ANISOTROPIC CAFFARELLI-KOHN-NIRENBERG INEQUALITIES WITH HIGHER-ORDER FRACTIONAL DERIVATIVES

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Abstract. We prove a family of anisotropic Caffarelli-Kohn-Nirenberg interpolation inequalities involving higher-order fractional derivatives and weights of the form $|x'|^{\theta_1}|x|^{\theta_2}|x_n|^{\theta_3}$.

1. Introduction and main result

In their foundational work [1], L. A. Caffarelli, R. Kohn, and L. Nirenberg characterized the indices $1 \leq p, q \leq \infty$, $0 < r \leq \infty$, and exponents $\alpha, \beta, \gamma \in \mathbb{R}$ such that, for each $\theta \in [0,1]$ the first-order weighted interpolation inequality

$$\||x|^{\gamma} f\|_{L^r} \lesssim \||x|^{\alpha}|Df|\|_{L^p}^{\theta} \||x|^{\beta} f\|_{L^q}^{1-\theta} \tag{1.1}$$

holds true for every f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. Later on, in [5] C.-S. Lin extended (1.1) as to include higher-order derivatives by providing necessary and sufficient conditions on $1 \leq p, q \leq \infty$, $0 < r \leq \infty$, $\alpha, \beta, \gamma \in \mathbb{R}$, $j \in \mathbb{N}_0$, $m \in \mathbb{N}$, with $j \leq m$, for the validity of the higher-order weighted interpolation inequality

$$\||x|^{\gamma}|D^j f|\|_{L^r} \lesssim \||x|^{\alpha}|D^m f|\|_{L^p}^{\theta} \||x|^{\beta} f\|_{L^q}^{1-\theta}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n), \theta \in [j/m, 1].$$

More recently, in [2, Theorem 1.6] R. Duarte and J. Drumond Silva extended Lin's result by identifying indices $1 < p, q, r < \infty$; $0 \leq t < s$ (smoothness indices, not necessarily integers), $t/s \leq \theta \leq 1$; $\alpha, \beta, \gamma \in \mathbb{R}$, such that the higher-order fractional-derivative weighted interpolation inequality

$$\||x|^{\gamma}|D^t f|\|_{L^r} \lesssim \||x|^{\alpha}|D^s f|\|_{L^p}^{\theta} \||x|^{\beta} f\|_{L^q}^{1-\theta},$$

holds true for every $f \in \mathcal{S}(\mathbb{R}^n)$. When $n \geq 2$ and in the context of asymptotic stability of solutions to the Navier-Stokes equations, in [4, Theorem 1.1] Y. Y. Li and X. Yan introduced an anisotropic version of the Caffarelli-Kohn-Nirenberg inequality (1.1) by determining the indices $1 \leq p, q \leq \infty$, $0 < r \leq \infty$, and exponents $\gamma_1, \gamma_2, \gamma_3, \alpha, \mu, \beta$ such that

$$\||x|^{\gamma_1}|x'|^{\alpha} f\|_{L^r} \lesssim \||x|^{\gamma_2}|x'|^{\mu}|\nabla f|\|_{L^p}^{\theta} \||x|^{\gamma_3}|x'|^{\beta} f\|_{L^q}^{1-\theta}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n), \theta \in [0,1],$$

where $x = (x', x_n) \in \mathbb{R}^n$ and $x' := (x_1, \ldots, x_{n-1})$. Always with $n \geq 2$, in their study on the regularity of solutions to p-Laplace equations, in [7] C. X. Miao and Z. W. Zhao

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identified indices $1 < p < q$ and exponents $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ yielding the anisotropic Sobolev inequality

$$\|f\|_{L^q(u_{\theta_1,\theta_2,\theta_3})} \lesssim \|\nabla f\|_{L^p(u_{\theta_1,\theta_2,\theta_3})}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n), \tag{1.2}$$

where

$$u_{\theta_1,\theta_2,\theta_3}(x) := |x'|^{\theta_1}|x|^{\theta_2}|x_n|^{\theta_3}, \tag{1.3}$$

see [7, Corollary 2.9]. As a key element in their approach, given $1 < p < \infty$, Miao and Zhao introduced in [7] the exponent classes

$$\begin{cases} \mathcal{A} := \{(\theta_1,\theta_2,\theta_3) : \theta_1 > -(n-1),\ \theta_2 \geq 0,\ \theta_3 > -1\}, \\ \mathcal{B} := \{(\theta_1,\theta_2,\theta_3) : \theta_1 > -(n-1),\ \theta_2 < 0,\ \theta_3 > -1,\ \theta_1+\theta_2+\theta_3 > -n\}, \\ \mathcal{C}_p := \{(\theta_1,\theta_2,\theta_3) : \theta_1 < (n-1)(p-1),\ \theta_2 \leq 0,\ \theta_3 < p-1\}, \\ \mathcal{D}_p := \{(\theta_1,\theta_2,\theta_3) : \theta_1 < (n-1)(p-1),\ \theta_2 > 0,\ \theta_3 < p-1,\ \theta_1+\theta_2+\theta_3 < n(p-1)\}, \end{cases}$$

to describe the membership of the weight $u_{\theta_1,\theta_2,\theta_3}$ as defined in (1.3) to the Muckenhoupt class $A_p(\mathbb{R}^n)$. More precisely, by [7, Theorem 3.2], $u_{\theta_1,\theta_2,\theta_3} \in A_p(\mathbb{R}^n)$ if and only if $(\theta_1,\theta_2,\theta_3) \in (\mathcal{A}\cup\mathcal{B}) \cap (\mathcal{C}_p \cup \mathcal{D}_p)$, see Section 2 below. On the other hand, as proved in Lemma 2.5 in Section 2, $(\theta_1,\theta_2,\theta_3) \in \mathcal{A}\cup\mathcal{B}$ means exactly that $u_{\theta_1,\theta_2,\theta_3} \in L^1_{\text{loc}}(\mathbb{R}^n)$. For the sake of coherence, we have kept the notations used in [7], although it is easy to prove that, in fact:

$$\mathcal{A}\cup\mathcal{B} = \{(\theta_1,\theta_2,\theta_3) : \theta_1 > -(n-1),\ \theta_3 > -1,\ \theta_1+\theta_2+\theta_3 > -n\},$$
$$\mathcal{C}_p\cup\mathcal{D}_p = \{(\theta_1,\theta_2,\theta_3) : \theta_1 < (n-1)(p-1),\ \theta_3 < p-1,\ \theta_1+\theta_2+\theta_3 < n(p-1)\}.$$

The purpose of this work is to provide a full scale of anisotropic interpolation inequalities based on weights of the form $u_{\theta_1,\theta_2,\theta_3}$ and involving the homogeneous higher-order fractional derivative operator D^s (defined as $\widehat{D^s f} := |\cdot|^s \hat{f}$ for $s > 0$). This scale of anisotropic interpolation inequalities extends all the aforementioned ones. Our main results are the following Theorems 1.1, 1.2, and 1.4.

Theorem 1.1. *Fix $1 < p, q, r < \infty$ and $0 < t < s$. Set $\theta := t/s \in (0,1)$ and suppose that*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{(1-\theta)}{q}. \tag{1.4}$$

Let $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ satisfy

$$p(\alpha_1,\beta_1,\gamma_1) \in (\mathcal{A}\cup\mathcal{B}) \cap (\mathcal{C}_p\cup\mathcal{D}_p), \tag{1.5}$$
$$q(\alpha_0,\beta_0,\gamma_0) \in (\mathcal{A}\cup\mathcal{B}) \cap (\mathcal{C}_q\cup\mathcal{D}_q), \text{ and} \tag{1.6}$$
$$(\alpha_j,\beta_k,\gamma_\ell) \in \mathcal{A}\cup\mathcal{B}, \quad \text{for } j,k,\ell = 0,1. \tag{1.7}$$

Then,

$$\||x'|^{\alpha_\theta}|x|^{\beta_\theta}|x_n|^{\gamma_\theta} D^t f\|_{L^r} \lesssim \||x'|^{\alpha_1}|x|^{\beta_1}|x_n|^{\gamma_1} D^s f\|_{L^p}^{\theta} \||x'|^{\alpha_0}|x|^{\beta_0}|x_n|^{\gamma_0} f\|_{L^q}^{1-\theta}, \tag{1.8}$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, where $(\alpha_\theta,\beta_\theta,\gamma_\theta) := (\alpha_1,\beta_1,\gamma_1)\theta + (\alpha_0,\beta_0,\gamma_0)(1-\theta)$.

For the statement of our next theorems, given $1 < p \leq q < \infty$ and $0 < \alpha < n$ define

$$\Theta_{p,q}(\alpha) := \alpha + n\left(\frac{1}{q} - \frac{1}{p}\right). \tag{1.9}$$

As pointed out in Remark 2.1 below, we can assume $\Theta_{p,q}(\alpha) \geq 0$. The cases $\Theta_{p,q}(\alpha) = 0$ (that is, $q = np/(n-\alpha p)$) and $\Theta_{p,q}(\alpha) > 0$ (that is, $q < np/(n-\alpha p)$) are referred to as the *critical* and *subcritical* cases, respectively, for the indices p, q, α. Our next two theorems cover both the critical and subcritical cases. Firstly, we have the following weighted Sobolev inequality, which corresponds to the case $\theta = 1$ in the interpolation inequality.

Theorem 1.2. *Fix $1 < p \leq q < \infty$, $t < s < t + n$, and let $\theta_1, \theta_2, \theta_3, \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}$ satisfy*

$$-\frac{p'}{p}(\sigma_1, \sigma_2, \sigma_3), (\theta_1, \theta_2, \theta_3) \in \mathcal{A} \cup \mathcal{B}. \tag{1.10}$$

(i) If $\Theta_{p,q}(s-t) := s - t + n\left(\frac{1}{q} - \frac{1}{p}\right) = 0$ and

$$\frac{\theta_j}{q} = \frac{\sigma_j}{p}, \quad \text{for } j = 1, 2, 3; \tag{1.11}$$

(ii) or if $\Theta_{p,q}(s-t) := s - t + n\left(\frac{1}{q} - \frac{1}{p}\right) > 0$ and

$$\frac{\theta_1 + \theta_2 + \theta_3}{q} - \frac{\sigma_1 + \sigma_2 + \sigma_3}{p} = -\Theta_{p,q}(s-t), \tag{1.12}$$

$$\left(\frac{\theta_1}{q} - \frac{\sigma_1}{p}\right) > -\frac{(n-1)}{n}\Theta_{p,q}(s-t), \tag{1.13}$$

$$\left(\frac{\theta_3}{q} - \frac{\sigma_3}{p}\right) > -\frac{1}{n}\Theta_{p,q}(s-t); \tag{1.14}$$

then,

$$\|D^t f\|_{L^q(u_{\theta_1,\theta_2,\theta_3})} \lesssim \|D^s f\|_{L^p(u_{\sigma_1,\sigma_2,\sigma_3})}, \quad \forall f \in D^{>\tau}\mathcal{S}(\mathbb{R}^n), \tag{1.15}$$

with $\tau := \max\{-n, -t\}$, $D^{>\tau} := \{f \in L^\infty : f = D^b g$ for some $g \in \mathcal{S}(\mathbb{R}^n)$ and $b > \tau\}$, and the implicit constant depends only on p, q, s, t, $\theta_1, \theta_2, \theta_3, \sigma_1, \sigma_2, \sigma_3$, and n.

Remark 1.3. As pointed out in [2, Remark 3], if $t > 0$ then $\tau < 0$ and $D^{>\tau}$ contains the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. In particular, the weighted Sobolev inequality (1.15) extends (1.2) to the case of arbitrary fractional derivatives D^t and D^s with $0 < t < s < t + n$. Finally, notice that (1.15) can be written as

$$\|u_{\theta_1/q,\theta_2/q,\theta_3/q} D^t f\|_{L^q} \lesssim \|u_{\sigma_1/p,\sigma_2/p,\sigma_3/p} D^s f\|_{L^p}, \quad \forall f \in D^{>\tau}\mathcal{S}(\mathbb{R}^n). \tag{1.16}$$

Secondly, for an arbitrary $\theta \in (t/s, 1)$ we have

Theorem 1.4. *Fix $1 < p, q, r < \infty$, $0 < t < s$, and $t/s < \theta < 1$, and let $a \in (1, \infty)$ be defined by*

$$\frac{1}{a} := \frac{\theta}{p} + \frac{1-\theta}{q}. \tag{1.17}$$

Suppose that $a \leq r$, that $\theta s < t + n$, and let $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ satisfy

$$p(\alpha_1, \beta_1, \gamma_1) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_p \cup \mathcal{D}_p),$$
$$q(\alpha_0, \beta_0, \gamma_0) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_q \cup \mathcal{D}_q),$$
$$(\alpha_j, \beta_k, \gamma_\ell) \in \mathcal{A} \cup \mathcal{B}, \quad \text{for } j, k, \ell = 0, 1.$$

As before, $(\alpha_\theta, \beta_\theta, \gamma_\theta) := (\alpha_1, \beta_1, \gamma_1)\theta + (\alpha_0, \beta_0, \gamma_0)(1-\theta)$.

(i) If $\Theta_{a,r}(\theta s - t) := \theta s - t + n\left(\frac{1}{r} - \frac{1}{a}\right) = 0$ *and*

$$-a'(\alpha_\theta, \beta_\theta, \gamma_\theta), r(\alpha_\theta, \beta_\theta, \gamma_\theta) \in \mathcal{A} \cup \mathcal{B}, \tag{1.18}$$

then

$$\||x'|^{\alpha_\theta}|x|^{\beta_\theta}|x_n|^{\gamma_\theta} D^t f\|_{L^r} \lesssim \||x'|^{\alpha_1}|x|^{\beta_1}|x_n|^{\gamma_1} D^s f\|_{L^p}^{\theta} \||x'|^{\alpha_0}|x|^{\beta_0}|x_n|^{\gamma_0} f\|_{L^q}^{1-\theta}, \tag{1.19}$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

(ii) If $\Theta_{a,r}(\theta s - t) := \theta s - t + n\left(\frac{1}{r} - \frac{1}{a}\right) > 0$ *and if* $\alpha'_\theta, \beta'_\theta, \gamma'_\theta \in \mathbb{R}$ *satisfy*

$$(\alpha'_\theta - \alpha_\theta) + (\beta'_\theta - \beta_\theta) + (\gamma'_\theta - \gamma_\theta) = -\Theta_{a,r}(\theta s - t), \tag{1.20}$$

$$\alpha'_\theta - \alpha_\theta > -\frac{(n-1)}{n}\Theta_{a,r}(\theta s - t), \tag{1.21}$$

$$\gamma'_\theta - \gamma_\theta > -\frac{1}{n}\Theta_{a,r}(\theta s - t); \tag{1.22}$$

then

$$\||x'|^{\alpha'_\theta}|x|^{\beta'_\theta}|x_n|^{\gamma'_\theta} D^t f\|_{L^r} \lesssim \||x'|^{\alpha_1}|x|^{\beta_1}|x_n|^{\gamma_1} D^s f\|_{L^p}^{\theta} \||x'|^{\alpha_0}|x|^{\beta_0}|x_n|^{\gamma_0} f\|_{L^q}^{1-\theta}, \tag{1.23}$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

In the case $a = r$ from Theorem 1.4, the fact that $\Theta_{a,r}(\theta s - t) = \theta s - t > 0$ immediately yields the following corollary.

Corollary 1.5. *Fix* $1 < p, q < \infty$, $0 < t < s$, *and* $t/s < \theta < 1$ *with* $\theta s < t + n$. *Let* $r \in (1, \infty)$ *be defined by*

$$\frac{1}{r} := \frac{\theta}{p} + \frac{1-\theta}{q}. \tag{1.24}$$

Suppose that $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ *satisfy*

$$p(\alpha_1, \beta_1, \gamma_1) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_p \cup \mathcal{D}_p),$$
$$q(\alpha_0, \beta_0, \gamma_0) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_q \cup \mathcal{D}_q),$$
$$(\alpha_j, \beta_k, \gamma_\ell) \in \mathcal{A} \cup \mathcal{B}, \quad \text{for } j, k, \ell = 0, 1.$$

Fix $\alpha'_\theta, \beta'_\theta, \gamma'_\theta \in \mathbb{R}$ *such that*

$$(\alpha'_\theta - \alpha_\theta) + (\beta'_\theta - \beta_\theta) + (\gamma'_\theta - \gamma_\theta) = -(\theta s - t),$$
$$\alpha'_\theta - \alpha_\theta > -\frac{(n-1)}{n}(\theta s - t),$$
$$\gamma'_\theta - \gamma_\theta > -\frac{1}{n}(\theta s - t).$$

Then

$$\||x'|^{\alpha'_\theta}|x|^{\beta'_\theta}|x_n|^{\gamma'_\theta} D^t f\|_{L^r} \lesssim \||x'|^{\alpha_1}|x|^{\beta_1}|x_n|^{\gamma_1} D^s f\|_{L^p}^{\theta} \||x'|^{\alpha_0}|x|^{\beta_0}|x_n|^{\gamma_0} f\|_{L^q}^{1-\theta},$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$.

The rest of the article is organized as follows: Section 2 describes the relevant weight classes, sufficient criteria for membership in those classes, and basic facts on the anisotropic weight $u_{\theta_1,\theta_2,\theta_3}$. Section 3 lays out an abstract, unifying approach towards weighted inequalities; whereas Sections 4, 5, and 6 contain the proofs of Theorems 1.1, 1.2, and 1.4, respectively.

2. Preliminaries

2.1. **Weight classes.** For $1 < p < \infty$ a weight w in $\mathbb{R}^n$ (that is, $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $w \geq 0$ a.e. in $\mathbb{R}^n$) is said to belong to the *Muckenhoupt class* $A_p(\mathbb{R}^n)$ if

$$[w]_{A_p} := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{\frac{-1}{(p-1)}} \right)^{p-1} < \infty \tag{A_p}$$

where $Q \subset \mathbb{R}^n$ is a cube and $\fint_Q u$ denotes the average $\frac{1}{|Q|}\int_Q u$. For $p = 1$ and $p = \infty$,

$$[w]_{A_1} := \sup_Q \left(\fint_Q w \right) \left(\operatorname*{ess\,inf}_Q w \right)^{-1} < \infty \tag{A_1}$$

and

$$[w]_{A_\infty} := \sup_Q \left(\fint_Q w \right) \exp\left(-\fint_Q \ln w \right) < \infty. \tag{A_∞}$$

It follows that $A_\infty(\mathbb{R}^n) = \bigcup_{p\geq 1} A_p(\mathbb{R}^n)$ and, for $1 < p < \infty$,

$$w \in A_p(\mathbb{R}^n) \Leftrightarrow w, w^{-p'/p} \in A_\infty(\mathbb{R}^n), \tag{2.1}$$

(see for instance [3, Section 9.3 and Exercise 9.3.3]). As typical examples of A_p-weights, for $1 < p < \infty$, the weight $|x|^a \in A_p(\mathbb{R}^n)$ if and only if $-n < a < n(p-1)$, whereas $|x|^a \in A_1(\mathbb{R}^n)$ if and only if $-n < a \leq 0$ (see for instance [3, Section 9.1.2]).

For $1 < p, q < \infty$ and $\alpha \in \mathbb{R}$, two (measurable) nonnegative functions v and w are said to satisfy the $A^\alpha_{p,q}$ condition (see [2, Definition 2.3]), in symbols $(v, w) \in A^\alpha_{p,q}$, if

$$[(v,w)]_{A^\alpha_{p,q}} := \sup_Q |Q|^{\alpha/n-1} \left(\int_Q w \right)^{1/q} \left(\int_Q v^{-p'/p} \right)^{1/p'} < \infty.$$

From the definition of $\Theta_{p,q}(\alpha)$ in (1.9), the condition $(v, w) \in A^\alpha_{p,q}$ can be written as

$$[(v,w)]_{A^\alpha_{p,q}} := \sup_Q |Q|^{\Theta_{p,q}(\alpha)/n} \left(\fint_Q w \right)^{1/q} \left(\fint_Q v^{-p'/p} \right)^{1/p'} < \infty. \tag{2.2}$$

In the case $\Theta_{p,q}(\alpha) = 0$, let us define $(v, w) \in A_{p,q}$ (as in [2, Definition 1.5]) by

$$[(v,w)]_{A_{p,q}} := \sup_Q \left(\fint_Q w \right)^{1/q} \left(\fint_Q v^{-p'/p} \right)^{1/p'} < \infty. \tag{2.3}$$

Remark 2.1. If $\Theta_{p,q}(\alpha) < 0$, then the condition (2.2) along with the Lebesgue Differentiation Theorem yields $w \equiv 0$. Thus, it is only meaningful to consider $\Theta_{p,q}(\alpha) \geq 0$.

The following two results from [6] provide sufficient conditions on a pair (v, w) with $w, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ so that $(v, w) \in A^\alpha_{p,q}$ within the critical and subcritical cases for the indices. For the critical case $\Theta_{p,q}(\alpha) = 0$, that is, $1/q = 1/p - \alpha/n$, we have

Theorem 2.2 (Theorem 1.1 from [6])**.** *Fix $1 < p \leq q < \infty$ and $0 < \alpha < n$ such that $\Theta_{p,q}(\alpha) = 0$. Given (v, w) with $w, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ suppose that there exists $C_0 > 0$ with*

$$w(x)^{1/q} \leq C_0\, v(x)^{1/p}, \quad a.e.\ x \in \mathbb{R}^n. \tag{2.4}$$

Then (2.2) holds true with the estimate $[(v,w)]_{A^\alpha_{p,q}} \leq C_0 C_1$, where $C_1 > 0$ depends only on n, p, q, α, $[w]_{A_\infty(\mathbb{R}^n)}$, and $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$.

For the subcritical case $\Theta_{p,q}(\alpha) > 0$, that is, $1/q > 1/p - \alpha/n$, we have

Theorem 2.3 (Theorem 1.2 from [6]). *Fix $1 < p \le q < \infty$ and $0 < \alpha < n$ such that $\Theta_{p,q}(\alpha) > 0$. Given (v, w) with $w, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ the condition $w^{1/q}/v^{1/p} \in L^{n/\Theta_{p,q}(\alpha),\infty}(\mathbb{R}^n)$ implies (2.2) with the estimate*

$$[(v,w)]_{A^\alpha_{p,q}} \le C_2 \|w^{1/q}/v^{1/p}\|_{L^{n/\Theta_{p,q}(\alpha),\infty}(\mathbb{R}^n)}, \tag{2.5}$$

where $C_2 > 0$ depends only on n, α, p, q, $[w]_{A_\infty(\mathbb{R}^n)}$, and $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$.

2.2. **On the anisotropic weights $u_{\theta_1,\theta_2,\theta_3}$.** Given $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$, let $u_{\theta_1,\theta_2,\theta_3}$ be the anisotropic weight defined in (1.3).

Lemma 2.4. *Given $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ and $0 < p < \infty$, then $u_{\theta_1,\theta_2,\theta_3} \in L^{p,\infty}(\mathbb{R}^n)$ if and only if $\theta_1 + \theta_2 + \theta_3 = -n/p$, $\theta_1 > -(n-1)/p$, and $\theta_3 > -1/p$.*

Proof. Since $u_{\theta_1,\theta_2,\theta_3} \in L^{p,\infty}(\mathbb{R}^n)$ if and only if $u^p_{\theta_1,\theta_2,\theta_3} \in L^{1,\infty}(\mathbb{R}^n)$, it suffices to consider the case $p = 1$. We want to characterize when the distribution function of $u_{\theta_1,\theta_2,\theta_3}$ satisfies:

$$\Big|\Big\{x \in \mathbb{R}^n : u_{\theta_1,\theta_2,\theta_3}(x) > t\Big\}\Big| \lesssim t^{-1}. \tag{2.6}$$

For this purpose, we split this level set in two parts: when $|x'| \le |x_n|$, in which case $|x| \approx |x_n|$ and hence $u_{\theta_1,\theta_2,\theta_3}(x) \approx |x'|^{\theta_1}|x_n|^{\theta_2+\theta_3}$, or $|x_n| \le |x'|$, getting now that $|x| \approx |x'|$ and $u_{\theta_1,\theta_2,\theta_3}(x) \approx |x'|^{\theta_1+\theta_2}|x_n|^{\theta_3}$. Thus, to simplify the notations, we consider the function $u(x) = |x'|^\alpha |x_n|^\beta$, with $\alpha, \beta \in \mathbb{R}$, and our goal now is to estimate the following integrals:

$$I = \int_0^\infty \left(\int_{\left\{|x'|\le r : |x'|^\alpha r^\beta > t\right\}} dx'\right) dr \quad \text{and} \quad II = \int_{\mathbb{R}^{n-1}} \left(\int_{\left\{r \le |x'| : |x'|^\alpha r^\beta > t\right\}} dr\right) dx'. \tag{2.7}$$

Set $s := \alpha + \beta$. We start with I and distinguish several cases:

If $\alpha > 0$ and $s \ge 0$, it is easy to see that $I = \infty$. Assume now that $\alpha > 0$ and $s < 0$. Then

$$I \approx \int_0^\infty \left(\int_{\left\{|x'|\le r : |x'| > \left(\frac{t}{r^\beta}\right)^{1/\alpha}\right\}} dx'\right) dr \approx \int_0^{t^{1/s}} \left(r^{n-1} - \left(\frac{t}{r^\beta}\right)^{(n-1)/\alpha}\right) dr. \tag{2.8}$$

The first term gives $t^{n/s}$ and (2.6) implies the condition $s = -n$. For the second term, we need that

$$-\frac{\beta}{\alpha}(n-1) + 1 > 0 \iff \beta < \frac{s}{n} = -1,$$

and hence

$$\int_0^{t^{1/s}} \left(\frac{t}{r^\beta}\right)^{(n-1)/\alpha} dr \approx t^{n/s} = t^{-1}.$$

Therefore the integral in (2.8) is then comparable to t^{-1} if and only if $s = -n$ and $\beta < -1$.

If $\alpha = 0$, then $s = \beta$. As in the previous case, it is easy to see that $I = \infty$ when $s \ge 0$. Finally, if $s < 0$,

$$I \approx \int_0^\infty \left(\int_{\left\{|x'|\le r : r < t^{1/s}\right\}} dx'\right) \approx \int_0^{t^{1/s}} r^{n-1} \approx t^{n/s} \approx t^{-1} \iff s = -n.$$

If $\alpha < 0$, again $I = \infty$ when $s \geq 0$. If $s < 0$,

$$I \approx \int_0^{t^{1/s}} r^{n-1}\, dr + \int_{t^{1/s}}^{\infty} \left(\frac{t}{r^\beta}\right)^{(n-1)/\alpha} dr$$

The first term is comparable to t^{-1} if and only if $s = -n$ and the second term in integrable if and only if $-\frac{\beta}{\alpha}(n-1)+1 < 0 \iff \beta < -1$, and the integral is comparable to $t^{n/s} = t^{-1}$. Therefore, putting all these different cases together, we conclude that

$$\begin{gathered} I \approx \left|\left\{x \in \mathbb{R}^n : |x'| \leq |x_n| \text{ and } |x'|^\alpha |x_n|^\beta > t\right\}\right| \lesssim t^{-1} \\ \Updownarrow \\ \alpha + \beta = -n \text{ and } \beta < -1. \end{gathered} \tag{2.9}$$

We now proceed to estimate the integral II in (2.7). Assume first that $\beta > 0$. Again, when $s = 0$ it turns out that $II = \infty$. When $s < 0$,

$$II \approx \int_{\{|x'|<t^{1/s}\}} \left(|x'| - \left(\frac{t}{|x'|^\alpha}\right)^{1/\beta}\right) dx'.$$

The first term gives $t^{n/s} \approx t^{-1} \iff s = -n$, and the second integral is finite if and only if $-\frac{\alpha}{\beta}+n-1 > 0 \iff \beta > -1$, in which case, the integral is equal to $t^{1/\beta+(-\alpha/\beta+n-1)/s} = t^{-1}$. It is easy to see that $II = \infty$ whenever $\beta = 0$ and $\alpha \geq 0$. If $\beta = 0$ and $\alpha < 0$,

$$II \approx \int_{\{|x'|<t^{1/\alpha}\}} |x'|\, dx' \approx t^{n/\alpha} = t^{-1} \iff s = \alpha = -n.$$

Finally, if $\beta < 0$ and $s \geq 0$, the integral II is equal to infinity, and if $\beta < 0$ and $s < 0$,

$$II \approx \int_{\{|x'|<t^{1/s}\}} |x'|\, dx' + \int_{\{|x'|\geq t^{1/s}\}} \left(\frac{t}{|x'|^\alpha}\right)^{1/\beta} dx'.$$

The first term is comparable to $t^{n/s}$, which again gives that $s = -n$, while the second integral is finite if and only if $-\alpha/\beta + n - 1 < 0 \iff \beta > -1$, and hence the integral is comparable to $t^{1/\beta+(-\alpha/\beta+n-1)/s} = t^{-1}$. Therefore, by putting all these different cases together, we conclude that

$$\begin{gathered} II \approx \left|\left\{x \in \mathbb{R}^n : |x_| \leq |x'| \text{ and } |x'|^\alpha |x_n|^\beta > t\right\}\right| \lesssim t^{-1} \\ \Updownarrow \\ \alpha + \beta = -n \text{ and } \beta > -1. \end{gathered} \tag{2.10}$$

By applying (2.9) and (2.10) to the function $u_{\theta_1,\theta_2,\theta_3}$, for the integral I we have that $\alpha = \theta_1$ and $\beta = \theta_2 + \theta_3$ and similarly, for the second integral II, $\alpha = \theta_1 + \theta_2$ and $\beta = \theta_3$. Hence,

$$\begin{aligned} u_{\theta_1,\theta_2,\theta_3} \in L^{1,\infty}(\mathbb{R}^n) &\iff \theta_1 + \theta_2 + \theta_3 = -n,\ \theta_2 + \theta_3 < -1, \text{ and } \theta_3 > -1 \\ &\iff \theta_1 + \theta_2 + \theta_3 = -n,\ \theta_1 > -(n-1), \text{ and } \theta_3 > -1. \end{aligned}$$

□

Lemma 2.5. *$u_{\theta_1,\theta_2,\theta_3} \in L^1_{\text{loc}}(\mathbb{R}^n)$ if and only if $\theta_1 + \theta_2 + \theta_3 > -n$, $\theta_1 > -(n-1)$, and $\theta_3 > -1$. In other words, $u_{\theta_1,\theta_2,\theta_3} \in L^1_{\text{loc}}(\mathbb{R}^n)$ if and only if $(\theta_1, \theta_2, \theta_3) \in \mathcal{A} \cup \mathcal{B}$.*

Proof. Similarly to the argument used in Lemma 2.4, we fix $R > 0$ and split the integral of $u_{\theta_1,\theta_2,\theta_3}$ over the ball $B(0, R)$ in two cases: when $|x'| \leq |x_n|$, and hence $|x| \approx |x_n|$ and $u_{\theta_1,\theta_2,\theta_3}(x) \approx |x'|^{\theta_1}|x_n|^{\theta_2+\theta_3}$, or $|x_n| \leq |x'|$, getting now that $|x| \approx |x'|$ and $u_{\theta_1,\theta_2,\theta_3}(x) \approx |x'|^{\theta_1+\theta_2}|x_n|^{\theta_3}$. Thus, we need to characterize the exponents $\theta_1, \theta_2, \theta_3$ for which these two terms are finite:

$$I = \int_{\{x\in\mathbb{R}^n:|x|\leq R,\ |x'|\leq|x_n|\}} |x'|^{\theta_1}|x_n|^{\theta_2+\theta_3}\,dx \text{ and } II = \int_{\{x\in\mathbb{R}^n:|x|\leq R,\ |x_n|\leq|x'|\}} |x'|^{\theta_1+\theta_2}|x_n|^{\theta_3}\,dx.$$

Now

$$I \approx \int_0^R \left(\int_{\{|x'|<t\}} |x'|^{\theta_1}\,dx'\right) t^{\theta_2+\theta_3}\,dt \approx \int_0^R \left(\int_0^t \rho^{\theta_1+n-2}\,d\rho\right) t^{\theta_2+\theta_3}\,dt,$$

and the inner integral is finite if and only if $\theta_1 > -(n-1)$ and we get

$$I \approx \int_0^R t^{\theta_1+n-1+\theta_2+\theta_3}\,dt < \infty \iff \theta_1 > -(n-1) \text{ and } \theta_1+\theta_2+\theta_3 > -n. \tag{2.11}$$

Similarly,

$$II \approx \int_{\{|x'|<R\}} \left(\int_0^{|x'|} t^{\theta_3}\,dt\right) |x'|^{\theta_1+\theta_2}\,dx'$$

and the inner integral is finite if and only if $\theta_3 > -1$ and we get

$$II \approx \int_{\{|x'|<R\}} |x'|^{\theta_1+\theta_2+\theta_3+1}\,dx' < \infty \iff \theta_3 > -(n-1) \text{ and } \theta_1+\theta_2+\theta_3 > -n,$$

which, together with (2.11), proves de result. □

The next result comes from [7] and identifies the triples of exponents $(\theta_1, \theta_2, \theta_3)$ so that $u_{\theta_1,\theta_2,\theta_3} \in A_p(\mathbb{R}^n)$. Namely,

Theorem 2.6 (Theorem 3.2 from [7])**.** *Fix $1 < p < \infty$. Then, $u_{\theta_1,\theta_2,\theta_3} \in A_p(\mathbb{R}^n)$ if and only if $(\theta_1, \theta_2, \theta_3) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_p \cup \mathcal{D}_p)$.*

3. A unifying approach towards weighted inequalities

Our strategy for the proofs of Theorems 1.1 and 1.4 will be based on Theorems 3.2 and 3.3 below, which provide an abstract framework for weighted interpolation inequalities. In particular, Theorem 3.2 represents an improvement on [2, Corollary 2.10], whereas Theorem 3.3 is precisely Theorem 2.14 from [2] whose statement has been included for the reader's convenience. The aforementioned improvement in Theorem 3.2 relies on the following lemma.

Lemma 3.1. *Fix $S \subset \mathbb{S}^{n-1}$ with $\mathcal{H}^{n-1}(S) > 0$, $R > 0$, and define*

$$\Gamma_R^S := \{x \in \mathbb{R}^n : |x| \geq R \text{ and } x/|x| \in S\}.$$

Then, $w(\Gamma_R^S) = \infty$ for every $w \in A_\infty(\mathbb{R}^n)$.

Proof. It is well known that given $u \in A_\infty(\mathbb{R}^n)$ there exist constants $0 < C_1 \le C_2$ and $0 < \theta_2 \le \theta_1$ (depending only on $[u]_{A_\infty(\mathbb{R}^n)}$ and n) such that

$$C_1 \left(\frac{|E|}{|B|}\right)^{\theta_1} \le \frac{u(E)}{u(B)} \le C_2 \left(\frac{|E|}{|B|}\right)^{\theta_2} \tag{3.1}$$

for every ball B and every measurable subset $E \subset B$ (see for instance the proof of Lemma 9.2.1 as well as Theorem 9.3.3(d) from [3]). Fix $\alpha > -n$ and set $w_\alpha(x) := |x|^\alpha$ so that $w_\alpha \in A_\infty(\mathbb{R}^n)$ (see [3, Example 9.1.7]). The second inequality in (3.1) applied to w_α gives

$$\frac{w_\alpha(E)}{w_\alpha(B)} \le C_2 \left(\frac{|E|}{|B|}\right)^{\theta_2} \tag{3.2}$$

for some C_2, θ_2 depending only on α and n. Now, given $w \in A_\infty$, the first inequality in (3.1) applied to w gives

$$C_1 \left(\frac{|E|}{|B|}\right)^{\theta_1} \le \frac{w(E)}{w(B)} \tag{3.3}$$

where C_1, θ_1 depend only on $[w]_{A_\infty(\mathbb{R}^n)}$ and n. By combining (3.2) and (3.3) we obtain constants $C > 0$ and $\theta > 0$ such that

$$\frac{w(E)}{w(B)} \ge C \left(\frac{w_\alpha(E)}{w_\alpha(B)}\right)^{\theta} \tag{3.4}$$

for every ball B and every measurable subset $E \subset B$. On the other hand, the second inequality in (3.1) applied to w and to concentric balls $B(x, r) \subset B(x, s)$ for $x \in \mathbb{R}^n$ and $0 < r < s$ gives

$$\frac{w(B(x,s))}{w(B(x,r))} \ge c_1 \left(\frac{s}{r}\right)^{\varepsilon_1} \tag{3.5}$$

for $c_1, \varepsilon_1 > 0$ depending only on $[w]_{A_\infty(\mathbb{R}^n)}$ and n. Now, given $S \subset \mathbb{S}^{n-1}$ with $\mathcal{H}^{n-1}(S) > 0$, $R > 0$, and $R_1 > R$ set

$$E_{R,R_1} := \Gamma_R^S \cap B(0, R_1),$$

so that (3.4) applied to E_{R,R_1} and $B(0, R_1)$ implies

$$\frac{w(E_{R,R_1})}{w(B(0,R_1))} \ge C \left(\frac{w_\alpha(E_{R,R_1})}{w_\alpha(B(0,R_1))}\right)^{\theta}. \tag{3.6}$$

By changing to polar coordinates we can write

$$w_\alpha(E_{R,R_1}) = \int_{E_{R,R_1}} |x|^\alpha\, dx = \mathcal{H}^{n-1}(S) \int_R^{R_1} \rho^{\alpha+n-1}\, d\rho = \frac{\mathcal{H}^{n-1}(S)}{\alpha+n}(R_1^{\alpha+n} - R^{\alpha+n})$$

and

$$w_\alpha(B(0,R_1)) = \int_{B(0,R_1)} |x|^\alpha\, dx = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \int_0^{R_1} \rho^{\alpha+n-1}\, d\rho = \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{\alpha+n} R_1^{\alpha+n}.$$

Thus,

$$\left(\frac{w_\alpha(E_{R,R_1})}{w_\alpha(B(0,R_1))}\right)^{\theta} = \left(\frac{\mathcal{H}^{n-1}(S)}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}\right)^{\theta} \left(1 - \left(\frac{R}{R_1}\right)^{\alpha+n}\right)^{\theta}. \tag{3.7}$$

Finally, from the inequality (3.5) used with $x = 0$ and $0 < R < R_1$ we obtain

$$\frac{w(B(0,R_1))}{w(B(0,R))} \geq c_1 \left(\frac{R_1}{R}\right)^{\varepsilon_1},$$

which combined with (3.6) and (3.7) yields

$$\begin{aligned} w(E_{R,R_1}) &\geq C w(B(0,R_1)) \left(\frac{w_\alpha(E_{R,R_1})}{w_\alpha(B(0,R_1))}\right)^\theta \\ &\geq c_1 C \left(\frac{R_1}{R}\right)^{\varepsilon_1} w(B(0,R)) \left(\frac{w_\alpha(E_{R,R_1})}{w_\alpha(B(0,R_1))}\right)^\theta \\ &= c_1 C \left(\frac{R_1}{R}\right)^{\varepsilon_1} w(B(0,R)) \left(\frac{\mathcal{H}^{n-1}(S)}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}\right)^\theta \left(1 - \left(\frac{R}{R_1}\right)^{\alpha+n}\right)^\theta, \end{aligned}$$

and $w(\Gamma_R^S) = \infty$ follows by letting $R_1 \to \infty$. □

Theorem 3.2 (Case $\theta = 1$, weighted Sobolev embedding)**.** *Let $1 < p \leq q < \infty$, $t < s < t + n$. Suppose that $(v, w) \in A_{p,q}^{s-t}$, and that $w, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$. Then,*

$$\|D^t f\|_{L^q(w)} \lesssim \|D^s f\|_{L^p(v)}, \quad \forall f \in D^{>\tau}\mathcal{S}(\mathbb{R}^n),$$

with $\tau := \max\{-n, -t\}$ and $D^{>\tau} := \{f \in L^\infty : f = D^b g$ for some $g \in \mathcal{S}(\mathbb{R}^n)$ and $b > \tau\}$.

Proof. The conclusion of Theorem 3.2 has been proved in [2, Corollary 2.10] under the additional assumption that $w(\Gamma_R^S) = \infty$ for every $S \subset \mathbb{S}^{n-1}$ with $\mathcal{H}^{n-1}(S) > 0$ and $R > 0$. Now, in view of Lemma 3.1, such assumption is automatically fulfilled by the fact that $w \in A_\infty(\mathbb{R}^n)$. □

Theorem 3.3 (Theorem 2.14 from [2], case $\theta = t/s$)**.** *Let $1 < p, q, r < \infty$ and $0 < t < s$ be related by*

$$\frac{1}{r} = \frac{t}{sp} + \left(1 - \frac{t}{s}\right)\frac{1}{q}. \tag{3.8}$$

Let $\Sigma := \{z \in \mathbb{C} : 0 < \mathcal{R}(z) < 1\}$ and suppose there is a family of weights $\{w_z\}_{z\in\overline{\Sigma}}$ satisfying the following conditions:

(i) there is $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $|w_z| \leq h(x)$ for every $x \in \mathbb{R}^n$ and $z \in \overline{\Sigma}$,
(ii) for a.e. $x \in \mathbb{R}^n$ the function $z \mapsto w_z(x)$ is continuous in $\overline{\Sigma}$ and analytic in Σ,
(iii) and there are weights w_0, w_1 such that $|w_{i\tau}(x)| \leq w_0(x)$ and $|w_{1+i\tau}(x)| \leq w_1(x)$ for every $x \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$.

Moreover, assume that there are weights u, v, w such that $w = w_{t/s}$, $(u^p, w_1^p) \in A_{p,p}$, $(v^q, w_0^q) \in A_{q,q}$, and $u^{-p'}, w_1^p, v^{-q'}, w_0^q \in A_\infty(\mathbb{R}^n)$. Then,

$$\|wD^t f\|_{L^r} \lesssim \|uD^s f\|_{L^p}^{t/s} \|vf\|_{L^q}^{1-t/s}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

4. Proof of Theorem 1.1

Fix $1 < p, q, r < \infty$ and $0 < t < s$ related by (3.8). For $z \in \mathbb{C}$ with $0 \leq \mathcal{R}(z) \leq 1$ define

$$w_z(x) := |x'|^{z\alpha_1 + (1-z)\alpha_0} |x|^{z\beta_1 + (1-z)\beta_0} |x_n|^{z\gamma_1 + (1-z)\gamma_0}$$

so that

$$|w_z(x)| \leq |x'|^{\mathcal{R}(z)\alpha_1+(1-\mathcal{R}(z))\alpha_0}|x|^{\mathcal{R}(z)\beta_1+(1-\mathcal{R}(z))\beta_0}|x_n|^{\mathcal{R}(z)\gamma_1+(1-\mathcal{R}(z))\gamma_0}.$$

In particular, if $z = i\tau$ with $\tau \in \mathbb{R}$, then

$$|w_{i\tau}(x)| \leq w_0(x) = |x'|^{\alpha_0}|x|^{\beta_0}|x_n|^{\gamma_0}$$

and if $z = 1 + i\tau$ with $\tau \in \mathbb{R}$

$$|w_{1+i\tau}(x)| \leq w_1(x) = |x'|^{\alpha_1}|x|^{\beta_1}|x_n|^{\gamma_1}.$$

Let us check that the hypotheses from Theorem 3.3 hold true with the choice $u := w_1$ and $v := w_0$. From the definition of the classes $A_{p,q}$ in (2.3), notice that $(u^p, w_1^p) = (w_1^p, w_1^p) \in A_{p,p}$ means $w_1^p \in A_p(\mathbb{R}^n)$, which by (2.1) is equivalent to $w_1^p, w_1^{-p'} \in A_\infty(\mathbb{R}^n)$. Now, since $w_1^p(x) := |x'|^{\alpha_1 p}|x|^{\beta_1 p}|x_n|^{\gamma_1 p}$, by virtue of Theorem 2.6 $w_1^p \in A_p(\mathbb{R}^n)$ amounts to $p(\alpha_1, \beta_1, \gamma_1) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_p \cup \mathcal{D}_p)$, which is hypothesis (1.5). Similarly, the hypothesis (1.6), that is, $q(\alpha_0, \beta_0, \gamma_0) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_q \cup \mathcal{D}_q)$ guarantees $w_0^q \in A_q(\mathbb{R}^n)$ and therefore $(v^q, w_0^q) = (w_0^q, w_0^q) \in A_{q,q}$ as well as $w_0^{-q'}, w_0^q \in A_\infty(\mathbb{R}^n)$. Finally, let us find $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $|w_z| \leq h$ for every $z \in \overline{\Sigma}$. By Young's inequality,

$$\begin{aligned} |x'|^{\mathcal{R}(z)\alpha_1+(1-\mathcal{R}(z))\alpha_0} &\leq |x'|^{\alpha_1} + |x'|^{\alpha_0} =: h_\alpha(x), \\ |x|^{\mathcal{R}(z)\beta_1+(1-\mathcal{R}(z))\beta_0} &\leq |x|^{\beta_1} + |x|^{\beta_0} =: h_\beta(x), \\ |x_n|^{\mathcal{R}(z)\gamma_1+(1-\mathcal{R}(z))\gamma_0} &\leq |x_n|^{\gamma_1} + |x_n|^{\gamma_0} =: h_\gamma(x). \end{aligned}$$

Thus, $|w_z| \leq h_\alpha h_\beta h_\gamma =: h$ with $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ due to Lemma 2.5 and the hypothesis (1.7). All the hypotheses of Theorem 3.3 are then met and we obtain

$$\|w_{t/s} D^t f\|_{L^r} \lesssim \|w_1 D^s f\|_{L^p}^{t/s} \|w_0 f\|_{L^q}^{1-t/s}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n),$$

which is precisely (1.8). □

5. Proof of Theorem 1.2

We will use Theorems 2.2, 2.3, and 3.2 with $w := u_{\theta_1,\theta_2,\theta_3}$ and $v := u_{\sigma_1,\sigma_2,\sigma_3}$. Indeed, by Theorem 2.6, the hypothesis (1.10) means that $u_{\theta_1,\theta_2,\theta_3}, u_{\sigma_1,\sigma_2,\sigma_3}^{-p'/p} \in A_\infty(\mathbb{R}^n)$, that is, $w, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$. Next, we check that $(v, w) \in A^\alpha_{p,q}$ where $\alpha := s - t \in (0, n)$. Indeed, if $\Theta_{p,q}(s-t) = 0$, then (1.11) yields $w^{1/q} = v^{1/p}$ which together with Theorem 2.2 (with $C_0 = 1$) implies $[(v, w)]_{A^{s-t}_{p,q}} \leq C_1$, where $C_1 > 0$ depends only on n, p, q, s, t, $\theta_1, \theta_2, \theta_3, \sigma_1, \sigma_2, \sigma_3$. If $\Theta_{p,q}(s-t) > 0$, let us see that (1.12) yields $w^{1/q}/v^{1/p} \in L^{n/\Theta_{p,q}(s-t),\infty}(\mathbb{R}^n)$ and then (2.5) will follow from Theorem 2.3. Notice that $w^{1/q}/v^{1/p} \in L^{n/\Theta_{p,q}(s-t),\infty}(\mathbb{R}^n)$ is equivalent to $\left(w^{1/q}/v^{1/p}\right)^{n/\Theta_{p,q}(s-t)} \in L^{1,\infty}(\mathbb{R}^n)$ and by Lemma 2.4 we have that

$$\left(w^{1/q}/v^{1/p}\right)^{n/\Theta_{p,q}(s-t)}(x) = (|x'|^{\theta_1/q-\sigma_1/p}|x|^{\theta_2/q-\sigma_2/p}|x_n|^{\theta_3/q-\sigma_3/p})^{n/\Theta_{p,q}(s-t)} \in L^{1,\infty}(\mathbb{R}^n),$$

if and only if

$$\frac{n}{\Theta_{p,q}(s-t)}\left(\frac{\theta_1+\theta_2+\theta_3}{q}-\frac{\sigma_1+\sigma_2+\sigma_3}{p}\right)=-n,$$
$$\frac{n}{\Theta_{p,q}(s-t)}\left(\frac{\theta_1}{q}-\frac{\sigma_1}{p}\right)>-(n-1),$$
$$\frac{n}{\Theta_{p,q}(s-t)}\left(\frac{\theta_3}{q}-\frac{\sigma_3}{p}\right)>-1,$$

which are precisely the conditions (1.12), (1.13), and (1.14). □

6. Proof of Theorem 1.4

The proof is a combination of Theorems 1.1 and 1.2. By Theorem 1.1 applied to $1<p,q,a<\infty$ and $0<\theta s<s$, so that the hypothesis (1.17) becomes (1.4) (think "$r=s$" and "$t=\theta s$" in the notation from Theorem 1.1) and since $\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_0,\gamma_1\in\mathbb{R}$ satisfy (1.5), (1.6), and (1.7) we get

$$\|u_{\alpha_\theta,\beta_\theta,\gamma_\theta}D^{\theta s}f\|_{L^a}\lesssim\|u_{\alpha_1,\beta_1,\gamma_1}D^s f\|_{L^p}^{\theta}\|u_{\alpha_0,\beta_0,\gamma_0}f\|_{L^q}^{1-\theta},\quad \forall f\in\mathcal{S}(\mathbb{R}^n),\tag{6.1}$$

where $(\alpha_\theta,\beta_\theta,\gamma_\theta):=(\alpha_1,\beta_1,\gamma_1)\theta+(\alpha_0,\beta_0,\gamma_0)(1-\theta)$.

Next, the idea is to apply Theorem 1.2 with $1<a\le r<\infty$ and $t<\theta s<t+n$ (think "$p=a$", "$q=r$", and "$s=\theta s$" in the notation from Theorem 1.2) to obtain the inequality

$$\|u_{\theta_1/r,\theta_2/r,\theta_3/r}D^t f\|_{L^r}\lesssim\|u_{\sigma_1/a,\sigma_2/a,\sigma_3/a}D^{\theta s}f\|_{L^a},\tag{6.2}$$

(recall Remark 1.3) which then determines the σ_j's as

$$(\sigma_1,\sigma_2,\sigma_3):=a(\alpha_\theta,\beta_\theta,\gamma_\theta),\tag{6.3}$$

so that $\|u_{\alpha_\theta,\beta_\theta,\gamma_\theta}D^{\theta s}f\|_{L^a}=\|u_{\sigma_1/a,\sigma_2/a,\sigma_3/a}D^{\theta s}f\|_{L^a}$ and (6.2) can be linked with (6.1).

If $\Theta_{a,r}(\theta s-t)=0$, we use Theorem 1.2(i) since the hypothesis (1.18) gives

$$-\frac{a'}{a}(\sigma_1,\sigma_2,\sigma_3)=-a'(\alpha_\theta,\beta_\theta,\gamma_\theta)\in\mathcal{A}\cup\mathcal{B}$$

as well as

$$(\theta_1,\theta_2,\theta_3):=\frac{r}{a}(\sigma_1,\sigma_2,\sigma_3)=r(\alpha_\theta,\beta_\theta,\gamma_\theta)\in\mathcal{A}\cup\mathcal{B},$$

and (6.2) follows from Theorem 1.2(i). Thus, by combining (6.1) and (6.2), the inequality (1.19) is proved since $(\alpha_\theta,\beta_\theta,\gamma_\theta)=\frac{1}{r}(\theta_1,\theta_2,\theta_3)$ from the above definition of $(\theta_1,\theta_2,\theta_3)$.

If $\Theta_{a,r}(\theta s-t)>0$, given $\alpha'_\theta,\beta'_\theta,\gamma'_\theta$ as in (1.20), (1.21) and (1.22) now define

$$(\theta_1,\theta_2,\theta_3):=r(\alpha'_\theta,\beta'_\theta,\gamma'_\theta)$$

and use Theorem 1.2(ii) with (1.20), (1.21) and (1.22) playing the role of (1.12), (1.13), and (1.14) to obtain (6.2), always with the σ_j's as in (6.3). Hence, (1.23) follows from (6.1) and (6.2). □

References

[1] L. A. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. 53 (1984), no. 3, 259–275.

[2] R. Duarte and J. Drumond Silva, *Weighted Gagliardo-Nirenberg interpolation inequalities,* J. Funct. Anal. 285 (2023), 11009.

[3] L. Grafakos, *Modern Fourier analysis*, 2nd edition. Graduate Texts in Mathematics 250, Springer, New York, 2008.

[4] Y. Y. Li and X. Yan, *Anisotropic Caffarelli-Kohn-Nirenberg type inequalities*, Adv. Math. 419 (2023), paper No. 108958.

[5] C.-S. Lin, *Interpolation inequalities with weights*, Commun. Partial Differ. Equ. 11 (1986) 1515–1538.

[6] D. Maldonado and J. Soria, *On weighted estimates for fractional operators and applications to Hardy-type inequalities*, submitted, preprint available at https://arxiv.org/abs/2606.19126.

[7] C. X. Miao and Z. W. Zhao, *On a class of anisotropic Muckenhoupt weights and their applications to p-Laplace equations*, preprint available at https://arxiv.org/abs/2310.01359.

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