A WEIERSTRASS EXTREMAL FIELD THEORY FOR NONLOCAL ELLIPTIC FUNCTIONALS

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joint work with X. Cabré (ICREA-UPC-CRM) and I. U. Erneta (UPC-BGSMath)

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO Y MATEMÁTICA APLICADA

Seminario de Análisis Matemático y Matemática Aplicada - UCM

Madrid, March 16th, 2023

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- If $\mathcal{E}$ is (strictly) convex, then there is a unique extremal that turns out to be a minimizer
- Important models with noncovex functional:
  - Allen-Cahn energy
  - Bernoulli free boundary problem
  - Perimeter

Outline of the talk

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5. Applications to monotone solutions and the viscosity theory

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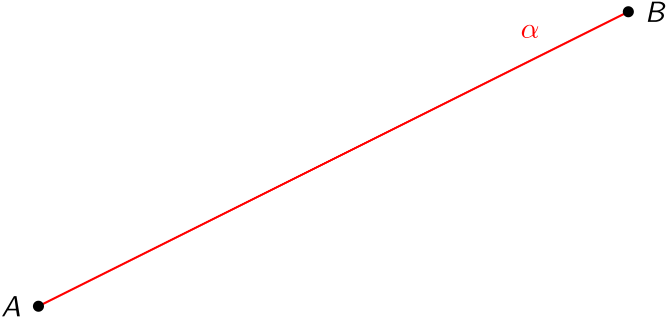

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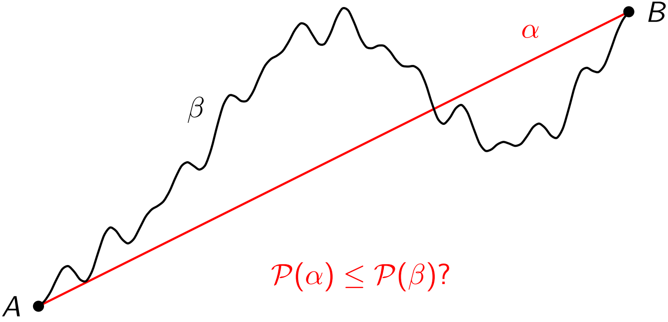

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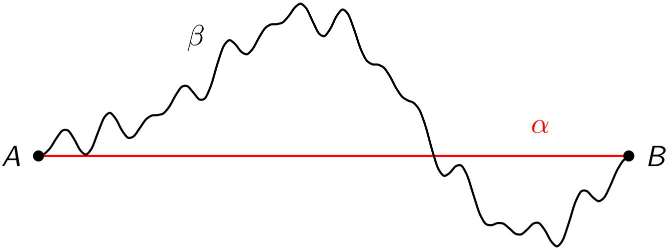

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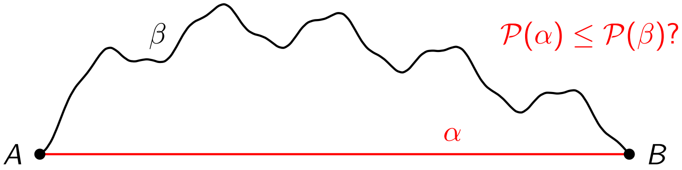

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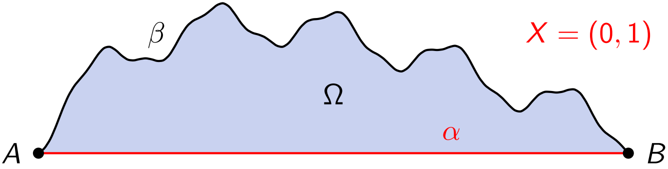

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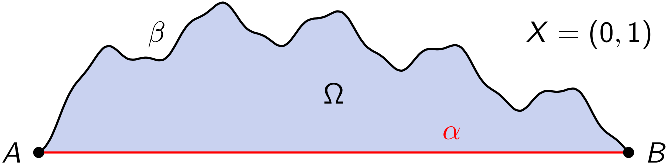

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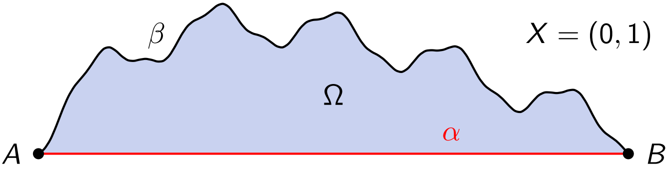

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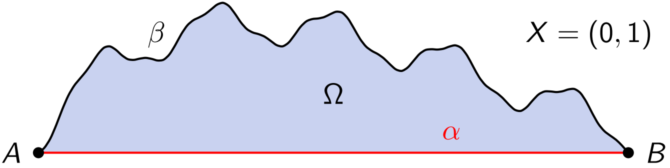

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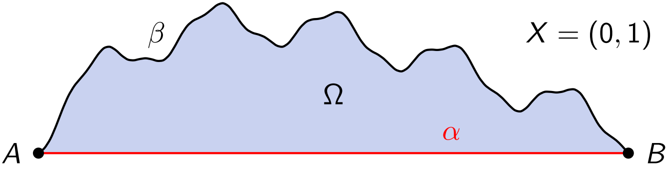

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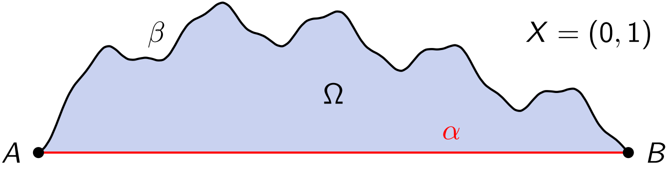

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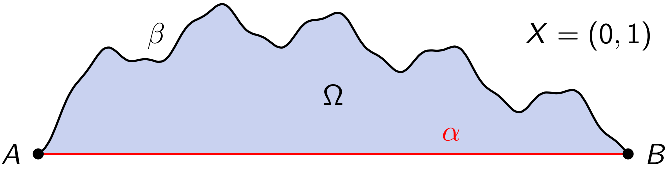

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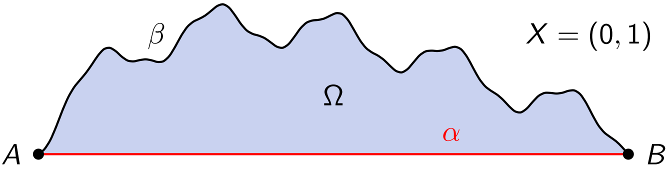

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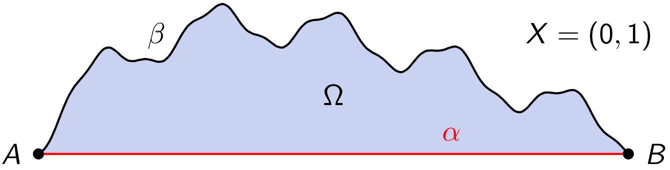

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The calibration functional

Definition

A functional $\mathcal{C}\colon \mathcal{A} \to \mathbb{R}$ is a *calibration* for $\mathcal{E}$ and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet condition as u.
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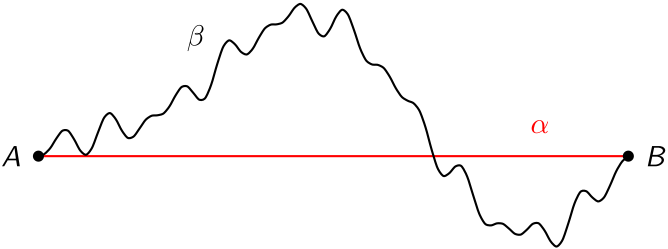

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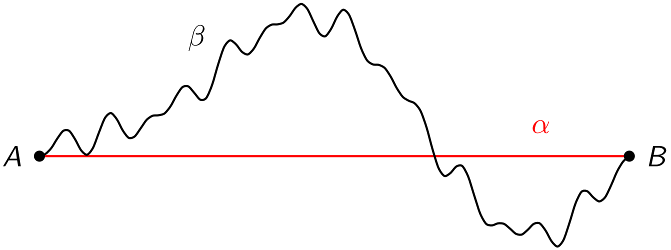

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provides a Calibration for the line α and the perimeter $\mathcal{P}$.

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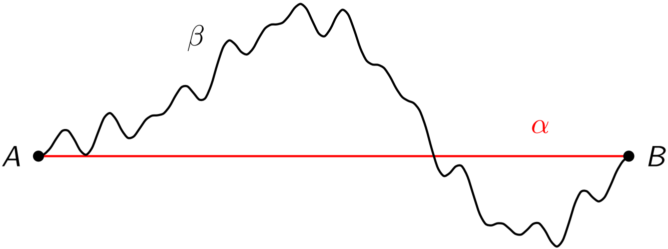

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When is it possible to find such a functional?

Classical theory of Calibrations

Necessary conditions: First order

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

First variation:

$$\delta_\eta \mathcal{E}_L(u) = \frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{E}_L(w + t\eta)$$

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u is minimizer of $\mathcal{E} \implies \mathcal{L}_L u = 0$ in Ω (in the weak sense)

Necessary conditions: Second order

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

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Necessary conditions: Legendre

Stability condition:

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Let u be a stable solution of the energy functional $\mathcal{E}$. Then, it satisfies

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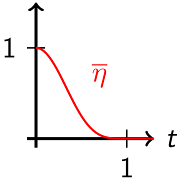

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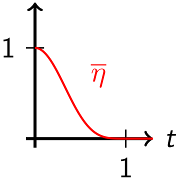

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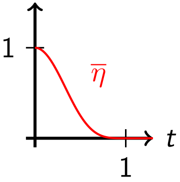

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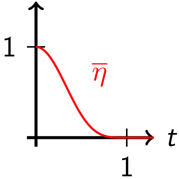

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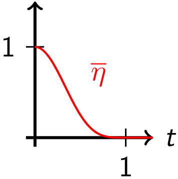

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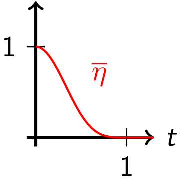

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$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

Weierstrass excess function:

$$E(x,\lambda,q,\tilde{q}) = G_L(x,\lambda,\tilde{q}) - G_L(x,\lambda,q) - \partial_q G_L(x,\lambda,q)\cdot(\tilde{q}-q)$$

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Theorem (Weiestrass Necessary Condition)

Let u be a minimizer of the energy functional $\mathcal{E}_L$ among functions with the same boundary data. Then, it satisfies

$$E(x,u(x),\nabla u(x),\xi) \geq 0 \qquad \text{for all } x\in\Omega,\ \xi\in\mathbb{R}^n.$$

Sufficient conditions: Convexity

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$G_L(x, \lambda, q)$ convex in (λ, q) $\implies$ $\mathcal{E}_L$ is convex

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Lemma (Comparison of energies)

Assume that $G_L(x,\lambda,q)$ is convex in (λ,q). Then, given $u, w \in H^1(\Omega)$, they satisfy

$$\mathcal{E}_L(w) \geq \mathcal{E}_L(u) + \int_\Omega \mathcal{L}_L u(x)\,\big(w(x) - u(x)\big)\,\mathrm{d}x + \int_{\partial\Omega} \mathcal{N}_L u(x)\,\big(w(x) - u(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

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It provides a Calibration in this convex framework for u and $\mathcal{E}$:

$$\mathcal{C}(w) = \mathcal{E}_L(u) + \int_\Omega \mathcal{L}_L u(x)\,\big(w(x)-u(x)\big)\,\mathrm{d}x + \int_{\partial\Omega} \mathcal{N}_L u(x)\,\big(w(x)-u(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

Sufficient conditions: Weierstrass extremal field theory

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

Theorem (Weierstrass Sufficient Condition)

Assume that $G_L(x, \lambda, q)$ is convex in q. If u is embedded in an extremal field, then it is a minimizer of $\mathcal{E}_L$.

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We say that a family of functions $\{u^t\}_{t\in\mathbb{R}}$ is a *field* if

- the functions $t \mapsto u^t(x)$ are increasing for each x
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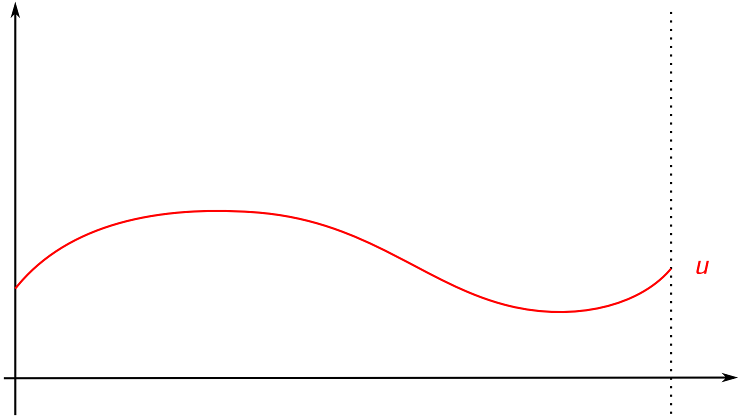
Moreover, it is an extremal field if each leaf u^t satisfies the E-L equation.

Proof via a touching argument



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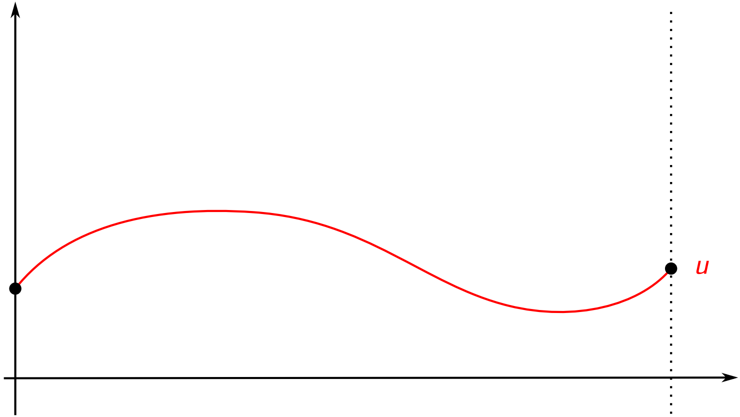

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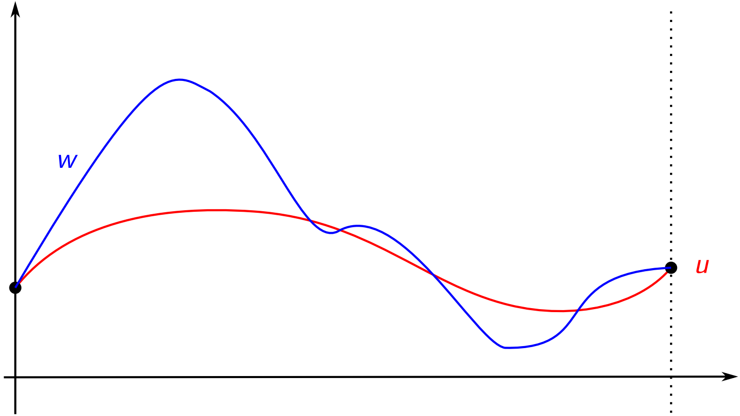

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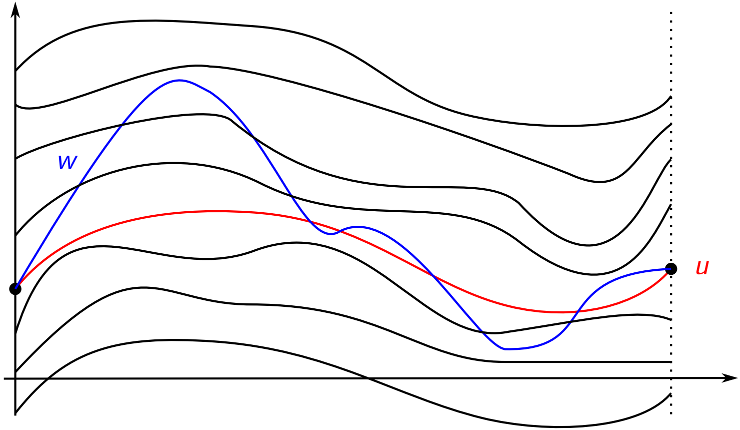

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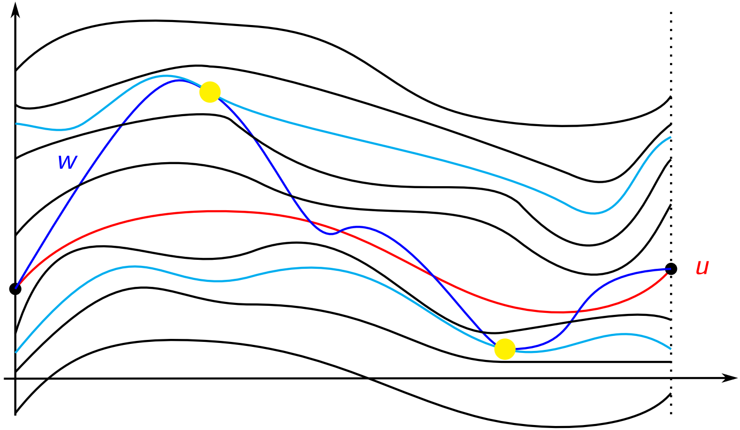

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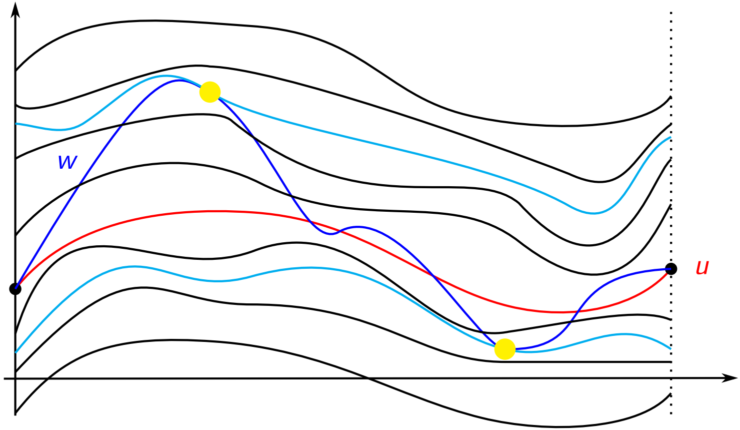
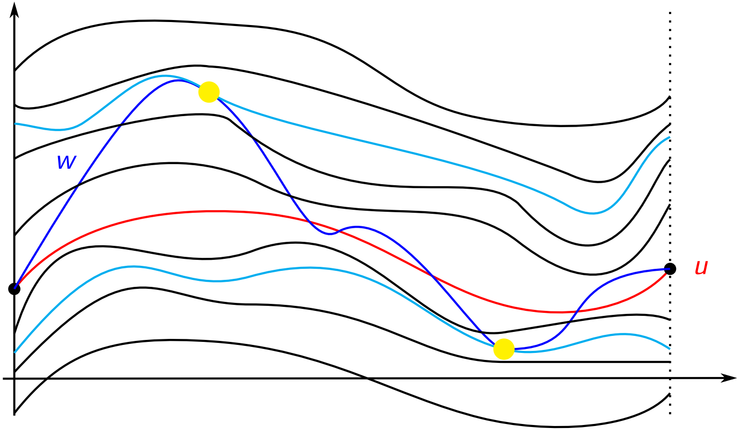


Tools:

- Strong Comparison Principle

Proof via a touching argument



Tools:

- Strong Comparison Principle (follows by the convexity assumption) ✓

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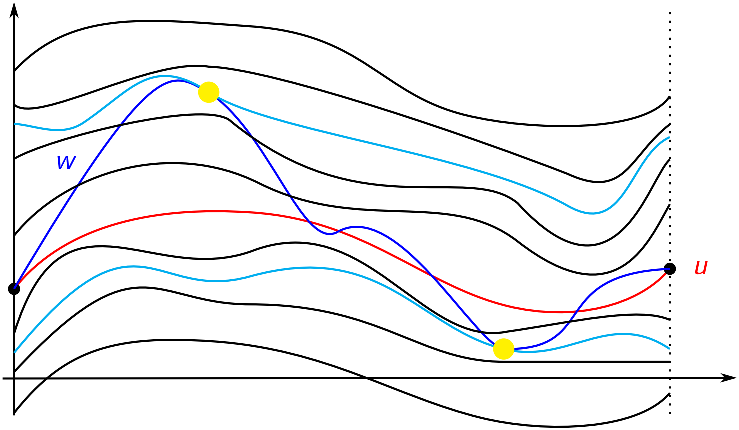
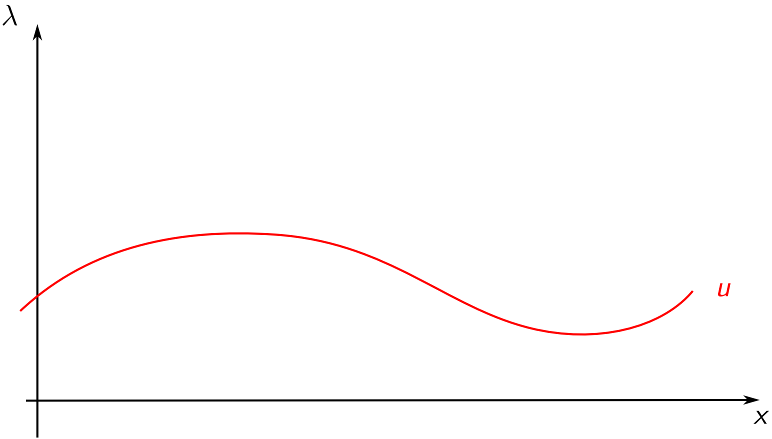


Tools:

- Strong Comparison Principle (follows by the convexity assumption) ✓
- Existence and regularity theory for minimizers ✗

The leaf-parameter function



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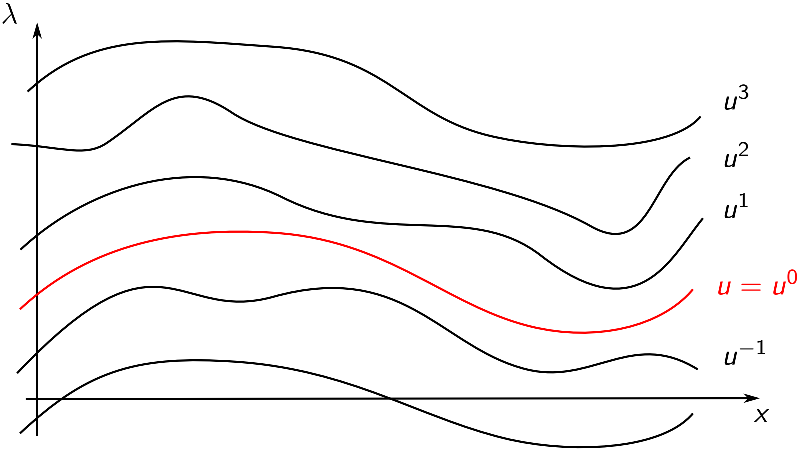

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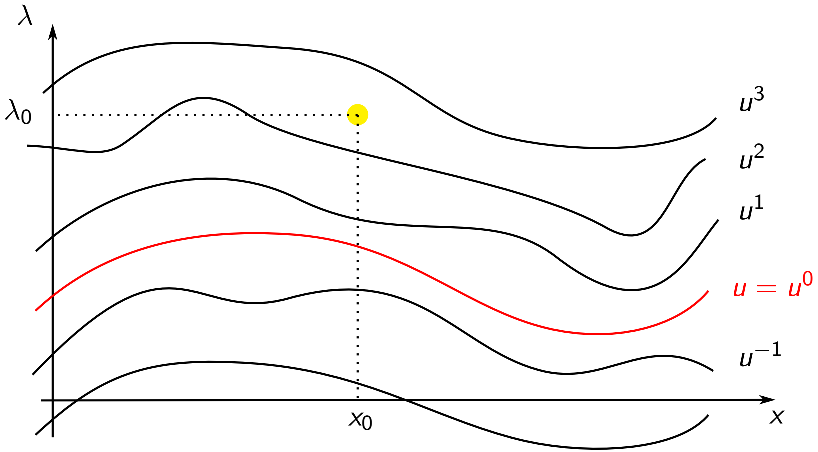

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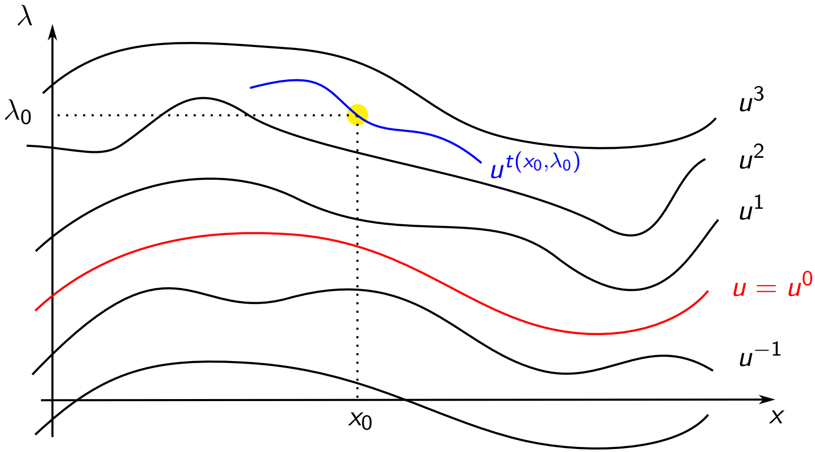

$t(x_0, \lambda_0)$ is the unique $\tau \in \mathbb{R}$ such that $u^\tau(x_0) = \lambda_0$

Proof via the construction of a calibration (1/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

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1. $\mathcal{C}_L(u) = \mathcal{E}_L(u)$?

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$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

1. $\mathcal{C}_L(u) = \mathcal{E}_L(u)$?

$$\mathcal{C}_L(u) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla u^0(x) - \nabla u^t(x)\big)\Big|_{t=t(x,u^0(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,u^0(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (1/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

1. $\mathcal{C}_L(u) = \mathcal{E}_L(u)$?

$$\mathcal{C}_L(u) = \int_\Omega \partial_q G_L(x, u^{\color{red}t}(x), \nabla u^{\color{red}t}(x))\big(\nabla u^0(x) - \nabla u^{\color{red}t}(x)\big)\Big|_{\color{red}t=0}\,\mathrm{d}x + \int_\Omega G_L(x, u^{\color{red}t}(x), \nabla u^{\color{red}t}(x))\Big|_{\color{red}t=0}\,\mathrm{d}x$$

Proof via the construction of a calibration (1/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$
$$+ \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

① $\mathcal{C}_L(u) = \mathcal{E}_L(u)$?

$$\mathcal{C}_L(u) = \int_\Omega \partial_q G_L(x, u^0(x), \nabla u^0(x))\big(\nabla u^0(x) - \nabla u^0(x)\big)\,\mathrm{d}x$$
$$+ \int_\Omega G_L(x, u^0(x), \nabla u^0(x))\,\mathrm{d}x$$

Proof via the construction of a calibration (1/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

1. $\mathcal{C}_L(u) = \mathcal{E}_L(u)$?

$$\mathcal{C}_L(u) = \int_\Omega \partial_q G_L(x, u^0(x), \nabla u^0(x))\big(\nabla u^0(x) - \nabla u^0(x)\big)\,\mathrm{d}x + \int_\Omega G_L(x, u^0(x), \nabla u^0(x))\,\mathrm{d}x$$

Proof via the construction of a calibration (1/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

1. $\mathcal{C}_L(u) = \mathcal{E}_L(u)$?

$$\mathcal{C}_L(u) = \int_\Omega G_L(x, u^0(x), \nabla u^0(x))\,\mathrm{d}x = \mathcal{E}_L(u)$$

Proof via the construction of a calibration (1/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L(x, w(x), \nabla w(x))\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

1. $\mathcal{C}_L(u) = \mathcal{E}_L(u)$? ✓

$$\mathcal{C}_L(u) = \int_\Omega G_L(x, u(x), \nabla u(x))\,\mathrm{d}x = \mathcal{E}_L(u)$$

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

2. $\mathcal{C}_L(w) \leq \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\begin{aligned}\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x \\ + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x\end{aligned}$$

2. $\mathcal{C}_L(w) \leq \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

$\mathcal{E}_L(w) - \mathcal{C}_L(w)$

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

2. $\mathcal{C}_L(w) \leq \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{E}_L(w) - \mathcal{C}_L(w) = \int_\Omega \Big\{ G_L\big(x, w(x), \nabla w(x)\big) - G_L(x, u^t(x), \nabla u^t(x)) - \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big\}\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

② $\mathcal{C}_L(w) \leq \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{E}_L(w) - \mathcal{C}_L(w) = \int_\Omega \Big\{ G_L\big(x, u^t(x), \nabla w(x)\big) - G_L(x, u^t(x), \nabla u^t(x)) - \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big\}\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

② $\mathcal{C}_L(w) \leq \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{E}_L(w) - \mathcal{C}_L(w) = \int_\Omega \Big\{ G_L\big(x, u^t(x), \nabla w(x)\big) - G_L(x, u^t(x), \nabla u^t(x)) - \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big\}\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\begin{aligned}\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x \\ + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x\end{aligned}$$

2. $\mathcal{C}_L(w) \leq \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{E}_L(w) - \mathcal{C}_L(w) = \int_\Omega E(x, u^t(x), \nabla u^t(x), \nabla w(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L(x, w(x), \nabla w(x))\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

2. $\mathcal{C}_L(w) \leq \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{E}_L(w) - \mathcal{C}_L(w) = \int_\Omega E(x, u^t(x), \nabla u^t(x), \nabla w(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

2. $\mathcal{C}_L(w) \le \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{E}_L(w) - \mathcal{C}_L(w) = \int_\Omega \color{red}{E(x, u^t(x), \nabla u^t(x), \nabla w(x))}\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (2/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\, \mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\, \mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\, \mathrm{d}x$$

2. $\mathcal{C}_L(w) \leq \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u? ✓

$$\mathcal{E}_L(w) - \mathcal{C}_L(w) = \int_\Omega E(x, u^t(x), \nabla u^t(x), \nabla w(x))\Big|_{t=t(x,w(x))}\, \mathrm{d}x$$

Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\begin{aligned}\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x \\ + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x\end{aligned}$$

3. $\mathcal{C}_L(u) = \mathcal{C}_L(w)$ for all w with the same Dirichlet condition as u?

Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\begin{aligned}\mathcal{C}_L(w) = &\int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x \\ &+ \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x\end{aligned}$$

3. $\mathcal{C}_L(u) = \mathcal{C}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_L(w) = \int_\Omega \begin{pmatrix} X^1(x,t) \\ X^2(x,t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

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$$\mathcal{C}_L(w) = \int_\Omega \begin{pmatrix} X^1(x,t) \\ X^2(x,t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

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Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

3. $\mathcal{C}_L(u) = \mathcal{C}_L(w)$ for all w with the same Dirichlet condition as u?

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Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

3. $\mathcal{C}_L(u) = \mathcal{C}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_L(w) = \int_\Omega \begin{pmatrix} X^1(x,t) \\ X^2(x,t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \big|_{t=t(x,w(x))}\,\mathrm{d}x$$

$$= \int_{\mathrm{graph}(w)} X \cdot \nu_w \,\mathrm{d}\mathcal{H}^n(x,\lambda)$$

Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

3. $\mathcal{C}_L(u) = \mathcal{C}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_L(w) = \int_\Omega \begin{pmatrix} X^1(x,t) \\ X^2(x,t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

$$= \int_{\text{graph}(w)} X \cdot \nu_w \,\mathrm{d}\mathcal{H}^n(x,\lambda), \quad \text{with} \quad \operatorname{div} X = \mathcal{L}_L(u^t)(x)\big|_{t=t(x,\lambda)}$$

Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x \\ + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

3. $\mathcal{C}_L(u) = \mathcal{C}_L(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_L(w) = \int_{\mathrm{graph}(w)} X \cdot \nu_w \,\mathrm{d}\mathcal{H}^n, \quad \text{where} \quad \operatorname{div} X = 0$$

Proof via the construction of a calibration (3/4)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

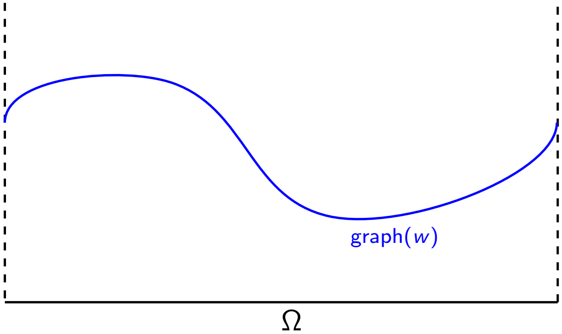
$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x \\ + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

3. $\mathcal{C}_L(u) = \mathcal{C}_L(w)$ for all w with the same Dirichlet condition as u? ✓

$$\mathcal{C}_L(w) = \int_{\mathrm{graph}(w)} X \cdot \nu_w\,\mathrm{d}\mathcal{H}^n, \quad \text{where} \quad \operatorname{div} X = 0$$

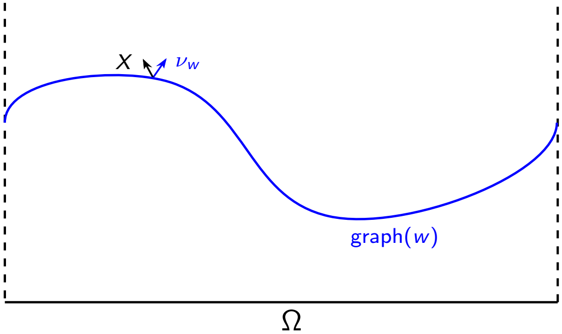
Proof via the construction of a calibration (4/4)

$$\mathcal{C}(w) = \int_{\mathrm{graph}(w)} X \cdot \nu_w \, \mathrm{d}\mathcal{H}^n, \quad \text{where} \quad \operatorname{div} X = 0$$



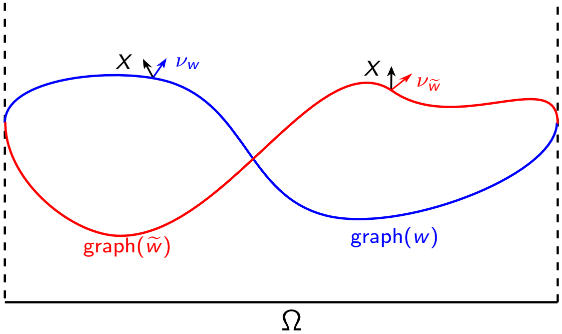
Proof via the construction of a calibration (4/4)

$$\mathcal{C}(w) = \int_{\mathrm{graph}(w)} X \cdot \nu_w \, \mathrm{d}\mathcal{H}^n, \quad \text{where} \quad \operatorname{div} X = 0$$



Proof via the construction of a calibration (4/4)

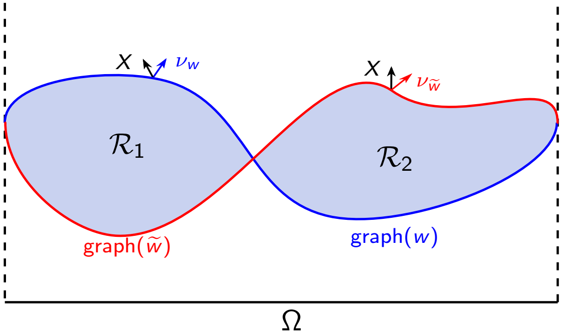
$$\mathcal{C}(w) = \int_{\mathrm{graph}(w)} X \cdot \nu_w \, \mathrm{d}\mathcal{H}^n, \quad \text{where} \quad \operatorname{div} X = 0$$



$$\mathcal{C}_L(w) - \mathcal{C}_L(\widetilde{w}) = \int_{\mathrm{graph}(w)} X \cdot \nu_w \, \mathrm{d}\mathcal{H}^n - \int_{\mathrm{graph}(\widetilde{w})} X \cdot \nu_{\widetilde{w}} \, \mathrm{d}\mathcal{H}^n$$

Proof via the construction of a calibration (4/4)

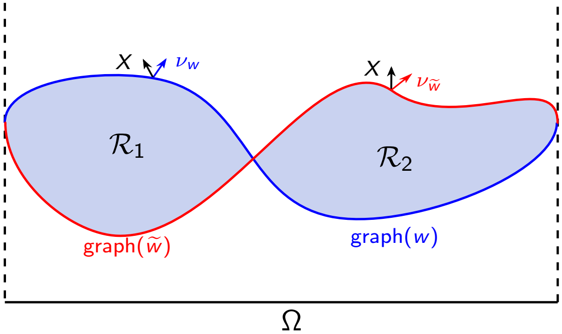
$$\mathcal{C}(w) = \int_{\mathrm{graph}(w)} X \cdot \nu_w \, \mathrm{d}\mathcal{H}^n, \quad \text{where} \quad \operatorname{div} X = 0$$



$$\mathcal{C}_L(w) - \mathcal{C}_L(\widetilde{w}) = \int_{\partial\mathcal{R}_1} X \cdot \nu \, \mathrm{d}\mathcal{H}^n - \int_{\partial\mathcal{R}_2} X \cdot \nu \, \mathrm{d}\mathcal{H}^n$$

Proof via the construction of a calibration (4/4)

$$\mathcal{C}(w) = \int_{\mathrm{graph}(w)} X \cdot \nu_w \, \mathrm{d}\mathcal{H}^n, \quad \text{where} \quad \operatorname{div} X = 0$$



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Proof via the construction of a calibration (4/4)

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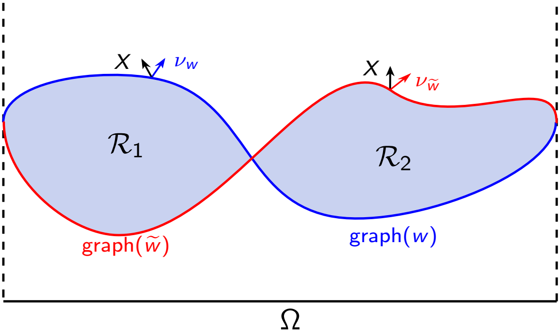

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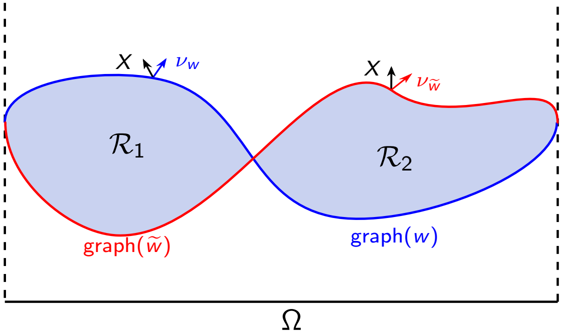

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This geometric strategy fails when applying to other problems

Embedding the extremal

When can u be embedded into an extremal field for $\mathcal{E}$?

- If there is translation invariance

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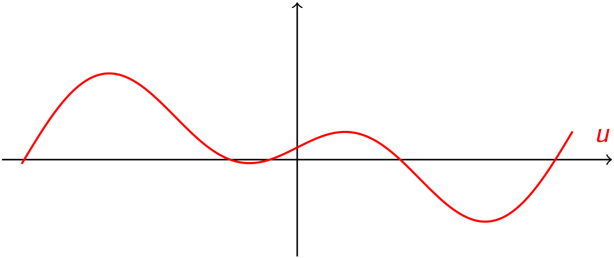

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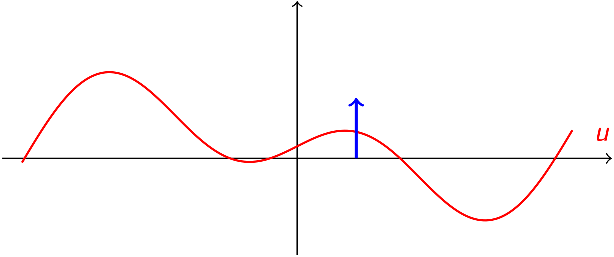

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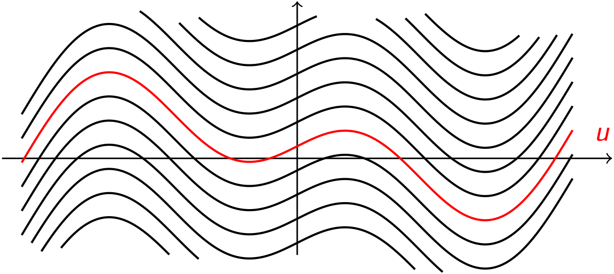

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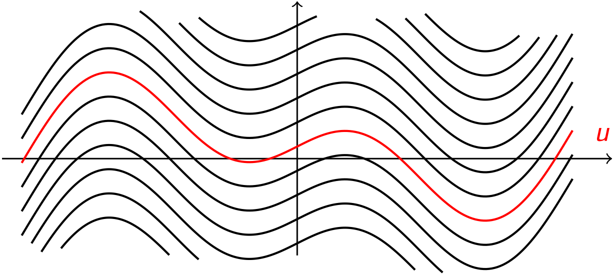

$u^t(x) := u(x) + t$

Nonlocal theory of Calibrations

Fractional energy functional

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

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First variation:

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \mathcal{E}_s(w + \varepsilon\eta) = \int_\Omega \left\{ (-\Delta)^s w(x) - F'(w(x)) \right\} \eta(x) \, \mathrm{d}x \\ + \int_{\Omega^c} \mathcal{N}_s w(x) \, \eta(x) \, \mathrm{d}x,$$

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where

$$(-\Delta)^s w(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x-y|^{n+2s}} \, \mathrm{d}y,$$

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and

$$\mathcal{N}_s w(x) = c_{n,s} \int_\Omega \frac{w(x) - w(y)}{|x - y|^{n+2s}}\,\mathrm{d}y.$$

Towards the fractional calibration: First attempts

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x$$

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- Replace the gradient terms by differences and integrals by double integrals

$$\begin{aligned} \mathcal{F}_s^1(w) = \frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} & \frac{(u^t(x) - u^t(y))(w(x) - w(y))}{|x-y|^{n+2s}} \Bigg|_{t=t(x,w(x))} \mathrm{d}x\, \mathrm{d}y \\ & - \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u^t(x) - u^t(y)|^2}{|x-y|^{n+2s}} \Bigg|_{t=t(x,w(x))} \mathrm{d}x\, \mathrm{d}y \\ & - \int_\Omega F(w(x))\, \mathrm{d}x \end{aligned}$$

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Revisit local theory

Can we find the same structure in the local theory? YES ✓

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$$\Downarrow$$

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$$+ \int_{\partial\Omega} \int_{u(x)}^{w(x)} \partial_\nu u^t(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1}(x) + \mathcal{E}_L(u).$$

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Theorem [Cabré, Erneta & F-N '22]

Let u be embedded in an extremal field $\{u^t\}_{t\in\mathbb{R}}$ such that $(x,t)\to u^t(x)$ is a bounded C^2 function, and let $\mathcal{C}_s$ be the functional

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Then, it follows that $\mathcal{C}_s$ is a calibration for the functional $\mathcal{E}_s$ and u.

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

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$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c \times \Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}} \Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

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- $\mathcal{C}_s(u) = \mathcal{C}_s(w)$ for all w with the same Dirichlet condition as u? ✓
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$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c \times \Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}} \Bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
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Fractional calibration: Proof II

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

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$$- \int_\Omega F(w(x)) \, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

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$$\lambda = u^t(x) \quad \Big\Downarrow \quad \text{Symmetrization in } x \text{ and } y$$

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$$\lambda = u^t(x) \quad \Big\Downarrow \quad \text{Symmetrization in } x \text{ and } y$$

$$\mathcal{C}_s(w) = -\frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}} \, \partial_t u^t(y) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y$$
$$+ \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Fractional calibration: Proof II

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y$$
$$- \int_\Omega F(w(x))\,\mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

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Now,

$$-\frac{1}{2} \int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y)) \partial_t u^t(y)\,\mathrm{d}t$$

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Now,

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t \leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

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Now,

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Now,

$$-\frac{1}{2} \int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y)) \partial_t u^t(y) \, \mathrm{d}t \leq -\frac{1}{2} \int_{t(x,w(x))}^{t(y,w(y))} (w(x) - u^t(y)) \partial_t u^t(y) \, \mathrm{d}t$$
$$= \frac{1}{4} \int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t} |w(x) - u^t(y)|^2 \, \mathrm{d}t$$

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Now,

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t \leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

$$= \frac{1}{4}\int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t}|w(x) - u^t(y)|^2\,\mathrm{d}t$$

$$= \frac{1}{4}|w(x) - w(y)|^2 - \frac{1}{4}|w(x) - u^{t(x,w(x))}(y)|^2$$

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- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\begin{aligned}\mathcal{C}_s(w) = &-\frac{c_{n,s}}{2}\iint_{(\Omega^c\times\Omega^c)^c}\int_{t(x,w(x))}^{t(y,w(y))}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\,\partial_t u^t(y)\,\mathrm{d}t\;\mathrm{d}x\,\mathrm{d}y\\ &+\frac{c_{n,s}}{4}\iint_{(\Omega^c\times\Omega^c)^c}\frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega F(w(x))\,\mathrm{d}x\end{aligned}$$

Now,

$$\begin{aligned}-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))}(u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t &\leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))}(w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t\\ &= \frac{1}{4}\int_{t(x,w(x))}^{t(y,w(y))}\frac{\mathrm{d}}{\mathrm{d}t}|w(x)-u^t(y)|^2\,\mathrm{d}t\\ &= \frac{1}{4}|w(x)-w(y)|^2-\frac{1}{4}|w(x)-u^{t(x,w(x))}(y)|^2\end{aligned}$$

Fractional calibration: Proof III

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u? ✓

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$$+ \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\, \mathrm{d}x\, \mathrm{d}y - \int_\Omega F(w(x))\, \mathrm{d}x$$

Now,

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Then, we conclude

$$\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$$

Nonlocal calibration

$$\mathcal{E}_N(w) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} G_N(x, y, w(x), w(y))\,\mathrm{d}x\,\mathrm{d}y$$

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Theorem [Cabré, Erneta & F-N '23]

Let u be embedded in an extremal field $\{u^t\}_{t\in\mathbb{R}}$ such that $(x,t) \to u^t(x)$ is a bounded C^2 function. Assume that G_N is a function of (x, y, a, b) satisfying $\partial^2_{ab} G_N \leq 0$ and let $\mathcal{C}_N$ be the functional

$$\mathcal{C}_N(w) = \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \partial_a G_N(x, y, u^t(x), u^t(y))\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y + \mathcal{E}_N(u)$$

Then, it follows that $\mathcal{C}_N$ is a calibration for the functional $\mathcal{E}_N$ and u.

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Theorem [Cabré, Erneta & F-N '23]

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$$\begin{aligned} \mathcal{C}_N(w) = \iint_{(\Omega^c \times \Omega^c)^c} \int_{u(x)}^{w(x)} \partial_a G_N(x, y, u^t(x), u^t(y)) \Big|_{t = t(x, \lambda)} \, \mathrm{d}\lambda \, \mathrm{d}x \, \mathrm{d}y \\ + \mathcal{E}_N(u) \end{aligned}$$

Then, it follows that $\mathcal{C}_N$ is a calibration for the functional $\mathcal{E}_N$ and u.

The condition $\partial^2_{ab} G_N \leq 0$ turns to be the natural ellipticity condition for the problem.

The ellipticity condition $\partial^2_{ab} G_N \leq 0$

$$\partial^2_{ab} G_N \leq 0 \quad \Longrightarrow \quad \mathcal{L}_N \ \text{ satisfies the SCP}$$

We say that the operator $\mathcal{L}_N$ satisfies the Strong Comp. Princ. (SCP) if

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$$G_N(x,y,a,b) = -\mathbb{1}_{\Omega\times\Omega}(x,y)K(x-y)\, a\, b$$
corresponds to convolution-type energies.

Application to monotone solutions

$$\mathcal{L}_N(u)(x) = \int_{\mathbb{R}^n} \partial_a G_N(x, y, u(x), u(y))\,\mathrm{d}y$$

Corollary

Let u be a sufficiently regular solution of $\mathcal{L}_N(u) = 0$ in $\mathbb{R}^n$ satisfying the monotonicity condition $\partial_{x_n} u > 0$ in $\mathbb{R}^n$.
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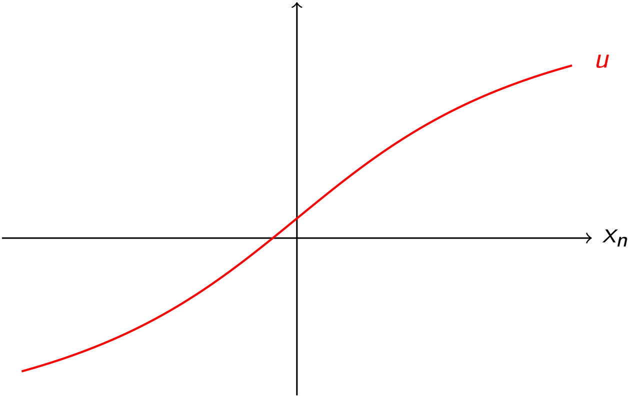
Then, for each bounded domain $\Omega \subset \mathbb{R}^n$, u is a minimizer of $\mathcal{E}_N$ in the set of functions w satisfying

$$\lim_{\tau \to -\infty} u(x', \tau) < w(x', x_n) < \lim_{\tau \to +\infty} u(x', \tau)$$

and such that $w \equiv u$ in Ω^c.

Application to monotone solutions: The field



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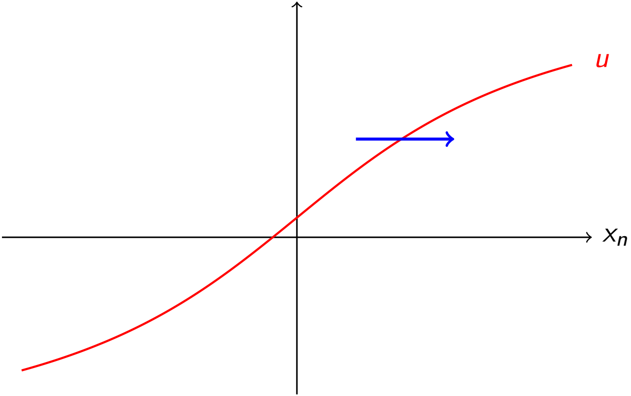

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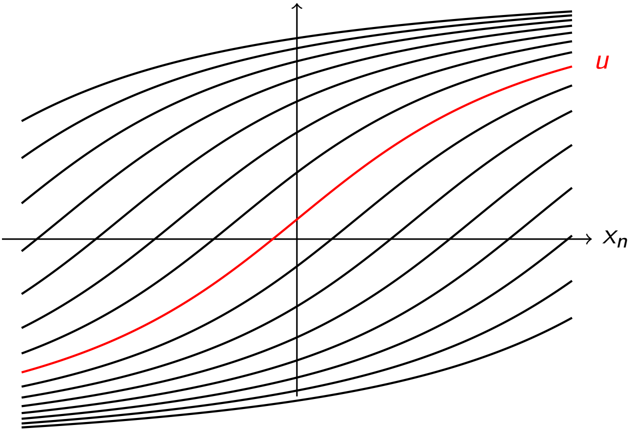

$$u^t(x) := u(x + t\, e_n)$$

Application to the viscosity theory

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Theorem (Cabré, Erneta & F-N '23)

Let u be a minimizer of $\mathcal{E}_N$ among functions with same Dirichlet conditions. Assume that the ellipticity condition $\partial_{ab} G_N \leq 0$ holds. Then, it is a viscosity solution of the associated equation $\mathcal{L}_N(u) = 0$ in Ω.

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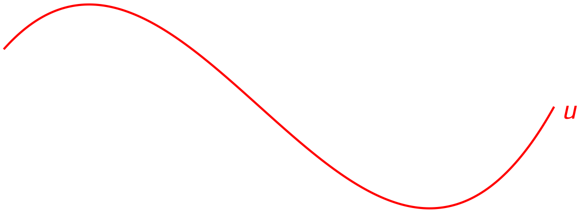
Application to the viscosity theory: Proof



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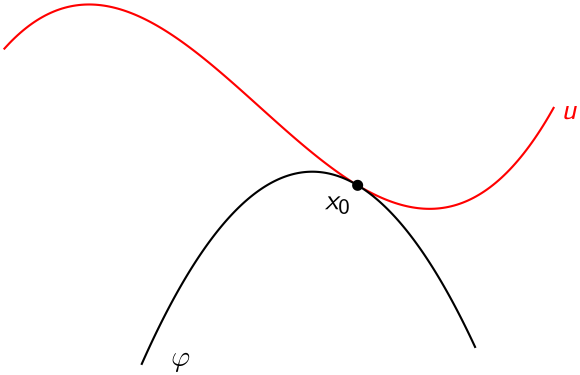
$$\mathcal{L}_N \varphi(x_0) \geq 0?$$



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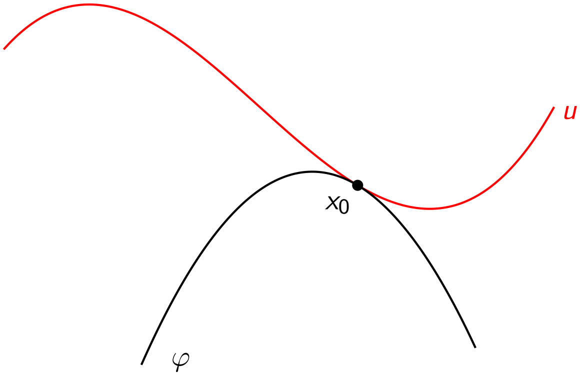
$\mathcal{L}_N \varphi(x_0) \geq 0$? Assume not



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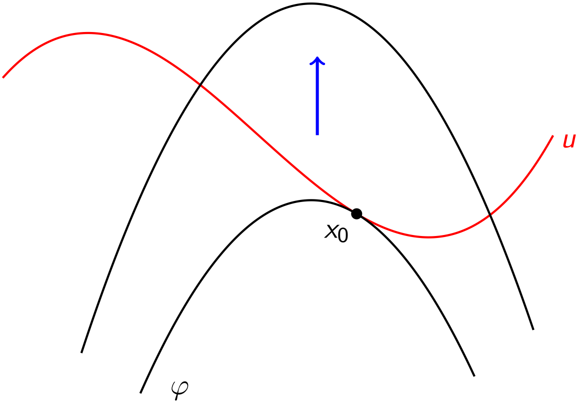

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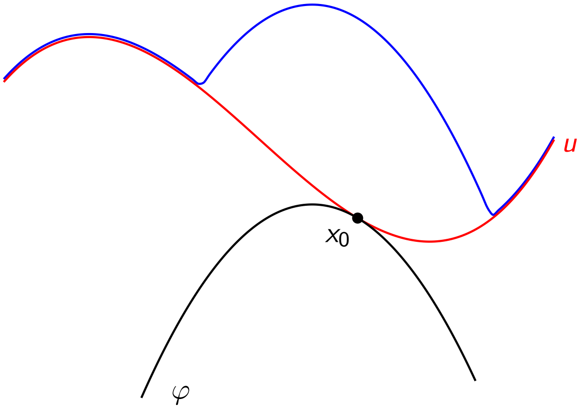

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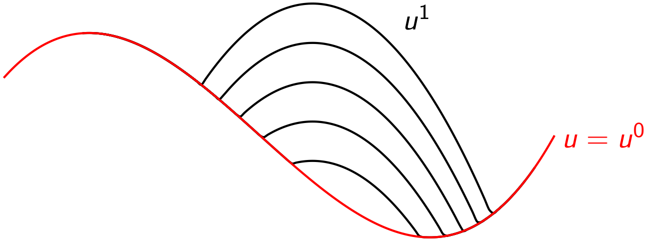

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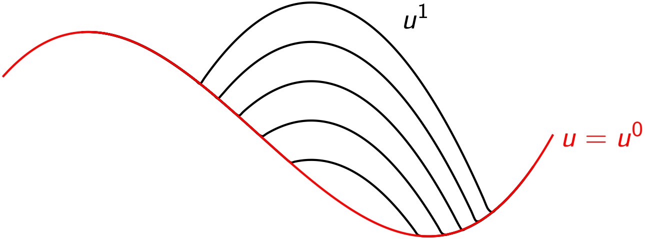

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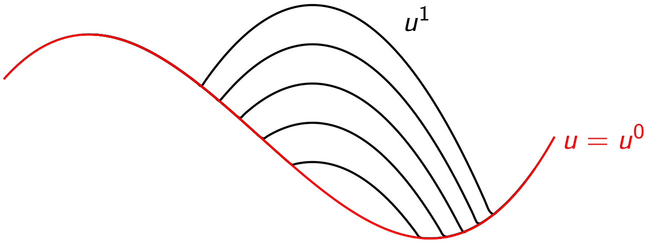

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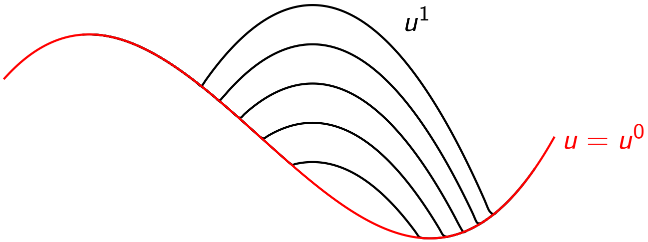

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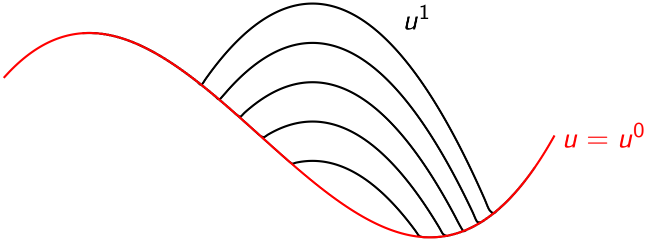

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Thank You

Towards the fractional calibration

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x))\, \mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals

$$\begin{aligned} \mathcal{F}_s^2(w) = \frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} & \frac{(u^\tau(x) - u^t(y))(w(x) - w(y))}{|x-y|^{n+2s}} \Bigg|_{\substack{t=t(x,w(x)) \\ \tau=t(y,w(y))}} \mathrm{d}x\, \mathrm{d}y \\ & - \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u^\tau(x) - u^t(y)|^2}{|x-y|^{n+2s}} \Bigg|_{\substack{t=t(x,w(x)) \\ \tau=t(y,w(y))}} \mathrm{d}x\, \mathrm{d}y \\ & - \int_\Omega F(w(x))\, \mathrm{d}x \end{aligned}$$

Nonlocal perimeter

$$\mathcal{P}(F) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} |\mathbb{1}_F(x) - \mathbb{1}_F(y)| K(x-y) \, \mathrm{d}x \, \mathrm{d}y$$

$$\Downarrow \quad \text{[Cabré '19]}$$

$$\begin{aligned} \mathcal{C}_{\mathcal{P}}(F) = & \int_\Omega \mathbb{1}_F(x) \, H_K[E^t](x) \Big|_{t=\phi(x)} \mathrm{d}x \\ & + \int_{\Omega^c} \mathbb{1}_F(x) \left\{ \int_\Omega \big(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y) \big) K(x-y) \, \mathrm{d}y \right\} \Bigg|_{t=\phi(x)} \mathrm{d}x \end{aligned}$$

Here the Euler-Lagrange operator is given by

$$H_K[E](x) = \int_{\mathbb{R}^n} \big(\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y) \big) K(x-y) \, \mathrm{d}y$$

The fractional Laplacian

The canonical example of integro-differential operator

$$(-\Delta)^s w(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x-y|^{n+2s}} \, \mathrm{d}y, \quad s \in (0,1).$$

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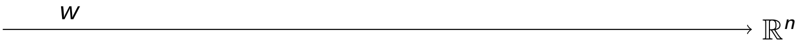
- It has an associated local extension problem
- Representation via the heat semigroup:

$$(-\Delta)^s w(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(\mathrm{e}^{t\Delta} w - w\right) \frac{dt}{t^{1+s}}$$

The extension problem for the fractional Laplacian



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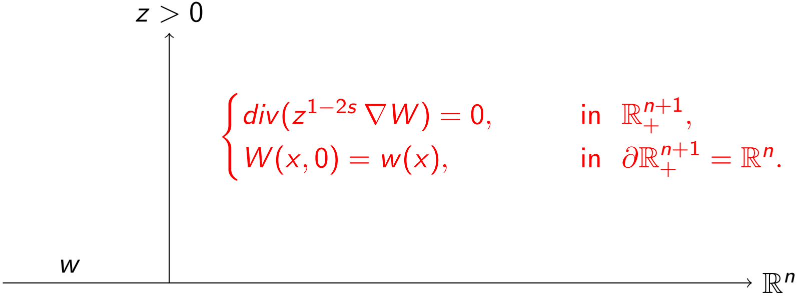

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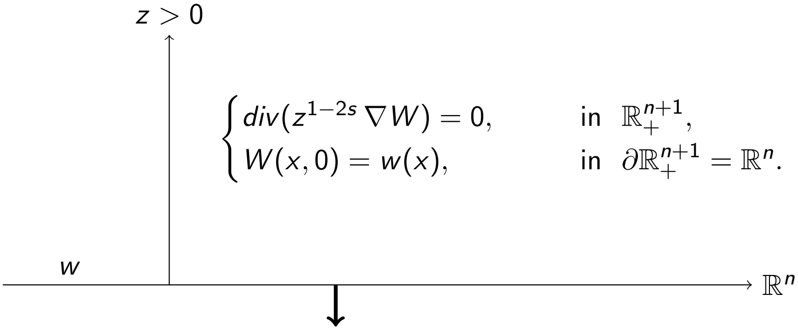

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$$-d_s \lim_{z \downarrow 0} z^{1-2s} \partial_z W = (-\Delta)^s w$$