SADDLE-SHAPED SOLUTION TO THE INTEGRO-DIFFERENTIAL ALLEN-CAHN EQUATION

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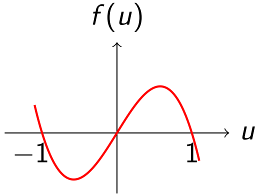
The Allen-Cahn equation

We consider

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n$$

where f is of bistable type.



The most classical example of nonlinearity is $f(u) = u - u^3$.

Motivation: Conjecture of De Giorgi

Conjecture 1 [De Giorgi; '78]

Let $u \in C^2(\mathbb{R}^n)$ be a solution of the Allen-Cahn equation

$$-\Delta u = u - u^3 \qquad \text{in } \mathbb{R}^n$$

such that

$$|u| \leq 1 \quad \text{and} \quad \partial_{x_n} u > 0$$

in the whole $\mathbb{R}^n$. Then, all level sets $\{u = \lambda\}$ of u are hyperplanes, at least if $n \leq 8$. Equivalently, u is a $1D$ solution, that is, a function depending only on one Euclidean variable.

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- $n = 2$ [Ghoussoub, Gui; '98]
- $n = 3$ [Ambrosio, Cabré; '00]
- $4 \leq n \leq 8$ [Savin; '09] with $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$
- $n \geq 9$ Counterexample [del Pino, Kowalczyk, Wei; '11]

Motivation: Conjecture of De Giorgi

Theorem 2 [Jerison, Monneau; '04]

A counterexample for the Conjecture of De Giorgi in $\mathbb{R}^{n+1}$ can be built from a bounded, even with respect to each coordinate, global minimizer of

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How to construct such a global minimizer?

Motivation: Connection with minimal surfaces

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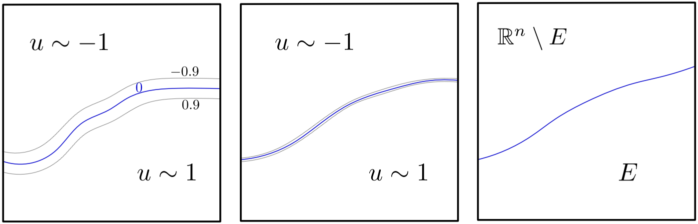
Consider the blow-down sequence $u_\varepsilon(x) = u(x/\varepsilon)$, where u is a minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$.



Then, $u_\varepsilon \to \chi_E - \chi_{\mathbb{R}^n \setminus E}$ in L^1_{loc} as $\varepsilon \to 0$, where E is a minimizer of the perimeter.

Motivation: Simons Cone

$$\mathcal{C} = \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \quad \text{st} \quad |x'| = |x''|\}$$

Theorem 4

For any given even dimension $n = 2m$, the Simons cone has zero mean curvature at every point outside the origin.

Theorem 5 [Bombieri, De Giorgi, Giusti; '69]

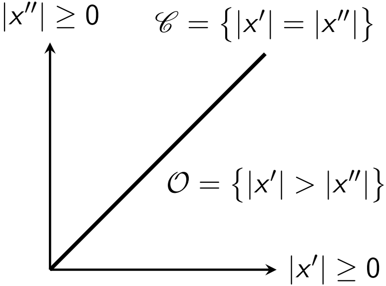
If $2m \geq 8$, the Simons cone is a global minimizer of the perimeter functional.

Saddle-shaped solutions

Definition 6

Let u be a bonded solution of the Allen-Cahn equation in $\mathbb{R}^{2m}$. We say the it is a saddle-shaped solution if

- It is doubly radial: $u = u(|x'|, |x''|)$.
- It is odd with respect to Simons Cone: $u(|x'|, |x''|) = -u(|x''|, |x'|)$.
- It is positive in $\mathcal{O} = \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \quad \text{st} \quad |x'| > |x''|\}$.



Known results

- Existence [Dang, Fife, Peletier; '92] [Cabré, Terra; '09]
- Uniqueness [Dang, Fife, Peletier; '92] [Cabré; '12]
  - Asymptotic behaviour at infinity
  - Maximum principle for linearized operator

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- Instability if $2m \leq 6$ [Shatzman; '95] [Cabré, Terra; '09,'10]
- Stability if $2m \geq 14$ [Cabré; '12]
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Stability in dimension 8, 10 and 12, and minimality when $2m \geq 8$ are open problems

The nonlocal problem

Study saddle-shaped solution to nonlocal Allen-Cahn equation

$$L_K u = f(u) \quad \text{in } \mathbb{R}^{2m}$$

with

$$L_K u(x) = \int_{\mathbb{R}^n} \{u(x) - u(y)\} K(x,y)\, \mathrm{d}y.$$

- Translation invariant:

$$K(x,y) = K(x-y)\,.$$

- Rotation invariant:

$$K(z) = K(|z|)\,.$$

- Uniformly elliptic:

$$\frac{\lambda}{|z|^{n+2s}} \leq K(z) \leq \frac{\Lambda}{|z|^{n+2s}}\,, \quad \text{with } \; 0 < \lambda \leq \Lambda.$$

The fractional Laplacian

Fractional Laplacian: canonical example of integro-differential operator.

$$K(z) = c_{n,s}\,|z|^{-n-2s}$$

It corresponds to radially symmetric Lévy process of order $2s$.

Theorem 7: Local extension problem [Caffarelli, Silvestre; '07]

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth bounded function, we can recover its fractional Laplacian as:

$$(-\Delta)^s\, u = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{|z|^{n+2s}}\,\mathrm{d}z = -d_s \lim_{y\to 0} y^{1-2s}\, U_y(x,y),$$

where U is the solution of the problem

$$\begin{cases} div(y^{1-2s}\,\nabla U) = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) = u(x), & \text{in } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

Results

In [Cinti; '13, '17]

- Existence
- Monotonicity properties
- Asymptotic behavior
- Instability if $2m \leq 6$ ($2m = 2$ in [Cabré, Solà-Morales; '05])

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We establish in [F-N, Sanz-Perela; '18]

- Uniqueness
- Stability in dimensions $2m \geq 14$ for all $s \in (0, 1)$

New result

An important consequence of our stability result:

Corollary 8: [F-N, Sanz-Perela; '18]

The Simons cone is a stable $2s$-minimal surface in dimensions $2m \geq 14$ for all $s \in (0, 1/2)$.

First analytical proof of a stability result for the Simons cone in any dimension (in the nonlocal setting).

General kernels

Rewrite the operator for doubly radial odd functions in the form:

$$L_K w(x) = \int_{\mathcal{O}} \{w(x)-w(y)\}\{\overline{K}(x,y)-\overline{K}(x,y^\star)\}\,\mathrm{d}y + 2w(x)\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y\,,$$

where $\overline{K}$ is doubly radial in both variables and is defined by

$$\overline{K}(x,y) := \fint_{O(m)^2} K(|Rx-y|)\,\mathrm{d}R\,.$$

We establish a necessary and sufficient condition on the kernel K such that a theory of existence and uniqueness of saddle-shaped solutions can be developed.

This condition turns out to be

$$K(\sqrt{\tau}) \text{ is a strictly convex function of } \tau\,.$$

New results

We establish in [F-N, Sanz-Perela; '19]

- Existence
  - Monotone iteration method
  - Variational techniques
- Asymptotic behaviour
- Uniqueness

New results: key ingredients

To establish the previous results for the saddle-shaped solution, we need to prove the following results:

- An energy estimate for doubly radial minimizers.
- A Liouville type result for nonnegative bounded solutions to $L_K u = f(u)$ in $\mathbb{R}^n$.
- A one-dimensional symmetry result for positive solutions to $L_K u = f(u)$ in a half-space $\mathbb{R}^n \cap \{x_n > 0\}$.
- A maximum principle in "narrow" sets.

Results: Summary

		Local case	Nonlocal: With extension	Nonlocal: General Kernels
Energy estimate	$2m \geq 2$	[Cabré-Terra, '09]	[Cinti, '11, '17]	[F-N, Sanz-Perela]
Existence	$2m = 2$	[Dang, Fife, Peletier, '92]	[Cinti, '11, '17]	[F-N, Sanz-Perela]
	$2m \geq 4$	[Cabré-Terra, '09]		
Maximum principle for the linearized operator in O	$2m \geq 2$	[Cabré, '12]	[F-N, Sanz-Perela]	[F-N, Sanz-Perela]
Asymptotic behavior	$2m = 2$	[Dang, Fife, Peletier, '92]	[Cinti, '11, '17]	[F-N, Sanz-Perela]
	$2m \geq 4$	[Cabré-Terra, '10]		
Uniqueness	$2m = 2$	[Dang, Fife, Peletier, '92]	[F-N, Sanz-Perela]	[F-N, Sanz-Perela]
	$2m \geq 4$	[Cabré, '12]		
Monotonicity properties	$2m = 2$	[Dang, Fife, Peletier, '92]	[Cinti, '11, '17] [F-N, Sanz-Perela]	
	$2m \geq 4$	[Cabré-Terra, '10] [Cabré, '12]		
Unstability	$2m = 2$	[Shatzman, '95]	[Cinti, '11, '17]	
	$2m = 4, 6$	[Cabré-Terra, '09, '10]		
Stability	$2m \geq 12$	[Cabré, '12]	[F-N, Sanz-Perela]	
Minimality	$2m \geq 8$			

Conclusion: Importance of saddle-shaped solutions

- They could be used to construct a counterexample for the nonlocal analogue of the conjecture by De Giorgi on the one-dimensional symmetry of monotone solutions.

- They are expected to be, in high dimensions, the simplest example of minimizer to $L_K u = f(u)$ which is not one-dimensional.

- Saddle-shaped solutions can be used to prove the stability/minimality of the Simons cone as a nonlocal minimal surface.

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Thank You