

# THE NEUMANN PROBLEM FOR THE FRACTIONAL LAPLACIAN: BOUNDARY REGULARITY

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joint work with A. Audrito (ETH Zurich) and X. Ros-Oton (ICREA-UB)



Universitat Politècnica de Catalunya - Barcelona Graduate School of Mathematics

**Online Workshop on Nonlocal PDEs and Applications**

Red de EDPs no locales y aplicaciones

**10 - 11th December 2020**

# The local framework

We consider the energy functional

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# Regularity for the local case

- Interior regularity:  $-\Delta u = f$  in  $B_1$ , with  $f \in L^\infty(B_1)$

It is well known by **Schauder estimates** that  $u \in C^{1,\alpha}(B_{1/2})$  for every  $\alpha \in (0, 1)$ .

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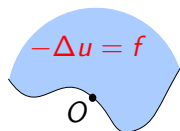
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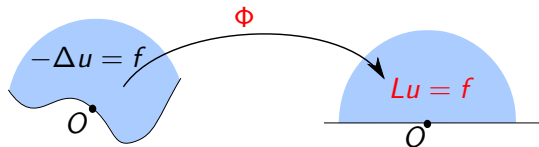
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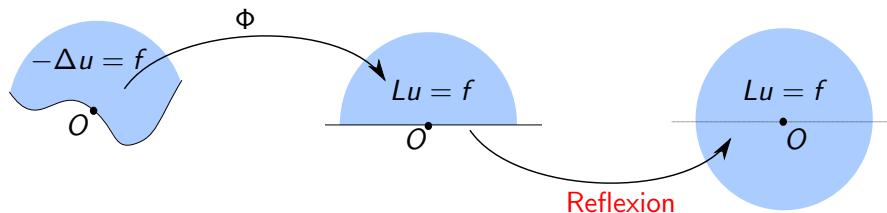
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# The fractional Laplacian

Defined for  $s \in (0, 1)$  by

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}} dz.$$

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- It has an associated local extension problem.

When working with the fractional Laplacian in bounded domains, the natural functional space to work is

$$\mathcal{H}^s(\Omega) = \{u \in L^2(\Omega) \text{ such that } [u]_{\mathcal{H}^s(\Omega)} < +\infty\},$$

where

$$[u]_{\mathcal{H}^s(\Omega)} := \left( \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2s}} dx dz \right)^{1/2},$$

and  $\mathcal{H}^s(\Omega)$  is a Banach space with associated norm

$$\|u\|_{\mathcal{H}^s(\Omega)} = \|u\|_{L^2(\Omega)} + [u]_{\mathcal{H}^s(\Omega)}$$

# The Dirichlet problem for the fractional Laplacian

We consider the **nonlocal** energy functional

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This problem has a nice **probabilistic interpretation** in terms of **Levy flights**.

# Regularity of the fractional Dirichlet problem

The **regularity** of solutions to the Dirichlet problem is nowadays **very well understood**

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**Theorem (Ros-Oton, Serra, 2014)**

*Assume  $\partial\Omega \in C^2$  and  $f \in L^\infty(\Omega)$ . Then,  $u \in C^s(\bar{\Omega})$ .*

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Assume  $\partial\Omega \in C^2$  and  $f \in L^\infty(\Omega)$ . Then,  $u \in C^s(\bar{\Omega})$ .

This **regularity** result is **optimal** since explicit solutions can be constructed:

$$\left. \begin{cases} (-\Delta)^s u = 1 & \text{in } B_1, \\ u = 0 & \text{in } B_1^c. \end{cases} \right\} \implies u(x) = c(1 - |x|^2)_+^s$$

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$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \Omega^c, \end{cases}$$

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**Existence** and **uniqueness** of weak solutions can be found in [Dipierro, Ros-Oton and Valdinoci, 2017] ([Du-Gunzburger-Lehoucq-Zhou, '13])

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- Integration by parts: If  $u, v$  are bounded  $C^2$  functions in  $\mathbb{R}^n$ , then

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- Natural probabilistic interpretation
- Conservation of mass in the associated heat equation
- **Convergence** to the classical Neumann problem as  $s \uparrow 1$

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## Theorem (Audrito, F-N, Ros-Oton, '20)

Let  $\Omega \subset \mathbb{R}^N$  be any bounded  $C^1$  domain and  $f \in L^\infty(\Omega)$  and  $\int_\Omega f = 0$ . Then, *weak solutions are bounded*, and

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C (\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}),$$

for some  $\alpha > 0$ .

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This is the **first regularity result** for the Neumann problem. Even the boundedness of solutions is new.

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- Classical methods do not work (flatten the boundary and even reflection)
- The values of  $u$  are not known neither inside nor outside  $\Omega$
- The methods from the Dirichlet case, based on **barriers**, do not work either

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On the other hand, it is also clear that

$$\iint_{(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R}_- \times \mathbb{R}_-)} \frac{(w(x) - w(z))(\varphi(x) - \varphi(z))}{|x - z|^{n+2s}} dx dz \neq 0$$

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This phenomenon is due to a **missing regularity condition** in the strong formulation that  $w$  is **does not satisfy**.

# Equivalent formulation

It is equivalent to minimize the nonlocal energy

$$\mathcal{E}(u) := \frac{c_{n,s}}{4} \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2s}} dx dz - \int_{\Omega} f u$$

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where  $K_{\Omega}$  is an explicit kernel satisfying

$$K_{\Omega}(x, z) \asymp \frac{1 + \log^{-} \left( \frac{\min\{d(x), d(z)\}}{|x-z|} \right)}{|x-z|^{n+2s}} \quad \text{for all } x, z \in \Omega,$$

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$$\bar{\mathcal{E}}(u) := \frac{c_{n,s}}{4} \iint_{\Omega \times \Omega} |u(x) - u(z)|^2 K_{\Omega}(x, z) dx dz - \int_{\Omega} f u$$

where  $K_{\Omega}$  is an explicit kernel satisfying

$$K_{\Omega}(x, z) \asymp \frac{1 + \log^{-} \left( \frac{\min\{d(x), d(z)\}}{|x-z|} \right)}{|x - z|^{n+2s}} \quad \text{for all } x, z \in \Omega,$$

It was first noted in the **strong form** by [Abatangelo, '20].

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$$K_{\Omega}(x, z) = \frac{1}{|x - z|^{n+2s}} + \int_{\Omega^c} \frac{dy}{|x - y|^{n+2s} |y - z|^{n+2s}} \int_{\Omega} \frac{dw}{|y - w|^{n+2s}}.$$

# The regional fractional Laplacian

Motivated by the equivalent formulation, we first study the minimization of the functional

$$\tilde{\mathcal{E}}(u) := \frac{c_{n,s}}{4} \iint_{\Omega \times \Omega} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2s}} dx dz - \int_{\Omega} f u$$

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In a more recent work, M. Fall proves  $C^{2s-\varepsilon}$  regularity up to the boundary by using the extension problem.

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$$\Downarrow \varphi \approx u^p + \text{Sob. Ineq.}$$

$$u \in L^\infty(\Omega)$$

$$\Downarrow \varphi \approx u^{-p} + \text{J-N Ineq.}$$

$$u \in C^\alpha(\bar{\Omega}) \text{ for some small } \alpha$$

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$$\left\{ \begin{array}{ll} (-\Delta)^s v_j = f_j & \text{in } \Omega_j, \\ \mathcal{N}_s v_j = 0 & \text{in } \Omega_j^c, \\ v_j(0) = 0, \\ \|v_j\|_{L^\infty(B_1)} \geq 1, \\ \|v_j\|_{L^\infty(B_R)} \leq R^\sigma. \end{array} \right.$$

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Thus, we arrive at a **contradiction** if we show that a **weak solution** with certain growth in a half-space **is constant**.

# From dimension $n$ to dimension 1

$$\left. \begin{aligned} (-\Delta)^s v &= 0 && \text{in } \mathbb{R}_+^n, \\ \mathcal{N}_s v &= 0 && \text{in } \mathbb{R}_-^n, \\ \|v\|_{L^\infty(B_R)} &\leq R^\sigma, && \text{with } \sigma < 1. \end{aligned} \right\} \implies v \equiv w(x_n)$$

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- We construct the auxiliary function

$$v_1(x) = \frac{v(x + he) - v(x)}{|h|^\alpha} \quad \text{for any given } e = (\tilde{e}, 0) \in S^{n-1} \text{ and } h > 0.$$

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- Since it **grows less than linear**, it should be **constant** in that directions

# 1D Liouville theorem

We want to prove

$$\left. \begin{aligned} (-\Delta)^s w &= 0 && \text{in } (0, \infty), \\ \mathcal{N}_s w &= 0 && \text{in } (-\infty, 0), \\ |w(x)| &\leq 1 + |x|^\sigma && \text{in } \mathbb{R}. \end{aligned} \right\} \implies w \equiv \text{ctt}$$

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- We conclude that  $w \equiv 0$ .

Thank You