SADDLE-SHAPED SOLUTIONS TO THE FRACTIONAL ALLEN-CAHN EQUATION

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joint work with Tomás Sanz-Perela (BCAM)

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V CONGRESO DE JÓVENES INVESTIGADORES DE LA RSME

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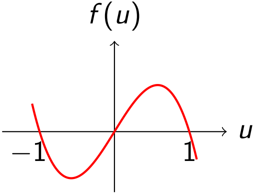
Sesión de Ecuaciones en derivadas parciales no lineales y no locales

The Allen-Cahn equation

We consider

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n$$

where f is of bistable type.



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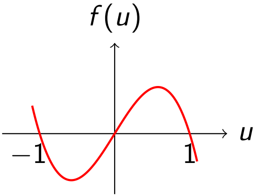

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$$E(u) = \int \left\{ \frac{1}{2}|\nabla u|^2 + G(u) \right\} \mathrm{d}x,$$

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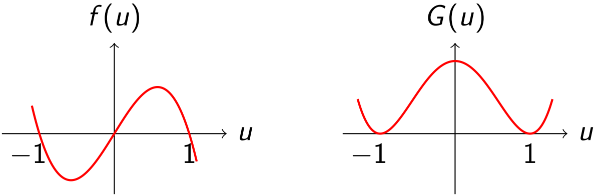

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- Solutions are critical points of the energy functional

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} E(u+\varepsilon\xi) = 0 \quad \forall\, \xi \in C_c^\infty(\mathbb{R}^n)$$

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- Minimizers are global minima of the energy functional

$$E(u + \xi) \geq E(u) \quad \forall\, \xi \in C_c^\infty(\mathbb{R}^n)$$

Connection with minimal surfaces

Theorem [Modica, Mortola; '77]

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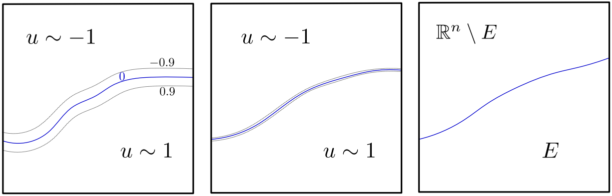

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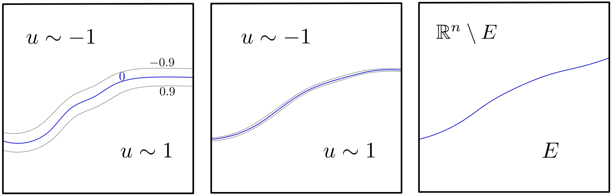

Consider the blow-down sequence $u_\varepsilon(x) = u(x/\varepsilon)$, where u is a minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$.

Then, $u_\varepsilon \to \chi_E - \chi_{\mathbb{R}^n \setminus E}$ in L^1_{loc} as $\varepsilon \to 0$, where E is a minimizer of the perimeter functional.

Minimizing minimal surfaces and the Simons cone

Theorem [Simons, 1968]

Let $E \in \mathbb{R}^n$ be an open set such that ∂E is a minimizing cone. Then, if $n \leq 7$, ∂E is a hyperplane.

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Simons Cone

$$\mathscr{C} = \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \quad \text{st} \quad |x'| = |x''|\}$$

Theorem [Bombieri, De Giorgi, and Giusti, 1969]

For any given even dimension $n = 2m$, the Simons cone has zero mean curvature at every point outside the origin. Moreover, if $2m \geq 8$, the Simons cone is a minimizer of the perimeter functional.

Minimizers of the Allen-Cahn equation

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Let u be a minimizer of the Allen-Cahn equation. Then, if $n \leq 7$, u is a one-dimensional solution. That is, the level sets are hyperplanes.

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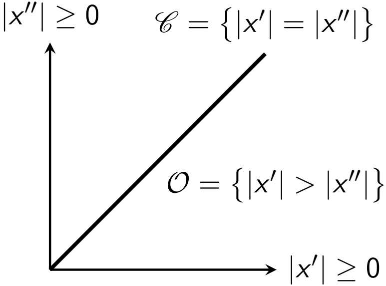
Is there an analogous object to the Simons cone for the semilinear problem?

Saddle-shaped solutions

Definition

Let u be a bonded solution of the Allen-Cahn equation in $\mathbb{R}^{2m}$. We say the it is a saddle-shaped solution if

- It is doubly radial: $u = u(|x'|, |x''|)$.
- It is odd with respect to Simons Cone: $u(|x'|, |x''|) = -u(|x''|, |x'|)$.
- It is positive in $\mathcal{O} = \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \quad \text{st} \quad |x'| > |x''|\}$.



Known results

- Existence [Dang, Fife, and Peletier, 1992] [Cabré, and Terra, 2009]
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Minimality when $2m \geq 8$ is an open problem

The fractional problem

Study saddle-shaped solution to fractional Allen-Cahn equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^{2m},$$

where

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, \mathrm{d}y.$$

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The energy functional associated to this equation is

$$\mathcal{E}(u) = \frac{c_{n,s}}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^n} G(u) \, \mathrm{d}x,$$

with $G' = -f$.

Connection to minimal surfaces

Theorem [González, 2009] [Savin, and Valdinoci, 2015]

The energy functional of a rescaled version of the fractional Allen-Cahn equation Γ-converges to the perimeter functional when $s \in [1/2, 1)$ and to the fractional perimeter functional when $s \in (0, 1/2)$.

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The fractional perimeter

$$Per_{2s}(E) := c_{n,s} \int_E \int_{\mathbb{R}^n \setminus E} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n+2s}}, \quad \text{for} \quad s \in (0, 1/2)$$

was introduced in [Caffarelli, Roquejoffre, and Savin, 2010].

Nonlocal minimal surfaces

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Proposition

For any given even dimension $n = 2m$ and $s \in (0, 1/2)$, the Simons cone has zero $2s$-mean curvature at every point.

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For any given even dimension $n = 2m$ and $s \in (0, 1/2)$, the Simons cone has zero $2s$-mean curvature at every point.

Simons cone is the candidate to nonplanar minimizing nonlocal minimal cone in high dimensions.

Why studying saddle-shaped solutions

- They are expected to be, in high dimensions, the simplest example of minimizer to $(-\Delta)^s u = f(u)$ which is not one-dimensional.

- Saddle-shaped solutions can be used to prove the stability/minimality of the Simons cone as a nonlocal minimal surface.

Results

In [Cinti, 2013, 2017]

- Existence
- Monotonicity properties
- Asymptotic behavior
- Instability if $2m \leq 6$ ($2m = 2$ in [Cabré, and Solà-Morales, 2005])

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J.C. Felipe-Navarro, T. Sanz-Perela, *Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation*, to appear in Revista Matemática Iberoamericana.

- Uniqueness
- Stability in dimensions $2m \geq 14$ for all $s \in (0, 1)$

New result

An important consequence of our stability result:

Corollary 8: [F-N, Sanz-Perela]

The Simons cone is a stable $2s$-minimal surface in dimensions $2m \geq 14$ for all $s \in (0, 1/2)$.

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Corollary 8: [F-N, Sanz-Perela]

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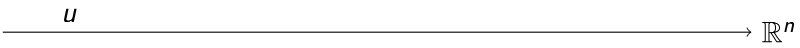
First analytical proof of a stability result for the Simons cone in any dimension (in the nonlocal setting).

The extension problem for the fractional Laplacian

Main tool, by [Caffarelli, and Silvestre, 2007]



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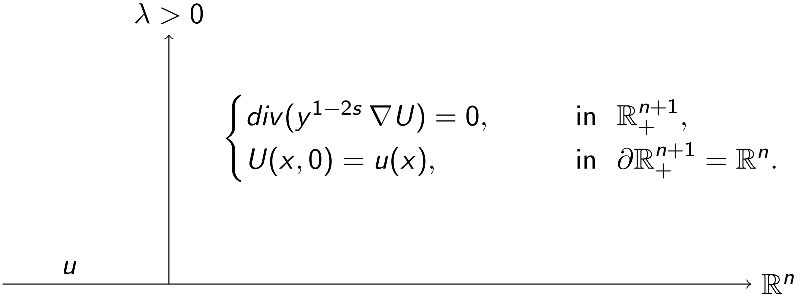

$$\begin{cases} div(y^{1-2s}\,\nabla U) = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) = u(x), & \text{in } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

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$\lambda > 0$

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u $\qquad \mathbb{R}^n$

$$-d_s \lim_{y\downarrow 0} \lambda^{1-2s}\partial_\lambda U = (-\Delta)^s u$$

Thank You