INTRODUCTION
TO
DIFFERENTIAL GEOMETRY

EXERCISES WITH SOLUCIONS

COURSE 21/22

MASTER IN MATHEMATICS AND STATISTICS

FACULTY OF SCIENCE - UNIVERSITY OF HELSINKI

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Introduction to differential geometry
Exercise 1, Solutions
24.1.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, January 24, 12:00 o'clock.

$*$**1.** Two metrics on a set are called *equivalent* if they induce the same metric topology.
Let (X, d) be a metric space. Prove that there exists a metric $\bar{d}$ on X, equivalent to d, such that $(X, \bar{d})$ is bounded.

[Hint: Consider the function $\bar{d}\colon X \times X \to \mathbb{R}$,

$$\bar{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Solution. Before anything let us show that $\bar{d}$ is a distance. In order to do it let us define the auxiliary increasing function $f(t) = t/(1+t)$, which gives a bijection between $[0, +\infty)$ and $[0, 1)$. Then, $\bar{d} = f(d)$. From being d a distance and the properties of f, it is trivial that $\bar{d}$ is positive definite and symmetric. Then, let us conclude that it is a distance by checking the triangle inequality. That is, by triangle inequality for d and the monotonicity of f we obtain

$$\begin{aligned}\bar{d}(x, y) = f(d(x, y)) \leq f(d(x, z) + d(z, y)) &= \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} \\ &= \frac{d(x, z)}{1 + d(x, z) + d(z, y)} + \frac{d(z, y)}{1 + d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} = \bar{d}(x, z) + \bar{d}(z, y)\end{aligned}$$

for any $x, y, z \in X$. Thus, $(X, \bar{d})$ is a metric space. In order to show that it is bounded it is enough to note that $\bar{d}(x, y) < 1$ for each $x, y \in X$.

Finally, let us show that (X, d) and $(X, \bar{d})$ induce the same topology. Since balls are a base in metric spaces, it is enough to check that balls with one metric are also balls in the other one. That is, if follows from the properties of f that

$$B_d(x, r) = B_{\bar{d}}\left(x, \frac{r}{1 + r}\right) \quad \text{and} \quad B_{\bar{d}}(x, \rho) = B_d\left(x, \frac{\rho}{1 - \rho}\right)$$

for any $x \in X$, $r \in [0, +\infty)$ and $\rho \in [0, 1)$.

$*$**2.** Prove that a topological space X is Hausdorff if and only if, for every $x \in X$, the intersection of all closed neighborhoods of x is the singleton $\{x\}$. [Note: a closed neighborhood of a point is a closed set containing some *open* neighborhood of the point.]

Solution. Given $x \in X$ let us define the set

$$N_x := \bigcap \{W : W \subset X \text{ is a closed neighborhood of } x\}.$$

Suppose that X is Hausdorff. Let $x, y \in X$ be distinct. Then, there exist U and V disjoint neighborhoods of x and y, respectively. In particular $X \setminus V$ is a closed neighborhood of x not containing y. Hence $y \notin N_x$. Since we can do it for any $y \neq x$ and $x \in N_x$ by definition, we conclude that $N_x = \{x\}$.

Let $x, y \in X$ be distinct. Since $N_x = \{x\}$ there exists W, a closed neighborhood of x not containing y. By definition of closed neighborhood, it means the existence of $U \subset W$, an open neighborhood of x. Therefore U and $X \setminus W$ are disjoint neighborhoods of x and y, respectively, and X is Hausdorff.

3. Let $(X_1, \mathcal{T}_1), \ldots, (X_k, \mathcal{T}_k)$ be topological spaces and let $X = X_1 \times \cdots \times X_k$ be equipped with the product topology. Prove:
   (a) If every $(X_i, \mathcal{T}_i)$ is Hausdorff, then also X is Hausdorff.
   (b) If every $(X_i, \mathcal{T}_i)$ is N_2, then also X is N_2.

Solution.

(a) Let $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in X$ be distinct. Then $x_j \neq y_j$ for some index j. Let U_j and V_j be disjoint neighbourhoods of x_j and y_j, respectively. Then the sets

$$U := X_1 \times \cdots \times U_j \times \cdots \times X_k$$

and

$$V := X_1 \times \cdots \times V_j \times \cdots \times X_k$$

are disjoint neighbourhoods of x and y, respectively.

(b) Let $\mathcal{U}_j$ be a countable base for X_j, $j = 1, \ldots, k$. Define

$$\mathcal{U} = \{U_1 \times \cdots \times U_k : U_j \in \mathcal{U}_j,\ j = 1, \ldots, k\}.$$

This set is countable. Moreover, we claim that it is also a base for the product topology of X. That is, let us take an open set $U \subset X$. For every $x = (x_1, \ldots, x_k) \in U$ we may find an open set $x \in V_1^x \times \cdots \times V_k^x \subset U$ by the definition of the product topology. For each index j we may also find a base element $U_j^x \in \mathcal{U}_j$ so that $x_j \in U_j^x \subset V_j^x$. Consequently $x \in U_1^x \times \cdots \times U_k^x \subset U$ and we may write U as a union of elements in $\mathcal{U}$:

$$U = \bigcup_{x \in U} U_1^x \times \cdots \times U_k^x.$$

4. Let $\mathcal{T}_{cc} = \{U \subset \mathbb{R} \colon \mathbb{R} \setminus U \text{ is countable}\} \cup \{\emptyset\}$.
   (a) Prove that $\mathcal{T}_{cc}$ is a topology in $\mathbb{R}$ (co-countable topology).

(b) Prove that $(\mathbb{R}, \mathcal{T}_{cc})$ is not Hausdorff.
(c) Prove that limits of convergent sequences are unique in $(\mathbb{R}, \mathcal{T}_{cc})$.

Solution.

(a) Clearly $\emptyset, \mathbb{R} \in \mathcal{T}_{cc}$ (since $\mathbb{R} \setminus \mathbb{R} = \emptyset$). Let $\{U_j\}_{j \in J} \subset \mathcal{T}_{cc}$. Since we want to prove that $\bigcup_j U_j \in \mathcal{T}_{cc}$, we may assume without loss of generality that $U_j \neq \emptyset$ for all $j \in J$. Then $\mathbb{R} \setminus U_j$ is countable for all $j \in J$. Since the intersection of an arbitrary family of countable sets is also countable, we have that

$$\mathbb{R} \setminus \left(\bigcup_{j \in J} U_j \right) = \bigcap_{j \in J} \mathbb{R} \setminus U_j$$

is a countable set. Hence $\bigcup\limits_{j \in J} U_j \in \mathcal{T}_{cc}$.

Similarly if $U, V \in \mathcal{T}_{cc} \setminus \{\emptyset\}$, then $\mathbb{R} \setminus (U \cap V) = \mathbb{R} \setminus U \cup \mathbb{R} \setminus V$ is countable, implying $U \cap V \in \mathcal{T}_{cc}$. These arguments show that $\mathcal{T}_{cc}$ is a topology.

(b) Any two non-empty sets $U, V \in \mathcal{T}_{cc}$ have uncountable intersection, since $\mathbb{R} \setminus (U \cap V)$ is countable. In particular no such intersection can be empty and hence no pair of distinct points can have disjoint neighborhoods.

(c) Suppose $(x_k) \subset \mathbb{R}$ is a sequence converging in $\mathcal{T}_{cc}$ to x and to y. Suppose $x \neq y$. Then $\mathbb{R} \setminus \{x\}$ is a neighbourhood of y and there exists n_0 so that $x_k \in \mathbb{R} \setminus \{x\}$ for all $k \geq k_0$. In other words $x_k \neq x$ for all $k \geq k_0$. Thus the set $\mathbb{R} \setminus \{x_{k_0}, x_{k_0+1}, \ldots\}$ is a neighbourhood of x so there should exist k_1 so that

$$x_k \in \mathbb{R} \setminus \{x_{k_0}, x_{k_0+1}, \ldots\}$$

whenever $k \geq k_1$. However this is clearly impossible. Thus we conclude that x must equal y.

5. Let M be the set of all continuous functions $f \colon [0,1] \to \mathbb{R}$. Define

$$d(f,g) = \sup\{|f(x) - g(x)| \colon 0 \leq x \leq 1\}$$

for $f, g \in M$. Prove that:
(a) d is a metric in M.
(b) $d(f_n, f) \to 0 \quad \Longleftrightarrow \quad f_n \to f$ uniformly.
(c) (M, d) is complete.

Solution.

(a) Clearly $d(f,f) = 0$ and conversely, if $d(f,g) = 0$ then

$$|f(t) - g(t)| \leq d(f,g) = 0$$

for all $t \in [0,1]$ so that $f = g$.

Also

$$d(f,g) = \sup_{0\le t\le 1} |f(t)-g(t)| = \sup_{0\le t\le 1} |g(t)-f(t)| = d(g,f)$$

so that d is symmetric.

Finally, the triangle inequality holds:

$$\begin{aligned} d(f,g) &= \sup_{0\le t\le 1} |f(t)-g(t)| \le \sup_{0\le t\le 1} (|f(t)-h(t)| + |h(t)-g(t)|) \\ &\le \sup_{0\le t\le 1} (|f(t)-g(t)| + d(h,g)) = d(f,h) + d(h,g) \end{aligned}$$

(b) Suppose $f_n \to f$ uniformly. Given $\varepsilon > 0$ there is n_0 so that $|f_n(t) - f(t)| < \varepsilon/2$ for all $t \in [0,1]$ whenever $n \ge n_0$. Thus

$$d(f_n, f) = \sup_{t\in[0,1]} |f_n(t) - f(t)| \le \varepsilon/2 < \varepsilon$$

whenever $n \ge n_0$. Thus $d(f_n, f) \to 0$ as $n \to \infty$.

Next assume that $d(f_n, f) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ and take n_0 so that $d(f_n, f) < \varepsilon$ whenever $n \ge n_0$. Then

$$|f_n(t) - f(t)| \le d(f_n, f) < \varepsilon$$

for all $t \in [0,1]$ whenever $n \ge n_0$. In other words $f_n \to f$ uniformly.

(c) Let (f_n) be a Cauchy sequence in M. Then $(f_n(t))$ is a Cauchy sequence in $\mathbb{R}$ having limit $f(t)$ for each t. We must prove that f is continuous (i.e. belongs to M) and that $d(f_n, f) \to 0$ as $n \to \infty$. To accomplish these it suffices to show that $f_n \to f$ uniformly.

For this let $\varepsilon > 0$ and take n_0 so that $d(f_n, f_m) < \varepsilon$ whenever $n, m \ge n_0$. Then

$$|f_n(t) - f(t)| = \lim_{m\to\infty} |f_n(t) - f_m(t)| \le \limsup_{m\to\infty} d(f_m, f_n) \le \varepsilon$$

for all $t \in [0,1]$ whenever $n \ge n_0$. Thus $f_n \to f$ uniformly as $n \to \infty$ and we are done.

6. Let M be as above. Define a mapping $T\colon M \to M$ by setting (for all $f \in M$ and $x \in [0,1]$)

$$T(f)(x) = a + \int_0^x K(x,y) f(y)\, dy,$$

where a is a constant and $K\colon [0,1]\times[0,1] \to \mathbb{R}$ is continuous. Let

$$k = \sup\left\{ \int_0^x |K(x,y)|\, dy \colon 0 \le x \le 1 \right\}$$

and suppose that $k < 1$.

(a) Prove that T is a contraction, i.e. $d\big(T(f), T(g)\big) \le L d(f,g)$, where $L < 1$.

(b) Prove that the integral equation

$$f(x) = a + \int_0^x K(x,y)f(y)\,dy$$

has a unique solution.

Solution. Before anything let us show that $Tf \in M$ for each $f \in M$ (i.e. Tf is continuous whenever f is). Take a sequence x_n converging to x in $[0,1]$. Then

$$\lim_{n\to\infty} \chi_{[0,x_n]}(y)K(x_n,y) = \chi_{[0,x]}(y)K(x,y)$$

pointwise except possibly at $y = x$. By the dominated convergence theorem

$$\begin{aligned}\lim_{n\to\infty} Tf(x_n) &= \lim_{n\to\infty} \int_0^1 \chi_{[0,x_n]}(y)K(x_n,y)f(y)\,dy + a\\ &= \int_0^1 \chi_{[0,x]}(y)K(x,y)f(y)\,dy + a = Tf(x)\end{aligned}$$

so that Tf is, indeed, continuous.

(a) Let us estimate

$$\begin{aligned}d(Tf,Tg) &= \sup_{0\le x\le 1} \left|\int_0^x K(x,y)(f(y)-g(y))\,dy\right|\\ &\le \sup_{0\le x\le 1} \int_0^x |K(x,y)||f(y)-g(y)|\,dy\\ &\le \sup_{0\le x\le 1} \int_0^x |K(x,y)|d(f,g)\,dy\\ &= kd(f,g).\end{aligned}$$

Thus T is a contraction with $L = k < 1$.

(b) By the previous results $T : M \to M$ is a contractive self mapping of a complete metric space. The Banach fixed point theorem implies the existence of a unique element $f \in M$ so that $T(f) = f$. By definition this is the unique solution of the equation

$$f(x) = a + \int_0^x K(x,y)f(y)\,dy, \quad x \in [0,1].$$

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 2, Solutions
31.1.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, January 31, 12:00 o'clock.

1. Let $M(m\times n,\mathbb{R})$ denote the space of all (real) $m\times n$-matrices. Give $M(m\times n,\mathbb{R})$ a (natural) topology by identifying a matrix $A=(a^i_j)\in M(m\times n,\mathbb{R})$ with a point $(a^1_1,a^1_2,\dots,a^1_n,a^2_1,\dots,a^2_n,\dots,a^m_1,\dots a^m_n)\in\mathbb{R}^{mn}$ and verify that $M(m\times n,\mathbb{R})$ is a topological mn-manifold. We have defined a topology of $M(m\times n,\mathbb{R})$ also by a norm $|A|=\sup\{|Ah|\colon |h|=1\}$. Show that these two topologies are the same.

   Solution. Define $\varphi\colon M(m\times n,\mathbb{R})\to\mathbb{R}^{nm}$, by setting

$$\varphi\big((a^j_i)\big)=(a^1_1,a^1_2,\dots,a^1_n,a^2_1,\dots,a^2_n,\dots,a^m_1,\dots a^m_n).$$

Then φ is a linear bijection. Let

$$\mathcal{T}=\{\varphi^{-1}U\colon U\subset\mathbb{R}^{nm}\text{ open}\}.$$

Then $\mathcal{T}$ is a topology on $M(m\times n,\mathbb{R})$, more precisely the topology induced by φ. Since φ is a bijection, it follows from the definition of $\mathcal{T}$ that $U\subset M(m\times n,\mathbb{R})$ is open if and only if $\varphi U\subset\mathbb{R}^{nm}$ is open. Hence φ is a homeomorphism.

Since φ is a homeomorphism and $\mathbb{R}^{nm}$ is Hausdorff and N_2, also $M(m\times n,\mathbb{R})$ is Hausdorff and N_2. It follows that $M(m\times n,\mathbb{R})$ is a topological nm-manifold with a global chart φ.

Next we prove that $\mathcal{T}$ is determined by a norm. Define a norm $|\cdot|_2$ on the vector space $M(m\times n,\mathbb{R})$ by setting $|A|_2=|\varphi(A)|$, where $|\varphi(A)|$ is the standard norm of $\varphi(A)\in\mathbb{R}^{nm}$. Then the mapping φ between normed spaces $\big(M(m\times n,\mathbb{R}),|\cdot|_2\big)$ and $\mathbb{R}^{nm}$ is norm preserving, therefore φ is a linear isomorphism. Consequently, if $\mathcal{T}'$ is the topology on $M(m\times n,\mathbb{R})$ determined by the norm $|\cdot|_2$, the mapping φ is a homeomorphism $\big(M(m\times n,\mathbb{R}),\mathcal{T}'\big)\to\mathbb{R}^{nm}$. Since φ is also a homeomorphism $\big(M(m\times n,\mathbb{R}),\mathcal{T}\big)\to\mathbb{R}^{nm}$, it follows that $\mathcal{T}'=\mathcal{T}$.

Let us still prove that the operator norm $|\cdot|$ and $|\cdot|_2$ are equivalent, so that the norm $|\cdot|$ determines the same topology $\mathcal{T}$. Let $A=(a^i_j)\in M(m\times n,\mathbb{R})$ be arbitrary and let $h\in\mathbb{R}^n$, with $|h|=1$. Denoting by A^i the ith row of A, we get

$$\begin{aligned}|Ah|^2 &= \sum_{i=1}^m (Ah)_i^2 = \sum_{i=1}^m \big((A^i)^T \cdot h\big)^2 \le \sum_{i=1}^m \big|(A^i)^T\big|^2 |h|^2 \\ &= \sum_{i=1}^m \big|(A^i)^T\big|^2 = \sum_{i=1}^m \sum_{j=1}^n (a_j^i)^2 = |A|_2^2.\end{aligned}$$

hence $|A| \le |A|_2$. On the other hand,

$$|A|_2^2 = \sum_{i=1}^m \sum_{j=1}^n (a_j^i)^2 = \sum_{i=1}^m \sum_{j=1}^n (Ae_j)_i^2 = \sum_{j=1}^n |Ae_j|^2 \le n|A|^2,$$

so $|A| \le |A|_2 \le \sqrt{n}|A|$. Hence the norms $|\cdot|$ and $|\cdot|_2$ are equivalent.

Note: In general, if E is a finite dimensional vector space, all norms on E are equivalent.

2. Prove that $GL(n,\mathbb{R})$ is a disconnected topological n^2-manifold.

   Solution. Let $\varphi\colon M(n\times n,\mathbb{R}) \to \mathbb{R}^{n^2}$ be as in the previous exercise, so φ is a homeomorphism between topological n^2-manifolds. Let us prove that the determinant $\det\colon M(n\times n,\mathbb{R}) \to \mathbb{R}$,

   $$\det\big((A_j^i)\big) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) A^1_{\sigma(1)} A^2_{\sigma(2)} \cdots A^n_{\sigma(n)},$$

   is continuous. Here S_n is the set of permutations of $\{1,2,\dots,n\}$ and $\operatorname{sgn}\colon S_n \to \{-1,1\}$ is the sign of permutation. Let $f = \det\circ\varphi^{-1}$ and $x = (x_1, x_2, \dots, x_{n^2}) \in \mathbb{R}^{n^2}$. Then

   $$f(x) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{n+\sigma(2)} x_{2n+\sigma(3)} \cdots x_{(n-1)n+\sigma(n)},$$

   and therefore f is continuous. Since φ is a homeomorphism, also $\det = f\circ\varphi$ is continuous.

   Since det is continuous and

   $$GL(n,\mathbb{R}) = \overset{-1}{\det}(\mathbb{R}\setminus\{0\}),$$

   we see that $GL(n,\mathbb{R})$ is an open subset of the topological n^2-manifold $M(n\times n,\mathbb{R})$. In particular, $GL(n,\mathbb{R})$ equipped with the relative topology is a topological n^2-manifold. The image of a connected set under a continuous mapping is connected. Since $\mathbb{R}\setminus\{0\}$ is disconnected, the same holds true for $GL(n,\mathbb{R})$.

3. Let $sl(n,\mathbb{R})$ be the set of those $n\times n$ matrices whose trace vanishes:

   $$sl(n,\mathbb{R}) = \{A = (a_j^i) \in M(n\times n,\mathbb{R})\colon\ \operatorname{tr} A = \sum_{i=1}^n a_i^i = 0\}.$$

   Prove that $sl(n,\mathbb{R})$ is a topological manifold.

Solution. Note that $\mathrm{tr} : M(m \times n, \mathbb{R}) \to \mathbb{R}$ is a linear map and

$$sl(n, \mathbb{R}) = \ker \mathrm{tr}$$

is thus a closed linear subspace of $M(m \times n, \mathbb{R})$ of dimension

$$\dim\big(\ker(\mathrm{tr})\big) = \dim\big(M(n \times n, \mathbb{R})\big) - \dim\big(\mathrm{Im}(\mathrm{tr})\big) = n^2 - 1.$$

Since all finite dimensional vector spaces over $\mathbb{R}$ are homeomorphic to some $\mathbb{R}^k$ and hence topological manifolds it follows that $sl(n, \mathbb{R})$, in particular, is a topological manifold (of dimension $n^2 - 1$).

4. Prove that in the implicit function theorem (Theorem 1.14 in the lecture notes)

$$\varphi'(x_0) = -\big(u'(y_0)\big)^{-1} v'(x_0),$$

where $v(x) = f(x, y_0)$.

Solution. Adopting the notation of Theorem 1.14 we have

$$f(x, \varphi(x)) = 0$$

in a neighbourhood of x_0. Let us denote $h : \mathbb{R}^m \to \mathbb{R}^{n+m}$, $h(x) = (x, \varphi(x))$. Differentiating this at x_0 using the chain rule we obtain

$$0 = f'(x_0, \varphi(x_0))h'(x_0).$$

Note that $\varphi(x_0) = y_0$ so that $0 = f'(x_0, y_0)h'(x_0)$. Inspecting the Jacobian matrix $f'(x_0, y_0)$ we note that the first m columns make up the Jacobian matrix for $v'(x_0)$ and the next n columns correspond to $u'(y_0)$. Similarly the first m rows of $h'(x_0)$ form the identity matrix $I_{m\times m}$ and the remaining n rows the Jacobian matrix for $\varphi'(x_0)$. Using the matrix block shorthand we may write

$$0 = (v'(x_0) \quad u'(y_0)) \begin{pmatrix} I_{m\times m} \\ \varphi'(x_0) \end{pmatrix}$$

from which we get

$$0 = v'(x_0) + u'(y_0)\varphi'(x_0).$$

Since $u'(y_0)$ is invertible the claim follows from this by rearranging terms and applying $(u'(x_0))^{-1}$ to both sides.

5. Prove lemma 1.7 in the lecture notes: Let $G \subset \mathbb{R}^m$ be open and $J \subset G$ be a closed line segment with endpoints a and b. Let $f : G \to \mathbb{R}^n$ be differentiable at each point of J. Then there exists, for all $v \in \mathbb{R}^n$, a point $x_v \in J$ such that $v \cdot (f(b) - f(a)) = v \cdot Df(x_v)(b - a)$. In particular, if $|Df(x)| \leq M$ for all $x \in J$, we have $|f(b) - f(a)| \leq M|b - a|$.

Solution. Let $\gamma : [0, 1] \to G$ be the map $\gamma(t) = a + t(b - a)$, in other words $\gamma([0, 1]) = J$, and $g_v : \mathbb{R}^n \to \mathbb{R}$ the map $g_v(x) = v \cdot x$ where $v \cdot x = v^T x$ in terms of matrix multiplication. Now, by the mean value theorem for real valued functions there exists a point $\xi \in (0,1)$ such that

$$\begin{aligned} v \cdot (f(b) - f(a)) &= v \cdot (f(\gamma(1)) - f(\gamma(0))) \\ &= (g_v \circ f \circ \gamma)(1) - (g_v \circ f \circ \gamma)(0) \\ &= (g_v \circ f \circ \gamma)'(\xi). \end{aligned}$$

On the other hand,

$$\begin{aligned} (g_v \circ f \circ \gamma)'(\xi) &= D(g_v \circ f)(\gamma(\xi))\gamma'(\xi) \\ &= ((Dg_v \circ f)Df)(\gamma(\xi))\gamma'(\xi) \\ &= v \cdot Df((\gamma(\xi)))(b-a). \end{aligned}$$

Choose $x_v = \gamma(\xi)$, and the first claim follows. In particular, as this holds for all $v \in \mathbb{R}^n$, choose $v = f(b) - f(a)$ and suppose that $|Df(x)| \leq M$ for all $x \in J$. Now,

$$\begin{aligned} |v|^2 = |v \cdot v| &= |v \cdot (f(b) - f(a))| = |v \cdot f'(x_v)(b-a)| \\ &= |v||f'(x_v)||b-a| \leq |v|M|b-a| \end{aligned}$$

and

$$|f(b) - f(a)| \leq M|b-a|.$$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x, g(x,y))$ where

$$g(x,y) = \begin{cases} y - x^2 & \text{for} \quad y > x^2 \\ 0 & \text{for} \quad |y| \leq x^2 \\ y + x^2 & \text{for} \quad y < -x^2 \end{cases}$$

Prove that f is continuous everywhere and differentiable at the origin. Compute $J_f(0)$. Is f locally injective at the origin?

Solution. Clearly g is continuous for $|y| \neq x^2$. Let $|y| = x^2$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ such that $\varepsilon_1, \varepsilon_2 > 0$. Now,

$$|g(x+\varepsilon_1, y+\varepsilon_2) - g(x,y)| \leq ||y+\varepsilon_2| - (x+\varepsilon_1)^2| \longrightarrow 0$$

as $|\varepsilon| \longrightarrow 0$. Hence g is continuous everywhere. Since the first component of f is also continuous, f is continuous everywhere. Let ε be as before. If $|\varepsilon_2| \leq \varepsilon_1^2$, then $|f(\varepsilon) - f(0) - \varepsilon| = |(\varepsilon_1, 0) - (\varepsilon_1, \varepsilon_2)| = |\varepsilon_2| \leq \varepsilon_1^2$. If, on the other hand $\varepsilon_1^2 < |\varepsilon_2|$, then $|f(\varepsilon) - f(0) - \varepsilon| = |(\varepsilon_1, \varepsilon_2 \pm \varepsilon_1^2) - (\varepsilon_1, \varepsilon_2)| = \varepsilon_1^2$. So, for all $\varepsilon > 0$

$$\frac{|f(\varepsilon) - f(0) - \varepsilon|}{|\varepsilon|} \leq \frac{\varepsilon_1^2}{|\varepsilon|} \leq |\varepsilon_1| \longrightarrow 0$$

as $|\varepsilon| \longrightarrow 0$. Thus, f is differentiable at the origin. By simple substitution, $J_f(0) = \det I = 1$, however, since $g(h,h) = g(h,-h^2)$ for all $h > 0$, f is not locally injective at the origin.

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 3, Solutions
7.2.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, February 7, 12:00 o'clock.

1. Let M be the following set

$$M = \{(x, y, |x|) \in \mathbb{R}^3 \colon (x, y) \in \mathbb{R}^2\}.$$

Construct a C^∞ atlas $\mathcal{A}$ on M so that $(M, \bar{\mathcal{A}})$ becomes a differentiable 2-manifold. Is it diffeomorphic with $\mathbb{R}^2$? If yes, what is a diffeomorphism $f \colon M \to \mathbb{R}^2$.

Solution. Let $\varphi \colon M \to \mathbb{R}^2$, $\varphi(x, y, |x|) = (x, y)$. The mapping φ is a homeomorphism when M has the relative topology. Thus M is a topological 2-manifold. Let $\mathcal{A} = \{(M, \varphi)\}$ and $\bar{\mathcal{A}}$ be the maximal C^∞-atlas containing (M, φ). Then $(M, \bar{\mathcal{A}})$ is a 2-dimensional C^∞ manifold. It is diffeomorphic to $\mathbb{R}^2$ since $f = \varphi$ is a diffeomorphic between $(M, \bar{\mathcal{A}})$ and $\mathbb{R}^2$. That is, each local representation of f is a C^∞-diffeomorphism.

2. Let M be a differentiable manifold, $p \in M$, $I \subset \mathbb{R}$ an open interval, and $0 \in I$.

(a) Let $\gamma \colon I \to M$ be a C^∞-path such that $\gamma(0) = p$. Show that $\dot{\gamma}_0 \in T_pM$.
(b) Let $v \in T_pM$. Show that there exists a C^∞-path $\gamma \colon I \to M$ such that $\dot{\gamma}_0 = v$.

Solution.

(a) We have to show that the functional $\dot{\gamma}_0 : C^\infty(p) \to \mathbb{R}$ is linear and satisfies the so called Leibniz rule. For the first we have

$$\begin{aligned}\dot{\gamma}_0(af + bg) &= \big((af + bg) \circ f\big)'(0) = (af \circ \gamma + bg \circ \gamma)'(0) \\ &= a(f \circ \gamma)'(0) + b(g \circ \gamma)'(0) = a\dot{\gamma}_0(f) + b\dot{\gamma}_0(g)\end{aligned}$$

and for the second similarly

$$\begin{aligned}\dot{\gamma}_0(fg) =& (f \circ \gamma \cdot g \circ \gamma)'(0) \\ =& f \circ \gamma(0)(g \circ \gamma)'(0) + g \circ \gamma(0)(f \circ \gamma)'(0) \\ =& f(p)\dot{\gamma}_0(g) + g(p)\dot{\gamma}_0(f),\end{aligned}$$

given any $f, g \in C^\infty(p)$ and $a, b \in \mathbb{R}$.

(b) Let us take a chart (U, x) at p. Given a vector $v \in T_pM$, it may be written as a sum of the basis elements, i.e.,

$$v = \sum_i^n a^i (\partial_i)_p.$$

Set $w = (a^1, \dots, a^n) \in \mathbb{R}^n$ and consider the path $\alpha\colon \mathbb{R} \to \mathbb{R}^n$,

$$\alpha(t) = x(p) + wt, \; t \in \mathbb{R}.$$

For a sufficiently small neighborhood I of 0 we have $\alpha(I) \subset xU$ so the composition $\gamma(t) = x^{-1} \circ \alpha(t)$ yields a C^∞-path defined on I. Now the chain rule gives

$$\begin{aligned} \dot\gamma_0(f) &= (f \circ \gamma)'(0) = (f \circ x^{-1} \circ \alpha)'(0) \\ &= \alpha'(0) \cdot \nabla\big(f \circ x^{-1}\big)\big(\alpha(0)\big) \\ &= \sum_{i=1}^n a^i D_i(f \circ x^{-1})(\alpha(0)) \\ &= \sum_i^n a^i (\partial_i)_p f = v(f) \end{aligned}$$

for every $f \in C^\infty(p)$.

3. (a) Let M, N, and L be differentiable manifolds and let $f\colon M \to N$ and $g\colon N \to L$ be C^∞-mappings. Show that

$$(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$$

for all $p \in M$.

(b) Let $f\colon M \to M$ be the identity mapping $f = id$. Show that

$$f_{*p} = id\colon T_pM \to T_pM$$

for every $p \in M$.

Solution.

(a) Given any $v \in T_pM$ and $h \in C^\infty\big((g \circ f)(p)\big)$,

$$(g \circ f)_{*p} v(h) = v(h \circ g \circ f) = f_{*p}v(h \circ g) = g_{*f(p)}(f_{*p}v)(h).$$

(b) Let us compute

$$id_{*p}v(h) = v(h \circ id) = v(h), \quad v \in T_pM, \quad h \in C^\infty(p),$$

thus $id_{*p}v = v$ for all $v \in T_pM$.

4. Let M^m and N^n be differentiable manifolds, $p \in M^m$, and $f\colon M^m \to N^n$ a smooth mapping such that $f_*\colon T_pM^m \to T_{f(p)}N^n$ is a linear isomorphism. Prove that f is a local diffeomorphism at $p \in M^m$.

Solution. First, since f_{*p} is a linear isomorphism, the tangent spaces T_pM^m and $T_{f(p)}N^n$ are isomorphic and hence $\dim M = \dim N$.

Let (U, x) be a chart at p and (V, y) a chart at $f(p)$. Now $y \circ f \circ x^{-1} \colon xU \to yV$ is C^∞ and $\det\big(y \circ f \circ x^{-1}\big)'(x(p)) \neq 0$ since f_{*p} is invertible. By the inverse mapping theorem there exist a neighborhood U_0 of p and a neighborhood V_0 of $f(p)$ such that the mapping $y \circ f \circ x^{-1}|xU_0$ has a C^1 smooth inverse mapping $g \colon yV_0 \to xU_0$. Since $y \circ f \circ x^{-1}$ has continuous partial derivatives of every order and $\det\big(y \circ f \circ x^{-1}\big)'(q) \neq 0,\ q \in xU_0$, we can conclude by using Cramer's rule that g has continuous partial derivatives of every order. Hence f is a local C^∞ diffeomorphism.

$*$**5.** (Embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.) Let $\mathbb{R}P^2$ be the real 2-dimensional projective space (see Example 0.12.5 in lecture notes for the definition). Define a mapping $F \colon \mathbb{R}^3 \to \mathbb{R}^4$,

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere and let $\varphi = F|\mathbb{S}^2$. Consider the mapping $\tilde{\varphi} \colon \mathbb{R}P^2 \to \mathbb{R}^4$,

$$\varphi\big([p]\big) = \varphi(p),\ p = (x, y, z).$$

Prove that:
(a) $\tilde{\varphi}$ is well-defined,
(b) $\tilde{\varphi}$ is injective,
(c) $\tilde{\varphi}$ is an immersion.

Solution.
(a) By definition of $\mathbb{R}P^2$, the antipodal points of S^2 are identified: $(x, y, z) \sim (-x, -y, -z)$. Clearly $\tilde{\varphi}$ is well-defined since $F(x, y, z) = F(-x, -y, -z)$.
(b) Suppose $F(x_1, y_1, z_1) = F(x_2, y_2, z_2)$. Then, one can easily conclude that $(x_2, y_2, z_2) = (x_1, y_1, z_1)$ or $(-x_1, -y_1, -z_1)$.
(c) Since the quotient map $\pi : S^2 \to \mathbb{R}P^2$ is a local diffeomorphism, it suffices to check that the map $F|_{S^2} : S^2 \to \mathbb{R}^4$ is an immersion. Consider the differential of F:

$$dF = \begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}.$$

Let $p = (x, y, z) \in \mathbb{S}^2$ and consider first the case $x^2 + y^2 \neq 0$. Then $\operatorname{rank} dF = 3$ since there exists non-singular 3×3 minor of dF. If $x = y = 0$, then $z^2 = 1$ and $\operatorname{rank} dF = 2$. However, in that case

$$dF = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \\ 0 & z & 0 \end{bmatrix}$$

and hence dF maps the xy-plane $= T_p\mathbb{S}^2$ to the zt-plane $\subset \mathbb{R}^4 = T_{F(p)}\mathbb{R}^4$. Hence in any case dF is injective.
Since $\mathbb{R}P^2$ is compact, the map $\tilde{\varphi}$ is closed (smooth injective), which shows that $\tilde{\varphi}$ is a homeomorphism to its image and hence an embedding.

$*$**6.** If $v \in T_pM$ is a (non-zero) vector and f is a smooth function in a neighborhood of p, the (directional) derivative $D_vf := vf$ makes sense. Suppose that we are given two linearly independent vectors $v, w \in T_pM$. What problems (if any) do you face if you want to define a nice "second order directional derivative", say D^2_{vw}?

Solution. Note that if we only use that $v, w \in T_pM$, then, wvf is not well-defined since vf is only defined at p. If using two vector fields V, W in a neighborhood of p, then WVf, VWf are well-defined, but $VWf \neq WVf$ in general.

Consider two linearly independent vectors $v, w \in T_pM$. Then we may find a chart $x = (x_1, \ldots, x_n)$ at p such that $v = \partial_1$ and $w = \partial_2$ (verify this!). Now a second order partial derivative of a function f, say $\partial^2_{ij}f(p)$ could be defined as the corresponding second order partial derivative of the local representation of f, but this is not well-defined! It depends on the choice of the chart. So it is not determined solely by the vectors v, w. This is where we need connection!

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 4, Solutions
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1. Let M, M_1, M_2, N, N_1, and N_2 be differentiable manifolds.

(a) Let $\pi_1\colon M\times N\to M$ and $\pi_2\colon M\times N\to N$ be the canonical projections. For $(p,q)\in M\times N$, define

$$\tau\colon T_{(p,q)}(M\times N)\to T_pM\oplus T_qN$$

by setting

$$\tau v=\pi_{1*}v+\pi_{2*}v.$$

Show that τ is an isomorphism. Thus we may identify $T_{(p,q)}(M\times N)=T_pM\oplus T_qN$. We use this identification in the sequel.

(b) Let $f\colon M_1\times M_2\to N$ be a smooth mapping. For $(p,q)\in M_1\times M_2$, we define mappings $f_p\colon M_2\to N$ and $f^q\colon M_1\to N$,

$$f_p(q)=f^q(p)=f(p,q).$$

Show that

$$f_{*(p,q)}(v+w)=(f^q)_{*p}v+(f_p)_{*q}w$$

for all $v\in T_pM_1$ and $w\in T_qM_2$.

(c) Let $f_i\colon M\to N_i$, $i=1,2$, be smooth mappings. Show that the mapping $(f_1,f_2)\colon M\to N_1\times N_2$ is smooth and that

$$(f_1,f_2)_{*p}v=f_{1*p}v+f_{2*p}v$$

for all $v\in T_pM$, $p\in M$.

(d) Let $f_i\colon M_i\to N_i$, $i=1,2$, be smooth mappings. Show that the mapping $f_1\times f_2\colon M_1\times M_2\to N_1\times N_2$, $(f_1\times f_2)(p,q)=\big(f_1(p),f_2(q)\big)$, is smooth and that

$$(f_1\times f_2)_{*(p,q)}(v+w)=f_{1*p}v+f_{2*2}w$$

for all $v\in T_pM$, $w\in T_qM_2$, $(p,q)\in M_1\times M_2$.

Solution. Denote $m_i=\dim M_i$, $n_i=\dim N_i$, $i=1,2$, and $m=\dim M$, $n=\dim N$.

(a) Let (U,x) and (V,y) be charts at $p\in M$ and $q\in N$ respectively, and $(W,z)=(U\times V,(x\circ\pi_1,y\circ\pi_2))$ be the induced chart on $M\times N$ at (p,q). Then, it suffices to show that

$$\tau\left(\frac{\partial}{\partial z^i}\right)_{(p,q)}=\begin{cases}\left(\frac{\partial}{\partial x^i}\right)_p & 1\le i\le m\\ \left(\frac{\partial}{\partial y^{i-m}}\right)_q & m<i\le m+n\end{cases}\tag{1}$$

for all $1 \leq i \leq m+n$ since τ maps a basis of $T_{(p,q)}(M \times N)$ to a basis of $T_pM \oplus T_qN$ and is therefore an isomorphism. Let us prove (1). We assume that $i \leq m$. If $h \in C^\infty(p)$, then

$$\begin{aligned}\left(\pi_{1*}\left(\frac{\partial}{\partial z^i}\right)_{(p,q)}\right) h &= \left(\frac{\partial}{\partial z^i}\right)_{(p,q)} (h \circ \pi_1)\\ &= D_i(h \circ \pi_1 \circ z^{-1})(z(p,q))\\ &= \big(h \circ \pi_1 \circ z^{-1} \circ (t \mapsto z(p,q) + te_i)\big)'(0)\\ &\overset{1 \leq m}{=} \big(h \circ x^{-1} \circ (t \mapsto x(p) + te_i)\big)'(0)\\ &= D_i(h \circ x^{-1})(x(p)) = \left(\frac{\partial}{\partial x^i}\right)_p h.\end{aligned}$$

Similarly, if $h \in C^\infty(q)$, then

$$\begin{aligned}\left(\pi_{2*}\left(\frac{\partial}{\partial z^i}\right)_{(p,q)}\right) h &= \big(h \circ \pi_2 \circ z^{-1} \circ (t \mapsto z(p,q) + te_i)\big)'(0)\\ &\overset{1 \leq m}{=} \big(t \mapsto h(q)\big)'(0) = 0.\end{aligned}$$

Hence

$$\tau\left(\frac{\partial}{\partial z^i}\right)_{(p,q)} = \pi_{1*}\left(\frac{\partial}{\partial z^i}\right)_{(p,q)} + \pi_{2*}\left(\frac{\partial}{\partial z^i}\right)_{(p,q)} = \left(\frac{\partial}{\partial x^i}\right)_p + 0$$

and (1) holds. The case $i > m$ follows in a similar way.

(b) Denote the projections $\pi_i \colon M_1 \times M_2 \to M_i$, $i = 1,2$. Take charts (U,x), (V,y) at p and q, respectively, and let $(W,z) = (U \times V, (x \circ \pi_1, y \circ \pi_2))$ be the induced chart at (p,q). By linearity it is enough to show that the claim holds if

$$v + w = \left(\frac{\partial}{\partial z^i}\right)_{(p,q)},$$

where $1 \leq i \leq m_1 + m_2$. For $1 \leq i \leq m_1$ and $h \in C^\infty\big(f(p,q)\big)$

$$\begin{aligned}\left(f_*\left(\frac{\partial}{\partial z^i}\right)_{(p,q)}\right) h = \left(\frac{\partial}{\partial z^i}\right)_{(p,q)} (h \circ f) &= D_i(h \circ f \circ z^{-1})(z(p,q))\\ &=(h \circ f \circ z^{-1} \circ (t \mapsto z(p,q) + te_i))'(0)\\ &=(h \circ f^q \circ x^{-1} \circ (t \mapsto x(p) + te_i))'(0)\\ &=D_i(h \circ f^q \circ x^{-1})(x(p)) = f^q_*\left(\frac{\partial}{\partial x^i}\right)_p h.\end{aligned}$$

The case $m < i \leq m+n$ is similar.

(c) Denote the projections $\pi_i \colon N_1 \times N_2 \to N_i$, $i = 1,2$. We write $f = (f_1, f_2) \colon M \to N_1 \times N_2$. Then $\pi_i \circ f = f_i$, $i = 1,2$. To show smoothness of f, let $p \in M$, $(q_1, q_2) \in N_1 \times N_2$, and let (U,x) be a chart on M at $p \in M$ and (V_i, y_i) be a chart on N_i at q_i, $i = 1,2$.

Then $(W, z) = (V_1 \times V_2, (y_1 \circ \pi_1, y_2 \circ \pi_2))$ is a chart on $N_1 \times N_2$ at (q_1, q_2). Now

$$\begin{aligned} z \circ f \circ x^{-1} &= (y_1 \circ \pi_1 \circ f \circ x^{-1}, y_2 \circ \pi_2 \circ f \circ x^{-1}) \\ &= (y_1 \circ f_1 \circ x^{-1}, y_2 \circ f_2 \circ x^{-1}) \end{aligned}$$

is smooth since f_1 and f_2 are. Hence f is smooth. Since $\pi_i \circ f = f_i$ we use the identification $\tau = \mathrm{id}$ from (a) to get

$$f_* v = \pi_{1*} f_* v + \pi_{2*} f_* v = f_{1*} v + f_{2*} v.$$

(d) To show the smoothness, fix $(p_1, p_2) \in M_1 \times M_2$ and $(q_1, q_2) \in N_1 \times N_2$. Let (U_i, x_i) be a chart on M_i at p_i and let (V_i, y_i) be a chart of N_i at q_i, $i = 1, 2$. Then $(W, z) = (U_1 \times U_2, (x_1 \circ \pi_1, x_2 \circ \pi_2))$ is a chart at $(p_1, p_2) \in M_1 \times M_2$ and $(R, w) = (V_1 \times V_2, (y_1 \circ \pi_1, y_2 \circ \pi_2))$ is a chart at $(q_1, q_2) \in N_1 \times N_2$. Now

$$\begin{aligned} &w \circ (f_1 \times f_2) \circ z^{-1} \\ &\qquad = \big(y_1 \circ \pi_1 \circ (f_1 \times f_2) \circ z^{-1}, y_2 \circ \pi_2 \circ (f_1 \times f_2) \circ z^{-1}\big) \\ &\qquad = \big(y_1 \circ f_1 \circ \pi_1 \circ z^{-1}, y_2 \circ f_2 \circ \pi_2 \circ z^{-1}\big) \\ &\qquad = \big(y_1 \circ f_1 \circ x_1^{-1} \circ \pi_1, y_2 \circ f_2 \circ x_2^{-1} \circ \pi_2\big) \end{aligned}$$

is a smooth map. This shows that $f_1 \times f_2$ is smooth. By the previous results, we get

$$\begin{aligned} (f_1 \times f_2)_{*(p,q)}(v + w) &\overset{(b)}{=} ((f_1 \times f_2)^q)_{*p} v + ((f_1 \times f_2)_p)_{*q} w \\ &= \big(s \mapsto (f_1(s), f_2(q))\big)_{*p} v + \big(t \mapsto (f_1(p), f_2(t))\big)_{*q} w \\ &\overset{(c)}{=} f_{1*p} v + (s \mapsto f_2(s))_{*p} v + f_{2*q} w + (t \mapsto f_1(p))_{*q} w \\ &= f_{1*p} v + f_{2*q} w. \end{aligned}$$

2. Suppose that M_i is a submanifold of N_i, $i = 1, 2$. Prove that $M_1 \times M_2$ is a submanifold of $N_1 \times N_2$.

Solution. Denote by $i_k \colon M_k \to N_k$ the immersions, $k = 1, 2$. Since they are homeomorphisms onto their images it follows that $i_1 \times i_2 \colon M_1 \times M_2 \to N_1 \times N_2$ is a homeomorphism onto its image. It suffices to see that $(i_1 \times i_2)_{*(p,q)}$ is an injection. For this let $(v, w) \in T_p M_1 \oplus T_q M_2$ be such that $(i_1 \times i_2)_{*(p,q)}(v, w) = 0$. Then

$$0 = (i_1 \times i_2)_{*(p,q)}(v, w) \overset{(1d)}{=} (i_{1*p} v, i_{2*q} w),$$

thus $i_{1*p} v = 0$ and $i_{2*q} w = 0$. Since i_{1*p} and i_{2*q} are injections we have that $(v, w) = 0$.

3. Let G be a Lie group, $e \in G$ the neutral element, and $i \colon G \to G$, $i(g) = g^{-1}$. Prove that $i_{*e} v = -v$ for all $v \in T_e G$.

Solution. The idea is to "differentiate both sides" of the identity $gg^{-1} = e$ with respect to g. For $g \in G$, define $I_g \colon G \to G \times G$ and $I^g \colon G \to G \times G$ as

$$I_g(h) = (g,h), \quad I^g(h) = (h,g).$$

Next we define the multiplication $m \colon G \times G \to G$, $m(g,h) = gh$ and also the left-translation $L_g \colon G \to G$ and the right-translation $R_g \colon G \to G$ by

$$L_g(h) = gh, \quad R_g(h) = hg.$$

Let us compute the tangent map of m. If $g, h \in G$, $v \in T_gG$, and $w \in T_hG$, then by (1b) we have

$$(v,w) = \big(\mathrm{id}_{G\times G}\big)_{*(g,h)}(v,w) = (I^h)_{*g}v + (I_g)_{*h}w.$$

Therefore

$$\begin{aligned} m_{*(g,h)}(v,w) &= m_{*(g,h)}\big((I^h)_{*g}v + (I_g)_{*h}w\big) \\ &= m_{*(g,h)}\big((I^h)_{*g}v\big) + m_{*(g,h)}(v,w)\big((I_g)_{*h}w\big) \\ &= \big(m \circ I^h\big)_{*g}v + \big(m \circ I_g\big)_{*h}w \\ &= (R_h)_{*g}v + (L_g)_{*h}w. \end{aligned}$$

Since $gg^{-1} = e$ for every $g \in G$, $m \circ (\mathrm{id}_g, i)$ is a constant map, so its tangent map is zero. Combining the above with (1c) we get

$$\begin{aligned} 0 &= \big(m \circ (\mathrm{id}_G, i)\big)_{*g}v = m_{*(g,g^{-1})}\big((\mathrm{id}_G, i)_{*g}v\big) = m_{*(g,g^{-1})}\big(v, i_{*g}v\big) \\ &= \big(R_{g^{-1}}\big)_{*g}v + \big(L_g\big)_{*g^{-1}}(i_{*g}v). \end{aligned}$$

Since $L_{g^{-1}} \circ L_g = \mathrm{id}_G$, operating with $\big(L_{g^{-1}}\big)_{*e}$ on both sides of the previous identity we get

$$0 = \big(L_{g^{-1}}\big)_{*e}\big(R_{g^{-1}}\big)_{*g}v + i_{*g}v.$$

Hence

$$i_{*g}v = -\big(L_{g^{-1}}\big)_{*e}\big(R_{g^{-1}}\big)_{*g}v,$$

and taking $g = e$ we get the desired formula since $L_e = \mathrm{id}_G = R_e$.

4. Let V be a vector field on M. Show that the following conditions are equivalent:

(a) $V \in \mathcal{T}(M)$.

(b) If $v^i \colon U \to \mathbb{R}$, $i = 1, \dots, n$, are the component functions of V with respect to a chart (U,x), $x = (x^1, \dots, x^n)$, i.e.

$$V|U = v^i\partial_i,$$

then $v^i \in C^\infty(U)$.

(c) If $U \subset M$ is open and $f \colon U \to \mathbb{R}$ is a smooth function, then the function $Vf \colon U \to \mathbb{R}$, $(Vf)(p) = V_pf$, is smooth.

Solution. Recall that if (U, x) is a chart on M, then $(TU, \bar{x})$ is a chart on TM. Here $\bar{x}\colon TU \to xU \times \mathbb{R}^n$ and

$$\bar{x}(p, v) = (x^1(p), \dots, x^n(p), vx^1, \dots, vx^n)$$

for $p \in U$ and $v \in T_pM$.

$(a) \iff (b)$: Let (U, x) be a chart. Note that given $s \in \mathbb{R}^n$

$$\begin{aligned}
&\overline{x} \circ V \circ x^{-1}(s) \\
&\qquad = (s, V_{x^{-1}(s)}(x^1), \dots, V_{x^{-1}(s)}(x^n)) \\
&\qquad = (s, v^i \circ x^{-1}(s)(\partial_i)_{x^{-1}(s)}x^1, \dots, v^i \circ x^{-1}(s)(\partial_i)_{x^{-1}(s)}x^n) \\
&\qquad = (s, v^1 \circ x^{-1}(s), \dots, v^n \circ x^{-1}(s))
\end{aligned}$$

where $n = \dim M$. Therefore $\overline{x} \circ V \circ x^{-1}$ is smooth if and only if $v^i \circ x^{-1}$ is for $i = 1, \dots, n$. Thus $V \in \mathcal{T}(M)$ if and only if each $v^i|U$ is smooth.

$(b) \implies (c)$: Let $U \subset M$ be open and $f \in C^\infty(U)$. Thus if (W, x) is a chart at $p \in U$ (with $W \subset U$) it follows that $f \circ x^{-1}$ is a smooth function, thus $D_i(f \circ x^{-1})$ is a smooth map as well. We have

$$Vf(p) = v^i(p)(\partial_i)_p f = v^i(p) D_i(f \circ x^{-1})(x(p))$$

and since, by assumption the functions v^i are smooth it follows that Vf is smooth.

$(c) \implies (b)$: Let (U, x) be a chart. Since x^k is smooth, by assumption we obtain that

$$Vx^k = v^i \partial_i x^k = v^k$$

is a smooth map, $k = 1, \dots, \dim M$.

$*$**5.** (a) Compute $[X, Y]$, $[X, Z]$ and $[Y, Z]$ when $X, Y, Z \in \mathcal{T}(\mathbb{R}^3)$ are vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \quad \text{and} \quad Z = \frac{\partial}{\partial z}.$$

(b) Does there exists a smooth submanifold S of $\mathbb{R}^3$ such that the vectors X_p and Y_p form a basis of T_pS for all $p \in S$?

(c) Does there exists a smooth submanifold M of $\mathbb{R}^3$ such that the vectors X_p and Z_p form a basis of T_pM for all $p \in M$?

Solution. (a):

$$\begin{aligned}
[X,Y]f &= X(Yf) - Y(Xf) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y} + \frac{x}{2}\frac{\partial f}{\partial z}\right) - \frac{y}{2}\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y} + \frac{x}{2}\frac{\partial f}{\partial z}\right) \\
&\quad - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} - \frac{y}{2}\frac{\partial f}{\partial z}\right) - \frac{x}{2}\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x} - \frac{y}{2}\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{2}\frac{\partial f}{\partial z} + \frac{x}{2}\frac{\partial^2 f}{\partial x \partial z} - \frac{y}{2}\frac{\partial^2 f}{\partial z \partial y} - \frac{xy}{4}\frac{\partial^2 f}{\partial^2 z} \\
&\quad - \frac{\partial^2 f}{\partial y \partial x} + \frac{1}{2}\frac{\partial f}{\partial z} + \frac{y}{2}\frac{\partial^2 f}{\partial y \partial z} - \frac{x}{2}\frac{\partial^2 f}{\partial z \partial x} + \frac{xy}{4}\frac{\partial^2 f}{\partial^2 z} \\
&= \frac{\partial f}{\partial z} = Zf.
\end{aligned}$$

Thus, $[X,Y] = Z$. Similarly,

$$\begin{aligned}
[X,Z]f &= X(Zf) - Z(Xf) \\
&= \frac{\partial^2 f}{\partial x \partial z} - \frac{y}{2}\frac{\partial^2 f}{\partial^2 z} - \frac{\partial^2 f}{\partial z \partial x} + \frac{y}{2}\frac{\partial^2 f}{\partial^2 z} = 0,
\end{aligned}$$

and

$$\begin{aligned}
[Y,Z]f &= Y(Zf) - Z(Yf) \\
&= \frac{\partial^2 f}{\partial y \partial z} + \frac{x}{2}\frac{\partial^2 f}{\partial^2 z} - \frac{\partial^2 f}{\partial z \partial y} - \frac{x}{2}\frac{\partial^2 f}{\partial^2 z} = 0.
\end{aligned}$$

Therefore $[X,Z] = [Y,Z] = 0$.
(b): Since $[X,Y] = Z$ and Z_p is not a linear combination of X_p and Y_p for any p, such submanifold S cannot exits (cf. Theorem 3.21 in the lecture notes).
(c): Yes, for example

$$M = \{(x,y,z) \in \mathbb{R}^3 \colon y = 0\}$$

is a submanifold of $\mathbb{R}^3$ and $X_p, Z_p \in T_pM$ for every $p \in M$.

***6.** Let $\mathbb{S}^3 = \{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4 \colon \sum_{i=1}^4 (x^i)^2 = 1\}$ and

$$\begin{aligned}
X &= -x^2\partial_1 + x^1\partial_2 + x^4\partial_3 - x^3\partial_4, \\
Y &= -x^3\partial_1 - x^4\partial_2 + x^1\partial_3 + x^2\partial_4, \\
Z &= -x^4\partial_1 + x^3\partial_2 - x^2\partial_3 + x^1\partial_4, \\
V &= x^1\partial_1 + x^2\partial_2 + x^3\partial_3 + x^4\partial_4,
\end{aligned}$$

where $\partial_i = \partial/\partial x^i$ for $i = 1, \ldots, 4$, are the standard coordinate vectors in $\mathbb{R}^4$. Show that:

(a) X_p, Y_p, Z_p and V_p are mutually orthogonal unit vectors of $\mathbb{R}^4$ at each point $p \in \mathbb{S}^3$.

(b) X, Y, $Z \in \mathcal{T}(\mathbb{S}^3)$.

(c) $\mathbb{S}^3$ is parallelizable.

Solution. (a): Computing the standard inner products (”dot products”) in $\mathbb{R}^4$ we immediately see that, for every $p \in \mathbb{S}^3$:

$$X_p \cdot X_p = Y_p \cdot Y_p = Z_p \cdot Z_p = V_p \cdot V_p = 1$$

and

$$X_p \cdot Y_p = X_p \cdot Z_p = X_p \cdot V_p = Y_p \cdot Z_p = Y_p \cdot V_p = Z_p \cdot V_p = 0.$$

(b): First we observe that $T_p\mathbb{S}^3$ is a 3-dimensioal vector subspace of $\mathbb{R}^4$ $(= T_p\mathbb{R}^4)$ for every $p \in \mathbb{S}^3$. Hence $\big(T_p\mathbb{S}^3\big)^\perp$, the orthogonal complement with respect to the standard inner product in $\mathbb{R}^4$, is 1-dimensional.

Let $f\colon \mathbb{R}^4 \to \mathbb{R}$ be $f(x) = \sum_{i=1}^4 (x^i)^2 - 1$, for $x = (x^1, \dots, x^4)$. By direct computation we have $X_p f = Y_p f = Z_p f = 0$ and $V_p f = 2$ for every $p \in \mathbb{S}^3$.

To prove that $X, Y, Z \in T\mathbb{S}^3$ we decompose orthogonally an arbitrary $W \in T_p\mathbb{R}^4$ into $W_p = W_p^\top + W_p^\perp$, where $W_p^\top \in T_p\mathbb{S}^3$ and $W_p^\perp \in \big(T_p\mathbb{S}^3\big)^\perp$. Since $f|\mathbb{S}^3 = 0$, we have $W_p^\top f = 0$ by Lemma 3.22 in the lecture notes. Above we noticed that $V_p f = 2$ for every $p \in \mathbb{S}^3$, so $V_p \notin T_p\mathbb{S}^3$, and therefore the $\big(T_p\mathbb{S}^3\big)^\perp$-component of V_p, $V_p^\perp$, is non-zero (in fact, soon we will see that $V_p = V_p^\perp$). Since $\big(T_p\mathbb{S}^3\big)^\perp$ is 1-dimensional, it is spanned by $V_p^\perp \neq 0$ for every $p \in \mathbb{S}^3$. Furthermore,

$$2 = V_p f = (V_p^\top + V_p^\perp) f = \underbrace{V_p^\top f}_{=0} + V_p^\perp f,$$

so $V_p^\perp f = 2$ for every $p \in \mathbb{S}^3$. Above we noticed that $X_p f = 0$, so

$$0 = X_p f = (X_p^\top + X_p^\perp) f = \underbrace{X_p^\top f}_{=0} + X_p^\perp f,$$

where $X_p^\top f = 0$ by Lemma 3.22. Hence also $X_p^\perp f = 0$. Since $X_p^\perp \in \big(T_p\mathbb{S}^3\big)^\perp$ and $\big(T_p\mathbb{S}^3\big)^\perp$ is spanned by $V_p^\perp$, we have

$$X_p^\perp = \underbrace{\lambda(p)}_{\in\mathbb{R}} V_p^\perp$$

for every $p \in \mathbb{S}^3$. Then

$$0 = X_p^\perp f = \lambda(p) V_p^\perp f = 2\lambda(p),$$

so $\lambda(p) = 0$ and consequently $X_p^\perp = 0$ for every $p \in \mathbb{S}^3$. This proves that $X_p \in T_p\mathbb{S}^3$ $\forall p \in \mathbb{S}^3$. Similarly, we can prove that $Y_p, Z_p \in T_p\mathbb{S}^3$ $\forall p \in \mathbb{S}^3$.

Finally, $X, Y, Z \in \mathcal{T}(\mathbb{S}^3)$, i.e. are smooth vector fields on $\mathbb{S}^3$, by Lemma 3.3.(b).

(c): By (a) and (b), the smooth vector fields $X, Y, Z \in \mathcal{T}(\mathbb{S}^3)$ form a global frame on $\mathbb{S}^3$, hence $\mathbb{S}^3$ is parallelizable.

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 5, Solution
21.2.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, February 21, 12:00 o'clock.

1. Let $X, Y, T \in \mathcal{T}(\mathbb{R}^3)$ be as in Example 3.12, that is

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

Define paths $\alpha_i, \beta \colon \mathbb{R} \to \mathbb{R}^3, i = 1, \ldots, 4$, as follows: For each (fixed) $t > 0$, let α_1 be the integral curve of X starting at $0 \in \mathbb{R}^3$, i.e. $\alpha_1(0) = 0$, let α_2 be the integral curve of Y starting at $\alpha_1(t)$ ($\alpha_2(0) = \alpha_1(t)$), let α_3 be the integral curve of $-X$ starting at $\alpha_2(t)$ ($\alpha_3(0) = \alpha_2(t)$), and let α_4 be the integral curve of $-Y$ starting at $\alpha_3(t)$ ($\alpha_4(0) = \alpha_3(t)$). Finally, let $\beta(t) = \alpha_4(\sqrt{t})$. (The values $t \leq 0$ are treated in an obvious way). Verify that $\dot{\beta}_0 = T (= [X, Y])$.

Solution. Fix $t > 0$. Then one can easily deduce that

$$\begin{cases} \alpha_1'(s) &= X_{\alpha_1(s)} \\ \alpha_1(0) &= (0,0,0) \end{cases} \implies \alpha_1(s) = (s, 0, 0) \implies \alpha_1(t) = (t, 0, 0);$$

$$\begin{cases} \alpha_2'(s) &= Y_{\alpha_2(s)} \\ \alpha_2(0) &= (t,0,0) \end{cases} \implies \alpha_2(s) = (t, s, ts) \implies \alpha_2(t) = (t, t, t^2);$$

$$\begin{cases} \alpha_3'(s) &= -X_{\alpha_3(s)} \\ \alpha_3(0) &= (t,t,t^2) \end{cases} \implies \alpha_3(s) = (t-s, t, t^2) \implies \alpha_3(t) = (0, t, t^2);$$

$$\begin{cases} \alpha_4'(s) &= -Y_{\alpha_4(s)} \\ \alpha_4(0) &= (0,t,t^2) \end{cases} \implies \alpha_4(s) = (0, t-s, t^2) \implies \alpha_4(t) = (0, 0, t^2).$$

Hence $\beta(t) = \alpha_4(\sqrt{t}) = (0, 0, t)$ and therefore $\dot{\beta}_0 = \beta'(0) = (0, 0, 1) = T$.

Moreover, one can check that $[X, Y] = T$. That is, given any smooth function f we have

$$\begin{aligned}
[X,Y](f) &= X(Y(f)) - Y(X(f)) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial t}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) - x\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial t} + x\frac{\partial^2 f}{\partial x \partial t} - \frac{\partial^2 f}{\partial y \partial x} - x\frac{\partial^2 f}{\partial t \partial x} \\
&= \frac{\partial f}{\partial t} = T(f).
\end{aligned}$$

2. A path $\gamma\colon \mathbb{R} \to M$ is *periodic* if there exists $T > 0$ such that $\gamma(t + kT) = \gamma(t)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let $V \in \mathcal{T}(M)$ and let γ be a maximal integral curve of V. Prove that γ has exactly one of the following properties:

(a) γ is a constant path,
(b) γ is injective,
(c) γ is periodic and non-constant.

Solution. It is clear that if the maximal integral curve has one of the listed properties it cannot have another. Hence, it is sufficient to show that if it is non-constant and non-injective, it is periodic. That is, suppose $\gamma : I \to M$ is non-constant and non-injective. Hence, there are $a, b \in I$, $a < b$, such that $\gamma(a) = \gamma(b)$. If we define $\gamma_a : I - a \to M$, $\gamma_a(t) = \gamma(t + a)$, it follows that it is a maximal integral curve of V starting at $\gamma(a)$. Similarly $\gamma_b : I - b \to M$, $\gamma_b(t) = \gamma(t+b)$ is a maximal integral curve of V starting at $\gamma(b)$. Since $\gamma(a) = \gamma(b)$ we obtain by uniqueness that $\gamma_a = \gamma_b$. In particular, $I - a = I - b$, so $I = I + a - b$ means that $I = \mathbb{R}$. Let us denote $T = b - a > 0$. Then, for all $t \in \mathbb{R}$

$$\begin{aligned}
\gamma(t) &= \gamma_a(t - a) = \gamma_b(t - a) \\
&= \gamma_b(t - a + b - b) = \gamma(t - a + b) \\
&= \gamma(t + T).
\end{aligned}$$

Finally, let us show by induction that the previous identity implies periodicity. Let

$$S = \{k \in \mathbb{N} \mid \gamma(t) = \gamma(t + kT)\ \forall t\}.$$

It is trivial that $0 \in S$. Moreover, we proved above that $1 \in S$. Now, let us suppose that $k \in S$, then

$$\gamma(t + (k+1)T) = \gamma((t + T) + kT) = \gamma(t + T) = \gamma(t),$$

so $k + 1 \in S$. Hence $\gamma(t) = \gamma(t + kT)$ for all $k \in \mathbb{N}$. But now also $\gamma(t) = \gamma(t - kT + kT) = \gamma((t - kT) + kT) = \gamma(t - kT)$ for every $k \in \mathbb{N}$, so γ is periodic.

3. Let G be a Lie group and $X \in \mathcal{T}(G)$ a left-invariant vector field. Prove that X is complete.

Solution. Let $e \in G$ denote the neutral element and $L_g : G \to G$, $h \mapsto gh$ the left multiplication. Let us recall that X being left-invariant means that $(L_g)_{*h} X_h = X_{gh}$ for all $g, h \in G$.

Given $g \in G$, let $\theta^g \colon I_g \to G$ be the maximal integral curve of X starting at g. Let us define $\gamma = L_g \circ \theta^e : I_e \to G$. First, it satisfies

$$\gamma(0) = g\theta^e(0) = g\theta(0, e) = ge = g,$$

so γ is a path starting at g. Moreover, for all $t \in I_e$ and $f \in C^\infty$, by left-invariance,

$$\begin{aligned} X_{\gamma(t)} f = X_{g\theta(t,e)} &= (L_g)_{*\theta(t,e)} X_{\theta(t,e)} f \\ &= X_{\theta(t,e)}(f \circ L_g) = \dot{\theta}^e_t (f \circ L_g) = (f \circ L_g \circ \theta^e)'(t) \\ &= (f \circ \gamma)'(t) = \dot{\gamma}_t f. \end{aligned}$$

Hence, $X_{\gamma(t)} = \dot{\gamma}_t$ and γ is an integral curve of X starting at g.

By maximality of θ^g, $I_e \subset I_g$ for all $g \in G$. It remains to show that $I_e = \mathbb{R}$. By part (b) of Theorem 3.36, $I_e = t + I_{\theta(t,e)}$, for all $t \in I_e$. Thus, since $I_e \subset I_{\theta(t,e)}$ it follows that $t + I_e \subset t + I_{\theta(t,e)} = I_e$, and so I_e can not be bounded from below or above. Since it is an interval, $I_e = \mathbb{R}$.

4. Prove that $\theta \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$,

$$\theta\big(t, (x, y)\big) = \big(\frac{1}{2}t^2 + yt + x, t + y\big),$$

is a flow and find its infinitesimal generator.

Solution. Clearly θ is a smooth mapping with flow domain $\mathbb{R} \times \mathbb{R}^2$. Let $p = (x_0, y_0) \in \mathbb{R}^2$, and $t, s \in \mathbb{R}$. It is clear that $\theta(0, p) = p$. Moreover,

$$\begin{aligned} \theta(t, \theta(s, p)) &= \theta(t, (s^2/2 + y_0 s + x_0, s + y_0)) \\ &= ((t + s)^2/2 + y_0(t + s) + x_0, (t + s) + y_0) \\ &= \theta(t + s, p), \end{aligned}$$

and hence θ is a flow on $\mathbb{R}^2$.

We know that θ^p is the integral curve of V starting at p if

$$V((\theta^p(0))^i)\,(\partial_i)_p = ((\theta^p)^i)'(0)\,(\partial_i)_p = y_0\,(\partial_x)_p + (\partial_y)_p.$$

Hence, the infinitesimal generator of θ is $V = y\,\partial_x + \partial_y$.

$*$5. Let $V \in \mathcal{T}(M)$ be a smooth compactly supported vector field, that is, the set

$$\operatorname{supp} V = \overline{\{p \in M \colon V_p \neq 0\}}$$

is compact. Prove that V is complete.

Solution. Let V be as above. By the "Fundamental Theorem of Flows", Theorem 3.37 of the lecture notes, let θ be its maximal smooth flow and given $p \in M$, $\theta^p : I_p \to M$ the corresponding maximal integral curve of V starting from p. Now, either $\theta^p(I_p) \subseteq \mathrm{suppV}$ or $\theta^p(I_p) \nsubseteq \mathrm{suppV}$. We will show that in both scenarios $I_p = \mathbb{R}$. On the one hand, if $\theta^p(I_p) \subseteq \mathrm{suppV}$, it follows by the Escape lemma, Lemma 3.47 of the lecture notes, by compactness of suppV, that $I_p = \mathbb{R}$. On the other hand, assume that $\theta^p(I_p) \nsubseteq \mathrm{suppV}$. Then, there exists $\widetilde{t} \in I_p$ such that $q = \theta^p(\widetilde{t}) \notin \mathrm{suppV}$. Let $\gamma : \mathbb{R} \to M$ be the constant path $\gamma(t) = q$. Thus, $\gamma(0) = q$ and $\dot{\gamma}(t) = 0 = V_q = V_{\gamma(t)}$. Hence, γ is the maximal integral curve of V starting from q, and by uniqueness $\gamma = \theta^q$. Furthermore, now

$$\begin{aligned} p = \theta^p(0) &= \theta(0,p) = \theta(-t, \theta(t,p)) \\ &= \theta(-t, q) = \theta^q(-t) = \gamma(-t) \\ &= q, \end{aligned}$$

so $p = q$ and, hence, $\theta^p = \theta^q = \gamma$ implying that $I_p = \mathbb{R}$. Since we have proven that $I_p = \mathbb{R}$ for all $p \in M$ we conclude that V is complete.

$*$6. Let $V \in \mathcal{T}(\mathbb{R}^2)$ be the vector field

$$V_{(x,y)} = (1+x^2)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Determine the flow $\theta \colon \mathcal{D}(V) \to \mathbb{R}^2$ of V. Is V complete?

Solution. Fix $p = (x_0, y_0) \in \mathbb{R}^2$. Let $\gamma : J_p \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$, be an integral curve of V starting from p. Then, $\dot{\gamma}_t = V_{\gamma(t)}$, or equivalently,

$$\begin{cases} x'(t) = 1 + x(t)^2, & x(0) = x_0 \\ y'(t) = 1, & y(0) = y_0. \end{cases}$$

Thus, by integration

$$\begin{cases} x(t) & = \tan(t + \arctan(x_0)) = \frac{x_0 \cos t + \sin t}{\cos t - x_0 \sin t} \\ y(t) & = t + y_0. \end{cases}$$

for all t s.t. $|t + \arctan(x_0)| < \pi/2$.

Let $J_p = (-\pi/2 - \arctan(x_0), \pi/2 - \arctan(x_0))$. Then $\gamma : J_p \to \mathbb{R}^2, \gamma(t) = \big(\tan(t + \arctan(x_0)), t + y_0\big)$ is an integral curve starting at p such that $x(t) \to -\infty$ as $t \to -\pi/2 - \arctan(x_0)$, and $x(t) \to \infty$ as $t \to \pi/2 - \arctan(x_0)$. Hence γ can not be continuously extended to a larger interval than J_p, so it is maximal, and by uniqueness $\gamma = \theta^p$. In particular, as $I_p = J_p \neq \mathbb{R}$, it follows that $\theta : \mathcal{D}(V) = \cup_{p\in\mathbb{R}^2}(I_p \times \{p\}) \to \mathbb{R}^2$, $\theta(t,(x,y)) = (\tan(t + \arctan(x)), t + y)$ is not a global flow.

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 6, Solution
28.2.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, February 28, 12:00 o'clock.

1. Let $X, Y \in \mathcal{T}(\mathbb{R}^3)$ be smooth vector fields defined by

$$X_p = \frac{\partial}{\partial x}, \quad Y_p = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad p = (x, y, z) \in \mathbb{R}^3.$$

Compute the Lie derivative $L_X Y$ directly by using the definition (3.61) in the lecture notes. Verify then that $L_X Y = [X, Y]$ as it should be.

Solution. First, it is easy to check that the flow of X starting at $p = (x_0, y_0, z_0)$ is the mapping $\theta\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$,

$$\theta\big(t, (x_0, y_0, z_0)\big) = (x_0 + t, y_0, z_0).$$

Then, $\theta_{-t}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the mapping

$$\theta_{-t}(p) = (x_0 - t, y_0, z_0)$$

Next, given $f \in C^\infty(p)$ we have

$$\begin{aligned}\theta_{(-t)*\theta(t,p)} Y_{\theta(t,p)} f &= Y_{\theta(t,p)}(f \circ \theta_{-t}) = Y_{\theta(t,p)}(f(x - t, y, z)) \\ &= \left(\left(\frac{\partial}{\partial y}\right)_{\theta(t,p)} + (x_0 + t)\left(\frac{\partial}{\partial z}\right)_{\theta(t,p)}\right) f(x - t, y, z) \\ &= \frac{\partial f}{\partial y}(x_0, y_0, z_0) + (x_0 + t)\frac{\partial f}{\partial z}(x_0, y_0, z_0).\end{aligned}$$

Hence,

$$\begin{aligned}(L_X Y)_p f &= \lim_{t\to 0} \frac{\theta_{(-t)*\theta(t,p)} Y_{\theta(t,p)} f - Y_p f}{t} \\ &= \lim_{t\to 0} \frac{t\frac{\partial}{\partial z}}{t} = \frac{\partial f}{\partial z}(x_0, y_0, t_0) = \left(\frac{\partial}{\partial z}\right)_p f,\end{aligned}$$

which means that $L_X Y = \frac{\partial}{\partial z}$.

On the other hand, we computed in Example 3.12 (lecture notes) that $[X, Y] = \partial/\partial z$. Thus, $L_X Y = [X, Y]$.

2. Let $V, W, X \in \mathcal{T}(M)$ and $h \in C^\infty(M)$. Prove that:
(a) $L_V W = -L_W V$.
(b) $L_V[W, X] = [L_V W, X] + [W, L_V X]$.
(c) $L_{[V,W]} X = L_V L_W X - L_W L_V X$.

(d) $L_V(hW) = (Vh)W + hL_VW$.
(e) If $f\colon M \to N$ is a diffeomorphism, then $f_*(L_VW) = L_{f_*V}(f_*W)$.

Solution. Rewriting the Lie derivative as the Lie bracket (Theorem 3.62) and applying Lemma 3.10 we get:

(a) Using the antisymmetry of the Lie bracket we have

$$L_VW = [V,W] = -[W,V] = -L_WV.$$

(b) From Jacobi identity and the antisymmetry of the Lie bracket we get

$$\begin{aligned} L_V[W,X] = [V,[W,X]] &= -[W,[X,V]] - [X,[V,W]] \\ &= [W,[V,X]] + [[V,W],X] \\ &= [W,L_VX] + [L_VW,X] \end{aligned}$$

(c) Again by applying the Jacobi identity and the antisymmetry of the Lie bracket we conclude

$$\begin{aligned} L_{[V,W]}X = [[V,W],X] &= [X,[W,V]] \\ &= -[W,[V,X]] - [V,[X,W]] \\ &= -[W,[V,X]] + [V,[W,X]] \\ &= -L_W[V,X] + L_V[W,X] \\ &= L_VL_WX - L_WL_VX. \end{aligned}$$

(d) From property (d) in Lemma 3.10 we deduce

$$L_V(hW) = [V,hW] = h[V,W] + (Vh)W = hL_VW + (Vh)W.$$

(e) Let us define $\widetilde{V} = f_*V$ and $\widetilde{W} = f_*W$. By Lemma 3.12 we know that for any $g \in C^\infty(N)$

$$V(g\circ f) = (\widetilde{V}g)\circ f \quad \text{and} \quad W(g\circ f) = (\widetilde{W}g)\circ f.$$

Then, given $p \in M$ we can apply the previous identities to get

$$V_p(W(g\circ f)) = V_p(\widetilde{W}(g)\circ f) = \widetilde{V}_{f(p)}(\widetilde{W}g)$$

and

$$W_p(V(g\circ f)) = W_p(\widetilde{V}(g)\circ f) = \widetilde{W}_{f(p)}(\widetilde{V}g).$$

Next, by using the definition of the tangent mapping we deduce

$$\begin{aligned} (f_{*p}[V,W]_p)\, g = [V,W]_p(g\circ f) & \\ &= V_p(W(g\circ f)) - W_p(V(g\circ f)) \\ &= \widetilde{V}_{f(p)}(\widetilde{W}g) - \widetilde{W}_{f(p)}(\widetilde{V}g) \\ &= [\widetilde{V},\widetilde{W}]_{f(p)}g = [f_*V, f_*W]_{f(p)}g. \end{aligned}$$

Finally, we arrive at

$$f_*(L_VW) = f_*[V,W] = [f_*V, f_*W] = L_{f_*V}(f_*W).$$

3. Let (U, x), $x = (x^1, \dots, x^n)$, be a chart at p and let $\partial_1, \dots, \partial_n$ be the corresponding coordinate vector fields.

(a) Prove that $dx^i_p(\partial_j) = \delta^i_j$.

(b) Prove that $df_p = (\partial_i)_p f dx^i_p$ is the differential of a function $f \in C^\infty(p)$ at p.

Solution.

(a) Since $x^i \circ x^{-1}$ is the ith component of the identity mapping we obtain

$$dx^i_p(\partial_j) = (\partial_j)_p x^i = D_j(x^i \circ x^{-1})(x(p)) = \delta^i_j.$$

(b) Given $v \in T_pM$ we know that it can be written as

$$v = \sum_{i=1}^n v^i (\partial_i)_p,$$

where $v^i = vx^i = dx^i_p v$. Then,

$$df_p v = vf = \sum_{i=1}^n (vx^i (\partial_i)_p) f = \sum_{i=1}^n (dx^i_p v)(\partial_i)_p f.$$

Hence $df_p = (\partial_i)_p f \, dx^i_p$.

4. Let $f, g \in C^\infty(M)$. Prove that:

(a) $d(af + bg) = adf + bdg$ if $a, b \in \mathbb{R}$.

(b) $d(fg) = fdg + gdf$.

(c) $d(f/g) = (gdf - fdg)/g^2$ in the set where $g \neq 0$.

(d) Let $J \subset \mathbb{R}$ be an open interval such that $f(M) \subset J$ and let $h \colon J \to \mathbb{R}$ be a smooth function. Then $d(h \circ f) = (h' \circ f)df$.

(e) If f is a constant function, then $df = 0$.

Solution.

(a) For all $p \in M$ and for all $v \in T_pM$,

$$d(af + bg)_p v = v(af + bg) = avf + bvg = adf_p v + bdg_p v.$$

Hence, $d(af + bg) = adf + bdg$.

(b) For all $p \in M$ and for all $v \in T_pM$,

$$\begin{aligned} d(fg)_p v = v(fg) &= g(p)vf + f(p)vg \\ &= g(p)df_p v + f(p)dg_p v. \end{aligned}$$

Hence, $d(fg) = fdg + gdf$.

(c) Let $p \in M$ such that $g(p) \neq 0$, then there is a neighbourhood A such that $p \in A \subset g^{-1}(\mathbb{R} \setminus \{0\})$. By apliying (b) in A we obtain

$$df = d((f/g)g) = (f/g)dg + gd(f/g).$$

Hence, $d(f/g) = (1/g)df - (f/g^2)dg = (gdf - fdg)/g^2$ in A.

(d) For all $p \in M$ and for all $v \in T_pM$, let γ be a smooth path such that $\dot{\gamma_0} = v$. Then,

$$\begin{aligned} d(h \circ f)_p v &= v(h \circ f) = \dot{\gamma}_0(h \circ f) = (h \circ f \circ \gamma)'(0) \\ &= h'(f(\gamma(0)))(f \circ \gamma)'(0) = (h' \circ f)(p)\dot{\gamma}_0 f \\ &= (h' \circ f)(p) v f = (h' \circ f)(p) df_p v \end{aligned}$$

Thus, $d(h \circ f) = (h' \circ f) df$.

(e) For all $p \in M$ and for all $v \in T_pM$, if f is constant, then

$$df_p v = vf = 0,$$

and so $df = 0$.

5. Suppose that X, $Y \in \mathcal{T}(M)$ are smooth vector fields and that θ is the flow of X and ψ the flow of Y. Suppose, furthermore, that X and Y are complete, that is $\mathcal{D}(X) = \mathcal{D}(Y) = \mathbb{R} \times M$. Prove that the following are equivalent:

(a) $[X, Y] = 0$,
(b) $(\theta_t)_* Y_p = Y_{\theta(t,p)}$ for all $(t, p) \in \mathcal{D}(X)$,
(c) $(\psi_t)_* X_p = X_{\psi(t,p)}$ for all $(t, p) \in \mathcal{D}(Y)$,
(d) $\theta_t \circ \psi_s = \psi_s \circ \theta_t$.

Solution.

(b)(c) $\Rightarrow$ (a)

If (b) holds, then $(\theta_{-t})_* Y_{\theta(t,p)} = Y_p$. Thus, we get from the definition of Lie derivative that

$$[X, Y]_p = \big(L_X Y\big)_p = \lim_{t \to 0} \frac{(\theta_{-t})_* Y_{\theta(t,p)} - Y_p}{t} = 0.$$

The same reasoning applied to (c) yields $[Y, X]_p = 0$.

(d) $\Rightarrow$ (b)(c)

Consider the path

$$\gamma : t \mapsto \theta_t \circ \psi_s(p).$$

On the one hand we have

$$\dot{\gamma}_0 = \dot{\theta}_0^{\psi_s(p)} = X_{\psi_s(p)}.$$

On the other hand, using (d) we have alternatively that $\gamma(t) = \psi_s \circ \theta_t(p)$. Hence

$$\dot{\gamma}_0 = \psi_{s*} \dot{\theta}_0^p = \psi_{s*} X_p.$$

Thus we get (c). The same reasoning applied to the path

$$\sigma \colon s \mapsto \psi_s \circ \theta_t(p)$$

yields (b).

(b) $\Rightarrow$ (d)

Let us take $t \in \mathbb{R}$ and consider the flow

$$\sigma(s, p) = \theta_{-t} \circ \psi_s \circ \theta_t(p),$$

whose infinitesimal generator is

$$\dot{\sigma}_0^p = (\theta_{-t})_* \dot{\psi}_0^{\theta_t(p)} = (\theta_{-t})_* Y_{\theta_t(p)}.$$

Next, using (b) we get

$$\dot{\sigma}_0^p = (\theta_{-t})_* (\theta_t)_* Y_p = Y_p$$

which, by the uniqueness of flows, means that $\sigma = \psi$. This proves (d).

$\underline{\text{(a)} \Rightarrow \text{(b)}}$

Let $p \in M$ and $V : \mathbb{R} \to T_pM$, $V(t) = (\theta_{-t})_* Y_{\theta(t,p)}$. Since $V(0) = Y_p$ it is enough to show that $V(t)$ is constant. That is, let $t_0 \in \mathbb{R}$ and $s = t - t_0$, then

$$\begin{aligned} V'(t_0) &= \frac{d}{dt}\big((\theta_{-t})_* Y_{\theta(t,p)}\big)_{|t=t_0} = \frac{d}{ds}\big((\theta_{-t_0-s})_* Y_{\theta(s+t_0,p)}\big)_{|s=0} \\ &= (\theta_{-t_0})_* \frac{d}{ds}\big((\theta_{-s})_* Y_{\theta(s,\theta(t_0,p))}\big)_{|s=0} \\ &= (\theta_{-t_0})_* (L_X Y)_{\theta(t_0,p)} = 0. \end{aligned}$$

Thus (b) follows.

$*$6. Prove Lemma 5.5 in the lecture notes: If $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^\ell(V)$, then

$$\alpha \wedge \beta(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in Sh(k,\ell)} (\operatorname{sgn} \sigma) \alpha\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) \beta\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big)$$

for every $v_1, \ldots, v_{k+\ell} \in V$.

Solution. Let us define the equivalence relation $\sim$ on $S_{k,l}$ by setting $\sigma \sim \tilde{\sigma}$ if

$$\{\sigma(1), ..., \sigma(k)\} = \{\tilde{\sigma}(1), ..., \tilde{\sigma}(k)\}$$

and

$$\{\sigma(k+1), ..., \sigma(k+l)\} = \{\tilde{\sigma}(k+1), ..., \tilde{\sigma}(k+l)\}.$$

It is clear that the number of elements in each class is $k!l!$. Moreover, there exists a unique shuffle in each class. Thus, we can take them as the representatives of the classes.

Next, we define $h : S_{k,l} \to \mathbb{R}$ as

$$h(\sigma) = \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

Let us check that $h(\sigma) = h(\tilde{\sigma})$ if $\sigma \sim \tilde{\sigma}$. That is, given $\sigma \sim \tilde{\sigma}$ we know that there exits $\pi, \tau \in S_{k,l}$ such that $\tilde{\sigma} = \pi\tau\sigma$ with

$$\tau(\sigma(i)) = \sigma(i) \quad \forall i \in \{1, ..., k\}$$

and

$$\pi(\sigma(i)) = \pi(i) \quad \forall i \in \{k+1, ..., k+l\}.$$

Then,

$$\begin{aligned} h(\sigma') &= \operatorname{sgn}(\sigma')\alpha(v_{\sigma'(1)},\dots,v_{\sigma'(k)})\beta(v_{\sigma'(k+1)}\dots,v_{\sigma'(k+l)}) \\ &= \operatorname{sgn}(\pi\tau\sigma)\alpha(v_{\pi\sigma(1)},\dots,v_{\pi\sigma(k)})\beta(v_{\tau\sigma(k+1)}\dots,v_{\tau\sigma(k+l)}) \\ &= \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})\beta(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}) \\ &= h(\sigma). \end{aligned}$$

Here we are using the fact that $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^\ell(V)$ are alternating as well as the properties of the sign function.

Finally we obtain

$$\begin{aligned} &\alpha\wedge\beta(v_1,\dots,v_{k+l}) \\ &\quad= \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})\beta(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}) \\ &\quad= \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}h(\sigma) = \frac{1}{k!l!}\sum_{\tilde\sigma\in S_{k+l}}\sum_{[\sigma]=[\tilde\sigma]}h(\sigma) \\ &\quad= \frac{1}{k!l!}\sum_{\tilde\sigma\in S_{k+l}}\sum_{[\sigma]=[\tilde\sigma]}h(\tilde\sigma) \\ &\quad= \frac{1}{k!l!}\sum_{\tilde\sigma\in S_{k+l}}h(\tilde\sigma)\left|\{\sigma\in S_{k,l} : [\sigma]=[\tilde\sigma]\}\right| = \sum_{\tilde\sigma\in S_{k+l}}h(\tilde\sigma) \\ &\quad= \sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\tilde\sigma)\alpha(v_{\tilde\sigma(1)},\dots,v_{\tilde\sigma(k)})\beta(v_{\tilde\sigma(k+1)},\dots,v_{\tilde\sigma(k+l)}). \end{aligned}$$

*7. Let M be a smooth manifold and let $g \in \mathcal{T}^2(M)$ be a Riemannian metric tensor field; see Example 4.18 in the lecture notes. The *gradient* of a smooth function $f \in C^\infty(U)$, $U \subset M$ open, is the vector field $\nabla f \in \mathcal{T}(U)$ defined by

$$g_p(\nabla f(p), X_p) = X_p f$$

for every $X_p \in T_pM$ and for every $p \in U$. Show that, for any $p \in U$, among all unit vectors $X_p \in T_pM$, the directional derivative X_pf is the greatest when X_p is parallel to $\nabla f(p)$ and the norm of $\nabla f(p)$ is equal to the value of the directional derivative in that direction.

Solution. Let us take $p \in U \subset M$. We distinguish two cases:

- If $\nabla f(p) = 0$, then for any $X_p \in T_pM$

  $$X_pf = g_p(\nabla f(p), X_p) = 0,$$

  and both claims follow.
- If $\nabla f(p) \neq 0$, let us take the unitary vector parallel to $\nabla f(p)$

  $$Y_p = \frac{\nabla f(p)}{|\nabla f(p)|}.$$

On the one hand we have

$$Y_pf = g_p(\nabla f(p), Y_p) = g_p\left(\nabla f(p), \frac{\nabla f(p)}{|\nabla f(p)|}\right) = |\nabla f(p)|.$$

On the other hand, for any unitary vector $X_p \in T_pM$ we obtain by Cauchy-Schwarz inequality that

$$X_pf = g_p(\nabla f(p), X_p) \leq |X_p||\nabla f(p)| = |\nabla f(p)|.$$

Thus, both claims are proved.

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 7, Solution
14.3.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, March 14, 12:00 o'clock.

1. Let V be a finite dimensional vector space.

(a) Prove that a sequence (α_j) in $T^k(V)$ converges to $\alpha \in T^k(V)$ (with respect to the topology induced by any norm of $T^k(V)$) if and only if

$$\alpha_j(v_1, \dots, v_k) \to \alpha(v_1, \dots, v_k)$$

for all $v_1, \dots, v_k \in V$.

(b) Prove that the wedge product is continuous in the following sense: if $\alpha_j \to \alpha$ in $T^k(V)$ and $\beta_j \to \beta$ in $T^l(V)$, then $\alpha_j \wedge \beta_j \to \alpha \wedge \beta$ in $T^{k+l}(V)$.

Solution.

(*a*) First, note that without loss of generality we can assume that $\alpha = 0$. Let $(e_1, \dots, , e_n)$ be a basis for V and $(\varepsilon^1, \dots, \varepsilon^n)$ the dual basis. Then, given any $\nu \in T^k(V)$, it can be written as

$$\nu = \sum_{1 \le i_j \le n} \nu_{i_1, \dots, i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

where $\nu_{i_1, \dots, i_k} = \nu(e_{i_1}, \dots, e_{i_k})$. We set $\|\nu\| = \sum_{1 \le i_j \le n} |\nu_{i_1, \dots, i_k}|$, which clearly defines a norm in $T^k(V)$.

On the one hand, suppose that $\alpha_j \to 0$, i.e., $\|\alpha_j\| \to 0$. It means that $\alpha_{j\,i_1, \dots, i_k} = \alpha_j(e_{i_1}, \dots, e_{i_k}) \to 0$. Hence, by multilinearity, given any $v_1, \dots, v_k \in V$ we conclude that $\alpha_j(v_1, \dots, v_k) \to 0$.

On the other hand, if $\alpha_j(v_1, \dots, v_k) \to 0$ for all $v_1, \dots, v_k \in V$, then in particular $\alpha_{j_{i_1, \dots, i_k}} = \alpha_j(e_{i_1}, \dots, e_{i_k}) \to 0$. Thus, $\|\alpha_j\| \to 0$.

(*b*) Given $v_1, \dots, v_{k+l} \in V$,

$$\begin{aligned}
&(\alpha_j \wedge \beta_j)(v_1, \dots, v_{k+l}) \\
&\quad = \sum_{\sigma \in Sh(k,l)} (\mathrm{sgn}\sigma) \alpha_j(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta_j(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\
&\quad \to \sum_{\sigma \in Sh(k,l)} (\mathrm{sgn}\sigma) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\
&\quad = (\alpha \wedge \beta)(v_1, \dots, v_{k+l}),
\end{aligned}$$

by (a). This means that $\alpha_j \wedge \beta_j \to \alpha \wedge \beta$.

2. Prove Lemma 4.8 in the lecture notes: If $(v_1, \dots, v_n)$ is a basis for V and $(\omega^1, \dots, \omega^n)$ the corresponding dual basis for V^* $(\omega^i(v_j) = \delta^i_j)$, then tensors

$$\omega^{i_1} \otimes \cdots \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_l}, \quad 1 \le j_p, i_q \le n,$$

form a basis for $T^k_l(V)$. In particular, $\dim T^k_l(V) = n^{k+l}$.

Solution. Denote

$$\mathcal{B} = \{\omega^{i_1} \otimes \cdots \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_l} : 1 \le j_p, i_q \le n\}.$$

First we claim that $\mathcal{B}$ spans $T^k_\ell(V)$. Suppose that $T \in T^k_\ell(V)$, then for every $(k+\ell)$-tuple $(i_1, \dots, i_k, j_1, \dots, j_\ell)$ we define

$$T^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} = T(v_{i_1}, \dots, v_{i_k}, \omega^{j_1}, \dots, \omega^{j_\ell}).$$

We claim

$$T = \sum_{1 \le j_p, i_q \le n} T^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} \, \omega^{i_1} \otimes \cdots \otimes \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_\ell}.$$

For each $(k+\ell)$-tuples $(v_{\alpha_1}, \dots, v_{\alpha_k}, \omega^{\beta_1}, \dots, \omega^{\beta_\ell})$ we have

$$\begin{aligned}
&\sum_{1 \le j_p, i_q \le n} T^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} \omega^{i_1} \otimes \cdots \otimes \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_\ell}(v_{\alpha_1}, \dots, v_{\alpha_k}, \omega^{\beta_1}, \dots, \omega^{\beta_\ell}) \\
&\quad = \sum_{1 \le j_p, i_q \le n} T^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} \omega^{i_1}(v_{\alpha_1}) \cdots \omega^{i_k}(v_{\alpha_k}) v_{j_1}(\omega^{\beta_1}) \cdots v_{j_\ell}(\omega^{\beta_\ell}) \\
&\quad = \sum_{1 \le j_p, i_q \le n} T^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} \delta^{i_1}_{\alpha_1} \cdots \delta^{i_k}_{\alpha_k} \delta^{\beta_1}_{j_1} \cdots \delta^{\beta_\ell}_{j_\ell} \\
&\quad = T^{\beta_1 \cdots \beta_\ell}_{\alpha_1 \cdots \alpha_k} \\
&\quad = T(v_{\alpha_1}, \dots, v_{\alpha_k}, \omega^{\beta_1}, \dots, \omega^{\beta_\ell}),
\end{aligned}$$

so the claim follows from multilinearity since a (k,ℓ)-tensor is determined by its action on $(k+\ell)$-tuples of basis vectors.

Next, to prove that $\mathcal{B}$ is independent suppose that there exist coefficients $a^{j_1 \cdots j_\ell}_{i_1 \cdots i_k}$ such that

$$\sum_{1 \le j_p, i_q \le n} a^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} \, \omega^{i_1} \otimes \cdots \otimes \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_\ell} = 0.$$

Applying this to (any) $(v_{\alpha_1}, \dots, v_{\alpha_k}, \omega^{\beta_1}, \dots, \omega^{\beta_\ell})$ gives $a^{\beta_1 \cdots \beta_\ell}_{\alpha_1 \cdots \alpha_k} = 0$, i.e. every coefficient is zero and $\mathcal{B}$ is independent.

3. Let M and N be smooth manifolds, $f\colon M \to N$ and $h\colon N \to \mathbb{R}$ smooth mappings, $\sigma \in \mathcal{T}^k(N)$, and $\tau \in \mathcal{T}^l(N)$. Prove that:
(a) $f^*dh = d(h \circ f)$.
(b) $f^*(\sigma \otimes \tau) = f^*\sigma \otimes f^*\tau$.

Solution.

(a) Given any $v \in T_pM$, by using the definition of the pull-back, the differential and the tangent map we get

$$f^*dh(v) = dh(f_*v) = f_*v(h) = v(h \circ f) = d(h \circ f)(v).$$

(b) Again, for any $v_1, \cdots, v_{k+l} \in T_pM$, we can use the definition of the pull-back and the tensor product to show

$$\begin{aligned} f^*(\sigma \otimes \tau)(v_1, \ldots, v_{k+l}) &= \sigma \otimes \tau(f_*v_1, \ldots, f_*v_{k+l}) \\ &= \sigma(f_*v_1, \ldots, f_*v_k)\tau(f_*v_{k+1}, \ldots, f_*v_{k+l}) \\ &= f^*\sigma(v_1, \ldots, v_k) f^*\tau(v_{k+1}, \ldots, v_{k+l}) \\ &= f^*\sigma \otimes f^*\tau(v_1, \ldots, v_{k+l}) \end{aligned}$$

4. Let $f\colon M \to N$ be a smooth mapping and $\sigma \in \mathcal{T}^k(N)$. Show that $f^*\sigma \in \mathcal{T}^k(M)$, that is, a smooth k-covariant tensor field on M.

Solution. Let us take $p \in M$ and let (V, y^i) be a chart (of N) around $f(p)$. We know that σ can be written in local coordinates as

$$\sigma = \sum_{1 \le i_q \le n} \sigma_{i_1 \cdots i_k}\, dy^{i_1} \otimes \cdots \otimes dy^{i_k}.$$

Then, by the previous exercise

$$f^*\sigma = \sum_{1 \le i_q \le n} (\sigma_{i_1 \cdots i_k} \circ f)\, d(y^{i_1} \circ f) \otimes \cdots \otimes d(y^{i_k} \circ f).$$

Since f is smooth, the differentials $p \mapsto d(y^{i_j} \circ f)_p$, $1 \le j \le k$, are smooth. One way to see this is to use Lemma 4.11 from the lecture notes: if X is a vector field defined on an open set of V Then $d(y^{i_j} \circ f)_p X_p = X_p(y^{i_j} \circ f)$ is a smooth map. Hence $f^*\sigma$ is smooth.

5. Let $M = \mathbb{R}^3$. Determine which of the following differential forms are closed and which are exact.

(a) $\alpha = yzdx + xzdy + xydz$.
(b) $\beta = xdx + x^2y^2dy + yzdz$.
(c) $\eta = 2xy^2dx \wedge dy + zdy \wedge dz$.

Solution.

(a) We see that $\alpha = d(xyz)$ so it is both closed and exact.

(b) Computing

$$\begin{aligned} d\beta &= dx \wedge dx + d(x^2y^2) \wedge dy + d(yz) \wedge dz \\ &= (2xy^2dx + 2yx^2dy) \wedge dy + (zdy + ydz) \wedge dz \\ &= 2xy^2dx \wedge dy + zdy \wedge dz \end{aligned}$$

we see that β is not closed. Therefore it cannot be exact either.

(c) From the previous computations we have $\eta = d\beta$ so that η is exact and thus closed.

6. Let $\sigma \in \mathcal{T}^k(M)$ and $Y_1, \ldots, Y_k \in \mathcal{T}(M)$. Prove that

$$\begin{aligned} L_X\big(\sigma(Y_1, \ldots, Y_k)\big) = \big(L_X\sigma\big)(Y_1, \ldots, Y_k) + \sigma(L_X Y_1, Y_2, \ldots, Y_k) + \\ \cdots + \sigma(Y_1, \ldots, Y_{k-1}, L_X Y_k). \end{aligned}$$

Prove the claim in detail in case $k = 2$ and then discuss the general case.

Solution. Let $\sigma \in \mathcal{T}^k(M)$ and $X, Y, Z \in \mathcal{T}(M)$. Note that $p \mapsto \sigma_p(Y_p, Z_p)$ and $p \mapsto \big(L_X(Y, Z)\big)_p$ are C^∞-functions. If θ is the flow of X, then

$$\begin{aligned}
\big(L_X\big(\sigma(Y, Z)\big)\big)_p &= \lim_{t \to 0} \frac{1}{t}\left[\big(\theta_t^*\big(\sigma(Y, Z)\big)\big)_p - \big(\sigma(Y, Z)\big)_p\right] \\
&= \lim_{t \to 0} \frac{1}{t}\left[\big(\sigma(Y, Z)\big) \circ \theta_t^p - \big(\sigma(Y, Z)\big)_p\right] \\
&= \lim_{t \to 0} \frac{1}{t}\left[\sigma_{\theta_t^p}\big(Y_{\theta_t^p}, Z_{\theta_t^p}\big) - \sigma_p(Y_p, Z_p)\right] \\
&= \lim_{t \to 0} \frac{1}{t}\left[\theta_t^*\sigma\big(\theta_{-t*}Y_{\theta_t^p}, \theta_{-t*}Z_{\theta_t^p}\big) - \sigma_p(Y_p, Z_p)\right] \\
&= \lim_{t \to 0} \frac{1}{t}\Big[\theta_t^*\sigma\big(\theta_{-t*}Y_{\theta_t^p}, \theta_{-t*}Z_{\theta_t^p}\big) - \sigma_p\big(\theta_{-t*}Y_{\theta_t^p}, \theta_{-t*}Z_{\theta_t^p}\big) \\
&\quad + \sigma_p\big(\theta_{-t*}Y_{\theta_t^p}, \theta_{-t*}Z_{\theta_t^p}\big) - \sigma_p\big(Y_p, \theta_{-t*}Z_{\theta_t^p}\big) \\
&\quad + \sigma_p\big(Y_p, \theta_{-t*}Z_{\theta_t^p}\big) - \sigma_p(Y_p, Z_p)\Big] \\
&= \lim_{t \to 0} \frac{1}{t}\Big[\theta_t^*\sigma\big(\theta_{-t*}Y_{\theta_t^p}, \theta_{-t*}Z_{\theta_t^p}\big) - \sigma_p\big(\theta_{-t*}Y_{\theta_t^p}, \theta_{-t*}Z_{\theta_t^p}\big)\Big] \\
&\quad + \lim_{t \to 0} \frac{1}{t}\Big[\sigma_p\big(\theta_{-t*}Y_{\theta_t^p}, \theta_{-t*}Z_{\theta_t^p}\big) - \sigma_p\big(Y_p, \theta_{-t*}Z_{\theta_t^p}\big)\Big] \\
&\quad + \lim_{t \to 0} \frac{1}{t}\Big[\sigma_p\big(Y_p, \theta_{-t*}Z_{\theta_t^p}\big) - \sigma_p(Y_p, Z_p)\Big] \\
&= \big(L_X\sigma\big)_p(Y_p, Z_p) + \sigma_p\big((L_X Y)_p, Z_p\big) + \sigma_p\big(Y_p, (L_X z)_p\big).
\end{aligned}$$

We can obtain the general case in a similar way by adding and subtracting terms corresponding to each variable Y_i.

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 8, Solution
21.3.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, March 21, 12:00 o'clock.

1. Prove Lemma 5.13 of the lecture notes:
If $f\colon M \to N$ is smooth, then:
(a) $f^*\colon \mathcal{A}^k(N) \to \mathcal{A}^k(M)$ is linear.
(b) $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$.
(c) If (U, y), $y = (y^1, \dots, y^n)$, is a chart in N, then

$$f^*\Big(\sum_{i_1<\dots<i_k} \omega_{i_1\cdots i_k} dy^{i_1} \wedge \dots \wedge dy^{i_k}\Big) = \sum_{i_1<\dots<i_k} \big(\omega_{i_1\cdots i_k} \circ f\big) d\big(y^{i_1} \circ f\big) \wedge \dots \wedge d\big(y^{i_k} \circ f\big).$$

Solution.
(a) First, we know from the previous lists of exercises that the pullback $f^*\colon \mathcal{A}^k(N) \to \mathcal{A}^k(M)$. Next, given $a, b \in \mathbb{R}$ and $\alpha, \beta \in \mathcal{A}^k(N)$, it satisfies

$$\begin{aligned}(f^*(a\alpha + b\beta))_p(v_1, \dots, v_k) &= (a\alpha + b\beta)_{f(p)}(f_*v_1, \dots, f_*v_k)\\ &= a\alpha_{f(p)}(f_*v_1, \dots, f_*v_k) + b\beta(f_*v_1, \dots, f_*v_k)\\ &= a(f^*\alpha)_p(v_1, \dots, v_k) + b(f^*\beta)(v_1, \dots, v_k)\\ &= (af^*\alpha + bf^*\beta)_p(v_1, \dots, v_k)\end{aligned}$$

for all $p \in M$ and $v_1, \dots, v_k \in T_pM$. Therefore, f^* is linear.

(b) Given $p \in M$ and $v_1, \dots, v_{k+l} \in T_pM$, and using the definitions of pullback, wedge product and tensor product, we obtain

$$\begin{aligned}&(f^*(\alpha \wedge \beta))_p(v_1, \dots, v_{k+l}) = (\alpha \wedge \beta)_{f(p)}(f_*v_1, \dots, f_*v_{k+l})\\ &= \sum_{\sigma \in Sh(k,l)} (\operatorname{sgn}\sigma)(\alpha_{f(p)} \otimes \beta_{f(p)})(f_*v_{\sigma(1)}, \dots, f_*v_{\sigma(k+l)})\\ &= \sum_{\sigma \in Sh(k,l)} (\operatorname{sgn}\sigma)\, \alpha_{f(p)}(f_*v_{\sigma(1)}, \dots, f_*v_{\sigma(k)})\, \beta_{f(p)}(f_*v_{\sigma(k+1)}, \dots, f_*v_{\sigma(k+l)})\\ &= \sum_{\sigma \in Sh(k,l)} (\operatorname{sgn}\sigma)\, (f^*\alpha)_p(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, (f^*\beta)_p(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})\\ &= \sum_{\sigma \in Sh(k,l)} (\operatorname{sgn}\sigma)(f^*\alpha \otimes f^*\beta)_p(v_{\sigma(1)}, \dots, v_{\sigma(k+l)})\\ &= (f^*\alpha \wedge f^*\beta)_p(v_{\sigma(1)}, \dots, v_{\sigma(k+l)}).\end{aligned}$$

(c) First, observe that

$$\begin{aligned} f^*\left(\omega_{i_1\dots i_k} dy^{i_1}\wedge\cdots\wedge dy^{i_k}\right) &= (\omega_{i_1\dots i_k}\circ f) f^*(dy^{i_1}\wedge\cdots\wedge dy^{i_k}) \\ &= (\omega_{i_1\dots i_k}\circ f)(f^*dy^{i_1})\wedge\cdots\wedge(f^*dy^{i_k}) \\ &= (\omega_{i_1\dots i_k}\circ f) d(y^{i_1}\circ f)\wedge\cdots\wedge d(y^{i_k}\circ f), \end{aligned}$$

where the first equality follows by Lemma 4.15(b) of the lecture notes, the second by (b) above, and the last by Exercise 7/3 (a). The result now follows by linearity of the pullback.

2. (a) Let $\omega^1,\omega^2,\dots,\omega^5\in\Lambda^1(\mathbb{R}^5)$,

$$\alpha = 2\omega^1\wedge\omega^3+\omega^2\wedge\omega^3-3\omega^3\wedge\omega^4\in\Lambda^2(\mathbb{R}^5),$$

and

$$\beta = -\omega^1\wedge\omega^2\wedge\omega^5+2\omega^1\wedge\omega^3\wedge\omega^4\in\Lambda^3(\mathbb{R}^5).$$

Compute $\alpha\wedge\beta$.

(b) Let $\alpha = dx - xdy\in\mathcal{A}^1(\mathbb{R}^3)$ and $\beta = ydx\wedge dz - dy\wedge dz\in\mathcal{A}^2(\mathbb{R}^3)$ Compute $\alpha\wedge\beta$.

Solution.

(a) By Theorem 5.6 (c) of the lecture notes $\omega^i\wedge\omega^j = -\omega^j\wedge\omega^i$ for $1\le i,j\le 5$. In particular $\omega^i\wedge\omega^i = 0$. Together with the fact that $\wedge$ is bilinear and associative this gives

$$\begin{aligned} \alpha\wedge\beta &= 3\omega^3\wedge\omega^4\wedge\omega^1\wedge\omega^2\wedge\omega^5 = 3\omega^1\wedge\omega^3\wedge\omega^4\wedge\omega^2\wedge\omega^5 \\ &= 3\omega^1\wedge\omega^2\wedge\omega^3\wedge\omega^4\wedge\omega^5. \end{aligned}$$

(b) Reasoning as before, '

$$\begin{aligned} \alpha\wedge\beta &= (dx - xdy)\wedge(ydx\wedge dz - dy\wedge dz) \\ &= -dx\wedge dy\wedge dz - xy\ dy\wedge dx\wedge dz \\ &= (xy-1)\ dx\wedge dy\wedge dz. \end{aligned}$$

3. Let $f\colon\mathbb{R}^3\to\mathbb{R}^2$, $f(x,y,z) = (x^2,yz)$, $\alpha = y^2\,dx+dy\in\mathcal{A}^1(\mathbb{R}^2)$, and $\beta = xy\,dx\wedge dy\in\mathcal{A}^2(\mathbb{R}^2)$. Compute $\alpha\wedge\beta$, $f^*\alpha$, $f^*\beta$, and $f^*(\alpha\wedge\beta)$.

Solution. As usual, let $(x,y)\mapsto(x,y)$ denote the standard coordinates on $\mathbb{R}^2$. Since $\alpha\wedge\beta\in\mathcal{A}^3(\mathbb{R}^2) = \{0\}$, $\alpha\wedge\beta = 0$, and so $f^*(\alpha\wedge\beta) = f^*0 = 0$ by linearity of the pullback. By Exercise 1 (c),

$$\begin{aligned} f^*\alpha &= f^*(y^2dx)+f^*(dy) = (y^2\circ f)d(x\circ f)+d(y\circ f) \\ &= y^2z^2d(x^2)+d(yz) = 2xy^2z^2dx+zdy+ydz. \end{aligned}$$

Similarly,

$$\begin{aligned} f^*\beta &= f^*(xydx\wedge dy) = (xy\circ f)d(x\circ f)\wedge d(y\circ f) \\ &= x^2yz\,d(x^2)\wedge d(yz) = 2x^3y^2zdx\wedge dz+2x^3yz^2dx\wedge dy. \end{aligned}$$

Note that one can also check directly that $f^*(\alpha)\wedge f^*(\beta) = 0$.

4. Let $\alpha \in \mathcal{A}^1(\mathbb{R}^2)$,

$$\alpha = f(x,y)dx + g(x,y)dy.$$

Compute $d\alpha$ and verify the connection between the Green's theorem (in vector calculus) and the Stokes' theorem.

Solution. First, we have

$$\begin{aligned} d\alpha &= d(fdx) + d(gdy) = df \wedge dx + dg \wedge dy \\ &= (\partial_x f dx + \partial_y f dy) \wedge dx + (\partial_x g dx + \partial_y g dy) \wedge dy \\ &= \partial_y f\, dy \wedge dx + \partial_x g\, dx \wedge dy = (\partial_x g - \partial_y f)\ dx \wedge dy \end{aligned}$$

Given M a smooth 2-manifold, Stokes' theorem implies

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

In the particular case $\alpha = f(x,y)dx + g(x,y)dy$ we get

$$\int_M (\partial_x g - \partial_y f)\ dx \wedge dy = \int_{\partial M} f(x,y)dx + g(x,y)dy.$$

Hence, if we take $M = \Omega \subset \mathbb{R}^2$ a smooth bounded domain we recover Green's Theorem in vector calculus.

5. Let $\omega \in \mathcal{A}^2(\mathbb{R}^3)$,

$$\omega = xdy \wedge dz + ydz \wedge dx + zdz \wedge dy.$$

Write ω in spherical coordinates (ρ, φ, θ),

$$\begin{cases} x = \rho \sin\varphi \cos\theta, \\ y = \rho \sin\varphi \sin\theta, \\ z = \rho \cos\theta. \end{cases}$$

Solution. First, let us compute dx, dy and dz. That is,

$$dx = \sin\varphi\cos\theta\ d\rho + \rho\cos\varphi\cos\theta\ d\varphi - \rho\sin\varphi\sin\theta\ d\theta,$$

$$dy = \sin\varphi\sin\theta\ d\rho + \rho\cos\varphi\sin\theta\ d\varphi + \rho\sin\varphi\cos\theta\ d\theta$$

and

$$dz = \cos\varphi\ d\rho - \rho\sin\varphi\ d\varphi.$$

Next, let us compute $dy \wedge dz$, $dz \wedge dx$ and $dx \wedge dy$

$$\begin{aligned} dy \wedge dz = &-\rho\cos^2\varphi\sin\theta\ d\rho \wedge d\varphi + \rho\cos^2\varphi\sin\theta\ d\varphi \wedge d\rho \\ &+ \rho\sin\varphi\cos\varphi\cos\theta\ d\theta \wedge d\rho - \rho^2\sin^2\varphi\cos\theta\ d\theta \wedge d\varphi \\ = &-\rho\sin\theta\ d\rho \wedge d\varphi + \rho^2\sin^2\varphi\cos\theta\ d\varphi \wedge d\theta \\ &+ \rho\sin\varphi\cos\varphi\cos\theta\ d\theta \wedge d\rho, \end{aligned}$$

$$\begin{aligned} dz \wedge dx &= \rho \cos^2 \varphi \cos \theta \ d\rho \wedge d\varphi - \rho \sin \varphi \cos \varphi \sin \theta \ d\rho \wedge d\theta \\ &\quad - \rho \sin^2 \varphi \cos \theta \ d\varphi \wedge d\rho + \rho^2 \sin^2 \varphi \sin \theta \ d\varphi \wedge d\theta \\ &= \rho \cos \theta \ d\rho \wedge d\varphi + \rho^2 \sin^2 \varphi \sin \theta \ d\varphi \wedge d\theta \\ &\quad + \rho \sin \varphi \cos \varphi \sin \theta \ d\theta \wedge d\rho \end{aligned}$$

and

$$\begin{aligned} dx \wedge dy &= \rho \cos \varphi \sin \varphi \cos \theta \sin \theta \ d\rho \wedge d\varphi + \rho \sin^2 \varphi \cos^2 \theta \ d\rho \wedge d\theta \\ &\quad + \rho \cos \varphi \sin \varphi \cos \theta \sin \theta \ d\varphi \wedge d\rho + \rho^2 \sin \varphi \cos \varphi \cos^2 \theta \ d\varphi \wedge d\theta \\ &\quad - \rho \sin^2 \varphi \sin^2 \theta \ d\theta \wedge d\rho - \rho^2 \sin \varphi \cos \varphi \sin^2 \theta \ d\theta \wedge d\varphi \\ &= \rho^2 \sin \varphi \cos \varphi \ d\varphi \wedge d\theta - \rho \sin^2 \varphi \ d\theta \wedge d\rho. \end{aligned}$$

Hence,

$$\begin{aligned} \omega &= x dy \wedge dz + y dz \wedge dx + z dz \wedge dy \\ &= \left(-\rho^2 \sin \varphi \sin \theta \cos \theta + \rho^2 \sin \varphi \sin \theta \cos \theta\right) d\rho \wedge d\varphi \\ &\quad + \left(\rho^3 \sin^3 \varphi \cos^2 \theta + \rho^3 \sin^3 \varphi \sin^2 \theta + \rho^3 \sin \varphi \cos^2 \varphi\right) d\varphi \wedge d\theta \\ &\quad + \left(\rho^2 \sin^2 \varphi \cos \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \cos \varphi \sin^2 \theta - \rho^2 \sin^2 \varphi \cos \varphi\right) d\theta \wedge d\rho \\ &= \rho^3 \sin \varphi \ d\varphi \wedge d\theta. \end{aligned}$$

6. Let $X \in \mathcal{T}(M)$ be a smooth vector field and let σ and τ be smooth covariant tensors fields. Prove that

$$L_X(\sigma \otimes \tau) = (L_X \sigma) \otimes \tau + \sigma \otimes (L_X \tau).$$

Solution. Let $p \in M$ and denote the flow of X by θ. Then,

$$\begin{aligned} (L_X(\sigma \otimes \tau))_p &= \lim_{t \to 0} \frac{(\theta_t^*(\sigma \otimes \tau))_p - (\sigma \otimes \tau)_p}{t} \\ &= \lim_{t \to 0} \frac{(\theta_t^* \sigma)_p \otimes (\theta_t^* \tau)_p - \sigma_p \otimes \tau_p}{t} \\ &= \lim_{t \to 0} \frac{((\theta_t^* \sigma)_p - \sigma_p) \otimes (\theta_t^* \tau)_p + \sigma_p \otimes ((\theta_t^* \tau)_p - \tau_p)}{t} \\ &= \lim_{t \to 0} \left(\frac{(\theta_t^* \sigma)_p - \sigma_p)}{t} \otimes (\theta_t^* \tau)_p \right) + \lim_{t \to 0} \left(\sigma_p \otimes \frac{((\theta_t^* \tau)_p - \tau_p)}{t} \right) \\ &= (L_X \sigma)_p \otimes \tau_p + \sigma_p \otimes (L_X \tau)_p. \end{aligned}$$

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 9, Solution
28.3.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, March 28, 12:00 o'clock.

1. Let $X, Y \in \mathcal{T}(M)$ and $\omega \in \mathcal{A}^k(M)$. Prove, by using the definition of Lie derivative, that

$$L_X(i_Y\omega) = i_{(L_XY)}\omega + i_Y(L_X\omega).$$

Solution. Let us use the other notation $Y \lrcorner\, \omega = i_Y\omega$ for the contraction. Let θ be the flow of X. By direct computation we have

$$\begin{aligned}
L_X(Y \lrcorner\, \omega) &= \lim_{t\to 0} \frac{1}{t}\Big(\theta_t^*(Y \lrcorner\, \omega) - Y \lrcorner\, \omega\Big) \\
&= \lim_{t\to 0} \frac{1}{t}\Big(\theta_{-t*}Y \lrcorner\, \theta_t^*\omega - Y \lrcorner\, \omega\Big) \\
&= \lim_{t\to 0} \frac{1}{t}\Big(\theta_{-t*}Y \lrcorner\, \theta_t^*\omega - Y \lrcorner\, \theta_t^*\omega + Y \lrcorner\, \theta_t^*\omega - Y \lrcorner\, \omega\Big) \\
&= \lim_{t\to 0} \frac{1}{t}\Big(\big(\theta_{-t*}Y - Y\big) \lrcorner\, \theta_t^*\omega + Y \lrcorner\, \big(\theta_t^*\omega - \omega\big)\Big) \\
&= \lim_{t\to 0} \frac{1}{t}\big(\theta_{-t*}Y - Y\big) \lrcorner\, \theta_t^*\omega + Y \lrcorner\, \lim_{t\to 0} \frac{1}{t}\big(\theta_t^*\omega - \omega\big) \\
&= (L_XY) \lrcorner\, \omega + Y \lrcorner\, (L_X\omega).
\end{aligned}$$

Another proof by using Theorem 5.28(g) of the lecture notes: We have

$$\begin{aligned}
L_X(\sigma(Z_1, \dots, Z_l)) &= (L_X\sigma)(Z_1, \dots, Z_l) \\
&\qquad + \sigma(L_XZ_1, Z_2, \dots, Z_l) + \dots + \sigma(Z_1, \dots, Z_{l-1}, L_XZ_l),
\end{aligned}$$

where $\sigma \in \mathcal{T}^l(M)$ and $Z_1, \dots, Z_l$ are smooth vector fields. Since $\sigma(Z_1, \dots, Z_l)$ is a smooth real valued function, solving for $L_X\sigma$ gives the useful identity

$$\begin{aligned}
&(L_X\sigma)(Z_1, \dots, Z_l) \\
&\qquad = X(\sigma(Z_1, \dots, Z_l)) - \sum_{i=1}^{l} \sigma(Z_1, \dots, [X, Z_i], \dots, Z_l),
\end{aligned}$$

where we have used Theorem 3.61 from the lecture notes.

Let ω be given as in the statement. First, if $k = 0$ the claim is trivial. Next, if $k > 0$ then $i_Y\omega \in \mathcal{A}^{k-1}(M)$ and

$$\begin{aligned} L_X(i_Y\omega)(Y_1,\dots,Y_{k-1}) &= X((i_Y\omega)(Y_1,\dots,Y_{k-1})) \\ &\quad - \sum_{i=1}^{k-1}(i_Y\omega)(Y_1,\dots,[X,Y_i],\dots,Y_{k-1}) \\ &= X(\omega(Y,Y_1,\dots,Y_{k-1})) - \sum_{i=1}^{k-1}\omega(Y,Y_1,\dots,[X,Y_i],\dots,Y_{k-1}). \end{aligned}$$

On the other hand,

$$\begin{aligned} i_Y(L_X\omega)(Y_1,\dots,,Y_{k-1}) &= L_X\omega(Y,Y_1,\dots,Y_{k-1}) \\ &= X(\omega(Y,Y_1,\dots,Y_{k-1})) - \omega([X,Y],Y_1,\dots,Y_{k-1}) \\ &\quad - \sum_{i=1}^{k-1}\omega(Y,Y_1,\dots,[X,Y_i],\dots,Y_{k-1}). \end{aligned}$$

Subtracting $i_Y(L_X\omega)$ from $L_X(i_Y\omega)$ proves the claim.

2. Let $\alpha \in \mathcal{A}^k(M)$ be a closed differential k-form (that is, $d\alpha = 0$) and $X \in \mathcal{T}(M)$ such that $di_X\alpha = 0$. Prove that

$$(\theta_t^*\alpha)_p = \alpha_p$$

for all $(t,p) \in \mathcal{D}(X)$. [We say that α is invariant under the flow of X.]

Solution. By assumption, it follows from the Cartan's magic formula that $L_X\alpha = i_Xd\alpha + di_X\alpha = 0$. Given $p \in M$, let $I_p = \{t \in \mathbb{R} \,|\, (t,p) \in \mathcal{D}(X)\}$ and $F : I_p \to T^k(T_pM)$, $t \mapsto (\theta_t^*\alpha)_p$, where θ is the unique maximal flow of X. Since $F(0) = \alpha_p$, the claim follows if F is constant for all $t \in I_p$:

$$\begin{aligned} F'(t) &= \frac{d}{ds}(\theta_s^*\alpha)_p|_{s=t} = \frac{d}{ds}((\theta_{s-t}\circ\theta_t)^*\alpha)_p|_{s=t} = \frac{d}{ds}(\theta_t^*(\theta_{s-t}^*\alpha))_p|_{s=t} \\ &= \frac{d}{dr}(\theta_t^*(\theta_r^*\alpha))_p|_{r=0} = (\theta_t^*(\frac{d}{dr}(\theta_r^*\alpha)|_{r=0}))_p = (\theta_t^*(L_X\alpha))_p = 0. \end{aligned}$$

3. Prove that every smooth manifold M admits a Riemannian metric, i.e. a smooth symmetric 2-covariant tensor field g that is positive definite at each point $p \in M$.

Solution. Let us take a chart $\{(U_j, x_j)\}$ in M. First, let us use the coordinates x_j to construct a 2-covariant tensor field g_j in each coordinate domain U_j. That is, let us define

$$g_j = \delta_{kl}dx^k \otimes dx^l,$$

which is clearly a smooth 2-tensor field on U_j. Moreover, given $p \in U_j$ and $X, Y \in T_pM$, it is satisfied

$$g_{j_p}(X,Y) = \delta_{kl} dx^k(X) \otimes dx^l(Y) = \delta_{kl} X^k Y^l = g_{j_p}(Y,X),$$

and

$$g_{j_p}(X,X) = X_k X^k > 0$$

if X is nonzero. Thus, g_j is symmetric and positive definite.

Now, let us use these tensor fields in the coordinate domains to construct the global one. By Theorem 6.7 of the lecture notes, M admits a smooth partition of unity $(h_j)_{j\in J}$ subordinate to $\{U_j\}_{j\in J}$. Then, let us define a 2-tensor field on M by

$$g = \sum_J \tilde{g}_j h_j,$$

where $\tilde{g}_j = g_j$ in U_j and 0 otherwise. Since g_j is symmetric, one can easily deduce that g is symmetric as well. Moreover, given $p \in M$ and $X \in T_pM$ nonzero, then

$$g_p(X,X) = \sum_J \tilde{g}_j|_p(X,X) h_j(p) > 0$$

as each term is nonnegative and at least one $h_j(p)$ is strictly positive and $g_j|_p(X,X) > 0$. Thus, g is a symmetric 2-tensor field on M that is positive definite at each point $p \in M$.

It only remains to show that the tensor field g we have built is smooth. Since $\tilde{g}_j h_j$ is smooth in U_j for each $j \in J$ and $\tilde{g}_j h_j \equiv 0$ in $M \setminus \operatorname{spt} h_j$, it is clear that $\tilde{g}_j h_j$ is smooth in M. Next, as the partition of unity is locally finite, given any $p \in M$ there exists a neighborhood W of p for which $J' = \{j \in J \,|\, W \cap \operatorname{spt} h_j \neq \emptyset\}$ is finite. Thus,

$$g|_W = \sum_J \tilde{g}_j h_j\,|_W = \sum_{J'} \tilde{g}_j h_j\,|_W$$

is smooth in W, and the claim follows.

4. Let M be an oriented smooth manifold and ω an orientation form that determines the orientation. Let $f \in C^\infty(M)$ such that $f(p) \neq 0$ for all $p \in M$. Prove that ω and $f\omega$ determine the same orientation on M if and only if $f(p) > 0$ for all $p \in M$.

Solution. Let us suppose that $f > 0$. The result follows by repeating the proof of Theorem 7.2 of the lecture notes for the n-form $f\omega$ noting that $\omega^\alpha > 0$ if and only if $f\omega^\alpha > 0$ as f was assumed strictly positive.

Suppose now that ω and $f\omega$ determine the same orientation and that M is connected. Hence M has exactly two orientations determined by ω and $-\omega$. If it would have k components, it would have

2^k orientations. Since f is continuous, then $f < 0$ or $f > 0$. Let us assume by contradiction that $f < 0$. Then by the result already proved ω and $-\omega$ would determine the same orientation, which is not possible. Thus, $f > 0$. For the general case suppose $p \in M$ and let U be a connected neighborhood. Now $\omega|_U$ and $f\omega|_U$ define the same orientation on U. In particular by the previous $f|_U > 0$ so $f(p) > 0$.

Recall that the *divergence* of a smooth vector field X with respect to an orientation n-form ω is the function $\mathrm{Div}_\omega X\colon M \to \mathbb{R}$ s.t. $(\mathrm{Div}_\omega X)\omega = L_X\omega$.

***5.** Let $\alpha \in \mathcal{A}^3(\mathbb{R}^3\backslash\{0\})$ be given in spherical coordinates $(\varrho, \varphi, \vartheta)$, $\varrho > 0$, $\varphi \in [0, \pi]$, $\vartheta \in [0, 2\pi]$, (see Exercise 8/5) by

$$\alpha = \varrho^2 \sin\varphi \, d\varrho \wedge d\varphi \wedge d\vartheta.$$

Let $\frac{\partial}{\partial\varrho} \in \mathcal{T}(\mathbb{R}^3\backslash\{0\})$ be the smooth coordinate vector field associated to the spherical coordinates and let $V = \varrho^{-2}\frac{\partial}{\partial\varrho} \in \mathcal{T}(\mathbb{R}^3 \setminus \{0\})$. Compute $\mathrm{Div}_\alpha V$ in $\mathbb{R}^3 \setminus \{0\}$.

Solution. We note that $d\alpha = 0$, in fact, $\alpha = d\omega$ where

$$\omega = \frac{1}{3}\varrho^3 \, \sin\varphi \, d\varphi \wedge d\theta.$$

By Cartan's magic formula and the use of the properties of the contraction (Theorem 5.27 in the lecture notes) we obtain

$$\begin{aligned}
L_V\alpha &= i_V d\alpha + d i_V \alpha = d i_V \alpha \\
&= d i_{\varrho^{-2}\frac{\partial}{\partial\rho}}(\varrho^2 \sin\varphi \, d\varrho \wedge d\varphi \wedge d\vartheta) \\
&= d\left(\varrho^{-2} \, i_{\frac{\partial}{\partial\varrho}}(\varrho^2 \sin\varphi \, d\varrho \wedge d\varphi \wedge d\vartheta)\right) \\
&= d\left(\sin\varphi \, i_{\frac{\partial}{\partial\varrho}}(d\varrho \wedge d\varphi \wedge d\vartheta)\right) \\
&= d\left(\sin\varphi \, \left[(i_{\frac{\partial}{\partial\varrho}} d\varrho)\, d\varphi \wedge d\vartheta - (i_{\frac{\partial}{\partial\varrho}} d\varphi)\, d\varrho \wedge d\vartheta \right.\right. \\
&\qquad\qquad \left.\left. + \ (i_{\frac{\partial}{\partial\varrho}} d\vartheta)\, d\varrho \wedge d\varphi\right]\right) \\
&= d\left(\sin\varphi \, d\varphi \wedge d\vartheta\right) \\
&= \cos\varphi \, d\varphi \wedge d\varphi \wedge d\vartheta \\
&= 0
\end{aligned}$$

in $\mathbb{R}^3 \setminus \{0\}$. Since $(Div_\alpha V)\alpha = L_V\alpha = 0$ we conclude that

$$Div_\alpha V = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\}.$$

Let us note that α is the standard volume form of R^3 in spherical coordinates and $V = \nabla g$ with $g(x) = -|x|^{-1}$. Then, since α is the volume form and g is harmonic in $\mathbb{R}^3 \setminus \{0\}$ one obtains

$$Div_\alpha V = \mathrm{div}\nabla g = \Delta g = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\}.$$

$*$**6.** Using the spherical coordinates as above, let $\omega = \frac{1}{3}\varrho^3 \sin\varphi\, d\varphi \wedge d\vartheta \in \mathcal{T}(\mathbb{R}^3 \setminus \{0\})$. Apply Stokes's theorem with ω and $d\omega$ in the closed ball $M = \bar{B}^3(0,R) \subset \mathbb{R}^3$.

Solution. On the one hand we have

$$\begin{aligned}\int_{\partial M} \omega &= \int_{\partial \overline{B}(0,R)} \frac{1}{3}\varrho^3 \sin\varphi\, d\varphi \wedge d\vartheta = \frac{R^3}{3}\int_0^{2\pi}\int_0^{\pi} \sin\varphi\, d\varphi\, d\vartheta \\ &= \frac{4}{3}\pi R^3.\end{aligned}$$

On the other hand, since $d\omega = \varrho^2 \sin\varphi\, d\varrho \wedge d\varphi \wedge d\vartheta$, we obtain

$$\begin{aligned}\int_M d\omega &= \int_{\overline{B}(0,R)} \varrho^2 \sin\varphi\, d\varrho \wedge d\varphi \wedge d\vartheta \\ &= \int_0^R \int_0^{2\pi} \int_0^{\pi} \varrho^2 \sin\varphi\, d\varphi\, d\vartheta\, d\varrho = \frac{4}{3}\pi R^3.\end{aligned}$$

Hence, $\int_M d\omega = \int_{\partial M} \omega$ as it should be. Let us point out that since the differential forms involved are defined just in $\mathbb{R}^3 \setminus \{0\}$, in order to be precise we should integrate over the annulus $\overline{B}(0,R) \setminus B(0,\varepsilon)$ and let $\varepsilon \to 0$.

We also note that $\frac{4}{3}\pi R^3$ is the measure (volume) of the ball $B(0,R)$ since $d\omega$ is the volume form in R^3 in spherical coordinates.

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 10
4.4.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, April 4, 12:00 o'clock.

1. Let G be a Lie group. A differential form α is called left invariant if $\alpha = L_p^*\alpha$ for every $p \in G$. More precisely, $(L_p^*\alpha)_q = \alpha_q$ for every $p, q \in G$. Prove that there exists an orientation form on G and that a left invariant orientation form is unique up to a multiplicative constant. [Here $L_p\colon G \to G$ is the left translation $L_p(q) = pq$.]

   Solution. Let be $n = \dim G$ and e the neutral element of G. Choose $\omega_1, \dots, \omega_n \in T^1(T_eG)$ s.t. $\omega_e := \omega_1 \wedge \omega_2 \wedge \dots \wedge \omega_n \neq 0$ (i.e. spans $\Lambda^n(T_eG)$ from being a one-dimensional linear space). Then, we define $\omega \in \Lambda^n G$ as
$$\omega_p = (L_p^{-1})^*\omega_e, \quad p \in G.$$
   We will show that ω is a left invariant smooth orientation form.

   First, given $p, q \in G$ we have
$$\begin{aligned}(L_p^*\omega)_q(v_1, \dots, v_n) &= \omega_{pq}\big(L_{p*q}v_1, \dots, L_{p*q}v_n\big)\\ &= (L_{pq}^{-1})^*\omega_e\big(L_{p*q}v_1, \dots, L_{p*q}v_n\big)\\ &= \omega_e\big((L_{pq}^{-1})_*L_{p*q}v_1, \dots, (L_{pq}^{-1})_*L_{p*q}v_n\big)\\ &= \omega_e\big((L_q^{-1})_{*q}v_1, \dots, (L_q^{-1})_{*q}v_n\big)\\ &= \omega_q(v_1, \dots, v_n)\end{aligned}$$
   for every $v_1, \dots, v_n \in T_qG$. Hence ω is left invariant. Moreover, since $\omega_e \neq 0$, there are $v_1, \dots, v_n \in T_eG$ (e.g. duals to $\omega_1, \dots, \omega_n$) such that $\omega_e(v_1, \dots, v_n) \neq 0$. Then, by left invariance
$$\begin{aligned}\omega_p\big(L_{p*}v_1, \dots, L_{p*}v_n\big) &= (L_p^{-1})^*\omega_e\big(L_{p*}v_1, \dots, L_{p*}v_n\big)\\ &= \omega_e(v_1, \dots, v_n) \neq 0,\end{aligned}$$
   i.e. $\omega_p \neq 0$.

   Next, we show that ω is smooth. To see this, we note that
$$\omega_p = (L_p^{-1})^*\omega_e = (L_p^{-1})^*\omega_1 \wedge \dots \wedge (L_p^{-1})^*\omega_n.$$
   Now each $p \mapsto (L_p^{-1})^*\omega_i, i = 1, \dots, n$, is a smooth 1-form. This can be seen by taking an arbitrary smooth vector field $X \in \mathcal{T}(G)$ and noticing that the function $p \mapsto (L_p^{-1})^*\omega_i(X_p) = (\omega_i)_e\big((L_p^{-1})_*X_p\big)$ is smooth as a composition of the linear functional $(\omega_i)_e$ and the smooth map $p \mapsto (L_p^{-1})_*X_p \in T_eG$. Hence, $\omega \in \mathcal{A}^n(G)$, and in particular is an orientation form.

Finally, if $\tilde{\omega}$ is another left invariant orientation from, then it is clear that $\tilde{\omega}_e = c\omega_e$ for some constant $c \neq 0$. From this relation and the left invariance (of both $\tilde{\omega}$ and ω), we conclude that $\tilde{\omega}_p = c\omega_p$ for every $p \in G$.

2. Let M and N be connected smooth oriented manifolds and $f : M \to N$ a local diffeomorphism. Prove that f is either orientation preserving or orientation reversing.

   Solution. Let ω be an orientation form on N associated to the orientation of N. Then $f^*\omega$ is a smooth n-form on M. Let η be an orientation form on M associated to the orientation of M. Now $f^*\omega = g\eta$, where $g \in C^\infty(M)$. Since f is a local diffeomorphism, we will show that the function g never vanishes. Let us take $\{(U_\alpha, x_\alpha)\}$ and $\{(V_\beta, y_\beta)\}$ orientations in M and N respectively. Given $p \in M$ we take $i \in I$ and $j \in J$ such that $p \in U_{\alpha_i}$ and $f(p) \in V_{\beta_j}$. Moreover, note that we can always assume that $f(U_{\alpha_i}) = V_{\beta_j}$. Then, we can write ω and η in coordinates as

   $$\omega|_{V_{\beta_j}} = \omega_j \, dy^1_{\beta_j} \wedge dy^2_{\beta_j} ... \wedge dy^n_{\beta_j}$$

   and

   $$\eta|_{U_{\alpha_i}} = \eta_i \, dx^1_{\alpha_i} \wedge dx^2_{\alpha_i} ... \wedge dx^n_{\alpha_i}$$

   with ω_j and η_i positive smooth functions. By using the relation $f^*\omega = g\eta$ we obtain that

   $$g\eta_i = (\omega_j \circ f) \det \left(y \circ f \circ x^{-1}\right)'.$$

   Since f is a local diffeomorphism we have $\det \left(y \circ f \circ x^{-1}\right)'$ never vanishes. Thus from the previous identity we conclude that g never vanishes. Finally, from being M connected, g is either positive everywhere (and f sense preserving) or g is negative everywhere (and f sense reversing).

3. Let M be an oriented n-manifold with boundary and an orientation form ω, and let V be an outward-pointing smooth vector field on ∂M. Prove that the induced orientation on ∂M, determined by $(i_V\omega)|\partial M$, is independent of the choice of the orientation form ω and the outward-pointing vector field V.

   Solution. Let $\omega^1, \omega^2 \in \mathcal{A}^n(M)$ be two orientation forms on M and $V_1, V_2 : \partial M \to TM$ be two smooth outward-pointing vector fields. We will show that $i_{V_1}\omega^1$ and $i_{V_2}\omega^2$ determine the same orientation restricted to ∂M. To this end, let $p \in \partial M$ and (U, x) a chart at p where $x = (x^1, \dots, x^n) : U \to \bar{\mathbb{H}}^n$. Then, we can write the vector fields in coordinates as $V_{1p} = \sum_{i=1}^n V_1^i (\partial_i)_p$ and $V_{2p} = \sum_{i=1}^n V_2^i (\partial_i)_p$. By Lemma 7.7 in the lecture notes we know that $V_1^n, V_2^n < 0$. Hence there exists a constant $c > 0$ such that $V_1^n = cV_2^n$. On the other hand, by Exercise 9/4 we know that $\omega^1 = f\omega^2$ for some $f > 0$ as can be seen writing out the orientation forms locally. Hence,

$$\begin{aligned}
&(i_{V_1}\omega^1)_p((\partial_1)_p, \ldots, (\partial_{n-1})_p) = (i_{V_1}(f\omega^2))_p((\partial_1)_p, \ldots, (\partial_{n-1})_p) \\
&= f(p)\omega^2_p(V_1, (\partial_1)_p, \ldots, (\partial_{n-1})_p) \\
&= f(p)\omega^2_p(V_1^n(\partial_n)_p, (\partial_1)_p, \ldots, (\partial_{n-1})_p) \\
&= cf(p)\omega^2_p(V_2^n(\partial_n)_p, (\partial_1)_p, \ldots, (\partial_{n-1})_p) \\
&= cf(p)\omega^2_p(V_2, (\partial_1)_p, \ldots, (\partial_{n-1})_p) \\
&= cf(p)(i_{V_2}\omega^2)_p((\partial_1)_p, \ldots, (\partial_{n-1})_p).
\end{aligned}$$

As $cf(p) > 0$ for all $p \in \partial M$ the result follows by Exercise 9/4.

$*$4. Let M be a smooth manifold, $\alpha \in \mathcal{A}^k(M)$, $X, Y \in \mathcal{T}(M)$ and $f \in C^\infty(M)$. Prove that

(a)

$$L_{fX}\alpha = fL_X\alpha + df \wedge i_X\alpha,$$

(b)

$$i_{[X,Y]}\alpha = L_X i_Y\alpha - i_Y L_X\alpha.$$

Solution.

(a) Given $Z_1, ..., Z_{k-1}$, if we use Exercise 7/6 and the properties of the Lie bracket, on the one hand we obtain

$$\begin{aligned}
&L_{fX}\alpha(Z_1, ..., Z_k) \\
&\qquad = L_{fX}\left(\alpha(Z_1, ..., Z_k)\right) - \sum_{i=1}^k \alpha(Z_1, ..., [fX, Z_i], ..., Z_k) \\
&\qquad = fL_X\left(\alpha(Z_1, ..., Z_k)\right) - f\sum_{i=1}^k \alpha(Z_1, ..., [X, Z_i], ..., Z_k) \\
&\qquad\qquad + \sum_{i=1}^k Z_i(f)\,\alpha(Z_1, ..., X, ..., Z_k) \\
&\qquad = fL_X\alpha(Z_1, ..., Z_k) + \sum_{i=1}^k Z_i(f)\,\alpha(Z_1, ..., X, ..., Z_k)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&df \wedge i_X\alpha(Z_1, ..., Z_k) \\
&\quad = \sum_{i=1}^k (-1)^{i-1} df(Z_i)\, i_X\alpha(Z_1, ..., Z_{i-1}, Z_{i+1}, ..., Z_k) \\
&\quad = \sum_{i=1}^k Z_i(f)\,\alpha(Z_1, ..., X, ..., Z_k)
\end{aligned}$$

Hence, the identity follows.

(b) Given $Z_1, ..., Z_k$, if we use again Exercise 7/6 and the properties of the Lie bracket, on the one hand we obtain

$$
\begin{aligned}
&L_X i_Y \alpha(Z_1, ..., Z_{k-1}) \\
&\quad = L_X \left(i_Y \alpha(Z_1, ..., Z_{k-1}) \right) - \sum_{i=1}^{k-1} i_Y \alpha(Z_1, ..., [X, Z_i], ..., Z_{k-1}) \\
&\quad = X \left(\alpha(Y, Z_1, ..., Z_{k-1}) \right) - \sum_{i=1}^{k-1} \alpha(Y, Z_1, ..., [X, Z_i], ..., Z_{k-1})
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
i_Y L_X \alpha(Z_1, ..., Z_{k-1}) &= L_X \alpha(Y, Z_1, ..., Z_{k-1}) \\
&= L_X(\alpha(Y, Z_1, ..., Z_{k-1})) - \alpha([X, Y], Z_1, ..., Z_{k-1}) \\
&\quad - \sum_{i=1}^{k-1} \alpha(Y, Z_1, ..., [X, Z_i], ..., Z_{k-1}) \\
&= X(\alpha(Y, Z_1, ..., Z_{k-1})) - i_{[X,Y]} \alpha(Z_1, ..., Z_{k-1}) \\
&\quad - \sum_{i=1}^{k-1} \alpha(Y, Z_1, ..., [X, Z_i], ..., Z_{k-1})
\end{aligned}
$$

Hence, the identity follows by subtracting both expressions.

Recall that the *divergence* of a smooth vector field X with respect to an orientation n-form ω is the function $\mathrm{Div}_\omega X \colon M \to \mathbb{R}$ s.t. $(\mathrm{Div}_\omega X)\omega = L_X \omega$.

*5. Let M be a smooth oriented n-manifold and let ω be an orientation form (volume form) on M. Suppose that $X \in \mathcal{T}(M)$ and $f, g \in C^\infty(M)$, with $f > 0$. Prove that

$$\mathrm{Div}_\omega(gX) = g\mathrm{Div}_\omega X + Xg$$

and

$$\mathrm{Div}_{f\omega} X = \mathrm{Div}_\omega X + \frac{Xf}{f}.$$

Solution. By definition,

$$L_{gX}\omega = \mathrm{Div}_\omega(gX)\omega.$$

We compute the left hand side by using Cartan's magic formula:

$$\begin{aligned} L_{gX}\omega &= i_{gX}d\omega + di_{gX}\omega = di_{gX}\omega \\ &= d\big(gi_X\omega\big) = dg \wedge i_X\omega + gdi_X\omega \\ &\overset{(*)}{=} i_Xdg \wedge \omega + gdi_X\omega \\ &= (Xg)\omega + gL_X\omega \\ &= \Big(Xg + d\mathrm{Div}_\omega(X)\Big)\omega. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Div}_\omega(gX) &= Xg + g\mathrm{Div}_\omega(X) \\ &= L_Xgdi_{gX}\omega. \end{aligned}$$

Equality $(*)$ follows from

$$0 = i_X\big(\underbrace{dg \wedge \omega}_{\equiv 0}\big) = i_Xdg \wedge \omega - dg \wedge i_X\omega.$$

For the other equation, we compute

$$\begin{aligned} \big(\mathrm{Div}_{f\omega}X\big)f\omega &= L_X(f\omega) = fL_X\omega + (Xf)\omega \\ &= f\big(\mathrm{Div}_\omega X\big)\omega + \frac{Xf}{f}(f\omega) \\ &= \big(\mathrm{Div}_\omega X + \frac{Xf}{f}\big)(f\omega). \end{aligned}$$

Department of Mathematics and Statistics
Introduction to differential geometry
Exercise 11, Solution
11.4.2022

Return your written solutions to the *bonus problems* (marked by $*$) to the Moodle area by Monday, April 11, 12:00 o'clock.

These are the last exercises!

1. Let M be a smooth oriented n-manifold and let ω be an orientation form (volume form) on M. Suppose that $X, Y \in \mathcal{T}(M)$. Prove that
$$\mathrm{Div}_\omega[X,Y] = X\big(\mathrm{Div}_\omega Y\big) - Y\big(\mathrm{Div}_\omega X\big).$$

Solution. By using Exercise 2 below and Theorem 5.28 (c)(a) from the lecture notes we obtain
$$\begin{aligned}\big(\mathrm{Div}_\omega[X,Y]\big)\omega &= L_{[X,Y]}\omega\\ &= L_X L_Y\omega - L_Y L_X\omega\\ &= L_X\big((\mathrm{Div}_\omega Y)\omega\big) - L_Y\big((\mathrm{Div}_\omega X)\omega\big)\\ &= X\big(\mathrm{Div}_\omega Y\big)\omega + \big(\mathrm{Div}_\omega Y\big)L_X\omega\\ &\quad - Y\big(\mathrm{Div}_\omega X\big)\omega - \big(\mathrm{Div}_\omega X\big)L_Y\omega\\ &= X\big(\mathrm{Div}_\omega Y\big)\omega + \big(\mathrm{Div}_\omega Y\big)\big(\mathrm{Div}_\omega X\big)\omega\\ &\quad - Y\big(\mathrm{Div}_\omega X\big)\omega - \big(\mathrm{Div}_\omega X\big)\big(\mathrm{Div}_\omega Y\big)\omega\\ &= \big(X(\mathrm{Div}_\omega Y) - Y(\mathrm{Div}_\omega X)\big)\omega.\end{aligned}$$

$*$2. Let M be a smooth manifold, $\alpha \in \mathcal{A}^k(M)$ and $X, Y \in \mathcal{T}(M)$. Prove that
$$L_{[X,Y]}\alpha = L_X L_Y\alpha - L_Y L_X\alpha.$$

Solution. By Cartan's magic formula, Exercise 10/4(b) and Theorem 5.28 (b) from the lecture notes we get
$$\begin{aligned}L_{[X,Y]}\alpha &= d i_{[X,Y]}\alpha + i_{[X,Y]}d\alpha\\ &= d\left(L_X i_Y\alpha - i_Y L_X\alpha\right) + L_X i_Y d\alpha - i_Y L_X d\alpha\\ &= L_X d i_Y\alpha - d i_Y L_X\alpha + L_X i_Y d\alpha - i_Y L_X d\alpha\\ &= L_X\left(d i_Y\alpha + i_Y d\alpha\right) - \left(d i_Y L_X\alpha + i_Y d L_X\alpha\right)\\ &= L_X L_Y\alpha - L_Y L_X\alpha.\end{aligned}$$

3. Suppose M and N are oriented n-manifolds and $f\colon M \to N$ an orientation preserving diffeomorphism. Prove: If $\omega \in \mathcal{A}^n(N)$ has compact support, then $f^*\omega \in \mathcal{A}^n(M)$ has compact support and
$$\int_N \omega = \int_M f^*\omega.$$

Solution. Let us take $\{(V_\alpha, y_\alpha)\}$ an orientation atlas of N. Since f is a sense preserving diffeomorphism, then $\{(U_\alpha := f^{-1}V_\alpha, x_\alpha := y_\alpha \circ f)\}$ is an orientation atlas of M. Furthermore, let $\{\lambda_i\}$ be a smooth partition of unity subordinate to $\{V_\alpha\}$. This clearly means that $\{\lambda_i \circ f\}$ is a smooth partition of unity subordinate to $\{U_\alpha\}$. Now,
$$\begin{aligned}\int_N \omega &= \int_N \left(\sum_i \lambda_i\right)\omega = \sum_i \int_N \lambda_i\omega = \sum_i \int_{V_{\alpha_i}} \lambda_i\omega \\ &= \sum_i \int_{\mathbb{R}^n} (y_{\alpha_i}^{-1})^*(\lambda_i\omega) = \sum_i \int_{\mathbb{R}^n} (f \circ x_{\alpha_i}^{-1})^*(\lambda_i\omega) \\ &= \sum_i \int_{\mathbb{R}^n} (x_{\alpha_i}^{-1})^*\left(f^*(\lambda_i\omega)\right) = \sum_i \int_{U_{\alpha_i}} f^*(\lambda_i\omega) \\ &= \sum_i \int_{U_{\alpha_i}} (\lambda_i \circ f)f^*\omega = \sum_i \int_M (\lambda_i \circ f)f^*\omega \\ &= \int_M \sum_i (\lambda_i \circ f)f^*\omega = \int_M f^*\omega.\end{aligned}$$

Note that the smoothness of $f^*\omega$ is clear, so $f^*\omega \in \mathcal{A}^n(M)$. Since f is a diffeomorphism and ω is compactly supported, also $f^*\omega$ is compactly supported.

Recall from Real Analysis: Let M be an oriented n-manifold and $\omega \in \mathcal{A}^n(M)$ an orientation (volume) form. By Riesz representation theorem, there exists a unique measure m_ω (defined in the σ-algebra of all Borel subsets of M) such that
$$\int_M u\, dm_\omega = \int_M u\omega$$
for every compactly supported continuous function u.

Let M and N be oriented n-manifolds with orientation forms ω_M and ω_N, respectively. Let $f\colon M \to N$ be a diffeomorphism. We define the Jacobian of f, denoted by J_f, with respect to ω_M and ω_N by the formula
$$f^*\omega_N = J_f\omega_M.$$

$*4$. Let M be an oriented n-manifold with an orientation form $\omega \in \mathcal{A}^n(M)$ and the corresponding Borel measure m_ω. Let $X \in \mathcal{T}(M)$ be a complete vector field and θ its flow. Prove that the following are equivalent:

(i) $\mathrm{Div}_\omega X \equiv 0$,
(ii) $J_{\theta_t} \equiv 1\ \forall t \in \mathbb{R}$,
(iii) $\theta_t^* \omega = \omega$,
(iv)

$$\int_M u\, dm_\omega = \int_M (u \circ \theta_t)\, dm_\omega$$

for all compactly supported continuous functions u on M and for all $t \in \mathbb{R}$.

Solution. First, note that by definition of divergence and Jacobian, and Exercise 9/2 we obtain directly the equivalence of (*i*), (*ii*) and (*iii*). Furthermore, these are equivalent to (*iv*) since by Exercise 3 we have

$$\begin{aligned}\int_M u dm_\omega &= \int_M u\omega = \int_M \theta_t^*\,(u\omega) = \int_M (u \circ \theta_t)\theta_t^*\omega \\ &= \int_M (u \circ \theta_t)\omega = \int_M (u \circ \theta_t) dm_\omega.\end{aligned}$$

Let us point out that this is a far reaching generalization of the translation invariance of the Lebesgue measure in $\mathbb{R}^n$. Note that a translation $x \mapsto x + a$, where $a \in \mathbb{R}^n$, is the flow of the parallel vector field $X_p = a$ for all $p \in \mathbb{R}^n$.