NULL-LAGRANGIANS AND CALIBRATIONS FOR NONLOCAL FUNCTIONALS

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joint work with X. Cabré (ICREA-UPC-CRM) and I. U. Erneta (UPC-BGSMath)

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Seminario de Ecuaciones en Derivadas Parciales UAM-ICMAT

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The problem

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- If $\mathcal{E}$ is (strictly) convex, then there is a unique extremal that turns out to be a minimizer
- Important models with noncovex functional:
  - Allen-Cahn energy
  - Bernoulli free boundary problem
  - Perimeter

Outline of the talk

1. Motivation through a "toy example"

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5. Applications to monotone solutions and the viscosity theory

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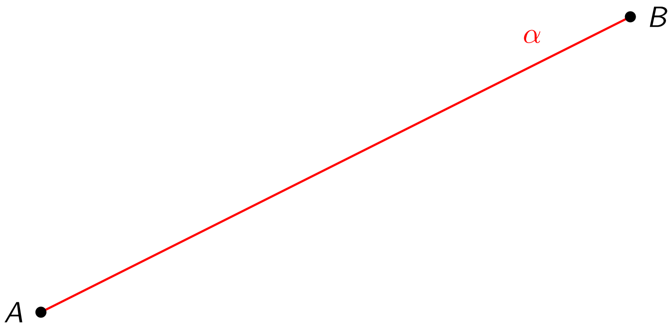

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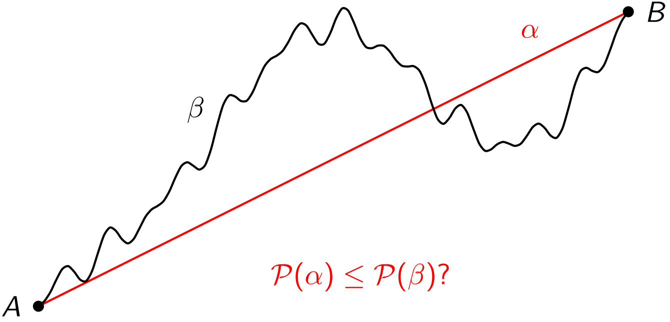

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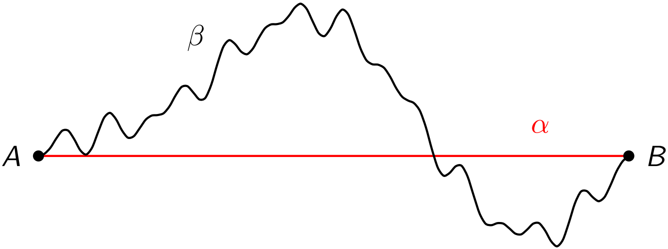

$\mathcal{P}(\alpha) \leq \mathcal{P}(\beta)$?

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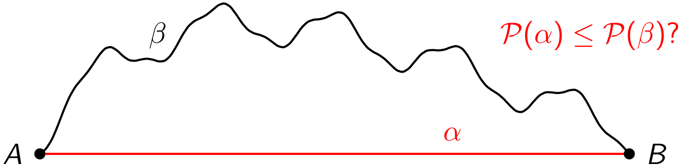

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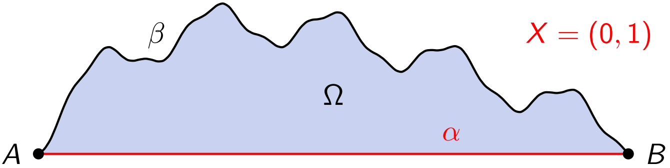

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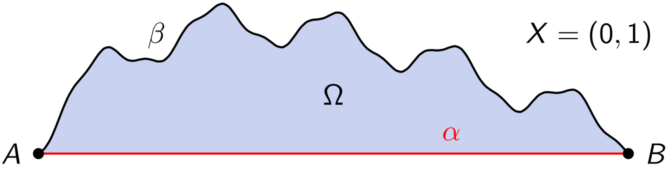

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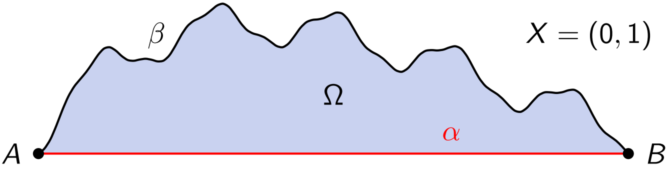

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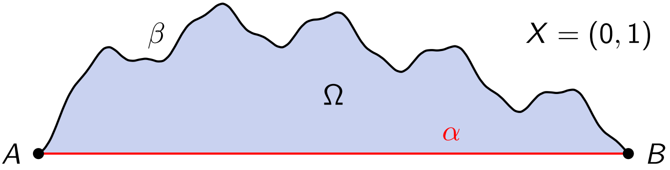

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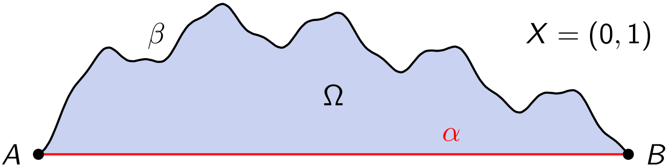

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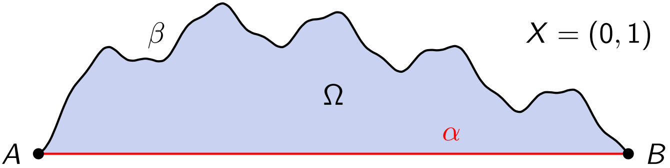

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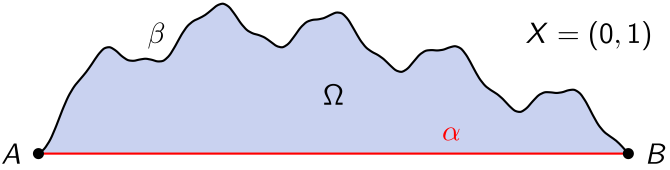

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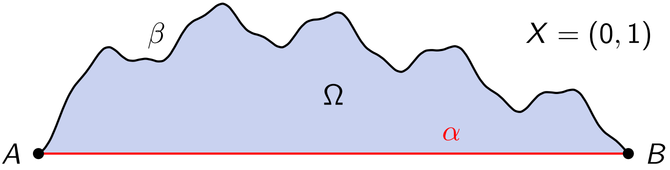

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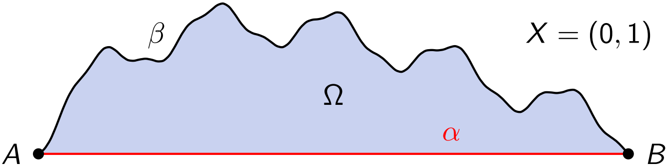

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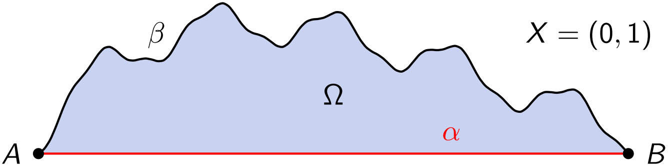

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$$\mathcal{C}(\gamma) = \int_\gamma X \cdot \nu_\gamma \, \mathrm{dl}, \quad \text{where} \quad |X| = 1 \quad \text{and} \quad \operatorname{div} X = 0$$

The calibration functional

Definition

A functional $\mathcal{C}\colon \mathcal{A} \to \mathbb{R}$ is a *calibration* for $\mathcal{E}$ and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet condition as u.
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When is it possible to find such a functional?

Classical theory of Calibrations

Necessary conditions: First order

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

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First variation:

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_L(w+\varepsilon\eta) = \int_\Omega \partial_\lambda G_L\big(x, w(x), \nabla w(x)\big)\,\eta(x) + \partial_q G_L\big(x, w(x), \nabla w(x)\big)\cdot\nabla\eta(x)\,\mathrm{d}x$$

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&= \int_\Omega \Big\{-div\big(\partial_q G_L(x, w(x), \nabla w(x))\big) + \partial_\lambda G_L(x, w(x), \nabla w(x))\Big\}\,\eta(x)\,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega} \partial_q G_L(x, w(x), \nabla w(x))\cdot\nu(x)\,\eta(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \\
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\end{aligned}$$

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\end{aligned}$$

u is minimizer of $\mathcal{E}_L \implies \mathcal{L}_L u = 0$ in Ω (in the weak sense)

Necessary conditions: Second order

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

Second variation:

$$\begin{aligned}\frac{\partial^2}{\partial\varepsilon^2}\Big|_{\varepsilon=0}\mathcal{E}_L(w+\varepsilon\eta) = \int_\Omega \Big\{&\partial^2_{\lambda\lambda} G_L(x, w(x), \nabla w(x))\,\eta^2(x)\\ &+ 2\partial^2_{\lambda q} G_L(x, w(x), \nabla w(x))\cdot\eta(x)\nabla\eta(x)\\ &+ \nabla\eta(x)\cdot\partial^2_{qq} G_L(x, w(x), \nabla w(x))\nabla\eta(x)\Big\}.\end{aligned}$$

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u is minimizer of $\mathcal{E}_L$ $\implies$ u is a stable solution

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u is minimizer of $\mathcal{E}_L \implies u$ is a stable solution

Theorem (Legendre Condition)

Let u be a minimizer of the energy functional $\mathcal{E}_L$ among functions with the same boundary data. Then, it satisfies

$$\partial^2_{qq} G_L(x, u(x), \nabla u(x)) \geq 0 \quad \textit{in } \Omega.$$

Necessary conditions: Weierstrass

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

Weierstrass excess function:

$$E(x, \lambda, q, \tilde{q}) = G_L(x, \lambda, \tilde{q}) - G_L(x, \lambda, q) - \partial_q G_L(x, \lambda, q) \cdot (\tilde{q} - q)$$

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Theorem (Weiestrass Necessary Condition)

Let u be a minimizer of the energy functional $\mathcal{E}_L$ among functions with the same boundary data. Then, it satisfies

$$E(x,u(x),\nabla u(x),\xi) \geq 0 \quad \text{for all } x\in\Omega,\ \xi\in\mathbb{R}^n.$$

Sufficient conditions: Convexity

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$G_L(x, \lambda, q)$ convex in (λ, q) $\implies$ $\mathcal{E}_L$ is convex

Sufficient conditions: Convexity

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$$G_L(x,\lambda,q) \text{ convex in } (\lambda, q) \implies \mathcal{E}_L \text{ is convex}$$

Lemma (Comparison of energies)

Assume that $G_L(x,\lambda,q)$ is convex in (λ, q). Then, given $u, w \in H^1(\Omega)$, they satisfy

$$\mathcal{E}_L(w) \geq \mathcal{E}_L(u) + \int_\Omega \mathcal{L}_L u(x)\,\big(w(x) - u(x)\big)\,\mathrm{d}x + \int_{\partial\Omega} \mathcal{N}_L u(x)\,\big(w(x) - u(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

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It provides a Calibration in this convex framework.

Sufficient conditions: Weierstrass extremal field theory

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

Theorem (Weierstrass Sufficient Condition)

Assume that $G_L(x, \lambda, q)$ is convex in q. If u is embedded in an extremal field, then it is a minimizer of $\mathcal{E}_L$.

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We say that a family of functions $\{u^t\}_{t\in\mathbb{R}}$ is a *field* if

- the functions $t \mapsto u^t(x)$ are increasing for each x
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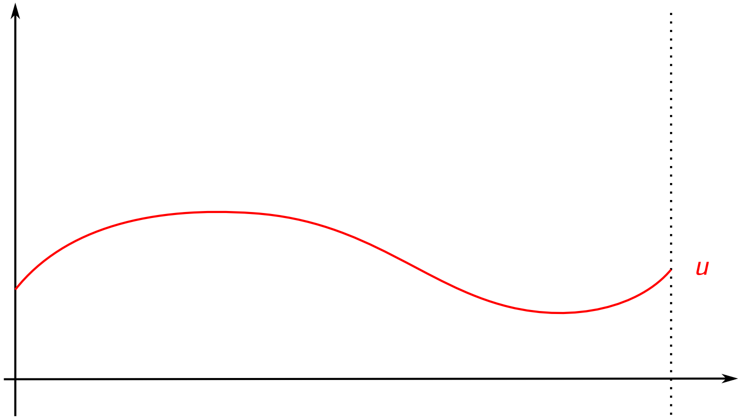
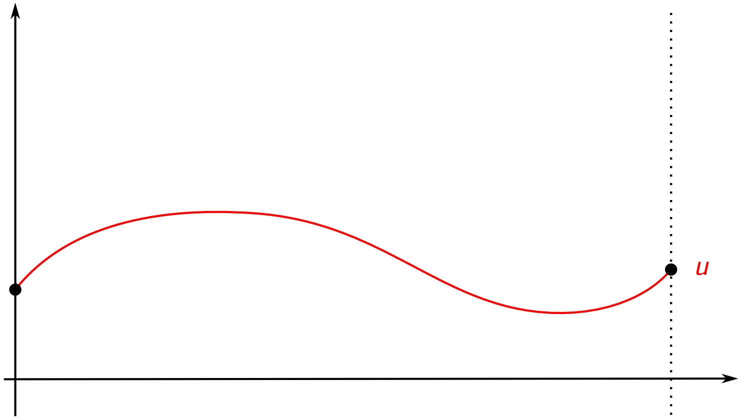
Moreover, it is an extremal field if each leaf u^t satisfies the E-L equation.

Proof via a touching argument



Proof via a touching argument



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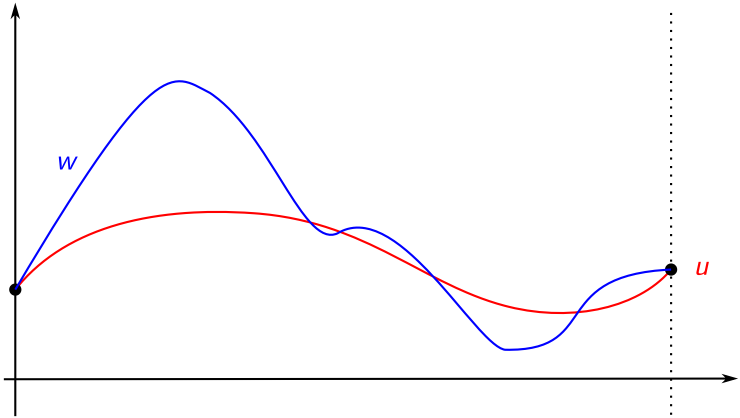

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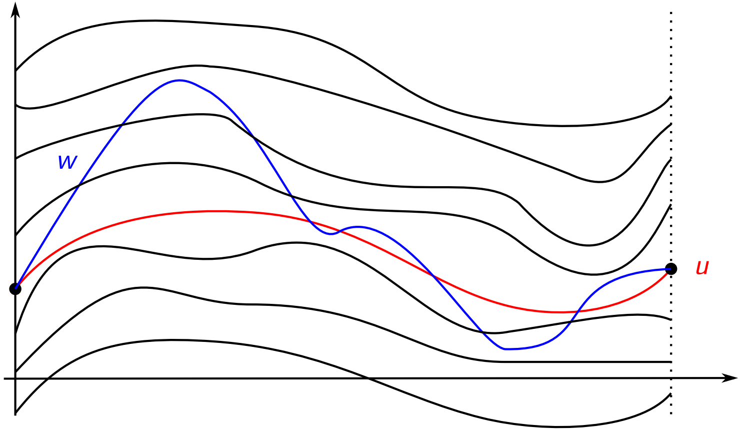

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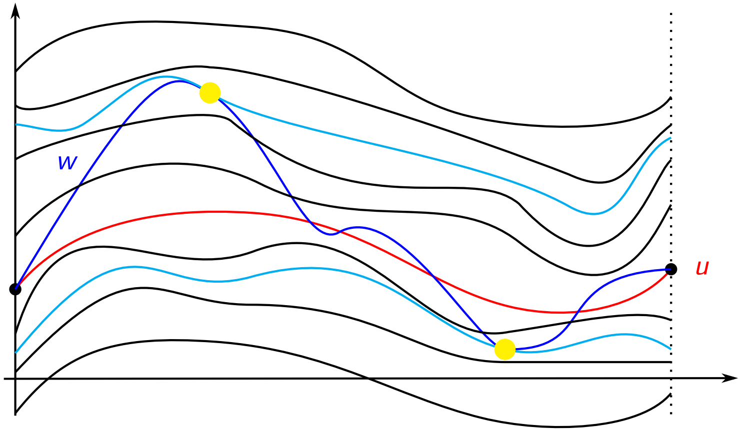

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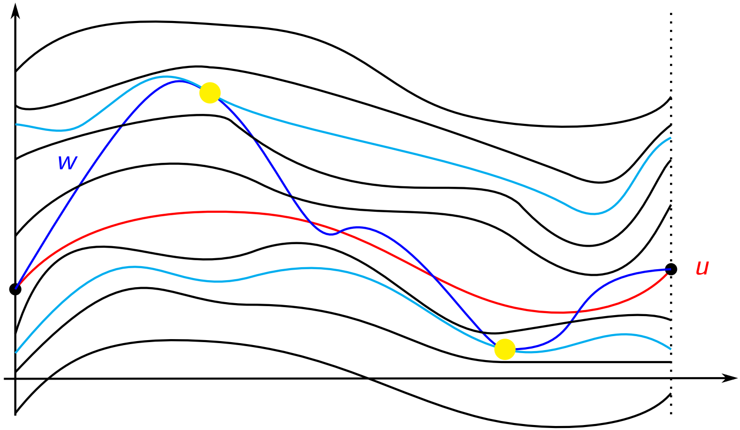
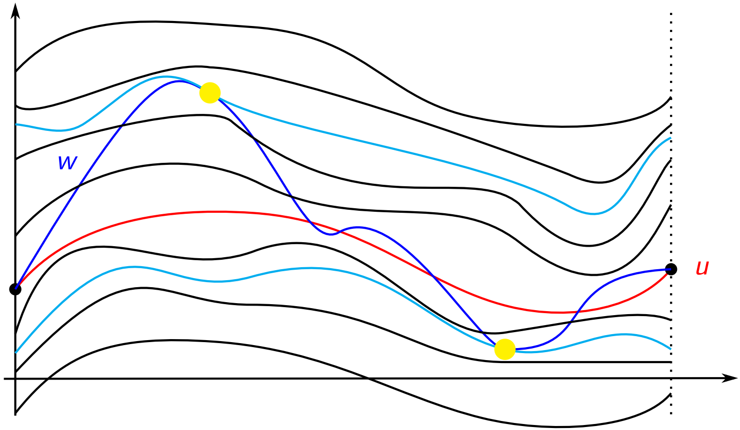


Tools:

- Strong Comparison Principle

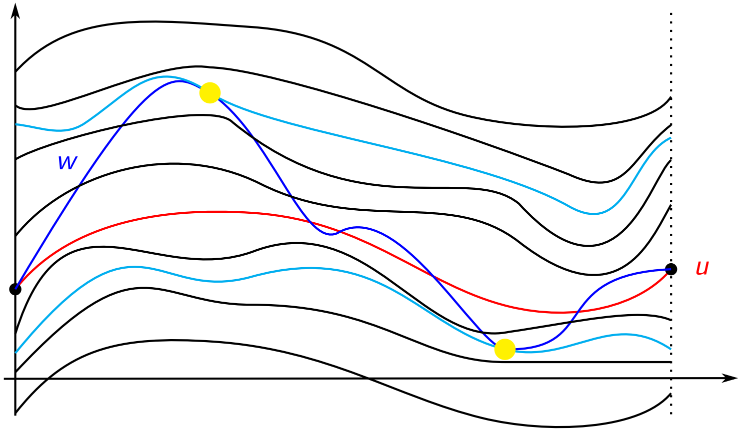
Proof via a touching argument



Tools:

- Strong Comparison Principle (follows by the convexity assumption) ✓

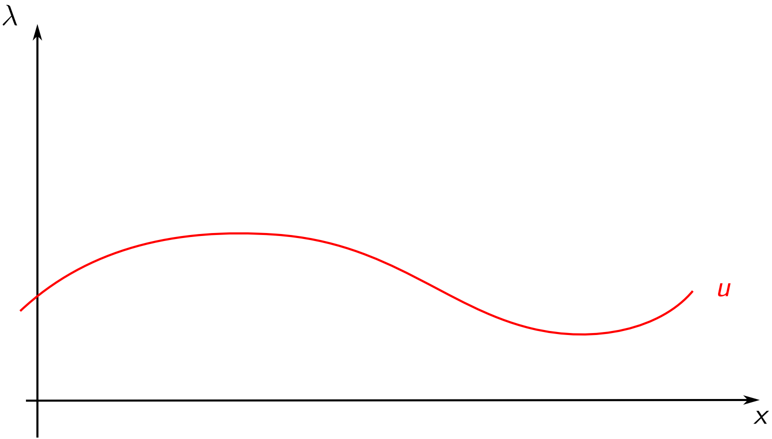
Proof via a touching argument



Tools:

- Strong Comparison Principle (follows by the convexity assumption) ✓
- Existence and regularity theory for minimizers ✗

The leaf-parameter function



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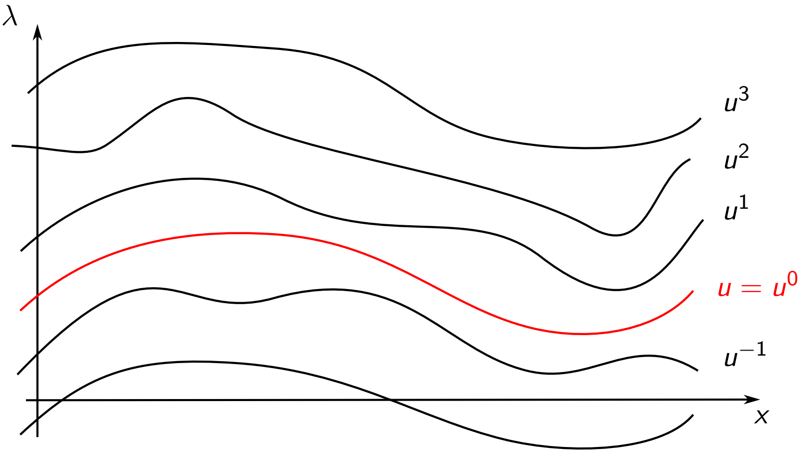

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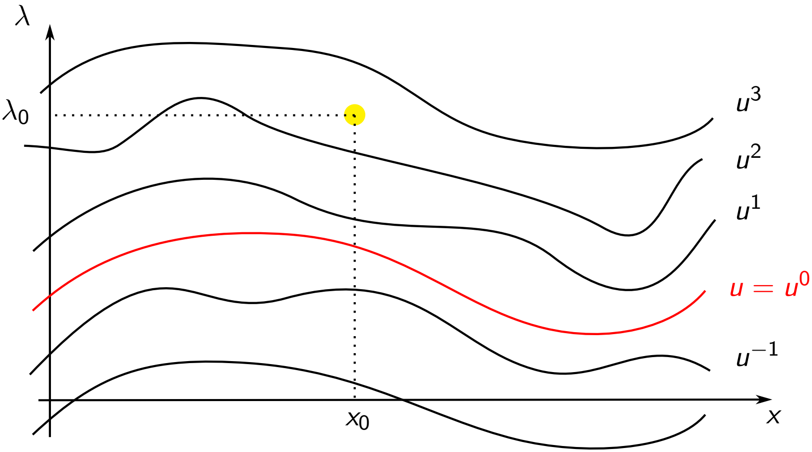

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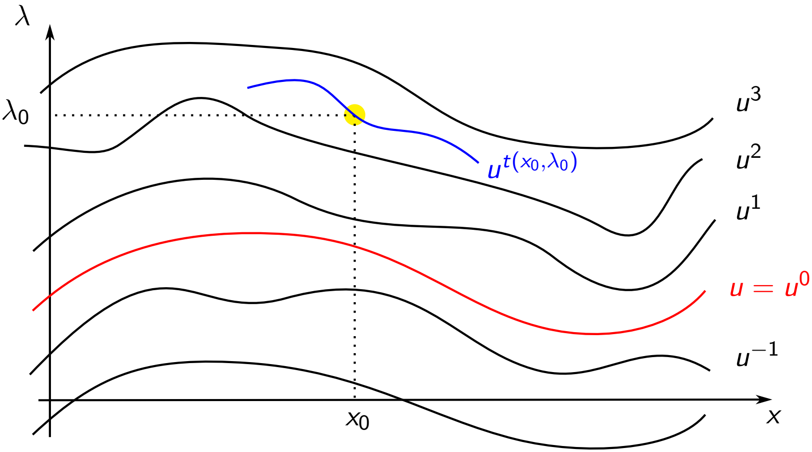

$t(x_0, \lambda_0)$ is the unique $\tau \in \mathbb{R}$ such that $u^\tau(x_0) = \lambda_0$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

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1. $\mathcal{C}_L(u) = \mathcal{E}_L(u)$?

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Proof via the construction of a calibration (1/3)

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Proof via the construction of a calibration (1/3)

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$$\mathcal{C}_L(u) = \int_\Omega G_L(x, u^0(x), \nabla u^0(x))\,\mathrm{d}x = \mathcal{E}_L(u)$$

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$$\mathcal{C}_L(u) = \int_\Omega G_L(x, u(x), \nabla u(x))\,\mathrm{d}x = \mathcal{E}_L(u)$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

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2. $\mathcal{C}_L(w) \le \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u?

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$\mathcal{E}_L(w) - \mathcal{C}_L(w)$

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2. $\mathcal{C}_L(w) \le \mathcal{E}_L(w)$ for all w with the same Dirichlet condition as u? ✓

$$\mathcal{E}_L(w) - \mathcal{C}_L(w) = \int_\Omega E(x, u^t(x), \nabla u^t(x), \nabla w(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_L(w) = \int_\Omega \partial_q G_L(x, u^t(x), \nabla u^t(x))\big(\nabla w(x) - \nabla u^t(x)\big)\big|_{t=t(x,w(x))}\,\mathrm{d}x + \int_\Omega G_L(x, u^t(x), \nabla u^t(x))\big|_{t=t(x,w(x))}\,\mathrm{d}x$$

3. $\mathcal{C}_L(u) = \mathcal{C}_L(w)$ for all w with the same Dirichlet condition as u?

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$$\mathcal{C}_L(w) = \int_\Omega \begin{pmatrix} X^1(x,t) \\ X^2(x,t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x,w(x))}\,\mathrm{d}x$$

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Proof via the construction of a calibration (3/3)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

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$$= \int_{\text{graph}(w)} X \cdot \nu_w \,\mathrm{d}\mathcal{H}^n(x,\lambda), \quad \text{with} \quad \operatorname{div} X = \mathcal{L}_L(u^t)(x)\big|_{t=t(x,\lambda)}$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

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Nonlocal theory of Calibrations

Fractional energy functional

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Fractional energy functional

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

First variation:

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_s(w + \varepsilon\eta) = \int_\Omega \left\{(-\Delta)^s w(x) - F'(w(x))\right\} \eta(x) \, \mathrm{d}x + \int_{\Omega^c} \mathcal{N}_s w(x) \, \eta(x) \, \mathrm{d}x,$$

Fractional energy functional

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x \,\mathrm{d}y - \int_\Omega F(w(x)) \,\mathrm{d}x$$

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$$(-\Delta)^s w(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x-y|^{n+2s}} \,\mathrm{d}y,$$

Fractional energy functional

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

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where

$$(-\Delta)^s w(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x-y|^{n+2s}} \, \mathrm{d}y,$$

and

$$\mathcal{N}_s w(x) = c_{n,s} \int_\Omega \frac{w(x) - w(y)}{|x-y|^{n+2s}} \, \mathrm{d}y.$$

Towards the fractional calibration: First attempts

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x$$

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- Replace the gradient terms by differences and integrals by double integrals

Towards the fractional calibration: First attempts

$$\mathcal{C}_1(w) = \int_\Omega \left\{\nabla u^t(x)\cdot\nabla w(x) - \frac{1}{2}|\nabla u^t(x)|^2\right\}\Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x))\,\mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals

$$\begin{aligned}\mathcal{F}_s^1(w) = \frac{c_{n,s}}{2} \iint_{(\Omega^c\times\Omega^c)^c} &\frac{(u^t(x)-u^t(y))(w(x)-w(y))}{|x-y|^{n+2s}}\Big|_{t=t(x,w(x))} \mathrm{d}x\,\mathrm{d}y \\ &- \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u^t(x)-u^t(y)|^2}{|x-y|^{n+2s}}\Big|_{t=t(x,w(x))} \mathrm{d}x\,\mathrm{d}y \\ &- \int_\Omega F(w(x))\,\mathrm{d}x\end{aligned}$$

Towards the fractional calibration: First attempts

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x))\,\mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals ✗

$$\mathcal{F}_s^1(w) = \frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} \frac{(u^t(x) - u^t(y))(w(x) - w(y))}{|x-y|^{n+2s}} \Big|_{t=t(x,w(x))} \mathrm{d}x\,\mathrm{d}y - \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u^t(x) - u^t(y)|^2}{|x-y|^{n+2s}} \Big|_{t=t(x,w(x))} \mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

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- Replace the gradient terms by differences and integrals by double integrals ✗

- Replace the gradients by fractional ones

Towards the fractional calibration: First attempts

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x))\, \mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals ✗

- Replace the gradients by fractional ones $\nabla \rightsquigarrow \nabla^s$

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- Replace the gradient terms by differences and integrals by double integrals ✗
- Replace the gradients by fractional ones $\nabla \rightsquigarrow \nabla^s$

$$\nabla^s w(x) = \tilde{c}_{n,s} \int_{\mathbb{R}^n} \frac{w(x)-w(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|}\, dy$$

Towards the fractional calibration: First attempts

$$\mathcal{C}_1(w) = \int_\Omega \left\{\nabla u^t(x)\cdot\nabla w(x) - \frac{1}{2}|\nabla u^t(x)|^2\right\}\Big|_{t=t(x,w(x))}\,\mathrm{d}x - \int_\Omega F(w(x))\,\mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals ✗

- Replace the gradients by fractional ones ✗

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$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals ✗

- Replace the gradients by fractional ones ✗

- Use the extension problem

Towards the fractional calibration: First attempts

$$\mathcal{C}_1(w) = \int_\Omega \left\{\nabla u^t(x)\cdot\nabla w(x) - \frac{1}{2}|\nabla u^t(x)|^2\right\}\Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x))\,\mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals ✗
- Replace the gradients by fractional ones ✗
- Use the extension problem ✗

Towards the fractional calibration: Key point

- Understand the calibration for the fractional perimeter by Cabré

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$$\text{Calibration} = \boxed{} + \boxed{}$$

Towards the fractional calibration: Key point

- Understand the calibration for the fractional perimeter by Cabré

Calibration	=	Euler-Lagrange on the leafs in Ω	+	

Towards the fractional calibration: Key point

- Understand the calibration for the fractional perimeter by Cabré

Calibration	$=$	Euler-Lagrange on the leafs in Ω	$+$	Exterior terms

Towards the fractional calibration: Key point

- Understand the calibration for the fractional perimeter by Cabré

$$\text{Calibration} = \boxed{\begin{array}{c}\text{Euler-Lagrange on}\\ \text{the leafs in } \Omega\end{array}} + \boxed{\text{Exterior terms}}$$

Can we find the same structure in the local theory?

Revisit local theory

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Can we find the same structure in the local theory?

$$\mathcal{E}_1(w) = \frac{1}{2}\int_\Omega |\nabla w(x)|^2 \, \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Revisit local theory

Can we find the same structure in the local theory? YES ✓

$$\mathcal{E}_1(w) = \frac{1}{2}\int_\Omega |\nabla w(x)|^2\,\mathrm{d}x - \int_\Omega F(w(x))\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{C}_1(w) = \int_\Omega \int_{u(x)}^{w(x)} \Big\{ -\Delta u^t(x) - F'(u^t(x)) \Big\}\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\partial\Omega}\int_{u(x)}^{w(x)} \partial_\nu u^t(x)\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1}(x) + \mathcal{E}_L(u).$$

Fractional calibration

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Fractional calibration

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Theorem [Cabré, Erneta & F-N '22]

Let u be embedded in an extremal field $\{u^t\}_{t\in\mathbb{R}}$ such that $(x,t) \to u^t(x)$ is a bounded C^2 function, and let $\mathcal{C}_s$ be the functional

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\} \Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x) \Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u).$$

Fractional calibration

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Theorem [Cabré, Erneta & F-N '22]

Let u be embedded in an extremal field $\{u^t\}_{t\in\mathbb{R}}$ such that $(x,t) \to u^t(x)$ is a bounded C^2 function, and let $\mathcal{C}_s$ be the functional

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u).$$

Then, it follows that $\mathcal{C}_s$ is a calibration for the functional $\mathcal{E}_s$ and u.

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{(-\Delta)^s u^t(x) - F'(u^t(x))\right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \mathrm{d}\lambda\, \mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$?

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\} \Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x) \Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$?

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$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
- $\mathcal{C}_s(u) = \mathcal{C}_s(w)$ for all w with the same Dirichlet condition as u?

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
- $\mathcal{C}_s(u) = \mathcal{C}_s(w)$ for all w with the same Dirichlet condition as u?

Fractional calibration: Proof I

$$
\begin{aligned}
\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{(-\Delta)^s u^t(x) - F'(u^t(x))\right\}\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x \\
+ \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_s(u)
\end{aligned}
$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
- $\mathcal{C}_s(u) = \mathcal{C}_s(w)$ for all w with the same Dirichlet condition as u? ✓

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

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$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\, \mathrm{d}x\, \mathrm{d}y$$
$$- \int_\Omega F(w(x))\, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \mathrm{d}x\, \mathrm{d}y.$$

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Fractional calibration: Proof I

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Fractional calibration: Proof II

- $\mathcal{C}_s(w) \le \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \left.\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\right|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y$$

$$- \int_\Omega F(w(x))\,\mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

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$\lambda = u^t(x)$ $\Big\Downarrow$ Symmetrization in x and y

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$\lambda = u^t(x)$ $\Downarrow$ Symmetrization in x and y

$$\mathcal{C}_s(w) = -\frac{c_{n,s}}{2} \iint_{(\Omega^c\times\Omega^c)^c} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\,\partial_t u^t(y)\,\mathrm{d}t\;\mathrm{d}x\,\mathrm{d}y$$
$$+ \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

Fractional calibration: Proof II

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y$$
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$$+\frac{c_{n,s}}{4}\iint_{(\Omega^c\times\Omega^c)^c}\frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

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$$+ \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Now,

$$-\frac{1}{2} \int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y)) \partial_t u^t(y) \, \mathrm{d}t$$

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Now,

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

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$$+\frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

Now,

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

$$= \frac{1}{4}\int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t}|w(x)-u^t(y)|^2\,\mathrm{d}t$$

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$$= \frac{1}{4}|w(x)-w(y)|^2 - \frac{1}{4}|w(x)-u^{t(x,w(x))}(y)|^2$$

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- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

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$$+\frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

Now,

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

$$= \frac{1}{4}\int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t}|w(x)-u^t(y)|^2\,\mathrm{d}t$$

$$= \frac{1}{4}|w(x)-w(y)|^2 - \frac{1}{4}|w(x)-u^{t(x,w(x))}(y)|^2$$

Fractional calibration: Proof III

- $\mathcal{C}_s(w) \le \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = -\frac{c_{n,s}}{2}\iint_{(\Omega^c\times\Omega^c)^c}\int_{t(x,w(x))}^{t(y,w(y))}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\,\partial_t u^t(y)\,\mathrm{d}t\,\,\mathrm{d}x\,\mathrm{d}y$$
$$+\frac{c_{n,s}}{4}\iint_{(\Omega^c\times\Omega^c)^c}\frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

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$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))}(u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \le -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))}(w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$
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$$=\frac{1}{4}|w(x)-w(y)|^2-\frac{1}{4}|w(x)-u^{t(x,w(x))}(y)|^2$$

Fractional calibration: Proof III

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u? ✓

$$
\begin{aligned}
\mathcal{C}_s(w) = &-\frac{c_{n,s}}{2}\iint_{(\Omega^c\times\Omega^c)^c}\int_{t(x,w(x))}^{t(y,w(y))}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\,\partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\\
&+\frac{c_{n,s}}{4}\iint_{(\Omega^c\times\Omega^c)^c}\frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega F(w(x))\,\mathrm{d}x
\end{aligned}
$$

Now,

$$
\begin{aligned}
-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))}(u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t &\leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))}(w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t\\
&= \frac{1}{4}\int_{t(x,w(x))}^{t(y,w(y))}\frac{\mathrm{d}}{\mathrm{d}t}|w(x)-u^t(y)|^2\,\mathrm{d}t\\
&= \frac{1}{4}|w(x)-w(y)|^2-\frac{1}{4}|w(x)-u^{t(x,w(x))}(y)|^2
\end{aligned}
$$

Then, we conclude

$$\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$$

Nonlocal calibration

$$\mathcal{E}_N(w) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} G_N(x, y, w(x), w(y)) \, \mathrm{d}x \, \mathrm{d}y$$

Nonlocal calibration

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Theorem [Cabré, Erneta & F-N '22]

Let u be embedded in an extremal field $\{u^t\}_{t\in\mathbb{R}}$ such that $(x,t) \to u^t(x)$ is a bounded C^2 function. Assume that G_N is a function of (x, y, a, b) satisfying $\partial^2_{ab} G_N \leq 0$ and let $\mathcal{C}_N$ be the functional

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The condition $\partial^2_{ab} G_N \leq 0$ turns to be the natural ellipticity condition for the problem.

The ellipticity condition $\partial^2_{ab} G_N \leq 0$

$$\partial^2_{ab} G_N \leq 0 \quad \Longrightarrow \quad \mathcal{L}_N \ \text{ satisfies the SCP}$$

We say that the operator $\mathcal{L}_N$ satisfies the Strong Comp. Princ. (SCP) if

$$\begin{cases} \mathcal{L}_N w \leq \mathcal{L}_N v & \text{in } \Omega \\ w \leq v & \text{in } \mathbb{R}^n \\ w(x_0) = v(x_0) & \text{for some } x_0 \in \Omega \end{cases} \qquad \Longrightarrow \quad w \equiv v \text{ in } \Omega$$

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Examples of nonlocal Lagrangians

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- The case
$$G_N(x,y,a,b) = -\mathbb{1}_{\Omega\times\Omega}(x,y)K(x-y)\,a\,b$$
corresponds to convolution-type energies.

Application to monotone solutions

$$\mathcal{L}_N(u)(x) = \int_{\mathbb{R}^n} \partial_a G_N(x, y, u(x), u(y))\,\mathrm{d}y$$

Corollary

Let u be a sufficiently regular solution of $\mathcal{L}_N(u) = 0$ in $\mathbb{R}^n$ satisfying the monotonicity condition $\partial_{x_n} u > 0$ in $\mathbb{R}^n$.
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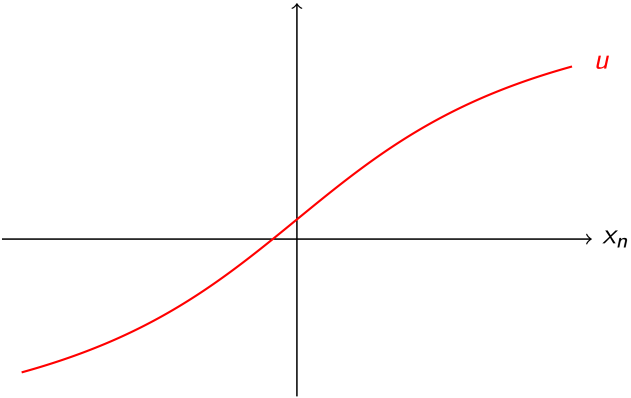
Then, for each bounded domain $\Omega \subset \mathbb{R}^n$, u is a minimizer of $\mathcal{E}_N$ in the set of functions w satisfying

$$\lim_{\tau \to -\infty} u(x', \tau) < w(x', x_n) < \lim_{\tau \to +\infty} u(x', \tau)$$

and such that $w \equiv u$ in Ω^c.

Application to monotone solutions: The field



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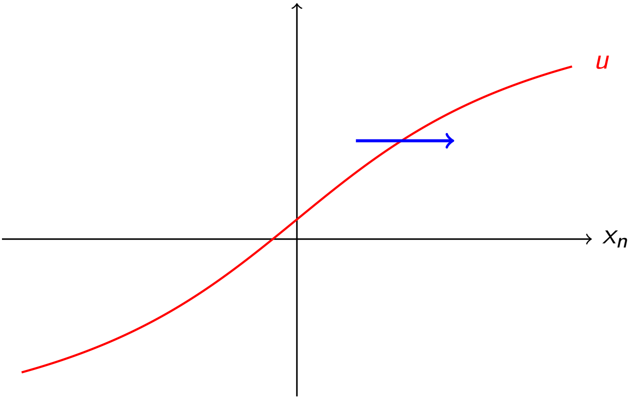

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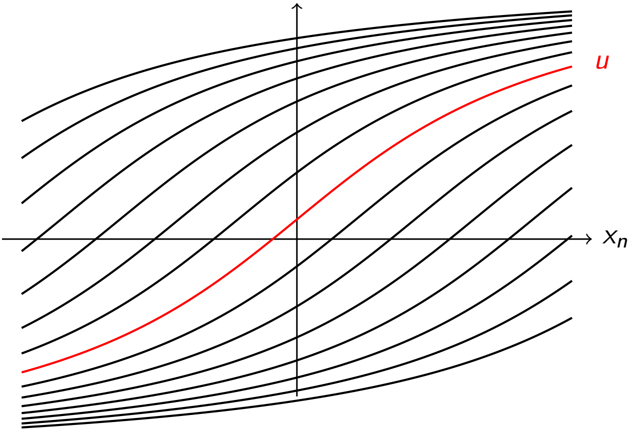

$$u^t(x) := u(x + t\, e_n)$$

Application to the viscosity theory

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Let u be a minimizer of $\mathcal{E}_N$ among functions with same Dirichlet conditions. Assume that the ellipticity condition $\partial_{ab} G_N \leq 0$ holds. Then, it is a viscosity solution of the associated equation $\mathcal{L}_N(u) = 0$ in Ω.

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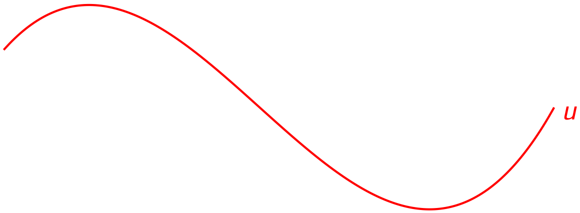
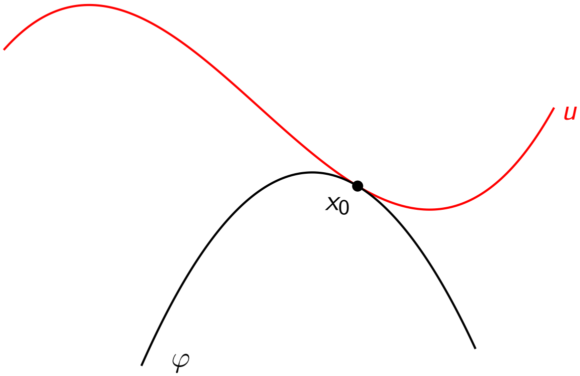
- We do not need a weak comparison principle
- We only treat minimizers, not general weak solutions

Application to the viscosity theory: Proof



Application to the viscosity theory: Proof

$$\mathcal{L}_N\varphi(x_0) \geq 0?$$



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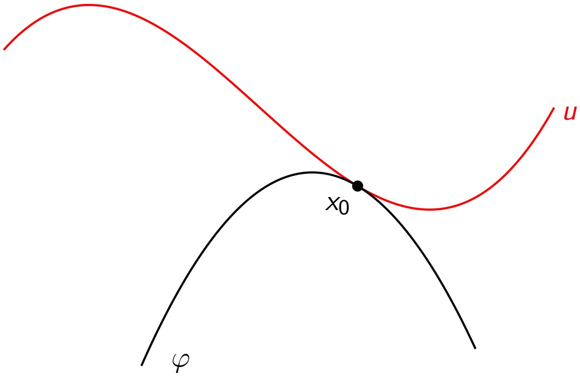

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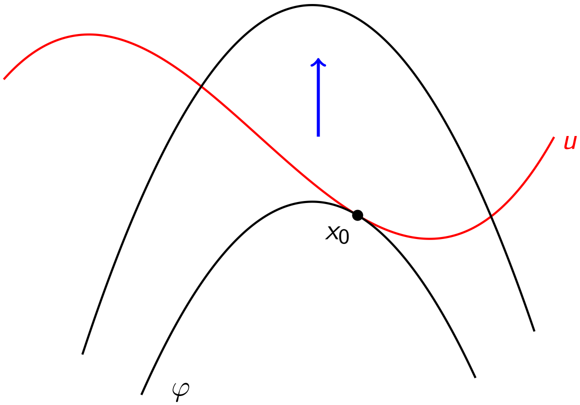

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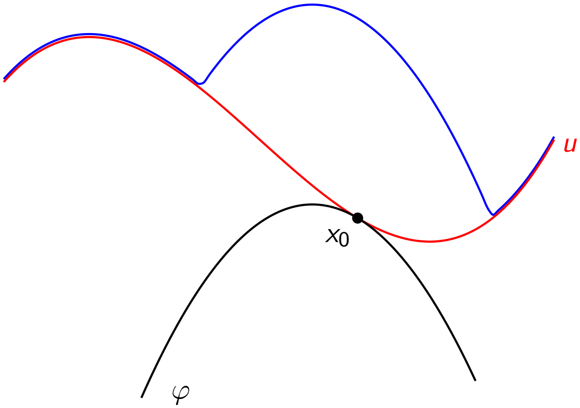

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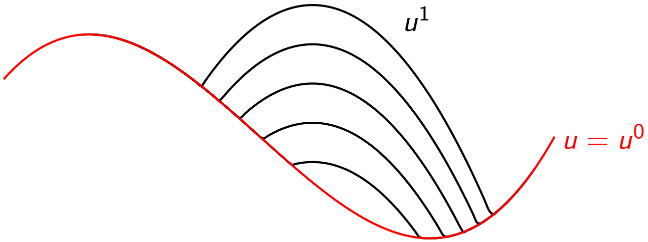

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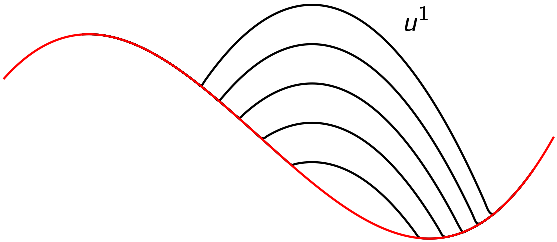

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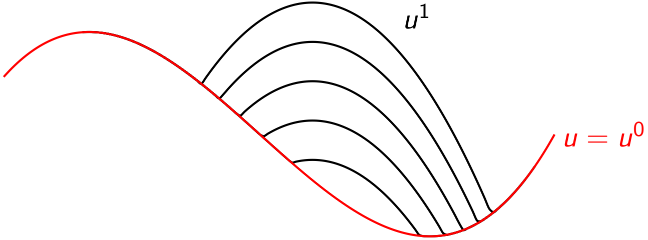

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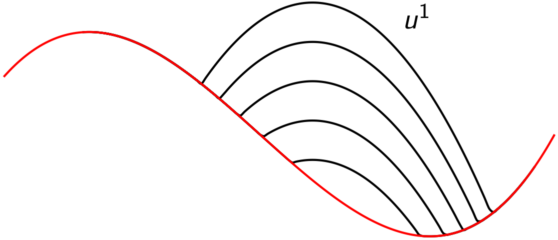

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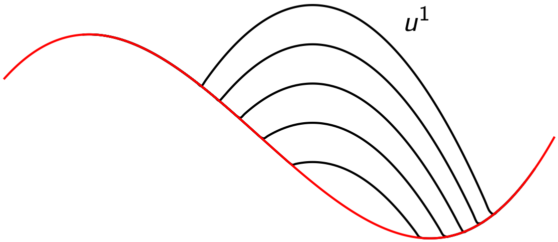

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Thank You

Towards the fractional calibration

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \,\mathrm{d}x - \int_\Omega F(w(x))\,\mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals

$$\begin{aligned}
\mathcal{F}_s^2(w) = {} & \frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} \frac{(u^\tau(x) - u^t(y))(w(x) - w(y))}{|x-y|^{n+2s}} \Bigg|_{\substack{t=t(x,w(x)) \\ \tau=t(y,w(y))}} \,\mathrm{d}x\,\mathrm{d}y \\
& - \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u^\tau(x) - u^t(y)|^2}{|x-y|^{n+2s}} \Bigg|_{\substack{t=t(x,w(x)) \\ \tau=t(y,w(y))}} \,\mathrm{d}x\,\mathrm{d}y \\
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Nonlocal perimeter

$$\mathcal{P}(F) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} |\mathbb{1}_F(x) - \mathbb{1}_F(y)| K(x-y)\, \mathrm{d}x\, \mathrm{d}y$$

$$\Downarrow \quad \text{[Cabré '19]}$$

$$\mathcal{C}_{\mathcal{P}}(F) = \int_\Omega \mathbb{1}_F(x)\, H_K[E^t](x)\Big|_{t=\phi(x)} \mathrm{d}x + \int_{\Omega^c} \mathbb{1}_F(x) \left\{ \int_\Omega \big(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y)\big) K(x-y)\, \mathrm{d}y \right\} \Bigg|_{t=\phi(x)} \mathrm{d}x$$

Here the Euler-Lagrange operator is given by

$$H_K[E](x) = \int_{\mathbb{R}^n} \big(\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)\big) K(x-y)\, \mathrm{d}y$$

The fractional Laplacian

The canonical example of integro-differential operator

$$(-\Delta)^s w(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x-y|^{n+2s}} \, \mathrm{d}y, \quad s \in (0,1).$$

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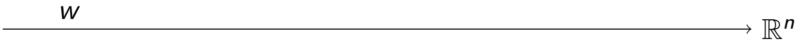
- Representation via the heat semigroup:

$$(-\Delta)^s w(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta} w - w\right) \frac{dt}{t^{1+s}}$$

The extension problem for the fractional Laplacian



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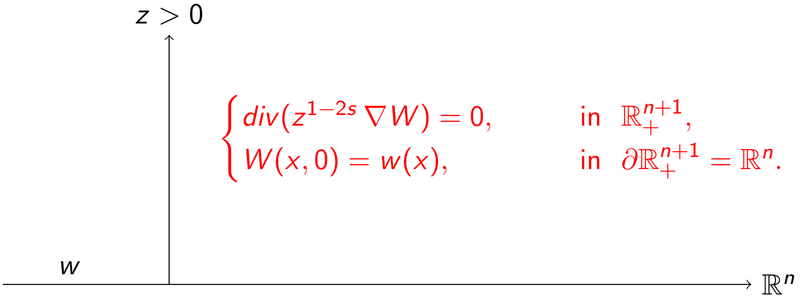

$$\begin{cases} div(z^{1-2s}\nabla W) = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ W(x,0) = w(x), & \text{in } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

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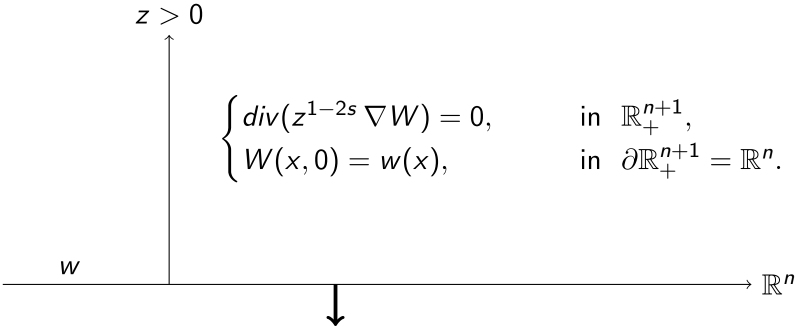

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$$-d_s \lim_{z\downarrow 0} z^{1-2s}\partial_z W = (-\Delta)^s w$$