right)\chi_E(x)$$

For $x_0 > \frac{e+1}{e-1}\frac{1}{n}$, $-1+\log\Big(\dfrac{x_0+1/n}{x}\Big) < 0$. Then

$$\begin{aligned}\|H^*f-f\|_{L^1(0,\infty)} &= \log\Big(\frac{x_0+1/n}{x_0-1/n}\Big)\Big(x_0-1/n\Big)+\int_{x_0-1/n}^{x_0+1/n}dt\\ &\qquad -\int_{x_0-1/n}^{x_0+1/n}\log\Big(\frac{x_0+1/n}{t}\Big)dt\\ &= (x_0-1/n)\log\Big(\frac{x_0+1/n}{x_0-1/n}\Big)+\frac{2}{n}-\frac{2}{n}\log(x_0+1/n)\\ &\qquad +\int_{x_0-1/n}^{x_0+1/n}\log t\,dt\end{aligned}$$

Using the inequality

$$\|H^*f-f\|_{L^1(0,\infty)} \lesssim \|f\|_{L^1(w)}$$

and multiplying both sides by $n/2$ we get

$$\begin{aligned}\ln\Big(\frac{x_0+1/n}{x_0-1/n}\Big)^{\frac{nx_0-1}{2}}+1-\ln(x_0+1/n)+\frac{n}{2}\int_{x_0-1/n}^{x_0+1/n}\ln t dt\\ \le \frac{n}{2}c\int_{x_0-1/n}^{x_0+1/n}w(t)dt,\end{aligned}$$

for some $c>0$. Letting $n\to\infty$ we obtain $w(x_0)\ge 2/c$ a.e.$(i.e., w(x)\gtrsim 1)$.

To prove the sufficiency, we assume that $w(x)\gtrsim 1$. Then we get $L^1(w)\subset L^1(0,\infty)$. Consequently, we obtain

$$\|H^*f-f\|_{L^1(0,\infty)} \le \|H^*f\|_{L^1(0,\infty)}+\|f\|_{L^1(0,\infty)} \lesssim \|f\|_{L^1(0,\infty)} \lesssim \|f\|_{L^1(w)},$$

since

$$\begin{aligned}\|H^*f\|_{L^1(0,\infty)} &\le \int_0^\infty\int_x^\infty \frac{|f(t)|}{t}dtdx = \int_0^\infty\int_0^t \frac{|f(t)|}{t}dxdt\\ &= \int_0^\infty |f(t)|dt = \|f\|_{L^1(0,\infty)}.\end{aligned}$$

By considering the function $f(x)=x/(x+1)^2$ we can show that the inclusion is strict. It is clear that $f\in \mathrm{Dom}[H^*-\mathrm{I}, L^1(0,\infty)]$ but $f\notin L^1(0,\infty)$.

To prove *(i)-(b)*, first let us show that $\mathrm{Dom}[H^* - \mathrm{I}, L^1(0,\infty)] \subset L^{1,\infty}(0,\infty)$. Let $f \in \mathrm{Dom}[H^* - \mathrm{I}, L^1(0,\infty)]$. That is, $g(x) := H^*f(x) - f(x) \in L^1(0,\infty)$. By Lemma 3.2-*(ii)*,

$$f(x) = Hg(x) - g(x) + \frac{\beta}{x},$$

where $\beta = \lim_{s\to 0} sH^*f(s)$. If g^* is the decreasing rearrangement of g, then

$$\begin{aligned}\|Hg\|_{L^{1,\infty}(0,\infty)} &\le \|Hg^*\|_{L^{1,\infty}}(0,\infty) = \sup_{t>0} t \cdot \frac{1}{t}\int_0^t g^*(x)dx \\ &= \|g^*\|_{L^1(0,\infty)} = \|g\|_{L^1(0,\infty)} < \infty\end{aligned}$$

Since $\frac{\beta}{x} \in L^{1,\infty}(0,\infty)$, we get $f \in L^{1,\infty}(0,\infty)$. To show that the inclusion is strict we consider $f(x) = \frac{1}{x}\sum_{k=1}^{\infty}(-1)^k \chi_{(k-1,k]}(x)$. Since $|f(x)| = \frac{1}{x}$, we have $f \in L^{1,\infty}(0,\infty)$. But, $f \notin \mathrm{Dom}[H^* - \mathrm{I}, L^1(0,\infty]$. In fact, for $x \in (2k-2, 2k-1], k \in \mathbb{N}$, we obtain

$$\begin{aligned}H^*f(x) &= \int_0^\infty \frac{f(t)}{t}dt \\ &= -\int_x^{2k-1} \frac{1}{t^2}dt + \int_{2k-1}^{2k}\frac{1}{t^2}dt - \int_{2k}^{2k+1}\frac{1}{t^2}dt + \int_{2k+1}^{2k+2}\frac{1}{t^2}dt - \cdots \\ &= -\frac{1}{x} + \frac{1}{2k-1} + \sum_{n=1}^{\infty}(-1)^{2(k-1)+n+1}\int_{2(k-1)+n}^{2(k-1)+n+1}\frac{1}{t^2}dt \\ &= -\frac{1}{x} + \frac{1}{2k-1} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2(k-1)+n)(2(k-1)+n+1)} \\ &= -\frac{1}{x} + 2\sum_{n=2k-1}^{\infty}\frac{(-1)^{n+1}}{n},\end{aligned}$$

because the jth partial sum S_j of the series is

$$S_j = \frac{1}{2k-1} - \frac{(-1)^{j+1}}{2(k-1)+j+1} + 2\sum_{m=2k}^{j}\frac{(-1)^{m+1}}{m}$$

and $\lim_{j\to\infty} S_j = \frac{1}{2k-1} + 2\sum_{m=2k}^{\infty} \frac{(-1)^{m+1}}{m} < \infty$. Thus

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2(k-1)+n)(2(k-1)+n+1} = \frac{1}{2k-1} + 2\sum_{m=2k}^{\infty}\frac{(-1)^{m+1}}{m}$$

Then

$$\begin{aligned} H^*f(x) - f(x) &= 2\sum_{n=2k-1}^{\infty}\frac{(-1)^{n+1}}{n} = 2\Big(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} - \sum_{n=1}^{2k-2}\frac{(-1)^{n+1}}{n}\Big) \\ &= 2\Big(\log 2 - \sum_{n=1}^{2k-2}\frac{(-1)^{n+1}}{n}\Big) \geq 0 \end{aligned}$$

Again for $x \in (2k-1, 2k], k \in \mathbb{N}$,

$$H^*f(x) = \frac{1}{x} - 2\sum_{n=2k}^{\infty}\frac{(-1)^n}{n} = \frac{1}{x} + 2\sum_{n=2k}^{\infty}\frac{(-1)^{n+1}}{n}$$

Then

$$\begin{aligned} H^*f(x) - f(x) &= 2\sum_{n=2k}^{\infty}\frac{(-1)^{n+1}}{n} = 2\Big(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} - \sum_{n=1}^{2k-1}\frac{(-1)^{n+1}}{n}\Big) \\ &= 2\Big(\log 2 - \sum_{n=1}^{2k-1}\frac{(-1)^{n+1}}{n}\Big) \leq 0. \end{aligned}$$

Thus,

$$\begin{aligned}
&\|H^*f - f\|_{L^1(0,\infty)} \\
&\qquad = \int_0^\infty |H^*f(x) - f(x)|dx \\
&\qquad = \sum_{k=1}^\infty \Big(\int_{2k-2}^{2k} |H^*f(x) - f(x)|dx \Big) \\
&\qquad = 2\sum_{k=1}^\infty \Big(\sum_{n=2k-1}^\infty \frac{(-1)^{n+1}}{n} \int_{2k-2}^{2k-1} dx - \sum_{n=2k}^\infty \frac{(-1)^{n+1}}{n} \int_{2k-1}^{2k} dx \Big) \\
&\qquad = 2\sum_{k=1}^\infty \Big(\sum_{n=2k-1}^\infty \frac{(-1)^{n+1}}{n} - \sum_{n=2k}^\infty \frac{(-1)^{n+1}}{n} \Big) \\
&\qquad = 2\sum_{k=1}^\infty \frac{1}{2k-1} = \infty. \qquad (54)
\end{aligned}$$

Hence $f \notin \mathrm{Dom}[H^* - I, L^1(0,\infty)]$. This completes the proof of *(i)-(b)*.

Since the operators $H, H^* : L^p(0,\infty) \to L^p(0,\infty)$ are bounded for $1 < p < \infty$ [35, Theorem 327 and 328], *(ii)* follows directly from Lemma 3.4.

To prove the equality of the domains in *(iii)*, we assume first that $f \in \mathrm{Dom}[H^* - \mathrm{I}, L^\infty(0,\infty)]$; that is, $g(x) := H^*f(x) - f(x) \in L^\infty(0,\infty)$. Since $H : L^\infty(0,\infty) \to L^\infty(0,\infty)$ is bounded, by Lemma 3.2-*(ii)* we obtain that $f - \beta/x \in L^\infty(0,\infty)$, where $\beta/x \in \ker[H^* - I]$. Moreover,

$$H^*(f - \beta/x) - (f - \beta/x) = H^*f - f \in L^\infty(0,\infty).$$

implies that $H^*(f-\beta/x) \in L^\infty(0,\infty)$. That is, $f-\beta/x \in \mathrm{Dom}[H^*, L^\infty(0,\infty)]$. Hence $f \in (\mathrm{Dom}[H^*, L^\infty(0,\infty)] \cap L^\infty(0,\infty)) + \ker[H^* - \mathrm{I}]$. To prove the reverse inclusion, we let

$$f + \beta/x \in (\mathrm{Dom}[H^*, L^\infty(0,\infty)] \cap L^\infty(0,\infty)) + \ker[H^* - \mathrm{I}],$$

where $\beta/x \in \ker(H^* - \mathrm{I})$. Then

$$H^*f - f = H^*(f + \beta/x) - (f + \beta/x) \in L^\infty(0,\infty),$$

and hence $f \in \mathrm{Dom}[H^* - \mathrm{I}, L^\infty(0,\infty)]$.

To prove the sufficiency of *(iii)-(a)*, let $w(x) \in W$. Since $w(x) \gtrsim 1$, for all $x > 0$, we get $L^\infty(w) \subset L^\infty(0,\infty)$. Again, if $w(x)$ satisfies $\int_0^\infty \frac{1}{tw(t)}dt < \infty$, from Proposition 3.12 we obtain that $L^\infty(w) \subset \mathrm{Dom}[H^*, L^\infty(0,\infty)]$, and hence the result follows.

Conversely, let $L^\infty(w) \subset (\mathrm{Dom}[H^*, L^\infty(0,\infty)] \cap L^\infty(0,\infty))$. Then $L^\infty(w) \subset \mathrm{Dom}[H^*, L^\infty(0,\infty)]$ and by Proposition 3.12 we obtain $\int_0^\infty \frac{1}{tw(t)}dt < \infty$.

Moreover, $L^\infty(w) \subset L^\infty(0,\infty)$ implies that $w(x) \gtrsim 1$. Consequently, we obtain $w(x) \in W$.

To prove the necessity of *(iii)-(b)*, fix x_0 and take $f_n(x) = xe^{-x}\chi_{E_n}(x)$, where $E_n = [x_0 - 1/n, x_0 + 1/n]$, for $0 < 1/n < x_0 < \infty$. Then

$$
\begin{aligned}
H^* f_n(x) &= \left(\int_{x_0-1/n}^{x_0+1/n} e^{-t}dt\right)\chi_{(0,x_0-1/n]}(x) + \left(\int_{x}^{x_0+1/n} e^{-t}dt\right)\chi_{E_n}(x) \\
&= \left(e^{-x_0+1/n} - e^{-x_0-1/n}\right)\chi_{(0,x_0-1/n]}(x) + \left(e^{-x} - e^{-x_0-1/n}\right)\chi_{E_n}(x),
\end{aligned}
$$

from which we get

$$
\|H^* f_n\|_{L^\infty(0,\infty)} = e^{-x_0+1/n} - e^{-x_0-1/n}.
$$

Moreover, for $0 < 1/n < x_0 < 1 - 1/n$, $n > 2$, $f_n(x) = xe^{-x}$ on E_n and it is increasing. Hence we get

$$
\left(x_0 - 1/n\right)e^{-x_0+1/n} \sup_{x\in E_n} w(x) \le \sup_{x\in E_n} |f_n(x)|w(x) = \|f_n\|_{L^\infty(w)},
$$

and $\|f_n\|_{L^\infty(0,\infty)} = \left(x_0 + 1/n\right)e^{-x_0-1/n}$. Then, from the embedding

$$
\|f_n\|_{L^\infty(w)} \lesssim \|H^* f_n\|_{L^\infty(0,\infty)} + \|f_n\|_{L^\infty(0,\infty)}
$$

we obtain

$$
\sup_{x\in E_n} w(x) \lesssim \frac{1}{x_0 - 1/n}\left(1 + \left(x_0 + 1/n - 1\right)e^{-2/n}\right)
$$

Letting $n \to \infty$, we get $w(x_0) \lesssim 1$ for $0 < x_0 < 1$. Since x_0 is arbitrary, we have that $w(x) \lesssim 1$, for $0 < x < 1$. Again, for $1 + 1/n \le x_0 < \infty$,

$f_n(x) = xe^{-x}$ on E_n and it is decreasing. Hence we get

$$\Big(x_0 + 1/n\Big)e^{-x_0-1/n} \sup_{x\in E_n} w(x) \leq \|f_n\|_{L^\infty(w,(0,\infty))},$$

and $\|f_n\|_{L^\infty(0,\infty)} = \Big(x_0 - 1/n\Big)e^{-x_0+1/n}$. Following similar procedures as above and letting $n \to \infty$, we get $w(x) \lesssim 1$, a.e. $x \geq 1$.

Again from the embedding

$$\|f_n\|_{L^\infty(w)} \lesssim \|H^* f_n\|_{L^\infty(0,\infty)} + \|f_n\|_{L^\infty(0,\infty)}$$

we obtain

$$\sup_{x\in E_n} w(x) \lesssim \frac{1}{x_0 + 1/n}\Big(-1 + \Big(x_0 - 1/n + 1\Big)e^{2/n}\Big)$$

Letting $n \to \infty$, we get $w(x_0) \lesssim 1$ for $x_0 \geq 1$. Thus, $w(x) \lesssim 1$, for $x \geq 1$. Consequently, $w(x) \lesssim 1$, a.e. $x > 0$.

Conversely, if $w(x) \lesssim 1$ a.e., we get $L^\infty(0,\infty) \subset L^\infty(w)$, and hence we get

$$\|f\|_{L^\infty(w)} \lesssim \|f\|_{L^\infty(0,\infty)} \leq \|H^* f\|_{L^\infty(0,\infty)} + \|f\|_{L^\infty(0,\infty)}.$$

That the inclusion

$$\Big(\mathrm{Dom}[H^*, L^\infty(0,\infty)] \cap L^\infty(0,\infty)\Big) \subset L^\infty(0,\infty).$$

is strict, follows by considering $f \equiv 1$. □

Remark 3.14. Observe that, in fact,

$$\frac{x}{(x+1)^2} \in \mathrm{Dom}[H^* - I, L^1(0,\infty)] \setminus (L^1(0,\infty) + \ker(H^* - I)),$$

and hence, Theorem 3.13 *(i)* tells us that

$$L^1(0,\infty) + \ker(H^* - I) \subsetneq \mathrm{Dom}[H^* - I, L^1(0,\infty)] \subsetneq L^{1,\infty}(0,\infty).$$

Moreover, if $f \in \mathrm{Dom}[H^* - I, L^1(0,\infty)]$ and $g(x) := H^* f(x) - f(x) \in L^1(0,\infty)$, since

$$\int_a^b H^* f(x) - f(x)dx = bH^* f(b) - aH^* f(a),$$

and

$$\Big|\int_a^b H^*f(x) - f(x)dx\Big| \le \int_a^b |H^*f(x) - f(x)|dx = \int_a^b |g(x)|dx,$$

we get that

$$|bH^*f(b) - aH^*f(a)| \le \int_0^\infty |g(x)|dx < \infty,$$

for any $0 < a < b < \infty$. From here we can easily conclude that $xH^*f(x)$ is a function of bounded variation on $(0, \infty)$.

Remark 3.15. The set W in Theorem 3.13 *(iii)-(a)* can not be satisfied, in the sense that both conditions defining W are independent. With $W \equiv 1$, we trivially get that the integral condition does not hold. On the other hand, there exists a weight $w > 0$ such that $\int_0^\infty \frac{dt}{tw(t)} < \infty$ but $w(x) \gtrsim 1$ fails. In fact, we consider I_k in (51) and w in (52). It is clear that $w(x) \gtrsim 1$ fails because $w(x) = \dfrac{1}{r^k} \xrightarrow{k\to\infty} 0$, $x \in I_k$. On the other hand, $\int_0^\infty \frac{dt}{tw(t)} < \infty$ as shown in (53).

Remark 3.16. $\mathrm{Dom}[H^* - \mathrm{I}, L^p(0,\infty)]$, $1 \le p \le \infty$, does not satisfy the lattice property. For $p = 1$, we consider $f(x) = \frac{1}{x}\sum_{k=1}^\infty (-1)^k \chi_{(k-1,k]}(x)$ and $h(x) = \frac{1}{x}$. It is clear that $|f(x)| = |h(x)|$ and $h \in \mathrm{Dom}[H^* - \mathrm{I}, L^1(0,\infty)]$ but $f \notin \mathrm{Dom}[H^* - \mathrm{I}, L^1(0,\infty)]$ as shown in (54).

For $1 < p < \infty$, we consider $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{\sqrt[p]{x}}\chi_{(0,1)}(x)$. Clearly, $|g(x)| \le |f(x)|$ for all $x > 0$. By Theorem 3.13 *(ii)*, it suffices to show that $f(x)$ belongs to $L^p(0,\infty) + \ker(H^* - \mathrm{I})$ but $g(x)$ does not.

For $p = \infty$, we take $f(x) = \frac{1}{x}$ and $g(x) = \chi_{(0,1)}(x)$. Here $|g| \le |f|$ and $f \in \mathrm{Dom}[H^* - \mathrm{I}, L^\infty(0,\infty)]$ but $g \notin \mathrm{Dom}[H^* - \mathrm{I}, L^\infty(0,\infty)]$. In fact,

$$H^*g(x) = \int_x^\infty \frac{1}{t}\chi_{(0,1)}(t)dt = \Big(\int_x^1 \frac{1}{t}dt\Big)\chi_{(0,1)}(x) = -\log x\chi_{(0,1)}(x).$$

Then $H^*g(x) - g(x) = -(1 + \log x)\chi_{(0,1)}(x)$. Consequently,

$$\begin{aligned}\|H^*g - g\|_{L^\infty(0,\infty)} &= \operatorname*{ess\,sup}_{x>0} |1 + \log x|\chi_{(0,1)}(x) \\ &= \operatorname*{ess\,sup}_{0<x<1} |1 + \log x| = \infty.\end{aligned}$$

4 Boundary-domain integral equations for variable-coefficient Helmholtz BVPs in 2D

4.1 Introduction

Many problems of mathematical physics and engineering such as the ones associated with steady-state oscillations (mechanical, acoustic, electromagnetic, etc.) lead to the Helmholtz equation. Since the fundamental solution of the constant-coefficient Helmholtz equation is known explicitly, the boundary value problems (BVPs) for this equation can be reduced to the boundary integral equations (BIEs), which have the advantage that the dimension of the problem is reduced by one and the BIEs could be effectively solved numerically.

In applications, such as seismic or medical imaging, the coefficients in the Helmholtz equation become variable [67]. For such partial differential equations (PDEs) with variable coefficients a fundamental solution is generally not available in explicit form, preventing reduction of BVPs for such PDEs to explicit BIEs. Instead, one can use a parametrix (Levi function), which is more widely available, to reduce the variable-coefficient BVPs to either segregated or united direct boundary-domain integral or integro-differential equations [57], BDIEs or BDIDEs. These equations are well studied for Dirichlet, Neumann and Mixed (Dirichlet-Neumann) BVPs for variable-coefficient second order scalar elliptic PDE

$$Au(x) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \qquad x \in \Omega \tag{55}$$

in 3D [19, 20, 21, 22, 59, 62, 63] as well as in 2D [5, 6, 31].

However, this is not the case for the parametrix-based system of BDIEs for variable-coefficient Helmholtz equation

$$Au(x) + k^2(x)u(x) = f(x), \qquad x \in \Omega \tag{56}$$

where $k(x)$ is a real function of x, $a(x)$ is a known variable coefficient, u is unknown function and $f \in L^2(\Omega)$ is a given function. Note that when $\Omega = \mathbb{R}^n$

and $k(x)$ is constant outside a bounded domain, (56) can be reduced to the Lippmann-Schwinger type integral equation, see e.g. [23, Section 8] for the case when $a(x)$ is a constant in $\mathbb{R}^n$, and [25, 42, 53] for the case when $a(x)$ is a constant only outside a bounded domain in $\mathbb{R}^n$. In both cases the integral equations can be considered as special cases of BDIEs. We also mention [1], where the numerical solutions of BDIE and BDIDE of the mixed problem for PDE (56) is given (without analysis of the equivalence to the BVP or the solution existence and uniqueness). Applying the previously developed techniques for the operator A in (55), in this paper we shall construct and investigate BDIE systems for the Dirichlet and mixed (Dirichlet-Neumann) BVPs associated with PDE (56) in appropriate function spaces in the two-dimensional case. The BDIEs in the n-dimensional cases with $n \geq 3$ can also be analyzed in a similar way, although the scaling with the parameter r_0 in the parametrix will not be needed in such cases because the invertibility of the standard single layer potential operator will not depend on the domain size then.

4.2 Preliminaries

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$. The set of all infinitely differentiable functions on Ω with compact support is denoted by $\mathcal{D}(\Omega)$. The function space $\mathcal{D}'(\Omega)$ consists of all continuous linear functionals over $\mathcal{D}(\Omega)$. The space of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^2)$ is denoted by $\mathcal{D}(\overline{\Omega})$. The space $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, denotes the Bessel potential space, and $H^{-s}(\mathbb{R}^2)$ is the dual space to $H^s(\mathbb{R}^2)$. We define $H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = U|_\Omega$ for some $U \in H^s(\mathbb{R}^2)\}$, and $H^1_0(\Omega)$ is the space of functions in $H^1(\Omega)$ with zero traces on $\partial\Omega$. By $H^s(\partial\Omega)$ we denote the Bessel potential spaces on the boundary $\partial\Omega$ (cf., e.g., [56]).

For the scalar elliptic differential operator A given by

$$A = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial}{\partial x_i}\right] \tag{57}$$

we consider the Helmholtz equation

$$Au(x) + k^2(x)u(x) = f(x), \qquad x \in \Omega$$

where $k(x)$ is a real function of x, $a(x)$ is a known variable coefficient, u is unknown function and f is a given function in Ω. We assume that $a, k \in C^\infty(\overline{\Omega})$ and $0 < a_0 < a(x) < a_1 < \infty$ for some constants a_0 and a_1, for all $x \in \Omega$.

Let us denote $A_k := A + k^2$. Following the definition given, e.g., in [24, 34, 60], for $s \in \mathbb{R}$ the subspace $H^{s,0}(\Omega; A_k)$ of $H^s(\Omega)$ is defined as

$$H^{s,0}(\Omega; A_k) := \{g \in H^s(\Omega) : \ A_k g \in L^2(\Omega)\}, \tag{58}$$

with the norm $\|g\|^2_{H^{s,0}(\Omega;A_k)} := \|g\|^2_{H^s(\Omega)} + \|A_k g\|^2_{L^2(\Omega)}$. Since $A_k u - Au = k^2 u \in L^2(\Omega)$ for $u \in H^1(\Omega)$, we get $H^{1,0}(\Omega; A_k) = H^{1,0}(\Omega; A)$. Moreover, if $s_2 \leq s_1$, then we have the embedding $H^{s_1,0}(\Omega; A_k) \subset H^{s_2,0}(\Omega; A_k)$.

For $u \in H^s(\Omega)$, $s > 3/2$, the corresponding classical co-normal derivative operator on $\partial\Omega$ in the sense of traces denoted by T^{c+} is given by

$$T^{c+}u(x) = \sum_{i=1}^{2} a(x)n_i(x)\gamma^+ \frac{\partial u(x)}{\partial x_i}, \tag{59}$$

where $n(x)$ is the outward (to Ω) unit normal vector at the point $x \in \partial\Omega$, and γ^+ is the trace operator.

For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, from the Gauss-Ostrogradsky theorem, we get

$$\int_\Omega v(x)Au(x)dx = -\sum_{i=1}^{2}\int_\Omega a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}dx + \int_{\partial\Omega} T^{c+}u(x)\gamma^+ v(x)dS_x.$$

From this, we obtain the first Green identity:

$$\mathcal{E}_k(u, v) = -\int_\Omega v(x)A_k u(x)dx + \int_{\partial\Omega} T^{c+}u(x)\gamma^+ v(x)dS_x, \tag{60}$$

where

$$\mathcal{E}_k(u, v) := \int_\Omega a(x)\nabla u(x) \cdot \nabla v(x)dx - \int_\Omega k^2(x)u(x)v(x)dx$$

is the symmetric bilinear form.

Even though the classical co-normal derivative is, generally, not defined for $u \in H^s(\Omega)$, $s < 3/2$, (some examples are provided in [62, Appendix A]) there exists the following continuous extension of this definition of the classical conormal derivative hinted by the first Green identity (60), for $u \in H^{s,0}(\Omega; A_k)$, $1/2 < s < 3/2$ (see e.g., [24], [56, Lemma 4.3],[60, 61]).

Definition 4.1. For $u \in H^{s,0}(\Omega; A_k)$, $1/2 < s < 3/2$, the (canonical) co-normal derivative $T^+u \in H^{s-\frac{3}{2}}(\partial\Omega)$ is defined in the following weak form:

$$\begin{aligned} \langle T^+u, w\rangle_{\partial\Omega} &:= \langle A_k u, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_k(u, \gamma^{-1}w) \\ &= \langle Au, \gamma^{-1}w\rangle_\Omega + \mathcal{E}_0(u, \gamma^{-1}w), \qquad \forall w \in H^{\frac{3}{2}-s}(\partial\Omega). \end{aligned} \tag{61}$$

In (61) and further on, $\gamma^{-1} : H^{\frac{3}{2}-s}(\partial\Omega) \to H^{2-s}(\Omega)$ is a bounded right inverse to the trace operator $\gamma : H^{2-s}(\Omega) \to H^{\frac{3}{2}-s}(\partial\Omega)$, the notation $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{s-\frac{3}{2}}(\partial\Omega)$ and $H^{\frac{3}{2}-s}(\partial\Omega)$, while $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality brackets between the spaces $H^{s-1}(\Omega)$ and $H^{1-s}(\Omega)$, extending the usual L^2-inner products.

The operator $T^+ : H^{s,0}(\Omega; A_k) \to H^{s-3/2}(\partial\Omega)$ is continuous for $s > 1/2$. Moreover, as we observe from [60, Corollary 3.14],

$$T^+u = T^{c+}u \text{ for } u \in H^s(\Omega), \quad s > 3/2. \tag{62}$$

By [24, Lemma 3.4], the first Green identity (60) in the form

$$\langle T^+u, \gamma^+v\rangle_{\partial\Omega} = \mathcal{E}_k(u, v) + \langle A_k u, v\rangle_\Omega. \tag{63}$$

holds for $u \in H^{1,0}(\Omega; A_k)$ and $v \in H^1(\Omega)$.

Interchanging the roles of u and v in the first Green identity (63) for $u \in H^1(\Omega)$ and $v \in H^{1,0}(\Omega; A_k)$, we obtain the first Green identity for v,

$$\langle T^+v, \gamma^+u\rangle_{\partial\Omega} = \mathcal{E}_k(v, u) + \langle A_k v, u\rangle_\Omega. \tag{64}$$

Then subtracting (64) from (63), we obtain the *second Green identity* for $u, v \in H^{1,0}(\Omega; A_k)$,

$$\langle A_k u, v\rangle_\Omega - \langle A_k v, u\rangle_\Omega = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega}. \tag{65}$$

4.3 Parametrix-based potential operators

Definition 4.2. A function $P(x, y)$ is a *parametrix* for the operator A_k if

$$(A_k)_x P(x, y) = \delta(x - y) + R_k(x, y),$$

where δ is the Dirac-delta distribution, while $R_k(x, y)$ is a remainder possessing at most a weak singularity at $x = y$.

Based on [57], the function

$$P(x, y) = \frac{1}{a(y)} P_\Delta(x, y) = \frac{1}{2\pi a(y)} \ln\Big(\frac{|x - y|}{r_0}\Big), \quad x, y \in \mathbb{R}^2,$$

where $r_0 > 0$ is a constant parameter, is a parametrix for the operator A. Note that

$$P_\Delta(x, y) = \frac{1}{2\pi} \ln\left(\frac{|x - y|}{r_0}\right), \quad r_0 > 0, \quad x, y \in \mathbb{R}^2 \tag{66}$$

is a fundamental solution of the Laplace operator, Δ, cf., e.g., [56, Theorem 8.1]. We can also take $P(x, y)$ as a parametrix for the operator A_k. Then the corresponding remainder function becomes

$$R_k(x, y) = k^2(x) P(x, y) + R(x, y), \qquad x, y \in \mathbb{R}^2, \tag{67}$$

where

$$R(x, y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \qquad x, y \in \mathbb{R}^2,$$

is the remainder function for the operator A and is weakly singular due to the smoothness of the function $a(x)$. Hence $R_k(x, y)$ is also weakly singular, and thus, $P(x, y)$ is, indeed, a parametrix for the operator A_k.

4.3.1 Surface potentials

The single and the double layer surface potential operators corresponding to the parametrix $P(x, y)$, are defined for $y \notin \partial\Omega$ as

$$Vg(y) := -\int_{\partial\Omega} P(x, y) g(x) dS_x,$$

$$Wg(y) := -\int_{\partial\Omega} \left[T_x^+ P(x, y)\right] g(x) dS_x$$

where the integrals are understood as the appropriate dual products if the scalar density function g is not integrable.

The corresponding boundary integral (pseudodifferential) operators of direct surface values of the single layer potential $\mathcal{V}$ and of the double layer potential $\mathcal{W}$, and the co-normal derivatives of the single layer potential $\mathcal{W}'$, and of the double layer potential $\mathcal{L}^+$, for $y \in \partial\Omega$, are

$$\begin{aligned}
\mathcal{V}g(y) &:= -\int_{\partial\Omega} P(x,y)g(x)dS_x,\\
\mathcal{W}g(y) &:= -\int_{\partial\Omega} \left[T_x^+ P(x,y)\right] g(x)dS_x,\\
\mathcal{W}'g(y) &:= -\int_{\partial\Omega} \left[T_y^+ P(x,y)\right] g(x)dS_x,\\
\mathcal{L}^+ g(y) &:= T^+ W g(y).
\end{aligned} \tag{68}$$

Let $V_\Delta, W_\Delta, \mathcal{V}_\Delta, \mathcal{W}_\Delta$ and $\mathcal{L}^+_\Delta$ denote the potentials and the boundary operators corresponding to the Laplace operator Δ. That is, the subscript Δ means that the corresponding surface potentials are constructed by means of the fundamental solution (66) of the Laplace operator Δ. Then the following relations hold in 2D (cf. [31]).

$$Vg = \frac{1}{a}V_\Delta g, \qquad Wg = \frac{1}{a}W_\Delta(ag) \tag{69}$$

$$\mathcal{V}g = \frac{1}{a}\mathcal{V}_\Delta g, \qquad \mathcal{W}g = \frac{1}{a}\mathcal{W}_\Delta(ag), \tag{70}$$

$$\mathcal{W}'g = \mathcal{W}'_\Delta g + \left[a\frac{\partial}{\partial n}\Big(\frac{1}{a}\Big)\right]\mathcal{V}_\Delta g, \tag{71}$$

$$\mathcal{L}^+ g = \mathcal{L}^+_\Delta(ag) + \left[a\frac{\partial}{\partial n}\Big(\frac{1}{a}\Big)\right]\gamma^+ W_\Delta(ag). \tag{72}$$

The following two theorems are proved in [31, Theorem 1 and Theorem 2].

Theorem 4.3. *Let $u \in H^{-\frac{1}{2}}(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$. Then the following* *relations hold for $y \in \partial\Omega$,*

$$\gamma^+ V u(y) = \mathcal{V} u(y), \tag{73}$$

$$\gamma^+ W v(y) = -\frac{1}{2} v(y) + \mathcal{W} v(y), \tag{74}$$

$$T^+ V u(y) = \frac{1}{2} u(y) + \mathcal{W}' u(y). \tag{75}$$

Theorem 4.4. *For $s \in \mathbb{R}$, the following operators are continuous,*

$$V : H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega),$$
$$W : H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega),$$
$$\mathcal{V}, \mathcal{W}, \mathcal{W}' : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega).$$

These theorems imply the following assertion.

Corollary 4.5. *The following operators are continuous,*

$$V : H^s(\partial\Omega) \to H^{s+\frac{3}{2},0}(\Omega; A_k), \qquad s \geq -\frac{1}{2},$$
$$W : H^s(\partial\Omega) \to H^{s+\frac{1}{2},0}(\Omega; A_k), \qquad s \geq \frac{1}{2}.$$

Proof. For $g \in H^s(\partial\Omega)$, from Theorem 4.4 we get $Vg \in H^{s+\frac{3}{2}}(\Omega)$. Then,

$$\begin{aligned} A(Vg) &= \Delta(aVg) - \sum_{i=1}^{2} \partial_i (Vg \partial_i a) \\ &= \Delta(V_\Delta g) - \sum_{i=1}^{2} \partial_i (Vg \partial_i a) = -\sum_{i=1}^{2} \partial_i (Vg \partial_i a) \end{aligned}$$

belongs to $L^2(\Omega)$ if $s \geq -\frac{1}{2}$. A similar proof holds for the operator W as well. □

The compactness of the following surface potential operators in Corollary 4.6 follow directly from Theorem 4.4 and Rellich compact embedding theorem [56, Theorem 3.27].

Corollary 4.6. *For $s \in \mathbb{R}$, the following operators are compact,*

$$\mathcal{V}, \mathcal{W}, \mathcal{W}' : H^s(\partial\Omega) \to H^s(\partial\Omega).$$

For $s \in \mathbb{R}$, $\Gamma_1 \subset \partial\Omega$, let us define the following subspaces of the space $H^s(\partial\Omega)$, (see e.g., [70, pp 147]):

$$\widetilde{H}^s(\Gamma_1) := \{\psi \in H^s(\partial\Omega) : \operatorname{supp}\psi \subset \overline{\Gamma}_1\},$$
$$H^s_{**}(\partial\Omega) := \{\psi \in H^s(\partial\Omega) : \langle\psi, 1\rangle_{\partial\Omega} = 0\},$$
$$\widetilde{H}^s_{**}(\Gamma_1) := \{\psi \in \widetilde{H}^s(\Gamma_1) : \langle\psi, 1\rangle_{\partial\Omega} = 0\}.$$

Corollary 4.6 implies the following assertion.

Theorem 4.7. *Let Γ_1 and Γ_2 be non-empty smooth pieces of a curve $\partial\Omega$. Then the operators*

$$r_{\Gamma_2}\mathcal{V}, r_{\Gamma_2}\mathcal{W}, r_{\Gamma_2}\mathcal{W}' : \widetilde{H}^s(\Gamma_1) \longrightarrow H^s(\Gamma_2). \tag{76}$$

are compact for $s \in \mathbb{R}$.

In (76) and further on, r_{Γ_1}, r_{Γ_2}, etc. denote the corresponding restriction operators.

4.3.2 Invertibility of single layer potential operator in 2D

It is well known that the kernel of the operator

$$\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \tag{77}$$

with the parameter $r_0 = 1$ is non-zero for some domains in 2D (see, e.g., [70, Theorem 6.22 proof]). Then the first relation in (70) and scaling imply a non-zero kernel also for $\mathcal{V}$ with $r_0 > 0$, for some domains Ω.

The following result is proved in [31, Theorem 4].

Theorem 4.8. *Let $\psi \in H^{-1/2}_{**}(\partial\Omega)$. If $\mathcal{V}\psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

On the other hand, choosing for a given Ω an appropriate parameter r_0, one can get the zero kernel for $\mathcal{V}$ not only on the subspace $H^{-1/2}_{**}(\partial\Omega)$ but also on the entire space $H^{-1/2}(\partial\Omega)$ and then prove the following invertibility assertion.

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^2$ with $r_0 > \mathrm{diam}(\Omega)$. Then, the single layer potential operator*

$$\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega) \tag{78}$$

is invertible.

Proof. For $r_0 = 1$, the assertion is available in [31, Theorem 5]. For arbitrary $r_0 > \mathrm{diam}(\Omega)$, the invertibility of operator (77) can be obtained by scaling the result for $r_0 = 1$, e.g., from Theorem 6.23 and reasoning following it in [70]. Then, the first relation in (70) implies the invertibility of operator (78) as well (Cf. also [3, Theorem 5.2] and [4, Theorem 6]. □

Similar to [6, Corollary 2.7], we obtain the following assertion.

Corollary 4.10. *Let Γ_1 be non-empty relatively open connected part of a curve $\partial\Omega$. Then, the operator*

$$r_{_{\Gamma_1}}\mathcal{V} : \ \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$$

is bounded and Fredholm of index zero.

Theorem 4.11. *Let Γ_1 be a non-empty relatively open connected part of a curve $\partial\Omega$ with $r_0 > \mathrm{diam}(\Gamma_1)$. Then, the operator $r_{_{\Gamma_1}}\mathcal{V} : \ \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ has a bounded inverse.*

Proof. Taking into account the condition $r_0 > \mathrm{diam}(\Gamma_1)$, we can follow the proof of [6, Corollary 2.9]. □

Due to (74) and the second relation in (69), relation (72) can also be written as

$$\widehat{\mathcal{L}}g = \Big[\mathcal{L}^+ + \frac{\partial a}{\partial n}\Big(-\frac{1}{2}I + \mathcal{W}\Big)\Big]g, \quad \text{on } \partial\Omega, \tag{79}$$

where $\widehat{\mathcal{L}}g := \mathcal{L}^+_\Delta(ag)$.

The following assertion is available, e.g., in [6, Theorem 2.10] (cf. [19, Theorem 3.6] in the 3D case).

Theorem 4.12. *Let Γ_1 be nonempty open smooth part of $\partial\Omega$.*

(i) Then, the operator

$$r_{\Gamma_1}\widehat{\mathcal{L}} : \widetilde{H}^{\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)$$

is continuously invertible.

(ii) Moreover, the operator

$$r_{\Gamma_1}(\mathcal{L}^+ - \widehat{\mathcal{L}}) : \widetilde{H}^{\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$$

is bounded, and the operator

$$r_{\Gamma_1}(\mathcal{L}^+ - \widehat{\mathcal{L}}) : \widetilde{H}^{\frac{1}{2}}(\Gamma_1) \to H^{-\frac{1}{2}}(\Gamma_1)$$

is compact.

4.3.3 Volume potentials

Similar to [5, 19, 57], we define the parametrix-based logarithmic and remainder volume potential operators, respectively, as

$$\mathcal{P}g(y) := \int_\Omega P(x,y)g(x)dx, \quad \mathcal{R}_k g(y) := \int_\Omega R_k(x,y)g(x)dx, \quad y \in \mathbb{R}^2.$$

Remark 4.13. As for the layer potentials, let $\mathcal{P}_\Delta$ denote the logarithmic potential for the operator Δ, that is,

$$\mathcal{P}_\Delta g(y) := \int_\Omega P_\Delta(x,y)g(x)dx, \quad y \in \mathbb{R}^2,$$

where P_Δ is the fundamental solution (66). Then,

$$\mathcal{P}g = \frac{1}{a}\mathcal{P}_\Delta g, \quad \mathcal{R}_k g = \mathcal{P}(k^2 g) + \mathcal{R}g, \tag{80}$$

where $\mathcal{R}$ is the parametrix-based remainder volume potential operator for the remainder function $R(x,y)$ and, see [31, 57],

$$\mathcal{R}g = -\frac{1}{a}\sum_{i=1}^{2} \partial_i[\mathcal{P}_\Delta\,(g\partial_i a)],$$

where $\partial_i = \partial/\partial x_i$.

Theorem 4.14. *Let Ω be a bounded open region in $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. The following operators are continuous.*

$$\mathcal{P} : H^s(\Omega) \longrightarrow H^{s+2}(\Omega), \qquad s > -\frac{1}{2}; \tag{81}$$

$$\mathcal{R} : H^s(\Omega) \longrightarrow H^{s+1}(\Omega), \qquad s > -\frac{1}{2}; \tag{82}$$

$$\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s+1}(\Omega), \qquad s > -\frac{1}{2}; \tag{83}$$

$$\gamma^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > -\frac{1}{2}; \tag{84}$$

$$T^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\partial\Omega), \qquad s > \frac{1}{2}. \tag{85}$$

Proof. For (81) and (82), we refer to [31, Theorem 3]. From the second relation in (80), together with (81) and (82) we obtain the continuity of (83). The continuity of the operators (84) and (85) are the direct consequences of the trace theorem, Definition 4.1 of the co-normal derivative and relation (62). □

Corollary 4.15. *The following operators are continuous.*

$$\mathcal{P} : H^s(\Omega) \longrightarrow H^{s+2,0}(\Omega; A_k), \qquad s \geq 0; \tag{86}$$

$$\mathcal{R} : H^s(\Omega) \longrightarrow H^{s+1,0}(\Omega; A_k), \qquad s \geq 1; \tag{87}$$

$$\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s+1,0}(\Omega; A_k), \qquad s \geq 1. \tag{88}$$

Proof. Using the continuity of operators (81)–(83) and the space definition (58), we obtain the continuity of operators (86)–(88). □

Corollary 4.16. *The following operators are compact.*

$$\mathcal{R}_k : H^s(\Omega) \longrightarrow H^s(\Omega), \qquad s > -\frac{1}{2}; \tag{89}$$

$$\gamma^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}; \tag{90}$$

$$T^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{3}{2}}(\partial\Omega) \qquad s > \frac{1}{2}. \tag{91}$$

Proof. The compactness of operators (89)–(91) follows from (83)–(85) and the Rellich compact embedding theorem. □

Corollary 4.17. *The operator*

$$\mathcal{R}_k - \mathcal{R} : H^s(\Omega) \longrightarrow H^{s,0}(\Omega; A_k), \quad s > 0, \tag{92}$$

is compact.

Proof. From the second equation in (80) we see that $\mathcal{R}_k g - \mathcal{R} g = \mathcal{P}(k^2 g)$. Then by (81) for $s > -1/2$, the operator $\mathcal{R}_k - \mathcal{R} : H^s(\Omega) \to H^{s+2}(\Omega)$ is continuous, and the operator $\mathcal{R}_k - \mathcal{R} : H^s(\Omega) \to H^s(\Omega)$ is compact. Hence, the operator $\Delta(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to H^s(\Omega)$ is also continuous for $s > -1/2$, and the operator $\Delta(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to L^2(\Omega)$ is compact for $s > 0$.

Further, $A_k(\mathcal{R}_k - \mathcal{R}) = a\Delta(\mathcal{R}_k - \mathcal{R}) + \sum_{j=1}^{2}(\partial_j a)\partial_j(\mathcal{R}_k - \mathcal{R}) + k^2(\mathcal{R}_k - \mathcal{R})$. The operator $\partial_j(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to H^{s+1}(\Omega)$ is continuous, and hence, the operator $\partial_j(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to H^0(\Omega)$ is compact for $s > -1/2$. Thus, the operator $A_k(\mathcal{R}_k - \mathcal{R}) : H^s(\Omega) \to L^2(\Omega)$ is compact for the operator A_k with infinitely smooth coefficients, for $s > 0$. Hence, the compactness of operator (92) follows from the space definition (58). □

Corollary 4.18. *Let* Γ_1 *and* Γ_2 *be non-empty, non-intersecting parts of* $\partial\Omega$ *such that* $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$*. Then the operators*

$$r_{\Gamma_1}\gamma^+\mathcal{R}, r_{\Gamma_1}\gamma^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma_1),$$
$$r_{\Gamma_1}T^+\mathcal{R}, r_{\Gamma_1}T^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{3}{2}}(\Gamma_1),$$

are compact for $s > \frac{1}{2}$*.*

Proof. Theorem (4.14) implies that the following operators are continuous for $s > \frac{1}{2}$:

$$r_{\Gamma_1}\gamma^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s+\frac{1}{2}}(\Gamma_1),$$
$$r_{\Gamma_1}T^+\mathcal{R}_k : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma_1).$$

Then, the proof follows from the compactness of the embeddings $H^{s+\frac{1}{2}}(\Gamma_1) \subset H^{s-\frac{1}{2}}(\Gamma_1)$ and $H^{s-\frac{1}{2}}(\Gamma_1) \subset H^{s-\frac{3}{2}}(\Gamma_1)$. The proof holds true also for $k = 0$. □

4.4 The Third Green Identity

As e.g., in [5, 6, 19, 31]), for $u \in H^{1,0}(\Omega; A_k)$, we substitute $P(x, y)$ for $v(x)$ in Green's second identity (65) for $\Omega \setminus \overline{B_\epsilon}(y)$, where $B_\epsilon(y)$ is a disc of radius ϵ centered at y and take the limit as $\epsilon \to 0$ to arrive at the parametrix-based third Green identity

$$u + \mathcal{R}_k u - VT^+u + W\gamma^+u = \mathcal{P}A_k u \quad \text{in } \Omega. \tag{93}$$

Taking the trace of (93) and using relations (73) and (74), we obtain

$$\frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}_k u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}A_k u \quad \text{on } \partial\Omega. \tag{94}$$

From Corollaries 4.5 and 4.15, we see that each term of (93) belongs to $H^{1,0}(\Omega; A_k)$. Now, taking the co-normal derivative of (93) and using relation (75), we get

$$\frac{1}{2}T^+u + T^+\mathcal{R}_k u - \mathcal{W}'T^+u + T^+W\gamma^+u = T^+\mathcal{P}A_k u \quad \text{on } \partial\Omega. \tag{95}$$

If $u \in H^1(\Omega)$ is a solution of equation $A_k u = f$ in Ω, where $f \in L^2(\Omega)$, then (93) becomes

$$u + \mathcal{R}_k u - VT^+u + W\gamma^+u = \mathcal{P}f \quad \text{in } \Omega. \tag{96}$$

For some functions f, Ψ and Φ, let us consider a more general indirect integral relation associated with (96),

$$u + \mathcal{R}_k u - V\Psi + W\Phi = \mathcal{P}f \quad \text{in } \Omega. \tag{97}$$

Lemma 4.19. *Let $u \in H^1(\Omega), f \in L^2(\Omega), \Psi \in H^{-\frac{1}{2}}(\partial\Omega), \Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy (97). Then u belongs to $H^{1,0}(\Omega; A_k)$ and is a solution of PDE $A_k u = f$ in Ω, and*

$$V(\Psi - T^+u)(y) - W(\Phi - \gamma^+u)(y) = 0, \quad y \in \Omega. \tag{98}$$

Proof. As in [19, Lemma 4.1] in the 3D case for $k = 0$, from Corollaries 4.5 and 4.15, we conclude that all terms in (97) except u belong to $H^{1,0}(\Omega; A_k)$.

Then, (97) implies that u belongs to $H^{1,0}(\Omega; A_k)$ as well. Now, let us prove the remaining results.

Subtracting (97) from (93), we obtain

$$V\Psi^* - W\Phi^* = \mathcal{P}[A_k u - f] \quad \text{in } \Omega, \tag{99}$$

where $\Psi^* := T^+u - \Psi$ and $\Phi^* := \gamma^+u - \Phi$. Multiplying equality (99) by $a(y)$ and using relation (69) and (80), we get

$$V_\Delta \Psi^* - W_\Delta(a\Phi^*) = \mathcal{P}_\Delta[A_k u - f], \quad \text{in } \Omega. \tag{100}$$

The application of the Laplace operator Δ to (100) gives

$$A_k u - f = 0 \quad \text{in } \ \Omega. \tag{101}$$

This shows that u solves differential equation $A_k u = f$ in Ω.

Substituting (101) into (99) leads to (98). □

Lemma 4.20.

(i) Let $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \operatorname{diam}(\Omega)$. If $V\Psi^* = 0$ in Ω, then $\Psi^* = 0$ on $\partial\Omega$.

(ii) Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ and $r_0 > 0$. If $W\Phi^* = 0$ in Ω, then $\Phi^* = 0$ on $\partial\Omega$.

Proof. The assertion was proved in [31, Lemma 2] for $r_0 = 1$. Taking into account Theorem 4.9, we follow the proof of [31, Lemma 2] almost word for word to obtain the assertion for arbitrary $r_0 > 0$. □

Lemma 4.21. *Let $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_1 and Γ_2 are non-empty, non-intersecting relatively open parts of the boundary curve $\partial\Omega$. Let $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\Gamma_2)$ and $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_1)$ with $r_0 > \operatorname{diam}(\Gamma_1)$. If*

$$V\Psi^*(y) - W\Phi^*(y) = 0, \quad y \in \Omega, \tag{102}$$

then $\Psi^* = 0$ and $\Phi^* = 0$ on $\partial\Omega$.

Proof. Keeping in mind [56, Theorem 8.16] we follow the proof of [19, Lemma 4.2 (iii)] (See also [3, Lemma 5.8], [4, Lemma 3], [6, Lemma 2.12]). □

Remark 4.22. The results of Lemma 4.20 and Lemma 4.21 with no restriction on the parameter r_0 can be similarly obtained if $\Psi^* \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ and $\Psi^* \in \widetilde{H}_{**}^{-\frac{1}{2}}(\Gamma_1)$, respectively.

4.5 Boundary-Domain Integral Equations of the Dirichlet BVP

Consider the Dirichlet BVP

$$\begin{aligned} A_k u &= f \qquad \text{in } \Omega, \\ \gamma^+ u &= \varphi_0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{103}$$

for unknown function $u \in H^1(\Omega)$, where $f \in L^2(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ are given functions. The first equation is understood in the distribution sense.

Let us derive and analyze BDIE systems for the Dirichlet BVP (103).

To reduce the variable-coefficient Dirichlet BVP (103) to segregated BDIE systems, we denote the unknown co-normal derivative as $\psi := T^+u$ and further consider ψ as formally independent of u.

4.5.1 BDIE System (D1)

We substitute $A_k u$ and $\gamma^+ u$ from the Dirichlet BVP (103) into (93) and into its trace (94) to reduce the Dirichlet BVP (103) to the BDIE system (D1) with the unknowns u and ψ:

$$\begin{aligned} u + \mathcal{R}_k u - V\psi &= F_0 \qquad \text{in } \Omega, \\ \gamma^+\mathcal{R}_k u - \mathcal{V}\psi &= \gamma^+ F_0 - \varphi_0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{D1}$$

where

$$F_0 = \mathcal{P}f - W\varphi_0 \qquad \text{in } \Omega. \tag{104}$$

The matrix form of system (D1) is $\mathcal{A}_k^1\mathcal{U} = \mathcal{F}^1$, where $\mathcal{U} = (u,\psi)^t \in H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega)$,

$$\mathcal{A}_k^1 = \begin{bmatrix} I+\mathcal{R}_k & -V \\ \gamma^+\mathcal{R}_k & -\mathcal{V} \end{bmatrix}, \qquad \mathcal{F}^1 = \begin{bmatrix} F_0 \\ \gamma^+F_0 - \varphi_0 \end{bmatrix}. \tag{105}$$

From the mapping properties of $\mathcal{P}$ and W provided in Section 4.3, we get $F_0 \in H^{1,0}(\Omega; A_k)$. Moreover, the trace theorem implies that $\gamma^+F_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathcal{F}^1 \in H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in (105) (see Section 4.3), the following operators are bounded:

$$\mathcal{A}_k^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{106}$$

$$\mathcal{A}_k^1 : H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega). \tag{107}$$

Remark 4.23. $\mathcal{F}^1 = \mathbf{0}$ if and only if $(f,\varphi_0) = \mathbf{0}$.

Proof. If $\mathcal{F}^1 = \mathbf{0}$, then $F_0 = 0$ and $\gamma^+F_0 + \varphi_0 = 0$. Consequently, $\varphi_0 = 0$ on $\partial\Omega$. From this and $F_0 = 0$ we obtain that $\mathcal{P}f = 0$ in Ω, and hence, $f = 0$ in Ω. The reverse implication is trivial. □

4.5.2 BDIE System (D2)

This system is obtained by substituting $A_k u$ and γ^+u from the Dirichlet BVP (103) into (93) and into its co-normal derivative (95), with the unknowns u and ψ:

$$\begin{aligned} u + \mathcal{R}_k u - V\psi &= F_0 && \text{in } \Omega, \\ \frac{1}{2}\psi + T^+\mathcal{R}_k u - \mathcal{W}'\psi &= T^+F_0 && \text{on } \partial\Omega, \end{aligned} \tag{D2}$$

where F_0 is the relation (104). The system (D2) can be written in matrix form as

$$\mathcal{A}_k^2\mathcal{U} = \mathcal{F}^2,$$

where

$$\mathcal{A}_k^2 := \begin{bmatrix} I+\mathcal{R}_k & -V \\ T^+\mathcal{R}_k & \frac{1}{2}I-\mathcal{W}' \end{bmatrix}, \qquad \mathcal{F}^2 = \begin{bmatrix} F_0 \\ T^+F_0 \end{bmatrix},$$

and F_0 is given by (104). The following operators are bounded:

$$\mathcal{A}_k^2 : H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega), \tag{108}$$

$$\mathcal{A}_k^2 : H^{1,0}(\Omega;A_k)\times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{1,0}(\Omega;A_k)\times H^{-\frac{1}{2}}(\partial\Omega). \tag{109}$$

Remark 4.24. $\mathcal{F}^2 = \mathbf{0}$ if and only if $(f,\varphi_0)=\mathbf{0}$.

Proof. If $\mathcal{F}^2=\mathbf{0}$, then $F_0=0$. From which we get

$$0=\Delta(aF_0)=\Delta(\mathcal{P}_\Delta f)+\Delta W_\Delta(\varphi_0)=f \quad \text{in } \Omega.$$

Then, the condition $F_0=0$ gives $W_\Delta(\varphi_0)=0$ and Lemma 4.20 *(ii)* implies that $\varphi_0=0$ on $\partial\Omega$. The reverse implication is trivial. □

4.6 Equivalence, Fredholm Properties, and Invertibility for BDIEs of the Dirichlet BVP

In this section, we first prove the equivalence of the Dirichlet BVP (103) to the BDIE systems (D1) and (D2), and then we show the necessary conditions for the invertibility of the two corresponding operators to the BDIE systems.

Theorem 4.25. *Let $\varphi_0\in H^{\frac{1}{2}}(\partial\Omega)$ and $f\in L^2(\Omega)$.*

(i) If some $u\in H^1(\Omega)$ solves the BVP (103), then the pair $(u,\psi)^t$, where

$$\psi = T^+u \in H^{-\frac{1}{2}}(\partial\Omega), \tag{110}$$

solves BDIE systems (D1) and (D2).

(ii) Let $r_0>\operatorname{diam}(\Omega)$. If a pair $(u,\psi)^t\in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1), then u solves BVP (103) and ψ satisfies (110).

(iii) Let $r_0>0$. If a pair $(u,\psi)^t\in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), then u solves BVP (103), and ψ satisfies (110).

Proof. To prove *(i)*, we let $u \in H^1(\Omega)$ be a solution of the BVP (103). Since $A_k u = f \in L^2(\Omega)$, we get $u \in H^{1,0}(\Omega; A_k)$. Setting $\psi = T^+u$ and recalling how BDIE system (D1) and (D2) are constructed, we obtain that the couple $(u, \psi)^t$ solves the systems.

To prove *(ii)* and *(iii)*, let us assume first that a pair $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves system (D1) or (D2). Due to the first equation in the BDIE systems, the hypotheses of Lemma 4.19 are satisfied implying that u belongs to $H^{1,0}(\Omega; A_k)$ and solves the PDE in the BVP (103) in Ω. Moreover, the equation

$$W(\varphi_0 - \gamma^+u)(y) - V(\psi - T^+u)(y) = 0, \quad y \in \Omega, \tag{111}$$

holds.

To prove the remaining parts of *(ii)*, we let $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D1). Taking the trace of the first equation in (D1) and subtracting the second equation from it, we get the Dirichlet boundary condition

$$\gamma^+u = \varphi_0 \quad \text{on } \partial\Omega,$$

and substituting this in equation (111) we obtain

$$V(\psi - T^+u)(y) = 0, \qquad y \in \Omega.$$

Since $r_0 > \operatorname{diam}(\Omega)$, from Lemma 4.20 *(i)* we get $\psi = T^+u$.

To complete *(iii)*, we let $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solve system (D2). It is already shown that $u \in H^{1,0}(\Omega; A_k)$. Moreover, all the remaining terms in the first equation of (D2) belong to $H^{1,0}(\Omega; A_k)$ due to the mapping properties of the operators involved (see Section 4.3). Then, taking the co-normal derivative of the first equation in (D2) and subtracting the second one from it, we get

$$\psi = T^+u \quad \text{on } \partial\Omega.$$

Then, inserting this in (111) gives

$$W(\varphi_0 - \gamma^+u)(y) = 0, \quad y \in \Omega,$$

and Lemma 4.20 *(ii)* implies $\varphi_0 = \gamma^+u$ on $\partial\Omega$. □

Theorem 4.25 implies the following two corollaries.

Corollary 4.26. *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L^2(\Omega)$.*

(i) Let $r_0 > \mathrm{diam}(\Omega)$. If a pair $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1), it solves BDIE system (D2).

(ii) Let $r_0 > 0$. If a pair $(u, \psi)^t \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2), it solves BDIE system (D1).

Corollary 4.27.

(i) Let $r_0 > \mathrm{diam}(\Omega)$. The homogeneous counterpart of BDIE system (D1) has a non-trivial solution in $H^1 \times H^{-\frac{1}{2}}(\partial\Omega)$ if and only if the homogeneous counterpart of the Dirichlet problem (103) has a non-trivial solution in $H^1(\Omega)$.

(ii) Let $r_0 > 0$. The homogeneous counterpart of BDIE system (D2) has a non-trivial solution in $H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ if and only if the homogeneous counterpart of the Dirichlet problem (103) has a non-trivial solution in $H^1(\Omega)$.

Let us now analyse the Fredholm properties of operators (106), (107), (108) and (109). As a bi-product we also prove the invertibility of the corresponding operators for $k = 0$.

Theorem 4.28.

(i) If $r_0 > \mathrm{diam}(\Omega)$, then operator (106) is Fredholm with zero index.

(ii) If $r_0 > 0$, then operator (108) is Fredholm with zero index.

Proof. **(i)** Let $r_0 > \mathrm{diam}(\Omega)$. Let us consider the auxiliary operator

$$\mathcal{A}^1_* := \begin{bmatrix} I & -V \\ 0 & -\mathcal{V} \end{bmatrix}.$$

Then, the operator $\mathcal{A}^1_* : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to the invertibility of its diagonal operators

$$I : H^1(\Omega) \to H^1(\Omega) \quad \text{and} \quad \mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega),$$

see Theorem 4.9. Due to the mapping properties of the operators involved, the operator $\mathcal{A}^1_k - \mathcal{A}^1_* : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ where

$$\mathcal{A}^1_k - \mathcal{A}^1_* = \begin{bmatrix} \mathcal{R}_k & 0 \\ \gamma^+\mathcal{R}_k & 0 \end{bmatrix},$$

is compact. Thus, operator (106) is Fredholm with index zero.

(ii) The operator $\mathcal{A}^2_* : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, where

$$\mathcal{A}^2_* = \begin{bmatrix} I & -V \\ 0 & \frac{1}{2}I \end{bmatrix}$$

is bounded. It is also invertible due to the invertibility of its diagonal operators

$$I : H^1(\Omega) \longrightarrow H^1(\Omega) \quad \text{and} \quad I : H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega).$$

By Corollaries 4.6 and 4.16, the operator

$$\mathcal{A}^2_k - \mathcal{A}^2_* : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega),$$

where

$$\mathcal{A}^2_k - \mathcal{A}^2_* = \begin{bmatrix} \mathcal{R}_k & 0 \\ T^+\mathcal{R}_k & -\mathcal{W}' \end{bmatrix},$$

is compact. This implies that operator (108) is a Fredholm operator of index zero. □

Let us consider the particular cases of operators (106), (107), (108) and (109), for $k = 0$, that is,

$$\mathcal{A}^1_0 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{112}$$

$$\mathcal{A}^1_0 : H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega), \tag{113}$$

$$\mathcal{A}^2_0 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{114}$$

$$\mathcal{A}^2_0 : H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{115}$$

where

$$\mathcal{A}_0^1 = \begin{bmatrix} I+\mathcal{R} & -V \\ \gamma^+\mathcal{R} & -\mathcal{V} \end{bmatrix}, \quad \mathcal{A}_0^2 = \begin{bmatrix} I+\mathcal{R} & -V \\ T^+\mathcal{R} & \frac{1}{2}I-\mathcal{W}' \end{bmatrix}.$$

Theorem 4.29.

(i) If $r_0 > \operatorname{diam}(\Omega)$*, then operators* (112) *and* (113) *are invertible.*

(ii) If $r_0 > 0$*, then operators* (114) *and* (115) *are invertible.*

Proof. The theorem for $r_0 = 1$ was proved in [31, Theorems 7 and 8]. Here, we update the proof for arbitrary $r_0 > 0$.

It is well known that the homogeneous Dirichlet problem (103) with $k = 0$, that is, with $A_k = A$, where the operator A is given by (57) and $0 < a_0 < a(x) < a_1 < \infty$, has only the trivial solution in $H^{1,0}(\Omega; A)$ and $H^1(\Omega)$. This can be obtained, e.g., from the first Green identity (63). Then, the equivalence Theorem 4.25 implies that operators (112), (113), (114) and (115) are injective. By Theorem 4.28, operators (112) and (114) are a Fredholm operator with zero index. Then, the injectivity of operators (112) and (114) implies their invertibility (see e.g. [56, Theorem 2.27]).

To prove invertibility of operator (113), we remark that for any $\mathcal{F}^1 \in H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega)$, a solution of the equation $\mathcal{A}_0^1\mathcal{U} = \mathcal{F}^1$ can be written as $\mathcal{U} = (\mathcal{A}_0^1)^{-1}\mathcal{F}^1$, where $(\mathcal{A}_0^1)^{-1} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is the continuous inverse to operator (112). But due to Lemma 4.19 the first equation of system (D1) with $k = 0$ implies that $\mathcal{U} = (\mathcal{A}_0^1)^{-1}\mathcal{F}^1 \in H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega)$ and moreover, the operator

$$(\mathcal{A}_0^1)^{-1} : H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega)$$

is continuous, which implies invertibility of operator (113).

The invertibility of operator (115) is proved in a similar fashion. □

Now, we are in the position to prove an analogue of Theorem 4.28 for operators (107) and (109).

Theorem 4.30.

(i) If $r_0 > \operatorname{diam}(\Omega)$, then operator (107) is Fredholm with zero index.

(ii) If $r_0 > 0$, then operator (109) is Fredholm with zero index.

Proof. By Theorem 4.29, we see that operators (113) and (115) are invertible. Due to Corollary 4.17, the operators

$$\mathcal{A}_k^1 - \mathcal{A}_0^1 : H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega),$$
$$\mathcal{A}_k^2 - \mathcal{A}_0^2 : H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega),$$

where

$$\mathcal{A}_k^1 - \mathcal{A}_0^1 = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 \\ \gamma^+(\mathcal{R}_k - \mathcal{R}) & 0 \end{bmatrix}, \quad \mathcal{A}_k^2 - \mathcal{A}_0^2 = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 \\ T^+(\mathcal{R}_k - \mathcal{R}) & 0 \end{bmatrix},$$

are compact, implying that operators (107) and (109) are Fredholm operator with index zero. □

Corollary 4.31.

(i) Let $r_0 > \operatorname{diam}(\Omega)$. The homogeneous counterpart of the Dirichlet problem (103) has only the trivial solution in $H^1(\Omega)$ if and only if operators (106) and (107) are invertible.

(ii) Let $r_0 > 0$. The homogeneous counterpart of the Dirichlet problem (103) has only the trivial solution in $H^1(\Omega)$ if and only if operators (108) and (109) are invertible.

Proof. If the homogeneous counterpart of the Dirichlet problem (103) has only the trivial solution in $H^1(\Omega)$, by Corollary 4.27 (i) the operators (106) and (107) will be injective. Hence, these operators become invertible due to Theorem 4.28.

Conversely, if the operator (106) or (107) is invertible, the homogeneous counterpart of BDIE system (D1) can have only the trivial solution in $H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, and hence, the result follows from Corollary 4.27 (i).

For operators (108) and (109), the proof is similar. □

4.7 Boundary Domain Integral Equations of the Mixed BVP

Let $\partial\Omega = \overline{\partial\Omega}_D \cup \overline{\partial\Omega}_N$, where $\partial\Omega_D$ and $\partial\Omega_N$ are non-empty, relatively open, non-intersecting parts of $\partial\Omega$. We will derive and analyze the system of BDIEs for the following mixed BVP

$$\begin{aligned} A_k u &= f \qquad &\text{in} \quad &\Omega, \\ \gamma^+ u &= \varphi_0 \qquad &\text{on} \quad &\partial\Omega_D, \\ T^+ u &= \psi_0 \qquad &\text{on} \quad &\partial\Omega_N, \end{aligned} \tag{116}$$

for unknown function $u \in H^1(\Omega)$, where $f \in L^2(\Omega)$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ are given functions.

Similar to the 3D case in [19] and the 2D case with $k = 0$ in [6], we let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of the given function φ_0 from $\partial\Omega_D$ to $\partial\Omega$ and ψ_0 from $\partial\Omega_N$ to $\partial\Omega$, respectively. Then, an arbitrary extension $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ preserving the function space can be represented as $\Phi = \Phi_0 + \varphi$ with $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$; and $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$ as $\Psi = \Psi_0 + \psi$ with $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$.

Considering (93), and restrictions of either (94) or (95) on the appropriate parts of $\partial\Omega$, we reduce the BVP (116) to four different BDIE systems. In each case, we substitute f for $A_k u$, $\Phi = \Phi_0 + \varphi$ for the boundary trace $\gamma^+ u$ and $\Psi = \Psi_0 + \psi$ for the co-normal derivative $T^+ u$, where Φ_0 and Ψ_0 are considered known while the triple $(u, \psi, \varphi) \in H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ is to be found.

4.7.1 BDIE System (M11)

This system is obtained by considering the third Green identity (93) in Ω, the restriction of its trace (94) on $\partial\Omega_D$, and the restriction of its co-normal derivative (95) on $\partial\Omega_N$, with respect to the unknowns u, ψ, and φ:

$$\begin{aligned} u + \mathcal{R}_k u - V\psi + W\varphi &= F_0, \qquad &\text{in} \quad &\Omega, \\ \gamma^+ \mathcal{R}_k u - \mathcal{V}\psi + \mathcal{W}\varphi &= \gamma^+ F_0 - \varphi_0, \quad &\text{on} \quad &\partial\Omega_D, \\ T^+ \mathcal{R}_k u - \mathcal{W}'\psi + \mathcal{L}^+\varphi &= T^+ F_0 - \psi_0, \quad &\text{on} \quad &\partial\Omega_N, \end{aligned} \tag{M11}$$

where

$$F_0 = \mathcal{P}f + V\Psi_0 - W\Phi_0 \qquad \text{in} \;\; \Omega. \tag{117}$$

The BDIE system (M11) can be rewritten in matrix form as

$$\mathcal{M}_k^{11}\mathscr{U} = \mathcal{F}^{11}, \tag{118}$$

where $\mathscr{U} = (u, \psi, \varphi)^t \in H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and

$$\mathcal{M}_k^{11} = \begin{bmatrix} I + \mathcal{R}_k & -V & W \\ r_{\partial\Omega_D}\gamma^+\mathcal{R}_k & -r_{\partial\Omega_D}\mathcal{V} & r_{\partial\Omega_D}\mathcal{W} \\ r_{\partial\Omega_N}T^+\mathcal{R}_k & -r_{\partial\Omega_N}\mathcal{W}' & r_{\partial\Omega_N}\mathcal{L}^+ \end{bmatrix},$$

$$\mathcal{F}^{11} = \begin{bmatrix} F_0 \\ r_{\partial\Omega_D}\gamma^+F_0 - \varphi_0 \\ r_{\partial\Omega_N}T^+F_0 - \psi_0 \end{bmatrix}.$$

Due to Corollaries 4.5 and 4.15, we get $F_0 \in H^{1,0}(\Omega; A_k)$. Then we have $\mathcal{F}^{11} \in H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N)$ and the operators

$$\mathcal{M}_k^{11} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N), \tag{119}$$

$$\mathcal{M}_k^{11} : H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N) \tag{120}$$

are bounded.

Taking into account Lemma 4.21, we prove the following remark in the same way as [19, Remark 5.1].

Remark 4.32. Let $r_0 > \operatorname{diam}(\Omega)$. $\mathcal{F}^{11} = \mathbf{0}$ if and only if $(f, \Phi_0, \Psi_0) = \mathbf{0}$.

4.7.2 BDIE System (M12)

By taking the third Green identity (93) in Ω and its trace (94) on the whole boundary $\partial\Omega$, we arrive at the system (M12):

$$\begin{aligned} u + \mathcal{R}_k u - V\psi + W\varphi &= F_0 && \text{in} \;\; \Omega, \\ \frac{1}{2}\varphi + \gamma^+\mathcal{R}_k u - \mathcal{V}\psi + \mathcal{W}\varphi &= \gamma^+F_0 - \Phi_0, && \text{on} \;\; \partial\Omega, \end{aligned} \tag{M12}$$

where F_0 is given by the relation (117). System (M12) can be rewritten in matrix form as

$$\mathcal{M}_k^{12}\mathscr{U} = \mathcal{F}^{12}, \tag{121}$$

where $\mathscr{U} = (u, \psi, \varphi)^t \in H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and

$$\mathcal{M}_k^{12} = \begin{bmatrix} I + \mathcal{R}_k & -V & W \\ \gamma^+\mathcal{R}_k & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}, \qquad \mathcal{F}^{12} = \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \Phi_0 \end{bmatrix}.$$

Note that $\mathcal{F}^{12} \in H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved (see Corollaries 4.5 and 4.15, Theorem 4.14 and [31, Theorem 1]), we see that the operators

$$\mathcal{M}_k^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{122}$$

$$\mathcal{M}_k^{12} : H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega) \tag{123}$$

are bounded.

Remark 4.33. Let $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ with $r_0 > \operatorname{diam}(\Omega)$. Then, $\mathcal{F}^{12} = \mathbf{0}$ if and only if $(f, \Phi_0, \Psi_0) = \mathbf{0}$.

Indeed, the latter obviously implies the former. Conversely, let $\mathcal{F}^{12} = (F_0, \gamma^+ F_0 - \Phi_0) = \mathbf{0}$. From $F_0 = 0$, we get $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω. Again from $\gamma^+ F_0 - \Phi_0 = 0$, we get $\Phi_0 = 0$ on $\partial\Omega$. Hence, we obtain $V\Psi_0 = 0$ in Ω, and the result follows from Lemma 4.20 *(i)*.

4.7.3 BDIE System (M21)

We obtain this system by using the third Green identity (93) on Ω and its co-normal derivative (95) on the whole boundary $\partial\Omega$:

$$\begin{aligned} u + \mathcal{R}_k u - V\psi + W\varphi &= F_0 \quad \text{in } \Omega, \\ \frac{1}{2}\psi + T^+\mathcal{R}_k u - \mathcal{W}'\psi + \mathcal{L}^+\varphi &= T^+ F_0 - \Psi_0 \quad \text{on } \partial\Omega, \end{aligned} \tag{M21}$$

where F_0 is given by (117). We rewrite the system (M21) in matrix form as

$$\mathcal{M}_k^{21}\mathscr{U} = \mathcal{F}^{21},$$

where $\mathscr{U} = (u, \psi, \varphi)^t \in H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and

$$\mathcal{M}_k^{21} = \begin{bmatrix} I + \mathcal{R}_k & -V & W \\ T^+\mathcal{R}_k & \frac{1}{2}I - \mathcal{W}' & \mathcal{L}^+ \end{bmatrix}, \qquad \mathcal{F}^{21} = \begin{bmatrix} F_0 \\ T^+F_0 - \Psi_0 \end{bmatrix}.$$

Here, $\mathcal{F}^{21} \in H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathcal{M}_k^{21}$, the following operators are bounded.

$$\mathcal{M}_k^{21} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \quad (124)$$

$$\mathcal{M}_k^{21} : H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega). \quad (125)$$

Remark 4.34. Let $r_0 > 0$. $\mathcal{F}^{21} = \mathbf{0}$ if and only if $(f, \Phi_0, \Psi_0) = \mathbf{0}$.

We prove this remark in the same way as Remark 4.33.

4.7.4 BDIE System (M22)

Here, we use the third Green identity (93) in Ω, the restriction of its trace (94) on $\partial\Omega_N$ and the restriction of its conormal derivative (95) on $\partial\Omega_D$ to get the system (M22),

$$\begin{aligned} u + \mathcal{R}_k u - V\psi + W\varphi &= F_0 \quad &&\text{in} \quad \Omega, \\ \frac{1}{2}\psi + T^+\mathcal{R}_k u - \mathcal{W}'\psi + \mathcal{L}^+\varphi &= T^+F_0 - \Psi_0 \quad &&\text{on} \quad \partial\Omega_D, \\ \frac{1}{2}\varphi + \gamma^+\mathcal{R}_k u - \mathcal{V}\psi + \mathcal{W}\varphi &= \gamma^+F_0 - \Phi_0 \quad &&\text{on} \quad \partial\Omega_N, \end{aligned} \quad \text{(M22)}$$

where F_0 is given by (117). Let us write the system (M22) in matrix form as

$$\mathcal{M}_k^{22}\mathscr{U} = \mathcal{F}^{22},$$

where $\mathscr{U} = (u, \psi, \varphi)^t \in H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$, and

$$\mathcal{M}_k^{22} = \begin{bmatrix} I + \mathcal{R}_k & -V & W \\ r_{\partial\Omega_D} T^+ \mathcal{R}_k & r_{\partial\Omega_D} \left(\frac{1}{2} I - \mathcal{W}'\right) & r_{\partial\Omega_D} \mathcal{L}^+ \\ r_{\partial\Omega_N} \gamma^+ \mathcal{R}_k & -r_{\partial\Omega_N} \mathcal{V} & r_{\partial\Omega_N} \left(\frac{1}{2} I + \mathcal{W}\right) \end{bmatrix},$$

$$\mathcal{F}^{22} = \begin{bmatrix} F_0 \\ r_{\partial\Omega_D}(T^+ F_0 - \Psi_0) \\ r_{\partial\Omega_N}(\gamma^+ F_0 - \Phi_0) \end{bmatrix}.$$

From the mapping properties of the operators involved, $\mathcal{F}^{22} \in H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N)$ and the following operators are bounded.

$$\mathcal{M}_k^{22} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N), \tag{126}$$

$$\mathcal{M}_k^{22} : H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N). \tag{127}$$

Taking into account Lemma 4.21, we prove the following remark in the same way as [19, Remark 5.11].

Remark 4.35. Let $r_0 > \mathrm{diam}(\Omega)$. $\mathcal{F}^{22} = \mathbf{0}$ if and only if $(f, \Phi_0, \Psi_0) = \mathbf{0}$.

4.8 Equivalence, Fredholm Properties and Invertibility for BDIE operators of the mixed BVP

Let us prove that the mixed BVP (116) is equivalent to the BDIE systems (M11), (M12), (M21), and (M22).

Theorem 4.36. *Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$, respectively, and let $f \in L^2(\Omega)$.*

(i) If some $u \in H^1(\Omega)$ solves the mixed BVP (116), then the triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$, where

$$\psi = T^+ u - \Psi_0, \qquad \varphi = \gamma^+ u - \Phi_0 \qquad \text{on } \partial\Omega, \tag{128}$$

solves the BDIE systems (M11), (M12), (M21) and (M22).

(ii) Let $r_0 > \operatorname{diam}(\Omega)$. *If a triple* $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ *solves one of the BDIE systems (M11) or (M12) or (M22), then* u *solves BVP* (116)*, and relations* (128) *hold.*

(iii) Let $r_0 > 0$. *If a triple* $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ *solves the BDIE system (M21), then* u *solves BVP* (116)*, and relations* (128) *hold.*

Proof. To prove (i), we let $u \in H^1(\Omega)$ be a solution to BVP (116). Then, for ψ and φ defined by (128), we get $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$. Recalling how the four BDIE systems were constructed, the result immediately follows from relations (93)–(95).

To prove (ii) and (iii), let us first assume that a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves either the BDIE system (M11) or (M12) or (M21) or (M22). The first equation of each system and Lemma 4.19 with $\Psi = \psi + \Psi_0$ and $\Phi = \varphi + \Phi_0$ imply that u solves the PDE $A_k u = f$ on Ω the relation

$$V\Psi^* - W\Phi^* = 0 \qquad \text{in } \Omega \tag{129}$$

holds for

$$\Psi^* = \Psi_0 + \psi - T^+u \quad \text{and} \quad \Phi^* = \Phi_0 + \varphi - \gamma^+u. \tag{130}$$

Whenever in the remaining proof we take the trace or co-normal derivative of the first equation of each system, we make use of relations (73)–(75) and the last equation in (68).

Proof for (M11). Let a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves the BDIE system (M11). Taking the trace of the first equation in (M11) on $\partial\Omega_D$ and subtracting the second equation from it, we obtain

$$\gamma^+u = \varphi_0 \qquad \text{on } \partial\Omega_D, \tag{131}$$

i.e., u satisfies the Dirichlet condition in (116). We now take the co-normal derivative of the first equation in (M11) on $\partial\Omega_N$ and subtract the third equation from it to get

$$T^+u = \psi_0 \quad \text{on } \partial\Omega_N, \tag{132}$$

i.e., u satisfies the Neumann condition in (116). Taking into account that $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, equations (131) and (132) imply that the first equation in (128) is satisfied on $\partial\Omega_N$ and the second equation in (128) on $\partial\Omega_D$. From this and relation (130), we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$, $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$. Since relation (129) holds and $r_0 > \mathrm{diam}(\partial\Omega_D)$, from Lemma 4.21 we get $\Psi^* = \Phi^* = 0$, which completes the proof of conditions (128).

Proof for (M12). Now, let a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves BDIE system (M12). Taking trace of the first equation in (M12) on $\partial\Omega$ and subtracting the second one from it, we obtain

$$\gamma^+u = \Phi_0 + \varphi \quad \text{on } \partial\Omega, \tag{133}$$

which means that the second equation in (128) holds. Since $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$, we see that the Dirichlet condition in (116) is satisfied.

Due to (133), the second term in (129) vanishes and by Lemma 4.20 (i), we obtain

$$\Psi_0 + \psi - T^+u = 0 \quad \text{on } \partial\Omega, \tag{134}$$

which shows that the first equation of (128) is satisfied as well. Since $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, (134) implies that u satisfies the Neumann boundary condition in (116).

Proof for (M22). Now, let a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves the BDIE system (M22). Taking the co-normal derivative of the first equation in (M22) on $\partial\Omega_D$ and subtracting it from the second equation, we obtain

$$\psi = T^+u - \Psi_0 \quad \text{on } \partial\Omega_D. \tag{135}$$

Taking the trace of the first equation in (M22) on $\partial\Omega_N$ and subtracting it from the third equation yields

$$\varphi = \gamma^+ u - \Phi_0 \qquad \text{on } \partial\Omega_N. \tag{136}$$

Equation (135) and (136) imply that the first equation in (128) is satisfied on $\partial\Omega_D$ and the second one on $\partial\Omega_N$. Due to (135) and (136), we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_N)$, $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_D)$ in (129) and (130). Then, Lemma 4.21 with $\Gamma_1 = \partial\Omega_N$ and $\Gamma_2 = \partial\Omega_D$ implies that $\Psi^* = \Phi^* = 0$, which completes the proof of conditions (128) on the whole boundary $\partial\Omega$. Taking into account that $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0, \Psi_0 = \psi_0$ on $\partial\Omega_N$, (128) implies the boundary conditions in the mixed BVP (116).

Proof for (M21). Let a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves the BDIE system (M21). We take the co-normal derivative of the first equation in (M21) on $\partial\Omega$ and subtract the second equation from it to obtain

$$\psi + \Psi_0 - T^+ u = 0 \qquad \text{on } \partial\Omega, \tag{137}$$

which is the first equation of (128). Since $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, we see that u satisfies the Neumann condition in (116).

Due to (137), the first term in (129) vanishes and, by Lemma 4.20 (ii), we obtain

$$\Phi_0 + \varphi - \gamma^+ u = 0 \qquad \text{on } \partial\Omega, \tag{138}$$

which means that the second condition in (128) holds as well. Since $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$, from (138) we see that u satisfies the Dirichlet boundary condition in (116). □

Corollary 4.37. *Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$, respectively, and let $f \in L^2(\Omega)$.*

(i) Let $r_0 > \operatorname{diam}(\Omega)$. If a triple $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves the BDIE system (M11) or (M12) or (M22), then it solves all the other three BDIE systems.

(ii) Let $r_0 > 0$*. If a triple* $(u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ *solves the BDIE system (M21), then it solves (M11), (M12) and (M22).*

Corollary 4.38.

(i) Let $r_0 > \mathrm{diam}(\Omega)$*. The homogeneous counterpart of BDIE system (M11) or (M12) or (M22) has a non-trivial solution in* $H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ *if and only if the homogeneous counterpart of the mixed problem* (116) *has a non-trivial solution in* $H^1(\Omega)$*.*

(ii) Let $r_0 > 0$*. The homogeneous counterpart of BDIE system (M21) has a non-trivial solution in* $H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ *if and only if the homogeneous counterpart of the mixed problem* (116) *has a non-trivial solution in* $H^1(\Omega)$*.*

Now, we prove the Fredholm property of the corresponding operators of the BDIE system (M11), (M12), and (M21).

Theorem 4.39.

(i) If $r_0 > \mathrm{diam}(\Omega)$*, operators* (119) *and* (122) *are Fredholm with index zero.*

(ii) If $r_0 > 0$*, operator* (124) *is Fredholm with index zero.*

Proof. Here we follow the arguments similar to the ones used in [19, for 3D case].

Operator (119)**.** To prove the Fredholm property of operator (119), let us consider the operator

$$\mathcal{M}^{11}_{*} := \begin{bmatrix} I & -V & W \\ 0 & -r_{\partial\Omega_D}\mathcal{V} & 0 \\ 0 & 0 & r_{\partial\Omega_N}\widehat{\mathcal{L}} \end{bmatrix},$$

where $\widehat{\mathcal{L}}$ is given by (79).

The operator $\mathcal{M}_*^{11}$ is an upper triangular matrix operator with the following scalar diagonal operators,

$$\begin{aligned} I : H^1(\Omega) &\longrightarrow H^1(\Omega), \\ r_{\partial\Omega_D}\mathcal{V} : \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) &\longrightarrow H^{\frac{1}{2}}(\partial\Omega_D), \\ r_{\partial\Omega_N}\widehat{\mathcal{L}} : \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) &\longrightarrow H^{-\frac{1}{2}}(\partial\Omega_N), \end{aligned}$$

that are invertible (due to Theorems 4.11 and 4.12 (i) for the second and third operators). Along with the mapping properties of the operators V and W (see Theorem 4.4), the operator

$$\mathcal{M}_*^{11} : H^1(\Omega)\times\widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)\times\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega)\times H^{\frac{1}{2}}(\partial\Omega_D)\times H^{-\frac{1}{2}}(\partial\Omega_N)$$

is invertible. The operator

$$\begin{aligned} \mathcal{M}_k^{11} - \mathcal{M}_*^{11} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \\ \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N), \end{aligned}$$

where

$$\mathcal{M}_k^{11} - \mathcal{M}_*^{11} := \begin{bmatrix} \mathcal{R}_k & 0 & 0 \\ r_{\partial\Omega_D}\gamma^+\mathcal{R}_k & 0 & r_{\partial\Omega_D}\mathcal{W} \\ r_{\partial\Omega_N}T^+\mathcal{R}_k & -r_{\partial\Omega_N}\mathcal{W}' & r_{\partial\Omega_N}\left(\mathcal{L}^+ - \widehat{\mathcal{L}}\right) \end{bmatrix}.$$

is compact due to Corollaries 4.16 and 4.18 as well as Theorems 4.7 and 4.12 (ii). Hence, (119) is a Fredholm operator with zero index.

Operator (122). Let us denote

$$\mathcal{M}_*^{12} := \begin{bmatrix} I & -V & W \\ 0 & -\mathcal{V} & \frac{1}{2}I \end{bmatrix}.$$

Then

$$\mathcal{M}_*^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is bounded. To show the invertibility of $\mathcal{M}_*^{12}$, taking into account Theorem 4.11, we follow the proof for 3D case in [19]. Consider the equation

$$\mathcal{M}_*^{12}\mathscr{U} = \widetilde{F} \tag{139}$$

with an unknown vector $\mathscr{U} = (u, \psi, \varphi)^t \in H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and a given vector $\widetilde{F} := (\widetilde{F}_1, \widetilde{F}_2)^t \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Rewrite (124) componentwise as

$$u - V\psi + W\varphi = \widetilde{F}_1 \quad \text{in } \Omega, \tag{140}$$

$$\frac{1}{2}\varphi - \mathcal{V}\psi = \widetilde{F}_2 \quad \text{on } \partial\Omega. \tag{141}$$

The restriction of (141) on $\partial\Omega_D$ gives

$$-r_{\partial\Omega_D}\mathcal{V}\psi = r_{\partial\Omega_D}\widetilde{F}_2. \tag{142}$$

Due to Theorem 4.11, (142) is uniquely solvable, i.e., for arbitrary $\widetilde{F}_2 \in H^{\frac{1}{2}}(\partial\Omega)$ there exists a unique $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ satisfying (142). Moreover,

$$\left[\mathcal{V}\psi + \widetilde{F}_2\right] \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N). \tag{143}$$

Then, (141) along with (143) yield that φ is defined also uniquely as

$$\varphi = 2\left[\mathcal{V}\psi + \widetilde{F}_2\right] \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N).$$

Hence, (141) with arbitrary $\widetilde{F}_2 \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega)$ defines $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ uniquely. Since $V\psi,\ W\varphi \in H^1(\Omega)$, from (140) we obtain that

$$u = V\psi - W\varphi + \widetilde{F}_1 \quad \text{in } \Omega,$$

showing that the function $u \in H^1(\Omega)$ is also defined uniquely. The above arguments show that operator $\mathcal{M}_*^{12}$ is invertible.

Due to Corollaries 4.6 and 4.16, the operator

$$\mathcal{M}_k^{12} - \mathcal{M}_*^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

where

$$\mathcal{M}_k^{12} - \mathcal{M}_*^{12} := \begin{bmatrix} \mathcal{R}_k & 0 & 0 \\ \gamma^+\mathcal{R}_k & 0 & \mathcal{W} \end{bmatrix},$$

is compact. Then, operator (122) is Fredholm of index zero.

Operator (124)**.** The proof for operator (124) follows by the arguments similar to those in the proof for operator (122). Let

$$\mathcal{M}_*^{21} := \begin{bmatrix} I & -V & W \\ 0 & \frac{1}{2}I & \widehat{\mathcal{L}} \end{bmatrix}.$$

Then

$$\mathcal{M}_*^{21} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$$

is bounded. Since the operators

$$I : H^1(\Omega) \to H^1(\Omega) \quad \text{and} \quad \widehat{\mathcal{L}} : \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \to H^{-\frac{1}{2}}(\partial\Omega)$$

are invertible, using similar arguments as in the proof of the operator (122), we can show that $\mathcal{M}_*^{21}$ is invertible.

Due to the mapping properties of the operators involved, the operator

$$\mathcal{M}_k^{21} - \mathcal{A}_*^{21} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega),$$

where

$$\mathcal{M}_k^{21} - \mathcal{M}_*^{21} := \begin{bmatrix} \mathcal{R}_k & 0 & 0 \\ T^+\mathcal{R}_k & -\mathcal{W}' & \left(\mathcal{L}^+ - \widehat{\mathcal{L}}\right) \end{bmatrix}$$

is compact implying that $\mathcal{M}_k^{21}$ is Fredholm operator of index zero. $\square$

Let us consider the particular cases of operators (119), (120), (122), (123), (124) and (125), for $k = 0$, that is,

$$\mathcal{M}_0^{11} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N), \tag{144}$$

$$\mathcal{M}_0^{11} : H^{1,0}(\Omega; A) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N), \tag{145}$$

$$\mathcal{M}_0^{12} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \tag{146}$$

$$\mathcal{M}_0^{12} : H^{1,0}(\Omega; A) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega), \tag{147}$$

$$\mathcal{M}_0^{21} : H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \tag{148}$$

$$\mathcal{M}_0^{21} : H^{1,0}(\Omega; A) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \longrightarrow H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega). \tag{149}$$

where

$$\mathcal{M}_0^{11} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ r_{\partial\Omega_D}\gamma^+\mathcal{R} & -r_{\partial\Omega_D}\mathcal{V} & r_{\partial\Omega_D}\mathcal{W} \\ r_{\partial\Omega_N}T^+\mathcal{R} & -r_{\partial\Omega_N}\mathcal{W}' & r_{\partial\Omega_N}\mathcal{L}^+ \end{bmatrix},$$

$$\mathcal{M}_0^{12} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ \gamma^+\mathcal{R} & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix},$$

$$\mathcal{M}_0^{21} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ T^+\mathcal{R} & \frac{1}{2}I - \mathcal{W}' & \mathcal{L}^+ \end{bmatrix}.$$

Theorem 4.40.

(i) If $r_0 > \operatorname{diam}(\Omega)$*, then operators* (144)*,* (145)*,* (146) *and* (147) *are invertible.*

(ii) If $r_0 > 0$*, then operators* (148) *and* (149) *are invertible.*

Proof. This theorem for $r_0 = 1$ was proved in [30, Theorem 3.25]. Here we update the proof for arbitrary $r_0 > 0$ similar to Theorem 4.29 for the BDIE system of the Dirichlet problem.

It is well known that the homogeneous mixed problem (116) with $k = 0$, that is, with $A_k = A$, where the operator A is given by (57) and $0 < a_0 < a(x) < a_1 < \infty$, has only the trivial solution in $H^{1,0}(\Omega; A)$ and $H^1(\Omega)$. This can be obtained, e.g., from the first Green identity (63). Then the equivalence Theorem 4.36 implies that all operators (144)–(149) are injective. By Theorem 4.39, operators (144), (146) and (148) are Fredholm with zero index. Then the injectivity of operators (144), (146) and (148) implies their invertibility (see e.g. [56, Theorem 2.27]).

To prove the invertibility of operator (145), we remark that for any

$$\mathcal{F}^{11} \in H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N),$$

a solution of the equation $\mathcal{M}_0^{11}\mathcal{U} = \mathcal{F}^{11}$ can be written as $\mathcal{U} = (\mathcal{M}_0^{11})^{-1}\mathcal{F}^{11}$, where

$$(\mathcal{M}_0^{11})^{-1} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N) \to H^1(\Omega) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$$

is the continuous inverse to operator (144). But due to Lemma 4.19, the first equation of system (M11) with $k = 0$ implies that

$$\mathcal{U} = (\mathcal{M}_0^{11})^{-1}\mathcal{F}^{11} \in H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N),$$

and moreover, the operator

$$(\mathcal{M}_0^{11})^{-1} : H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N) \\ \to H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$$

is continuous, which implies invertibility of operator (145).

The invertibility of operators (147) and (149) is proved in a similar fashion. □

Now, we are in the position to prove an analogue of Theorem 4.39 for operators (120), (123), and (125).

Theorem 4.41.

(i) If $r_0 > \operatorname{diam}(\Omega)$*, operators* (120) *and* (123) *are Fredholm with index zero.*

(ii) If $r_0 > 0$*, operator* (125) *is Fredholm with index zero.*

Proof. By Theorem 4.40, we see that operators (145), (147), and (149) are invertible. Due to Corollaries 4.17, the operators

$$
\begin{aligned}
\mathcal{M}_k^{11} - \mathcal{M}_0^{11} &: H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \\
&\qquad\longrightarrow H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N), \\
\mathcal{M}_k^{12} - \mathcal{M}_0^{12} &: H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \\
&\qquad\longrightarrow H^{1,0}(\Omega; A_k) \times H^{\frac{1}{2}}(\partial\Omega), \\
\mathcal{M}_k^{21} - \mathcal{M}_0^{21} &: H^{1,0}(\Omega; A_k) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N) \\
&\qquad\longrightarrow H^{1,0}(\Omega; A_k) \times H^{-\frac{1}{2}}(\partial\Omega).
\end{aligned}
$$

where

$$
\mathcal{M}_k^{11} - \mathcal{M}_0^{11} = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 & 0 \\ r_{\partial\Omega_D}\gamma^+(\mathcal{R}_k - \mathcal{R}) & 0 & 0 \\ r_{\partial\Omega_N}T^+(\mathcal{R}_k - \mathcal{R}) & 0 & 0 \end{bmatrix},
$$

$$
\mathcal{M}_k^{12} - \mathcal{M}_0^{12} = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 & 0 \\ \gamma^+(\mathcal{R}_k - \mathcal{R}) & 0 & 0 \end{bmatrix},
$$

$$
\mathcal{M}_k^{21} - \mathcal{M}_0^{21} = \begin{bmatrix} \mathcal{R}_k - \mathcal{R} & 0 & 0 \\ T^+(\mathcal{R}_k - \mathcal{R}) & 0 & 0 \end{bmatrix}.
$$

are compact, implying that operators (120), (123), and (125) are Fredholm operators with index zero. □

Due to Corollary 4.38 and Theorem 4.39, we obtain the following assertion.

Corollary 4.42.

(i) Let $r_0 > \operatorname{diam}(\Omega)$*. The homogeneous counterpart of the mixed problem* (116) *has only the trivial solution in* $H^1(\Omega)$ *if and only if the operators* (119)*,* (120)*,* (122)*, and* (123) *are invertible.*

(ii) Let $r_0 > 0$. The homogeneous counterpart of the mixed problem (116) *has only the trivial solution in $H^1(\Omega)$ if and only if the operators* (124) *and* (125) *are invertible.*

Remark 4.43. Equivalence, Fredholm properties, and invertibility for BDIE operators (126) and (127), for $\mathcal{M}_k^{22}$, are not analysed in Section 4.8. Note that they can be considered using a different approach similar to [22, Theorem 7.1], [30, Theorem 3.31], cf. also [19, Theorems 5.15, 5.19].

5 Conclusion

The Hardy operator plays a significant role in the analysis of partial differential equations. So, a better understanding of this operator is important. Moreover, analysis of BDIEs for the variable coefficient BVPs is also crucial for the numerical approximation of their solutions.

Determining the best possible constants in Hardy-type inequalities and finding sharp versions of these inequalities is an ongoing challenge. In paper I, operator norms of some matrix operators on cones is found. Consequently, the best constants for the inequalities relating the Cesàro and Copson operators are obtained. As a by product, the end-point norm estimates of Cesàro, Copson operators, and the Cesàro and Copson operators minus identity have been found.

In paper II, we characterized the optimal non-absolute domain for the Hardy operator (and its dual) minus the identity, on Banach function spaces in general and in the Lebesgue space $L^p(0,\infty)$, $1 \leq p \leq \infty$ in particular.

In paper III, appropriate parametrix is used to formulate and analyze BDIEs for the Dirichlet and Mixed (Dirichlet-Neumann) BVPs for a two-dimensional variable coefficient Helmholtz equation

$$Au(x) + k^2(x)u(x) = f(x), \qquad x \in \Omega,$$

where

$$A = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big[a(x)\frac{\partial}{\partial x_i}\Big],$$

$k(x)$ is a real function of x, $a(x)$ is a known variable coefficient, u is unknown function and $f \in L^2(\Omega)$ is a given function, with the assumption that $a, k \in C^\infty(\overline{\Omega})$ and $0 < a_0 < a(x) < a_1 < \infty$, for some constants a_0 and a_1, for all $x \in \Omega$. The equivalence of these BVPs and the formulated BDIE systems is proved. Fredholm properties, invertibility and unique solvability of BDIE systems are investigated in appropriate Sobolev spaces.

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