

Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas



TESIS DOCTORAL

Optimal Constants for Classical Operators
on Lebesgue Spaces

Constantes Óptimas para Operadores
Clásicos en Espacios de Lebesgue

Memoria para optar al grado de doctor presentada por

Achraf Ben Said Abid

Directores:

Santiago Boza Rocho

F. Javier Soria de Diego

Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas

Programa de Doctorado en Investigación Matemática



TESIS DOCTORAL

Optimal Constants for Classical Operators
on Lebesgue Spaces

Constantes Óptimas para Operadores
Clásicos en Espacios de Lebesgue

Memoria para optar al grado de doctor presentada por

Achraf Ben Said Abid

Directores:

Santiago Boza Rocho

F. Javier Soria de Diego



U N I V E R S I D A D
COMPLUTENSE
M A D R I D

**DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD DE LA TESIS
PRESENTADA PARA OBTENER EL TÍTULO DE DOCTOR**

D./Dña. Achraf Ben Said Abid,
estudiante en el Programa de Doctorado en Investigación Matemática,
de la Facultad de Ciencias Matemáticas de la Universidad Complutense de
Madrid, como autor/a de la tesis presentada para la obtención del título de Doctor y
titulada:

Optimal Constants for Classical Operators on Lebesgue Spaces
Constantes Óptimas para Operadores Clásicos en Espacios de Lebesgue

y dirigida por: Santiago Boza Rocho y F. Javier Soria de Diego

DECLARO QUE:

La tesis es una obra original que no infringe los derechos de propiedad intelectual ni los derechos de propiedad industrial u otros, de acuerdo con el ordenamiento jurídico vigente, en particular, la Ley de Propiedad Intelectual (R.D. legislativo 1/1996, de 12 de abril, por el que se aprueba el texto refundido de la Ley de Propiedad Intelectual, modificado por la Ley 2/2019, de 1 de marzo, regularizando, aclarando y armonizando las disposiciones legales vigentes sobre la materia), en particular, las disposiciones referidas al derecho de cita.

Del mismo modo, asumo frente a la Universidad cualquier responsabilidad que pudiera derivarse de la autoría o falta de originalidad del contenido de la tesis presentada de conformidad con el ordenamiento jurídico vigente.

En Madrid, a 13 de abril de 2026

Fdo.: _____

Esta DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD debe ser insertada en
la primera página de la tesis presentada para la obtención del título de Doctor.

أولاً الحمد لله، ثم الحمد لله، ثم الحمد لله،

ثم أشكر والديّ،
حمو وعبي ولالة رقية أحجاوي،

ثم أشكر والديّ،
ربحة عبيد،

ثم أشكر والديّ،
عزيز ماينا و ناتاليا أونزاين،

ثم أشكر والديّ،
المحجوب القاري وفاطمة التونسي.

Acknowledgments

Quizás resulte irónico comenzar agradeciendo a Dios en una monografía de tesis doctoral en una ciencia donde, literalmente, cada paso requiere justificación. Sin embargo, siento que debo hacerlo, pues la probabilidad de que yo llegara a aspirar al grado de Doctor en Matemáticas, dada la vida que he vivido, era prácticamente nula. Así que, o bien se trata de Dios, o bien del azar. Me hace feliz creer en lo primero y, por encima de todo, eso es lo que realmente importa.

Agradezco, por tanto, a Dios por haber puesto en mi camino a tantas personas de corazón noble que, de una u otra forma, han contribuido no solo a mi vida profesional, sino también a mi vida personal, ayudándome a ser quien soy hoy y quien aspiro a ser. Muchas de vuestras buenas acciones hacia mí me han motivado a no conformarme nunca y a luchar contra las dificultades, con el deseo de hacerlos sentir orgullosos. De nuevo, gracias de corazón.

Volviendo ahora a mi vida profesional y, sobre todo, a mi formación como matemático, hay cuatro personas que han sido esenciales, a quienes he bautizado como “los tres Javieres de mi vida” y don Santiago Boza. El primero de ellos es Javier Rojo Lozano, del instituto IES Pablo Picasso de Málaga, quien me presentó los números como amigos, amigos con los que contaré toda mi vida. El segundo Javier es Francisco Javier Martín Reyes, también llamado Kiko, a quien conocí durante mis estudios en la Universidad de Málaga. Fue el primer profesor que me mostró el Análisis Matemático y, en particular, la teoría de la integración, como un mundo ideal, bello y acogedor. Dicha teoría es la base de esta tesis. El tercer Javier es don Francisco Javier Soria de Diego, a quien conocí, casi por “casualidad”, en la UCM, siendo ambos nuevos en la institución, y quien me brindó la oportunidad más hermosa: aspirar a ser doctor. En el segundo año de mi doctorado se unió don Santiago Boza Rocho, a quien también agradezco profundamente su dedicación y su empeño en mi mejora. A estos dos últimos quiero decirles que considero que juntos hemos hecho muy buena matemática, y espero que así continúe, pues trabajar con vosotros ha sido un verdadero placer. Muchas gracias de corazón.

Agradezco también al Departamento de Análisis Matemático y Matemática Aplicada por acogerme de la manera en que lo hizo y por hacerme sentir uno más de la familia. En especial, a María Jesús Carro Rossell, por su cercanía y humanidad, y a Teresa Luque, por encontrar siempre soluciones a mi agenda docente. Muchas gracias de corazón.

Ahora toca agradecer a mis amigos. Siempre me he considerado introvertido,

pero al intentar poner nombres a mis amistades me he dado cuenta de que quizá no lo soy tanto, pues la lista sería muy larga. Dada mi mala memoria, prefiero no mencionar nombres. Pero si estás leyendo esto, es porque probablemente te considero amigo. Muchas gracias de corazón a ti también.

Esta tesis, y los trabajos contenidos en ella, han recibido financiación de entidades públicas. Por lo tanto, me siento agradecido de que se haya confiado en mí para recibir dinero público y así llevar a cabo mi trabajo. En concreto, mi doctorado ha sido financiado por el Ministerio de Ciencia e Innovación a través de un contrato predoctoral para la formación de doctores (MCIN/AEI/10.13039/501100011033) por medio de la Universidad Complutense de Madrid, y la mayor parte de las actividades y viajes que he hecho han sido financiados por los proyectos PID2020-113048GB-I00 y PID2024-155917NB-I00 del mismo Ministerio.

Finally, I would like to thank Prof. Gord Sinnamon for hosting my PhD research visit at Western University. Working with you has been truly inspiring for me.

Contents

Acknowledgments	i
Abstract	v
Resumen	vii
1 Introduction	1
2 General notations and preliminaries	11
3 Sharp bounds for the Cesàro operator	21
3.1 Inequalities in $\ell^p(\mathbb{N})$ for $1 < p < \infty$	24
3.2 Inequalities in $\ell_+^p(\mathbb{N})$ for $1 < p < \infty$	29
3.3 Endpoint inequalities	35
3.4 Tables	40
4 Sharp bounds for the Copson operator	43
4.1 Inequalities in $\ell^p(\mathbb{N})$ for $1 < p < \infty$	45
4.2 Inequalities in $\ell_{\text{dec}}^p(\mathbb{N})$ for $1 < p < \infty$	57
4.3 Inequalities in $\ell_+^p(\mathbb{N})$ for $1 < p < \infty$	61
4.4 Endpoint inequalities	62
4.5 Tables	77
5 Sharp bounds for the Hardy operator	79
5.1 Inequalities in $L^p(\mathbb{R}^+)$ for $1 < p < \infty$	81
5.2 Inequalities in $L_+^p(\mathbb{R}^+)$ for $1 < p < \infty$	83
5.3 Endpoint inequalities	85
5.4 Tables	87
6 Sharp bounds for the adjoint of Hardy operator	89
6.1 Inequalities in $L^p(\mathbb{R}^+)$ for $1 < p < \infty$	90
6.2 Inequalities in $L_{\text{dec}}^p(\mathbb{R}^+)$ for $1 < p < \infty$	92
6.3 Inequalities in $L_+^p(\mathbb{R}^+)$ for $1 < p < \infty$	97
6.4 Endpoint inequalities	97
6.5 Tables	103

7	Sharp bounds related to powers of H and H^*	105
7.1	Auxiliary statements	107
7.2	The norm of a polynomial of H and H^*	109
7.3	Application	118
7.4	Equivalent norms on Cesàro and Copson spaces.	120
	Bibliography	131

Abstract

In recent years, several authors have studied the problem of determining optimal constants in the L^p -norm for Hardy-type inequalities (see, for instance, [9, 16, 24, 34, 44] and the references therein). In this thesis, we continue this line of research by focusing on operators of fundamental importance in Analysis, such as the Hardy operator H , the adjoint of Hardy operator H^* , the Cesàro operator C , the Copson operator C^* , the left-shift operator S , and the right-shift operator S^* .

More specifically, we study the sharp lower and upper bounds for the p -norm of the difference between the Cesàro operator and the identity, $C - I$, when acting on various cones like $\ell^p(\mathbb{N})$ itself, the cone of nonnegative sequences $\ell^p_+(\mathbb{N})$, and the cone of nonnegative and nonincreasing sequences $\ell^p_{\text{dec}}(\mathbb{N})$, for the whole range $1 \leq p \leq \infty$. The same study is conducted for the difference between the Cesàro operator and the left-shift operator, $C - S$, as well as for the difference between the Copson operator minus the identity, $C^* - I$, and the difference between the Copson operator minus the right-shift operator, $C^* - S^*$. As an application, we identify norms on $\ell^p(\mathbb{N})$ that are equivalent to their standard p -norms, and we provide the exact optimal constants for these equivalences.

A similar analysis is carried out in the continuous setting (see, for example, [18, 34, 44, 46] and the references therein), where we investigate the sharp lower and upper bounds in the $L^p(\mathbb{R}^+)$ -norm for the Hardy operator minus the identity, $H - I$, and its adjoint, $H^* - I$, on cones analogous to those considered in the discrete setting, now within the Lebesgue space $L^p(\mathbb{R}^+)$, for $1 \leq p \leq \infty$. Again, as an application, we identify norms on $L^p(\mathbb{R}^+)$ that are equivalent to their standard p -norms, and we provide the exact optimal constants for these equivalences.

Finally, we extend some classical results of G. H. Hardy, J. Littlewood, and G. Pólya related to the operator p -norm of a polynomial in H and H^* , and we provide several applications of this extension. Additionally, we study new norms that are equivalent to the standard norms of the classical Cesàro spaces, $\text{Ces}_p(\mathbb{R}^+)$, for $1 < p \leq \infty$, and the classical Copson spaces, $\text{Cop}_p(\mathbb{R}^+)$, for $1 \leq p < \infty$, providing the sharp constants in these equivalences.

Keywords: Hardy-type inequalities, Hardy operator, Cesàro operator, Copson operator, Shift operator, Cesàro function spaces, Copson function spaces, sharp bounds, optimal constants.

Resumen

En los últimos años, se ha prestado considerable atención al problema de determinar constantes óptimas en la norma L^p para desigualdades de tipo Hardy (véase, por ejemplo, [9, 16, 24, 34, 44] y las referencias allí citadas). En esta tesis, continuamos esta línea de investigación centrándonos en operadores de importancia fundamental en Análisis, tales como el operador de Hardy H , el adjunto del operador de Hardy H^* , el operador de Cesàro C , el operador de Copson C^* , el operador de desplazamiento a la izquierda S y el operador de desplazamiento a la derecha S^* .

Más específicamente, estudiamos los límites inferiores y superiores exactos para la norma p de la diferencia entre el operador de Cesàro y la identidad, $C - I$, actuando sobre distintos conos como $\ell^p(\mathbb{N})$, el cono de sucesiones no negativas $\ell_+^p(\mathbb{N})$, y el cono de sucesiones no negativas y no crecientes $\ell_{\text{dec}}^p(\mathbb{N})$, para todo el rango $1 \leq p \leq \infty$. El mismo estudio se realiza para la diferencia entre el operador de Cesàro y el operador de desplazamiento a la izquierda, $C - S$, así como para la diferencia entre el operador de Copson menos la identidad, $C^* - I$, y la diferencia entre el operador de Copson menos el operador de desplazamiento a la derecha, $C^* - S^*$. Como aplicación, identificamos normas en $\ell^p(\mathbb{N})$ que son equivalentes a sus normas p estándar, y proporcionamos las constantes óptimas exactas para estas equivalencias.

Un análisis similar se realiza en el contexto continuo (véase, por ejemplo, [18, 34, 44, 46] y las referencias allí citadas), donde investigamos los límites inferiores y superiores exactos en la norma estándar de $L^p(\mathbb{R})$ para el operador $H - I$, y su adjunto, $H^* - I$, sobre conos análogos a los considerados en el caso discreto, ahora dentro del espacio de Lebesgue $L^p(\mathbb{R}^+)$, para $1 \leq p \leq \infty$. Nuevamente, como aplicación, identificamos normas en $L^p(\mathbb{R}^+)$ equivalentes a sus normas p estándar, y proporcionamos las constantes óptimas exactas para estas equivalencias.

Finalmente, extendemos algunos resultados clásicos de G. H. Hardy, J. Littlewood y G. Pólya relacionados con la norma p de un polinomio en H y H^* , y presentamos varias aplicaciones de esta extensión. Adicionalmente, estudiamos nuevas normas que son equivalentes a las normas estándar de los espacios clásicos de Cesàro, $\text{Ces}_p(\mathbb{R}^+)$, para $1 < p \leq \infty$, y de los espacios clásicos de Copson, $\text{Cop}_p(\mathbb{R}^+)$, para $1 \leq p < \infty$, proporcionando las constantes exactas en estas equivalencias.

Chapter 1

Introduction

Inequalities have always played a crucial role in the development of many branches of mathematics (e.g., functional analysis, the theory of differential and integral equations, interpolation theory, homogenization theory, etc.) and other sciences (e.g., signal processing, composite engineering, mechanics, physics, etc.). In particular, in recent years, several authors have investigated Hardy-type inequalities and the problem of determining optimal constants in the L^p -norm for this class of problems (see, for instance, [9, 16, 24, 34, 44] and the references therein). These results take the form of L^p -norm estimates involving the *Cesàro operator* or its adjoint in the discrete setting, or the classical *Hardy operator* or its adjoint in the continuous setting.

For a sequence $\{x(n)\}_{n \geq 1} \subset \mathbb{R}$, the *Cesàro operator* C , the *left-shift operator* S , and their transposes are defined by

$$Cx(n) = \frac{1}{n} \sum_{k=1}^n x(k), \quad C^*x(n) = \sum_{k=n}^{\infty} \frac{x(k)}{k},$$

$$Sx(n) = x(n+1), \quad \text{and} \quad S^*x(n) = x(n-1),$$

with the convention $S^*x(1) = 0$. The operator C^* is also referred to as the *Copson operator*, while S^* is called the *right-shift operator*.

The continuous counterparts of C and C^* are the *classical Hardy averaging operator* H and its adjoint H^* . These are defined by

$$Hf(t) = \frac{1}{t} \int_0^t f(s) ds, \quad \text{and} \quad H^*f(t) = \int_t^{\infty} \frac{f(s)}{s} ds,$$

whenever the integrals are well defined for a function f on $(0, \infty)$.

These mappings play a fundamental role in Analysis. For instance, H has a deep connection with one of the most important objects in Harmonic Analysis, namely the classical *Hardy-Littlewood maximal operator*, defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x , and $|Q|$ denotes the Lebesgue measure of Q . This link arises from the fact that

$$(Mf)^* \simeq Hf^*, \quad (1.1)$$

where f^* denotes the classical decreasing rearrangement with respect to the Lebesgue measure. The relationship in (1.1) implies that the problem of determining the boundedness of the rather complex operator M on rearrangement-invariant spaces is equivalent to studying the boundedness of the much simpler operator H on the cone of nonnegative, nonincreasing functions in some Lebesgue space (see [7] for standard notations and results).

The boundedness of H on $L^p(\mathbb{R}^+)$, for $1 < p \leq \infty$, follows from Hardy's inequality (cf. [24, p. 240]):

$$\|Hf\|_{L^p(\mathbb{R}^+)} \leq p' \|f\|_{L^p(\mathbb{R}^+)}, \quad (1.2)$$

where $p' = \frac{p}{p-1}$ if $1 < p < \infty$, and $p' = 1$ if $p = \infty$. Moreover, p' is the best possible constant in (1.2). This inequality has been widely generalized in many different ways; for instance, in [17, 19], the authors provide sharp estimates for the norm of H in the context of weighted Lebesgue spaces. More specifically, they studied the sharp constant in the inequality

$$\int_0^\infty (Hf(x))^p w(x) dx \leq C_w \int_0^\infty f(x)^p w(x) dx,$$

where w is a weight, as well as related questions.

One important property of H is that it is not invertible on $L^p(\mathbb{R}^+)$ (see [40]), so no constant $c(p)$ depending only on p exists for which the reverse inequality

$$\|Hf\|_{L^p(\mathbb{R}^+)} \geq c(p) \|f\|_{L^p(\mathbb{R}^+)}, \quad (1.3)$$

holds in general, even for nonnegative functions. Nevertheless, if f is positive and nonincreasing on \mathbb{R}^+ , inequality (1.3) holds with $c(p) = (p')^{1/p}$, where p' is defined as above. Furthermore, this constant is sharp (cf. [8], [42]).

An analogous result is obtained in [48] when the L^p -norm is replaced with the BMO-norm. Specifically, the author proved that if

$$\|f\|_{\text{BMO}(\mathbb{R}^+)} = \sup_{I \subset \mathbb{R}^+} \frac{1}{|I|} \int_I |f(x) - f_I| dx,$$

where the supremum is taken over all subintervals I of \mathbb{R}^+ , and f_I denotes the mean value of f on I ,

$$f_I = \frac{1}{|I|} \int_I f(x) dx,$$

with $|I|$ being the length of I , then H is bounded on $\text{BMO}(\mathbb{R}^+)$, which consists of all functions f with $\|f\|_{\text{BMO}(\mathbb{R}^+)} < \infty$. Moreover, the author showed that

$$\frac{1}{17} \|f\|_{\text{BMO}(\mathbb{R}^+)} \leq \|Hf\|_{\text{BMO}(\mathbb{R}^+)},$$

for all nonnegative, nonincreasing functions $f \in L^1_{\text{loc}}(\mathbb{R}^+)$.

A similar situation occurs with the Cesàro operator C . In general, there is no constant $k(p)$ depending only on p such that

$$\|Cx\|_{\ell^p(\mathbb{N})} \geq k(p) \|x\|_{\ell^p(\mathbb{N})}$$

holds for all sequences $x \in \ell^p(\mathbb{N})$. However, when x is positive and nonincreasing, the inequality holds with $k(p) = \zeta(p)^{1/p}$ for $1 < p < \infty$, and $k(p) = 1$ when $p = \infty$, where $\zeta(p)$ denotes the Riemann zeta function evaluated at p . Moreover, $k(p)$ is also optimal (cf. [8], [39], [42]).

It is worth noting that these constants differ between the discrete and continuous settings, whereas in the classical inequalities they coincide: $\|H\|_{L^p(\mathbb{R}^+)} = p'$ (cf. [24, p. 240]) and $\|C\|_{\ell^p(\mathbb{N})} = p'$ (see [37]).

The questions addressed in this thesis are closely related to the examples above, focusing on the computation of optimal constants in L^p -norm inequalities involving the operators C , S , H , and their adjoints. The investigation is organized into two main parts: the first addresses problems in the discrete setting (Chapters 3 and 4), and the second deals with the continuous setting (Chapters 5, 6, and 7).

Chapters 3, 4, and 6 are each divided into several sections. The first three address the main problem within a specific cone of $X = L^p(\mathbb{R}^+)$ or $\ell^p(\mathbb{N})$. The first section treats the question in the full space X , the second considers X_{dec} , and the third focuses on X_+ . Although $X_{\text{dec}} \subset X_+ \subset X$, the study of X_+ is presented last, since many results follow directly from the previous two cases. The interest in studying the behavior of the norm on X_+ lies in the fact that estimates in this setting are related, for instance, to renorming certain spaces, as we shall see in Chapter 7. The reason for analysis of the norm on X_{dec} is related, for example, to the classical nonincreasing rearrangement function f^* of a given function f . By definition, f^* is nonnegative and nonincreasing (see [7]). Surprisingly, as we will see, the norm of the operators studied in this work changes significantly when evaluated on these different cones.

Chapters 3 and 5 do not include a section devoted to the study of the main question in the cone X_{dec} . The reason is that, in the discrete case $X = \ell^p_{\text{dec}}(\mathbb{N})$, the analysis is essentially complete, and the main results are already presented in the introduction of the corresponding chapter. In the continuous case, $X = L^p(\mathbb{R}^+)$, the problem is fully solved, and the principal results in this direction are likewise summarized in the introduction.

Furthermore, each of Chapters 3–6 contains a section entitled *Tables*, in which all known constants associated with the operator studied in that chapter are collected. The results obtained in this dissertation are highlighted, and estimates for the remaining unknown quantities are also provided.

The main results of this thesis have been published in the following articles:

- (a) A. Ben Said, S. Boza, and J. Soria, *The norm of Hardy-type oscillation operators in the discrete and continuous setting*. To appear in *Analysis and Applications*.
- (b) A. Ben Said, S. Boza, and J. Soria, *On sharp reverse Hardy-type inequalities for oscillation operators in the discrete and continuous settings*. Preprint. Available at: <https://www.ucm.es/garf/file/bensaid-boza-soria-reverse>.
- (c) A. Ben Said and G. Sinnamon, *Best bounds for the dual Hardy operator minus identity on decreasing functions*, *Ann. Mat. Pura Appl.* 205 (2026), 313–325.
- (d) A. Ben Said, S. Boza, and J. Soria, *Sharp bounds for equivalent norms in the classical Cesàro and Copson function spaces*. Preprint. Available at: <https://www.ucm.es/garf/file/CesCop>.
- (e) A. Ben Said, *Optimal constants for certain Hardy-type integral operators*. Preprint. Available at: <https://www.ucm.es/garf/file/polynomialhardy>.

Let $p \geq 1$. The classical Cesàro sequence spaces, $\text{ces}_p(\mathbb{N})$, and the classical Copson sequence spaces, $\text{cop}_p(\mathbb{N})$, are defined respectively by

$$\text{ces}_p(\mathbb{N}) = \{x : \|Cx\|_p < \infty\},$$

and

$$\text{cop}_p(\mathbb{N}) = \{x : \|C^*x\|_p < \infty\}.$$

In [9], G. Bennett proved that

$$\text{ces}_p(\mathbb{N}) = \text{cop}_p(\mathbb{N})$$

for all $1 < p < \infty$, and he was interested in the norms, $\alpha(p)$ and $\beta(p)$, of the inclusions

$$\text{ces}_p(\mathbb{N}) \xrightarrow{\alpha(p)} \text{cop}_p(\mathbb{N}) \quad \text{and} \quad \text{cop}_p(\mathbb{N}) \xrightarrow{\beta(p)} \text{ces}_p(\mathbb{N}).$$

To address this question, he observed that the operators C and C^* are closely related in the sense that

$$C = (C - S)C^*, \quad \text{and} \quad C = (C^* - I)S^*C,$$

from which it follows that $\alpha(p) \leq \|C - S\|_p$ and $\beta(p) \leq \|C^* - I\|_p$ for all $1 < p < \infty$. He then asked what the norm of the operator $C - I$ is, or equivalently, using the duality principle, what the norm of the operator $C^* - I$ is. Closely related problems are investigated in Chapters 3 and 4, which form the first part of this thesis.

In Chapter 3, we focus on establishing some of the exact values of the optimal constants $A(p)$, $B(p)$, $U(p)$, and $V(p)$, all nonnegative, such that

$$A(p)\|(C - I)x\|_p \leq \|x\|_p \leq B(p)\|(C - I)x\|_p, \quad (1.4)$$

and

$$U(p)\|(C - S)x\|_p \leq \|x\|_p \leq V(p)\|(C - S)x\|_p, \quad (1.5)$$

for all $x \in \ell^p(\mathbb{N})$, $\ell_+^p(\mathbb{N})$, or $\ell_{\text{dec}}^p(\mathbb{N})$, and for all $1 \leq p \leq \infty$.

Starting from (1.4) for a general $x \in \ell^p(\mathbb{N})$, as mentioned earlier, G. Bennett formulated in [9] the problem of determining the exact value of $A(p)$ in that inequality. Using Hilbert space methods, it was previously shown that $A(2) = 1$ in the case of $\ell^2(\mathbb{N})$ [21]. G. J. O. Jameson [28] conjectured that $A(p) = p - 1$ for $p \in (1, 2]$ and proved the result for $p = 4/3$. More recently, G. Sinnamon extended Jameson's method to prove the conjecture and completely resolved Bennett's question for the full range $p \in (1, \infty]$, with a different value of $A(p)$ for $p \in (2, \infty]$ (see [44]). The exact constant $B(p)$ in (1.4) for $p \in (1, 2]$ is established in Theorem 3.10. The endpoint estimates $A(\infty)$, $B(1)$, and $B(\infty)$ are provided in Theorem 3.19.

Continuing the study of the inequalities in (1.4), we now consider $\ell_+^p(\mathbb{N})$ the cone of nonnegative sequences in $\ell^p(\mathbb{N})$. The optimal constant $A(p)$ for $x \in \ell_+^p(\mathbb{N})$ is established in Theorem 3.15 for $p \in (1, 2]$ and in Theorem 3.18 for $p \in [2, \infty)$. The endpoint values $A(1)$, $A(\infty)$, $B(1)$, and $B(\infty)$ are provided in Theorem 3.19.

Regarding the inequalities (1.4) for $x \in \ell_{\text{dec}}^p(\mathbb{N})$, the optimal estimate $A(p)$ for $p \in (1, \infty)$ is given in [20, Theorem 2.2]. The sharp constant $A(\infty)$ is provided in [20, Proposition 2.4]. The remaining endpoint values $A(1)$, $B(1)$, and $B(\infty)$ are established in Theorem 3.19.

Now we turn to (1.5) and begin by considering the best constants $U(p)$ and $V(p)$ in these inequalities for $x \in \ell^p(\mathbb{N})$, with $p \in (1, \infty)$. The optimal constant $U(p)$ for $p \in [2, \infty)$ is established in Theorem 3.24. The sharp estimate $V(p)$ is given in Theorem 3.12 for all $p \in (1, \infty)$. The endpoint values $U(1)$, $U(\infty)$, $V(1)$, and $V(\infty)$ are provided in Theorem 3.19.

Concerning the inequalities (1.5) for nonnegative and nonincreasing sequences $x \in \ell_{\text{dec}}^p(\mathbb{N})$, the exact value of $U(p)$ for $p \in [2, \infty)$ is provided in [20, Corollary 2.3]. For $p \in (1, 2)$, the authors of [20] obtained the following estimates:

$$\frac{(p-1)^{1/p'}}{\zeta(p)^{1/p}} \leq U(p) \leq p-1.$$

Determining the exact value of $U(p)$ for $p \in (1, 2)$ remains a major open problem and has been unsolved since 1996, when G. Bennett posed an equivalent question in [9]. The exact value of $V(p)$ is established in [9, Corollary 10.13] for $p \in (1, 2]$, while for $p \in [2, \infty)$ it follows as a direct consequence of [33, Theorem 3.1]. The endpoint value $U(\infty)$ is given in [20, Proposition 2.4], while $U(1)$, $V(1)$, and $V(\infty)$ are established in Theorem 3.19.

All known constants related to the operators $C - I$ and $C - S$ are collected in Section 3.4, where the estimates established in this thesis are highlighted.

In Chapter 4, we study inequalities related to the adjoints of the operators $C - I$ and $C - S$ considered in the previous chapter; that is, inequalities involving $C^* - I$ and $C^* - S^*$.

We focus on determining the exact values of the optimal constants $A(p)$, $B(p)$, $U(p)$, $V(p) \geq 0$ such that

$$A(p)\|(C^* - I)x\|_p \leq \|x\|_p \leq B(p)\|(C^* - I)x\|_p, \quad (1.6)$$

and

$$U(p)\|(C^* - S^*)x\|_p \leq \|x\|_p \leq V(p)\|(C^* - S^*)x\|_p, \quad (1.7)$$

for all $x \in \ell^p(\mathbb{N})$, $\ell_+^p(\mathbb{N})$, or $\ell_{\text{dec}}^p(\mathbb{N})$, and for all $1 \leq p \leq \infty$.

For a general sequence $x \in \ell^p(\mathbb{N})$, the sharp constant $A(p)$ in inequalities (1.6) for $p \in [1, \infty)$ follows as a consequence of the results in [44], while the exact value of $B(p)$ in (1.6) for $p \in [1, \infty]$ is given in (4.5). The endpoint value $A(\infty)$ is established in Theorem 4.11.

Continuing the study of the inequalities in (1.6), we now consider $\ell_+^p(\mathbb{N})$ the cone of nonnegative sequences in $\ell^p(\mathbb{N})$. The optimal constant $A(p)$ for $x \in \ell_+^p(\mathbb{N})$ is established in Theorem 4.7 for $p \in (1, \infty)$, and the sharp estimate $B(p)$ in (1.6) for $p \in [1, \infty]$ is given in (4.5). The endpoint value $A(1)$ is provided in Theorem 4.8, while $A(\infty)$ is established in Theorem 4.11.

Regarding the inequalities (1.6) for $x \in \ell_{\text{dec}}^p(\mathbb{N})$, the best constant $A(p)$ for $p \in [2, \infty)$ is given in Theorem 4.6. The estimate of $B(p)$ in (1.6) for $p \in [1, \infty]$ is given in (4.5). The endpoint quantity $A(1)$ is provided in Theorem 4.10, and $A(\infty)$ is established in Theorem 4.11.

Now we turn to (1.7) and begin by considering the exact values of $U(p)$ and $V(p)$ in these inequalities for $x \in \ell^p(\mathbb{N})$, with $p \in (1, \infty)$. The optimal constant $U(p)$ for $p \in [1, 2]$ is established in Proposition 3.9. The optimal constant $V(p)$ is given in Theorem 4.4 for all $p \in (1, \infty)$. The endpoint value $U(\infty)$ is established in Theorem 4.11, while $V(1)$ and $V(\infty)$ are provided in Theorems 4.12 and 4.13, respectively.

For the inequalities (1.7) in the case where $x \in \ell_+^p(\mathbb{N})$ is a nonnegative sequence, the best estimate $U(p)$ for $p \in [1, 2]$ is established in Proposition 3.9. The optimal constant $V(p)$ is given in Theorem 4.7 for all $p \in (1, \infty)$. The endpoint value $U(\infty)$ is established in Theorem 4.11. Finally, $V(1)$ and $V(\infty)$ are provided in Theorems 4.12 and 4.13, respectively.

In the setting of the inequalities (1.7), for $x \in \ell_{\text{dec}}^p(\mathbb{N})$ being a nonnegative and nonincreasing sequence, the sharp constant $U(p)$ for $p \in [1, 2]$ is established in Proposition 3.9. The optimal estimate $V(p)$ is given in Theorem 4.6 for $p \in (1, 2]$. The endpoint value $U(\infty)$ is established in Theorem 4.11, while $V(1)$ and $V(\infty)$ are provided in Theorems 4.12 and 4.13, respectively.

All known constants related to the operators $C^* - I$ and $C^* - S^*$ are collected in Section 4.5, where the estimates established in this thesis are highlighted.

We now turn to the second part of this thesis, which focuses on questions similar to those addressed in the previous part, but in the continuous setting, studying

estimates for $H - I$ and its adjoint. It is a well-known fact that operator norms often behave more smoothly in the continuous case than in the discrete case. Many results obtained here are very similar to those in the previous part, but the proofs are generally much simpler than in the discrete setting. One reason for this simplicity is the following identity:

$$HH^* = H^*H = H + H, \quad (1.8)$$

(see [30]). This property does not hold in the discrete setting; indeed (see [9, 20]),

$$CC^* = C + SC^* = CS^* + C^* \neq C + C^*.$$

The operator $H - I$ controls the oscillation of a function with respect to its average. It arises naturally in various contexts; for instance, the well-known Iwaniec's conjecture regarding the norm of the complex Beurling-Ahlfors transform B on $L^p(\mathbb{C})$, since its norm for radial functions $F(z) = f(|z|^2)$ is equivalent to the norm of $H - I$ for general functions f (see [3, 26, 46]). When $p = 2$, $H - I$ is a Hilbert space isomorphism that is unitarily equivalent to the unilateral shift (see [21]). It has also been linked to Laguerre polynomials in [36]. In the theory of interpolation of operators, it acts on rearrangements of functions (nonnegative, nonincreasing functions) to produce equivalent norms in Lorentz spaces (see [7, p. 384]).

Chapter 5 is dedicated to the study of inequalities in the L^p -norm involving the operator $H - I$. Recent work on this operator can be found, for instance, in [20, 18, 34, 44, 46, 47] and the references therein.

In this chapter, we investigate the best constants $A(p), B(p) \geq 0$ in the inequalities

$$A(p) \|(H - I)f\|_p \leq \|f\|_p \leq B(p) \|(H - I)f\|_p, \quad (1.9)$$

for all $f \in L^p(\mathbb{R}^+)$, $L^p_+(\mathbb{R}^+)$, or $L^p_{\text{dec}}(\mathbb{R}^+)$, with $1 \leq p \leq \infty$.

When $f \in L^p(\mathbb{R}^+)$ is a general function, the authors in [44, 46, 47] independently computed the best constant $A(p)$ in (1.9) for $p \in (1, \infty]$. In Theorem 5.4, we calculate $B(p)$ in (1.9) for all $p \in (1, \infty)$. The endpoint cases $A(1)$, $B(1)$, and $B(\infty)$ are given in Theorem 5.9.

For nonnegative functions $f \in L^p_+(\mathbb{R}^+)$, the authors in [18] determined the best constant $A(p)$ for $p \in [2, \infty]$. The value of $A(p)$ for $p \in (1, 2]$ is given in Theorem 5.8. The sharp estimate $B(p)$ is provided in Theorem 5.7 for $p \in (1, 2]$ and in Theorem 5.8 for $p \in [2, \infty)$. The optimal quantities $A(1)$, $B(1)$, and $B(\infty)$ are provided in Theorem 5.9.

Concerning the case of nonnegative, nonincreasing functions $f \in L^p_{\text{dec}}(\mathbb{R}^+)$, V. Kolyada in [34] completed the work of [36] by determining $A(p)$ and $B(p)$ in (1.9) for all $p \in (1, \infty)$. The number $A(\infty)$ is given in [18]. The endpoint estimates $A(1)$, $B(1)$, and $B(\infty)$ are provided in Theorem 5.9.

We emphasize that in this chapter we complete the investigation related to $H - I$ by providing all remaining optimal constants. All known estimates associated with

the operator $H - I$ are collected in Section 5.4, where the results established in this thesis are highlighted.

In Chapter 6, we conduct a study similar to that in Chapter 5, but we replace the operator $H - I$ with its adjoint $H^* - I$. We investigate the best constants $A(p), B(p) \geq 0$ in the inequalities

$$A(p) \|(H^* - I)f\|_p \leq \|f\|_p \leq B(p) \|(H^* - I)f\|_p, \quad (1.10)$$

for all $f \in L^p(\mathbb{R}^+)$, $L_+^p(\mathbb{R}^+)$, or $L_{\text{dec}}^p(\mathbb{R}^+)$, with $1 \leq p \leq \infty$. Many of the results obtained in this chapter are a direct consequence of the duality principle, while others follow from the identities in (1.8).

For a general function $f \in L^p(\mathbb{R}^+)$, the sharp constant $A(p)$ in (1.10) for the range $p \in [1, \infty)$ follows from the results in [44, 46, 47], while the best estimate $B(p)$ for $p \in (1, \infty)$ is given in Theorem 6.2. The endpoint quantities $A(\infty)$, $B(1)$, and $B(\infty)$ are provided in Theorem 6.10.

Regarding the case of nonnegative functions $f \in L_+^p(\mathbb{R}^+)$, the exact values of $A(p)$ and $B(p)$ in (1.10) for $p \in (1, \infty)$ are established in Theorem 6.7. The optimal constant $A(1)$ is given in Theorem 6.8, while $A(\infty)$, $B(1)$, and $B(\infty)$ are provided in Theorem 6.10.

In the case of nonnegative, nonincreasing functions $f \in L_{\text{dec}}^p(\mathbb{R}^+)$, the precise values of $A(p)$ and $B(p)$ for $p \in (1, \infty)$ are established in Theorems 6.4 and 6.5. The sharp constant $A(1)$ is obtained in Theorem 6.9, while $A(\infty)$, $B(1)$, and $B(\infty)$ appear in Theorem 6.10.

We emphasize again that in this chapter we complete the investigation related to $H^* - I$ by providing all remaining optimal constants. All known estimates associated with the operator $H^* - I$ are collected in Section 6.5, where the results established in this thesis are highlighted.

In the first part of Chapter 7, we generalize the classical results of G. H. Hardy, J. Littlewood, and G. Pólya (see [23, 24]):

$$\begin{aligned} \|H\|_{L^p(\mathbb{R}^+)} &= p', & \text{for } 1 < p \leq \infty, \\ \|H^*\|_{L^p(\mathbb{R}^+)} &= p, & \text{for } 1 \leq p < \infty, \\ \|H + H^*\|_{L^p(\mathbb{R}^+)} &= p + p', & \text{for } 1 < p < \infty. \end{aligned}$$

This generalization is presented in Theorem 7.3, where we prove that

$$\begin{aligned} \|Q(H, H^*)\|_{L_{\text{dec}}^p(\mathbb{R}^+)} &= \|Q(H, H^*)\|_{L_+^p(\mathbb{R}^+)} = \|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} \\ &= Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = Q(p', p), \end{aligned}$$

for $1 < p < \infty$ and any real polynomial $Q \in \mathbb{R}[X, Y]$ with nonnegative coefficients. Several applications of Theorem 7.3 are presented, including Corollary 7.5 and Theorem 7.10.

In the second part of Chapter 7, we investigate various norms on the classical Cesàro function spaces $\text{Ces}_p(\mathbb{R}^+)$ and the classical Copson function spaces $\text{Cop}_p(\mathbb{R}^+)$, defined respectively by

$$\text{Ces}_p(\mathbb{R}^+) = \{f : \|f\|_{\text{Ces}_p(\mathbb{R}^+)} = \|H|f|\|_p < \infty\}, \quad 1 < p \leq \infty,$$

and

$$\text{Cop}_p(\mathbb{R}^+) = \{f : \|f\|_{\text{Cop}_p(\mathbb{R}^+)} = \|H^*|f|\|_p < \infty\}, \quad 1 \leq p < \infty.$$

In [9, Theorem 21.1], G. Bennett showed that $\text{Ces}_p(\mathbb{R}^+)$ and $\text{Cop}_p(\mathbb{R}^+)$ coincide for $1 < p < \infty$. Note that (1.2) together with (1.3) implies the strict inclusion

$$L^p(\mathbb{R}^+) \subsetneq \text{Ces}_p(\mathbb{R}^+),$$

for all $1 < p \leq \infty$.

K. Lešnika and L. Maligranda [38] established the inequalities

$$\frac{p'}{e} \|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq \|Hf\|_{\text{Ces}_p(\mathbb{R}^+)} \leq p' \|f\|_{\text{Ces}_p(\mathbb{R}^+)}, \quad (1.11)$$

which hold for any nonnegative measurable function f and $1 < p \leq \infty$. Our first contribution is Theorem 7.14, where we improve the *lower bound in (1.11)* by determining the *sharp constant* on the left-hand side for $1 < p < \infty$.

We also introduce two alternative norms for $\text{Cop}_p(\mathbb{R}^+)$, which Theorems 7.12 and 7.15 show to be equivalent to the standard norm, with the optimal constants in the relations between the norms also provided.

Chapter 2

General notations and preliminaries

This chapter is devoted to establishing several propositions that are essential for the development of the results presented later. We begin by introducing the notations and conventions that will be used consistently throughout the text, followed by a summary of the preliminary concepts and results required in the subsequent chapters. Any additional, more specialized background will be introduced at the appropriate points in the exposition.

Our study will be carried out in the spaces $X = L^p(\mathbb{R}^+)$ or $X = \ell^p(\mathbb{N})$, with $1 \leq p \leq \infty$. For such space X we consider the following cones:

$$X_+ = \{x \in X : x \text{ is nonnegative}\},$$

and

$$X_{\text{dec}} = \{x \in X_+ : x \text{ is nonincreasing}\}.$$

Let $T : X \rightarrow X$ be a bounded operator. We introduce the following notation:

$$\|T\|_X := \sup \{\|Tx\|_X : \|x\|_X = 1\},$$

$$\|T\|_{X_+} := \sup \{\|Tx\|_X : x \in X_+, \|x\|_X = 1\},$$

$$\|T\|_{X_{\text{dec}}} := \sup \{\|Tx\|_X : x \in X_{\text{dec}}, \|x\|_X = 1\}.$$

When $T : X \rightarrow X$ is unbounded, there exists a sequence $\{x_n\}_{n \geq 1}$ with $\|x_n\|_X = 1$ for all $n \geq 1$, such that

$$\lim_{n \rightarrow \infty} \|Tx_n\|_X = \infty.$$

In this situation, we define

$$\|T\|_X = \infty.$$

If, in addition, the sequence $\{x_n\}_{n \geq 1}$ belongs to X_{dec} or X_+ , we denote

$$\|T\|_{X_{\text{dec}}} = \infty, \quad \text{or} \quad \|T\|_{X_+} = \infty,$$

respectively.

Clearly, for every operator $T : X \rightarrow X$ the following inequalities hold:

$$0 \leq \|T\|_{X_{\text{dec}}} \leq \|T\|_{X_+} \leq \|T\|_X \leq \infty. \quad (2.1)$$

We recall the well known definition of what it means for an operator $T : X \rightarrow Y$ between two normed spaces to be bounded below.

Definition 2.1. An operator $T : X \rightarrow Y$ between normed spaces is said to be **bounded below** if there exists a constant $\alpha > 0$ such that

$$\|Tx\|_Y \geq \alpha \|x\|_X$$

for each $x \in X$. Equivalently, T is bounded below if there exists some $\alpha > 0$ satisfying $\|Tx\|_Y \geq \alpha$ for all $x \in X$ with $\|x\|_X = 1$.

The negation of the preceding definition can be formulated in terms of sequences as follows.

Lemma 2.2. [1, Lemma 2.2]. *An operator $T : X \rightarrow Y$ between two normed spaces fails to be bounded below if and only if there exists a sequence of unit vectors $\{x_n\}_{n \geq 1}$ in X such that $\|Tx_n\|_Y \rightarrow 0$.*

Let $T : X \rightarrow X$ be a bounded below operator. We define

$$\begin{aligned} [T]_X &:= \sup \left\{ \frac{1}{\|Tx\|_X} : \|x\|_X = 1 \right\}, \\ [T]_{X_+} &:= \sup \left\{ \frac{1}{\|Tx\|_X} : x \in X_+, \|x\|_X = 1 \right\}, \\ [T]_{X_{\text{dec}}} &:= \sup \left\{ \frac{1}{\|Tx\|_X} : x \in X_{\text{dec}}, \|x\|_X = 1 \right\}. \end{aligned}$$

In the case where T is not bounded below, by Lemma 2.2 there exists a sequence $\{x_n\}_{n \geq 1} \subset X$ with $\|x_n\|_X = 1$ for all $n \geq 1$ such that

$$\lim_{n \rightarrow \infty} \|Tx_n\|_X = 0.$$

In this case, we simply set

$$[T]_X = \infty.$$

If, additionally, the sequence $\{x_n\}_{n \geq 1}$ lies in X_{dec} or in X_+ , we write, respectively,

$$[T]_{X_{\text{dec}}} = \infty \quad \text{or} \quad [T]_{X_+} = \infty.$$

If $Tx \notin X$ for all $x \in X_{\text{dec}} \setminus \{0\}$, $x \in X_+ \setminus \{0\}$, or $x \in X \setminus \{0\}$, we denote

$$[T]_{X_{\text{dec}}} = 0, \quad [T]_{X_+} = 0, \quad \text{or} \quad [T]_X = 0,$$

respectively.

It is clear that for every operator $T : X \rightarrow X$, the following inequalities hold:

$$0 \leq [T]_{X_{\text{dec}}} \leq [T]_{X_+} \leq [T]_X \leq \infty. \quad (2.2)$$

In several of the arguments below, it is important that the power functions involved be defined on all of \mathbb{R} . For this purpose, we let

$$\mathbb{F} = \left\{ \frac{2i}{2j+1} : i, j \in \mathbb{N} \right\} \cap (1, 2),$$

and

$$\mathbb{E} = \left\{ \frac{2i}{2j+1} : i, j \in \mathbb{N} \right\} \cap (2, \infty).$$

The sets \mathbb{F} and \mathbb{E} are dense in $(1, 2]$ and $[2, \infty)$ respectively. The set \mathbb{E} was first introduced by G. Sinnamon in his work [44]. If $p \in \mathbb{F} \cup \mathbb{E}$, it is easy to see that the function $t \mapsto t^p$ is well defined on \mathbb{R} since for $a < 0$ we have that $a^p = |a|^p = (-a)^p$. Note that if $a = 0$, then $a^{p-1} = 0$, whereas if $a \neq 0$, then

$$a^{p-1} = \frac{a^p}{a} = \frac{|a|^p}{a}.$$

We have also that $t \mapsto t^p$ is continuously differentiable on \mathbb{R} . Its derivative is strictly increasing, and we have the following inequality (see [44]):

$$pa^{p-1}(b-a) \leq b^p - a^p \leq pb^{p-1}(b-a), \quad (2.3)$$

which holds for all $a, b \in \mathbb{R}$.

Let $1 \leq p < \infty$ and $f_p : [0, 1/2] \rightarrow \mathbb{R}$ be the function given by

$$f_p(t) = pt^{p-1} + (1-t)^p - t^p. \quad (2.4)$$

Note that when $p \in \mathbb{F} \cup \mathbb{E}$, the function f_p extends to the entire real line \mathbb{R} .

The following two lemmas related to the function f_p are of importance.

Lemma 2.3. [44, Lemma 7] *If $2 < p < \infty$, then the function f_p has a unique critical point t_p in $(0, \frac{1}{2})$, $m_p = f_p(t_p)$ and m_p is a continuous function of p . If, in addition, $p \in \mathbb{E}$, then t_p is the unique critical point of f_p on all of \mathbb{R} and $f_p(t) \geq m_p$ for all $t \in \mathbb{R}$.*

Lemma 2.4. *If $1 < p < 2$, then the function f_p has a unique critical point t_p in $(0, \frac{1}{2})$, $M_p = f_p(t_p)$ and M_p is a continuous function of p . If, in addition, $p \in \mathbb{F}$, then t_p is the unique critical point of f_p on all of \mathbb{R} and $f_p(t) \leq M_p$ for all $t \in \mathbb{R}$.*

Proof. On $(0, \frac{1}{2})$, $f'_p(t) = p((p-1)t^{p-2} - (1-t)^{p-1} - t^{p-1})$, which can be extended to be continuous on $(0, \frac{1}{2}]$, with $f'_p(\frac{1}{2}) = p(p-2)2^{2-p} < 0$. Since

$$\lim_{t \rightarrow 0^+} f'_p(t) = +\infty,$$

there is some $\epsilon > 0$ such that,

$$f'_p(t) > 0, \quad \text{for every } t \in (0, \epsilon).$$

We compute the second derivative of f ,

$$f''_p(t) = p(p-1)((p-2)t^{p-3} + (1-t)^{p-2} - t^{p-2}), \quad t \in (0, 1/2).$$

Let $t \in (0, 1/2)$. On the one hand, we have

$$0 < t < 1-t \implies \frac{1}{(1-t)} < \frac{1}{t} \implies \frac{1}{(1-t)^{2-p}} < \frac{1}{t^{2-p}}.$$

On the other hand, we get

$$0 < 2-p \implies 0 < \frac{2-p}{t} \implies 1 < \frac{2-p}{t} + 1 = t^{-1}((2-p) + t),$$

and thus

$$\frac{1}{(1-t)^{2-p}} < \frac{1}{t^{2-p}} t^{-1}((2-p) + t) = -t^{p-3}((p-2) - t).$$

Equivalently,

$$t^{p-3}((p-2) - t) + \frac{1}{(1-t)^{2-p}} < 0,$$

which means,

$$f''_p(t) < 0,$$

and that is true for all $t \in (0, 1/2)$. Therefore f'_p is strictly decreasing on $(0, 1/2]$, f_p has a unique critical point t_p in $(0, 1/2)$, and $f_p(t_p) = M_p$ is the maximum value of f_p . For every $p_0 < 2$, the function $(p, t) \rightarrow f_p(t)$ is uniformly continuous on $[p_0, 2] \times [0, \frac{1}{2}]$. It follows that $p \rightarrow M_p$ is continuous on $(1, 2]$.

If $p \in \mathbb{F}$, then f_p is defined on \mathbb{R} , $(1-t)^p = (t-1)^p$, and

$$f'_p(t) = p((p-1)t^{p-2} + (t-1)^{p-1} - t^{p-1}) = p(p-1) \int_{t-1}^t (t^{p-2} - s^{p-2}) ds.$$

If $t < 0$, $t-1 < s < t$ then $|t| < |s|$, so $s^{p-2} < t^{p-2}$ and we get $f'_p(t) > 0$. Thus f_p is strictly increasing on $(-\infty, 0]$. If $t \geq 1/2$ then we get $t-1 < s < t$ so $t^{p-2} < s^{p-2}$ and $f'_p(t) < 0$. Thus f_p is strictly decreasing on $[1/2, \infty)$. It follows that t_p is the unique critical point of f_p on \mathbb{R} and $f_p(t) \leq M_p$, for all $t \in \mathbb{R}$.

We are now going to show that the map $p \mapsto M_p$ is continuous on $(1, 2)$. First observe that the function

$$F(p, t) := f_p(t) = pt^{p-1} + (1-t)^p - t^p$$

is continuous on $(1, 2) \times [0, 1/2]$: for fixed t , each term is continuous in p , and for fixed p the function is continuous in t . Hence F is jointly continuous.

Fix $p_0 \in (1, 2)$ and let $p_n \rightarrow p_0$. For each n , choose $t_n \in [0, 1/2]$ such that

$$M_{p_n} = F(p_n, t_n).$$

Since $[0, 1/2]$ is compact, the sequence t_n has a convergent subsequence t_{n_k} . Let $t^* \in [0, 1/2]$ be such that $t_{n_k} \rightarrow t^* \in [0, 1/2]$. By the joint continuity of F , we obtain

$$\lim_{k \rightarrow \infty} M_{p_{n_k}} = \lim_{k \rightarrow \infty} F(p_{n_k}, t_{n_k}) = F(p_0, t^*) \leq M_{p_0},$$

so $\limsup_{n \rightarrow \infty} M_{p_n} \leq M_{p_0}$.

Next, let $t_0 \in [0, 1/2]$ be such that $M_{p_0} = F(p_0, t_0)$ (which exists by compactness). Then for every n ,

$$M_{p_n} \geq F(p_n, t_0),$$

and by continuity of F ,

$$F(p_n, t_0) \rightarrow F(p_0, t_0) = M_{p_0}.$$

Hence $\liminf_{n \rightarrow \infty} M_{p_n} \geq M_{p_0}$. Combining the two inequalities gives

$$\lim_{n \rightarrow \infty} M_{p_n} = M_{p_0}.$$

Since $p_n \rightarrow p_0$ was arbitrary, the function $p \mapsto M_p$ is continuous on $(1, 2)$. \square

Let m_p and M_p be the constants defined by

$$m_p = \min_{t \in [0, 1/2]} f_p(t) \quad \text{and} \quad M_p = \max_{t \in [0, 1/2]} f_p(t). \quad (2.5)$$

An important property of the constants $m_p^{-1/p}$, which we will use later, is that

$$m_{p'}^{-1/p'} = \max_{t \in [0, 1/2]} f_{p'}(t)^{-1/p'} = \max_{t \in [0, 1/2]} f_p(t)^{1/p} = M_p^{1/p}, \quad (2.6)$$

where $1 < p \leq 2$, p' denotes the conjugate exponent of p , and f_p is defined as in (2.4). The proof of the identity (2.6) is nontrivial and is given in Remark 4.3.

The following result is due to G. Sinnamon.

Lemma 2.5. [44, Lemma 1]. *Let $p \in \mathbb{E}$. For all $t \in \mathbb{R}$,*

$$(p-1)p^{p-2}t^p + (t+1)^p - p(t+1)^{p-1}t \geq p^{p-2}(p-1)^{1-p}.$$

Let $p \in \mathbb{F} \cup \mathbb{E}$ and let $F_p : [0, \infty) \rightarrow \mathbb{R}$ be the function defined by

$$F_p(t) = (p-1)t^p - pt^{p-1} - (t-1)^p + 1.$$

The next result gives us some important inequalities that involve F_p .

Lemma 2.6. *Let $p \in \mathbb{F} \cup \mathbb{E}$. Then*

$$F_p(t) \geq 0, \quad t \in [0, \infty),$$

if $p \in \mathbb{E}$, and

$$F_p(t) \leq 0, \quad t \in [0, \infty),$$

if $p \in \mathbb{F}$.

Proof. Let $p \in \mathbb{E}$. We have that,

$$\begin{aligned} F'_p(t) &= p(p-1)t^{p-1} - p(p-1)t^{p-2} - p(t-1)^{p-1} \\ &= p((p-1)(t^{p-1} - t^{p-2}) - (t-1)^{p-1}), \end{aligned}$$

for all $t > 1$. Let $t > 1$. Since $p-2 > 0$ we obtain that

$$(p-1)t^{p-2} > t^{p-2} > (t-1)^{p-2} > 0,$$

which means

$$(p-1)t^{p-2} > (t-1)^{p-2} = (t-1)^{p-1} \frac{1}{(t-1)}.$$

Therefore

$$(p-1)(t^{p-1} - t^{p-2}) = (p-1)t^{p-2}(t-1) > (t-1)^{p-1},$$

so we have

$$(p-1)(t^{p-1} - t^{p-2}) - (t-1)^{p-1} > 0.$$

Then

$$F'_p(t) = p((p-1)(t^{p-1} - t^{p-2}) - (t-1)^{p-1}) > 0,$$

for all $t > 1$. Thus F_p is strictly increasing in the interval $(1, \infty)$. So, we have $F_p(t) > F_p(1) = 0$, for all $t > 1$.

Now let us analyze the case where $0 < t < 1$. Since $p \in \mathbb{E}$, we have that $(t-1)^p = (1-t)^p$. Then

$$F_p(t) = (p-1)t^p - pt^{p-1} - (1-t)^p + 1.$$

Computing the derivative of F_p yields

$$\begin{aligned} F'_p(t) &= p(p-1)t^{p-1} - p(p-1)t^{p-2} + p(1-t)^{p-1} \\ &= p((p-1)(t^{p-1} - t^{p-2}) + (1-t)^{p-1}) \\ &= p((p-1)t^{p-2}(t-1) + (1-t)^{p-1}) \\ &= p(-(p-1)t^{p-2}(1-t) + (1-t)^{p-1}) \\ &= p((1-t)((1-t)^{p-2} - (p-1)t^{p-2})) \\ &= p(1-t)G_p(t), \end{aligned}$$

where $G_p(t) = (1-t)^{p-2} - (p-1)t^{p-2}$. Let us see that F'_p has only one zero in the interval $(0, 1)$. For that we need to check that G_p has only one zero in $(0, 1)$. Indeed,

$$G'_p(t) = (p-2)(1-t)^{p-3}(-1) - (p-1)(p-2)t^{p-3} < 0,$$

for all $t \in (0, 1)$. Then G_p is strictly decreasing in $(0, 1)$. We have $G_p(0) = 1 > 0$ and also $G_p(1) = -(p-1) < 0$. So G_p has only one zero in $(0, 1)$. Hence F_p has only one local extremum in $(0, 1)$. Since $F_p(1/2) = (2^p - p - 2)/2^p > 0$ and $F_p(0) = F_p(1) = 0$, we can conclude that F_p has a local maximum in $(0, 1)$ and

$$F_p(t) \geq 0, \quad \text{for every } t \in [0, 1].$$

In the case where $p \in \mathbb{F}$, the proof is analogous, except now G_p is strictly increasing on $(0, 1)$, we have

$$\lim_{t \rightarrow 0^+} G_p(t) = -\infty \quad \text{and} \quad \lim_{t \rightarrow 1^-} G_p(t) = +\infty.$$

It follows that G_p has just one zero in $(0, 1)$, and hence F_p has exactly one critical point there. Moreover,

$$F_p\left(\frac{1}{2}\right) = \frac{2^p - p - 2}{2^p} < 0, \quad F_p(0) = F_p(1) = 0,$$

and by continuity this critical point is a local minimum. Therefore, for all $t \in [0, 1]$ we have

$$F_p(t) \leq 0.$$

This concludes the proof. \square

We use the differentiation formula $\frac{d}{dz}|z|^q = q|z|^{q-2}z$ in the proof. It is valid for all nonzero z and all real q . Note that if $q > 1$, the right-hand side extends continuously to $z = 0$.

Lemma 2.7. *Let $u > 0$ and $0 \leq t \leq 1$. For $r \geq 0$, set*

$$h(r) = e^r \int_0^{e^{-r}} |1 + \ln x|^p dx.$$

If $1 \leq p \leq 2$, then

$$h(t^{1/p}u) \leq (1-t)h(0) + th(u). \quad (2.7)$$

If $p > 2$, then

$$h(t^{1/p}u) \geq (1-t)h(0) + th(u). \quad (2.8)$$

Proof. We will prove only the inequality (2.7), since (2.8) admits an analogous proof.

We will use $h(r)$ in a different form. Substitute $x = e^{-y}$ to obtain

$$h(r) = e^r \int_r^\infty |y-1|^p e^{-y} dy.$$

Now fix $u > 0$ and let $r(t) = t^{1/p}u$. We will write r , r' , and r'' instead of $r(t)$, $r'(t)$, and $r''(t)$. Observe that $r''r = (1-p)(r')^2$.

Since

$$\frac{d}{dy}(|y-1|^p e^{-y}) = p|y-1|^{p-2}(y-1)e^{-y} - |y-1|^p e^{-y}$$

is continuous, we may integrate both sides and simplify to get

$$pe^r \int_r^\infty |y-1|^{p-2}(y-1)e^{-y} dy = h(r) - |r-1|^p. \quad (2.9)$$

Also, since $|y-1|^{p-2}(y-1)e^{-y}$ extends to be continuous at $y=1$ and its derivative,

$$\begin{aligned} \frac{d}{dy}(|y-1|^{p-2}(y-1)e^{-y}) \\ &= (p-2)|y-1|^{p-4}(y-1)^2 e^{-y} + |y-1|^{p-2}e^{-y} - |y-1|^{p-2}(y-1)e^{-y} \\ &= (p-1)|y-1|^{p-2}e^{-y} - |y-1|^{p-2}(y-1)e^{-y}, \end{aligned}$$

is continuous except at $y=1$ we may integrate both sides and simplify to get

$$(p-1)e^r \int_r^\infty |y-1|^{p-2}e^{-y} dy = e^r \int_r^\infty |y-1|^{p-2}(y-1)e^{-y} dy - |r-1|^{p-2}(r-1). \quad (2.10)$$

Next, we compute two derivatives of $h(r)$ with respect to t . By (2.9),

$$\frac{d}{dt}h(r) = (h(r) - |r-1|^p)r' = \left(pe^r \int_r^\infty |y-1|^{p-2}(y-1)e^{-y} dy \right) r'.$$

By (2.10), and the observation $r''r = (1-p)(r')^2$,

$$\begin{aligned} \frac{d^2}{dt^2}h(r) &= p \left(e^r \int_r^\infty |y-1|^{p-2}(y-1)e^{-y} dy - |r-1|^{p-2}(r-1) \right) (r')^2 \\ &\quad + \left(pe^r \int_r^\infty |y-1|^{p-2}(y-1)e^{-y} dy \right) r'' \\ &= \frac{p}{r} (p-1)e^r (r')^2 \left(r \int_r^\infty |y-1|^{p-2}e^{-y} dy - \int_r^\infty |y-1|^{p-2}(y-1)e^{-y} dy \right). \end{aligned}$$

Now let

$$g(s) = s \int_s^\infty |y-1|^{p-2}e^{-y} dy - \int_s^\infty |y-1|^{p-2}(y-1)e^{-y} dy.$$

Taking $r=0$ in (2.9) shows

$$g(0) = - \int_0^\infty |y-1|^{p-1}(y-1)e^{-y} dy = \frac{1}{p}(1-h(0)).$$

But Hölder's inequality and a bit of calculus shows that

$$h(0)^{1/p} = \left(\int_0^1 |1+\ln x|^p dx \right)^{1/p} \leq \left(\int_0^1 |1+\ln x|^2 dx \right)^{1/2} = 1,$$

so $0 \leq g(0)$. In particular, if $p = 2$, then $g(0) = 0$.

Clearly,

$$\lim_{s \rightarrow \infty} \int_s^{\infty} |y - 1|^{p-2} (y - 1) e^{-y} dy = 0.$$

Also, by L'Hôpital's rule

$$\lim_{s \rightarrow \infty} \frac{\int_s^{\infty} |y - 1|^{p-2} e^{-y} dy}{s^{-1}} = \lim_{s \rightarrow \infty} \frac{-|s - 1|^{p-2} e^{-s}}{-s^{-2}} = 0.$$

Thus, $g(s) \rightarrow 0$ as $s \rightarrow \infty$.

If $s \neq 1$,

$$g'(s) = \int_s^{\infty} |y - 1|^{p-2} e^{-y} dy - |s - 1|^{p-2} e^{-s} \quad \text{and} \quad g''(s) = (p - 2)(1 - s)|s - 1|^{p-4} e^{-s}.$$

If $p = 2$, $g'(s) = 0$ and $g(s) = g(0) = 0$. If $1 < p < 2$, g is concave on $(0, 1)$ and convex on $(1, \infty)$. It follows that g decreases to zero on $(1, \infty)$, that $g(1) > 0$, that g has no local minimum in $(0, 1)$ and that $g \geq 0$ on $(0, 1)$. If $p > 2$, g is convex on $(0, 1)$ and concave on $(1, \infty)$. It follows that g increases to zero on $(1, \infty)$, that $g(1) < 0$, that g has no local maximum on $(0, 1)$ and that $g \leq 0$ on $(0, 1)$. We have shown that $0 \leq g(s)$ for $0 < s < \infty$ and, taking $s = r$, that $0 \leq \frac{d^2}{dt^2} h(r)$.

We conclude that $h(r)$ is a convex function of t on $[0, 1]$ when $1 < p \leq 2$ and $h(r)$ is a concave function of t on $[0, 1]$ when $p \geq 2$. When $t = 0$, $r = 0$ and when $t = 1$, $r = u$. Thus, for any $t \in [0, 1]$ we have $h(t^{1/p}u) = h(r) \leq (1 - t)h(0) + th(u)$ as required. \square

Chapter 3

Sharp bounds for the Cesàro operator

Given a sequence $x = \{x(n)\}_{n \geq 1}$, the *Cesàro operator* C is defined by

$$Cx(n) = \frac{1}{n} \sum_{k=1}^n x(k), \quad n \geq 1,$$

and the *left shift operator* is given by

$$Sx(n) = x(n+1), \quad n \geq 1.$$

G. Bennett formulated in [9] the question of determining the exact norm of $C - I$ as a map acting in $\ell^p(\mathbb{N})$. This operator controls the oscillation of a sequence with respect to its average. By means of Hilbert space methods, that operator norm was previously shown to be 1 in the case of $\ell^2(\mathbb{N})$ (see [21]). G. J. O. Jameson [28] conjectured that $\|C - I\|_p = 1/(p-1)$, for $1 < p \leq 2$ and he obtained the result for $p = 4/3$. Very recently, G. Sinnamon [44] used Jameson's method to prove the conjecture and, in fact, he was able to answer Bennett's question in the full range $1 < p \leq \infty$ (with a different value of the norm for $2 < p \leq \infty$). More specifically, he proved that

$$\|C - I\|_{\ell^p(\mathbb{N})} = \begin{cases} 1/(p-1), & \text{if } 1 < p \leq 2, \\ m_p^{-1/p}, & \text{if } 2 \leq p < \infty, \\ 2, & \text{if } p = \infty, \end{cases} \quad (3.1)$$

where m_p is defined in (2.5).

S. Boza and J. Soria [20] also contributed to this line of research and proved the following result.

Theorem 3.1. [20, Theorem 2.2]. *Let $1 < p < \infty$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then the following inequalities are sharp:*

$$(p-1)\|(C-I)x\|_p \leq \|x\|_p,$$

if $1 < p \leq 2$, and

$$(p-1)^{1/p}\|(C-I)x\|_p \leq \|x\|_p,$$

if $2 < p < \infty$.

With regard to sharp reverse inequalities for the operator $C - I$, we have the following result due to G. J. O. Jameson:

Theorem 3.2. [29, Corollary 5] *Let $x \in \ell^2(\mathbb{N})$. Then the following inequality is sharp:*

$$\|x\|_2 \leq \sqrt{2}\|(C-I)x\|_2.$$

On the norm of the operator $C - S$ on $\ell^2(\mathbb{N})$, the author in [29] proved the following result:

Theorem 3.3. [29, Theorem 6] *Let $x \in \ell^2(\mathbb{N})$. Then the following inequality is sharp:*

$$\|(C-S)x\|_2 \leq \sqrt{2}\|x\|_2.$$

Concerning the norm of the operator $C - S$ acting on the cone of nonnegative, nonincreasing sequences in $\ell^p(\mathbb{N})$, S. Boza and J. Soria proved the following result. Here ζ denotes the Riemann zeta function, defined for $p > 1$ by

$$\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}.$$

Theorem 3.4. [20, Corollary 2.3] *Let $2 \leq p < \infty$ and $x \in \ell^p(\mathbb{N})$ be a nonnegative and nonincreasing sequence. Then the following inequality is sharp:*

$$\zeta(p)^{-1/p}\|(C-S)x\|_p \leq \|x\|_p. \quad (3.2)$$

Regarding the case $1 < p < 2$ in inequality (3.2), the authors of [20] provided estimates for the best constant, namely the norm of the operator $C - S$ on $\ell_{\text{dec}}^p(\mathbb{N})$:

$$\frac{1}{p-1} \leq \|C-S\|_{\ell_{\text{dec}}^p(\mathbb{N})} \leq \frac{\zeta(p)^{1/p}}{(p-1)^{1/p'}}.$$

Determining the exact value of $\|C-S\|_{\ell_{\text{dec}}^p(\mathbb{N})}$ for $1 < p < 2$ is a major open problem in this area and has been unsolved since 1996, when G. Bennett posed an equivalent question in [9].

In [9], the author established the following reverse inequality relating the operators $C - S$ and I on the cone of nonincreasing, nonnegative sequences in $\ell^p(\mathbb{N})$, for $1 < p \leq 2$.

Theorem 3.5. [9, Corollary 10.13]. *Let $1 < p \leq 2$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then the following inequality is sharp:*

$$\|x\|_p \leq (p-1)^{1/p} \|(C-S)x\|_p. \quad (3.3)$$

The case $2 \leq p < \infty$ of the inequality (3.3) was completed by V. Kolyada. More precisely, he proved that if C^* denotes the dual operator of the Cesàro operator, defined by

$$C^*x(n) = \sum_{k=n}^{\infty} \frac{x(k)}{k}, \quad n \geq 1,$$

then the following result holds.

Theorem 3.6. [33, Theorem 3.1]. *Let $2 \leq p < \infty$ and let x be a nonnegative sequence. Then*

$$\|C^*x\|_p \leq (p-1)\|Cx\|_p.$$

The constant is optimal.

Now, by using the following identity (see [9, 20])

$$C \circ C^* = C + SC^*, \quad (3.4)$$

and taking into account that, for a positive sequence x , the sequence C^*x is positive and nonincreasing, it follows that Theorem 3.6 is equivalent to the following result.

Theorem 3.7. *Let $2 \leq p < \infty$ and let $x = \{x(n)\}_{n \geq 1} \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then*

$$\|x\|_p \leq (p-1)\|(C-S)x\|_p. \quad (3.5)$$

The constant is the best possible.

The case $p = 2$ in (3.5) was obtained independently by G. J. O. Jameson in [29].

Regarding sharp estimates at the endpoints for the operators $C - I$ and $C - S$ acting on the cone of nonnegative, nonincreasing sequences in $\ell^\infty(\mathbb{R}^+)$, the following inequalities have been established in [20, Proposition 2.4]:

$$\|(C - I)x\|_\infty \leq \|x\|_\infty, \quad (3.6)$$

and

$$\|(C - S)x\|_\infty \leq \|x\|_\infty, \quad (3.7)$$

for all nonnegative, nonincreasing sequences in $\ell^\infty(\mathbb{R}^+)$. Moreover, both inequalities (3.6) and (3.7) are sharp.

Motivated by these works, the goal of this chapter is to determine some of the constants $A(p)$, $B(p)$, $U(p)$, and $V(p)$ in the inequalities

$$A(p) \|(C - I)x\|_p \leq \|x\|_p \leq B(p) \|(C - I)x\|_p, \quad (3.8)$$

and

$$U(p) \|(C - S)x\|_p \leq \|x\|_p \leq V(p) \|(C - S)x\|_p, \quad (3.9)$$

on the three cones $\ell^p(\mathbb{N})$, $\ell_+^p(\mathbb{N})$, and $\ell_{\text{dec}}^p(\mathbb{N})$, for the whole range $1 \leq p \leq \infty$.

The main results of this chapter have been published in [11, 12].

Chapter Overview

This chapter is divided into three sections. The first two focus on the study of inequalities (3.8) and (3.9) on $\ell^p(\mathbb{N})$ and $\ell_+^p(\mathbb{N})$ for the full range $1 < p < \infty$. The study of inequalities (3.8) and (3.9) on $\ell_{\text{dec}}^p(\mathbb{N})$ for $1 < p < \infty$ is essentially complete and the main results are given in Theorem 3.1, Theorem 3.4, and Theorem 3.7. The third section is devoted to the analysis of (3.8) and (3.9) on various cones of $\ell^p(\mathbb{N})$ at the endpoints $p \in \{1, \infty\}$. In the final section, we present in tables all optimal constants related to the operators $C - I$ and $C - S$, indicating which constants were established in this chapter.

• 3.1 Inequalities in $\ell^p(\mathbb{N})$ for $1 < p < \infty$	24
• 3.2 Inequalities in $\ell_+^p(\mathbb{N})$ for $1 < p < \infty$	29
• 3.3 Endpoint inequalities	35
• 3.4 Tables	40

3.1 Inequalities in $\ell^p(\mathbb{N})$ for $1 < p < \infty$

Let $A(p)$, $B(p)$, $U(p)$, and $V(p)$ denote the optimal constants for which the inequalities

$$A(p) \|(C - I)x\|_p \leq \|x\|_p \leq B(p) \|(C - I)x\|_p, \quad (3.10)$$

and

$$U(p) \|(C - S)x\|_p \leq \|x\|_p \leq V(p) \|(C - S)x\|_p, \quad (3.11)$$

hold for all $x \in \ell^p(\mathbb{N})$ and for the full range $1 < p < \infty$.

Regarding the first inequality in (3.10), we have seen in (3.1), that

$$A(p) = \begin{cases} p - 1, & \text{if } 1 < p \leq 2, \\ m_p^{1/p}, & \text{if } 2 \leq p < \infty, \end{cases}$$

where m_p is defined in (2.5).

Although the exact value of $B(p)$ in (3.10) is known only for $p = 2$, Theorem 3.2 implies that

$$B(2) = \sqrt{2}.$$

Our next goal is to determine the exact value of $B(p)$ in (3.10) for $1 < p < 2$. Let S^* be the operator defined by

$$S^*x(n) = x(n-1),$$

where we set $S^*x(1) = 0$. The following lemma will be needed.

Lemma 3.8. *The following identities holds:*

$$(C^* - I)S^*(C - S) = I, \quad (3.12)$$

and

$$(C - I)(C - S)^* = S^*. \quad (3.13)$$

Proof. The operator C^* can be expressed as (see [9])

$$C^* = (C^* - I)S^*C. \quad (3.14)$$

Let $e_1 = (1, 0, 0, \dots)$. Since $(C^* - I)e_1 = 0$, we have

$$(C^* - I)S^*S = (C^* - I). \quad (3.15)$$

Combining (3.14) and (3.15), we obtain

$$(C^* - I)S^*(C - S) = (C^* - I)S^*C - (C^* - I)S^*S = C^* - (C^* - I) = I,$$

which proves (3.12).

As for (3.13), it follows from $CC^* = CS^* + C^*$ (see [9]), that

$$(C - I)(C - S)^* = (C - I)(C^* - S^*) = CC^* - CS^* - C^* + S^* = S^*.$$

This completes the proof. □

The following result will be needed.

Proposition 3.9. *Let $1 \leq p \leq 2$, and let $x \in \ell^p(\mathbb{N})$. Then*

$$\|(C - S)^*x\|_p \leq 2^{1/p}\|x\|_p. \quad (3.16)$$

Moreover, the constant is the best possible.

Proof. Let $1 \leq p \leq 2$. First of all, observe that by considering the sequence e_1 , we have

$$(C - S)^*e_1 = (1, -1, 0, 0, \dots).$$

It follows that

$$\frac{\|(C - S)^*e_1\|_p}{\|e_1\|_p} = 2^{1/p}.$$

This establishes the optimality of the constant in (3.16).

Now, on the one hand, it is well known (see [24]) that $\|C^*\|_1 = 1$. On the other hand, we trivially have $\|S^*\|_1 = 1$. Therefore, for all $x \in \ell^1(\mathbb{N})$,

$$\begin{aligned} \|(C - S)^*x\|_1 &= \|C^*x - S^*x\|_1 \leq \|C^*x\|_1 + \|S^*x\|_1 \\ &\leq (\|C^*\|_1 + \|S^*\|_1) \|x\|_1 = 2 \|x\|_1. \end{aligned} \quad (3.17)$$

For the case $p = 2$, it was shown in [29] that

$$\|(C - S)^*x\|_2 \leq \sqrt{2} \|x\|_2, \quad x \in \ell^2(\mathbb{N}). \quad (3.18)$$

Now, let $1 < p < 2$. We have, by taking $\theta = 2(1 - 1/p)$, that

$$\frac{1 - \theta}{1} + \frac{\theta}{2} = 1 - 2\left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{p}\right) = 1 - \left(1 - \frac{1}{p}\right) = \frac{1}{p}.$$

By applying the Riesz-Thorin theorem [7, Chapter 4, Corollary 2.3] and taking into account (3.17) and (3.18), we obtain that

$$\|(C - S)^*\|_p \leq \|(C - S)^*\|_1^{1-\theta} \|(C - S)^*\|_2^\theta = 2^{1-\theta} 2^{\theta/2} = 2^{1/p}.$$

□

The next theorem gives us the exact value of $B(p)$ in (3.10) for $1 < p \leq 2$. In particular, it extends Theorem 3.2.

Theorem 3.10. *Let $1 < p \leq 2$, and let x be an arbitrary sequence. Then the following inequality is sharp:*

$$\|x\|_p \leq 2^{1/p} \|(C - I)x\|_p. \quad (3.19)$$

Proof. Recall the equality (3.12):

$$(C^* - I)S^*(C - S) = I.$$

Taking the transpose of this identity yields

$$(C - S)^*S(C - I) = I. \quad (3.20)$$

Now, let x be an arbitrary sequence. If $(C - I)x \notin \ell^p(\mathbb{N})$, then (3.19) holds trivially. Otherwise, assuming that the sequence $(C - I)x \in \ell^p(\mathbb{N})$, it follows that $S(C - I)x \in \ell^p(\mathbb{N})$. Using (3.20), together with (3.16) and the fact that $\|S\|_p = 1$, we obtain

$$\|x\|_p = \|(C - S)^*S(C - I)x\|_p \leq \|(C - S)^*\|_p \|S\|_p \|(C - I)x\|_p = 2^{1/p} \|(C - I)x\|_p.$$

It remains to show that the constant $2^{1/p}$ in (3.19) is optimal. To this end, consider the sequence $x = (1, -1, 0, 0, \dots)$. Then

$$(C - I)x = (1, 0, 0, \dots),$$

and hence

$$\frac{\|x\|_p}{\|(C - I)x\|_p} = 2^{1/p}.$$

This completes the proof. □

Remark 3.11. From (3.1) and (3.19), it follows that

$$(p-1) \|(C-I)x\|_p \leq \|x\|_p \leq 2^{1/p} \|(C-I)x\|_p$$

for all $x \in \ell^p(\mathbb{N})$, $1 < p \leq 2$. It is now easy to see that $\|(C-I)x\|_p$ is an equivalent norm in $\ell^p(\mathbb{N})$.

Now, we pass to the second inequality in (3.11). Let M_p be as defined in (2.5). We will need the norm of the operator $C^* - I$ acting on $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$. By applying duality to (3.1) we obtain

$$\begin{aligned} \|C^* - I\|_p &= \|C - I\|_{p'} = \begin{cases} 2, & \text{if } p = 1, \\ m_{p'}^{-1/p'}, & \text{if } 1 < p \leq 2, \\ \frac{1}{p'-1}, & \text{if } 2 \leq p < \infty, \end{cases} \\ &= \begin{cases} 2, & \text{if } p = 1, \\ M_p^{1/p}, & \text{if } 1 < p \leq 2, \\ p-1, & \text{if } 2 \leq p < \infty, \end{cases} \end{aligned} \quad (3.21)$$

where we have used the fact that $\frac{1}{p'-1} = p-1$ and the identity $m_{p'}^{-1/p'} = M_p^{1/p}$ from (2.6).

The following theorem provides the best constant $V(p)$ in the inequality (3.11), for all $1 < p < \infty$.

Theorem 3.12. *Let $1 < p < \infty$ and let x be an arbitrary sequence. If $1 < p \leq 2$, then*

$$\|x\|_p \leq M_p^{1/p} \|(C-S)x\|_p, \quad (3.22)$$

and if $2 \leq p < \infty$, then

$$\|x\|_p \leq (p-1) \|(C-S)x\|_p. \quad (3.23)$$

Furthermore, the constants $M_p^{1/p}$ and $(p-1)$ are optimal in (3.22) and (3.23), respectively.

Proof. Let $1 < p < \infty$. We are going to prove only (3.22), since the proof of (3.23) is analogous. Assume that $1 < p \leq 2$, and let x be an arbitrary sequence. If $(C-S)x \notin \ell^p(\mathbb{N})$, then (3.22) holds trivially. Otherwise, assuming that the sequence $(C-S)x \in \ell^p(\mathbb{N})$, it follows that $S^*(C-S)x \in \ell^p(\mathbb{N})$. Using (3.12), together with (3.21) and the fact that $\|S^*\|_p = 1$, we obtain

$$\|x\|_p = \|(C^* - I)S^*(C-S)x\|_p \leq \|C^* - I\|_p \|S^*\|_p \|(C-S)x\|_p = M_p^{1/p} \|(C-S)x\|_p.$$

What remains is to verify the optimality of the constants in (3.22) and (3.23). We start with $M_p^{1/p}$ in (3.22). To show that (3.22) is sharp, we can simply use the exactness of the value from [44]. Let $0 < \varepsilon < 1$ and let y such that

$$\|(C^* - I)y\|_p > (1 - \varepsilon)M_p^{1/p}\|y\|_p.$$

This y can be chosen to be finitely non-zero, since such sequences are dense in $\ell^p(\mathbb{N})$, where $m \geq 1$ is a natural number. Note that we can also assume that $y(1) = 0$, since $(C^* - I)e_1 = 0$. If we can find x such that $S^*(C - S)x = y$, then $x = (C^* - I)y$ and $\|y\|_p = \|(C - S)x\|_p$, so

$$\|x\|_p \geq (1 - \varepsilon)M_p^{1/p}\|(C - S)x\|_p.$$

This will follow if for each $m \geq 1$, there exists x_m such that $(C - S)x_m = e_m$, hence $S^*(C - S)x_m = e_{m+1}$. This is achieved by

$$x_m(n) = \begin{cases} \frac{1}{m+1}, & \text{if } 1 \leq n \leq m, \\ -\frac{m}{m+1}, & \text{if } n = m+1, \\ 0, & \text{if } n > m+1. \end{cases}$$

Regarding the sharpness of the constant $(p - 1)$ in (3.23), observe that since $(p - 1)$ is optimal in (3.5), there exists a sequence $\{x_m\}_{m \geq 1} \subset \ell^p(\mathbb{N})$ of nonnegative, nonincreasing sequences such that

$$\lim_{m \rightarrow \infty} \frac{\|x_m\|_p}{\|(C - S)x_m\|_p} = p - 1,$$

for all $2 \leq p < \infty$. □

The following result provides the exact value of $U(p)$ in (3.11) for $2 \leq p < \infty$. In particular, it extends Theorem 3.3, which asserts that $U(2) = 1/\sqrt{2}$.

Theorem 3.13. *Let $2 \leq p < \infty$ and let $x \in \ell^p(\mathbb{N})$. Then the following inequality is sharp:*

$$2^{\frac{1-p}{p}} \|(C - S)x\|_p \leq \|x\|_p. \quad (3.24)$$

Proof. Let $2 \leq p < \infty$. By duality, we have

$$\|C - S\|_p = \|(C - S)^*\|_{p'} = 2^{1/p'},$$

where we have used the fact that

$$\|(C - S)^*\|_p = 2^{1/p} \quad \text{for all } 1 < p \leq 2,$$

as established in Proposition 3.9. □

Remark 3.14. From (3.23) and (3.24), it follows that

$$2^{\frac{1-p}{p}} \|(C - S)x\|_p \leq \|x\|_p \leq (p - 1) \|(C - S)x\|_p$$

for all $x \in \ell^p(\mathbb{N})$, $2 \leq p < \infty$. It is now easy to see that $\|(C - S)x\|_p$ is an equivalent norm in $\ell^p(\mathbb{N})$.

3.2 Inequalities in $\ell_+^p(\mathbb{N})$ for $1 < p < \infty$

Let $A(p)$, $B(p)$, $U(p)$, and $V(p)$ denote the optimal constants for which the inequalities

$$A(p) \|(C - I)x\|_p \leq \|x\|_p \leq B(p) \|(C - I)x\|_p, \quad (3.25)$$

$$U(p) \|(C - S)x\|_p \leq \|x\|_p \leq V(p) \|(C - S)x\|_p \quad (3.26)$$

hold for all $x \in \ell_+^p(\mathbb{N})$ and for the entire range $1 < p < \infty$.

We begin with a result that yields the exact value of $A(p)$ in (3.25) for $1 < p \leq 2$ and the exact value of $V(p)$ in (3.26) for $2 \leq p < \infty$.

Theorem 3.15. *Let $1 < p < \infty$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative sequence. Then*

$$(p - 1) \|(C - I)x\|_p \leq \|x\|_p, \quad (3.27)$$

if $1 < p \leq 2$, and

$$\|x\|_p \leq (p - 1) \|(C - S)x\|_p, \quad (3.28)$$

if $2 \leq p < \infty$. Moreover, the constant $(p - 1)$ is optimal in both inequalities (3.27) and (3.28).

Proof. We start with the case $1 < p \leq 2$. By (3.1), the following inequality is sharp for this range of p :

$$(p - 1) \|(C - I)x\|_p \leq \|x\|_p,$$

for all $x \in \ell^p(\mathbb{N})$. In [20], S. Boza and J. Soria showed that the same inequality holds for all nonnegative, nonincreasing sequences $x \in \ell^p(\mathbb{N})$, and that $(p - 1)$ is the best possible constant in this cone. Combining these results and applying (2.1), we conclude that the inequality (3.27) holds and that the constant $p - 1$ is optimal in the cone of nonnegative sequences in $\ell^p(\mathbb{N})$.

For the case $2 \leq p < \infty$, we have already seen in (3.23) that the following inequality is sharp:

$$\|x\|_p \leq (p - 1) \|(C - S)x\|_p,$$

for all $x \in \ell^p(\mathbb{N})$. Additionally, (3.5) shows that the same inequality is sharp:

$$\|x\|_p \leq (p - 1) \|(C - S)x\|_p,$$

for all nonnegative, nonincreasing $x \in \ell^p(\mathbb{N})$. By combining the results from (3.5) and (3.23) and taking into account (2.2), we obtain that the inequality (3.28) holds and is sharp. \square

Before proceeding with the proofs of the remaining inequalities, we first state two useful lemmas.

Lemma 3.16. *Let $1 < p < \infty$ and $x = \{x(n)\}_{n \geq 1} \in \ell^p(\mathbb{N})$. Then*

$$\lim_{n \rightarrow \infty} n|Cx(n)|^p = 0.$$

Proof. Let $u = \{u(n)\}_{n \geq 1} \in \ell^p(\mathbb{N})$ with finite support. Then there exists $M \in \mathbb{N}$ such that

$$u(n) = 0,$$

for all $n \geq M$. Thus, for all $n > M$, we have

$$nCu(n) = u(1) + u(2) + \cdots + u(M).$$

Therefore

$$n|Cu(n)| \leq |u(1)| + |u(2)| + \cdots + |u(M)| = K, \quad \forall n > M.$$

Hence

$$n|Cu(n)|^p \leq \frac{K^p}{n^{p-1}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Now, let $x = \{x(n)\}_{n \geq 1} \in \ell^p(\mathbb{N})$ be an arbitrary real sequence and $\epsilon > 0$. We aim to show that there exists $N \geq 1$ such that

$$n|Cx(n)|^p < \epsilon, \quad \forall n \geq N.$$

Since finite sequences are dense in $\ell^p(\mathbb{N})$, there exists a sequence $u \in \ell^p(\mathbb{N})$ with finite support such that

$$\|x - u\|_p < \frac{\epsilon^{1/p}}{2^{1+1/p}}. \quad (3.29)$$

Using Hölder's inequality, we obtain

$$\begin{aligned} |C(x - u)(n)| &= \frac{1}{n} \left| \sum_{k=1}^n (x - u)(k) \right| \leq \frac{1}{n} \sum_{k=1}^n |(x - u)(k)| \\ &\leq \frac{1}{n} \left(\sum_{k=1}^n |(x - u)(k)|^p \right)^{1/p} \left(\sum_{k=1}^n 1 \right)^{1/q} \\ &= \left(\sum_{k=1}^n |(x - u)(k)|^p \right)^{1/p} n^{-1/p}, \quad \forall n \geq 1. \end{aligned}$$

Equivalently,

$$n^{1/p}|C(x - u)(n)| \leq \left(\sum_{k=1}^n |(x - u)(k)|^p \right)^{1/p} \leq \|x - u\|_p, \quad \forall n \geq 1.$$

So,

$$n|C(x-u)(n)|^p \leq \|x-u\|_p^p < \frac{\epsilon}{2^{p2}}, \quad \forall n \geq 1, \quad (3.30)$$

where we have used (3.29) in the last inequality. Since u has finite support, there exists $N_0 \geq 1$ such that

$$n|Cu(n)|^p < \frac{\epsilon}{2^{p2}}, \quad \forall n \geq N_0. \quad (3.31)$$

Now, for $n \geq 1$, we have

$$\begin{aligned} |Cx(n)| &= |Cu(n) + C(x-u)(n)| \leq |Cu(n)| + |C(x-u)(n)| \\ &\leq 2 \max\{|Cu(n)|, |C(x-u)(n)|\}, \end{aligned}$$

which implies

$$|Cx(n)|^p \leq 2^p \max\{|Cu(n)|, |C(x-u)(n)|\}^p \leq 2^p(|Cu(n)|^p + |C(x-u)(n)|^p).$$

Thus

$$n|Cx(n)|^p \leq 2^p n|Cu(n)|^p + 2^p n|C(x-u)(n)|^p, \quad \forall n \geq 1.$$

Letting $N = N_0$ and applying (3.30) and (3.31) for all $n \geq N_0$, we obtain

$$n|Cx(n)|^p \leq 2^p \cdot \frac{\epsilon}{2^{p2}} + 2^p \cdot \frac{\epsilon}{2^{p2}} = \epsilon,$$

which completes the proof. \square

Lemma 3.17. *Let $p \in \mathbb{F} \cup \mathbb{E}$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative sequence. Then*

$$\|x\|_p \leq \|(C-S)x\|_p, \quad (3.32)$$

if $p \in \mathbb{F}$, and

$$\|(C-I)x\|_p \leq \|x\|_p, \quad (3.33)$$

if $p \in \mathbb{E}$.

Proof. We start with the case $p \in \mathbb{F}$. Let $x = \{x(n)\}_{n \geq 1} \in \ell^p(\mathbb{N})$ and let $\{y(n)\}_{n \geq 1}$ be defined by $y(n) = Cx(n)$, for all $n \geq 1$. Clearly,

$$Sx(n) = x(n+1) = (n+1)y(n+1) - ny(n), \quad \forall n \geq 1,$$

and,

$$z(n) = y(n) - Sx(n) = (n+1)(y(n) - y(n+1)), \quad \forall n \geq 1.$$

Let $n \in \mathbb{N}$, by applying (2.3) we get that,

$$(n+1)(y(n)^p - y(n+1)^p) \leq (n+1)py(n)^{p-1}(y(n) - y(n+1)).$$

Thus,

$$\begin{aligned}
ny(n)^p - (n+1)y(n+1)^p &\leq (n+1)py(n)^{p-1}(y(n) - y(n+1)) - y(n)^p \\
&= py(n)^{p-1}(y(n) - Sx(n)) - y(n)^p \\
&= py(n)^p - py(n)^{p-1}Sx(n) - y(n)^p \\
&= (p-1)y(n)^p - py(n)^{p-1}Sx(n).
\end{aligned}$$

If $Sx(n) \neq 0$ we may let $t = y(n)/Sx(n)$ and apply the Lemma 2.6 in the form,

$$(p-1)t^p - pt^{p-1} \leq (t-1)^p - 1,$$

to get,

$$\begin{aligned}
(p-1)y(n)^p - py(n)^{p-1}Sx(n) &= Sx(n)^p((p-1)t^p - pt^{p-1}) \\
&\leq Sx(n)^p((t-1)^p - 1) \\
&= (y(n) - Sx(n))^p - Sx(n)^p.
\end{aligned}$$

If $Sx(n) = 0$, the same inequality holds as $(p-1)y(n)^p \leq y(n)^p$. Therefore,

$$(p-1)y(n)^p - py(n)^{p-1}Sx(n) \leq (y(n) - Sx(n))^p - Sx(n)^p, \quad \forall n \geq 1.$$

Summing over n , we have,

$$\begin{aligned}
y(1)^p - (N+1)y(N+1)^p &= \sum_{n=1}^N ny(n)^p - (n+1)y(n+1)^p \\
&\leq \sum_{n=1}^N (y(n) - Sx(n))^p - Sx(n)^p.
\end{aligned}$$

Letting $N \rightarrow \infty$ and using the Lemma 3.16, we have,

$$0 \leq y(1)^p \leq \sum_{n=1}^{\infty} (y(n) - Sx(n))^p - \sum_{n=1}^{\infty} Sx(n)^p.$$

Equivalently,

$$\sum_{n=1}^{\infty} x(n)^p \leq \sum_{n=1}^{\infty} (Cx(n) - Sx(n))^p.$$

Now, we turn to the case when $p \in \mathbb{E}$. We have that

$$x(n) = ny(n) - (n-1)y(n-1), \quad \text{for every } n \geq 1,$$

and

$$z(n) = y(n) - x(n) = (n-1)(y(n-1) - y(n)), \quad \text{for all } n \geq 1.$$

Let $n \in \mathbb{N}$. By applying (2.3) we get that,

$$(n-1)(y(n)^p - y(n-1)^p) \leq (n-1)py(n)^{p-1}(y(n) - y(n-1)).$$

Thus

$$\begin{aligned}
ny(n)^p - (n-1)y(n-1)^p &\leq y(n)^p + (n-1)py(n)^{p-1}(y(n) - y(n-1)) \\
&= y(n)^p - py(n)^{p-1}z(n) \\
&= y(n)^p - py(n)^{p-1}(y(n) - x(n)) \\
&= y(n)^p - py(n)^p + py(n)^{p-1}x(n) \\
&= py(n)^{p-1}x(n) - (p-1)y(n)^p.
\end{aligned}$$

If $x(n) \neq 0$ we may let $t = y(n)/x(n)$ and apply the Lemma 2.6 in the form

$$pt^{p-1} - (p-1)t^p \leq 1 - (t-1)^p,$$

to get

$$\begin{aligned}
py(n)^{p-1}x(n) - (p-1)y(n)^p &= x(n)^p(pt^{p-1} - (p-1)t^p) \\
&\leq x(n)^p(1 - (t-1)^p) \\
&= x(n)^p - (y(n) - x(n))^p.
\end{aligned}$$

If $x(n) = 0$, the same inequality holds. Therefore

$$ny(n)^p - (n-1)y(n-1)^p \leq x(n)^p - (y(n) - x(n))^p, \quad \text{for every } n \geq 1.$$

Summing over n , we get

$$Ny(N)^p = \sum_{n=1}^N ny(n)^p - (n-1)y(n-1)^p \leq \sum_{n=1}^N x(n)^p - \sum_{n=1}^N (y(n) - x(n))^p.$$

Letting $N \rightarrow \infty$ and by taking into account Lemma 3.16, we obtain

$$\sum_{n=1}^{\infty} (y(n) - x(n))^p \leq \sum_{n=1}^{\infty} x(n)^p.$$

□

Next, we extend Lemma 3.17 from $p \in \mathbb{F} \cup \mathbb{E}$ to all $p \in (1, \infty)$.

Theorem 3.18. *Let $1 < p < \infty$, and let $x \in \ell^p(\mathbb{N})$ be a nonnegative sequence. Then*

$$\|x\|_p \leq \|(C - S)x\|_p, \quad (3.34)$$

if $1 < p \leq 2$, and

$$\|(C - I)x\|_p \leq \|x\|_p, \quad (3.35)$$

if $2 < p < \infty$. Moreover, the constant 1 is optimal in both inequalities (3.34) and (3.35).

Proof. Let $p > 2$. By a density argument, we can restrict ourselves to prove the inequality for nonnegative sequences with finite support in $\ell^p(\mathbb{N})$. Let x be such a sequence. We have that $x \in \ell^q(\mathbb{N})$, for all $2 < q < \infty$ and $\|x\|_{\ell^q(\mathbb{N})}$ is a continuous function in the interval $(2, \infty)$ with respect to q . Let y be the sequence given by $y(n) = Cx(n) - x(n)$, for all $n \in \mathbb{N}$. Since $C - I$ is a bounded operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$, we have that

$$\|y\|_q \leq \|y\|_2 = \|Cx - x\|_2 \leq \|x\|_2 < \infty,$$

for all $2 < q < \infty$. Thus the map $q \mapsto \|y\|_q$ is well defined and continuous in $(2, \infty)$. Let $\{p(n)\}_{n \geq 1} \subset \mathbb{E}$ be a sequence such that $\lim_{n \rightarrow \infty} p(n) = p$. Then, by using (3.33), we get

$$\|Cx - x\|_p = \lim_{n \rightarrow \infty} \|Cx - x\|_{p(n)} \leq \lim_{n \rightarrow \infty} \|x\|_{p(n)} = \|x\|_p.$$

To prove (3.34), we proceed as we did in (3.35) by using (3.32) instead of (3.33).

It remains to verify that the constant in both (3.34) and (3.35) is optimal. To this end, consider the sequence defined by

$$x^m(n) = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

It is easy to see that $\|x^m\|_p = 1$ for all $m \geq 1$. A straightforward computation yields

$$|Cx^m(n) - x^m(n)|^p = \begin{cases} 0, & \text{if } n < m, \\ (1 - \frac{1}{n})^p, & \text{if } n = m, \\ \frac{1}{n^p}, & \text{if } n > m. \end{cases}$$

Hence

$$\frac{\|Cx^m - x^m\|_p^p}{\|x^m\|_p^p} = \left(1 - \frac{1}{m}\right)^p + \sum_{n=m+1}^{\infty} \frac{1}{n^p}.$$

Taking the limit as $m \rightarrow \infty$ gives

$$\lim_{m \rightarrow \infty} \frac{\|Cx^m - x^m\|_p^p}{\|x^m\|_p^p} = \lim_{m \rightarrow \infty} \left(1 - \frac{1}{m}\right)^p + \sum_{n=m+1}^{\infty} \frac{1}{n^p} = 1,$$

which shows that the constant 1 in (3.35) is indeed sharp.

Similarly, we compute

$$|Cx^m(n) - Sx^m(n)|^p = \begin{cases} 0, & \text{if } n < m - 1, \\ 1, & \text{if } n = m - 1, \\ \frac{1}{n^p}, & \text{if } n \geq m, \end{cases}$$

which leads to

$$\frac{\|x^m\|_p^p}{\|Cx^m - Sx^m\|_p^p} = \frac{1}{1 + \sum_{n=m}^{\infty} \frac{1}{n^p}}.$$

Taking the limit as $m \rightarrow \infty$, we obtain

$$\lim_{m \rightarrow \infty} \frac{\|x^m\|_p^p}{\|Cx^m - Sx^m\|_p^p} = 1.$$

This confirms that the constant 1 in (3.34) is also optimal, completing the proof. \square

3.3 Endpoint inequalities

Let $A(p)$, $B(p)$, $U(p)$, and $V(p)$ denote the optimal constants for which the inequalities

$$A(p) \|(C - I)x\|_p \leq \|x\|_p \leq B(p) \|(C - I)x\|_p,$$

$$U(p) \|(C - S)x\|_p \leq \|x\|_p \leq V(p) \|(C - S)x\|_p,$$

hold for all $x \in \ell^p(\mathbb{N})$, $\ell_+^p(\mathbb{N})$, or $\ell_{\text{dec}}^p(\mathbb{N})$ when $p = 1$ or $p = \infty$.

By (3.1), we have $\|C - I\|_{\ell^\infty(\mathbb{N})} = 2$. However, on the cone of positive, decreasing sequences we have seen in (3.6) and in (3.7) that

$$\|C - I\|_{\ell_{\text{dec}}^\infty(\mathbb{N})} = \|C - S\|_{\ell_{\text{dec}}^\infty(\mathbb{N})} = 1.$$

For the reader's convenience, all the estimates for $\|C - I\|_{\ell^\infty(\mathbb{N})}$, $\|C - S\|_{\ell_{\text{dec}}^\infty(\mathbb{N})}$, and $\|C - I\|_{\ell_{\text{dec}}^\infty(\mathbb{N})}$ mentioned above are included in the following theorem, which forms the main result of this section.

Theorem 3.19. *The following equalities hold:*

$$\|C - I\|_{\ell_{\text{dec}}^\infty(\mathbb{N})} = \|C - I\|_{\ell_+^\infty(\mathbb{N})} = 1, \quad \|C - I\|_{\ell^\infty(\mathbb{N})} = 2, \quad (3.36)$$

$$\|C - I\|_{\ell_{\text{dec}}^1(\mathbb{N})} = \|C - I\|_{\ell_+^1(\mathbb{N})} = \|C - I\|_{\ell^1(\mathbb{N})} = \infty, \quad (3.37)$$

$$[C - I]_{\ell_{\text{dec}}^1(\mathbb{N})} = [C - I]_{\ell_+^1(\mathbb{N})} = 0, \quad [C - I]_{\ell^1(\mathbb{N})} = 2, \quad (3.38)$$

$$[C - I]_{\ell_{\text{dec}}^\infty(\mathbb{N})} = [C - I]_{\ell_+^\infty(\mathbb{N})} = [C - I]_{\ell^\infty(\mathbb{N})} = \infty, \quad (3.39)$$

$$\|C - S\|_{\ell_{\text{dec}}^1(\mathbb{N})} = \|C - S\|_{\ell_+^1(\mathbb{N})} = \|C - S\|_{\ell^1(\mathbb{N})} = \infty, \quad (3.40)$$

$$\|C - S\|_{\ell_{\text{dec}}^\infty(\mathbb{N})} = \|C - S\|_{\ell_+^\infty(\mathbb{N})} = 1, \quad \|C - S\|_{\ell^\infty(\mathbb{N})} = 2, \quad (3.41)$$

$$[C - S]_{\ell_{\text{dec}}^\infty(\mathbb{N})} = [C - S]_{\ell_+^\infty(\mathbb{N})} = [C - S]_{\ell^\infty(\mathbb{N})} = \infty, \quad (3.42)$$

$$[C - S]_{\ell_{\text{dec}}^1(\mathbb{N})} = [C - S]_{\ell_+^1(\mathbb{N})} = 0, \quad [C - S]_{\ell^1(\mathbb{N})} = 2. \quad (3.43)$$

Proof. Proof of (3.36). Let $x = \{x(n)\}_{n \geq 1} \in \ell^\infty(\mathbb{N})$ be a nonnegative sequence with $\|x\|_\infty = 1$. It is easy to see that for every $a, b \geq 0$ we have that

$$|a - b| \leq \max\{a, b\}.$$

Hence,

$$|Cx(n) - x(n)| \leq \max\{|Cx(n)|, |x(n)|\},$$

for all $n \geq 1$. Thus

$$\|Cx - x\|_{\ell^\infty} \leq \max\{\|Cx\|_{\ell^\infty}, \|x\|_{\ell^\infty}\} \leq \|x\|_{\ell^\infty} = 1,$$

since $\|C\|_\infty = 1$ (see [24]). This implies that

$$\|Cx - x\|_{\ell^\infty} \leq 1. \quad (3.44)$$

In order to see that (3.44) is sharp, let $m \in \mathbb{N}$ and consider the nonnegative sequence $y^m = \{y^m(n)\}_{n \geq 1}$ given by

$$y^m(n) = \begin{cases} 1, & \text{if } n \leq m, \\ 0, & \text{if } n > m. \end{cases}$$

So,

$$Cy^m(n) = \begin{cases} 1, & \text{if } n \leq m, \\ \frac{m}{n}, & \text{if } n > m. \end{cases}$$

Therefore

$$|Cy^m(n) - y^m(n)| = \begin{cases} 0, & \text{if } n \leq m, \\ \frac{m}{n}, & \text{if } n > m. \end{cases}$$

Finally,

$$\frac{\|Cy^m - y^m\|_{\ell^\infty}}{\|y^m\|_{\ell^\infty}} = \frac{m}{m+1} \rightarrow 1,$$

as $m \rightarrow \infty$.

Proof of (3.37). Consider the sequence e_1 which is nonnegative, nonincreasing, and satisfies $\|e_1\|_1 = 1$. A straightforward computation shows

$$|(C - I)e_1(n)| = \begin{cases} 0, & \text{if } n = 1, \\ \frac{1}{n}, & \text{if } n > 1. \end{cases}$$

Note that $(C - S)e_1 \notin \ell^1(\mathbb{N})$. This shows that $\|C - I\|_{\ell^1_{\text{dec}}(\mathbb{N})} = \infty$ and, using (2.1), we obtain (3.37).

Proof of (3.38). The proofs of $[C - I]_{\ell^1_{\text{dec}}(\mathbb{N})} = [C - I]_{\ell^1_+(\mathbb{N})} = 0$ have analogous arguments to those of $[C - S]_{\ell^1_{\text{dec}}(\mathbb{N})} = [C - S]_{\ell^1_+(\mathbb{N})} = 0$ in (3.43), while the proof of $[C - I]_{\ell^1(\mathbb{N})} = 2$ is similar to that of the inequality (3.19).

Proof of (3.39). Consider the same sequence z_ϵ from (3.46). As observed in (3.47), we have $\|z_\epsilon\|_\infty = 1$ for all $0 < \epsilon < 1$, and from (3.48), applying the Cesàro operator to z_ϵ gives $Cz_\epsilon(n) = n^{-\epsilon}$, for all $n \geq 1$.

Hence, we can write

$$(C - I)z_\epsilon(n) = n^{-\epsilon} - n^{1-\epsilon} + (n-1)^{1-\epsilon} = (n-1)((n-1)^{-\epsilon} - n^{-\epsilon}),$$

for all $n \geq 1$. Define the function $h_\epsilon : [1, \infty) \rightarrow \mathbb{R}$ by

$$h_\epsilon(t) = (t-1)((t-1)^{-\epsilon} - t^{-\epsilon}).$$

Note that h_ϵ is nonnegative, satisfies $h_\epsilon(1) = 0$, and $\lim_{t \rightarrow \infty} h_\epsilon(t) = 0$. Therefore, there exists $t_0 > 1$ such that $h_\epsilon(t) \leq h_\epsilon(t_0)$ for all $t \geq 1$. In particular, this gives $0 \leq \|(C - I)z_\epsilon\|_\infty \leq h_\epsilon(t_0) = (t_0 - 1)((t_0 - 1)^{-\epsilon} - t_0^{-\epsilon}) \rightarrow 0$ as $\epsilon \rightarrow 0^+$. Thus, $\lim_{\epsilon \rightarrow 0^+} \|(C - I)z_\epsilon\|_\infty = 0$. By Lemma 2.2, this shows that $(C - I)$ is not bounded below on $\ell^\infty(\mathbb{N})$. Since each z_ϵ is nonnegative and nonincreasing, it follows that $[C - I]_{\ell^\infty(\mathbb{N})} = \infty$. Finally, combining this with (2.2) yields (3.39).

Proof of (3.40). Consider the sequence e_1 which is nonnegative, nonincreasing, and satisfies $\|e_1\|_1 = 1$. A straightforward computation shows

$$|(C - S)e_1(n)| = \frac{1}{n},$$

for all $n \geq 1$. Note that $(C - S)x \notin \ell^1(\mathbb{N})$. This shows that $\|C - S\|_{\ell^1_{\text{dec}}(\mathbb{N})} = \infty$ and, using (2.1), we obtain (3.40).

Proof of (3.41). The equality $\|C - S\|_{\ell^\infty(\mathbb{N})} = 2$ follows from the fact that $\|(C - S)^*\|_{\ell^1(\mathbb{N})} = 2$ in (3.16), together with a standard duality argument.

The corresponding inequality for $\|C - S\|_{\ell^{\infty}_+(\mathbb{N})} = 1$ can be established analogously to the proof of (3.44), and the optimality of the constant 1 follows from the fact that $\|(C - S)e_1\|_\infty = 1$.

Proof of (3.42). We start with the equality

$$[C - S]_{\ell^\infty_{\text{dec}}(\mathbb{N})} = \infty. \quad (3.45)$$

Let $0 < \epsilon < 1$, and define the sequence $z_\epsilon = \{z_\epsilon(n)\}_{n \geq 1}$ by

$$z_\epsilon(n) = n^{1-\epsilon} - (n-1)^{1-\epsilon}. \quad (3.46)$$

Consider the function

$$f(t) = t^{1-\epsilon} - (t-1)^{1-\epsilon}.$$

Its derivative is

$$f'(t) = (1 - \epsilon)(t^{-\epsilon} - (t-1)^{-\epsilon}) < 0, \quad \text{for all } t > 1,$$

since $0 < \epsilon < 1$. Moreover, using L'Hôpital's rule, we compute

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} t^{1-\epsilon} \left(1 - \left(1 - \frac{1}{t} \right)^{1-\epsilon} \right) = \lim_{t \rightarrow \infty} \frac{1 - \left(1 - \frac{1}{t} \right)^{1-\epsilon}}{1/t^{1-\epsilon}} = \lim_{t \rightarrow \infty} (t-1)^{-\epsilon} = 0.$$

Hence the function f is strictly decreasing on $[1, \infty)$, with

$$0 = \lim_{t \rightarrow \infty} f(t) < f(t) < f(1) = 1.$$

This shows that the sequence z is nonnegative, nonincreasing, and satisfies

$$\|z_\epsilon\|_\infty = 1. \quad (3.47)$$

Now consider the Cesàro operator C applied to z :

$$Cz_\epsilon(n) = \frac{1}{n} \sum_{k=1}^n (k^{1-\epsilon} - (k-1)^{1-\epsilon}) = n^{-\epsilon}. \quad (3.48)$$

Also, observe that

$$Sz_\epsilon(n) = z_\epsilon(n+1) = (n+1)^{1-\epsilon} - n^{1-\epsilon},$$

so that

$$(C - S)z_\epsilon(n) = n^{-\epsilon} - [(n+1)^{1-\epsilon} - n^{1-\epsilon}] = (n^{-\epsilon} - (n+1)^{-\epsilon})(n+1).$$

Define

$$g(t) = (t^{-\epsilon} - (t+1)^{-\epsilon})(t+1).$$

Clearly, $g(t) > 0$ for all $t > 0$. Moreover, its derivative is

$$g'(t) = -\epsilon t^{-\epsilon-1}(t+1) + (\epsilon-1)(t+1)^{-\epsilon} + t^{-\epsilon}.$$

Alternatively, using an integral representation:

$$g'(t) = -\epsilon t^{-\epsilon-1} - \epsilon(\epsilon-1) \int_t^{t+1} u^{-\epsilon-1} du \leq -\epsilon t^{-\epsilon-1} - \epsilon(\epsilon-1)t^{-\epsilon-1} = -\epsilon^2 t^{-\epsilon-1} < 0,$$

for all $t > 1$. Thus g is strictly decreasing on $(1, \infty)$ and its maximum is attained at $t = 1$:

$$\|(C - S)z_\epsilon\|_\infty = (C - S)z_\epsilon(1) = 2(1 - 2^{-\epsilon}) \rightarrow 0,$$

as $\epsilon \rightarrow 0^+$. By Lemma 2.2, this implies that $(C - S)$ is not bounded below on $\ell^\infty(\mathbb{N})$. Since z is nonnegative and nonincreasing, we get (3.45). Now, by taking into account (2.2), we obtain (3.42).

Proof of (3.43). We start with the equality

$$[C - S]_{\ell^1(\mathbb{N})} = 2. \quad (3.49)$$

Equivalently, we have to show that the following inequality is sharp

$$\|x\|_1 \leq 2\|(C - S)x\|_1, \quad (3.50)$$

for all $x \in \ell^1(\mathbb{N})$. Let x be an arbitrary sequence. If $(C - S)x \notin \ell^1(\mathbb{N})$, then (3.50) holds trivially. Otherwise, assuming that the sequence $(C - S)x \in \ell^1(\mathbb{N})$, it follows

that $S^*(C - S)x \in \ell^1(\mathbb{N})$. Note that there exist nonzero elements $x \in \ell^1(\mathbb{N})$ such that $(C - S)x \neq 0$; for instance, the sequences x_m defined below have this property. Now, by using (3.12), together with (3.21) and the fact that $\|S^*\|_1 = 1$, we obtain

$$\begin{aligned} \|x\|_1 &= \|(C^* - I)S^*(C - S)x\|_1 \leq \|C^* - I\|_1 \|S^*\|_1 \|(C - S)x\|_1 \\ &= \|C^* - I\|_1 \|(C - S)x\|_1 = 2\|(C - S)x\|_1. \end{aligned}$$

This proves the inequality (3.50). To establish the sharpness of the constant 2 in (3.50), we consider the sequence

$$x_m(n) = \begin{cases} 1, & \text{if } 1 \leq n \leq m, \\ -m, & \text{if } n = m + 1, \\ 0, & \text{if } n > m + 1, \end{cases}$$

where $m \geq 1$ is a natural number. Then

$$Sx_m(n) = \begin{cases} 1, & \text{if } 1 \leq n < m, \\ -m, & \text{if } n = m, \\ 0, & \text{if } n \geq m + 1, \end{cases}$$

and

$$Cx_m(n) = \begin{cases} 1, & \text{if } 1 \leq n \leq m, \\ 0, & \text{if } n \geq m + 1. \end{cases}$$

Thus

$$(C - S)x_m(n) = \begin{cases} 0, & \text{if } 1 \leq n < m, \\ 1 + m, & \text{if } n = m, \\ 0, & \text{if } n > m. \end{cases}$$

Hence

$$\frac{\|x_m\|_1}{\|(C - S)x_m\|_1} = \frac{2m}{m + 1} \rightarrow 2,$$

as $m \rightarrow \infty$. This ends the proof of (3.49).

Now, let $x \in \ell_+^1(\mathbb{N})$ be a nonnegative sequence with $\|x\|_1 = 1$. We argue by contradiction and assume that $(C - S)x \in \ell^1(\mathbb{N})$. Then

$$\|Cx\|_1 = \|(C - S)x + Sx\|_1 \leq \|(C - S)x\|_1 + \|Sx\|_1 \leq \|(C - S)x\|_1 + \|x\|_1 < \infty.$$

Hence $Cx \in \ell^1(\mathbb{N})$. By [5, Proposition 2.1], this implies that $x = 0$, which contradicts the assumption $\|x\|_1 = 1$. Hence $(C - S)x \notin \ell^1(\mathbb{N})$ for all nonnegative sequence $x \in \ell_+^1(\mathbb{N}) \setminus \{0\}$. This proves that

$$[C - S]_{\ell_+^1(\mathbb{N})} = 0.$$

In particular, by taking into account (2.2), we have

$$0 \leq [C - S]_{\ell_{\text{dec}}^1(\mathbb{N})} \leq [C - S]_{\ell_+^1(\mathbb{N})} = 0.$$

Thus

$$[C - S]_{\ell_{\text{dec}}^1(\mathbb{N})} = 0.$$

This completes the proof. \square

3.4 Tables

In this section, we present in tables all known constants related to the operators $C - I$ and $C - S$ acting on different cones of $\ell^p(\mathbb{N})$, for $1 \leq p \leq \infty$. The constants displayed in bold have been proved in this chapter.

$\ C - I\ _X$	$\ell^p(\mathbb{N})$	$\ell_+^p(\mathbb{N})$	$\ell_{\text{dec}}^p(\mathbb{N})$
$p = 1$	∞	∞	∞
$1 < p \leq 2$	$1/(p - 1)$	$1/(p - 1)$	$1/(p - 1)$
$2 \leq p < \infty$	$m_p^{-1/p}$	1	$1/(p - 1)^{1/p}$
$p = \infty$	2	1	1

$[C - I]_X$	$\ell^p(\mathbb{N})$	$\ell_+^p(\mathbb{N})$	$\ell_{\text{dec}}^p(\mathbb{N})$
$p = 1$	2	0	0
$1 < p \leq 2$	$2^{1/p}$?	?
$2 < p < \infty$?	?	?
$p = \infty$	∞	∞	∞

$\ C - S\ _X$	$\ell^p(\mathbb{N})$	$\ell_+^p(\mathbb{N})$	$\ell_{\text{dec}}^p(\mathbb{N})$
$p = 1$	∞	∞	∞
$1 < p < 2$?	?	?
$2 \leq p < \infty$	$2^{1/p'}$?	$\zeta(p)^{1/p}$
$p = \infty$	2	1	1

$[C - S]_X$	$\ell^p(\mathbb{N})$	$\ell^p_+(\mathbb{N})$	$\ell^p_{\text{dec}}(\mathbb{N})$
$p = 1$	2	0	0
$1 < p \leq 2$	$M_p^{1/p}$	1	$(p - 1)^{1/p}$
$2 \leq p < \infty$	$p - 1$	$p - 1$	$p - 1$
$p = \infty$	∞	∞	∞

Remark 3.20. Regarding the unknown constants $\|C - S\|_{\ell^p_{\text{dec}}(\mathbb{N})}$, $\|C - S\|_{\ell^p_+(\mathbb{N})}$, and $\|C - S\|_{\ell^p(\mathbb{N})}$ for $1 < p < 2$, we use the fact that $\|C\|_{\ell^p_{\text{dec}}(\mathbb{N})} = p'$ (see [24, p. 239]). It follows that

$$p' - 1 \leq \|C\|_{\ell^p_{\text{dec}}(\mathbb{N})} - \|S\|_{\ell^p_{\text{dec}}(\mathbb{N})} \leq \|C - S\|_{\ell^p_{\text{dec}}(\mathbb{N})}.$$

Consequently, we obtain the estimates

$$\begin{aligned} \frac{1}{p - 1} &\leq \|C - S\|_{\ell^p_{\text{dec}}(\mathbb{N})} \leq \|C - S\|_{\ell^p_+(\mathbb{N})} \\ &\leq \|C - S\|_{\ell^p(\mathbb{N})} \leq \|C\|_{\ell^p(\mathbb{N})} + \|S\|_{\ell^p(\mathbb{N})} = p' + 1, \end{aligned}$$

for $1 < p < 2$. A better upper estimate for $\|C - S\|_{\ell^p_{\text{dec}}(\mathbb{N})}$ for $1 < p < 2$ is given in [20], where the authors proved the following bounds:

$$\frac{1}{p - 1} \leq \|C - S\|_{\ell^p_{\text{dec}}(\mathbb{N})} \leq \frac{\zeta(p)^{1/p}}{(p - 1)^{1/p'}}.$$

On the other hand, consider the sequence

$$x^m = \sum_{k=1}^m e_k,$$

where $m \geq 1$ is a natural number. A straightforward calculation shows that

$$\frac{\|x^m\|_p^p}{\|(C - I)x^m\|_p^p} = \frac{1}{m^{p-1} \sum_{k=m+1}^{\infty} \frac{1}{k^p}}, \quad m \geq 1.$$

Taking into account [20, Lemma 2.1], we obtain

$$\sup_{m \in \mathbb{N}} \frac{\|x^m\|_p^p}{\|(C - I)x^m\|_p^p} = \frac{\|x^1\|_p^p}{\|(C - I)x^1\|_p^p} = \frac{1}{\zeta(p) - 1} \leq [C - I]_{\ell^p_{\text{dec}}(\mathbb{N})}. \quad (3.51)$$

Concerning the estimate of $[C - I]_{\ell^p(\mathbb{N})}$, Theorem 3.10 shows

$$[C - I]_{\ell^p(\mathbb{N})} = 2^{1/p}, \quad 1 < p \leq 2. \quad (3.52)$$

For $2 < p < \infty$, we seek the best constant $C(p) > 0$ in

$$\|x\|_p \leq C(p) \|(C - I)x\|_p, \quad x \in \ell^p(\mathbb{N}).$$

If $(C - I)x \notin \ell^p(\mathbb{N})$, the inequality holds trivially. Otherwise, assuming that the sequence $(C - I)x \in \ell^p(\mathbb{N})$, it follows that $S(C - I)x \in \ell^p(\mathbb{N})$. Now, by using (4.24) we have

$$\|(C - S)^*\|_p \leq \|C^*\|_p + \|S^*\|_p = p + 1. \quad (3.53)$$

Combining (3.20), (3.53), and $\|S\|_p = 1$ gives

$$\|x\|_p = \|(C - S)^*S(C - I)x\|_p \leq \|(C - S)^*\|_p \|S\|_p \|(C - I)x\|_p \leq (p + 1) \|(C - I)x\|_p,$$

so that

$$[C - I]_{\ell^p(\mathbb{N})} \leq p + 1, \quad 2 < p < \infty. \quad (3.54)$$

Now, (3.51) together with (3.52) imply

$$\frac{1}{(\zeta(p) - 1)^{1/p}} \leq [C - I]_{\ell_{\text{dec}}^p(\mathbb{N})} \leq [C - I]_{\ell_+^p(\mathbb{N})} \leq [C - I]_{\ell^p(\mathbb{N})} \leq 2^{1/p}, \quad 1 < p \leq 2,$$

while (3.51) combined with (3.54) yields

$$\frac{1}{(\zeta(p) - 1)^{1/p}} \leq [C - I]_{\ell_{\text{dec}}^p(\mathbb{N})} \leq [C - I]_{\ell_+^p(\mathbb{N})} \leq [C - I]_{\ell^p(\mathbb{N})} \leq p + 1, \quad 2 < p < \infty.$$

Regarding the unknown constant $\|C - S\|_{\ell_+^p(\mathbb{N})}$ for $2 \leq p < \infty$, by combining (2.1) with the bounds presented in the third table, we deduce

$$\zeta(p)^{1/p} = \|C - S\|_{\ell_{\text{dec}}^p(\mathbb{N})} \leq \|C - S\|_{\ell_+^p(\mathbb{N})} \leq \|C - S\|_{\ell^p(\mathbb{N})} = 2^{1/p'}.$$

Chapter 4

Sharp bounds for the Copson operator

Let $x = \{x(n)\}_{n \geq 1}$ be a sequence. The adjoint of the Cesàro operator, called the *Copson operator*, is defined by

$$C^*x(n) = \sum_{k=n}^{\infty} \frac{x(k)}{k}, \quad n \geq 1,$$

whenever the above series is well defined for the sequence x . The *right shift operator* is given by

$$S^*x(n) = x(n-1), \quad n \geq 1,$$

with the convention that $S^*x(1) = 0$.

The operators C and C^* are closely related. Indeed, recalling the identity (3.4),

$$CC^* = C + SC^*,$$

we observe that this equality is equivalent to

$$S = (C - S)(C^* - I). \quad (4.1)$$

Taking the transpose of (4.1), we obtain

$$S^* = (C - I)(C - S)^* = (C - I)(C^* - S^*). \quad (4.2)$$

The identity (4.2) plays a crucial role in the development of this chapter. Expanding the right-hand side of (4.2) gives

$$S^* = CC^* - CS^* - C^* + S^*,$$

and therefore

$$CC^* = CS^* + C^*. \quad (4.3)$$

In particular, this implies that

$$C^* = C(C^* - S^*).$$

We have seen in (3.14) that the operator C^* also can be expressed as

$$C^* = (C^* - I)S^*C.$$

By applying duality to (3.1), we obtain the operator p -norm of $C^* - I$ on $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$:

$$\|C^* - I\|_{\ell^p(\mathbb{N})} = \|C - I\|_{\ell^{p'}(\mathbb{N})} = \begin{cases} 2, & \text{if } p = 1, \\ m_{p'}^{-1/p'}, & \text{if } 1 < p \leq 2, \\ p - 1, & \text{if } 2 \leq p < \infty. \end{cases} \quad (4.4)$$

Note that, in contrast to the operator $C - I$, the operator $C^* - I$ is not bounded below. That is, there is no constant $B(p) > 0$, with $1 \leq p \leq \infty$, such that

$$\|x\|_p \leq B(p) \|(C^* - I)x\|_p$$

for all $x \in \ell^p(\mathbb{N})$. In fact, no such constant exists even when x is restricted to the cone of nonnegative, nonincreasing sequences in $\ell^p(\mathbb{N})$. This lack of a lower bound is a consequence of the non-injectivity of the operator $C^* - I$. Indeed, if such a constant $B(p) > 0$ existed, then evaluating at e_1 would yield

$$1 = \|e_1\|_p \leq B(p) \|(C^* - I)e_1\|_p = 0,$$

which is a contradiction. Therefore

$$[C^* - I]_{\ell_{\text{dec}}^p(\mathbb{N})} = [C^* - I]_{\ell_+^p(\mathbb{N})} = [C^* - I]_{\ell^p(\mathbb{N})} = \infty, \quad (4.5)$$

for all $1 \leq p \leq \infty$.

However, instead of the operator $C^* - I$, we consider the transpose of $C - S$, introduced in Chapter 3, and obtain lower bounds.

Motivated by the work done in Chapter 3, the main goal of this chapter is to determine some of the remaining optimal constants $A(p)$, $B(p)$, $U(p)$ and $V(p)$ in the inequalities

$$A(p) \|(C^* - I)x\|_p \leq \|x\|_p \leq B(p) \|(C - I)^*x\|_p, \quad (4.6)$$

$$U(p) \|(C^* - S)x\|_p \leq \|x\|_p \leq V(p) \|(C - S)^*x\|_p, \quad (4.7)$$

on the three cones $\ell^p(\mathbb{N})$, $\ell_+^p(\mathbb{N})$, and $\ell_{\text{dec}}^p(\mathbb{N})$, for the whole range $1 \leq p \leq \infty$.

The results of this chapter have appeared in [11, 12].

Chapter Overview

This chapter is divided into four sections. The first three focus on the study of inequalities (4.6) and (4.7) on a specific cone of $\ell^p(\mathbb{N})$ for the range $1 < p < \infty$. The fourth section is dedicated to the analysis of the same inequalities on various cones of $\ell^p(\mathbb{N})$ at the endpoints $p \in \{1, \infty\}$. In the final section, we present in tables all optimal constants related to the operators $C^* - I$ and $C^* - S^*$, indicating which constants were established in this chapter.

• 4.1 Inequalities in $\ell^p(\mathbb{N})$ for $1 < p < \infty$	45
• 4.2 Inequalities in $\ell_{\text{dec}}^p(\mathbb{N})$ for $1 < p < \infty$	57
• 4.3 Inequalities in $\ell_+^p(\mathbb{N})$ for $1 < p < \infty$	61
• 4.4 Endpoint inequalities	87
• 4.5 Tables	77

4.1 Inequalities in $\ell^p(\mathbb{N})$ for $1 < p < \infty$

Let $A(p)$, $B(p)$, $U(p)$, and $V(p)$ denote the optimal constants for which the inequalities

$$A(p) \|(C^* - I)x\|_p \leq \|x\|_p \leq B(p) \|(C^* - I)x\|_p, \quad (4.8)$$

$$U(p) \|(C - S)^*x\|_p \leq \|x\|_p \leq V(p) \|(C - S)^*x\|_p \quad (4.9)$$

hold for all $x \in \ell^p(\mathbb{N})$ and for the entire range $1 < p < \infty$.

Let $1 < p < \infty$. As shown in (4.4), the optimal constant $A(p)$ in (4.8) is given by

$$A(p) = \begin{cases} m_{p'}^{1/p'}, & \text{if } 1 < p \leq 2, \\ \frac{1}{p-1}, & \text{if } 2 \leq p < \infty. \end{cases}$$

On the other hand, by (4.5), the best constant $B(p)$ in (4.8) is not finite.

Concerning the sharp constant $U(p)$ in (4.9), it follows from Proposition 3.9 that

$$U(p) = 2^{-1/p}, \quad 1 < p \leq 2.$$

The first result in this direction is Theorem 4.2, which provides an alternative representation of the best constant $A(p)$ in (4.8) for the range $1 < p \leq 2$. The motivation for introducing this new expression is that it involves a constant depending

on p , rather than on p' (as in $m_p^{-1/p'}$). Moreover, this formulation will later allow us to determine the exact value of $\|C^* - I\|_{\ell_+^p(\mathbb{N})}$ in Theorem 4.7.

The following lemma we will required.

Lemma 4.1. *Let $p \in \mathbb{F}$, let $x = \{x(n)\}_{n \geq 1} \in \ell^p(\mathbb{N})$ be a real sequence, and set $y(n) = \sum_{k=n}^{\infty} \frac{x(k)}{k}$. Then,*

$$\sum_{k=1}^{\infty} (y(k) - x(k))^p \leq M_p \sum_{k=1}^{\infty} x(k)^p.$$

Proof. By Hölder's inequality, we get that

$$|y(N)| = \left| \sum_{k=N}^{\infty} \frac{x(k)}{k} \right| \leq \sum_{k=N}^{\infty} \frac{|x(k)|}{k} \leq \left(\sum_{k=N}^{\infty} x(k)^p \right)^{\frac{1}{p}} \left(\sum_{k=N}^{\infty} \frac{1}{k^{p'}} \right)^{1/p'}.$$

On the other hand, if $N \geq 2$, we have

$$\sum_{k=N}^{\infty} \frac{1}{k^{p'}} \leq \sum_{k=N}^{\infty} \int_{k-1}^k \frac{dt}{t^{p'}} = \int_{N-1}^{\infty} \frac{dt}{t^{p'}} = \frac{1}{(p'-1)(N-1)^{p'-1}} \leq \frac{2^{p'-1}}{(p'-1)N^{p'-1}}.$$

An analogous argument shows that

$$\frac{1}{(p'-1)^{1/p'}} N^{-1/p} \leq \left(\sum_{k=N}^{\infty} \frac{1}{k^{p'}} \right)^{1/p'}.$$

Combining the last two inequalities, we obtain that

$$\frac{1}{(p'-1)^{1/p'}} N^{-1/p} \leq \left(\sum_{k=N}^{\infty} \frac{1}{k^{p'}} \right)^{1/p'} \leq \frac{2^{1/p}}{(p'-1)^{1/p'}} N^{-1/p}.$$

Therefore

$$N^{1/p} |y(N)| \lesssim \left(\sum_{k=N}^{\infty} x(k)^p \right)^{\frac{1}{p}},$$

which means,

$$\sum_{n=1}^N (ny(n+1)^p - (n-1)y(n)^p) = Ny(N+1)^p \rightarrow 0, \quad (4.10)$$

when $N \rightarrow \infty$. Now, if $n \geq 1$, by applying (2.3), we obtain,

$$\begin{aligned} ny(n+1)^p - (n-1)y(n)^p &= n(y(n+1)^p - y(n)^p) + y(n)^p \\ &\geq npy(n)^{p-1}(y(n+1) - y(n)) + y(n)^p \\ &= -py(n)^{p-1}x(n) + y(n)^p. \end{aligned}$$

If $x(n) \neq 0$, we may let $t = y(n)/x(n)$ and apply Lemma 2.4 in the form,

$$t^p - pt^{p-1} \geq (t-1)^p - M_p,$$

to get

$$\begin{aligned} y(n)^p - py(n)^{p-1}x(n) &= x(n)^p(t^p - pt^{p-1}) \\ &\geq x(n)^p((t-1)^p - M_p) \\ &= (y(n) - x(n))^p - M_px(n)^p. \end{aligned}$$

If $x(n) = 0$, as before, the same inequality holds trivially. Therefore,

$$y(n)^p - py(n)^{p-1}x(n) \geq (y(n) - x(n))^p - M_px(n)^p, \quad \text{for every } n \geq 1.$$

Summing over n we get

$$\sum_{n=1}^N (y(n) - x(n))^p - M_p \sum_{n=1}^N x(n)^p \leq \sum_{n=1}^N ny(n+1)^p - (n-1)y(n)^p = Ny(N+1)^p.$$

Since $Ny(N+1)^p \rightarrow 0$ as $N \rightarrow \infty$ by (4.10), we obtain

$$\sum_{n=1}^{\infty} (y(n) - x(n))^p \leq M_p \sum_{n=1}^{\infty} x(n)^p.$$

□

Now, we pass from $p \in \mathbb{F}$ to all $1 < p \leq 2$ using the Riesz-Thorin theorem.

Theorem 4.2. *Let $1 < p \leq 2$. Then, for any real sequence $x \in \ell^p(\mathbb{N})$, the following inequality holds:*

$$\|(C^* - I)x\|_p \leq M_p^{1/p} \|x\|_p. \quad (4.11)$$

Furthermore, the constant $M_p^{1/p}$ in (4.11) is the best possible.

Proof. Let $1 < p < 2$. Lemma 4.1 shows that $\|C^* - I\|_{\ell^p(\mathbb{N})} \leq M_p^{1/p}$, for all $p \in \mathbb{F}$. By taking into account the Riesz-Thorin theorem [7, Chapter 4, Corollary 2.3] we have that if $1 < p_0 < p < p_1 < 2$, with $p_0, p_1 \in \mathbb{F}$, then for some $\theta \in (0, 1)$, depending on p_0, p and p_1 ,

$$\|C^* - I\|_{\ell^p(\mathbb{N})} \leq \|C^* - I\|_{\ell^{p_0}(\mathbb{N})}^{1-\theta} \|C^* - I\|_{\ell^{p_1}(\mathbb{N})}^{\theta} \leq \max\{M_{p_0}^{1/p_0}, M_{p_1}^{1/p_1}\}.$$

Letting p_0 and p_1 approach p through \mathbb{F} , the continuity of $p \rightarrow M_p$, (see Lemma 2.4) implies

$$\|C^* - I\|_{\ell^p(\mathbb{N})} \leq M_p^{1/p}.$$

It remains to show that the constant in (4.11) is optimal. Let $r = 1/t_p$, where t_p is the critical point from the Lemma 2.4, and fix an integer $m > 1$. Note that $r > 2$. Let us define

$$x(n) = \begin{cases} m^{-r}, & \text{if } n = 1, \\ 0, & \text{if } 1 < n \leq m, \\ n((n-1)^{-r} - n^{-r}), & \text{if } n > m. \end{cases}$$

Then, with $y(n) = \sum_{k=n}^{\infty} \frac{x(k)}{k}$ and $z(n) = |y(n) - x(n)|$,

$$y(n) = \begin{cases} 2m^{-r}, & \text{if } n = 1, \\ m^{-r}, & \text{if } 1 < n \leq m, \\ (n-1)^{-r}, & \text{if } n > m, \end{cases}$$

and

$$z(n) = \begin{cases} m^{-r}, & \text{if } n \leq m, \\ (n-1)^{1-r} - n^{1-r}, & \text{if } n > m. \end{cases}$$

Now, if $n \geq m+1$ we get

$$(n-1)^{-r} - n^{-r} = r \int_{(n-1)}^n t^{-r-1} dt \leq r(n-1)^{-r-1}.$$

Thus

$$n((n-1)^{-r} - n^{-r}) \leq rn(n-1)^{-r-1} = r \frac{n}{n-1} (n-1)^{-r} \leq r \frac{(m+1)}{m} (n-1)^{-r}.$$

As a consequence,

$$\begin{aligned} \sum_{n=m+1}^{\infty} |x(n)|^p &\leq r^p \frac{(m+1)^p}{m^p} \sum_{n=m+1}^{\infty} \frac{1}{(n-1)^{rp}} = r^p \frac{(m+1)^p}{m^p} \sum_{n=m}^{\infty} \frac{1}{n^{rp}} \\ &\leq r^p \frac{(m+1)^p}{m^p} \sum_{n=m}^{\infty} \int_{(n-1)}^n \frac{1}{x^{rp}} dx = r^p \frac{(m+1)^p}{m^p} \int_{(m-1)}^{\infty} \frac{1}{x^{rp}} dx \\ &= r^p \frac{(m+1)^p}{m^p} \frac{1}{rp-1} (m-1)^{1-rp}. \end{aligned}$$

So

$$\begin{aligned} \|x\|_{\ell^p}^p &\leq m^{-rp} + r^p \frac{(m+1)^p}{m^p} \frac{1}{rp-1} (m-1)^{1-rp} \\ &= \frac{(m-1)^{1-rp}}{rp-1} \left(\left(\frac{m}{m-1} \right)^{-rp} \frac{rp-1}{m-1} + r^p \left(\frac{m+1}{m} \right)^p \right). \end{aligned}$$

On the other hand, we have that if $n \geq m + 1$, then

$$z(n) = (n-1)^{1-r} - n^{1-r} = (r-1) \int_{(n-1)}^n t^{-r} dt \geq (r-1)n^{-r}.$$

Hence

$$\begin{aligned} \sum_{n=m+1}^{\infty} z(n)^p &\geq (r-1)^p \sum_{n=m+1}^{\infty} \frac{1}{n^{rp}} \\ &\geq (r-1)^p \sum_{n=m+1}^{\infty} \int_n^{n+1} \frac{1}{x^{rp}} dx = (r-1)^p \int_{(m+1)}^{\infty} \frac{1}{x^{rp}} dx \\ &= \frac{(r-1)^p}{rp-1} (m+1)^{1-rp}. \end{aligned}$$

Therefore

$$\begin{aligned} \|z\|_{\ell^p}^p &\geq m^{1-rp} + \frac{(r-1)^p}{rp-1} (m+1)^{1-rp} \\ &= \frac{(m-1)^{1-rp}}{rp-1} \left(\left(\frac{m}{m-1} \right)^{1-rp} (rp-1) + (r-1)^p \left(\frac{m+1}{m-1} \right)^{1-rp} \right). \end{aligned}$$

We conclude that

$$\frac{\sum_{n=1}^{\infty} |z(n)|^p}{\sum_{n=1}^{\infty} |x(n)|^p} \geq \frac{\left(\frac{m}{m-1} \right)^{1-rp} (rp-1) + (r-1)^p \left(\frac{m+1}{m-1} \right)^{1-rp}}{\left(\frac{m}{m-1} \right)^{-rp} \left(\frac{rp-1}{m-1} \right) + r^p \left(\frac{m+1}{m} \right)^p} \rightarrow \frac{rp-1 + (r-1)^p}{r^p},$$

as $m \rightarrow \infty$. Since $r = 1/t_p$, Lemma 2.4 shows that this last expression is M_p . This completes the proof. \square

Remark 4.3. Let $1 < p \leq 2$ and $p' = p/(p-1)$. From (4.4), we know that

$$\|C^* - I\|_{\ell^p(\mathbb{N})} = m_{p'}^{-1/p'},$$

where $m_{p'}$ is defined in (2.5). On the other hand, Theorem 4.2 shows that

$$\|C^* - I\|_{\ell^p(\mathbb{N})} = M_p^{1/p},$$

where M_p is also defined in (2.5). Consequently, we deduce that

$$m_{p'}^{-1/p'} = \max_{t \in [0, 1/2]} f_{p'}(t)^{-1/p'} = \max_{t \in [0, 1/2]} f_p(t)^{1/p} = M_p^{1/p}, \quad (4.12)$$

where f_p is defined as in (2.4) and p' is the conjugate exponent of p . An illustration of (4.12) is provided in Figure 4.1.

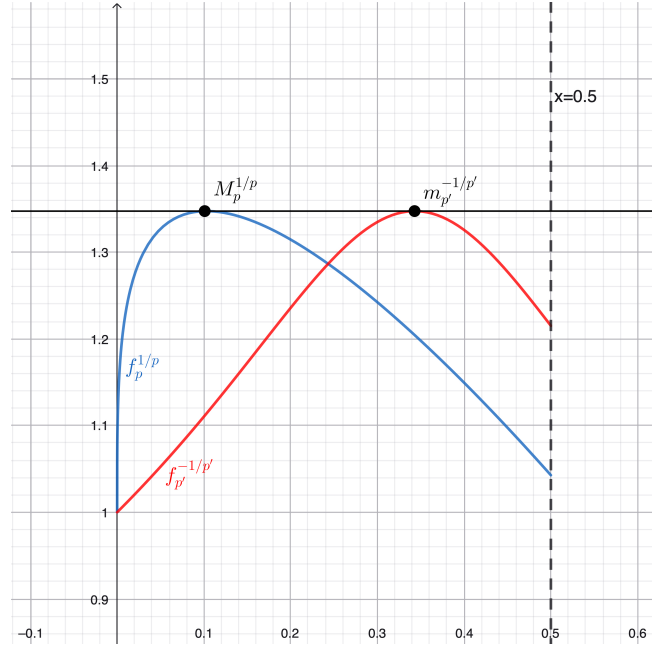


Figure 4.1 The plots of $f_p^{1/p}$ (in blue) and $f_{p'}^{-1/p'}$ (in red) for $p = 1.2$.

We are now ready to present one of the main results of this section, which provides the exact value of $V(p)$ in (4.9) for a general sequence x and $p \in (1, \infty)$.

Theorem 4.4. *Let $1 < p < \infty$ and let x be an arbitrary sequence. If $1 < p \leq 2$, then*

$$(p-1)\|x\|_p \leq \|(C-S)^*x\|_p, \quad (4.13)$$

and if $2 \leq p < \infty$, then

$$m_p^{1/p}\|x\|_p \leq \|(C-S)^*x\|_p. \quad (4.14)$$

Furthermore, all the constants in (4.13) and (4.14) are the best possible.

Proof. Let $1 < p < \infty$. We are going to prove only (4.13), since the proof of (4.14) is analogous. Assume $1 < p \leq 2$, and let x be an arbitrary sequence. If the sequence $(C-S)^*x \notin \ell^p(\mathbb{N})$, then (4.13) holds trivially. Otherwise, assuming $(C-S)^*x \in \ell^p(\mathbb{N})$, by taking into account (3.13) and applying (3.1) yields

$$\|x\|_p = \|S^*x\|_p = \|(C-I)(C-S)^*x\|_p \leq (p-1)^{-1} \|(C-S)^*x\|_p.$$

It remains to show that the constants in (4.13) and in (4.14) are optimal. We start with the optimality of $(p-1)^{-1}$ in (4.13). Let $m \geq 1$ be a natural number and $s = s(\epsilon) = 1/p' - \epsilon$, where $p' = p/(p-1)$ and $0 < \epsilon < 1/p'$. Consider the sequence $x_{\epsilon,m} = \{x_{\epsilon,m}(n)\}_{n \in \mathbb{N}}$ given by

$$x_{\epsilon,m}(n) = x(n) = \begin{cases} (m+1)^{s-1}, & \text{if } 1 \leq n \leq m, \\ n(n^{s-1} - (n+1)^{s-1}), & \text{if } n > m. \end{cases}$$

Let us now estimate the p -norm of x from below. Let $n > m$. We have that

$$n^{s-1} - (n+1)^{s-1} = (1-s) \int_n^{n+1} t^{s-2} dt \geq (1-s)(n+1)^{s-2}.$$

So,

$$n(n^{s-1} - (n+1)^{s-1}) \geq (1-s) \frac{n}{n+1} (n+1)^{s-1} \geq (1-s) \frac{m}{m+1} (n+1)^{s-1}.$$

Therefore,

$$\begin{aligned} \sum_{n=m+1}^{\infty} x(n)^p &\geq (1-s)^p \left(\frac{m}{m+1} \right)^p \sum_{n=m+1}^{\infty} \frac{1}{(n+1)^{p(1-s)}} \\ &= (1-s)^p \left(\frac{m}{m+1} \right)^p \sum_{n=m+2}^{\infty} \frac{1}{n^{p(1-s)}} \\ &\geq (1-s)^p \left(\frac{m}{m+1} \right)^p \sum_{n=m+2}^{\infty} \int_n^{n+1} \frac{dx}{x^{p(1-s)}} \\ &= (1-s)^p \left(\frac{m}{m+1} \right)^p \int_{m+2}^{\infty} \frac{dx}{x^{p(1-s)}} \\ &= (1-s)^p \left(\frac{m}{m+1} \right)^p \frac{1}{p(1-s)-1} \frac{1}{(m+2)^{p(1-s)-1}}. \end{aligned}$$

Hence,

$$\|x\|_p^p \geq m(m+1)^{p(s-1)} + (1-s)^p \left(\frac{m}{m+1} \right)^p \frac{1}{p(1-s)-1} \frac{1}{(m+2)^{p(1-s)-1}}.$$

Now, let $n > m$. We have that,

$$C^*x(n) = \sum_{k=n}^{\infty} \frac{x(k)}{k} = \sum_{k=n}^{\infty} (k^{s-1} - (k+1)^{s-1}) = n^{s-1}.$$

So, if $n > m+1$, we get

$$(C-S)^*x(n) = n^{s-1} - (n-1)^s + (n-1)n^{s-1} = n^s - (n-1)^s.$$

For $n=1$ we obtain that

$$\begin{aligned} (C-S)^*x(1) &= C^*x(1) = \sum_{k=1}^m \frac{x(k)}{k} + \sum_{k=m+1}^{\infty} \frac{x(k)}{k} \\ &= (m+1)^{s-1} \sum_{k=1}^m \frac{1}{k} + (m+1)^{s-1} = (m+1)^{s-1} \left(1 + \sum_{k=1}^m \frac{1}{k} \right), \end{aligned}$$

and

$$\begin{aligned}
(C - S)^*x(n) &= \sum_{k=n}^m \frac{x(k)}{k} + \sum_{k=m+1}^{\infty} \frac{x(k)}{k} \\
&= (m+1)^{s-1} \sum_{k=n}^m \frac{1}{k} + (m+1)^{s-1} - (m+1)^{s-1} \\
&= (m+1)^{s-1} \left(\sum_{k=n}^m \frac{1}{k} \right),
\end{aligned}$$

for all $2 \leq n \leq m$. Finally,

$$C^*x(m+1) - S^*x(m+1) = (m+1)^{s-1} - x(m) = (m+1)^{s-1} - (m+1)^{s-1} = 0.$$

Thus,

$$|(C - S)^*x(n)| = \begin{cases} (m+1)^{s-1} \left(1 + \sum_{k=1}^m \frac{1}{k} \right), & \text{if } n = 1, \\ (m+1)^{s-1} \left(\sum_{k=n}^m \frac{1}{k} \right), & \text{if } 2 \leq n \leq m, \\ 0, & \text{if } n = m+1, \\ n^s - (n-1)^s, & \text{if } n > m+1. \end{cases}$$

Let $n > m+1$. Then,

$$n^s - (n-1)^s = s \int_{n-1}^n t^{s-1} dt \leq s(n-1)^{s-1}.$$

So,

$$\begin{aligned}
\sum_{n=m+2}^{\infty} |(C - S)^*x(n)|^p &\leq s^p \sum_{n=m+2}^{\infty} \frac{1}{(n-1)^{p(1-s)}} = s^p \sum_{n=m+1}^{\infty} \frac{1}{n^{p(1-s)}} \\
&\leq s^p \sum_{n=m+1}^{\infty} \int_{n-1}^n \frac{dx}{x^{p(1-s)}} = s^p \int_m^{\infty} \frac{dx}{x^{p(1-s)}} \\
&= s^p \frac{1}{p(1-s)-1} \frac{1}{m^{p(1-s)-1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(C - S)^*x\|_p^p &\leq (m+1)^{p(s-1)} \left(1 + \sum_{k=1}^m \frac{1}{k} \right)^p + (m+1)^{p(s-1)} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k} \right)^p \\
&\quad + s^p \frac{1}{p(1-s)-1} \frac{1}{m^{p(1-s)-1}}.
\end{aligned}$$

Hence, if we define

$$\begin{aligned} I(m, s) &= (m+1)^{p(s-1)} \left(1 + \sum_{k=1}^m \frac{1}{k}\right)^p, \\ J(m, s) &= (m+1)^{p(s-1)} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k}\right)^p, \\ K(m, s) &= s^p \frac{1}{p(1-s) - 1} \frac{1}{m^{p(1-s)-1}}, \end{aligned}$$

we have

$$\begin{aligned} (p-1)^p &\leq \frac{\|(C-S)^*x\|_p^p}{\|x\|_p^p} \\ &\leq \frac{I(m, s) + J(m, s) + K(m, s)}{m(m+1)^{p(s-1)} + (1-s)^p \left(\frac{m}{m+1}\right)^p \frac{1}{p(1-s)-1} \frac{1}{(m+2)^{p(1-s)-1}}} \\ &= \frac{\left(\frac{m+2}{m+1}\right)^{p(1-s)} \frac{p(1-s)-1}{m+2} \left(1 + \sum_{k=1}^m \frac{1}{k}\right)^p}{\frac{m}{m+2} \left(\frac{m+2}{m+1}\right)^{p(1-s)} (p(1-s) - 1) + (1-s)^p \left(\frac{m}{m+1}\right)^p} \\ &\quad + \frac{\left(\frac{m+2}{m+1}\right)^{p(1-s)} \frac{p(1-s)-1}{m+2} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k}\right)^p + s^p \left(\frac{m+2}{m}\right)^{p(1-s)-1}}{\frac{m}{m+2} \left(\frac{m+2}{m+1}\right)^{p(1-s)} (p(1-s) - 1) + (1-s)^p \left(\frac{m}{m+1}\right)^p} \\ &\stackrel{\text{def}}{=} A(m, s) + B(m, s), \end{aligned} \tag{4.15}$$

for all $m \geq 1$ and $0 < \epsilon < 1/p'$. Note that

$$\begin{aligned} 0 &\leq \frac{1}{m+2} \left(1 + \sum_{k=1}^m \frac{1}{k}\right)^p = \frac{1}{m+2} \left(2 + \sum_{k=2}^m \frac{1}{k}\right)^p \leq \frac{1}{m+2} \left(2 + \sum_{k=2}^m \int_{k-1}^k \frac{dx}{x}\right)^p \\ &= \frac{1}{m+2} \left(2 + \sum_{k=2}^m \int_1^m \frac{dx}{x}\right)^p = \frac{1}{m+2} (2 + \ln(m))^p = \frac{\ln^p(e^2 m)}{m+2}, \end{aligned} \tag{4.16}$$

for all $m \geq 1$. Now, since

$$\lim_{m \rightarrow \infty} \frac{\ln^p(e^2 m)}{m+2} = 0,$$

we have by an application of the squeeze theorem in (4.16) that

$$\lim_{m \rightarrow \infty} \frac{1}{m+2} \left(1 + \sum_{k=1}^m \frac{1}{k}\right)^p = 0.$$

Therefore,

$$\begin{aligned} A(\infty, s) &= \lim_{m \rightarrow \infty} A(m, s) = \lim_{m \rightarrow \infty} \frac{\left(\frac{m+2}{m+1}\right)^{p(1-s)} \frac{p(1-s)-1}{m+2} \left(1 + \sum_{k=1}^m \frac{1}{k}\right)^p}{\frac{m}{m+2} \left(\frac{m+2}{m+1}\right)^{p(1-s)} (p(1-s) - 1) + (1-s)^p \left(\frac{m}{m+1}\right)^p} \\ &= \frac{(p(1-s) - 1) \cdot 0}{p(1-s) - 1 + (1-s)^p} = 0, \end{aligned}$$

for all $s = 1/p' - \epsilon$, where $0 < \epsilon < 1/p'$. On the other hand, if $2 \leq n \leq m$ are natural numbers, then

$$\sum_{k=n}^m \frac{1}{k} \leq \sum_{k=n}^m \int_{k-1}^k \frac{dx}{x} = \int_{n-1}^m \frac{dx}{x} = \ln \left(\frac{m}{n-1} \right)$$

and

$$\sum_{k=n}^m \frac{1}{k} \geq \sum_{k=n}^m \int_k^{k+1} \frac{dx}{x} = \int_n^{m+1} \frac{dx}{x} = \ln \left(\frac{m+1}{n} \right).$$

So,

$$\ln \left(\frac{m+1}{n} \right) \leq \sum_{k=n}^m \frac{1}{k} \leq \ln \left(\frac{m}{n-1} \right),$$

for all natural numbers $2 \leq n \leq m$. Thus,

$$\ln^p \left(\frac{m+1}{n} \right) \leq \left(\sum_{k=n}^m \frac{1}{k} \right)^p \leq \ln^p \left(\frac{m}{n-1} \right),$$

for all natural numbers $2 \leq n \leq m$. Therefore,

$$\frac{1}{m} \sum_{n=2}^m \ln^p \left(\frac{m+1}{n} \right) \leq \frac{1}{m} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k} \right)^p \leq \frac{1}{m} \sum_{n=2}^m \ln^p \left(\frac{m}{n-1} \right),$$

for all $m \geq 2$. Equivalently,

$$\frac{1}{m} \sum_{n=2}^{m+1} \ln^p \left(\frac{m+1}{n} \right) \leq \frac{1}{m} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k} \right)^p \leq \frac{1}{m} \sum_{n=1}^{m-1} \ln^p \left(\frac{m}{n} \right) = \frac{1}{m} \sum_{n=1}^m \ln^p \left(\frac{m}{n} \right),$$

for all $m \geq 2$. Hence,

$$\begin{aligned} -\frac{\ln^p(m+1)}{m} + \frac{m+1}{m} \frac{1}{m+1} \sum_{n=1}^{m+1} \ln^p \left(\frac{m+1}{n} \right) &\leq \frac{1}{m} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k} \right)^p \\ &\leq \frac{1}{m} \sum_{n=1}^m \ln^p \left(\frac{m}{n} \right), \end{aligned} \quad (4.17)$$

for all $m \geq 2$. Identifying these terms as suitable Riemann sums,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{n=1}^m \ln^p \left(\frac{m}{n} \right) &= \lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{n=1}^{m+1} \ln^p \left(\frac{m+1}{n} \right) = \int_0^1 \ln^p \left(\frac{1}{t} \right) dt \\ &= \int_0^\infty u^p e^{-u} du = \Gamma(p+1), \end{aligned} \quad (4.18)$$

where we have used the change of variable $u = -\ln(t)$ to arrive at the Γ function. Then, applying the squeeze theorem to (4.17) together with (4.18), we conclude that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k} \right)^p = \Gamma(p+1).$$

Consequently,

$$\lim_{m \rightarrow \infty} \frac{1}{m+2} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k} \right)^p = \lim_{m \rightarrow \infty} \frac{m}{(m+2)} \frac{1}{m} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k} \right)^p = \Gamma(p+1).$$

Then, using the above limit, we obtain

$$\begin{aligned} B(\infty, s) &= \lim_{m \rightarrow \infty} B(m, s) \\ &= \lim_{m \rightarrow \infty} \frac{\left(\frac{m+2}{m+1}\right)^{p(1-s)} \frac{p(1-s)-1}{m+2} \sum_{n=2}^m \left(\sum_{k=n}^m \frac{1}{k}\right)^p + s^p \left(\frac{m+2}{m}\right)^{p(1-s)-1}}{\frac{m}{m+2} \left(\frac{m+2}{m+1}\right)^{p(1-s)} (p(1-s)-1) + (1-s)^p \left(\frac{m}{m+1}\right)^p} \\ &= \frac{(p(1-s)-1)\Gamma(p+1) + s^p}{p(1-s)-1 + (1-s)^p}, \end{aligned}$$

for all $0 < \epsilon < 1/p'$, where $s = 1/p' - \epsilon$. Therefore, it follows that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} (A(\infty, s) + B(\infty, s)) &= \lim_{\epsilon \rightarrow 0^+} B(\infty, s) = \lim_{\epsilon \rightarrow 0^+} \frac{(p(1-s)-1)\Gamma(p+1) + s^p}{p(1-s)-1 + (1-s)^p} \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{(p(1/p + \epsilon) - 1)\Gamma(p+1) + (1/p' - \epsilon)^p}{p(1/p + \epsilon) - 1 + (1/p + \epsilon)^p} \\ &= (p/p')^p = (p-1)^p. \end{aligned} \tag{4.19}$$

By taking into account (4.19) and using the squeeze theorem in (4.15), we conclude that

$$(p-1)^p \leq \lim_{\epsilon \rightarrow 0^+} \lim_{m \rightarrow \infty} \frac{\|(C-S)^*x\|_p^p}{\|x\|_p^p} \leq \lim_{\epsilon \rightarrow 0^+} \lim_{m \rightarrow \infty} (A(m, s) + B(m, s)) = (p-1)^p,$$

which means that

$$\lim_{\epsilon \rightarrow 0^+} \lim_{m \rightarrow \infty} \frac{\|(C-S)^*x\|_p^p}{\|x\|_p^p} = (p-1)^p.$$

This proves that the constant $(p-1)$ is sharp in (4.13).

Now, let us see the optimality of the constant $m_p^{-1/p}$ in (4.14). By taking into account Lemma 2.3, we set $r = 1/t_p$, where $t_p \in (0, 1/2)$ is such that $f_p(t_p) = m_p$ and f_p is defined as in (2.4). Note that $r > 2$. Fix an integer $m > 1$. Let us define

$$y(n) = \begin{cases} 0, & \text{if } 1 \leq n \leq m, \\ n(n^{-r} - (n+1)^{-r}), & \text{if } n > m. \end{cases}$$

Then, with $C^*y(n) = \sum_{k=n}^{\infty} \frac{y(k)}{k}$ and $z(n) = |(C - S)^*y(n)|$,

$$C^*y(n) = \begin{cases} (m+1)^{-r}, & \text{if } 1 \leq n \leq m+1, \\ n^{-r}, & \text{if } n > m+1, \end{cases}$$

and

$$z(n) = \begin{cases} (m+1)^{-r}, & \text{if } n \leq m+1, \\ (n-1)^{1-r} - n^{1-r}, & \text{if } n > m+1. \end{cases}$$

Now, if $n \geq m+1$ we get

$$n^{-r} - (n+1)^{-r} = r \int_n^{n+1} t^{-r-1} dt \geq r(n+1)^{-r-1}.$$

Thus,

$$n(n^{-r} - (n+1)^{-r}) \geq rn(n+1)^{-r-1} = r \frac{n}{n+1} (n+1)^{-r} \geq r \frac{m}{m+1} (n+1)^{-r}.$$

As a consequence,

$$\begin{aligned} \|y\|_p^p &= \sum_{n=m+1}^{\infty} |y(n)|^p \geq r^p \frac{m^p}{(m+1)^p} \sum_{n=m+1}^{\infty} \frac{1}{(n+1)^{rp}} = r^p \frac{m^p}{(m+1)^p} \sum_{n=m+2}^{\infty} \frac{1}{n^{rp}} \\ &\geq r^p \frac{m^p}{(m+1)^p} \sum_{n=m+2}^{\infty} \int_n^{n+1} \frac{1}{x^{rp}} dx = r^p \frac{m^p}{(m+1)^p} \int_{m+2}^{\infty} \frac{1}{x^{rp}} dx \\ &= r^p \frac{m^p}{(m+1)^p} \frac{1}{rp-1} (m+2)^{1-rp}. \end{aligned} \quad (4.20)$$

On the other hand, we have that if $n > m+1$, then

$$z(n) = (n-1)^{1-r} - n^{1-r} = (r-1) \int_{n-1}^n t^{-r} dt \leq (r-1)(n-1)^{-r}.$$

Hence,

$$\begin{aligned} \sum_{n=m+2}^{\infty} z(n)^p &\leq (r-1)^p \sum_{n=m+2}^{\infty} \frac{1}{(n-1)^{rp}} \\ &= (r-1)^p \sum_{n=m+1}^{\infty} \frac{1}{n^{rp}} \\ &\leq (r-1)^p \sum_{n=m+1}^{\infty} \int_{n-1}^n \frac{1}{x^{rp}} dx = (r-1)^p \int_m^{\infty} \frac{1}{x^{rp}} dx \\ &= \frac{(r-1)^p}{rp-1} m^{1-rp}. \end{aligned} \quad (4.21)$$

Therefore,

$$\|z\|_p^p \leq (m+1)^{1-rp} + \frac{(r-1)^p}{rp-1} m^{1-rp}.$$

By using (4.20) and (4.21), we conclude that

$$\begin{aligned} m_p^{-1} &\geq \frac{\|y\|_p^p}{\|(C-S)^*y\|_p^p} \geq \frac{r^p \frac{m^p}{(m+1)^p} \frac{1}{rp-1} (m+2)^{1-rp}}{(m+1)^{1-rp} + \frac{(r-1)^p}{rp-1} m^{1-rp}} \\ &= \frac{r^p \left(\frac{m}{m+1}\right)^p}{(rp-1) \left(\frac{m+1}{m+2}\right)^{1-rp} + (r-1)^p \left(\frac{m}{m+2}\right)^{1-rp}} \\ &\rightarrow \frac{r^p}{rp-1 + (r-1)^p} = m_p^{-1}, \end{aligned}$$

as $m \rightarrow \infty$. This completes the proof. \square

4.2 Inequalities in $\ell_{\text{dec}}^p(\mathbb{N})$ for $1 < p < \infty$

Let $A(p)$, $B(p)$, $U(p)$, and $V(p)$ denote the optimal constants for which the inequalities

$$A(p) \|(C^* - I)x\|_p \leq \|x\|_p \leq B(p) \|(C^* - I)x\|_p, \quad (4.22)$$

$$U(p) \|(C - S)^*x\|_p \leq \|x\|_p \leq V(p) \|(C - S)^*x\|_p \quad (4.23)$$

hold for all nonnegative, nonincreasing $x \in \ell_{\text{dec}}^p(\mathbb{N})$ and for the range $1 < p < \infty$.

By (4.5), the best constant $B(p)$ in (4.22) is not finite.

Concerning the sharp constant $U(p)$ in (4.23), it follows from Proposition 3.9 that

$$U(p) = 2^{-1/p}, \quad 1 < p \leq 2.$$

Our next goal is to determine the exact value of the best constant $A(p)$ in (4.22) for $2 \leq p < \infty$, and the exact value of the optimal constant $V(p)$ in (4.23) for all $1 < p \leq 2$.

In [37], Landau proved that

$$\|C^*\|_{\ell^p(\mathbb{N})} = p, \quad (4.24)$$

for all $1 \leq p < \infty$. The following lemma shows that (4.24) holds when C^* is restricted to the cone of nonnegative, nonincreasing sequences in $\ell_{\text{dec}}^p(\mathbb{N})$, with $1 \leq p < \infty$.

Lemma 4.5. *Let $1 \leq p < \infty$. Then,*

$$\|C^*\|_{\ell_{\text{dec}}^p(\mathbb{N})} = p. \quad (4.25)$$

Proof. Let $1 \leq p < \infty$. We have, by (4.24), that

$$\|C^*\|_{\ell_{\text{dec}}^p(\mathbb{N})} \leq \|C^*\|_{\ell^p(\mathbb{N})} \leq p.$$

Now, let $\epsilon > 0$ and $x^\epsilon = \{x^\epsilon(n)\}_{n \geq 1}$ be the sequence given by

$$x^\epsilon(n) = \frac{1}{(n+1)^{\frac{1}{p}+\epsilon}}.$$

Observe that x^ϵ is a nonnegative, nonincreasing sequence that belongs to $\ell^p(\mathbb{N})$. Now, if $n \geq 1$ then

$$\begin{aligned} C^*x^\epsilon(n) &= \sum_{k=n}^{\infty} \frac{1}{k(k+1)^{\frac{1}{p}+\epsilon}} \geq \sum_{k=n}^{\infty} \frac{1}{(k+1)^{\frac{1}{p}+\epsilon+1}} \geq \sum_{k=n}^{\infty} \int_k^{k+1} \frac{1}{(t+1)^{\frac{1}{p}+\epsilon+1}} dt \\ &= \int_n^{\infty} \frac{1}{(t+1)^{\frac{1}{p}+\epsilon+1}} dt = \frac{1}{(\frac{1}{p}+\epsilon)} \frac{1}{(n+1)^{\frac{1}{p}+\epsilon}} = \frac{1}{(\frac{1}{p}+\epsilon)} x^\epsilon(n), \end{aligned}$$

which means that,

$$C^*x^\epsilon(n) \geq \frac{1}{(\frac{1}{p}+\epsilon)} x^\epsilon(n), \quad \text{for all } n \geq 1.$$

Taking the p -norm, we have that

$$\|C^*x^\epsilon\|_p \geq \frac{1}{(\frac{1}{p}+\epsilon)} \|x^\epsilon\|_p.$$

Finally, we conclude that

$$\frac{\|C^*x^\epsilon\|_p}{\|x^\epsilon\|_p} \geq \frac{1}{(\frac{1}{p}+\epsilon)} \rightarrow p,$$

as $\epsilon \rightarrow 0^+$. □

We are now ready to prove the main result of this section.

Theorem 4.6. *Let $1 < p < \infty$, and let $x \in \ell_{\text{dec}}^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then the following inequalities are sharp: if $1 < p \leq 2$, then*

$$\|x\|_p \leq (p-1)^{-1} \|(C-S)^*x\|_p,$$

and if $2 \leq p < \infty$, then

$$(p-1)^{-1} \|(C^*-I)x\|_p \leq \|x\|_p.$$

Proof. We start with the case $1 < p \leq 2$. We have seen in (4.13)

$$\|x\|_p \leq (p-1)^{-1} \|(C-S)^*x\|_p,$$

for all $x \in \ell^p(\mathbb{N})$. In particular, the previous inequality holds when $x \in \ell^p(\mathbb{N})$ is nonnegative and nonincreasing. In order to prove the optimality of the constant $(p-1)^{-1}$ in (4.13) we choose the sequence

$$x(n) = \begin{cases} (m+1)^{s-1}, & \text{if } 1 \leq n \leq m, \\ n(n^{s-1} - (n+1)^{s-1}), & \text{if } n > m, \end{cases}$$

where $m \geq 1$ is a natural number and $s = s(\epsilon) = \frac{1}{p'} - \epsilon$, with $0 < \epsilon < \frac{1}{p'}$. It is obvious that x is nonnegative. Let us see that x is nonincreasing sequence. We start with the case $n = m$. So, we have to prove that $x(m) \geq x(m+1)$.

$$\begin{aligned} x(m) \geq x(m+1) &\iff (m+1)^{s-1} \geq (m+1)((m+1)^{s-1} - (m+2)^{s-2}) \\ &\iff (m+1)(m+2)^{s-1} \geq m(m+1)^{s-1} \\ &\iff (m+1)^{2-s} \geq m(m+2)^{1-s} \\ &\iff \frac{(m+1)^2}{m+2} \geq m \left(\frac{m+1}{m+2} \right)^s, \end{aligned}$$

for all $m \geq 1$. Equivalently,

$$\frac{m^2}{m+1} \geq (m-1) \left(\frac{m}{m+1} \right)^s \iff \left(\frac{m}{m+1} \right)^{1-s} \geq \frac{m-1}{m},$$

for all $m \geq 2$. But since, for all $0 < a < b < 1$ and $0 < r < 1$, we have that $0 < a < a^r < b^r < 1$, then

$$\left(\frac{m}{m+1} \right)^{1-s} \geq \left(\frac{m-1}{m} \right)^{1-s} \geq \frac{m-1}{m},$$

for all $m \geq 2$, which means that $x(m) \geq x(m+1)$, for all $m \geq 1$. Now, consider the function $f : (1, \infty) \rightarrow \mathbb{R}^+$ defined by

$$f(t) = t(t^{s-1} - (t+1)^{s-1}) = t^s - t(t+1)^{s-1}.$$

Its derivative is given by

$$\begin{aligned} f'(t) &= st^{s-1} - (t+1)^{s-1} - (s-1)t(t+1)^{s-2} \\ &= (t+1)^{s-1} \left[s \left(\frac{t}{t+1} \right)^{s-1} - 1 - (s-1) \frac{t}{t+1} \right] \\ &= (1+t)^{s-1} h(t), \end{aligned}$$

where $h : (1, \infty) \rightarrow \mathbb{R}$ is defined by

$$h(t) = s \left(\frac{t}{t+1} \right)^{s-1} - 1 - (s-1) \frac{t}{t+1}.$$

To show that $f'(t) \leq 0$ for all $t > 1$, it suffices to prove that

$$g(t) = st^{s-1} - 1 - (s-1)t \leq 0, \quad \text{for all } t \in [1/2, 1],$$

where $t = \frac{x}{x+1}$. Now, observe that the second derivative of g is

$$g''(t) = \frac{d}{dt} \left(\frac{d}{dt} g(t) \right) = \frac{d}{dt} (s(s-1)t^{s-2} - (s-1)) = s(s-1)(s-2)t^{s-3} > 0,$$

for all $t \in (1/2, 1)$, since $s-1 = 1/p' - \epsilon - 1 < 0$. Therefore, g is strictly convex on $(1/2, 1)$. Note that

$$g(1) = s - 1 - (s-1) = 0.$$

If we prove that $g(1/2) \leq 0$, then by convexity we would have

$$g(t) \leq \max\{g(1), g(1/2)\} = 0,$$

for all $t \in [1/2, 1]$. We now show that $g(1/2) < 0$. Consider the function u defined on $[0, 1/2]$ by

$$u(x) = x2^{2-x} - 1 - x.$$

We compute

$$\frac{d^2}{dx^2} u(x) = \frac{d}{dx} \left(\frac{d}{dx} u(x) \right) = \frac{d}{dx} (2^{2-x}(1 - x \ln 2) - 1) = -2^{2-x} \ln 2 (2 - x \ln 2) < 0,$$

for all $x \in (0, 1/2)$. This means that u' is strictly decreasing. Since

$$u'(1/2) = 2^{3/2}(1 - \ln \sqrt{2}) - 1 \approx 0.8482 > 0,$$

we obtain

$$u'(x) > 0,$$

for all $x \in (0, 1/2)$. Hence, u is strictly increasing on $[0, 1/2]$. Since

$$u(1/2) = \sqrt{2} - 3/2 < 0,$$

we get

$$u(x) \leq u(1/2) < 0,$$

for all $x \in [0, 1/2]$. But

$$g(1/2) = \frac{1}{2} (s2^{2-s} - 1 - s) = \frac{1}{2} u(s) \leq 0,$$

since $0 < s = 1/p' - \epsilon < 1/p' \leq \frac{1}{2}$ for all $0 < \epsilon < 1/p'$. Thus f is nonnegative and nonincreasing on $(1, \infty)$. Therefore x is nonnegative and nonincreasing.

Now, we turn to the case $2 \leq p < \infty$. By using (4.4) and (4.25), we get that

$$p - 1 = \|C^*\|_{\ell_{\text{dec}}^p(\mathbb{N})} - \|I\|_{\ell_{\text{dec}}^p(\mathbb{N})} \leq \|C^* - I\|_{\ell_{\text{dec}}^p(\mathbb{N})} \leq \|C^* - I\|_{\ell^p(\mathbb{N})} = p - 1,$$

from which it follows

$$\|C^* - I\|_{\ell_{\text{dec}}^p(\mathbb{N})} = p - 1.$$

□

4.3 Inequalities in $\ell_+^p(\mathbb{N})$ for $1 < p < \infty$

Let $A(p)$, $B(p)$, $U(p)$, and $V(p)$ denote the optimal constants for which the inequalities

$$A(p) \|(C^* - I)x\|_p \leq \|x\|_p \leq B(p) \|(C^* - I)x\|_p, \quad (4.26)$$

$$U(p) \|(C - S)^*x\|_p \leq \|x\|_p \leq V(p) \|(C - S)^*x\|_p \quad (4.27)$$

hold for all nonnegative $x \in \ell_+^p(\mathbb{N})$ and for the range $1 < p < \infty$.

By (4.5), the best constant $B(p)$ in (4.26) is not finite.

Concerning the sharp constant $U(p)$ in (4.27), it follows from Proposition 3.9 that

$$U(p) = 2^{-1/p}, \quad 1 < p \leq 2.$$

The following result gives the best constant $A(p)$ in (4.26) and the optimal constant $V(p)$ in (4.27) for all $1 < p < \infty$.

Theorem 4.7. *Let $1 < p < \infty$ and $x \in \ell_+^p(\mathbb{N})$ be a nonnegative sequence. Then the following inequalities are sharp: if $1 < p \leq 2$, then*

$$M_p^{-1/p} \|(C^* - I)x\|_p \leq \|x\|_p \leq (p-1)^{-1} \|(C - S)^*x\|_p,$$

and if $2 \leq p < \infty$, then

$$(p-1)^{-1} \|(C^* - I)x\|_p \leq \|x\|_p \leq m_p^{-1/p} \|(C - S)^*x\|_p.$$

Proof. Let $1 < p < \infty$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative sequence. By applying Theorem 4.2 and Theorem 4.4, we obtain

$$M_p^{-1/p} \|(C^* - I)x\|_p \leq \|x\|_p \quad \text{if } 1 < p \leq 2,$$

and

$$\|x\|_p \leq m_p^{-1} \|(C^* - S^*)x\|_p \quad \text{if } 2 \leq p < \infty.$$

Regarding the optimality of the constants $M_p^{-1/p}$ and m_p^{-1} , the sequences used to establish their sharpness in Theorem 4.2 and Theorem 4.4, respectively, are in fact nonnegative. Now, from (4.4) we have

$$\|C^* - I\|_{\ell^p(\mathbb{N})} = p - 1, \quad \text{for } 2 \leq p < \infty,$$

and from Theorem 4.6 we have

$$\|C^* - I\|_{\ell_{\text{dec}}^p(\mathbb{N})} = p - 1, \quad \text{for } 2 \leq p < \infty.$$

Taking into account (2.1), we then obtain

$$\|C^* - I\|_{\ell_+^p(\mathbb{N})} = p - 1, \quad \text{for } 2 \leq p < \infty.$$

In Theorem 4.4 we have seen that

$$[(C - S)^*]_{\ell^p(\mathbb{N})} = \frac{1}{p-1}, \quad \text{for } 1 < p \leq 2,$$

and from Theorem 4.6 we have

$$[(C - S)^*]_{\ell_{\text{dec}}^p(\mathbb{N})} = \frac{1}{p-1}, \quad \text{for } 1 < p \leq 2.$$

By using (2.2), we obtain that

$$[(C - S)^*]_{\ell_+^p(\mathbb{N})} = \frac{1}{p-1}, \quad \text{for } 1 < p \leq 2.$$

□

4.4 Endpoint inequalities

Let $A(p)$, $B(p)$, $U(p)$, and $V(p)$ denote the optimal constants for which the inequalities

$$A(p) \|(C^* - I)x\|_p \leq \|x\|_p \leq B(p) \|(C^* - I)x\|_p, \quad (4.28)$$

$$U(p) \|(C - S)^*x\|_p \leq \|x\|_p \leq V(p) \|(C - S)^*x\|_p, \quad (4.29)$$

hold for all $x \in \ell^p(\mathbb{N})$, $\ell_+^p(\mathbb{N})$, or $\ell_{\text{dec}}^p(\mathbb{N})$ when $p = 1$ or $p = \infty$.

As established in (4.4), we have

$$\|C^* - I\|_{\ell^1(\mathbb{N})} = 2.$$

Turning to reverse inequalities for the operator $C^* - I$ on $\ell^p(\mathbb{N})$, at the endpoint cases $p = 1$ and $p = \infty$, it follows from (4.5) that

$$[C^* - I]_{\ell_{\text{dec}}^1(\mathbb{N})} = [C^* - I]_{\ell_+^1(\mathbb{N})} = [C^* - I]_{\ell^1(\mathbb{N})} = \infty,$$

and likewise,

$$[C^* - I]_{\ell_{\text{dec}}^\infty(\mathbb{N})} = [C^* - I]_{\ell_+^\infty(\mathbb{N})} = [C^* - I]_{\ell^\infty(\mathbb{N})} = \infty.$$

Concerning the value of the operator p -norm of $(C - S)^*$ on the various cones of $\ell^1(\mathbb{N})$, it follows from Proposition 3.9 that

$$\|(C - S)^*\|_{\ell_{\text{dec}}^1(\mathbb{N})} = \|(C - S)^*\|_{\ell_+^1(\mathbb{N})} = \|(C - S)^*\|_{\ell^1(\mathbb{N})} = 2.$$

The following result gives us the exact value of $\|C^* - I\|_{\ell^1(\mathbb{N})}$.

Theorem 4.8. *Let $x \in \ell_+^1(\mathbb{N})$ be a nonnegative sequence. Then the following inequality is sharp:*

$$\|(C^* - I)x\|_1 \leq 2\|x\|_1.$$

Proof. By (4.4) we have that

$$\|(C^* - I)x\|_1 \leq 2\|x\|_1,$$

for all $x \in \ell^1(\mathbb{N})$. In particular, the inequality holds for all nonnegative $x \in \ell^1_+(\mathbb{N})$.

Let us now see that the above inequality is sharp. Consider the sequence $\{e_m\}_{m \geq 1}$. We have that

$$|(C^* - I)e_m(n)| = \begin{cases} \frac{1}{m}, & \text{if } n < m, \\ 1 - \frac{1}{m}, & \text{if } n = m, \\ 0, & \text{if } n > m. \end{cases}$$

Hence

$$\frac{\|(C^* - I)e_m\|_1}{\|e_m\|_1} = \frac{m-1}{m} + \left(1 - \frac{1}{m}\right) = 2\left(1 - \frac{1}{m}\right) \rightarrow 2,$$

as $m \rightarrow \infty$. □

Now, we study the exact value of $\|C^* - I\|_{\ell^1_{\text{dec}}(\mathbb{N})}$. But before that, we need to prove the following technical but useful result related to harmonic numbers (see [32]).

Lemma 4.9. *If $n, m \in \mathbb{N}$. Then*

$$\sum_{k=n}^{n+m} \frac{1}{k} \neq 1.$$

Proof. We are going to proceed by contradiction. So, we assume that there are $n, m \in \mathbb{N}$ such that

$$\sum_{k=n}^{n+m} \frac{1}{k} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m} = 1. \quad (4.30)$$

Let $D(n, m) = \{n, n+1, \dots, n+m\}$ be the set of the denominators that appear in $\sum_{k=n}^{n+m} \frac{1}{k}$ and let 2^α be the largest power of 2 dividing one of the denominators $k \in D(n, m)$. The set $D(n, m)$ cannot have an even multiple of 2^α , which means that if $k = 2^\alpha r$, then r must be odd, and k is the unique element in $D(n, m)$ with this property, because if there exists another $l = 2^\alpha s$, with s being an odd number, and assuming without loss of generality that $k < l$, we would have that $r < s$ being both odd numbers. Thus, there is an even number t such $r < t < s$ and $2^\alpha t \in D(n, m)$ which is a contradiction. So, multiplying by $2^{\alpha-1}$ on both sides of (4.30) we obtain

$$\begin{aligned} 2^{\alpha-1} &= 2^{\alpha-1} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{k-1} + \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n+m} \right) \\ &= \frac{2^{\alpha-1}}{n} + \frac{2^{\alpha-1}}{n+1} + \cdots + \frac{2^{\alpha-1}}{k-1} + \frac{2^{\alpha-1}}{k} + \frac{2^{\alpha-1}}{k+1} + \cdots + \frac{2^{\alpha-1}}{n+m} \\ &= \frac{2^{\alpha-1}}{n} + \frac{2^{\alpha-1}}{n+1} + \cdots + \frac{2^{\alpha-1}}{k-1} + \frac{1}{2r} + \frac{2^{\alpha-1}}{k+1} + \cdots + \frac{2^{\alpha-1}}{n+m}. \end{aligned}$$

Let $\frac{a_q}{b_q}$ be the irreducible fraction of $\frac{2^{\alpha-1}}{q}$, where $q \in \{n, n+1, \dots, n+m\} \setminus \{k\}$. Hence, we get that every $\frac{a_q}{b_q}$ has an odd denominator for $q \in \{n, n+1, \dots, n+m\} \setminus \{k\}$. Therefore, we obtain that

$$\frac{a_n}{b_n} + \frac{a_{n+1}}{b_{n+1}} + \dots + \frac{a_{k-1}}{b_{k-1}} + \frac{a_{k+1}}{b_{k+1}} + \dots + \frac{a_{n+m}}{b_{n+m}} = \frac{a}{b},$$

where b is an odd number (being the product of odd numbers). Hence,

$$\begin{aligned} 2^{\alpha-1} &= \left(\frac{a_n}{b_n} + \frac{a_{n+1}}{b_{n+1}} + \dots + \frac{a_{k-1}}{b_{k-1}} + \frac{a_{k+1}}{b_{k+1}} + \dots + \frac{a_{n+m}}{b_{n+m}} \right) + \frac{1}{2r} \\ &= \frac{a}{b} + \frac{1}{2r} = \frac{2ar + b}{2rb}, \end{aligned}$$

and then

$$\frac{2ar + b}{2rb} = 2^{\alpha-1} \in \mathbb{N},$$

which is a contradiction because $\frac{2ar+b}{2rb}$ is a fraction of an odd number over an even number. \square

The following result is one of main results of this section and it provides the exact value of $\|C^* - I\|_{\ell_{\text{dec}}^1(\mathbb{N})}$.

Theorem 4.10. *Let $x \in \ell_{\text{dec}}^1(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then the following inequality is sharp:*

$$\|(C^* - I)x\|_1 \leq \left(\sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| \right) \|x\|_1.$$

Moreover,

$$\sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| = \frac{2}{e}.$$

Proof. An useful fact that we are going to use is the following

$$\|y\|_1 = \|C^*y\|_1, \tag{4.31}$$

for all $y \in \ell_+^1(\mathbb{N})$. Indeed, if $y \in \ell_+^1(\mathbb{N})$ is a nonnegative sequence then

$$\|C^*y\|_1 = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{y(k)}{k} = \sum_{k=1}^{\infty} \frac{y(k)}{k} \sum_{n=1}^k 1 = \sum_{k=1}^{\infty} y(k) = \|y\|_1.$$

Now, let $x = \{x(n)\}_{n \geq 1} \in \ell^1(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Observe that there is $y = \{y(n)\}_{n \geq 1}$ such that $C^*y = x$. Indeed, it suffices to take

$$y(n) = n(x(n) - x(n+1)) \geq 0, \quad n \in \mathbb{N}.$$

So, if $n \geq 1$ is a natural number,

$$C^*y(n) = \sum_{k=n}^{\infty} \frac{y(k)}{k} = \sum_{k=n}^{\infty} x(k) - \sum_{k=n}^{\infty} x(k+1) = \sum_{k=n}^{\infty} x(k) - \sum_{k=n+1}^{\infty} x(k) = x(n).$$

Now, we have that if $n \in \mathbb{N}$, then

$$(C^*)^2y(n) = \sum_{k=n}^{\infty} \frac{C^*y(k)}{k} = \sum_{k=n}^{\infty} \sum_{j=k}^{\infty} \frac{y(j)}{jk} = \sum_{j=n}^{\infty} \frac{y(j)}{j} \sum_{k=n}^j \frac{1}{k},$$

so

$$(C^*)^2y(n) - C^*y(n) = \sum_{j=n}^{\infty} \frac{y(j)}{j} \left(\sum_{k=n}^j \frac{1}{k} - 1 \right),$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} \|C^*x - x\|_1 &= \|(C^*)^2y - C^*y\|_1 = \sum_{n=1}^{\infty} \left| \sum_{j=n}^{\infty} \frac{y(j)}{j} \left(\sum_{k=n}^j \frac{1}{k} - 1 \right) \right| \\ &\leq \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \frac{y(j)}{j} \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| = \sum_{j=1}^{\infty} \frac{y(j)}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| \\ &\leq \left(\sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| \right) \sum_{j=1}^{\infty} y(j) = \|y\|_1 \sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| \\ &= \|C^*y\|_1 \sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| = \|x\|_1 \sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|, \end{aligned}$$

where we have used the equality (4.31). This means that

$$\|C^* - I\|_{\ell_{\text{dec}}^1(\mathbb{N})} \leq \sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|.$$

Now let us see that the constant

$$B(1) = \sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|$$

is optimal. For this purpose, we are going to take the sequence of sequences $\{x^j\}_{j \geq 1}$ given by

$$x^j(n) = \begin{cases} \frac{1}{j}, & \text{if } n \leq j, \\ 0, & \text{if } n > j. \end{cases}$$

Observe that $\|x^j\|_{\ell^1} = \frac{1}{j} \sum_{n=1}^j 1 = 1$, for all $j \in \mathbb{N}$. Thus, we have that

$$|C^* x^j(n) - x^j(n)| = \begin{cases} \frac{1}{j} \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|, & \text{if } n \leq j, \\ 0, & \text{if } n > j, \end{cases}$$

and therefore

$$\|C^* x^j - x^j\|_1 = \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|.$$

Then

$$\begin{aligned} \sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| &\geq \|C^* - I\|_{\ell_{\text{dec}}^1(\mathbb{N})} = \sup_{x \in \ell_{\text{dec}}^1(\mathbb{N}) \setminus \{0\}} \frac{\|C^* x - x\|_{\ell^1(\mathbb{N})}}{\|x\|_{\ell^1(\mathbb{N})}} \\ &\geq \sup_{j \in \mathbb{N}} \frac{\|C^* x^j - x^j\|_{\ell^1(\mathbb{N})}}{\|x^j\|_{\ell^1(\mathbb{N})}} = \sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|, \end{aligned}$$

and hence

$$\sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| = \|C^* - I\|_{\ell_{\text{dec}}^1(\mathbb{N})}.$$

Now, let $A = \{A(j)\}_{j=1}^{\infty}$ be the sequence given by

$$A(j) = \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|.$$

We begin by seeing that A is increasing. Observe that $A(1) = 0$ and for each natural number $j \geq 2$, there is $1 \leq n_j < j$ such that

$$\left| \sum_{k=n_j}^j \frac{1}{k} - 1 \right| = \sum_{k=n_j}^j \frac{1}{k} - 1 \quad \text{and} \quad \left| \sum_{k=n_j+1}^j \frac{1}{k} - 1 \right| = 1 - \sum_{k=n_j+1}^j \frac{1}{k}.$$

Observe that, for each $j \geq 2$, we have that n_{j+1} verifies that

$$n_{j+1} = \begin{cases} n_j, & \text{if } \left| \sum_{k=n_j+1}^{j+1} \frac{1}{k} - 1 \right| = 1 - \sum_{k=n_j+1}^{j+1} \frac{1}{k}, \\ n_j + 1, & \text{if } \left| \sum_{k=n_j+1}^{j+1} \frac{1}{k} - 1 \right| = \sum_{k=n_j+1}^{j+1} \frac{1}{k} - 1, \end{cases}$$

and the number n_{j+1} is well defined thanks to Lemma 4.9. Now, let $j \geq 2$ be a natural number. By changing the order of summation and separating the sums, we get

$$\begin{aligned}
\sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| &= \sum_{n=1}^{n_j} \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| + \sum_{n=n_j+1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| \\
&= \sum_{n=1}^{n_j} \left(\sum_{k=n}^j \frac{1}{k} - 1 \right) + \sum_{n=n_j+1}^j \left(1 - \sum_{k=n}^j \frac{1}{k} \right) \\
&= \left(\sum_{n=1}^{n_j} \sum_{k=n}^j \frac{1}{k} \right) - n_j + (j - n_j) - \sum_{n=n_j+1}^j \sum_{k=n}^j \frac{1}{k} \\
&= \sum_{n=1}^{n_j} \sum_{k=n}^{n_j} \frac{1}{k} + \sum_{n=1}^{n_j} \sum_{k=n_j+1}^j \frac{1}{k} - n_j + (j - n_j) - \sum_{k=n_j+1}^j \frac{1}{k} \sum_{n=n_j+1}^k 1 \\
&= n_j + n_j \sum_{k=n_j+1}^j \frac{1}{k} - n_j + (j - n_j) - \sum_{k=n_j+1}^j \frac{k - n_j}{k} \\
&= n_j \sum_{k=n_j+1}^j \frac{1}{k} + (j - n_j) - (j - n_j) + n_j \sum_{k=n_j+1}^j \frac{1}{k} = 2n_j \sum_{k=n_j+1}^j \frac{1}{k}.
\end{aligned}$$

Then, for all natural numbers $j \geq 2$, we have that

$$A(j) = \frac{2n_j}{j} \sum_{k=n_j+1}^j \frac{1}{k},$$

where n_j is defined above. Now let $j \geq 2$ be a natural number. If $n_{j+1} = n_j = n$, then

$$\begin{aligned}
\frac{j(j+1)}{2n} (A(j+1) - A(j)) &= j \sum_{k=n+1}^{j+1} \frac{1}{k} - (j+1) \sum_{k=n+1}^j \frac{1}{k} \\
&= 1 + (j - (j+1)) \sum_{k=n+1}^{j+1} \frac{1}{k} = 1 - \sum_{k=n_{j+1}+1}^{j+1} \frac{1}{k} > 0.
\end{aligned}$$

If $n_{j+1} = n_j + 1 = n$, then

$$\begin{aligned}
\frac{j(j+1)}{2} (A(j+1) - A(j)) &= nj \sum_{k=n+1}^{j+1} \frac{1}{k} - (n-1)(j+1) \sum_{k=n}^j \frac{1}{k} \\
&= n - 1 - j + (nj - (n-1)(j+1)) \sum_{k=n}^{j+1} \frac{1}{k} \\
&= (j+1 - n_{j+1}) \left(\sum_{k=n_{j+1}}^{j+1} \frac{1}{k} - 1 \right) > 0.
\end{aligned}$$

Thus A is increasing. Since $C^* - I$ is a bounded operator from $\ell^1(\mathbb{N})$ on itself, we have that

$$0 \leq A(j) \leq \sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| = \|C^* - I\|_{\ell^1_{\text{dec}}(\mathbb{N})} \leq \|C^* - I\|_{\ell^1(\mathbb{N})} < \infty,$$

for all $j \in \mathbb{N}$. Thus, A is an increasing and bounded sequence in \mathbb{R} which implies that A converges. Let $H_m = \sum_{k=1}^m \frac{1}{k}$ be the m -th harmonic number. It is well known (for example, see [31]) that

$$H_m = \ln m + \gamma + \frac{1}{2m} - \epsilon_m, \quad (4.32)$$

where γ is the Euler-Mascheroni constant and $0 \leq \epsilon_m \leq \frac{1}{8m^2}$, which approaches 0 as m goes to infinity. Let $1 \leq n \leq j$ be natural numbers. Then

$$\left| \sum_{k=n}^j \frac{1}{k} - 1 \right| = \left| H_j - H_n + \frac{1}{n} - 1 \right| = \left| \ln \left(\frac{j}{n} \right) - 1 + \frac{1}{2j} + \frac{1}{2n} + \epsilon_n - \epsilon_j \right|,$$

so

$$\begin{aligned} \left| \ln \left(\frac{j}{n} \right) - 1 \right| - \left(\frac{1}{2j} + \frac{1}{2n} \right) - \left| \epsilon_n - \epsilon_j \right| &\leq \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| \\ &\leq \left| \ln \left(\frac{j}{n} \right) - 1 \right| + \left(\frac{1}{2j} + \frac{1}{2n} \right) + \left| \epsilon_n - \epsilon_j \right|, \end{aligned} \quad (4.33)$$

for all natural numbers $1 \leq n \leq j$. Thus, by (4.32), we have

$$\frac{1}{j} \sum_{n=1}^j \left(\frac{1}{2j} + \frac{1}{2n} \right) = \frac{1 + H_j}{2j} \rightarrow 0, \quad (4.34)$$

as $j \rightarrow \infty$, and

$$\begin{aligned} 0 \leq \frac{1}{j} \sum_{n=1}^j \left| \epsilon_n - \epsilon_j \right| &\leq \frac{1}{j} \sum_{n=1}^j \epsilon_n + \frac{1}{j} \sum_{n=1}^j \epsilon_j \leq \frac{1}{8j^2} + \frac{1}{8j} \sum_{n=1}^j \frac{1}{n^2} \\ &\leq \frac{1}{8j^2} + \frac{1}{8j} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{8j^2} + \frac{\pi^2}{48j} \rightarrow 0, \end{aligned}$$

as $j \rightarrow \infty$, which means that

$$\frac{1}{j} \sum_{n=1}^j \left| \epsilon_n - \epsilon_j \right| \rightarrow 0, \quad (4.35)$$

as $j \rightarrow \infty$. On the other hand, identifying these terms as suitable Riemann sums, we obtain that

$$\lim_{j \rightarrow \infty} \frac{1}{j} \sum_{n=1}^j \left| \ln \left(\frac{j}{n} \right) - 1 \right| = \int_0^1 \left| \ln \left(\frac{1}{x} \right) - 1 \right| dx = \frac{2}{e}. \quad (4.36)$$

Now, by taking the limit as $j \rightarrow \infty$ in the inequality (4.33), and using the values of the limits in (4.34), (4.35), and (4.36), we obtain

$$\sup_{j \in \mathbb{N}} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| = \lim_{j \rightarrow \infty} \frac{1}{j} \sum_{n=1}^j \left| \sum_{k=n}^j \frac{1}{k} - 1 \right| = \frac{2}{e},$$

and this concludes the proof. \square

We now turn to the study of the first inequality in (4.28) and (4.29) in the case $p = \infty$.

Theorem 4.11. *The following equalities hold:*

$$\|C^* - I\|_{\ell_{\text{dec}}^{\infty}(\mathbb{N})} = \|C^* - I\|_{\ell_{+}^{\infty}(\mathbb{N})} = \|C^* - I\|_{\ell^{\infty}(\mathbb{N})} = \infty,$$

and

$$\|(C - S)^*\|_{\ell_{\text{dec}}^{\infty}(\mathbb{N})} = \|(C - S)^*\|_{\ell_{+}^{\infty}(\mathbb{N})} = \|(C - S)^*\|_{\ell^{\infty}(\mathbb{N})} = \infty.$$

Proof. We are going to consider the sequence $\{x_j\}_{j \geq 1}$ given by

$$x_j(n) = \begin{cases} 1, & \text{if } n \leq j, \\ 0, & \text{if } n > j. \end{cases} \quad (4.37)$$

Observe that $\|x_j\|_{\infty} = 1$, for all natural number $j \geq 1$. Thus, we have that

$$|(C^* - I)x_j(n)| = \begin{cases} \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|, & \text{if } n \leq j, \\ 0, & \text{if } n > j, \end{cases}$$

and therefore

$$\|(C^* - I)x_j\|_{\infty} \geq |(C^* - I)x_j(1)| = \sum_{k=2}^j \frac{1}{k} \rightarrow \infty,$$

as $j \rightarrow \infty$. This shows that the operator $C^* - I$ is not bounded on $\ell^{\infty}(\mathbb{N})$, even when restricted to the cone of nonnegative and nonincreasing sequences in $\ell^{\infty}(\mathbb{N})$. Hence

$$\|C^* - I\|_{\ell_{\text{dec}}^{\infty}(\mathbb{N})} = \|C^* - I\|_{\ell_{+}^{\infty}(\mathbb{N})} = \|C^* - I\|_{\ell^{\infty}(\mathbb{N})} = \infty.$$

By taking the same x_j as in (4.37), we obtain that

$$|(C - S)^*x_j(n)| = \begin{cases} \sum_{k=1}^j \frac{1}{k}, & \text{if } n = 1, \\ \left| \sum_{k=n}^j \frac{1}{k} - 1 \right|, & \text{if } 1 < n \leq j, \\ 1, & \text{if } n = j + 1, \\ 0, & \text{if } n > j + 1. \end{cases}$$

Hence

$$\|(C - S)^* x_j\|_\infty \geq |(C - S)^* x_j(1)| = \sum_{k=1}^j \frac{1}{k} \rightarrow \infty,$$

as $j \rightarrow \infty$. This shows that the operator $(C - S)^*$ is not bounded on $\ell^\infty(\mathbb{N})$, even when restricted to the cone of nonnegative and nonincreasing sequences in $\ell^\infty(\mathbb{N})$.

Hence

$$\|(C - S)^*\|_{\ell_{\text{dec}}^\infty(\mathbb{N})} = \|(C - S)^*\|_{\ell_+^\infty(\mathbb{N})} = \|(C - S)^*\|_{\ell^\infty(\mathbb{N})} = \infty.$$

□

Now, let us see what happens with the second inequality in (4.29) in case $p = 1$.

Theorem 4.12. *The following equalities hold:*

$$[(C - S)^*]_{\ell_{\text{dec}}^1(\mathbb{N})} = [(C - S)^*]_{\ell_+^1(\mathbb{N})} = [(C - S)^*]_{\ell^1(\mathbb{N})} = \infty.$$

Proof. Let $m \geq 2$ be a natural number and consider the sequence x_m given by

$$x_m(n) = \begin{cases} 1, & \text{if } n = 1, \\ \frac{1}{n-1}, & \text{if } 1 < n \leq m, \\ 0, & \text{if } n > m. \end{cases}$$

Note that x_m is a nonnegative, nonincreasing sequence, and

$$\|x_m\|_1 = 1 + \sum_{k=2}^m \frac{1}{k-1} = 1 + \sum_{k=1}^{m-1} \frac{1}{k}.$$

Now, on the one hand, we have

$$S^* x_m(n) = \begin{cases} 0, & \text{if } n = 1, \\ 1, & \text{if } n = 2, \\ \frac{1}{n-2}, & \text{if } 2 < n \leq m+1, \\ 0, & \text{if } n > m+1, \end{cases}$$

On the other hand,

$$\begin{aligned} C^* x_m(1) &= \sum_{k=1}^{\infty} \frac{x(k)}{k} = 1 + \sum_{k=2}^m \frac{1}{k(k-1)} = 1 + \sum_{k=2}^m \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ &= 1 + \sum_{k=1}^{m-1} \frac{1}{k} - \sum_{k=2}^m \frac{1}{k} = 2 - \frac{1}{m}, \end{aligned}$$

and, for $2 \leq n \leq m$, we get that

$$C^*x_m(n) = \sum_{k=1}^{\infty} \frac{x_m(k)}{k} = \sum_{k=n}^m \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{n-1} - \frac{1}{m}.$$

Thus

$$C^*x^m(n) = \begin{cases} 2 - \frac{1}{m}, & \text{if } n = 1, \\ 1 - \frac{1}{m}, & \text{if } n = 2, \\ \frac{1}{n-1} - \frac{1}{m}, & \text{if } 2 < n \leq m, \\ 0, & \text{if } n > m. \end{cases}$$

So

$$\begin{aligned} |(C - S)^*x_m(n)| &= \begin{cases} 2 - \frac{1}{m}, & \text{if } n = 1, \\ \frac{1}{m}, & \text{if } n = 2, \\ \left| \frac{1}{n-1} - \frac{1}{n-2} - \frac{1}{m} \right|, & \text{if } 2 < n \leq m, \\ \frac{1}{m-1}, & \text{if } n = m+1, \\ 0, & \text{if } n > m+1, \end{cases} \\ &= \begin{cases} 2 - \frac{1}{m}, & \text{if } n = 1, \\ \frac{1}{m}, & \text{if } n = 2, \\ \frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{m}, & \text{if } 2 < n \leq m, \\ \frac{1}{m-1}, & \text{if } n = m+1, \\ 0, & \text{if } n > m+1. \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} \|(C - S)^*x_m\|_1 &= 2 - \frac{1}{m} + \frac{1}{m} + \frac{1}{m-1} + \sum_{n=3}^m \left(\frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{m} \right) \\ &= 2 + \frac{1}{m-1} + \frac{m-2}{m} + 1 - \frac{1}{m-1} = 4 - \frac{2}{m}. \end{aligned}$$

Hence

$$0 \leq \frac{\|(C - S)^*x_m\|_1}{\|x_m\|_1} \leq \frac{4}{\sum_{k=1}^m \frac{1}{k}} \rightarrow 0,$$

as $m \rightarrow \infty$. By Lemma 2.2, this shows that $(C - S)^*$ is not bounded below on $\ell^1(\mathbb{N})$. Since each x_m is nonnegative and nonincreasing, it follows that

$$[(C - S)^*]_{\ell^1_{\text{dec}}(\mathbb{N})} = \infty.$$

Finally, combining this with (2.2) yields

$$[(C - S)^*]_{\ell_{\text{dec}}^1(\mathbb{N})} = [(C - S)^*]_{\ell_+^1(\mathbb{N})} = [(C - S)^*]_{\ell^1(\mathbb{N})} = \infty.$$

□

Now, let us see what happens with the second inequality in (4.29) in case $p = \infty$.

Theorem 4.13. *Let $x \in \ell^\infty(\mathbb{N})$, then the following sharp estimates hold:*

(i) *If x is a general sequence,*

$$\|x\|_\infty \leq 2\|(C - S)^*x\|_\infty. \quad (4.38)$$

(ii) *If x is a nonnegative sequence,*

$$\|x\|_\infty \leq 2\|(C - S)^*x\|_\infty. \quad (4.39)$$

(iii) *If x is a nonnegative and nonincreasing sequence,*

$$\|x\|_\infty \leq \|(C - S)^*x\|_\infty. \quad (4.40)$$

Proof. We have seen in (3.1) that

$$\|(C - I)y\|_\infty \leq 2\|y\|_\infty, \quad (4.41)$$

for all $y \in \ell^\infty(\mathbb{N})$. Let $x \in \ell^\infty(\mathbb{N})$. If $(C - S)^*x \notin \ell^\infty(\mathbb{N})$, inequality (4.38) holds trivially. If $(C - S)^*x \in \ell^\infty(\mathbb{N})$, then the inequality (4.41), with $y = (C - S)^*x$, and the identity (4.2) shows

$$\|x\|_\infty = \|S^*x\|_\infty = \|(C - I)(C - S)^*x\|_\infty \leq 2\|(C - S)^*x\|_\infty.$$

This means that (4.38) and (4.39) hold. Let us now see the optimality of the constant 2 in (4.38) and in (4.39). To carry out the proof, we consider the sequence x_m defined by

$$x_m(n) = \begin{cases} 0, & \text{if } n < m, \\ n((n-1)^{-2} - n^{-2}), & \text{if } n \geq m, \end{cases}$$

where $m \geq 2$ is a natural number. First, we compute the ℓ^∞ -norm of x_m . For $n \geq m$, we have

$$x_m(n) = \frac{n}{(n-1)^2} - \frac{1}{n} = \frac{2n-1}{n(n-1)^2}.$$

Next, consider the function $f : (1, \infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{2x-1}{x(x-1)^2}.$$

A straightforward computation gives

$$f'(x) = -\frac{4x^2 - 3x + 1}{x^2(x-1)^3} < 0 \quad \text{for all } x > 1.$$

Thus, f is decreasing on $(1, \infty)$, which implies that x_m is nonnegative and

$$\|x_m\|_\infty = x_m(m) = m((m-1)^{-2} - m^{-2}), \quad \text{for all } m \geq 2.$$

Now, on the one hand, we have that

$$C^*x^m(n) = \sum_{k=n}^{\infty} \frac{x^m(k)}{k} = \sum_{k=n}^{\infty} ((k-1)^{-2} - k^{-2}) = \sum_{k=n-1}^{\infty} k^{-2} - \sum_{k=n}^{\infty} k^{-2} = (n-1)^{-2},$$

for all $n \geq m$. So

$$C^*x_m(n) = \begin{cases} (m-1)^{-2}, & \text{if } n < m, \\ (n-1)^{-2}, & \text{if } n \geq m. \end{cases}$$

On the other hand,

$$S^*x_m(n) = \begin{cases} 0, & \text{if } n \leq m, \\ (n-1)((n-2)^{-2} - (n-1)^{-2}), & \text{if } n > m. \end{cases}$$

Therefore

$$\begin{aligned} |(C-S)^*x_m(n)| &= \begin{cases} (m-1)^{-2}, & \text{if } n \leq m, \\ |n(n-1)^{-2} - (n-1)(n-2)^{-2}|, & \text{if } n > m, \end{cases} \\ &= \begin{cases} (m-1)^{-2}, & \text{if } n \leq m, \\ (n-1)(n-2)^{-2} - n(n-1)^{-2}, & \text{if } n > m. \end{cases} \end{aligned}$$

Now, consider the function $g : (2, \infty) \rightarrow \mathbb{R}$ and $h : (1, \infty) \rightarrow \mathbb{R}$ given by

$$g(x) = (x-1)(x-2)^{-2} - x(x-1)^{-2},$$

and

$$h(x) = g(x+1) = \frac{x}{(x-1)^2} - \frac{x+1}{x^2}.$$

The derivative of h is given by

$$h'(x) = \frac{x+2}{x^3} - \frac{x+1}{(x-1)^3}.$$

Thus

$$\begin{aligned} h'(x) \leq 0 &\iff \frac{x+2}{x^3} \leq \frac{x+1}{(x-1)^3} \iff (x+2)(x-1)^3 \leq x^3(x+1) \\ &\iff q(x) := 2x^3 + 3x^2 - 5x + 2 \geq 0. \end{aligned}$$

Since

$$q'(x) = 6x^2 + 6x - 5 \geq 0 \quad \text{for all } x > 1,$$

we have q increasing on $(1, \infty)$, and thus

$$q(x) \geq q(1) = 2 + 3 - 5 + 2 = 2.$$

Therefore, $h'(x) \leq 0$ for all $x > 1$. This means that h is nonincreasing in $(1, \infty)$. Thus, g is not increasing in $(2, \infty)$. Therefore,

$$\|(C - S)^*x_m\|_\infty = \max\{(m-1)^{-2}, m(m-1)^{-2} - (m+1)m^{-2}\}.$$

Now, we have that

$$\begin{aligned} \frac{(m+1)(m-1)}{m^2} = \frac{m^2-1}{m^2} < 1 &\iff \frac{(m+1)(m-1)^2}{m^2} < m-1 \\ &\iff 1 < m - \frac{(m+1)(m-1)^2}{m^2} \\ &\iff (m-1)^{-2} < m(m-1)^{-2} - (m+1)m^{-2}, \end{aligned}$$

for all $m \geq 2$. Then

$$\|(C - S)^*x_m\|_\infty = m(m-1)^{-2} - (m+1)m^{-2},$$

for all $m \geq 2$. Finally,

$$\begin{aligned} \frac{\|x_m\|_\infty}{\|(C - S)^*x_m\|_\infty} &= \frac{m((m-1)^{-2} - m^{-2})}{m(m-1)^{-2} - (m+1)m^{-2}} = \frac{(m-1)^{-2} - m^{-2}}{(m-1)^{-2} - (m+1)m^{-3}} \\ &= \frac{1 - (1 - 1/m)^2}{1 - \frac{m+1}{m}(1 - 1/m)^2} = \frac{2 - 1/m}{m - (m+1)(1 + 1/m^2 - 2/m)} \\ &= \frac{2 - 1/m}{1 - 1/m^2 + 1/m} \rightarrow 2, \end{aligned}$$

as $m \rightarrow \infty$. Since x_m is nonnegative, we have that the constant 2 is optimal in (4.38) and in (4.39).

We now turn to the cone $\ell_{\text{dec}}^\infty(\mathbb{N})$. Let $x \in \ell_{\text{dec}}^\infty(\mathbb{N})$ be a nonnegative, nonincreasing sequence. We will distinguish between two cases: when $x \in c_0$ and when $x \notin c_0$. We begin with the case $x \notin c_0$. Let $x \in \ell_{\text{dec}}^\infty(\mathbb{N}) \setminus c_0$ be a nonnegative and nonincreasing sequence. Since x is monotone and bounded below, there exists some $\alpha > 0$ such that

$$\lim_{n \rightarrow \infty} x(n) = \alpha.$$

Let y be the sequence defined by

$$y(n) = \begin{cases} 1, & \text{if } n \leq m, \\ 0, & \text{if } n > m, \end{cases}$$

where $m \geq 1$ is a fixed natural number. Then,

$$\begin{aligned} m\|(C - S)^*x\|_\infty &= \|y\|_1\|(C - S)^*x\|_\infty \geq \|y(C - S)^*x\|_1 \\ &= \sum_{n=1}^{\infty} |y(n)(C - S)^*x(n)| = \sum_{n=1}^m |(C - S)^*x(n)| \\ &\geq \left| \sum_{n=1}^m (C - S)^*x(n) \right|. \end{aligned}$$

Thus,

$$\begin{aligned} \|(C - S)^*x\|_\infty &\geq \left| \frac{1}{m} \sum_{n=1}^m (C - S)^*x(n) \right| = |CC^*x(m) - CS^*x(m)| \\ &= |CS^*x(m) + C^*x(m) - CS^*x(m)| = |C^*x(m)|, \end{aligned}$$

for all $m \in \mathbb{N}$, where we have applied (4.3). Hence,

$$\|C^*x\|_\infty \leq \|(C - S)^*x\|_\infty.$$

Thus,

$$\|(C - S)^*x\|_\infty \geq \|C^*x\|_\infty \geq \sum_{k=1}^{\infty} \frac{x(k)}{k} \geq \alpha \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

We have proved that if $x \in \ell_{\text{dec}}^\infty(\mathbb{N}) \setminus c_0$, then $(C - S)^*x \notin \ell^\infty(\mathbb{N})$, and therefore the inequality (4.40) is satisfied trivially.

Let us now see what happens in the case when $x \in c_0$. Since the set of finitely supported sequences is dense in c_0 , we may assume without loss of generality that x has finite support. That means that there is $m \in \mathbb{N}$ such that

$$x(1) \geq x(2) \geq \cdots \geq x(m-1) \geq x(m) > 0,$$

and

$$x(n) = 0,$$

for all $n > m$. Observe that $\|x\|_\infty = x(1)$. We are going to need the following inequality

$$\left| x(n-1) - \sum_{k=n}^m \frac{x(k)}{k} \right| \leq \sum_{k=1}^m \frac{x(k)}{k}, \quad (4.42)$$

for all $2 \leq n \leq m$. Let $T(n)$ be

$$T(n) = \sum_{k=n}^m \frac{x(k)}{k},$$

where $1 \leq n \leq m$. On the one hand, trivially, we have that

$$T(1) = \sum_{k=1}^m \frac{x(k)}{k} \geq \sum_{k=n}^m \frac{x(k)}{k} = T(n),$$

for all $2 \leq n \leq m$. On the other hand,

$$T(1) = \sum_{k=1}^m \frac{x(k)}{k} \geq \sum_{k=1}^{n-1} \frac{x(k)}{k} \geq x(n-1) \sum_{k=1}^{n-1} \frac{1}{k} \geq x(n-1), \quad (4.43)$$

for all $2 \leq n \leq m$, where we have used the fact that x is nonnegative, and nonincreasing. Therefore, since for any nonnegative a, b we have the elementary inequality

$$\max\{a, b\} \geq |a - b|,$$

it follows that for $2 \leq n \leq m$,

$$T(1) \geq \max\{T(n), x(n-1)\} \geq |T(n) - x(n-1)|.$$

This establishes (4.42). Thus,

$$|(C - S)^*x(n)| = \begin{cases} \sum_{k=1}^m \frac{x(k)}{k}, & \text{if } n = 1, \\ \left| x(n-1) - \sum_{k=n}^m \frac{x(k)}{k} \right|, & \text{if } 2 \leq n \leq m, \\ x(m), & \text{if } n = m + 1, \\ 0, & \text{if } n > m + 1. \end{cases}$$

Hence, by using (4.42) and (4.43), we get that

$$\|(C - S)^*x\|_\infty = \sum_{k=1}^m \frac{x(k)}{k},$$

which means that

$$\frac{\|x\|_\infty}{\|(C - S)^*x\|_\infty} = \frac{x(1)}{\sum_{k=1}^m \frac{x(k)}{k}} \leq 1.$$

This proves (4.40). To show that the constant 1 is sharp in (4.40) we consider the sequence e_1 . A straightforward computation shows that $(C - S)^*e_1 = (1, -1, 0, 0, \dots)$. Therefore,

$$\|e_1\|_\infty = \|(C - S)^*e_1\|_\infty = 1.$$

□

4.5 Tables

In this section, we present in tables all known constants related to the operators $C^* - I$ and $C^* - S^*$ acting on different cones of $\ell^p(\mathbb{N})$, for $1 \leq p \leq \infty$. The constants displayed in bold have been proved in this chapter.

$\ C^* - I\ _X$	$\ell^p(\mathbb{N})$	$\ell_+^p(\mathbb{N})$	$\ell_{\text{dec}}^p(\mathbb{N})$
$p = 1$	2	2	2/e
$1 < p < 2$	$M_p^{1/p}$	$M_p^{1/p}$?
$2 \leq p < \infty$	$p - 1$	$p - 1$	$p - 1$
$p = \infty$	∞	∞	∞

$[C^* - I]_X$	$\ell^p(\mathbb{N})$	$\ell_+^p(\mathbb{N})$	$\ell_{\text{dec}}^p(\mathbb{N})$
$p = 1$	∞	∞	∞
$1 < p < 2$	∞	∞	∞
$2 \leq p < \infty$	∞	∞	∞
$p = \infty$	∞	∞	∞

$\ C^* - S^*\ _X$	$\ell^p(\mathbb{N})$	$\ell_+^p(\mathbb{N})$	$\ell_{\text{dec}}^p(\mathbb{N})$
$p = 1$	2	2	2
$1 < p \leq 2$	$2^{1/p}$	$2^{1/p}$	$2^{1/p}$
$2 < p < \infty$?	?	?
$p = \infty$	∞	∞	∞

$[C^* - S^*]_X$	$\ell^p(\mathbb{N})$	$\ell_+^p(\mathbb{N})$	$\ell_{\text{dec}}^p(\mathbb{N})$
$p = 1$	∞	∞	∞
$1 < p \leq 2$	$1/(p - 1)$	$1/(p - 1)$	$1/(p - 1)$
$2 < p < \infty$	$m_p^{-1/p}$	$m_p^{-1/p}$?
$p = \infty$	2	2	1

Remark 4.14. These tables and those in Section 6.5 indicate, in particular, that

$$\begin{aligned}\|C^* - I\|_{\ell^p(\mathbb{N})} &= \|H^* - I\|_{L^p(\mathbb{R}^+)}, \\ \|C^* - I\|_{\ell_+^p(\mathbb{N})} &= \|H^* - I\|_{L_+^p(\mathbb{R}^+)}, \\ [C^* - S^*]_{\ell^p(\mathbb{N})} &= [H^* - I]_{L^p(\mathbb{R}^+)}, \\ [C^* - S^*]_{\ell_+^p(\mathbb{N})} &= [H^* - I]_{L_+^p(\mathbb{R}^+)},\end{aligned}$$

for all $1 \leq p \leq \infty$. Moreover,

$$\|C^* - I\|_{\ell_{\text{dec}}^p(\mathbb{N})} = \|H^* - I\|_{L_{\text{dec}}^p(\mathbb{R}^+)},$$

for all $2 \leq p \leq \infty$, and

$$\|C^* - I\|_{\ell_{\text{dec}}^1(\mathbb{N})} = \|H^* - I\|_{L_{\text{dec}}^1(\mathbb{R}^+)},$$

while

$$[C^* - S^*]_{\ell_{\text{dec}}^p(\mathbb{N})} = [H^* - I]_{L_{\text{dec}}^p(\mathbb{R}^+)}$$

for all $1 \leq p \leq 2$. Also we note in the tables in Section 6.5 that

$$\|H^* - I\|_{L_{\text{dec}}^1(\mathbb{R}^+)} = \left(\int_0^1 |\ln(t) + 1|^p dt \right)^{1/p}, \quad 1 < p < 2,$$

and

$$[H^* - I]_{L_{\text{dec}}^p(\mathbb{R}^+)} = \left(\int_0^1 |\ln(t) + 1|^p dt \right)^{-1/p}, \quad 2 < p < \infty.$$

Let

$$A(p) = \|C^* - I\|_{\ell_{\text{dec}}^p(\mathbb{N})}, \quad 1 < p < 2,$$

and

$$V(p) = [C^* - S^*]_{\ell_{\text{dec}}^p(\mathbb{N})}, \quad 2 < p < \infty.$$

In light of the preceding observations, it is natural to conjecture that

$$A(p) = \left(\int_0^1 |\ln(t) + 1|^p dt \right)^{1/p}, \quad 1 < p < 2,$$

and

$$V(p) = \left(\int_0^1 |\ln(t) + 1|^p dt \right)^{-1/p}, \quad 2 < p < \infty.$$

Regarding the quantities $\|C^* - S^*\|_{\ell_{\text{dec}}^p(\mathbb{N})}$, $\|C^* - S^*\|_{\ell_+^p(\mathbb{N})}$, and $\|C^* - S^*\|_{\ell^p(\mathbb{N})}$ for $2 < p < \infty$, we make use of (4.25), from which it follows that

$$p - 1 \leq \|C^*\|_{\ell_{\text{dec}}^p(\mathbb{N})} - \|S^*\|_{\ell_{\text{dec}}^p(\mathbb{N})} \leq \|C^* - S^*\|_{\ell_{\text{dec}}^p(\mathbb{N})}.$$

Moreover,

$$\frac{\|(C - S)^* e_1\|_p}{\|e_1\|_p} = 2^{1/p}.$$

Consequently, we obtain the estimates

$$\begin{aligned}\max\{p - 1, 2^{1/p}\} &\leq \|C^* - S^*\|_{\ell_{\text{dec}}^p(\mathbb{N})} \leq \|C^* - S^*\|_{\ell_+^p(\mathbb{N})} \\ &\leq \|C^* - S^*\|_{\ell^p(\mathbb{N})} \leq \|C^*\|_{\ell^p(\mathbb{N})} + \|S^*\|_{\ell^p(\mathbb{N})} = p + 1.\end{aligned}$$

Chapter 5

Sharp bounds for the Hardy operator

In the continuous setting, the counterpart of the Cesàro operator is the classical *Hardy averaging operator*, denoted by H , and defined as

$$Hf(x) = \frac{1}{x} \int_0^x f(t) dt,$$

for any locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^+)$.

Following the studies carried out in Chapters 3 and 4, we investigate inequalities in Lebesgue spaces associated with the operator $H - I$.

A fundamental property of the operator H , which is not shared by the Cesàro operator C in the discrete setting, is the identity (see [30])

$$H \circ H^* = H^* \circ H = H + H^*, \quad (5.1)$$

where H^* denotes the adjoint of H , given by

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t} dt,$$

whenever the above integral is well defined for f .

In contrast, in the discrete setting we have seen in (3.4) and (4.3) that

$$C \circ C^* = C + S \circ C^* = C \circ S^* + C^* \neq C + C^*.$$

Thus, the continuous operator H satisfies a symmetry relation that fails in the discrete case.

Observe moreover that identity (5.1) can equivalently be written as

$$(H - I)(H^* - I) = (H^* - I)(H - I) = I. \quad (5.2)$$

This identity plays a central role in the development of the present chapter as well as in the next one.

The operator $H - I$ controls the oscillation of a function with respect to its average and arises naturally in several otherwise disparate contexts. For example, the complex Beurling–Ahlfors transform reduces to $H - I$ when restricted to radial functions; see [3, 46]. When $p = 2$, the operator $H - I$ is a Hilbert space isomorphism that is unitarily equivalent to the unilateral shift; see [21]. It has also been linked to Laguerre polynomials in [36]. In the theory of interpolation of operators, $H - I$ acts on rearrangements of functions (nonnegative, nonincreasing functions) and yields an equivalent norm in Lorentz spaces; see [7, p. 384].

The p -norm of the operator $H - I$ acting on $L^p(\mathbb{R}^+)$ for $1 < p \leq \infty$ was first obtained by M. Strzelecki in [46]. In [47], the same author extended this study to various cones of Lebesgue spaces, including weighted cases with power weights. Later, independently, G. Sinnamon derived the same result in [44]. In these works, it was shown that

$$\|H - I\|_{L^p(\mathbb{R}^+)} = \begin{cases} \frac{1}{p-1}, & \text{for } 1 < p \leq 2, \\ m_p^{-1/p}, & \text{for } 2 \leq p < \infty, \\ 2, & \text{for } p = \infty, \end{cases} \quad (5.3)$$

where m_p is defined in (2.5).

Regarding the cone of nonnegative functions in $L^p(\mathbb{R}^+)$, S. Boza and J. Soria proved the following result.

Theorem 5.1. [18, Theorem 3.2, Proposition 4.1] *Let $2 \leq p \leq \infty$ and $f \in L_+^p(\mathbb{R}^+)$ be a nonnegative function. Then the following inequality is sharp.*

$$\|(H - I)f\|_p \leq \|f\|_p. \quad (5.4)$$

With respect to the cone of nonnegative, nonincreasing sequences in $L^p(\mathbb{R}^+)$, V. Kolyada [34] completed earlier work in [16] and [36] by determining the optimal constants in the L^p -norm equivalence between Hf and H^*f for nonnegative functions. A central idea in this approach is its connection with the problem of finding sharp upper and lower bounds for the operator $H - I$ when restricted to nonnegative, nonincreasing functions.

Theorem 5.2. [34, Theorem 1.2] *Let $1 < p < \infty$ and let f be a nonnegative, nonincreasing function. If $1 < p \leq 2$, then*

$$(p - 1)\|(H - I)f\|_p \leq \|f\|_p \leq (p - 1)^{1/p}\|(H - I)f\|_p, \quad (5.5)$$

and if $2 \leq p < \infty$, then

$$(p - 1)^{1/p}\|(H - I)f\|_p \leq \|f\|_p \leq (p - 1)\|(H - I)f\|_p. \quad (5.6)$$

The constants $p - 1$ and $(p - 1)^{1/p}$ in (5.5) and (5.6) are the best possible.

S. Boza and J. Soria proved the following endpoint estimate.

Theorem 5.3. *Let $f \in L_{\text{dec}}^\infty(\mathbb{R}^+)$ be a nonnegative and nonincreasing function. Then the following inequality is sharp:*

$$\|(H - I)f\|_\infty \leq \|f\|_\infty.$$

Building on these results, our goal is to determine the missing constants $A(p)$ and $B(p)$ in the inequalities

$$A(p) \|(H - I)f\|_p \leq \|f\|_p \leq B(p) \|(H - I)f\|_p, \quad (5.7)$$

on the three cones $L^p(\mathbb{R}^+)$, $L_+^p(\mathbb{R}^+)$, and $L_{\text{dec}}^p(\mathbb{R}^+)$, for the whole range $1 \leq p \leq \infty$.

The results of this chapter have appeared in [11, 12].

Chapter Overview

This chapter is divided into four sections. The first two focus on the study of inequality (5.7) on $L^p(\mathbb{R}^+)$ and on $L_+^p(\mathbb{R}^+)$ for $1 < p < \infty$. The inequality (5.7) on $L_{\text{dec}}^p(\mathbb{R}^+)$ for $1 < p < \infty$ is addressed in Theorem 5.2. The third section examines this inequality on the three cones of interest in $L^p(\mathbb{R}^+)$ at the endpoints $p \in \{1, \infty\}$. In the final section, we present in tables all optimal constants related to the operator $H - I$, indicating which constants were established in this chapter.

• 5.1 Inequalities in $L^p(\mathbb{R}^+)$ for $1 < p < \infty$	81
• 5.2 Inequalities in $L_+^p(\mathbb{R}^+)$ for $1 < p < \infty$	83
• 5.3 Endpoint inequalities	85
• 5.4 Tables	87

5.1 Inequalities in $L^p(\mathbb{R}^+)$ for $1 < p < \infty$

Let $A(p)$ and $B(p)$ denote the optimal constants for which

$$A(p) \|(H - I)f\|_p \leq \|f\|_p \leq B(p) \|(H - I)f\|_p \quad (5.8)$$

holds for every nonnegative function $f \in L_+^p(\mathbb{R}^+)$ and for all $1 < p < \infty$.

Regarding the first inequality in (5.8), we have seen in (5.3) that

$$A(p) = \begin{cases} p - 1, & \text{if } 1 < p \leq 2, \\ m_p^{1/p}, & \text{if } 2 \leq p < \infty. \end{cases}$$

where m_p is defined in (2.5).

The result below provides the exact value of $B(p)$ in (5.8) for all $1 < p < \infty$ and for arbitrary functions $f \in L^p(\mathbb{R}^+)$.

Theorem 5.4. *Let $1 < p < \infty$ and let f be an arbitrary function. If $1 < p \leq 2$, then*

$$\|f\|_p \leq M_p^{1/p} \|(H - I)f\|_p, \quad (5.9)$$

where M_p is defined in (2.5), and if $2 \leq p < \infty$, then

$$\|f\|_p \leq (p - 1) \|(H - I)f\|_p. \quad (5.10)$$

Moreover, all the constants in (5.9) and (5.10) are the best possible.

Proof. Let $1 < p < \infty$. Taking duality into account and using (2.6) in (5.3), we obtain that

$$\|H^* - I\|_p = \|H - I\|_{p'} = \begin{cases} 2, & \text{if } p = 1, \\ M_p^{1/p}, & \text{if } 1 < p \leq 2, \\ p - 1, & \text{if } 2 \leq p < \infty. \end{cases} \quad (5.11)$$

We focus on proving only (5.9), since the proof of (5.10) is analogous. Assume that $1 < p \leq 2$, and let f be an arbitrary function. If $(H - I)f \notin L^p(\mathbb{R}^+)$, then (5.9) holds trivially. Otherwise, assuming $(H - I)f \in L^p(\mathbb{R}^+)$, by taking into account (5.2) and applying (5.11) yields

$$\|f\|_p = \|(H^* - I)(H - I)f\|_p \leq M_p^{1/p} \|(H - I)f\|_p.$$

Let us now prove the optimality of the constant $M_p^{1/p}$ in (5.9). By taking into account Lemma 2.4, let $t_p \in (0, 1/2)$ be such that $f_p(t_p) = M_p$, where f_p is defined as in (2.4). Let $r = 1/t_p$ and let g be the function given by

$$g(x) = -\chi_{(0,1)}(x) + (r - 1)x^{-r}\chi_{[1,\infty)}(x).$$

Then

$$Hg(x) = -\chi_{(0,1)}(x) - x^{-r}\chi_{[1,\infty)}(x).$$

Therefore

$$|(H - I)g(x)| = rx^{-r}\chi_{[1,\infty)}(x).$$

Hence

$$\frac{\|g\|_p^p}{\|(H - I)g\|_p^p} = \frac{rp - 1 + (r - 1)^p}{r^p} = pt_p^{p-1} - t_p + (1 - t_p^p) = M_p.$$

To see the optimality of the constant $(p - 1)$ we consider the function $f = \chi_{(0,1)}$, which is nonnegative, nonincreasing function. Then

$$Hf(x) = \chi_{(0,1)}(x) + \frac{1}{x}\chi_{[1,\infty)}(x) = f(x) + \frac{1}{x}\chi_{[1,\infty)}(x).$$

Therefore,

$$\frac{\|f\|_p}{\|(H-I)f\|_p} = \int_1^\infty x^{-p} dx = (p-1),$$

for all $1 < p < \infty$. This concludes the proof. \square

Remark 5.5. Note that (5.3), together with (5.9) and (5.10), implies the existence of optimal constants $c(p), C(p) > 0$ such that

$$c(p) \|(H-I)f\|_p \leq \|f\|_p \leq C(p) \|(H-I)f\|_p,$$

for all $f \in L^p(\mathbb{R}^+)$, with $1 < p < \infty$. It is now easy to see that $\|(H-I)f\|_p$ is an equivalent norm in $L^p(\mathbb{R}^+)$.

5.2 Inequalities in $L^p_+(\mathbb{R}^+)$ for $1 < p < \infty$

Let $A(p)$ and $B(p)$ denote the optimal constants for which

$$A(p) \|(H-I)f\|_p \leq \|f\|_p \leq B(p) \|(H-I)f\|_p \quad (5.12)$$

hold for every nonnegative function $f \in L^p_+(\mathbb{R}^+)$ and for all $1 < p < \infty$.

As shown in (5.4), the exact value of $A(p)$ in (5.12) for $2 \leq p < \infty$ is

$$A(p) = 1.$$

The aim of this section is to determine the exact value of $A(p)$ for $1 < p < 2$ and of $B(p)$ for $1 < p < \infty$ in (5.12).

The following lemma will be needed.

Lemma 5.6. *Let $p \in \mathbb{F}$ and let $f \in L^p_+(\mathbb{R}^+)$ be a nonnegative function. Then the following inequality holds:*

$$\int_0^\infty f(x)^p dx \leq \int_0^\infty (Hf(x) - f(x))^p dx.$$

Proof. It suffices to prove the result for functions f in a dense subset of $L^p_+(\mathbb{R}^+)$; hence, we may assume that f is continuous, nonnegative, and compactly supported in \mathbb{R}^+ . Then $Hf(x)$ satisfies

$$\lim_{x \rightarrow 0^+} x(Hf(x))^p = \lim_{x \rightarrow \infty} x(Hf(x))^p = 0,$$

and

$$\frac{d}{dx} x(Hf(x))^p = (1-p)(Hf(x))^p + p(Hf(x))^{p-1} f(x).$$

Thus,

$$\int_0^\infty \left((1-p)(Hf(x))^p + p(Hf(x))^{p-1} f(x) \right) dx = \int_0^\infty \frac{d}{dx} x(Hf(x))^p dx = 0.$$

Let $x > 0$ be such that $f(x) \neq 0$, and set $t = Hf(x)/f(x)$. Then, by Lemma 2.6, we have

$$\begin{aligned} (p-1)(Hf(x))^p - p(Hf(x))^{p-1}f(x) &= f(x)^p \left((p-1)t^p - pt^{p-1} \right) \\ &\leq f(x)^p \left((t-1)^p - 1 \right) \\ &= (Hf(x) - f(x))^p - f(x)^p, \end{aligned}$$

which also holds when $f(x) = 0$. Integrating over \mathbb{R}^+ then yields

$$\int_0^\infty f(x)^p dx \leq \int_0^\infty (Hf(x) - f(x))^p dx.$$

□

The following result gives the best constant $B(p)$ in (5.12) for $1 < p \leq 2$.

Theorem 5.7. *If $1 < p \leq 2$ and let $f \in L^p_+(\mathbb{R}^+)$ be a nonnegative function, then the following sharp estimate holds:*

$$\|f\|_p \leq \|(H - I)f\|_p. \quad (5.13)$$

Proof. Let $1 < p \leq 2$. It suffices to prove the result for functions f in a dense subset of $L^p_+(\mathbb{R}^+)$; hence, we may assume that f is continuous, nonnegative, and compactly supported in \mathbb{R}^+ . We have that $f \in L^q(\mathbb{R}^+)$ for all $q \in (1, 2)$ and $q \mapsto \|f\|_q$ is a continuous function in $(1, 2)$. Let $g = (H - I)f$. Since $H - I$ is an isometry on $L^2(\mathbb{R}^+)$ (see [21]), we get that

$$\|g\|_q \leq M\|g\|_2 = M\|f\|_2 < \infty,$$

for all $q \in (1, 2)$, where $M > 0$ depends only on f . Thus the map $q \mapsto \|g\|_q$ is also well defined and continuous in $(1, 2)$. Let $\{p(n)\}_{n \geq 1} \subset \mathbb{F}$ such that $\lim_{n \rightarrow \infty} p(n) = p$. Then, by taking into account Lemma (5.6), we obtain

$$\|f\|_p = \lim_{n \rightarrow \infty} \|f\|_{p(n)} \leq \lim_{n \rightarrow \infty} \|(H - I)f\|_{p(n)} = \|(H - I)f\|_p.$$

The function $f_\epsilon = \chi_{[1, 1+\epsilon]}$, where $\epsilon > 0$, shows the optimality of the constant 1 in (5.13) since $\lim_{\epsilon \rightarrow 0^+} \frac{\|f_\epsilon\|_p}{\|(H-I)f_\epsilon\|_p} = 1$. □

The following theorem determines the optimal constant $A(p)$ for $1 < p \leq 2$ and the sharp constant $B(p)$ for $2 \leq p < \infty$ in (5.12).

Theorem 5.8. *Let $1 < p < \infty$ and let $f \in L^p_+(\mathbb{R}^+)$ be a nonnegative function. If $1 < p \leq 2$, then*

$$(p-1)\|(H - I)f\|_p \leq \|f\|_p.$$

If $2 \leq p < \infty$, then

$$\|f\|_p \leq (p-1)\|(H - I)f\|_p.$$

Furthermore, the constant $(p-1)$ is the best possible in both inequalities.

Proof. We have by (5.3) that if $1 < p \leq 2$, then

$$(p-1)\|(H-I)f\|_p \leq \|f\|_p,$$

for all $f \in L^p(\mathbb{R}^+)$. In particular, the inequality holds for all nonnegative functions $f \in L_+^p(\mathbb{R}^+)$. Also, we have seen in (5.10) that if $2 \leq p < \infty$, then

$$\|f\|_p \leq (p-1)\|(H-I)f\|_p,$$

for all $f \in L_+^p(\mathbb{R}^+)$. In particular, the inequality holds for all nonnegative functions $f \in L_+^p(\mathbb{R}^+)$.

Regarding the optimality of the constant $(p-1)$, note that the function employed to establish the sharpness of $(p-1)$ in (5.10) of Theorem 5.4 is in fact nonnegative and nonincreasing. \square

5.3 Endpoint inequalities

The main goal of this section is to determine the optimal constants $A(p)$ and $B(p)$ in the inequalities

$$A(p)\|(H-I)f\|_p \leq \|f\|_p \leq B(p)\|(H-I)f\|_p,$$

valid for all $f \in L^p(\mathbb{R}^+)$, $L_+^p(\mathbb{R}^+)$, or $L_{\text{dec}}^p(\mathbb{R}^+)$ when $p = 1$ or $p = \infty$.

We observed in (5.3) that

$$\|H-I\|_{L^\infty(\mathbb{R}^+)} = 2,$$

and in Theorems 5.1 and 5.3 that

$$\|H-I\|_{L_{\text{dec}}^\infty(\mathbb{R}^+)} = \|H-I\|_{L_+^\infty(\mathbb{R}^+)} = 1.$$

For the reader's convenience, all the estimates for $\|H-I\|_{L^\infty(\mathbb{R}^+)}$, $\|H-I\|_{L_+^\infty(\mathbb{R}^+)}$, and $\|H-I\|_{L_{\text{dec}}^\infty(\mathbb{R}^+)}$ mentioned above are included in the following theorem, which forms the main result of this section.

Theorem 5.9. *The following equalities hold:*

$$\begin{aligned} \|H-I\|_{L_{\text{dec}}^\infty(\mathbb{R}^+)} &= \|H-I\|_{L_+^\infty(\mathbb{R}^+)} = 1, & \|H-I\|_{L^\infty(\mathbb{R}^+)} &= 2, \\ [H-I]_{L_{\text{dec}}^\infty(\mathbb{R}^+)} &= [H-I]_{L_+^\infty(\mathbb{R}^+)} = [H-I]_{L^\infty(\mathbb{R}^+)} = \infty, \end{aligned} \quad (5.14)$$

$$\|H-I\|_{L_{\text{dec}}^1(\mathbb{R}^+)} = \|H-I\|_{L_+^1(\mathbb{R}^+)} = \|H-I\|_{L^1(\mathbb{R}^+)} = \infty, \quad (5.15)$$

$$[H-I]_{L_{\text{dec}}^1(\mathbb{R}^+)} = [H-I]_{L_+^1(\mathbb{R}^+)} = 0, \quad [H-I]_{L^1(\mathbb{R}^+)} = 2. \quad (5.16)$$

Proof. We start proving (5.14). Consider the function $f = \chi_{(0,\infty)}$. Note that f is nonnegative, nonincreasing, and satisfies $\|f\|_\infty = 1$. A straightforward computation shows that

$$(H-I)f = 0.$$

By Lemma 2.2, this implies that $H - I$ is not bounded below on $L^\infty(\mathbb{R}^+)$. Since f is nonnegative and nonincreasing, it follows that

$$[H - I]_{L_{\text{dec}}^\infty(\mathbb{R}^+)} = \infty.$$

Finally, combining this with (2.2) gives

$$[H - I]_{L_{\text{dec}}^\infty(\mathbb{R}^+)} = [H - I]_{L_+^\infty(\mathbb{R}^+)} = [H - I]_{L^\infty(\mathbb{R}^+)} = \infty.$$

We now turn to (5.15). Let $g = \chi_{(0,1)}$. Note that g is nonnegative, nonincreasing, and satisfies $\|g\|_1 = 1$. A simple calculation gives

$$(H - I)g(x) = \begin{cases} 0, & 0 < x < 1, \\ \frac{1}{x}, & 1 \leq x < \infty. \end{cases}$$

Observe that $(H - I)g \notin L^1(\mathbb{R}^+)$. This shows that

$$\|H - I\|_{L_{\text{dec}}^1(\mathbb{R}^+)} = \infty.$$

Using (2.1), it follows that

$$\|H - I\|_{L_{\text{dec}}^1(\mathbb{R}^+)} = \|H - I\|_{L_+^1(\mathbb{R}^+)} = \|H - I\|_{L^1(\mathbb{R}^+)} = \infty.$$

To conclude, we proceed to prove (5.16). We start with the equality

$$[H - I]_{L^1(\mathbb{R}^+)} = 2. \tag{5.17}$$

Equivalently, we have to show that the following inequality is sharp

$$\|f\|_1 \leq 2\|(H - I)f\|_1, \tag{5.18}$$

for all $f \in L^1(\mathbb{R}^+)$. Let f be an arbitrary function. If $(H - I)f \notin L^1(\mathbb{R}^+)$, then (5.18) holds trivially. Otherwise, assuming that $(H - I)f \in L^1(\mathbb{R}^+)$, and using (5.2) together with (5.11), we obtain

$$\|f\|_1 = \|(H^* - I)(H - I)f\|_1 \leq \|H^* - I\|_1 \|(H - I)f\|_1 = 2\|(H - I)f\|_1.$$

This proves the inequality (5.18). To establish the sharpness of the constant 2 in (5.18), we consider the function

$$f_m(x) = \chi_{(0,m]}(x) - m\chi_{(m,m+1]}(x),$$

where $m \geq 1$ is a natural number. Note that $\|f_m\|_1 = 2m$. On the other hand, we have

$$Hf_m(x) = \begin{cases} 1, & \text{if } 0 < x \leq m, \\ m \cdot \frac{(m+1-x)}{x}, & \text{if } m < x \leq m+1, \\ 0, & \text{if } x > 0. \end{cases}$$

Therefore

$$(H - I)f_m(x) = \begin{cases} 0, & \text{if } 0 < x \leq m, \\ m \cdot \frac{(m+1)}{x}, & \text{if } m < x \leq m+1, \\ 0, & \text{if } x > m+1. \end{cases}$$

Hence

$$\frac{\|f_m\|_1}{\|(H - I)f_m\|_1} = \frac{2m}{m(m+1) \ln\left(\frac{m+1}{m}\right)} = \frac{2}{\ln\left(1 + \frac{1}{m}\right)^{m+1}} \rightarrow 2,$$

as $m \rightarrow \infty$. This concludes the proof of (5.17).

Now, we turn to the equality

$$[H - I]_{L^1_+(\mathbb{R}^+)} = 0.$$

Let $f \in L^1_+(\mathbb{R}^+)$ be a nonnegative function with $\|f\|_1 = 1$. We argue by contradiction and assume that $(H - I)f \in L^1(\mathbb{R}^+)$. Then

$$\|Hf\|_1 = \|(H - I)f + f\|_1 \leq \|(H - I)f\|_1 + \|f\|_1 < \infty.$$

Hence $Hf \in L^1(\mathbb{R}^+)$. By [4, Proposition 3.1], this implies that $f = 0$, which contradicts the assumption $\|f\|_1 = 1$. Hence $(H - I)f \notin L^1(\mathbb{R}^+)$ for all nonnegative $f \in L^1_+(\mathbb{R}^+) \setminus \{0\}$. This proves that

$$[H - I]_{L^1_+(\mathbb{R}^+)} = 0.$$

In particular, by taking into account (2.2), we have

$$0 \leq [H - I]_{L^1_{\text{dec}}(\mathbb{R}^+)} \leq [H - I]_{L^1_+(\mathbb{R}^+)} = 0.$$

Thus

$$[H - I]_{L^1_{\text{dec}}(\mathbb{R}^+)} = 0.$$

This completes the proof. \square

5.4 Tables

In this section, we present in tables all known constants related to the operator $H - I$ acting on different cones of $L^p(\mathbb{R}^+)$, for $1 \leq p \leq \infty$. The constants displayed in bold have been proved in this chapter.

$\ H - I\ _X$	$L^p(\mathbb{R}^+)$	$L^p_+(\mathbb{R}^+)$	$L^p_{\text{dec}}(\mathbb{R}^+)$
$p = 1$	∞	∞	∞
$1 < p \leq 2$	$1/(p - 1)$	$1/(p - 1)$	$1/(p - 1)$
$2 \leq p < \infty$	$m_p^{-1/p}$	1	$1/(p - 1)^{1/p}$
$p = \infty$	2	1	1

$[H - I]_X$	$L^p(\mathbb{R}^+)$	$L^p_+(\mathbb{R}^+)$	$L^p_{\text{dec}}(\mathbb{R}^+)$
$p = 1$	2	0	0
$1 < p \leq 2$	$M_p^{1/p}$	1	$(p - 1)^{1/p}$
$2 \leq p < \infty$	$p - 1$	$p - 1$	$p - 1$
$p = \infty$	∞	∞	∞

Chapter 6

Sharp bounds for the adjoint of Hardy operator

The adjoint of the Hardy operator, denoted by H^* , is defined by

$$H^* f(x) = \int_x^\infty \frac{f(t)}{t} dt,$$

whenever the above integral is well defined for the function f on $(0, \infty)$.

We have seen in (5.11) that

$$\|H^* - I\|_{L^p(\mathbb{R}^+)} = \begin{cases} 2, & \text{if } p = 1, \\ M_p^{1/p}, & \text{if } 1 < p \leq 2, \\ p - 1, & \text{if } 2 \leq p < \infty, \end{cases}$$

where M_p is defined in (2.5).

Motivated by the results established in Chapters 3, 4, and 5, the main goal of this chapter is to determine the missing constants $A(p)$ and $B(p)$ in the inequalities

$$A(p) \|(H^* - I)f\|_p \leq \|f\|_p \leq B(p) \|(H^* - I)f\|_p, \quad (6.1)$$

on the cones $L^p(\mathbb{R}^+)$, $L_+^p(\mathbb{R}^+)$, and $L_{\text{dec}}^p(\mathbb{R}^+)$, for the whole range $1 \leq p \leq \infty$.

The results of this chapter have appeared in [11, 12, 14].

Chapter Overview

This chapter is divided into six sections. The first three focus on the study of inequality (6.1) on a specific cone of $L^p(\mathbb{R}^+)$ for the range $1 < p < \infty$. The fourth section examines the inequality on the three cones of our interest in $L^p(\mathbb{R}^+)$ at the endpoints $p \in \{1, \infty\}$. In the concluding section, we compile and present in tables

all the optimal bounds associated with the operator $H^* - I$, specifying which of them were established in this chapter.

- **6.1** Inequalities in $L^p(\mathbb{R}^+)$ for $1 < p < \infty$ 90
- **6.2** Inequalities in $L^p_{\text{dec}}(\mathbb{R}^+)$ for $1 < p < \infty$ 92
- **6.3** Inequalities in $L^p_+(\mathbb{R}^+)$ for $1 < p < \infty$ 97
- **6.4** Endpoint inequalities 97
- **6.5** Tables 103

6.1 Inequalities in $L^p(\mathbb{R}^+)$ for $1 < p < \infty$

Let $A(p)$ and $B(p)$ denote the optimal constants for which

$$A(p) \|(H^* - I)f\|_p \leq \|f\|_p \leq B(p) \|(H^* - I)f\|_p, \tag{6.2}$$

hold for all $f \in L^p(\mathbb{R}^+)$ and for the full range $1 < p < \infty$.

Regarding the first inequality in (6.2), we have seen in (5.11) that $A(p)$ is given by

$$A(p) = \begin{cases} M_p^{1/p}, & \text{if } 1 < p \leq 2, \\ \frac{1}{p-1}, & \text{if } 2 \leq p < \infty. \end{cases}$$

Our next goal is to determine the exact value of $B(p)$ in (6.2) for all $1 < p < \infty$. Before doing so, we introduce a family of functions that will be used in Theorem 6.2.

Lemma 6.1. *Let $p > 1$, $g_q(s) = \frac{1}{q}s^{-1/q}(s)\chi_{(1,\infty)}(s)$ and $f_q = H^*g_q$ for $1 < q < p$. Then $g_q, f_q \in L^p_+(\mathbb{R}^+)$ are nonnegative, f_q is nonincreasing, and*

$$\lim_{q \rightarrow p^-} \frac{\|(H^* - I)f_q\|_p}{\|f_q\|_p} = p - 1.$$

Proof. Observe that

$$f_q(t) = \int_t^\infty g_q(s) \frac{ds}{s} = \frac{1}{q} \int_{\max(t,1)}^\infty s^{-1/q} \frac{ds}{s} = \chi_{(0,1)}(t) + t^{-1/q}\chi_{[1,\infty)}(t).$$

On the one hand,

$$\|f_q\|_p^p = \int_0^1 dt + \int_1^\infty t^{-p/q} dt = 1 - \frac{1}{1 - p/q} = \frac{p}{p - q}.$$

On the other hand, if $0 < x < 1$,

$$|H^* f_q(x) - f_q(x)| = \left| \int_x^1 \frac{dt}{t} + \int_1^\infty t^{-1/q} \frac{dt}{t} - 1 \right| = -\ln x + q - 1$$

and, if $1 \leq x < \infty$,

$$|H^* f_q(x) - f_q(x)| = \left| \int_x^\infty t^{-1/q} \frac{dt}{t} - x^{-1/q} \right| = (q-1)x^{-1/q}.$$

Therefore,

$$\begin{aligned} \lim_{q \rightarrow p^-} \frac{\|(H^* - I)f_q\|_p^p}{\|f_q\|_p^p} &= \lim_{q \rightarrow p^-} \frac{p-q}{p} \left(\int_0^1 (-\ln x + q - 1)^p dx + \int_1^\infty (q-1)^p x^{-p/q} dx \right) \\ &= \lim_{q \rightarrow p^-} \frac{p-q}{p} \left(\int_0^1 (-\ln x + q - 1)^p dx + (q-1)^p \frac{q}{p-q} \right) \\ &= (p-1)^p. \end{aligned}$$

Take p th roots to complete the proof. \square

Theorem 6.2. *Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^+)$. If $1 < p \leq 2$, then*

$$\|f\|_p \leq (p-1)^{-1} \|(H^* - I)f\|_p, \quad (6.3)$$

and if $2 \leq p < \infty$, then

$$\|f\|_p \leq m_p^{-1/p} \|(H^* - I)f\|_p. \quad (6.4)$$

Moreover, all the constants in (6.3) and (6.4) are the best possible.

Proof. We prove only (6.3), since (6.4) admits an analogous proof. Let $1 < p \leq 2$ and let $f \in L^p(\mathbb{R}^+)$. Define $g = (H^* - I)f$. Since $H^* - I$ is bounded on $L^p(\mathbb{R}^+)$, it follows that $g \in L^p(\mathbb{R}^+)$. Taking into account the identities (5.2) and (5.3), we obtain

$$\begin{aligned} \|f\|_p &= \|(H - I)(H^* - I)f\|_p = \|(H - I)g\|_p \\ &\leq \|H - I\|_{L^p(\mathbb{R}^+)} \|g\|_p = (p-1)^{-1} \|g\|_p \\ &= (p-1)^{-1} \|(H^* - I)f\|_p. \end{aligned}$$

We now address the optimality of the constants in (6.3) and (6.4). For the case $1 < p \leq 2$, Lemma 6.1 shows that the constant $(p-1)^{-1}$ in (6.3) is optimal.

Now, we are going to consider the case $2 \leq p < \infty$. Let t_p be as in Lemma 2.3 and $r = 1/t_p > 2$. Let f be the function defined on $(0, \infty)$ by

$$f(s) = r s^{-r} \chi_{(1, \infty)}(s).$$

We have that

$$\|f\|_p^p = r^p \int_1^\infty s^{-rp} ds = \frac{r^p}{rp-1}.$$

On the other hand,

$$H^*f(s) = \begin{cases} 1, & \text{if } 0 < s < 1, \\ s^{-r}, & \text{if } s \geq 1. \end{cases}$$

So

$$|H^*f(s) - f(s)| = \begin{cases} 1, & \text{if } 0 < s < 1, \\ (r-1)s^{-r}, & \text{if } s \geq 1. \end{cases}$$

Therefore

$$\frac{\|f\|_p^p}{\|H^*f - f\|_p^p} = \frac{\frac{r^p}{rp-1}}{1 + \frac{(r-1)^p}{rp-1}} = \frac{r^p}{rp-1 + (r-1)^p}.$$

Since $r = 1/t_p$, Lemma 2.3 shows that the last expression is m_p^{-1} . This ends the proof. \square

Remark 6.3. Note that (5.11), together with (6.3) and (6.4), implies the existence of optimal constants $c(p), C(p) > 0$ such that

$$c(p) \|(H^* - I)f\|_p \leq \|f\|_p \leq C(p) \|(H^* - I)f\|_p,$$

for all $f \in L^p(\mathbb{R}^+)$, with $1 < p < \infty$. It is now easy to see that $\|(H^* - I)f\|_p$ is an equivalent norm in $L^p(\mathbb{R}^+)$.

6.2 Inequalities in $L_{\text{dec}}^p(\mathbb{R}^+)$ for $1 < p < \infty$

Let $A(p)$ and $B(p)$ denote the optimal constants for which

$$A(p) \|(H^* - I)f\|_p \leq \|f\|_p \leq B(p) \|(H^* - I)f\|_p, \quad (6.5)$$

hold for all $f \in L_{\text{dec}}^p(\mathbb{R}^+)$, where f is a nonnegative, nonincreasing function, and for the range $1 < p < \infty$.

The following result gives the exact value of $B(p)$ in (6.5) for $1 < p \leq 2$, and the exact value of $A(p)$ in (6.5) for $2 \leq p < \infty$.

Theorem 6.4. *Let $1 < p < \infty$ and $f \in L_{\text{dec}}^p(\mathbb{R}^+)$ be a nonnegative, nonincreasing function. If $1 < p \leq 2$, then*

$$\|f\|_p \leq (p-1)^{-1} \|(H^* - I)f\|_p, \quad (6.6)$$

and if $2 \leq p < \infty$, then

$$(p-1)^{-1} \|(H^* - I)f\|_p \leq \|f\|_p. \quad (6.7)$$

Furthermore, all the constants in (6.6) and (6.7) are the best possible.

Proof. We begin with the case $1 < p \leq 2$. As seen in (6.3),

$$\|f\|_p \leq (p-1)^{-1} \|(H^* - I)f\|_p,$$

for all $f \in L^p(\mathbb{R}^+)$. In particular, this inequality holds for nonnegative, nonincreasing functions $f \in L^p_{\text{dec}}(\mathbb{R}^+)$, which establishes (6.6). The inequality (6.7) follows by an analogous argument, now using (5.11) in place of (6.3).

The optimality of the constant $(p-1)^{-1}$ in both (6.6) and (6.7) follows from Lemma 6.1. \square

Let $C_p = \int_0^1 |\ln(t) + 1|^p dt$. The following theorem provides the exact value of $A(p)$ in (6.5) for $1 < p \leq 2$, and the exact value of $B(p)$ in (6.5) for $2 \leq p < \infty$.

Theorem 6.5. *Let $1 < p < \infty$ and let $f \in L^p_{\text{dec}}(\mathbb{N})$ be a nonnegative, nonincreasing function. If $1 < p \leq 2$, then*

$$\|(H^* - I)f\|_p \leq C_p^{1/p} \|f\|_p, \quad (6.8)$$

and if $2 \leq p < \infty$, then

$$C_p^{1/p} \|f\|_p \leq \|(H^* - I)f\|_p. \quad (6.9)$$

In both cases, $C_p^{1/p}$ is the optimal constant for which the inequality holds.

Proof. We will prove only the inequality (6.8), since (6.9) admits an analogous proof. Taking $f = \chi_{(0,1)}$ reduces both inequalities above to equality, which will show that the constant C_p is optimal once we prove that both inequalities hold. As in Lemma 2.7, let

$$h(r) = e^r \int_0^{e^{-r}} |1 + \ln x|^p dx.$$

Observe that $h(0) = C_p$.

As mentioned earlier, the operator $H^* - I$ is bounded above and below on $L^p(\mathbb{R}^+)$. Thus, it suffices to prove the theorem for simple functions f of the following form: Let $a_0 = 0 < a_1 < \dots < a_N < \infty$, $b_1 > \dots > b_N > 0 = b_{N+1}$ and set $d_n = \ln a_n$ for $n = 0, \dots, N$, so $d_0 = -\infty < d_1 < \dots < d_N$. Define

$$f = \sum_{n=1}^N b_n \chi_{(a_{n-1}, a_n]}.$$

For convenience, set

$$A_n = \int_{a_{n-1}}^{a_n} \left| \int_x^\infty f(t) \frac{dt}{t} - f(x) \right|^p dx, \quad n = 1, 2, \dots, N.$$

Since $(H^* - I)f$ is supported on $(0, a_N]$,

$$\|(H^* - I)f\|_p^p = \sum_{n=1}^N A_n.$$

If $a_{n-1} < x \leq a_n$, then

$$\begin{aligned} \int_x^\infty f(t) \frac{dt}{t} - f(x) &= \int_x^{a_n} b_n \frac{dt}{t} + \int_{a_n}^{a_{n+1}} b_{n+1} \frac{dt}{t} + \cdots + \int_{a_{N-1}}^{a_N} b_N \frac{dt}{t} - b_n \\ &= b_n(d_n - \ln x) + b_{n+1}(d_{n+1} - d_n) + \cdots + b_N(d_N - d_{N-1}) - b_n \\ &= b_n(S_n - 1 - \ln x), \end{aligned}$$

where S_n is chosen so that

$$b_n S_n = b_n d_n + \sum_{k=n+1}^N b_k (d_k - d_{k-1}),$$

for $1 \leq n \leq N-1$ and $S_N = d_N$. Observe that $b_n(S_n - d_n) = b_{n+1}(S_{n+1} - d_n)$ for $n = 1, \dots, N-1$ and $b_N(S_N - d_N) = 0$. Let $\ln z = \ln x - S_n$ to get

$$\begin{aligned} \frac{A_n}{b_n^p} &= \int_{a_{n-1}}^{a_n} |S_n - 1 - \ln x|^p dx = e^{S_n} \int_{e^{d_{n-1}-S_n}}^{e^{d_n-S_n}} |1 + \ln z|^p dz \\ &= a_n h(S_n - d_n) - a_{n-1} h(S_n - d_{n-1}) \quad (6.10) \\ &= a_n h((b_{n+1}/b_n)(S_{n+1} - d_n)) - a_{n-1} h(S_n - d_{n-1}). \end{aligned}$$

Taking $t = (b_{n+1}/b_n)^p$ and $u = S_{n+1} - d_n$ in Lemma 2.7, we have

$$a_n h((b_{n+1}/b_n)(S_{n+1} - d_n)) \leq a_n (1 - (b_{n+1}/b_n)^p) C_p + a_n (b_{n+1}/b_n)^p h(S_{n+1} - d_n). \quad (6.11)$$

Multiplying by b_n^p in (6.11) and using (6.10) we get that

$$A_n \leq a_n (b_n^p - b_{n+1}^p) C_p + [b_{n+1}^p a_n h(S_{n+1} - d_n) - b_n^p a_{n-1} h(S_n - d_{n-1})].$$

When we sum over n , the bracketed terms telescope to zero, so

$$\sum_{n=1}^N A_n \leq C_p \sum_{n=1}^N a_n (b_n^p - b_{n+1}^p) = C_p \sum_{n=1}^N b_n^p (a_n - a_{n-1}) = C_p \int_0^\infty f(t)^p dt = C_p \|f\|_p^p.$$

This completes the proof. \square

In the following remark, we present some readily derived properties of the optimal constant $C_p^{1/p}$ appearing in Theorem 6.5.

Remark 6.6. Recall that

$$C_p = \int_0^1 |1 + \ln x|^p dx. \quad (6.12)$$

(a) The integral defining C_p converges for $0 < p < \infty$.

(b) A bit of calculus shows that $C_1 = 2/e$ and $C_2 = 1$.

- (c) $C_p^{1/p}$ is a strictly increasing function of p for $0 < p < \infty$: If $0 < p < q < \infty$, then Hölder's inequality with indices q/p and $q/(q-p)$ implies

$$C_p^{1/p} = \left(\int_0^1 |1 + \ln x|^p dx \right)^{1/p} \leq \left(\int_0^1 |1 + \ln x|^q dx \right)^{1/q} \left(\int_0^1 dx \right)^{(q-p)/(pq)} = C_q^{1/q}.$$

Since $|1 + \ln x|$ is not constant, the inequality is strict.

- (d) Well-known Hölder's inequality arguments, much like the one above, show that $\ln(C_p)$ and $\ln(C_{1/p}^p)$ are convex functions of p on $(0, \infty)$.

- (e) C_p is not an increasing function of p for $0 < p \leq 1$: If $0 < p \leq 1$, the inequality

$$|1 + \ln x|^p \leq (|1 + \ln x| + 1)^p \leq |1 + \ln x| + 1,$$

together with the dominated convergence theorem, shows that $\lim_{p \rightarrow 0^+} C_p = 1$. But $C_1 = 2/e < 1$.

- (f) $C_p^{1/p}$ does not tend to zero as $p \rightarrow 0$: by L'Hôpital's rule and a dominated convergence argument similar to the one in the previous item,

$$\lim_{p \rightarrow 0^+} \frac{\ln(C_p)}{p} = \lim_{p \rightarrow 0^+} \frac{\int_0^1 |1 + \ln x|^p \ln(|1 + \ln x|) dx}{\int_0^1 |1 + \ln x|^p dx} = \int_0^1 \ln(|1 + \ln x|) dx.$$

Thus,

$$\lim_{p \rightarrow 0^+} C_p^{1/p} = \exp \left(\int_0^1 \ln(|1 + \ln x|) dx \right).$$

- (g) C_p is a strictly increasing function of p for $p \geq 1$. Let $p > 0$. For each positive base $b \neq 1$, the exponential function $p \mapsto b^p$ is strictly convex. It follows that if $0 < p_0 < p_1$, $0 < \theta < 1$, and $p = (1-\theta)p_0 + \theta p_1$, then $b^p < (1-\theta)b^{p_0} + \theta b^{p_1}$. Taking $b = |1 + \ln x|$ and integrating with respect to x , we get $C_p < (1-\theta)C_{p_0} + \theta C_{p_1}$. That is, C_p is a strictly convex function for $0 < p < \infty$. (Because logarithmic convexity implies convexity, this follows from item (d), above, but we prefer the direct argument.) In particular, $\frac{d}{dp} C_p$ is strictly increasing for $0 < p < \infty$. Therefore, to conclude that C_p is nondecreasing on $1 \leq p < \infty$, it suffices to verify that $\frac{d}{dp} C_p \geq 0$ at $p = 1$.

For this, we split the integral defining C_p into two parts, letting $y = -1 - \ln x$ in the first and letting $y = 1 + \ln x$ in the second. So

$$C_p = \int_0^{1/e} (-1 - \ln x)^p dx + \int_{1/e}^1 (1 + \ln x)^p dx = \int_0^\infty y^p e^{-y} dy + \int_0^1 y^p e^y dy.$$

Recognizing the first term as a gamma function and expressing the second as a power series, we get

$$C_p = \Gamma(p+1) + \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 y^{k+p} dy = \Gamma(p+1) + \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{k+p+1}.$$

Differentiating the uniformly convergent sum term by term yields

$$\frac{d}{dp}C_p = \Gamma'(p+1) - \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(k+p+1)^2}.$$

Taking $p = 1$ in the first term we use the formula $\Gamma'(2) = 1 - \gamma$. Here γ is the Euler-Mascheroni constant, which is known to have approximate value $0.5772156649\dots$. Thus $\Gamma'(2) > 1 - 0.578 = 0.421$.

Taking $p = 1$ in the second term, we have

$$\sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(k+2)^2} < \frac{1}{4} + \frac{1}{9} + \sum_{k=2}^{\infty} \frac{1}{(k+2)!} = \frac{1}{4} + \frac{1}{9} + e - \left(1 + 1 + \frac{1}{2} + \frac{1}{6}\right).$$

A calculation shows this is less than 0.42 and we obtain $\frac{d}{dp}C_p > 0.421 - 0.42 > 0$ when $p = 1$.

(h) Let $1 \leq p < \infty$, and let M_p and m_p be defined as in (2.5). If $1 \leq p < 2$, then

$$C_p < M_p, \tag{6.13}$$

and if $2 < p < \infty$, then

$$m_p < C_p. \tag{6.14}$$

Moreover, $C_2 = M_2 = m_2 = 1$.

We prove only (6.13), since the proof of (6.14) is analogous. Recall that

$$M_p = \max_{t \in [0, 1/2]} (pt^{p-1} + (1-t)^p - t^p) = \max_{t \in [0, 1/2]} f_p(t).$$

Since

$$f_p(0) = 1,$$

it follows that

$$M_p \geq f_p(0) = 1,$$

for all $1 \leq p \leq 2$. On the other hand, as established in the previous item, the mapping

$$p \longmapsto C_p$$

is strictly increasing on $[1, \infty)$. Consequently,

$$C_p < C_2 = 1,$$

for all $1 \leq p < 2$. Combining these observations, we obtain

$$C_p < C_2 = 1 = f_p(0) \leq \max_{t \in [0, 1/2]} f_p(t) = M_p,$$

for all $1 \leq p < 2$, which proves (6.13). In fact, the argument shows more generally that

$$C_p < M_q,$$

for all $1 \leq p, q < 2$. Finally, a direct computation yields

$$C_2 = M_2 = m_2 = 1.$$

6.3 Inequalities in $L_+^p(\mathbb{R}^+)$ for $1 < p < \infty$

The purpose of this section is to determine the sharp constants $A(p)$ and $B(p)$ such that

$$A(p) \|(H^* - I)f\|_p \leq \|f\|_p \leq B(p) \|(H^* - I)f\|_p$$

for all nonnegative functions $f \in L_+^p(\mathbb{R}^+)$ and for $1 < p < \infty$. The following theorem establishes these values and constitutes the main result of the section.

Theorem 6.7. *Let $1 < p < \infty$ and $f \in L_+^p(\mathbb{R}^+)$ be a nonnegative function. If $1 < p \leq 2$, then*

$$M_p^{-1/p} \|(H^* - I)f\|_p \leq \|f\|_p \leq (p-1)^{-1} \|(H^* - I)f\|_p, \quad (6.15)$$

and if $2 \leq p < \infty$, then

$$(p-1)^{-1} \|(H^* - I)f\|_p \leq \|f\|_p \leq m_p^{-1/p} \|(H^* - I)f\|_p. \quad (6.16)$$

Moreover, all the constants in (6.15) and (6.16) are the best possible.

Proof. Let $1 < p < \infty$ and let $f \in L_+^p(\mathbb{R}^+)$ be nonnegative. In view of (5.11) and Theorem 6.2, we deduce that if $1 < p \leq 2$, then

$$M_p^{-1/p} \|(H^* - I)f\|_p \leq \|f\|_p \leq (p-1)^{-1} \|(H^* - I)f\|_p,$$

and if $2 \leq p < \infty$, then

$$(p-1)^{-1} \|(H^* - I)f\|_p \leq \|f\|_p \leq m_p^{-1/p} \|(H^* - I)f\|_p.$$

This proves that (6.15) and (6.16) hold.

We now prove that the constants in (6.15) and (6.16) are optimal. Observe that the functions used to show the optimality of $(p-1)^{-1}$ in (6.6) and (6.7), as well as of m_p^{-1} in (6.4), are nonnegative. Hence, the second inequality in (6.15) and the inequalities in (6.16) are sharp.

The optimality of the constant $M_p^{-1/p}$ in the first inequality of (6.15) is established by an argument analogous to that used to prove the sharpness of $m_p^{-1/p}$ in (6.4), replacing Lemma 2.3 with Lemma 2.4. This completes the proof. \square

6.4 Endpoint inequalities

The main goal of this section is to determine the optimal constants $A(p)$ and $B(p)$ in the inequalities

$$A(p) \|(H^* - I)f\|_p \leq \|f\|_p \leq B(p) \|(H^* - I)f\|_p,$$

valid for all $f \in L^p(\mathbb{R}^+)$, $L_+^p(\mathbb{R}^+)$ or $L_{\text{dec}}^p(\mathbb{R}^+)$, when $p = 1$ or $p = \infty$.

We have seen in (5.11) that $\|H^* - I\|_{L^1(\mathbb{R}^+)} = 2$. This means that

$$\|(H^* - I)f\|_1 \leq 2\|f\|_1, \quad (6.17)$$

for all $f \in L^1(\mathbb{R}^+)$, and the constant 2 is the best possible.

The following result gives us the exact value of $\|H^* - I\|_{L^1_+(\mathbb{R}^+)}$.

Theorem 6.8. *Let $f \in L^1_+(\mathbb{R}^+)$ be a nonnegative function. Then the following inequality is sharp*

$$\|(H^* - I)f\|_1 \leq 2\|f\|_1,$$

Proof. By using (6.17) we get that

$$\|(H^* - I)f\|_1 \leq 2\|f\|_1,$$

for all nonnegative $f \in L^1_+(\mathbb{R}^+)$.

Let us now examine the optimality of the constant 2 in the inequality above. Let $0 < \epsilon < 1$ and consider the nonnegative function f_ϵ defined by $f_\epsilon(x) = x \chi_{[1,1+\epsilon)}(x)$. Observe that

$$\|f_\epsilon\|_1 = \frac{1}{2}(\epsilon^2 + 2\epsilon),$$

for all $0 < \epsilon < 1$. On the other hand, we have

$$|H^*f_\epsilon(x) - f_\epsilon(x)| = \epsilon \chi_{(0,1)}(x) + (2x - 1 - \epsilon) \chi_{[1,1+\epsilon)}(x).$$

Therefore,

$$\frac{\|H^*f_\epsilon - f_\epsilon\|_1}{\|f_\epsilon\|_1} = \frac{4}{\epsilon + 2} \rightarrow 2 \quad \text{as } \epsilon \rightarrow 0^+,$$

showing that the constant 2 is indeed optimal. \square

The following theorem provides the exact value of $\|H^* - I\|_{L^1_{\text{dec}}(\mathbb{R}^+)}$. Although this result could be established by adapting the argument used in Theorem 6.5, we choose instead to present a more direct proof based on elementary methods.

Theorem 6.9. *Let $f \in L^1_{\text{dec}}(\mathbb{R}^+)$ be a nonnegative, nonincreasing function. Then, the following sharp estimate holds*

$$\|H^*f - f\|_1 \leq \frac{2}{e}\|f\|_1. \quad (6.18)$$

Proof. Without loss of generality we can restrict ourselves to the dense subset consisting of all simple, nonnegative, and nonincreasing functions. Let f_N be the function

$$f_N(x) = \sum_{j=1}^N b_j \chi_{(a_{j-1}, a_j]}(x),$$

where $0 = a_0 < a_1 < \cdots < a_N$ and $0 = b_{N+1} < b_N < \cdots < b_1$. On the one hand, we have

$$\|f_N\|_{L^1(\mathbb{R}^+)} = \sum_{j=1}^N b_j (a_j - a_{j-1}) = \sum_{j=1}^N (b_j - b_{j+1}) a_j.$$

On the other hand, if $a_{j-1} < x \leq a_j$, we get

$$\begin{aligned} H^* f_N(x) &= \int_x^\infty \frac{f_N(t)}{t} dt = b_j \int_x^{a_j} \frac{dt}{t} + \sum_{k=j+1}^N b_k \int_{a_{k-1}}^{a_k} \frac{dt}{t} \\ &= b_j \ln \left(\frac{a_j}{x} \right) + \sum_{k=j+1}^N b_k \ln \left(\frac{a_k}{a_{k-1}} \right) \\ &= b_j \ln \left(\frac{a_j}{x} \right) + \ln \left(\prod_{k=j+1}^N \left(\frac{a_k}{a_{k-1}} \right)^{b_k} \right) \\ &= b_j \ln \left(\frac{a_j}{x} \prod_{i=j+1}^N \left(\frac{a_i}{a_{i-1}} \right)^{\frac{b_i}{b_j}} \right). \end{aligned}$$

Hence,

$$H^* f_N(x) = \sum_{j=1}^N b_j \ln \left(\frac{a_j}{x} \prod_{i=j+1}^N \left(\frac{a_i}{a_{i-1}} \right)^{\frac{b_i}{b_j}} \right) \chi_{(a_{j-1}, a_j]}(x), \quad x \in (0, a_N],$$

where an empty product is understood to be 1. So

$$H^* f_N(x) - f_N(x) = \sum_{j=1}^N b_j \left(\ln \left(\frac{a_j}{x} \prod_{i=j+1}^N \left(\frac{a_i}{a_{i-1}} \right)^{\frac{b_i}{b_j}} \right) - 1 \right) \chi_{(a_{j-1}, a_j]}(x),$$

$x \in (0, a_N]$. By taking L^1 -norm, we obtain,

$$\begin{aligned} \|H^* f_N - f_N\|_{L^1(\mathbb{R}^+)} &= \sum_{j=1}^N b_j \int_{a_{j-1}}^{a_j} \left| \ln \left(\frac{a_j}{x} \prod_{i=j+1}^N \left(\frac{a_i}{a_{i-1}} \right)^{\frac{b_i}{b_j}} \right) - 1 \right| dx \\ &= \sum_{j=1}^N b_j \int_{a_{j-1}}^{a_j} \left| \ln \left(\frac{x}{a_j} \prod_{i=j+1}^N \left(\frac{a_{i-1}}{a_i} \right)^{\frac{b_i}{b_j}} \right) + 1 \right| dx \\ &= \sum_{j=1}^N b_j \int_{a_{j-1}}^{a_j} \left| \ln \left(x \prod_{i=j}^N \left(\frac{1}{a_i} \right)^{\frac{b_i - b_{i+1}}{b_j}} \right) + 1 \right| dx \\ &= \sum_{j=1}^N b_j \int_{a_{j-1}}^{a_j} \left| \ln \left(\prod_{i=j}^N \left(\frac{x}{a_i} \right)^{\frac{b_i - b_{i+1}}{b_j}} \right) + 1 \right| dx \\ &= \sum_{j=1}^N b_j \int_{a_{j-1}}^{a_j} \left| \sum_{i=j}^N \left(\frac{b_i - b_{i+1}}{b_j} \right) \ln \left(\frac{x}{a_i} \right) + \sum_{i=j}^N \left(\frac{b_i - b_{i+1}}{b_j} \right) \right| dx \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^N b_j \int_{a_{j-1}}^{a_j} \left| \sum_{i=j}^N \left(\frac{b_i - b_{i+1}}{b_j} \right) \left(\ln \left(\frac{x}{a_i} \right) + 1 \right) \right| dx \\
&= \sum_{j=1}^N \int_{a_{j-1}}^{a_j} \left| \sum_{i=j}^N (b_i - b_{i+1}) \left(\ln \left(\frac{x}{a_i} \right) + 1 \right) \right| dx \\
&\leq \sum_{j=1}^N \int_{a_{j-1}}^{a_j} \sum_{i=j}^N (b_i - b_{i+1}) \left| \ln \left(\frac{x}{a_i} \right) + 1 \right| dx \\
&= \sum_{j=1}^N \sum_{i=j}^N (b_i - b_{i+1}) \int_{a_{j-1}}^{a_j} \left| \ln \left(\frac{x}{a_i} \right) + 1 \right| dx \\
&= \sum_{j=1}^N \sum_{i=j}^N (b_i - b_{i+1}) a_i \int_{\frac{a_{j-1}}{a_i}}^{\frac{a_j}{a_i}} |\ln(x) + 1| dx \\
&= \sum_{i=1}^N \sum_{j=1}^i (b_i - b_{i+1}) a_i \int_{\frac{a_{j-1}}{a_i}}^{\frac{a_j}{a_i}} |\ln(x) + 1| dx \\
&= \sum_{i=1}^N (b_i - b_{i+1}) a_i \sum_{j=1}^i \int_{\frac{a_{j-1}}{a_i}}^{\frac{a_j}{a_i}} |\ln(x) + 1| dx \\
&= \sum_{i=1}^N (b_i - b_{i+1}) a_i \int_0^1 |\ln(x) + 1| dx \\
&= \|f_N\|_{L^1(\mathbb{R}^+)} \int_0^1 |\ln(x) + 1| dx.
\end{aligned}$$

Therefore,

$$\frac{\|H^* f_N - f_N\|_1}{\|f_N\|_1} \leq \int_0^1 |\ln(x) + 1| dx = \frac{2}{e}.$$

It remains to show that the constant in (6.18) is optimal. By taking the function $f = \chi_{(0,1)}$, we have that

$$\frac{\|H^* f - f\|_1}{\|f\|_1} = \int_0^1 |\ln(1/x) - 1| dx = \int_0^1 |\ln(x) + 1| dx = \frac{2}{e},$$

and this concludes the proof. \square

Theorem 6.10. *The following equalities hold:*

$$[H^* - I]_{L_{\text{dec}}^1(\mathbb{R}^+)} = [H^* - I]_{L_+^1(\mathbb{R}^+)} = [H^* - I]_{L^1(\mathbb{R}^+)} = \infty, \quad (6.19)$$

$$\|H^* - I\|_{L_{\text{dec}}^\infty(\mathbb{R}^+)} = \|H^* - I\|_{L_+^\infty(\mathbb{R}^+)} = \|H^* - I\|_{L^\infty(\mathbb{R}^+)} = \infty, \quad (6.20)$$

$$[H^* - I]_{L_{\text{dec}}^\infty(\mathbb{R}^+)} = 0, \quad [H^* - I]_{L_+^\infty(\mathbb{R}^+)} = [H^* - I]_{L^\infty(\mathbb{R}^+)} = 2. \quad (6.21)$$

Proof. We start proving (6.19). Let f_ϵ be the function defined by $f_\epsilon(x) = x^{\epsilon-1} \chi_{(0,1)}(x)$, where $0 < \epsilon < 1$. We have that f_ϵ is nonnegative, nonincreasing function, for all

$0 < \epsilon < 1$. By computing the L^1 -norm of f_ϵ we obtain that

$$\|f_\epsilon\|_1 = \frac{1}{\epsilon}.$$

On the other hand,

$$(H^* - I)f_\epsilon(x) = \left(\left(\frac{1}{1-\epsilon} - 1 \right) x^{\epsilon-1} - \frac{1}{1-\epsilon} \right) \chi_{(0,1)}(x).$$

By taking the L^1 -norm of $(H^* - I)f_\epsilon$, we get

$$\|(H^* - I)f_\epsilon\|_1 = 2 \left(\frac{1}{1-\epsilon} - 1 \right) \frac{1}{\epsilon} \frac{1}{\epsilon^{\epsilon/(\epsilon-1)}} - 2 \frac{1}{(1-\epsilon)\epsilon^{1/(\epsilon-1)}} + \frac{1}{1-\epsilon} - \left(\frac{1}{1-\epsilon} - 1 \right) \frac{1}{\epsilon}.$$

After some simplifications, we have that

$$\frac{\|(H^* - I)f_\epsilon\|_1}{\|f_\epsilon\|_1} = 2\epsilon^{\frac{1}{1-\epsilon}} \rightarrow 0,$$

as $\epsilon \rightarrow 0^+$. By Lemma 2.2, this implies that $H^* - I$ is not bounded below on $L^1(\mathbb{R}^+)$. Since f is nonnegative and nonincreasing, it follows that

$$[H^* - I]_{L^1_{\text{dec}}(\mathbb{R}^+)} = \infty.$$

Finally, combining this with (2.2) gives

$$[H^* - I]_{L^1_{\text{dec}}(\mathbb{R}^+)} = [H^* - I]_{L^1_+(\mathbb{R}^+)} = [H^* - I]_{L^1(\mathbb{R}^+)} = \infty.$$

We now turn to (6.20). Let $g(x) = \chi_{(0,1)}(x)$. We have that g is nonnegative, nonincreasing, and $\|g\|_\infty = 1$. On the other hand,

$$(H^* - I)g(x) = \left(\ln \left(\frac{1}{x} \right) - 1 \right) \chi_{(0,1)}(x).$$

Observe that $(H^* - I)g \notin L^\infty(\mathbb{R}^+)$. This shows that

$$\|H^* - I\|_{L^\infty_{\text{dec}}(\mathbb{R}^+)} = \infty.$$

Using (2.1), it follows that

$$\|H^* - I\|_{L^\infty_{\text{dec}}(\mathbb{R}^+)} = \|H^* - I\|_{L^\infty_+(\mathbb{R}^+)} = \|H^* - I\|_{L^\infty(\mathbb{R}^+)} = \infty.$$

To conclude, we proceed to prove (6.21). We start with the equality

$$[H^* - I]_{L^\infty(\mathbb{R}^+)} = 2. \tag{6.22}$$

Equivalently, we have to show that the following inequality is sharp

$$\|f\|_\infty \leq 2\|(H^* - I)f\|_\infty, \quad (6.23)$$

for all $f \in L^\infty(\mathbb{R}^+)$. Let f be an arbitrary function. If $(H^* - I)f \notin L^\infty(\mathbb{R}^+)$, then (6.23) holds trivially. Otherwise, assuming that $(H^* - I)f \in L^\infty(\mathbb{R}^+)$, and using (5.2) together with (5.3), we obtain

$$\|f\|_\infty = \|(H - I)(H^* - I)f\|_\infty \leq \|H - I\|_{L^\infty(\mathbb{R}^+)} \|(H^* - I)f\|_\infty = 2\|(H^* - I)f\|_\infty.$$

This proves the inequality (6.23). To establish the sharpness of the constant 2 in (6.23), we consider the function u given by $u(x) = 2x^{-2}\chi_{[1,\infty)}$. Note that u is nonnegative and $\|u\|_\infty = 2$. On the other hand we have that

$$|(H^* - I)u(x)| = \chi_{(0,1)}(x) + x^{-2}\chi_{[1,\infty)}(x).$$

Thus, $\|(H^* - I)u\|_\infty = 1$. Therefore,

$$\frac{\|u\|_\infty}{\|(H^* - I)u\|_\infty} = 2.$$

This establishes (6.22), and since u is nonnegative, it also shows that

$$[H^* - I]_{L^\infty_+(\mathbb{R}^+)} = 2.$$

Next, we consider the case of the cone $L^\infty_{\text{dec}}(\mathbb{R}^+)$. Let $f \in L^\infty(\mathbb{R}^+)$ be a nonnegative and nonincreasing function with $\|f\|_\infty > 0$ and let $y > 0$. Then,

$$\begin{aligned} y\|(H^* - I)f\|_\infty &= \|\chi_{(0,y)}\|_1 \|(H^* - I)f\|_\infty \geq \|\chi_{(0,y)}(H^* - I)f\|_1 \\ &= \int_0^\infty |\chi_{(0,y)}(x)(H^* - I)f(x)|dx = \int_0^y |(H^* - I)f(x)|dx \\ &\geq \left| \int_0^y (H^* - I)f(x)dx \right| = \left| \int_0^y \int_x^\infty \frac{f(t)}{t} dt dx - \int_0^y f(x)dx \right| \\ &= \left| \int_0^y \int_0^t \frac{f(t)}{t} dx dt + \int_y^\infty \int_0^y \frac{f(t)}{t} dx dt - \int_0^y f(x)dx \right| \\ &= \left| \int_0^y f(t)dt + y \int_y^\infty \frac{f(t)}{t} dt - \int_0^y f(x)dx \right| = y \int_y^\infty \frac{f(t)}{t} dt. \end{aligned}$$

Thus,

$$\int_y^\infty \frac{f(t)}{t} dt \leq \|(H^* - I)f\|_\infty,$$

for all $y > 0$. Hence,

$$\|H^* f\|_\infty \leq \|(H^* - I)f\|_\infty.$$

Let $\alpha > 0$ such that $f(\alpha) > 0$. Then,

$$\|H^* f\|_\infty \geq \int_\epsilon^\infty \frac{f(t)}{t} dt \geq \int_\epsilon^\alpha \frac{f(t)}{t} dt \geq f(\alpha) \int_\epsilon^\alpha \frac{1}{t} dt = f(\alpha) \ln \left(\frac{\alpha}{\epsilon} \right),$$

for all $0 < \epsilon < \alpha$. By taking limit when ϵ goes to 0^+ , we get that

$$\|(H^* - I)f\|_\infty \geq \|H^*f\|_\infty \geq \lim_{\epsilon \rightarrow 0^+} f(\alpha) \ln \left(\frac{\alpha}{\epsilon} \right) = \infty.$$

This shows that $(H^* - I)f \notin L^\infty(\mathbb{R}^+)$ for every nonnegative, nonincreasing function $f \in L^\infty_{\text{dec}}(\mathbb{R}^+) \setminus \{0\}$. Therefore,

$$[H^* - I]_{L^\infty_{\text{dec}}(\mathbb{R}^+)} = 0.$$

This completes the proof. \square

6.5 Tables

In this section, we present in tables all known constants related to the operator $H^* - I$ acting on different cones of $L^p(\mathbb{R}^+)$, for $1 \leq p \leq \infty$. The constants displayed in bold have been proved in this chapter.

$\ H^* - I\ _X$	$L^p(\mathbb{R}^+)$	$L^p_+(\mathbb{R}^+)$	$L^p_{\text{dec}}(\mathbb{R}^+)$
$p = 1$	2	2	2/e
$1 < p \leq 2$	$M_p^{1/p}$	$M_p^{1/p}$	$C_p^{1/p}$
$2 \leq p < \infty$	$p - 1$	$p - 1$	$p - 1$
$p = \infty$	∞	∞	∞

$[H^* - I]_X$	$L^p(\mathbb{R}^+)$	$L^p_+(\mathbb{R}^+)$	$L^p_{\text{dec}}(\mathbb{R}^+)$
$p = 1$	∞	∞	∞
$1 < p \leq 2$	$1/(p - 1)$	$1/(p - 1)$	$1/(p - 1)$
$2 \leq p < \infty$	$m_p^{-1/p}$	$m_p^{-1/p}$	$C_p^{-1/p}$
$p = \infty$	2	2	0

Chapter 7

Sharp bounds related to powers of Hardy operator and its adjoint

This chapter is devoted to the study of optimal constants in p -norm inequalities involving powers of the classical Hardy operator H and its adjoint H^* . By powers, we mean compositions in which H is applied n times and H^* is applied m times, where n and m are natural numbers.

The computation of the p -norm of a bounded operator T acting on $L^p(\mathbb{R}^+)$ is, in general, a challenging problem, as illustrated earlier. In this chapter, however, we show that the task can be considerably simplified when T is a polynomial with nonnegative coefficients in the Hardy operator H and its adjoint H^* . One of the main goals is to extend the following results due to G. H. Hardy, J. Littlewood, and G. Pólya (see [23, 24]):

$$\|H\|_{L^p(\mathbb{R}^+)} = p', \quad \text{for } 1 < p \leq \infty, \quad (7.1)$$

$$\|H^*\|_{L^p(\mathbb{R}^+)} = p, \quad \text{for } 1 \leq p < \infty, \quad (7.2)$$

$$\|H + H^*\|_{L^p(\mathbb{R}^+)} = p + p', \quad \text{for } 1 < p < \infty. \quad (7.3)$$

The operator $L := H + H^*$ is well-known in Analysis and is commonly referred to as Calderón's operator.

In the second part of this chapter, we focus on the classical Cesàro function spaces $\text{Ces}_p(\mathbb{R}^+)$, defined by

$$\text{Ces}_p(\mathbb{R}^+) = \{f : \|f\|_{\text{Ces}_p(\mathbb{R}^+)} = \|H|f|\|_p < \infty\},$$

where $1 < p \leq \infty$, and on the classical Copson function spaces $\text{Cop}_p(\mathbb{R}^+)$ given by

$$\text{Cop}_p(\mathbb{R}^+) = \{f : \|f\|_{\text{Cop}_p(\mathbb{R}^+)} = \|H^*|f|\|_p < \infty\},$$

where $1 \leq p < \infty$. These spaces are Banach spaces that have been extensively studied; see, for example, [2, 9, 27, 38, 45] and the references therein.

In [9, Theorem 21.1] G. Bennett showed that the Cesàro function spaces and the Copson function spaces coincide for $1 < p < \infty$. He also derived estimates for

the norms of the corresponding inclusion operators. Moreover, he proved that if $1 < p \leq 2$, then

$$\|f\|_{\text{Cop}_p(\mathbb{R}^+)} \leq (p-1)^{1/p} \|f\|_{\text{Ces}_p(\mathbb{R}^+)},$$

and if $2 \leq p < \infty$, then

$$(p-1)^{1/p} \|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq \|f\|_{\text{Cop}_p(\mathbb{R}^+)}.$$

Later, V. Kolyada completed the work started by G. Bennett proving the following result.

Theorem 7.1. [34, Theorem 1.1] *Let $1 < p < \infty$ and let f be a nonnegative measurable function. If $1 < p \leq 2$, then*

$$(p-1) \|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq \|f\|_{\text{Cop}_p(\mathbb{R}^+)} \leq (p-1)^{1/p} \|f\|_{\text{Ces}_p(\mathbb{R}^+)}, \quad (7.4)$$

and if $2 \leq p < \infty$, then

$$(p-1)^{1/p} \|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq \|f\|_{\text{Cop}_p(\mathbb{R}^+)} \leq (p-1) \|f\|_{\text{Ces}_p(\mathbb{R}^+)}. \quad (7.5)$$

Moreover, all constants in (7.4) and (7.5) are the best possible.

K. Lešnika and L. Maligranda [38] also contributed to this line of research by proving the following inequalities

$$\frac{p'}{e} \|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq \|Hf\|_{\text{Ces}_p(\mathbb{R}^+)} \leq p' \|f\|_{\text{Ces}_p(\mathbb{R}^+)}, \quad (7.6)$$

for any nonnegative measurable function f and $1 < p \leq \infty$. However, the constant p'/e is not optimal for $1 < p < \infty$. The sharp constant in the first inequality of (7.6) for this range of p is given in Theorem 7.14.

Motivated by the results of G. Bennett, V. Kolyada, K. Lešnika, and L. Maligranda, we define new norms on $\text{Ces}_p(\mathbb{R}^+)$ and $\text{Cop}_p(\mathbb{R}^+)$ that are equivalent to their standard norms and determine the exact values of the best constants in these equivalences.

The main results of this chapter have appeared in [10, 13].

Chapter Overview

This chapter is organized into four sections. The first presents several auxiliary results. The second is devoted to computing the norm of $Q(H, H^*)$, where $Q \in \mathbb{R}[X, Y]$ is a real polynomial. Next, we provide an application of the results obtained previously. Finally, we introduce different norms on the spaces $\text{Ces}_p(\mathbb{R}^+)$ and $\text{Cop}_p(\mathbb{R}^+)$ that are equivalent to their standard norms, and we determine the sharp constants in these equivalences.

- 7.1 Auxiliary statements 107
 - 7.2 The norm of a polynomial of H and H^* 109
 - 7.3 Application 118
 - 7.4 Equivalent norms on Cesàro and Copson spaces. 120
-

7.1 Auxiliary statements

This section focuses on results that will be useful for what follows. Let $n \in \mathbb{N}$ and let f be a measurable function defined on \mathbb{R}^+ . D. W. Boyd proved (see [15, Lemma 2]) the following equality

$$H^n f(x) = (H \circ H \circ \dots \circ H) f(x) = \frac{1}{\Gamma(n)} \frac{1}{x} \int_0^x f(y) \ln^{n-1} \left(\frac{x}{y} \right) dy. \quad (7.7)$$

Using analogous reasoning, we find

$$H^{*n} f(x) = (H^* \circ H^* \circ \dots \circ H^*) f(x) = \frac{1}{\Gamma(n)} \int_x^\infty \frac{f(y)}{y} \ln^{n-1} \left(\frac{y}{x} \right) dy, \quad (7.8)$$

whenever the defining integral exists almost everywhere. The following result provides a formula for the composition $H^n \circ H^{*m}$ expressed as a linear combination of powers of H and H^* , where m and n are natural numbers.

Proposition 7.2. *Let $m, n \geq 1$ be natural numbers and $f : (0, \infty) \rightarrow \mathbb{R}^+$ be a measurable function. Then*

$$(H^{*m} \circ H^n) f(x) = \sum_{k=0}^{m-1} \binom{n+k-1}{k} H^{*m-k} f(x) + \sum_{k=0}^{n-1} \binom{m+k-1}{k} H^{n-k} f(x), \quad (7.9)$$

for all $x > 0$.

Proof. Let $x > 0$. Then,

$$\begin{aligned} (H^{*m} \circ H^n) f(x) &= \frac{1}{\Gamma(m)} \int_x^\infty \frac{H^n f(y)}{y} \ln^{m-1} \left(\frac{y}{x} \right) dy \\ &= \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{1}{y^2} \int_0^y f(t) \ln^{n-1} \left(\frac{y}{t} \right) dt \ln^{m-1} \left(\frac{y}{x} \right) dy \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \int_0^y \frac{f(t)}{y^2} \ln^{n-1} \left(\frac{y}{t} \right) \ln^{m-1} \left(\frac{y}{x} \right) dt dy \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_0^x \int_x^\infty \frac{f(t)}{y^2} \ln^{n-1} \left(\frac{y}{t} \right) \ln^{m-1} \left(\frac{y}{x} \right) dy dt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \int_t^\infty \frac{f(t)}{y^2} \ln^{n-1} \left(\frac{y}{t} \right) \ln^{m-1} \left(\frac{y}{x} \right) dy dt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{1}{y^2} \ln^{n-1} \left(\frac{x}{t} y \right) \ln^{m-1}(y) dy dt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{1}{y^2} \ln^{n-1}(y) \ln^{m-1} \left(\frac{t}{x} y \right) dy dt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \ln^{n-1} \left(\frac{x}{t} y \right) dy dt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \ln^{m-1} \left(\frac{t}{x} y \right) dy dt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \left(\ln \left(\frac{x}{t} \right) + \ln y \right)^{n-1} dy dt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \left(\ln \left(\frac{t}{x} \right) + \ln y \right)^{m-1} dy dt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \sum_{k=0}^{n-1} \binom{n-1}{k} \ln^{n-k-1} \left(\frac{x}{t} \right) \ln^k(y) dy dt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \sum_{k=0}^{m-1} \binom{m-1}{k} \ln^{m-k-1} \left(\frac{t}{x} \right) \ln^k(y) dy dt \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \ln^{n-k-1} \left(\frac{x}{t} \right) \int_1^\infty \frac{\ln^{m+k-1}(y)}{y^2} dy dt \\
&\quad + \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \ln^{m-k-1} \left(\frac{t}{x} \right) \int_1^\infty \frac{\ln^{n+k-1}(y)}{y^2} dy dt \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(n)\Gamma(m+k)}{\Gamma(k+1)\Gamma(n-k)\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \ln^{n-k-1} \left(\frac{x}{t} \right) dt \\
&\quad + \sum_{k=0}^{m-1} \frac{\Gamma(m)\Gamma(n+k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \ln^{m-k-1} \left(\frac{t}{x} \right) dt \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(m+k)}{\Gamma(k+1)\Gamma(m)} H^{n-k} f(x) + \sum_{k=0}^{m-1} \frac{\Gamma(n+k)}{\Gamma(k+1)\Gamma(n)} H^{*m-k} f(x) \\
&= \sum_{k=0}^{n-1} \binom{m+k-1}{k} H^{n-k} f(x) + \sum_{k=0}^{m-1} \binom{n+k-1}{k} H^{*m-k} f(x),
\end{aligned}$$

where we used Tonelli's Theorem. We also applied (7.7) and (7.8) in the second to

last equality. This concludes the proof. \square

7.2 The norm of a polynomial of H and H^*

Let $1 \leq p \leq \infty$, and let $T : L^p(\mathbb{R}^+) \rightarrow L^p(\mathbb{R}^+)$ be a bounded linear operator. Let

$$F(x, y) = \sum_{i=0}^m \sum_{j=0}^n c_{ij} x^i y^j \in \mathbb{R}[X, Y].$$

We define the associated operator $F(T, T^*) : L^p(\mathbb{R}^+) \rightarrow L^p(\mathbb{R}^+)$ as follows

$$F(T, T^*) = \sum_{i=0}^m \sum_{j=0}^n c_{ij} (T^i \circ T^{*j}).$$

By closely examining equalities (7.1), (7.2), and (7.3), we note that

$$\begin{aligned} \|H\|_{L^p(\mathbb{R}^+)} &= \|P(H, H^*)\|_{L^p(\mathbb{R}^+)} = P(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p', \\ \|H^*\|_{L^p(\mathbb{R}^+)} &= \|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} = Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p, \\ \|H + H^*\|_{L^p(\mathbb{R}^+)} &= \|R(H, H^*)\|_{L^p(\mathbb{R}^+)} = R(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p + p', \end{aligned}$$

where $P(x, y) = x$, $Q(x, y) = y$, and $R(x, y) = x + y$.

It is important to note that, when working with H and H^* , two different polynomials can sometimes define the same associated operator. For instance, as seen in (5.1),

$$R(H, H^*) = V(H, H^*),$$

where R is as above and $V(x, y) = xy$. It is worth noting that,

$$\begin{aligned} p' + p &= R(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = \|R(H, H^*)\|_{L^p(\mathbb{R}^+)} = \|V(H, H^*)\|_{L^p(\mathbb{R}^+)} \\ &= V(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p'p, \end{aligned}$$

since p and p' are conjugate exponents.

The following theorem extends (7.1), (7.2), and (7.3), and constitutes one of the main results of this chapter.

Theorem 7.3. *Let $1 < p < \infty$ and let $Q \in \mathbb{R}[X, Y]$ be a polynomial with nonnegative coefficients. Then*

$$\begin{aligned} \|Q(H, H^*)\|_{L^p_{\text{dec}}(\mathbb{R}^+)} &= \|Q(H, H^*)\|_{L^p_+(\mathbb{R}^+)} = \|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} \\ &= Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = Q(p', p). \end{aligned}$$

Proof. Let $f \in L^p(\mathbb{R}^+)$ and let $Q \in \mathbb{R}[X, Y]$ be the polynomial given by

$$Q(x, y) = \sum_{i=0}^m \sum_{j=0}^n c_{ij} x^i y^j \in \mathbb{R}[X, Y],$$

where $c_{ij} \geq 0$ and $m, n \geq 1$ are natural numbers. By the triangle inequality and the submultiplicativity of the L^p -norm for operators, we get that

$$\begin{aligned} \|Q(H, H^*)f\|_p &\leq \|Q(H, H^*)|f|\|_p = \left\| \sum_{i=0}^m \sum_{j=0}^n c_{ij} (H^i \circ H^{*j})|f| \right\|_p \\ &\leq \sum_{i=0}^m \sum_{j=0}^n c_{ij} \|(H^i \circ H^{*j})|f|\|_p \\ &\leq \sum_{i=0}^m \sum_{j=0}^n c_{ij} \|H\|_p^i \|H^*\|_p^j \|f\|_p = Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) \|f\|_p. \end{aligned}$$

Thus

$$\|Q(H, H^*)\|_p \leq Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}).$$

Now, by using (7.1) with (7.2) we obtain that

$$\|Q(H, H^*)\|_p \leq Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = Q(p', p).$$

Now, consider the functions $f_\epsilon, g_\epsilon : (0, \infty) \rightarrow \mathbb{R}$ given by $f_\epsilon(x) = x^{\epsilon/p-1/p} \chi_{(0,1)}(x)$ and $g_\epsilon(x) = x^{\epsilon/p'-1/p'} \chi_{(0,1)}(x)$, where $0 < \epsilon < 1$. Then

$$\|f_\epsilon\|_p^p = \|g_\epsilon\|_{p'}^{p'} = \frac{1}{\epsilon}.$$

On the other hand, if $0 < x < 1$ and $n \in \mathbb{N}$, an easy computation shows that

$$H^n f_\epsilon(x) = x^{\epsilon/p-1/p} \frac{1}{(\epsilon/p + 1/p')^n}. \quad (7.10)$$

Now, as H and H^* are positive operators and $c_{ij} \geq 0$, it follows that $Q(H, H^*)$ is a positive operator. So

$$\int_0^\infty |Q(H, H^*)f_\epsilon(x) \cdot g_\epsilon(x)| dx = \int_0^\infty Q(H, H^*)f_\epsilon(x) \cdot g_\epsilon(x) dx.$$

Since g_ϵ has support equal to $[0, 1]$, we have that

$$\begin{aligned} \int_0^\infty Q(H, H^*)f_\epsilon(x) \cdot g_\epsilon(x) dx &= \int_0^1 Q(H, H^*)f_\epsilon(x) \cdot g_\epsilon(x) dx \\ &= \int_0^1 \left(\sum_{i=0}^m \sum_{j=0}^n c_{ij} H^i \circ H^{*j} \right) f_\epsilon(x) \cdot g_\epsilon(x) dx \\ &= \sum_{i=0}^m \sum_{j=0}^n c_{ij} \int_0^1 (H^i \circ H^{*j}) f_\epsilon(x) \cdot g_\epsilon(x) dx. \end{aligned}$$

Given that H and H^* commute, we obtain that

$$\int_0^1 (H^i \circ H^{*j}) f_\epsilon(x) \cdot g_\epsilon(x) dx = \int_0^1 (H^{*j} \circ H^i) f_\epsilon(x) \cdot g_\epsilon(x) dx,$$

for every $1 \leq i \leq m$ and $1 \leq j \leq n$. Observe that, as H^j is the adjoint of H^{*j} , we get that

$$\int_0^1 (H^{*j} \circ H^i) f_\epsilon(x) \cdot g_\epsilon(x) dx = \int_0^1 H^i f_\epsilon(x) \cdot H^j g_\epsilon(x) dx,$$

for every $1 \leq i \leq m$ and $1 \leq j \leq n$. Now, using (7.10), we have that

$$\begin{aligned} \int_0^1 H^i f_\epsilon(x) \cdot H^j g_\epsilon(x) dx &= \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \int_0^1 x^{\epsilon/p-1/p} x^{\epsilon/p'-1/p'} dx \\ &= \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \frac{1}{\epsilon}. \end{aligned}$$

Hence,

$$\int_0^\infty |Q(H, H^*) f_\epsilon(x) \cdot g_\epsilon(x)| dx = \frac{1}{\epsilon} \sum_{i=0}^m \sum_{j=0}^n c_{ij} \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j}.$$

Now, by using Hölder's inequality, we obtain

$$\begin{aligned} \|Q(H, H^*)\|_p &\geq \frac{\|Q(H, H^*) f_\epsilon\|_p}{\|f_\epsilon\|_p} \geq \frac{\int_0^\infty |Q(H, H^*) f_\epsilon(x) \cdot g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} \\ &= \sum_{i=0}^m \sum_{j=0}^n c_{ij} \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \rightarrow \sum_{i=0}^m \sum_{j=0}^n c_{ij} (p')^i p^j = Q(p', p), \end{aligned}$$

as $\epsilon \rightarrow 0^+$. Since the functions f_ϵ and g_ϵ are nonnegative and nonincreasing, the proof is concluded. \square

Remark 7.4. The condition that the coefficients of the polynomial Q be nonnegative in Theorem 7.3 is necessary, since without this hypothesis the conclusion of the theorem may fail. To illustrate this, consider the polynomial

$$Q(x, y) = y - 1,$$

for which the associated operator is $Q(H, H^*) = H^* - I$. We have seen in (5.11) that

$$\|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} = M_p^{1/p},$$

for $1 < p < 2$, where $M_p = \max_{t \in [0, 1/2]} f_p(t)$ and $f_p : [0, 1/2] \rightarrow \mathbb{R}$ is defined by (2.4). In particular,

$$\|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} = M_p^{1/p} \geq (f_p(0))^{1/p} = 1 > p - 1 = Q(p', p),$$

for all $1 < p < 2$.

The nonnegativity of the coefficients of Q is also necessary to ensure that

$$\|Q(H, H^*)\|_{L_{\text{dec}}^p(\mathbb{R}^+)} = \|Q(H, H^*)\|_{L_+^p(\mathbb{R}^+)} = \|Q(H, H^*)\|_{L^p(\mathbb{R}^+)},$$

since this equality may fail otherwise. For instance, consider again the operator $P(H, H^*)$ defined above. It was proved in Theorem 6.5 that

$$\|Q(H, H^*)\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|H^* - I\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \left(\int_0^\infty |\ln(x) + 1|^p dx \right)^{1/p} \stackrel{\text{def}}{=} C_p^{1/p},$$

for $1 < p \leq 2$. Moreover, it is shown in Remark 6.6 that the function $p \mapsto C_p^{1/p}$ is strictly increasing on \mathbb{R}^+ . Hence,

$$\begin{aligned} \|P(H, H^*)\|_{L^p_{\text{dec}}(\mathbb{R}^+)} &= C_p^{1/p} < C_2^{1/2} = \left(\int_0^\infty |\ln(x) + 1|^2 dx \right)^{1/2} \\ &= 1 = (f_p(0))^{1/p} < M_p^{1/p} = \|P(H, H^*)\|_{L^p_+(\mathbb{R}^+)} \\ &= \|P(H, H^*)\|_{L^p(\mathbb{R}^+)}, \end{aligned}$$

for all $1 < p < 2$, where we have taken into account (6.7) and Theorem 6.7.

Before stating our next result, we recall that by convention, $\binom{n}{0} = 1$ for all integers n ; in particular, $\binom{-1}{0} = 1$. We also set the sum equal to zero whenever the upper index is less than the lower index.

Corollary 7.5. *Let $m, n \in \mathbb{N} \cup \{0\}$ and $1 < p < \infty$. Then*

$$\begin{aligned} \|H^{*m} \circ H^n\|_{L^p_{\text{dec}}(\mathbb{R}^+)} &= \|H^{*m} \circ H^n\|_{L^p_+(\mathbb{R}^+)} = \|H^{*m} \circ H^n\|_{L^p(\mathbb{R}^+)} \\ &= \sum_{k=0}^{m-1} \binom{n+k-1}{k} p^{m-k} + \sum_{k=0}^{n-1} \binom{m+k-1}{k} (p')^{n-k} \quad (7.11) \\ &= p^m (p')^n. \end{aligned}$$

Proof. Let $P, Q \in \mathbb{R}[X, Y]$ be the polynomials defined by

$$P(x, y) = \sum_{k=0}^{n-1} \binom{m+k-1}{k} x^{n-k} + \sum_{k=0}^{m-1} \binom{n+k-1}{k} y^{m-k},$$

and

$$Q(x, y) = x^n y^m,$$

where $m, n \in \mathbb{N} \cup \{0\}$. Note that in the case $m = 0$, we have

$$P(x, y) = x^n = Q(x, y),$$

trivially, for any $n \in \mathbb{N} \cup \{0\}$. Similarly, in the case $n = 0$, we have

$$P(x, y) = y^m = Q(x, y),$$

for any $m \in \mathbb{N} \cup \{0\}$. This implies that

$$P(H, H^*) = Q(H, H^*) \quad \text{if } n = 0 \text{ or } m = 0.$$

In the case $m, n \geq 1$, we have seen in (7.9) that

$$P(H, H^*)f = Q(H, H^*)f.$$

for all measurable and nonnegative f . Hence,

$$P(H, H^*)f = Q(H, H^*)f, \quad (7.12)$$

for all $m, n \in \mathbb{N} \cup \{0\}$, and any measurable and nonnegative f . Therefore, taking into account Theorem 7.3 and (7.12), we obtain (7.11). \square

Remark 7.6. Let

$$A = \{H^{*m} \circ H^n : m, n \in \mathbb{N} \cup \{0\}\}.$$

Then (A, \circ) , where \circ denotes composition, is an abelian semigroup with identity. In particular, Corollary 7.5 implies that the mapping

$$\begin{aligned} \varphi : A &\longrightarrow [1, \infty), \\ T &\longmapsto \|T\|_{L^p(\mathbb{R}^+)} \end{aligned}$$

is a homomorphism.

Remark 7.7. Let $1 < p < \infty$ and let $n \geq 1$ be a natural number. On the one hand, equality (7.11) implies

$$p^n(p')^n = \sum_{k=0}^{n-1} \binom{n+k-1}{k} (p^{n-k} + (p')^{n-k}).$$

On the other hand, since p and p' are conjugate exponents, we have

$$p^n(p')^n = (pp')^n = (p + p')^n.$$

Hence

$$(p + p')^n = \sum_{k=0}^{n-1} \binom{n+k-1}{k} (p^{n-k} + (p')^{n-k}).$$

The coefficients $\binom{n+k-1}{k}$, for $k = 1, \dots, n$, are known as *generalized triangular numbers* (see [6]). For a fixed n , their arrangement within Pascal's Triangle is illustrated below:

For instance, when $n = 4$, we have

$$(p + p')^4 = \mathbf{1} (p^4 + (p')^4) + \mathbf{4} (p^3 + (p')^3) + \mathbf{10} (p^2 + (p')^2) + \mathbf{20} (p + p').$$

Let $a > 0$ be a real number. For a function f on \mathbb{R}^+ , the operators H^a and H^{*a} are given by

$$H^a f(x) = \frac{1}{\Gamma(a)} \frac{1}{x} \int_0^x f(t) \ln^{a-1} \left(\frac{x}{t} \right) dt,$$

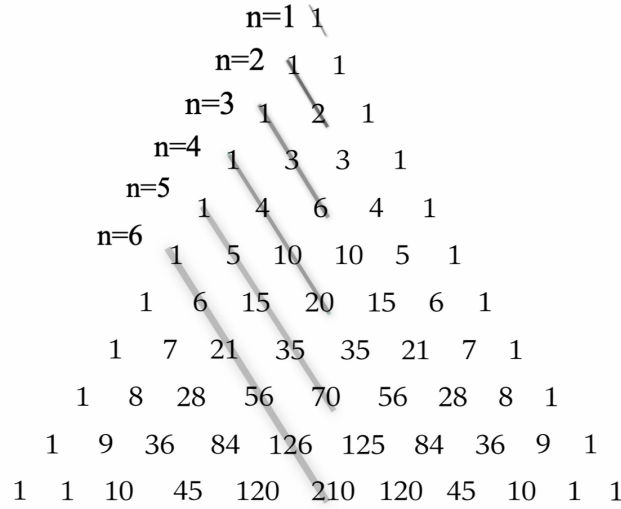


Figure 7.1 The corresponding coefficients $\binom{n+k-1}{k}$ for $n = 1, \dots, 6$ and $k = 1, \dots, n$.

and

$$H^{*a} f(x) = \frac{1}{\Gamma(a)} \int_x^\infty \frac{f(t)}{t} \ln^{a-1} \left(\frac{t}{x} \right) dt,$$

whenever these integrals are well defined. Note that H^{*a} is the adjoint operator of H^a .

Observe that for $a = 1$, the operators H^1 and H^{*1} reduce to the classical Hardy operator and its adjoint, respectively. Furthermore, if a is a natural number, H^a coincides with the a -fold composition of H , as shown in (7.7). A similar statement holds for H^{*a} .

The following result extends (7.1) and (7.2).

Theorem 7.8. *Let $1 \leq p \leq \infty$. Then*

$$\|H^a\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|H^a\|_{L^p_+(\mathbb{R}^+)} = \|H^a\|_{L^p(\mathbb{R}^+)} = (p')^a, \quad (1 < p \leq \infty), \quad (7.13)$$

and

$$\|H^{*a}\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|H^{*a}\|_{L^p_+(\mathbb{R}^+)} = \|H^{*a}\|_{L^p(\mathbb{R}^+)} = p^a, \quad (1 \leq p < \infty). \quad (7.14)$$

Proof. Let $a > 0$ and $f \in L^p(\mathbb{R}^+)$ with $1 < p \leq \infty$. Using the change of variables $t = xe^{-u}$, we obtain

$$H^a f(x) = \frac{1}{\Gamma(a)} \frac{1}{x} \int_0^x f(t) \ln^{a-1} \left(\frac{x}{t} \right) dt = \frac{1}{\Gamma(a)} \int_0^\infty f(xe^{-u}) u^{a-1} e^{-u} du.$$

Set

$$k(u) = \frac{1}{\Gamma(a)} u^{a-1} e^{-u}, \quad k(u) \geq 0, \quad \int_0^\infty k(u) du = 1.$$

We start with the case $p = \infty$. We have trivially that

$$\begin{aligned} |H^a f(x)| &= \left| \int_0^\infty f(xe^{-u})k(u)du \right| \leq \int_0^\infty |f(xe^{-u})|k(u)du \\ &\leq \|f\|_\infty \int_0^\infty k(u)du = \|f\|_\infty. \end{aligned}$$

For $1 < p < \infty$, by Minkowski's integral inequality,

$$\begin{aligned} \|H^a f\|_p &= \left(\int_0^\infty \left| \int_0^\infty f(xe^{-u})k(u) du \right|^p dx \right)^{1/p} \\ &\leq \int_0^\infty \left(\int_0^\infty |f(xe^{-u})|^p dx \right)^{1/p} k(u) du. \end{aligned}$$

For fixed u , using the change of variables $y = xe^{-u}$, $dx = e^u dy$,

$$\int_0^\infty |f(xe^{-u})|^p dx = \int_0^\infty |f(y)|^p e^u dy = e^u \|f\|_p^p.$$

Hence

$$\left(\int_0^\infty |f(xe^{-u})|^p dx \right)^{1/p} = e^{u/p} \|f\|_p.$$

Therefore

$$\|H^a f\|_p \leq \|f\|_p \int_0^\infty e^{u/p} k(u) du.$$

Now compute

$$\begin{aligned} \int_0^\infty e^{u/p} k(u) du &= \frac{1}{\Gamma(a)} \int_0^\infty u^{a-1} e^{-u(1-1/p)} du \\ &= \frac{1}{\Gamma(a)} \cdot \frac{\Gamma(a)}{(1-1/p)^a} = \left(\frac{p}{p-1} \right)^a = (p')^a. \end{aligned}$$

Thus,

$$\|H^a f\|_p \leq (p')^a \|f\|_p,$$

for all $1 < p \leq \infty$. We now show that the constant $(p')^a$ is optimal in the above inequality. To this end, consider the functions

$$f_\epsilon(x) = x^{\frac{\epsilon}{p} - \frac{1}{p}} \chi_{(0,1)}(x) \quad \text{and} \quad g_\epsilon(x) = x^{\frac{\epsilon}{p'} - \frac{1}{p'}} \chi_{(0,1)}(x),$$

where $1 \leq p < \infty$ and $0 < \epsilon < 1$. If $0 < x < 1$, then

$$H^a f_\epsilon(x) = x^{\frac{\epsilon}{p} - \frac{1}{p}} \frac{1}{\Gamma(a)} \int_0^\infty e^{-u(\frac{\epsilon}{p} + \frac{1}{p'})} u^{a-1} du = x^{\frac{\epsilon}{p} - \frac{1}{p}} \frac{1}{(\epsilon/p + 1/p')^a},$$

for all $1 < p < \infty$. Now, by using Hölder's inequality, we obtain

$$\|H^a\|_p \geq \frac{\|H^a f_\epsilon\|_p}{\|f_\epsilon\|_p} \geq \frac{\int_0^\infty |H^a f_\epsilon(x) \cdot g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} = \frac{1}{(\epsilon/p + 1/p')^a} \rightarrow (p')^a,$$

as $\epsilon \rightarrow 0^+$, for all $1 < p < \infty$. In the case when $p = \infty$, we consider the function $f = \chi_{(0,1)}$. A straightforward computation shows that

$$\|H^a f_\epsilon\|_\infty = \|f_\epsilon\|_\infty = 1.$$

This ends the proof of (7.13).

Now we turn to (7.14). By the duality argument, we obtain

$$\|H^{*a}\|_{L^p(\mathbb{R}^+)} = \|H^a\|_{L^{p'}(\mathbb{R}^+)} = p^a,$$

for all $1 \leq p < \infty$. To conclude the proof of (7.14), we consider the same functions f_ϵ and g_ϵ defined above. Then

$$\begin{aligned} \|H^{*a}\|_p &\geq \frac{\|H^{*a} f_\epsilon\|_p}{\|f_\epsilon\|_p} \geq \frac{\int_0^\infty |H^{*a} f_\epsilon(x) \cdot g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} \\ &= \frac{\int_0^\infty |f_\epsilon(x) \cdot H^a g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} = \frac{1}{(\epsilon/p + 1/p)^a} \rightarrow p^a, \end{aligned}$$

as $\epsilon \rightarrow 0^+$, for all $1 < p < \infty$. In the case $p = 1$, we consider the function $f = \chi_{(0,1)}$ again. By Tonelli's theorem, we get

$$\begin{aligned} \|H^{*a} f\|_1 &= \frac{1}{\Gamma(a)} \int_0^1 \int_x^1 \frac{1}{t} \ln^{a-1} \left(\frac{t}{x} \right) dt dx = \frac{1}{\Gamma(a)} \int_0^1 \frac{1}{t} \int_0^t \ln^{a-1} \left(\frac{t}{x} \right) dx dt \\ &= \frac{1}{\Gamma(a)} \int_0^1 \int_0^1 \ln^{a-1} \left(\frac{1}{x} \right) dx dt = \frac{1}{\Gamma(a)} \Gamma(a) = 1. \end{aligned}$$

This completes the proof. □

Let $C_c^1(\mathbb{R}^+)$ denote the space of continuously differentiable functions with compact support. The following result shows that, for every $f \in C_c^1(\mathbb{R}^+)$,

$$\lim_{a \rightarrow 0^+} H^a f = f, \quad \text{and} \quad \lim_{a \rightarrow 0^+} H^{*a} f = f.$$

Proposition 7.9. *Let $f \in C_c^1(\mathbb{R}^+)$. Then*

$$\lim_{a \rightarrow 0^+} \frac{1}{\Gamma(a)} \frac{1}{x} \int_0^x f(t) \ln^{a-1} \left(\frac{x}{t} \right) dt = f(x), \quad (7.15)$$

for all $x \in \mathbb{R}^+$, and

$$\lim_{a \rightarrow 0^+} \frac{1}{\Gamma(a)} \int_x^\infty \frac{f(t)}{t} \ln^{a-1} \left(\frac{t}{x} \right) dt = f(x), \quad (7.16)$$

for all $x \in \mathbb{R}^+$.

Proof. We start by proving (7.15). Let $f \in C_c^1(\mathbb{R}^+)$ and let $x > 0$. Then

$$\frac{1}{\Gamma(a)} \int_0^x f(t) \ln^{a-1} \left(\frac{x}{t} \right) dt = \frac{1}{\Gamma(a)} \int_0^x f(t) t \ln^{a-1} \left(\frac{x}{t} \right) \frac{dt}{t}.$$

Now, using integration by parts we get

$$\begin{aligned} & \frac{1}{\Gamma(a)} \int_0^x f(t) t \ln^{a-1} \left(\frac{x}{t} \right) \frac{dt}{t} \\ &= \frac{1}{\Gamma(a+1)} f(t) t \ln^a \left(\frac{x}{t} \right) \Big|_{t=x}^{t=0} + \frac{1}{\Gamma(a+1)} \int_0^x (f(t)t)' \ln^a \left(\frac{x}{t} \right) dt \\ &= \lim_{t \rightarrow 0^+} \frac{1}{\Gamma(a+1)} f(t) t \ln^a \left(\frac{x}{t} \right) + \frac{1}{\Gamma(a+1)} \int_0^x (f(t)t)' \ln^a \left(\frac{x}{t} \right) dt. \end{aligned}$$

So

$$\frac{1}{\Gamma(a)} \int_0^x f(t) t \ln^{a-1} \left(\frac{x}{t} \right) \frac{dt}{t} = \frac{1}{\Gamma(a+1)} \int_0^x (f(t)t)' \ln^a \left(\frac{x}{t} \right) dt,$$

for all $a > 0$.

Since $f \in C_c^1(\mathbb{R}^+)$, there exists $M > 0$ such that

$$|(f(t)t)'| \leq M, \quad \text{for all } t \in \mathbb{R}^+.$$

Consequently,

$$\left| \frac{1}{\Gamma(a+1)} (f(t)t)' \ln^a \left(\frac{x}{t} \right) \right| \leq M \frac{1}{\Gamma(a+1)} \ln^a \left(\frac{x}{t} \right),$$

for all $0 < t \leq x$. Assume now that $0 < a \leq 1$. Using the elementary inequality $u^a \leq 1 + u$ for all $u \geq 0$, we obtain

$$\frac{1}{\Gamma(a+1)} \ln^a \left(\frac{x}{t} \right) \leq \frac{1}{\Gamma(a+1)} \left(1 + \ln \left(\frac{x}{t} \right) \right) \leq K \left(1 + \ln \left(\frac{x}{t} \right) \right)$$

for all $0 < t \leq x$, where $K > 0$. The function $t \mapsto 1 + \ln \left(\frac{x}{t} \right)$ is integrable on $(0, x)$, since

$$\int_0^x \left(1 + \ln \left(\frac{x}{t} \right) \right) dt = 2x.$$

Therefore, by the Dominated Convergence Theorem,

$$\begin{aligned} \lim_{a \rightarrow 0^+} \frac{1}{\Gamma(a+1)} \int_0^x (f(t)t)' \ln^a \left(\frac{x}{t} \right) dt &= \int_0^x (f(t)t)' \lim_{a \rightarrow 0^+} \frac{\ln^a \left(\frac{x}{t} \right)}{\Gamma(a+1)} dt \\ &= \int_0^x (f(t)t)' dt = f(x)x, \end{aligned}$$

from which it follows (7.15) for continuously differentiable functions with compact support.

Now, we turn to (7.16). Let $a > 0$ and $x > 0$. Then, by applying integration by parts, we have

$$\begin{aligned} \frac{1}{\Gamma(a)} \int_x^\infty \frac{f(t)}{t} \ln^{a-1} \left(\frac{t}{x} \right) dt &= \frac{1}{\Gamma(a+1)} f(t) \ln^a \left(\frac{t}{x} \right) \Big|_{t=x}^{t=\infty} \\ &\quad - \frac{1}{\Gamma(a+1)} \int_x^\infty f'(t) \ln^a \left(\frac{t}{x} \right) dt. \end{aligned}$$

Thus

$$\frac{1}{\Gamma(a)} \int_x^\infty \frac{f(t)}{t} \ln^{a-1} \left(\frac{t}{x} \right) dt = -\frac{1}{\Gamma(a+1)} \int_x^\infty f'(t) \ln^a \left(\frac{t}{x} \right) dt,$$

for all $a > 0$. Now, using a similar justification as before to apply the Dominated Convergence Theorem, and invoking this result, we obtain

$$\begin{aligned} \lim_{a \rightarrow 0^+} \frac{1}{\Gamma(a+1)} \int_x^\infty \left(\frac{d}{dt} f(t) \right) \ln^a \left(\frac{t}{x} \right) dt &= \int_x^\infty \left(\frac{d}{dt} f(t) \right) \lim_{a \rightarrow 0^+} \frac{\ln^a \left(\frac{t}{x} \right)}{\Gamma(a+1)} dt \\ &= \int_x^\infty \left(\frac{d}{dt} f(t) \right) dt = -f(x), \end{aligned}$$

from which (7.16) follows for all $f \in C_c^1(\mathbb{R}^+)$ and $x > 0$. \square

7.3 Application

In this section, we investigate the norm of the exponential maps of H and H^* . Let $e^H, e^{H^*} : L^p(\mathbb{R}^+) \rightarrow L^p(\mathbb{R}^+)$ be the operators defined by

$$e^H = \sum_{n=0}^{\infty} \frac{1}{n!} H^n, \quad e^{H^*} = \sum_{n=0}^{\infty} \frac{1}{n!} H^{*n}.$$

The operators e^H and e^{H^*} are well defined and bounded on $L^p(\mathbb{R}^+)$ (see [41]).

Theorem 7.10. *Let $1 < p < \infty$. Then*

$$\|e^H\|_{L^p(\mathbb{R}^+)} = e^{\|H\|_{L^p(\mathbb{R}^+)}} = e^{p'}. \quad (7.17)$$

$$\|e^{H^*}\|_{L^p(\mathbb{R}^+)} = e^{\|H^*\|_{L^p(\mathbb{R}^+)}} = e^p. \quad (7.18)$$

Proof. We will prove only (7.17), since (7.18) can be established by an analogous argument. So,

$$\begin{aligned} \|e^H\|_{L^p(\mathbb{R}^+)} &= \left\| \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{1}{n!} H^n \right\|_{L^p(\mathbb{R}^+)} = \lim_{N \rightarrow \infty} \left\| \sum_{n=0}^N \frac{1}{n!} H^n \right\|_{L^p(\mathbb{R}^+)} \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{1}{n!} \|H\|_{L^p(\mathbb{R}^+)}^n = \sum_{n=0}^{\infty} \frac{1}{n!} \|H\|_{L^p(\mathbb{R}^+)}^n = e^{\|H\|_{L^p(\mathbb{R}^+)}} = e^{p'}, \end{aligned}$$

In the second equality, we have used the continuity of the operator L^p -norm, while in the third equality we apply Theorem 7.3. \square

Remark 7.11. Note that, in general, the identity

$$\|e^T\|_{L^p(\mathbb{R}^+)} = e^{\|T\|_{L^p(\mathbb{R}^+)}}$$

does not hold for every bounded operator $T : L^p(\mathbb{R}^+) \rightarrow L^p(\mathbb{R}^+)$. To illustrate this, consider the Volterra operator $V : L^2[0, 1] \rightarrow L^2[0, 1]$ defined by

$$Vf(x) = \int_0^x f(t) dt.$$

Halmos [22, Problem 188] computed the exact value of the operator norm of V . It is given by

$$\|V\|_{L^2[0,1]} = \frac{2}{\pi}.$$

The norm of the square of the Volterra operator was determined by Horgan (see [25, Table 1]), who showed that

$$\|V^2\|_{L^2[0,1]} = \frac{1}{\rho^2},$$

where ρ is the smallest positive solution of

$$(\cos \rho)(\cosh \rho) + 1 = 0.$$

Numerically,

$$\|V^2\|_{L^2[0,1]} \approx 0.28441289\dots$$

In particular,

$$\|V^2\|_{L^2[0,1]} < 0.285 < 0.4 < \left(\frac{2}{\pi}\right)^2 = \|V\|_{L^2[0,1]}^2.$$

Consequently, we obtain the estimate

$$\begin{aligned} \|e^V\|_{L^2[0,1]} &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \|V^n\|_{L^2[0,1]} \\ &\leq 1 + \|V\|_{L^2[0,1]} + \frac{1}{2} \|V^2\|_{L^2[0,1]} + \sum_{n=3}^{\infty} \frac{1}{n!} \|V\|_{L^2[0,1]}^n \\ &< 1 + \|V\|_{L^2[0,1]} + \frac{1}{2} \|V\|_{L^2[0,1]}^2 + \sum_{n=3}^{\infty} \frac{1}{n!} \|V\|_{L^2[0,1]}^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \|V\|_{L^2[0,1]}^n = e^{\|V\|_{L^2[0,1]}}. \end{aligned}$$

This shows that, in general,

$$\|e^T\|_{L^p(\mathbb{R}^+)} < e^{\|T\|_{L^p(\mathbb{R}^+)}}$$

may occur.

7.4 Equivalent norms on Cesàro and Copson spaces.

Let $1 \leq p \leq \infty$. For $1 < p < \infty$, [9, Theorem 21.1] shows that

$$\text{Ces}_p(\mathbb{R}^+) = \text{Cop}_p(\mathbb{R}^+).$$

This section aims to introduce alternative norms $\|\cdot\|_{\text{Ces}_p(\mathbb{R}^+)}$ on $\text{Ces}_p(\mathbb{R}^+)$ and $\|\cdot\|_{\text{Cop}_p(\mathbb{R}^+)}$ on $\text{Cop}_p(\mathbb{R}^+)$ such that

$$A(p) \|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq \|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq B(p) \|f\|_{\text{Ces}_p(\mathbb{R}^+)},$$

or

$$U(p) \|f\|_{\text{Cop}_p(\mathbb{R}^+)} \leq \|f\|_{\text{Cop}_p(\mathbb{R}^+)} \leq V(p) \|f\|_{\text{Cop}_p(\mathbb{R}^+)},$$

for all $f \in \text{Ces}_p(\mathbb{R}^+)$, where $A(p), B(p), U(p), V(p) > 0$ denote the best possible constants in the inequalities above.

The first result in this line of research is, in fact, a consequence of Theorem 6.4 together with Theorem 6.5. Let g be a nonnegative measurable function on $(0, \infty)$. Then, by (7.8), we have

$$H^{*2}g(x) = \int_x^\infty g(t) \ln\left(\frac{t}{x}\right) \frac{dt}{t}.$$

The theorem below shows that, for $1 < p < \infty$ and for all such functions $g \geq 0$ for which $(H^{*2} - H^*)g$ is well defined, the p -norms of $(H^* - I)H^*g$ and H^*g are equivalent.

Theorem 7.12. *Let $1 < p < \infty$ and let g be a nonnegative, measurable function such that $H^*g(x) < \infty$ for all $x > 0$. If $1 < p \leq 2$, then*

$$(p-1) \|g\|_{\text{Cop}_p(\mathbb{R}^+)} \leq \|(H^* - I)H^*g\|_p \leq C_p^{1/p} \|g\|_{\text{Cop}_p(\mathbb{R}^+)}, \quad (7.19)$$

and if $2 \leq p \leq \infty$, then

$$C_p^{1/p} \|g\|_{\text{Cop}_p(\mathbb{R}^+)} \leq \|(H^* - I)H^*g\|_p \leq (p-1) \|g\|_{\text{Cop}_p(\mathbb{R}^+)}. \quad (7.20)$$

The constants $p-1$ and $C_p^{1/p}$ are optimal in both (7.19) and (7.20), with C_p defined as in (6.12).

Proof. Set $f = H^*g$ and observe that f is nonnegative and nonincreasing. Since $(H^{*2} - H^*)g = (H^* - I)f$, Theorem 6.4 combined with Theorem 6.5 shows that inequalities (7.19) and (7.20) both hold.

Lemma 6.1 shows that if $p-1$ were replaced by a larger constant in (7.19) or by a smaller constant in (7.20), then that inequality would fail for some nonnegative function g_r for which $H^*g_r(x) = f_r(x) < \infty$ for all $x > 0$. This shows $p-1$ is optimal in both (7.19) and (7.20).

It remains to prove that the constant $C_p^{1/p}$ is optimal in both (7.19) and (7.20). Let $k_\varepsilon(s) = (s/\varepsilon)\chi_{(1-\varepsilon,1)}(s)$. Then $H^*k_\varepsilon(t) = \min(1, (1-t)/\varepsilon)\chi_{(0,1)}(t)$. A straightforward argument demonstrates that as ε goes to zero, the piecewise linear functions H^*k_ε increase pointwise to $\chi_{(0,1)}$. The Monotone Convergence Theorem implies that H^*k_ε converges to $\chi_{(0,1)}$ in $L^p(\mathbb{R}^+)$. But H^* and $H^* - I$ are continuous maps on $L^p(\mathbb{R}^+)$, so we have

$$\|H^*k_\varepsilon\|_p \rightarrow \|\chi_{(0,1)}\|_p = 1 \quad \text{and} \quad \|(H^{*2} - H^*)k_\varepsilon\|_p \rightarrow \|(H^* - I)\chi_{(0,1)}\|_p = C_p^{1/p}.$$

It follows that $C_p^{1/p}$ is the smallest constant for which (7.19) holds and the largest constant for which (7.20) holds. This completes the proof. \square

The following result will be needed. It was initiated by S. Boza and J. Soria in [18] and later extended by V. Kolyada in [35].

Theorem 7.13. ([18, Theorem 3.3], [35, Theorem 1.3]) *Let f be a nonnegative and nonincreasing function on \mathbb{R}^+ . If $1 < p \leq 2$, then*

$$(p-1)\|Hf\|_p \leq \|H^*f\|_p \leq \left(\frac{\Gamma(p+1)}{p'}\right)^{1/p} \|Hf\|_p,$$

and if $2 \leq p < \infty$, then

$$\left(\frac{\Gamma(p+1)}{p'}\right)^{1/p} \|Hf\|_p \leq \|H^*f\|_p \leq (p-1)\|Hf\|_p. \quad (7.21)$$

The constants in these inequalities are optimal.

We have seen in (7.6) that

$$\frac{p'}{e}\|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq \|Hf\|_{\text{Ces}_p(\mathbb{R}^+)} \leq p'\|f\|_{\text{Ces}_p(\mathbb{R}^+)},$$

for every nonnegative measurable function f and $1 < p \leq \infty$. However, the constant $\frac{p'}{e}$ in (7.6) is not optimal for $1 < p < \infty$. The following result provides an improvement of the inequalities (7.6).

Theorem 7.14. *Let $1 < p < \infty$ and f be a nonnegative function. Then the following inequalities are sharp:*

$$\frac{\Gamma(p+1)^{1/p}}{(p-1)}\|f\|_{\text{Ces}_p(\mathbb{R}^+)} \leq \|Hf\|_{\text{Ces}_p(\mathbb{R}^+)} \leq p'\|f\|_{\text{Ces}_p(\mathbb{R}^+)}, \quad (7.22)$$

and if $p = \infty$ the constants in (7.22) are, respectively, $\lim_{p \rightarrow \infty} \frac{\Gamma(p+1)^{1/p}}{(p-1)} = \frac{1}{e}$ and 1.

Proof. Let $1 < p \leq 2$ and let f be a nonnegative function. By using integration by parts we obtain that

$$\|Hf\|_{\text{Ces}_p(\mathbb{R}^+)}^p = \|H^2f\|_p^p = p' \int_0^\infty \left(\frac{1}{x} \int_0^x f(u)du \right) \left(\frac{1}{x} \int_0^x f(t) \ln \left(\frac{x}{t} \right) dt \right)^{p-1} dx.$$

By applying Jensen's inequality, we have that

$$\left(\frac{1}{x} \int_0^x f(t) \ln \left(\frac{x}{t} \right) dt \right)^{p-1} \geq \left(\frac{1}{x} \int_0^x f(t) dt \right)^{p-2} \frac{1}{x} \int_0^x f(t) \ln^{p-1} \left(\frac{x}{t} \right) dt,$$

for all $x > 0$. Now, by using Fubini's Theorem we get that

$$\int_0^x f(t) \ln \left(\frac{x}{t} \right)^{p-1} dt = (p-1) \int_0^x \left(\frac{1}{t} \int_0^t f(u)du \right) \ln^{p-2} \left(\frac{x}{t} \right) dt,$$

for all $x > 0$. Therefore,

$$\begin{aligned} \|Hf\|_{\text{Ces}_p(\mathbb{R}^+)}^p &\geq p \int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^{p-1} \frac{1}{x} \int_0^x \left(\frac{1}{t} \int_0^t f(u)du \right) \ln^{p-2} \left(\frac{x}{t} \right) dt dx \\ &= p \int_0^\infty \left(\frac{1}{t} \int_0^t f(u)du \right) \int_t^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^{p-1} \ln^{p-2} \left(\frac{x}{t} \right) \frac{dx}{x} dt \\ &\geq p \int_0^\infty \left(\frac{1}{t} \int_0^t f(u)du \right) \left(\int_0^t f(x) dx \right)^{p-1} \int_t^\infty \frac{1}{x^p} \ln^{p-2} \left(\frac{x}{t} \right) dx dt \\ &= p \int_0^\infty \left(\frac{1}{t} \int_0^t f(x) dx \right)^p \int_t^\infty \left(\frac{t}{x} \right)^p \ln^{p-2} \left(\frac{x}{t} \right) \frac{dx}{t} dt \\ &= p \frac{\Gamma(p-1)}{(p-1)^{p-1}} \int_0^\infty \left(\frac{1}{t} \int_0^t f(x) dx \right)^p dt = \frac{\Gamma(p+1)}{(p-1)^p} \|f\|_{\text{Ces}_p(\mathbb{R}^+)}^p. \end{aligned}$$

Now we turn to the case $2 \leq p < \infty$. Let $x > 0$ and let $\varphi_x : (0, x] \rightarrow \mathbb{R}^+$ be the map defined by

$$\varphi_x(v) = \left(\int_0^v f(t) \ln \left(\frac{x}{t} \right) dt \right)^{p-1}.$$

By applying the Fundamental Theorem of Calculus we obtain that

$$\begin{aligned} \varphi_x(x) &= \varphi_x(x) - \varphi_x(0) = \int_0^x \varphi'(v) dv \\ &= (p-1) \int_0^x \left(\int_0^v f(t) \ln \left(\frac{x}{t} \right) dt \right)^{p-2} f(v) \ln \left(\frac{x}{v} \right) dv \\ &\geq (p-1) \int_0^x \left(\int_0^v f(t) dt \right)^{p-2} f(v) \ln^{p-1} \left(\frac{x}{v} \right) dv. \end{aligned}$$

So, we have that

$$\left(\int_0^x f(t) \ln \left(\frac{x}{t} \right) dt \right)^{p-1} \geq (p-1) \int_0^x \left(\int_0^v f(t) dt \right)^{p-2} f(v) \ln^{p-1} \left(\frac{x}{v} \right) dv, \quad (7.23)$$

for all $x > 0$. Now, by using integration by parts and (7.23) we get that

$$\begin{aligned} \|H^2 f\|_p^p &= p' \int_0^\infty \left(\frac{1}{x} \int_0^x f(u) du \right) \left(\frac{1}{x} \int_0^x f(t) \ln^{p-1} \left(\frac{x}{t} \right) dt \right) dx \\ &\geq p \int_0^\infty \frac{1}{x^p} \left(\int_0^x f(u) du \right) \int_0^x \left(\int_0^v f(t) dt \right)^{p-2} f(v) \ln^{p-1} \left(\frac{x}{v} \right) dv dx \\ &= p \int_0^\infty \left(\int_0^v f(t) dt \right)^{p-2} f(v) \int_v^\infty \frac{1}{x^p} \left(\int_0^x f(u) du \right) \ln^{p-1} \left(\frac{x}{v} \right) dx dv, \end{aligned}$$

In the last equality, we applied Fubini's Theorem. Then, using integration by parts once more, we obtain that

$$\begin{aligned} &p \int_0^\infty \left(\int_0^v f(t) dt \right)^{p-2} f(v) \int_v^\infty \frac{1}{x^p} \left(\int_0^x f(u) du \right) \ln^{p-1} \left(\frac{x}{v} \right) dx dv \\ &= p \int_0^\infty \frac{1}{v} \left(\int_0^v f(t) dt \right)^{p-1} \int_v^\infty \frac{1}{x^p} \left(\int_0^x f(u) du \right) \ln^{p-2} \left(\frac{x}{v} \right) dx dv \\ &\geq p \int_0^\infty \frac{1}{v} \left(\int_0^v f(t) dt \right)^p \int_v^\infty \frac{1}{x^p} \ln^{p-2} \left(\frac{x}{v} \right) dx dv. \end{aligned}$$

Therefore

$$\begin{aligned} \|Hf\|_{\text{Ces}_p(\mathbb{R}^+)}^p &= \|H^2 f\|_p^p \geq p \int_0^\infty \frac{1}{v} \left(\int_0^v f(t) dt \right)^p \int_v^\infty \frac{1}{x^p} \ln^{p-2} \left(\frac{x}{v} \right) dx dv \\ &= p \int_0^\infty \left(\frac{1}{v} \int_0^v f(t) dt \right)^p \int_v^\infty \left(\frac{v}{x} \right)^p \ln^{p-2} \left(\frac{x}{v} \right) \frac{dx}{v} dv \\ &= p \frac{\Gamma(p-1)}{(p-1)^{p-1}} \int_0^\infty \left(\frac{1}{v} \int_0^v f(x) dx \right)^p dv = \frac{\Gamma(p+1)}{(p-1)^p} \|Hf\|_p^p \\ &= \frac{\Gamma(p+1)}{(p-1)^p} \|f\|_{\text{Ces}_p(\mathbb{R}^+)}^p. \end{aligned}$$

We now consider the case $p = \infty$ and begin by computing the limit below:

$$\lim_{x \rightarrow \infty} (2\pi x)^{\frac{1}{2x}} = e^{\lim_{x \rightarrow \infty} \frac{\ln(2\pi x)}{2x}} = e^{\lim_{x \rightarrow \infty} \frac{1}{2x}} = e^0 = 1, \quad (7.24)$$

where we have applied L'Hôpital's rule. Next, we use the following inequality due to Robbins [43], valid for all positive integers n :

$$\left(\frac{n}{e} \right)^n \sqrt{2\pi n} e^{\frac{1}{12n+1}} \leq n! \leq \left(\frac{n}{e} \right)^n \sqrt{2\pi n} e^{\frac{1}{12n}}.$$

Since the function $x \mapsto x^{-1/x}$ is strictly increasing on $[e, \infty)$, we have that

$$\frac{1}{\left(\frac{n}{e} \right) (2\pi n)^{\frac{1}{2n}} e^{\frac{1}{12n^2+n}}} \leq \frac{1}{(n!)^{1/n}} \leq \frac{1}{\left(\frac{n}{e} \right) (2\pi n)^{\frac{1}{2n}} e^{\frac{1}{12n}}},$$

for all $n \geq 3$. Hence,

$$\frac{n-1}{\left(\frac{n}{e}\right)(2\pi n)^{\frac{1}{2n}} e^{\frac{1}{12n^2+n}}} \leq \frac{n-1}{(n!)^{1/n}} \leq \frac{n-1}{\left(\frac{n}{e}\right)(2\pi n)^{\frac{1}{2n}} e^{\frac{1}{12n}}}, \quad (7.25)$$

for all $n \geq 3$. Finally, taking the limit as $n \rightarrow \infty$ in (7.25) and using (7.24), we get

$$\lim_{n \rightarrow \infty} \frac{n-1}{\Gamma(n+1)^{1/n}} = e. \quad (7.26)$$

Now, if $f \in L^\infty(\mathbb{R}^+)$ is a nonnegative function, then, by (7.26) and (7.6), we have

$$\left(\lim_{n \rightarrow \infty} \frac{\Gamma(n+1)^{1/n}}{(n-1)} \right) \|f\|_{\text{Ces}_\infty(\mathbb{R}^+)} = \frac{1}{e} \|f\|_{\text{Ces}_\infty(\mathbb{R}^+)} \leq \|Hf\|_{\text{Ces}_\infty(\mathbb{R}^+)}.$$

Regarding the right-hand side of (7.22), if $Hf \notin L^p(\mathbb{R}^+)$, the inequality holds trivially. Otherwise, it follows directly from (7.1).

Let us now examine the optimality of the constant on the left-hand side of (7.22). We begin with the case $1 < p < \infty$. Consider the sequence

$$f_\epsilon = \chi_{(1,1+\epsilon)}, \quad \epsilon > 0.$$

After some computations, we obtain that

$$Hf_\epsilon(x) = \frac{x-1}{x} \chi_{(1,1+\epsilon)}(x) + \frac{\epsilon}{x} \chi_{[1+\epsilon, \infty)}(x),$$

and

$$\begin{aligned} H^2 f_\epsilon(x) &= \left(\frac{x - \ln(x) - 1}{x} \right) \chi_{(1,1+\epsilon)}(x) \\ &\quad + \frac{\epsilon(\ln(x) + 1) - (\epsilon + 1)\ln(1 + \epsilon)}{x} \chi_{[1+\epsilon, \infty)}(x). \end{aligned}$$

By taking p -norm we have that

$$\begin{aligned} \frac{\|f_\epsilon\|_{\text{Ces}_p(\mathbb{R}^+)}^p}{\|Hf_\epsilon\|_{\text{Ces}_p(\mathbb{R}^+)}^p} &= \frac{\|Hf_\epsilon\|_p^p}{\|H^2 f_\epsilon\|_p^p} \\ &= \frac{\int_1^{1+\epsilon} \left(1 - \frac{1}{x}\right)^p dx + \frac{\epsilon^p}{(p-1)(1+\epsilon)^{p-1}}}{\int_1^{1+\epsilon} \left(1 - \frac{1}{x} - \frac{\ln(x)}{x}\right)^p dx + \int_{1+\epsilon}^\infty (\epsilon(\ln(x) + 1) - (\epsilon + 1)\ln(\epsilon + 1))^p \frac{dx}{x^p}} \\ &\geq \frac{\frac{\epsilon^p}{(p-1)(1+\epsilon)^{p-1}}}{\int_1^{1+\epsilon} \left(1 - \frac{1}{x} - \frac{\ln(x)}{x}\right)^p dx + \int_{1+\epsilon}^\infty (\epsilon(\ln(x) + 1) - (\epsilon + 1)\ln(\epsilon + 1))^p \frac{dx}{x^p}} \\ &\geq \frac{1/(p-1)}{\frac{(1+\epsilon)^{p-1}}{\epsilon^p} \int_1^{1+\epsilon} \left(1 - \frac{1}{x} - \frac{\ln(x)}{x}\right)^p dx + \int_1^\infty \left(\ln(x) + 1 - \frac{\ln(\epsilon + 1)}{\epsilon}\right)^p \frac{dx}{x^p}}. \end{aligned}$$

Now, on the one hand, by applying L'Hôpital's rule we get that

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon^p} \int_1^{1+\epsilon} \left(1 - \frac{1}{x} - \frac{\ln(x)}{x}\right)^p dx = 0,$$

and on the other hand, by using Dominated Convergence Theorem we obtain that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_1^\infty \left(\ln(x) + 1 - \frac{\ln(\epsilon + 1)}{\epsilon}\right)^p \frac{dx}{x^p} &= \int_1^\infty \lim_{\epsilon \rightarrow 0^+} \left(\ln(x) + 1 - \frac{\ln(\epsilon + 1)}{\epsilon}\right)^p \frac{dx}{x^p} \\ &= \int_1^\infty \frac{\ln^p(x)}{x^p} dx = \frac{\Gamma(p+1)}{(p-1)^{p+1}}. \end{aligned}$$

Therefore, using (7.22), it follows that

$$\frac{(p-1)^p}{\Gamma(p+1)} \geq \sup_{f \geq 0, f \neq 0} \frac{\|f\|_{\text{Ces}_p(\mathbb{R}^+)}^p}{\|Hf\|_{\text{Ces}_p(\mathbb{R}^+)}^p} = \sup_{f \geq 0, f \neq 0} \frac{\|Hf\|_p^p}{\|H^2 f\|_p^p} \geq \sup_{\epsilon > 0} \frac{\|Hf_\epsilon\|_p^p}{\|H^2 f_\epsilon\|_p^p} \geq \frac{(p-1)^p}{\Gamma(p+1)}.$$

In the case of $p = \infty$ we get that

$$\|Hf_\epsilon\|_\infty = Hf_\epsilon(1+\epsilon) = \frac{\epsilon}{\epsilon+1},$$

and

$$\|H^2 f_\epsilon\|_\infty = H^2 f_\epsilon\left((1+\epsilon)^{\frac{1+\epsilon}{\epsilon}}\right) = \frac{\epsilon}{(1+\epsilon)^{\frac{1+\epsilon}{\epsilon}}},$$

therefore,

$$\sup_{\epsilon > 0} \frac{\|f_\epsilon\|_{\text{Ces}_\infty(\mathbb{R}^+)}}{\|Hf_\epsilon\|_{\text{Ces}_\infty(\mathbb{R}^+)}} = \sup_{\epsilon > 0} \frac{\|Hf_\epsilon\|_\infty}{\|H^2 f_\epsilon\|_\infty} = \sup_{\epsilon > 0} (1+\epsilon)^{\frac{1}{\epsilon}} = \lim_{\epsilon \rightarrow 0^+} (1+\epsilon)^{\frac{1}{\epsilon}} = e.$$

This shows that the constant in the first inequality of (7.22) is the best possible for $1 < p \leq \infty$.

Now, we are going to see the sharpness of the constant p' in the right-hand side of (7.22). In order to do that, we consider the functions $f_\epsilon(x) = x^{\epsilon/p-1/p} \chi_{(0,1)}(x)$ and $g_\epsilon(x) = x^{\epsilon/p'-1/p'} \chi_{(0,1)}(x)$, where $1 < p < \infty$ and $0 < \epsilon < 1$. We have seen in (7.10) in that

$$H^n f_\epsilon(x) = x^{\epsilon/p-1/p} \frac{1}{(\epsilon/p + 1/p')^n},$$

for all $n \in \mathbb{N}$ and for all $0 < x < 1$. So, by using Hölder's inequality and (7.1), we get

$$\begin{aligned} \frac{\|Hf_\epsilon\|_{\text{Ces}_p(\mathbb{R}^+)}}{\|f_\epsilon\|_{\text{Ces}_p(\mathbb{R}^+)}} &= \frac{\|H^2 f_\epsilon\|_p}{\|Hf_\epsilon\|_p} \geq \frac{\int_0^1 H^2 f_\epsilon(x) \cdot g_\epsilon(x) dx}{\|Hf_\epsilon\|_p \|g_\epsilon\|_{p'}} \geq \frac{1}{p'} \frac{\int_0^1 H^2 f_\epsilon(x) \cdot g_\epsilon(x) dx}{\|f_\epsilon\|_p \|g_\epsilon\|_{p'}} \\ &= \frac{1}{p'} \frac{1}{(\epsilon/p + 1/p')^{2p}} \frac{\int_0^1 x^{\epsilon/p-1/p} x^{\epsilon/p'-1/p'} dx}{\frac{1}{\epsilon^{1/p}} \frac{1}{\epsilon^{1/p'}}} \\ &= \frac{1}{p'} \frac{1}{(\epsilon/p + 1/p')^{2p}} \rightarrow p', \end{aligned}$$

as $\epsilon \rightarrow 0^+$. In the case when $p = \infty$, we consider the function $f = \chi_{(0,1)}$ and an easy calculations show that

$$\|H^2 f\|_\infty = \|Hf\|_\infty = 1,$$

which means

$$\|Hf_\epsilon\|_{\text{Ces}_\infty(\mathbb{R}^+)} = \|f_\epsilon\|_{\text{Ces}_\infty(\mathbb{R}^+)} = 1.$$

This ends the proof. \square

The following result introduces a new norm on $\text{Cop}_p(\mathbb{R}^+)$ that is equivalent to the standard norm.

Theorem 7.15. *Let $1 \leq p < \infty$ and let f be a nonnegative function. Then the following inequalities are sharp:*

$$\Gamma(p+1)^{1/p} \|f\|_{\text{Cop}_p(\mathbb{R}^+)} \leq \|H^* f\|_{\text{Cop}_p(\mathbb{R}^+)} \leq p \|f\|_{\text{Cop}_p(\mathbb{R}^+)}. \quad (7.27)$$

Proof. Let $1 \leq p < \infty$ and let f be a nonnegative function. If $p = 1$, by Fubini's Theorem we get that

$$\begin{aligned} \|H^{*n} f\|_1 &= \frac{1}{\Gamma(n)} \int_0^\infty \int_x^\infty \frac{f(t)}{t} \ln^{n-1} \left(\frac{t}{x} \right) dt dx \\ &= \frac{1}{\Gamma(n)} \int_0^\infty \frac{f(t)}{t} \int_0^t \ln^{n-1} \left(\frac{t}{x} \right) dx dt \\ &= \frac{1}{\Gamma(n)} \int_0^1 \ln^{n-1} \left(\frac{1}{x} \right) dx \cdot \int_0^\infty f(t) dt = \|f\|_1. \end{aligned}$$

Since the choice of the natural number n was arbitrary, we obtain that

$$\|H^{*n} f\|_1 = \|H^{*m} f\|_1,$$

for all $n, m \in \mathbb{N}$. In particular, we have that

$$\|H^* f\|_{\text{Cop}_1(\mathbb{R}^+)} = \|H^{*2} f\|_1 = \|H^* f\|_1 = \|f\|_{\text{Cop}_1(\mathbb{R}^+)}.$$

Now, let us assume that $1 < p \leq 2$. Then by using integration by parts we get

$$\|H^* f\|_{\text{Cop}_p(\mathbb{R}^+)}^p = \|H^{*2} f\|_p^p = p \int_0^\infty \left(\int_x^\infty \frac{f(u)}{u} du \right) \left(\int_x^\infty \frac{f(t)}{t} \ln \left(\frac{t}{x} \right) dt \right)^{p-1} dx.$$

By using Jensen's inequality we get that

$$\left(\int_x^\infty \frac{f(t)}{t} \ln \left(\frac{t}{x} \right) dt \right)^{p-1} \geq \left(\int_x^\infty \frac{f(t)}{t} dt \right)^{p-2} \int_x^\infty \frac{f(t)}{t} \ln^{p-1} \left(\frac{t}{x} \right) dt,$$

for all $x > 0$. Therefore,

$$\begin{aligned} \|H^{*2}f\|_p^p &\geq p \int_0^\infty \left(\int_x^\infty \frac{f(u)}{u} du \right)^{p-1} \int_x^\infty \frac{f(t)}{t} \ln^{p-1} \left(\frac{t}{x} \right) dt dx \\ &= p \int_0^\infty \frac{f(t)}{t} \int_0^t \left(\int_x^\infty \frac{f(u)}{u} du \right)^{p-1} \ln^{p-1} \left(\frac{t}{x} \right) dx dt, \end{aligned}$$

where we have applied Fubini's Theorem in the last equality. Now, by using integration by parts we get

$$\begin{aligned} &\int_0^\infty \frac{f(t)}{t} \int_0^t \left(\int_x^\infty \frac{f(u)}{u} du \right)^{p-1} \ln^{p-1} \left(\frac{t}{x} \right) dx dt \\ &= (p-1) \int_0^\infty \left(\int_t^\infty \frac{f(u)}{u} du \right) \frac{1}{t} \int_0^t \left(\int_x^\infty \frac{f(u)}{u} du \right)^{p-1} \ln^{p-2} \left(\frac{t}{x} \right) dx dt \\ &\geq (p-1) \int_0^\infty \left(\int_t^\infty \frac{f(u)}{u} du \right)^p \frac{1}{t} \int_0^t \ln^{p-2} \left(\frac{t}{x} \right) dx dt \\ &= (p-1)\Gamma(p-1)\|H^*f\|_p^p = \Gamma(p+1)\|f\|_{\text{Cop}_p(\mathbb{R}^+)}^p. \end{aligned}$$

Thus

$$\|H^*f\|_{\text{Cop}_p(\mathbb{R}^+)}^p = \|H^{*2}f\|_p^p \geq \Gamma(p+1)\|H^*f\|_p^p = \Gamma(p+1)\|f\|_{\text{Cop}_p(\mathbb{R}^+)}^p.$$

Now we turn to the case $2 \leq p < \infty$. Observe that if g is a nonnegative and nonincreasing function, then

$$Hg(x) = \frac{1}{x} \int_0^x g(t) dt \geq g(x),$$

for all $x > 0$. Using integration by parts we obtain that

$$\begin{aligned} \|Hg\|_p^p &= \int_0^\infty \left(\frac{1}{x} \int_0^x g(t) dt \right)^p dx = p' \int_0^\infty g(x) \left(\frac{1}{x} \int_0^x g(t) dt \right)^{p-1} dx \\ &\geq p' \int_0^\infty g(x)^p dx = p' \|g\|_p^p. \end{aligned}$$

So

$$(p')^{1/p} \|g\|_p \leq \|Hg\|_p. \quad (7.28)$$

Now, since f is a nonnegative function we get that the function H^*f is nonnegative and nonincreasing. Then by (7.28) and the first inequality in (7.21) we get that

$$\begin{aligned} (p')^{1/p} \|f\|_{\text{Cop}_p(\mathbb{R}^+)} &= (p')^{1/p} \|H^*f\|_p \leq \|HH^*f\|_p \leq \left(\frac{p'}{\Gamma(p+1)} \right)^{1/p} \|H^{*2}f\|_p \\ &= \left(\frac{p'}{\Gamma(p+1)} \right)^{1/p} \|H^*f\|_{\text{Cop}_p(\mathbb{R}^+)}, \end{aligned}$$

from which it is follow that

$$\Gamma(p+1)^{1/p} \|f\|_{\text{Cop}_p(\mathbb{R}^+)} \leq \|H^* f\|_{\text{Cop}_p(\mathbb{R}^+)}.$$

This completes the proof of the first inequality in (7.27).

Concerning the second inequality in (7.27), if $H^* f \notin L^p(\mathbb{R}^+)$, then the inequality holds trivially. Otherwise, it follows directly from (7.2).

To see the optimality of the constant in the left-hand side of (7.27) we consider the sequence of nonnegative functions f_ϵ given by $f_\epsilon = \chi_{[1,1+\epsilon]}$, where $0 < \epsilon < 1$. Then

$$H^* f_\epsilon(x) = \begin{cases} \ln(1+\epsilon), & \text{if } 0 < x \leq 1, \\ \ln\left(\frac{1+\epsilon}{x}\right), & \text{if } 1 < x < 1+\epsilon, \end{cases}$$

and

$$H^{*2} f_\epsilon(x) = \begin{cases} \ln(1+\epsilon) \ln\left(\frac{1}{x}\right) + \frac{1}{2} \ln^2(1+\epsilon), & \text{if } 0 < x \leq 1, \\ \frac{1}{2} \ln^2\left(\frac{1+\epsilon}{x}\right), & \text{if } 1 < x < 1+\epsilon. \end{cases}$$

Therefore

$$\begin{aligned} \frac{\|H^* f_\epsilon\|_p^p}{\|H^{*2} f_\epsilon\|_p^p} &= \frac{\ln^p(1+\epsilon) + \int_1^{1+\epsilon} \ln^p\left(\frac{1+\epsilon}{x}\right) dx}{\ln^p(1+\epsilon) \int_0^1 \left(\ln\left(\frac{1}{x}\right) + \frac{1}{2} \ln(1+\epsilon)\right)^p dx + \frac{1}{2^p} \int_1^{1+\epsilon} \ln^{2p}\left(\frac{1+\epsilon}{x}\right) dx} \\ &\geq \frac{1}{\int_0^1 \left(\ln\left(\frac{1}{x}\right) + \frac{1}{2} \ln(1+\epsilon)\right)^p dx + 2^{-p} \frac{1+\epsilon}{\ln^p(1+\epsilon)} \int_{1/(1+\epsilon)}^1 \ln^{2p}\left(\frac{1}{x}\right) dx}, \end{aligned}$$

for all $0 < \epsilon < 1$. Now by applying L'Hôpital's rule we get that

$$\lim_{\epsilon \rightarrow 0^+} \frac{\int_{1/(1+\epsilon)}^1 \ln^{2p}\left(\frac{1}{x}\right) dx}{\ln^p(1+\epsilon)} = \lim_{\epsilon \rightarrow 0^+} \frac{\ln^{2p}(1+\epsilon)(1+\epsilon)^{-2}}{p \ln^{p-1}(1+\epsilon)(1+\epsilon)^{-1}} = \lim_{\epsilon \rightarrow 0^+} \frac{\ln^{p+1}(1+\epsilon)}{p(1+\epsilon)} = 0.$$

On the other hand, by using Dominated Convergence Theorem we obtain that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_0^1 \left(\ln\left(\frac{1}{x}\right) + \frac{1}{2} \ln(1+\epsilon)\right)^p dx &= \int_0^1 \lim_{\epsilon \rightarrow 0^+} \left(\ln\left(\frac{1}{x}\right) + \frac{1}{2} \ln(1+\epsilon)\right)^p dx \\ &= \Gamma(p+1). \end{aligned}$$

Hence

$$\frac{1}{\Gamma(p+1)} \geq \lim_{\epsilon \rightarrow 0^+} \frac{\|f_\epsilon\|_{\text{Cop}_p(\mathbb{R}^+)}^p}{\|H^* f_\epsilon\|_{\text{Cop}_p(\mathbb{R}^+)}^p} = \lim_{\epsilon \rightarrow 0^+} \frac{\|H^* f_\epsilon\|_p^p}{\|H^{*2} f_\epsilon\|_p^p} \geq \frac{1}{\Gamma(p+1)}.$$

Thus

$$\lim_{\epsilon \rightarrow 0^+} \frac{\|f_\epsilon\|_{\text{Cop}_p(\mathbb{R}^+)}^p}{\|H^* f_\epsilon\|_{\text{Cop}_p(\mathbb{R}^+)}^p} = \frac{1}{\Gamma(p+1)},$$

for all $1 < p < \infty$. This proves that the first inequality in (7.27) is sharp.

Now, let us see the optimality of the constant in the right-hand side of (7.27) for $1 < p < \infty$. In order to do that, we are going to consider the sequence of functions $f_\epsilon(x) = x^{-\epsilon/p-1/p} \chi_{[1,\infty)}(x)$ and $g_\epsilon(x) = x^{-\epsilon/p'-1/p'} \chi_{[1,\infty)}(x)$, where $0 < \epsilon < 1$. Let $1 \leq x < \infty$, then

$$\begin{aligned} H^{*2} f_\epsilon(x) &= \int_x^\infty t^{-\epsilon/p-1/p-1} \ln\left(\frac{t}{x}\right) dt = x^{-\epsilon/p-1/p} \int_x^\infty \left(\frac{t}{x}\right)^{-\epsilon/p-1/p-1} \ln\left(\frac{t}{x}\right) \frac{dt}{x} \\ &= x^{-\epsilon/p-1/p} \int_1^\infty t^{-\epsilon/p-1/p-1} \ln(t) dt = \frac{x^{-\epsilon/p-1/p}}{(\epsilon/p+1/p)^2} = \frac{f_\epsilon(x)}{(\epsilon/p+1/p)^2}. \end{aligned}$$

Now, by using (7.2) and Hölder's inequality, we obtain

$$\begin{aligned} \frac{\|H^* f_\epsilon\|_{\text{Cop}_p(\mathbb{R}^+)}}{\|f_\epsilon\|_{\text{Cop}_p(\mathbb{R}^+)}} &= \frac{\|H^{*2} f_\epsilon\|_p}{\|H^* f_\epsilon\|_p} \geq \frac{1}{p} \frac{\|H^{*2} f_\epsilon\|_p}{\|f_\epsilon\|_p} \geq \frac{1}{p} \frac{\int_0^\infty H^{*2} f_\epsilon(x) \cdot g_\epsilon(x) dx}{\|f_\epsilon\|_p \|g_\epsilon\|_{p'}} \\ &= \frac{1}{p(\epsilon/p+1/p)^2} \frac{\int_1^\infty x^{-\epsilon-1} dx}{\frac{1}{\epsilon^{1/p}} \frac{1}{\epsilon^{1/p'}}} = \frac{1}{p(\epsilon/p+1/p)^2} = \frac{p}{1+\epsilon} \rightarrow p, \end{aligned}$$

as $\epsilon \rightarrow 0^+$. This concludes the proof. \square

Remark 7.16. Let $1 < p \leq \infty$, and let

$$\|\cdot\|_{\text{Ces}_p(\mathbb{R}^+)} : \text{Ces}_p(\mathbb{R}^+) \rightarrow [0, \infty)$$

determined by

$$\|f\|_{\text{Ces}_p(\mathbb{R}^+)} = \|Hf\|_{\text{Ces}_p(\mathbb{R}^+)}.$$

Since $\|\cdot\|_{\text{Ces}_p(\mathbb{R}^+)}$ is a norm on $\text{Ces}_p(\mathbb{R}^+)$ and H is an injective linear operator, it follows that $\|\cdot\|_{\text{Ces}_p(\mathbb{R}^+)}$ defines a norm on $\text{Ces}_p(\mathbb{R}^+)$, which, by Theorem 7.14, is equivalent to the standard norm $\|\cdot\|_{\text{Ces}_p(\mathbb{R}^+)}$ on the classical Cesàro function space $\text{Ces}_p(\mathbb{R}^+)$.

An analogous construction holds for the Copson function spaces $\text{Cop}_p(\mathbb{R}^+)$, with $1 \leq p < \infty$. Let

$$\|\cdot\|_{\text{Cop}_p(\mathbb{R}^+)} : \text{Cop}_p(\mathbb{R}^+) \rightarrow [0, \infty)$$

given by

$$\|f\|_{\text{Cop}_p(\mathbb{R}^+)} = \|(H^* - I)H^* f\|_p.$$

This defines a norm that, by Theorem 7.12, is equivalent to the standard norm $\|\cdot\|_{\text{Cop}_p(\mathbb{R}^+)}$. Alternatively, one may define

$$\|f\|_{\text{Cop}_p(\mathbb{R}^+)} = \|H^* f\|_{\text{Cop}_p(\mathbb{R}^+)}.$$

This also defines a norm equivalent to $\|\cdot\|_{\text{Cop}_p(\mathbb{R}^+)}$, as established in Theorem 7.15.

Bibliography

- [1] Y. A. Abramovich and C. D. Aliprantis, *An Invitation to Operator Theory*, Amer. Math. Soc., Graduate Studies in Math., Vol. 50, Providence RI, 2002.
- [2] S. V. Astashkin and L. Maligranda, *Structure of Cesàro function spaces*, Indag. Math. (N.S.), 20 (2009), 329–379.
- [3] R. Bañuelos and P. Janakiraman, *L^p -bounds for the Beurling-Ahlfors transform*, Trans. Amer. Math. Soc., 360 (2008), 3603–3612.
- [4] S. Barza, B. M. Demissie, and J. Soria, *Optimal non-absolute domains for the Hardy operator minus identity*, J. Math. Anal. Appl. 538 (2024) 128324.
- [5] S. Barza and J. Soria, *Optimal non-absolute domains for the Cesàro operator minus the identity*. The Mathematical Heritage of Guido Weiss. In: Hernández, E., Peloso, M.M., Ricci, F., Soria, F., and Tabacco, A. (eds). Applied and Numerical Harmonic Analysis. Birkhäuser, 2025.
- [6] M. H. Baumann, *Generalized Triangular Numbers and Combinatorial Explanations*, J. Recreat. Math., 12 (2025), 103–119.
- [7] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.
- [8] G. Bennett, *Lower bounds for matrices*, Linear Algebra Appl., 82 (1986), 81–98.
- [9] G. Bennett, *Factorizing the Classical Inequalities*, Mem. Amer. Math. Soc., 120 (1996). Providence, RI.
- [10] A. Ben Said, *Optimal constants for certain Hardy-type integral operators*. Preprint. Available at: <https://www.ucm.es/garf/file/polynomialhardy>.
- [11] A. Ben Said, S. Boza, and J. Soria, *The norm of Hardy-type oscillation operators in the discrete and continuous setting*. To appear in Analysis and Applications.
- [12] A. Ben Said, S. Boza, and J. Soria, *On sharp reverse Hardy-type inequalities for oscillation operators in the discrete and continuous settings*. Preprint. Available at: <https://www.ucm.es/garf/file/bensaid-boza-soria-reverse>.

-
- [13] A. Ben Said, S. Boza, and J. Soria, *Sharp bounds for equivalent norms in the classical Cesàro and Copson function spaces*. Preprint. Available at: <https://www.ucm.es/garf/file/CesCop>.
- [14] A. Ben Said and G. Sinnamon, *Best bounds for the dual Hardy operator minus identity on decreasing functions*, Ann. Mat. Pura Appl. 205 (2026), 313–325.
- [15] D. W. Boyd, *The spectral radius of averaging operators*, Pacific J. Math., 24(1) (1986), 19–28.
- [16] S. Boza and J. Soria, *Solution to a conjecture on the norm of the Hardy operator minus the Identity*, J. Funct. Anal., 260 (2011), 1020–1028.
- [17] S. Boza and J. Soria, *Norm estimates for the Hardy operator in terms of B_p weights*, J. Proc. Amer. Math. Soc., 145 (2017), 2455–2465.
- [18] S. Boza and J. Soria, *Averaging operators on decreasing or positive functions: Equivalence and optimal bounds*, J. Approx. Theory 237 (2019), 135–152.
- [19] S. Boza and J. Soria, *Weak-type and end-point norm estimates for Hardy operators*, Ann. Mat. Pura Appl., 199 (2020), 2381–2393.
- [20] S. Boza and J. Soria, *The norm of the Cesàro operator minus identity and related operators acting on decreasing sequences*, J. Approx. Theory 292 (2023) 105911.
- [21] A. Brown, P. R. Halmos, and A. L. Shields, *Cesàro operators*, Acta Sci. Math. (Szeged), 26 (1965), 125–137.
- [22] P. R. Halmos, *A Hilbert space problem book*. Encyclopedia of Mathematics and its Applications. Vol. 17, 2nd Ed., 1982.
- [23] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z., 6 (1920), 314–317.
- [24] G. H. Hardy, J. Littlewood, and G. Pólya, *Inequalities*, 2nd Ed., Cambridge Univ. Press, 1967.
- [25] C. O. Horgan, *A note on a class of integral inequalities*, Proc. Camb. Philos. Soc., 74 (1973), 127–131.
- [26] T. Iwaniec, *Extremal inequalities in Sobolev spaces and quasiconformal mappings*, Z. Anal. Anwend. (1) (1982), 1–16.
- [27] A. A. Jagers, *A note on Cesàro sequence spaces*, Nieuw Arch. Wiskd., (3) 22 (1974), 113–124.
- [28] G. J. O. Jameson, *The ℓ_p -norm of $C - I$, where C is the Cesàro operator*, Math. Inequal. Appl., 24 (2021), 551–557.

-
- [29] G. J. O. Jameson, *An Equality Underlying Hardy's Inequality*, American Math. Monthly, 129 (2022), 582–586.
- [30] N. Kaiblinger, L. Maligranda, and L.-E. Persson, *Norms in weighted L^2 -spaces and Hardy operators*, *Function Spaces*, The Fifth Conference (Poznań, 1998), Lecture Notes in Pure and Appl. Math., 213, Dekker (2000), 205–216.
- [31] D. E. Knuth, *Euler's constant to 1271 places*, Math. Comput., 16 (1962), 275–280.
- [32] D. E. Knuth, *The Art of Computer Programming: Volume 1: Fundamental Algorithms (3rd ed.)* (1997), 75–79.
- [33] V. I. Kolyada, *On Cesàro and Copson norms of nonnegative sequences*, Ukrainian Math. J., 71 (2019), 241–258.
- [34] V. I. Kolyada, *Optimal relationships between L^p -norms for the Hardy operator and its dual*, Ann. Mat. Pura Appl., 193 (2014), 423–430.
- [35] V. Kolyada. *On the optimal relationships between L^p -norms for the Hardy operator and its dual for decreasing functions*, J. Approx. Theory 252 (2020) 105362.
- [36] N. Kruglyak and E. Setterqvist, *Sharp estimates for the identity minus Hardy operator on the cone of decreasing functions*, Proc. Amer. Math. Soc., 136 (2008), 2005–2013.
- [37] E. Landau, *A note on a theorem concerning series of positive terms: Extract from a letter from Prof. E. Landau to Prof. I. Schur (communicated by G. H. Hardy)*, J. Lond. Math. Soc., 1 (1926), 38–39.
- [38] K. Lešnika and L. Maligranda, *Abstract Cesàro spaces. Duality*, J. Math. Anal. Appl., 424 (2015), 932–951.
- [39] R. Lyons, *A lower bound on the Cesàro operator*, Proc. Amer. Math. Soc., 86 (1982), 694.
- [40] M. Milman, *A note on reversed Hardy inequalities and Gehring's lemma*, Comm. Pure Appl. Math., L (1997), 0311–0315.
- [41] J. Muscat, *Functional Analysis, An Introduction to Metric Spaces, Hilbert Spaces, and Banach algebras*, 2nd Ed., Springer, 2024.
- [42] P. F. Renaud, *A reversed Hardy inequality*, Bull. Austral. Math. Soc., 34 (1986), 225–232.
- [43] H. Robbins, *A Remark on Stirling's Formula*, American Math. Monthly. 62 (1955), 26–29.
- [44] G. Sinnamon, *Norm of the discrete Cesàro operator minus identity*, Math. Inequal. Appl., 25 (2022), 41–48.

-
- [45] G. Sinnamon, *The level functions in rearrangement invariant spaces*, *Pub. Mat.*, 45 (2001), 175–198.
- [46] M. Strzelecki, *The L^p -norms of the Beurling-Ahlfors transform on radial functions*, *Ann. Acad. Sci. Fenn. Math.*, 42 (2017), no. 1, 73–93.
- [47] M. Strzelecki, *Hardy's operator minus identity and power weights*, *J. Funct. Anal.*, 279 (2020), no. 2, 108532.
- [48] J. Xiao, *A Reverse BMO-Hardy Inequality*, *Real Anal. Exch.*, 25 (1996), no. 2, 673–678.