NULL-LAGRANGIANS AND CALIBRATIONS FOR NONLOCAL FUNCTIONALS

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joint work with X. Cabré (ICREA-UPC) and I. U. Erneta (UPC-BGSMath)

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Geometric and Functional Analysis Seminar

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- If $\mathcal{E}$ is convex, then there is a unique extremal that turns out to be a minimizer
- Important models with noncovex functional:
  - Allen-Cahn energy
  - Bernoulli free boundary problem
  - Perimeter

Outline of the talk

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5. Applications to monotone solutions and the viscosity theory

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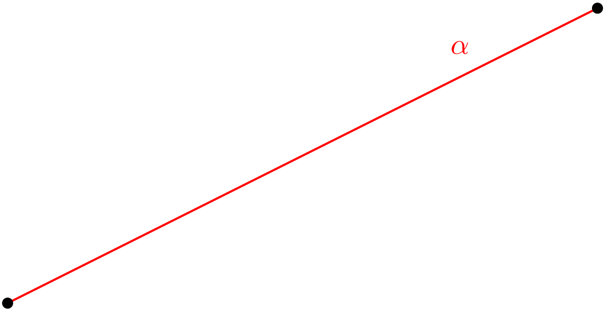

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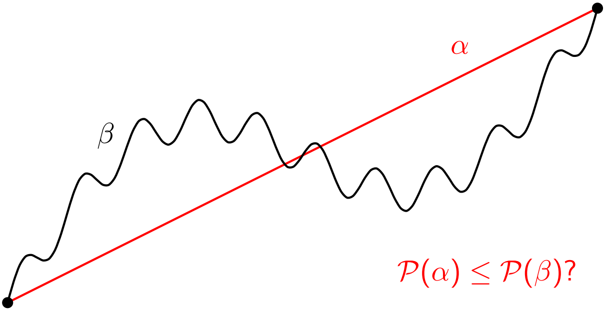

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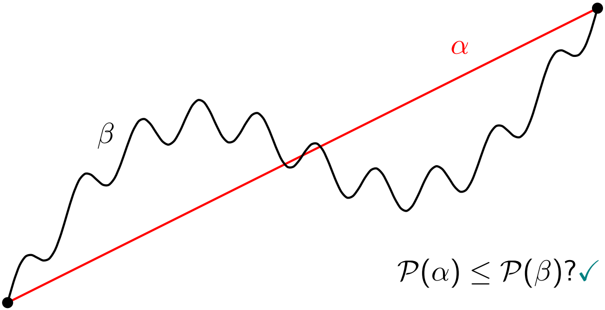

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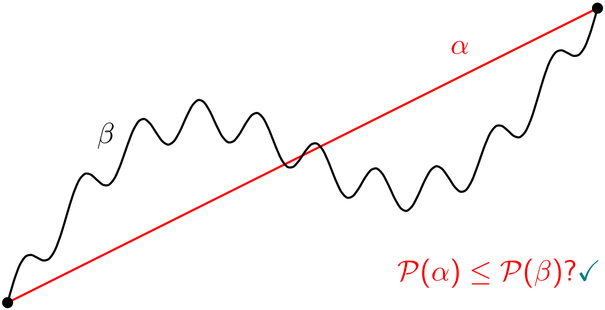

$\mathcal{P}(\alpha) \leq \mathcal{P}(\beta)?$✓

$$\mathcal{C}(\beta) = \int_\beta \nu_\beta \cdot X \,\mathrm{d}l, \quad \text{where} \quad \operatorname{div} X = 0$$

The calibration functional

Definition

A functional $\mathcal{C}\colon \mathcal{A} \to \mathbb{R}$ is a *calibration* for $\mathcal{E}$ and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet condition as u.
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When is it possible to find such a functional?

Classical theory of Calibrations

Necessary conditions: First order

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

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First variation:

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_L(w+\varepsilon\eta) = \int_\Omega \partial_\lambda G_L\big(x, w(x), \nabla w(x)\big)\,\eta(x) + \partial_q G_L\big(x, w(x), \nabla w(x)\big)\cdot\nabla\eta(x)\,\mathrm{d}x$$

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\end{aligned}$$

u is minimizer of $\mathcal{E}_L \implies \mathcal{L}_L u = 0$ in Ω (in the weak sense)

Necessary conditions: Second order

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

Second variation:

$$\begin{aligned}\frac{\partial^2}{\partial\varepsilon^2}\Big|_{\varepsilon=0}\mathcal{E}_L(w+\varepsilon\eta) = \int_\Omega \Big\{&\partial^2_{\lambda\lambda} G_L(x, w(x), \nabla w(x))\,\eta^2(x)\\ &+ 2\partial^2_{\lambda q} G_L(x, w(x), \nabla w(x)) \cdot \eta(x)\nabla\eta(x)\\ &\quad + \nabla\eta(x)\cdot\partial^2_{qq} G_L(x, w(x), \nabla w(x))\nabla\eta(x)\Big\}.\end{aligned}$$

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Theorem (Legendre Condition)

Let u be a minimizer of the energy functional $\mathcal{E}_L$ among functions with the same boundary data. Then, it satisfies

$$\partial^2_{qq} G_L(x, u(x), \nabla u(x)) \geq 0 \quad \textit{in } \Omega.$$

Necessary conditions: Weierstrass

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

Weierstrass excess function:

$$E(x, \lambda, q, \tilde{q}) = G_L(x, \lambda, \tilde{q}) - G_L(x, \lambda, q) - \partial_q G_L(x, \lambda, q) \cdot (\tilde{q} - q)$$

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$$E(x,u(x),\nabla u(x),\xi) \geq 0 \quad \text{for all } x\in\Omega,\ \xi\in\mathbb{R}^n.$$

Sufficient conditions: Convexity

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

$$G_L(x, \lambda, q) \text{ convex in } (\lambda, q) \implies \mathcal{E}_L \text{ is convex}$$

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Lemma (Comparison of energies)

Assume that $G_L(x,\lambda,q)$ is convex in (λ,q). Then, given $u, w \in H^1(\Omega)$, they satisfy

$$\mathcal{E}_L(w) \geq \mathcal{E}_L(u) + \int_\Omega \mathcal{L}_L u(x)\,\big(w(x) - u(x)\big)\,\mathrm{d}x + \int_{\partial\Omega} \mathcal{N}_L u(x)\,\big(w(x) - u(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}(x)$$

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Sufficient conditions: Weierstrass extremal field theory

$$\mathcal{E}_L(w) = \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\,\mathrm{d}x$$

Theorem (Weierstrass Sufficient Condition)

Assume that $G_L(x, \lambda, q)$ is convex in q. If u is embedded in an extremal field, then it is a minimizer of $\mathcal{E}_L$.

Sufficient conditions: Weierstrass extremal field theory

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Theorem [Weierstrass]

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We say that a family of functions $\{u^t\}_{t\in\mathbb{R}}$ is a *field* if

- the functions $t \mapsto u^t(x)$ are increasing for each x
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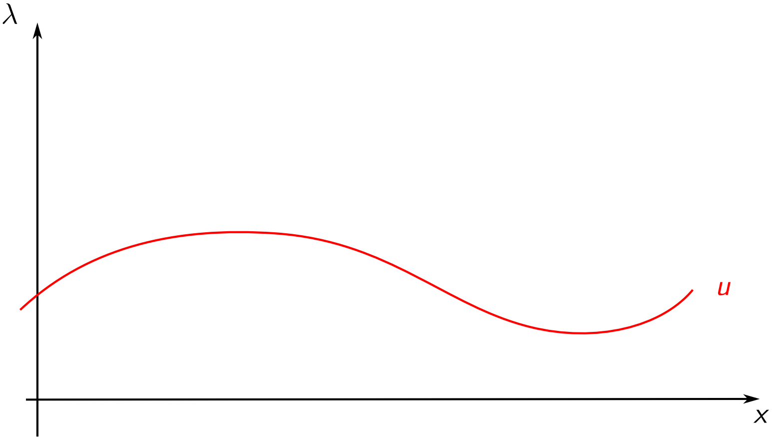
Moreover, it is an extremal field if each leaf u^t satisfies the E-L equation.

Field and the leaf-parameter function



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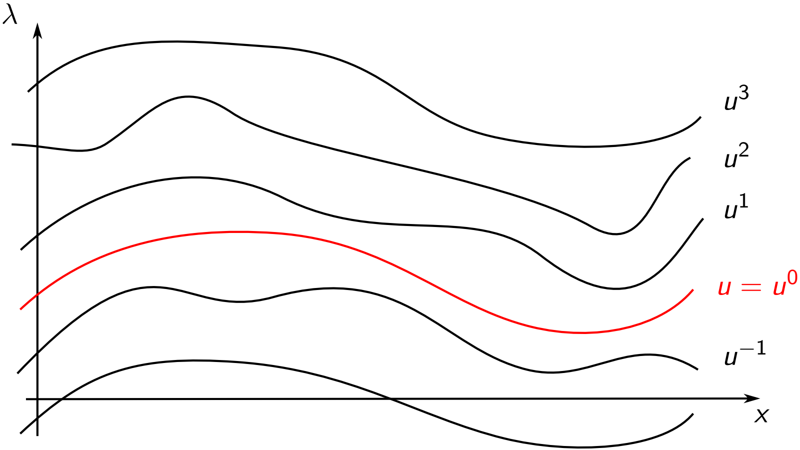

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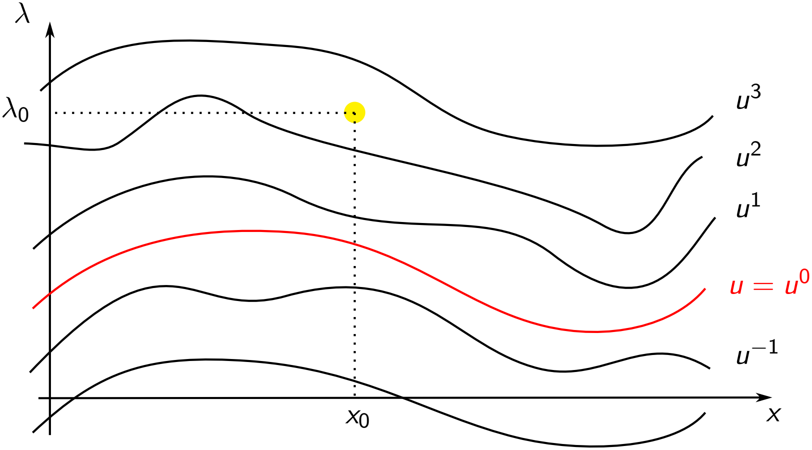

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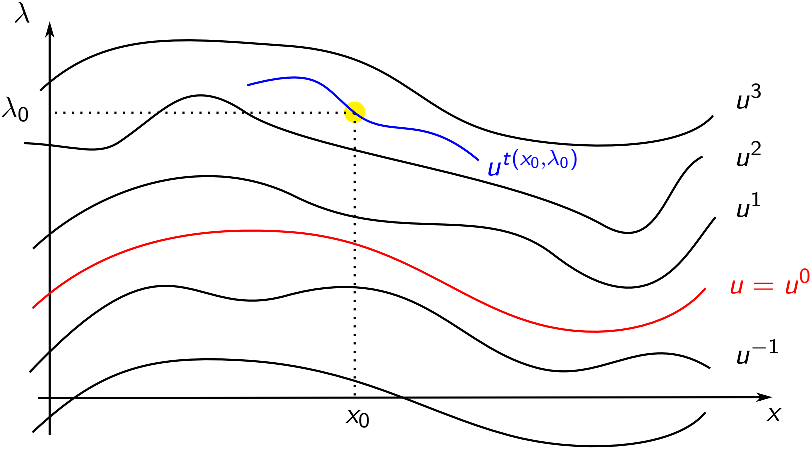

$t(x_0, \lambda_0)$ is the unique $\tau \in \mathbb{R}$ such that $u^\tau(x_0) = \lambda_0$

Proof via the construction of a calibration

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$$\Downarrow$$

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Nonlocal theory of Calibrations

Fractional energy functional

$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

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First variation:

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_s(w + \varepsilon\eta) = \int_\Omega \left\{(-\Delta)^s w(x) - F'(w(x))\right\} \eta(x) \, \mathrm{d}x + \int_{\Omega^c} \mathcal{N}_s w(x) \, \eta(x) \, \mathrm{d}x,$$

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The fractional Laplacian

The canonical example of integro-differential operator

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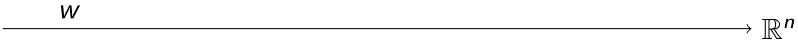
- Representation via the heat semigroup:

$$(-\Delta)^s w(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(\mathrm{e}^{t\Delta} w - w\right) \frac{dt}{t^{1+s}}$$

The extension problem for the fractional Laplacian



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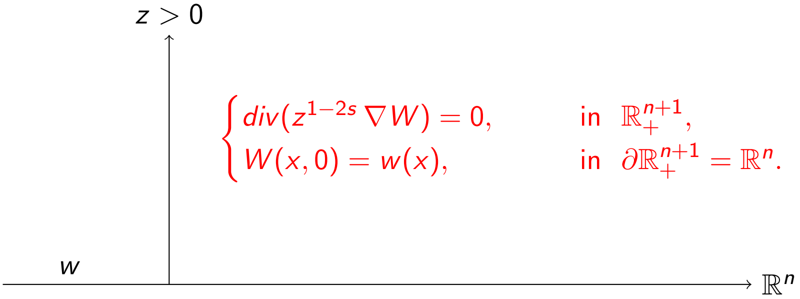

$$\begin{cases} div(z^{1-2s}\,\nabla W) = 0, & \text{in } \ \mathbb{R}^{n+1}_+, \\ W(x,0) = w(x), & \text{in } \ \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases}$$

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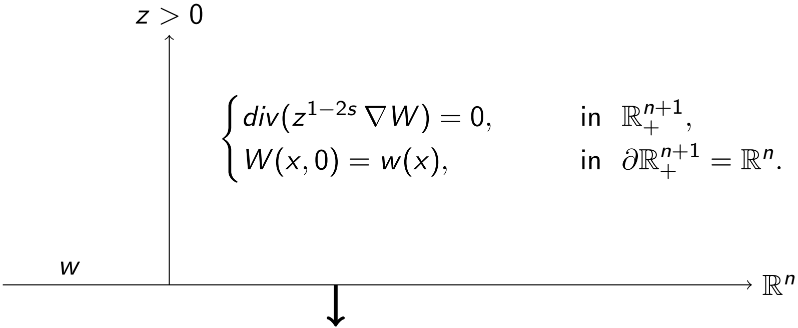

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$$-d_s \lim_{z\downarrow 0} z^{1-2s}\partial_z W = (-\Delta)^s w$$

Towards the fractional calibration: First attempts

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x))\, \mathrm{d}x$$

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Can we find the same structure in the local theory?

Revisit local theory

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$$\mathcal{E}_1(w) = \frac{1}{2}\int_\Omega |\nabla w(x)|^2 \,\mathrm{d}x - \int_\Omega F(w(x))\,\mathrm{d}x$$

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Revisit local theory

Can we find the same structure in the local theory? YES

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$$\mathcal{E}_s(w) = \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

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Theorem [Cabré, Erneta & F-N '21]

Let u be embedded in an extremal field $\{u^t\}_{t\in\mathbb{R}}$ such that $(x,t) \to u^t(x)$ is a bounded C^2 function, and let $\mathcal{C}_s$ be the functional

$$\begin{aligned} \mathcal{C}_s(w) = {} & \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\} \Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x \\ & + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x) \Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u). \end{aligned}$$

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Then, it follows that $\mathcal{C}_s$ is a calibration for the functional $\mathcal{E}_s$ and u.

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$?

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \, \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \, \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$?

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓

Fractional calibration: Proof I

$$\begin{aligned}\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x \\ + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_s(u)\end{aligned}$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
- $\mathcal{C}_s(u) = \mathcal{C}_s(w)$ for all w with the same Dirichlet condition as u?

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{(-\Delta)^s u^t(x) - F'(u^t(x))\right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

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- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
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Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \Big\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \Big\}\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
- $\mathcal{C}_s(u) = \mathcal{C}_s(w)$ for all w with the same Dirichlet condition as u? ✓
- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\} \Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x) \big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
- $\mathcal{C}_s(u) = \mathcal{C}_s(w)$ for all w with the same Dirichlet condition as u? ✓
- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}} \bigg|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y.$$

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_s(u)$$

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Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{(-\Delta)^s u^t(x) - F'(u^t(x))\right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
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Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{(-\Delta)^s u^t(x) - F'(\lambda)\right\}\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \mathcal{E}_s(u)$$

- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
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- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Fractional calibration: Proof I

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- $\mathcal{C}_s(u) = \mathcal{E}_s(u)$? ✓
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$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\Bigg|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y$$
$$- \int_\Omega F(w(x))\,\mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Fractional calibration: Proof I

$$\mathcal{C}_s(w) = \int_\Omega \int_{u(x)}^{w(x)} \left\{ (-\Delta)^s u^t(x) - F'(u^t(x)) \right\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \int_{\Omega^c} \int_{u(x)}^{w(x)} \mathcal{N}_s u^t(x)\big|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x + \mathcal{E}_s(u)$$

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- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c \times \Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

Fractional calibration: Proof II

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda\, \mathrm{d}x\, \mathrm{d}y$$

$$- \int_\Omega F(w(x))\, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, \mathrm{d}x\, \mathrm{d}y.$$

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$$- \int_\Omega F(w(x)) \, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

$$\color{red}{\lambda = u^t(x)} \quad \Big\Downarrow \quad \color{red}{\text{Symmetrization in } x \text{ and } y}$$

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$$- \int_\Omega F(w(x)) \, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y.$$

$$\lambda = u^t(x) \quad \Big\Downarrow \quad \text{Symmetrization in } x \text{ and } y$$

$$\mathcal{C}_s(w) = -\frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}} \, \partial_t u^t(y) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y$$

$$+ \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Fractional calibration: Proof II

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = c_{n,s} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u(x)}^{w(x)} \left.\frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\right|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y$$

$$- \int_\Omega F(w(x))\,\mathrm{d}x + \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

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Fractional calibration: Proof III

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$$\mathcal{C}_s(w) = -\frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}} \, \partial_t u^t(y) \, \mathrm{d}t \; \mathrm{d}x \, \mathrm{d}y$$
$$+ \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

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$$+\frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

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Now,

$$-\frac{1}{2} \int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y)) \partial_t u^t(y) \,\mathrm{d}t$$

Fractional calibration: Proof III

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$$+ \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}y - \int_\Omega F(w(x)) \, \mathrm{d}x$$

Now,

$$-\frac{1}{2} \int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y)) \partial_t u^t(y) \, \mathrm{d}t \leq -\frac{1}{2} \int_{t(x,w(x))}^{t(y,w(y))} (w(x) - u^t(y)) \partial_t u^t(y) \, \mathrm{d}t$$

Fractional calibration: Proof III

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = -\frac{c_{n,s}}{2} \iint_{(\Omega^c\times\Omega^c)^c} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\,\partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$$

$$+\frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|w(x)-u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$

Now,

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

Fractional calibration: Proof III

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Now,

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t \le -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

$$= \frac{1}{4}\int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t}|w(x)-u^t(y)|^2\,\mathrm{d}t$$

Fractional calibration: Proof III

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

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Now,

$$-\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t \leq -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} (w(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

$$= \frac{1}{4}\int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t}|w(x) - u^t(y)|^2\,\mathrm{d}t$$

$$= \frac{1}{4}|w(x) - w(y)|^2 - \frac{1}{4}|w(x) - u^{t(x,w(x))}(y)|^2$$

Fractional calibration: Proof III

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = -\frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}} \partial_t u^t(y)\, \mathrm{d}t \; \mathrm{d}x\, \mathrm{d}y$$
$$+ \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|w(x) - u^{t(x,w(x))}(y)|^2}{|x-y|^{n+2s}} \mathrm{d}x\, \mathrm{d}y - \int_\Omega F(w(x))\, \mathrm{d}x$$

Now,

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Fractional calibration: Proof III

- $\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$ for all w with the same Dirichlet condition as u?

$$\mathcal{C}_s(w) = -\frac{c_{n,s}}{2} \iint_{(\Omega^c\times\Omega^c)^c} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\, \partial_t u^t(y)\,\mathrm{d}t\,\,\mathrm{d}x\,\mathrm{d}y$$

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Then, we conclude

$$\mathcal{C}_s(w) \leq \mathcal{E}_s(w)$$

Nonlocal calibration

$$\mathcal{E}_N(w) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} G_N(x, y, w(x), w(y))\, \mathrm{d}x\, \mathrm{d}y$$

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Theorem [Cabré, Erneta & F-N '21]

Let u be embedded in an extremal field $\{u^t\}_{t\in\mathbb{R}}$ such that $(x,t)\to u^t(x)$ is a bounded C^2 function. Assume that G_N is a function of (x,y,a,b) satisfying $\partial^2_{ab}G_N \le 0$ and let $\mathcal{C}_N$ be the functional

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Then, it follows that $\mathcal{C}_N$ is a calibration for the functional $\mathcal{E}_N$ and u.

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The condition $\partial^2_{ab} G_N \leq 0$ turns to be the natural ellipticity condition for the problem.

The ellipticity condition $\partial^2_{ab} G_N \leq 0$

$$\partial^2_{ab} G_N \leq 0 \quad \Longrightarrow \quad \mathcal{L}_N \ \text{ satisfies the SCP}$$

We say that the operator $\mathcal{L}_N$ satisfies the Strong Comp. Princ. (SCP) if

$$\begin{cases} \mathcal{L}_N w \leq \mathcal{L}_N v & \text{in } \Omega \\ w \leq v & \text{in } \mathbb{R}^n \\ w(x_0) = v(x_0) & \text{for some } x_0 \in \Omega \end{cases} \quad \Longrightarrow \quad w \equiv v \text{ in } \Omega$$

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$$\partial^2_{ab} G_N \leq 0 \quad \not\Rightarrow \quad \mathcal{L}_N \ \text{ satisfies the WCP}$$

We say that the operator $\mathcal{L}_N$ satisfies the Weak Comp. Princ. (WCP) if

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Examples of nonlocal Lagrangians

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$$G_N(x, y, a, b) = \frac{|a-b|^p}{2p|x-y|^{n+ps}}$$

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where $s \in (0,1)$,
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$$G_N(x,y,a,b) = -\mathbb{1}_{\Omega\times\Omega}(x,y)K(x-y)\,a\,b$$

corresponds to convolution-type energies.

Application to monotone solutions

$$\mathcal{L}_N(u)(x) = \int_{\mathbb{R}^n} \partial_a G_N(x, y, u(x), u(y))\,\mathrm{d}y$$

Corollary

Let u be a sufficiently regular solution of $\mathcal{L}_N(u) = 0$ in $\mathbb{R}^n$ satisfying the monotonicity condition $\partial_{x_n} u > 0$ in $\mathbb{R}^n$.
Assume that the ellipticity condition $\partial_{ab} G_N \leq 0$ holds and that $\mathcal{L}_N$ is translation invariant.

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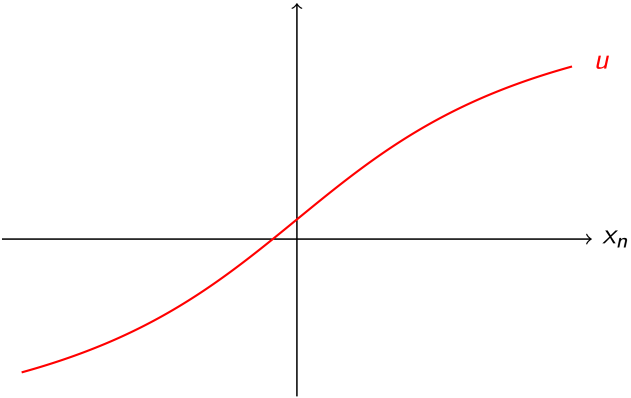
Then, for each bounded domain $\Omega \subset \mathbb{R}^n$, u is a minimizer of $\mathcal{E}_N$ in the set of functions w satisfying

$$\lim_{\tau \to -\infty} u(x', \tau) < w(x', x_n) < \lim_{\tau \to +\infty} u(x', \tau)$$

and such that $w \equiv u$ in Ω^c.

Application to monotone solutions: The field



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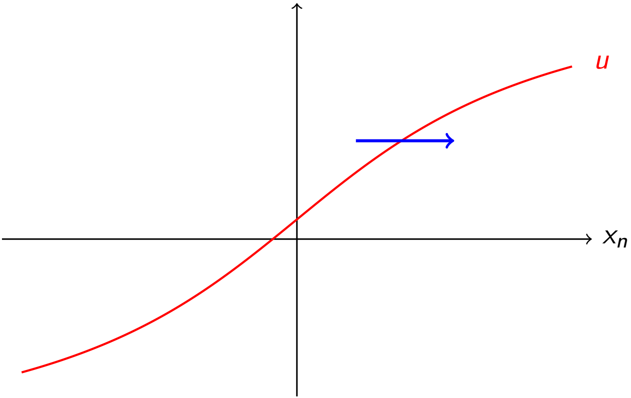

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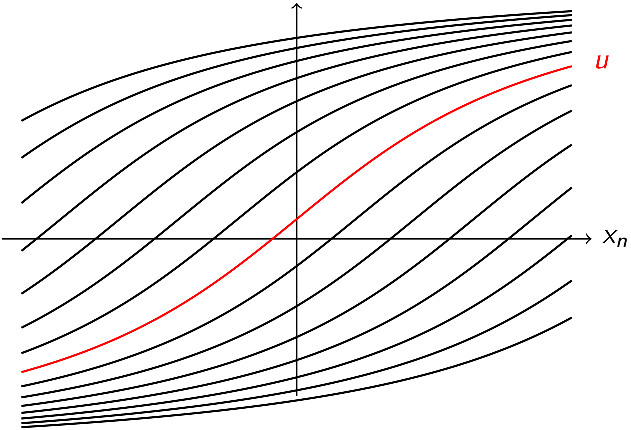

$$u^t(x) := u(x + t\,e_n)$$

Application to the viscosity theory

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Theorem (Cabré, Erneta & F-N '21)

Let u be a minimizer of $\mathcal{E}_N$ among functions with same Dirichlet conditions. Assume that the ellipticity condition $\partial_{ab} G_N \leq 0$ holds. Then, it is a viscosity solution of the associated equation $\mathcal{L}_N(u) = 0$ in Ω.

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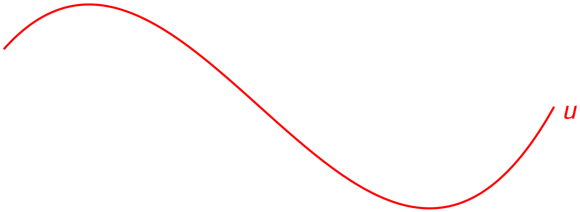
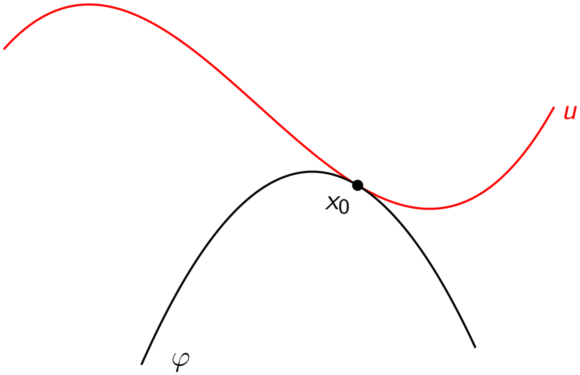
- We do not need a weak maximum principle
- We only treat minimizers, not general weak solutions

Application to the viscosity theory: Proof



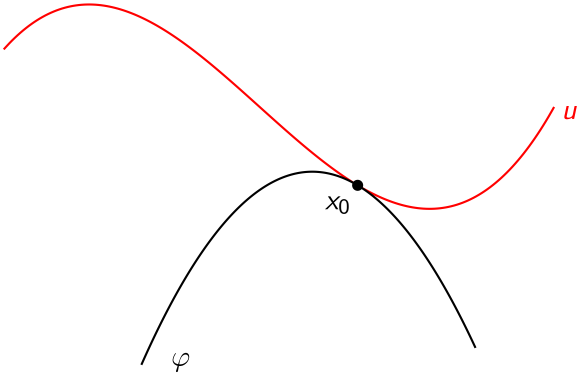
Application to the viscosity theory: Proof

$$\mathcal{L}_N\varphi(x_0) \geq 0?$$



Application to the viscosity theory: Proof

$\mathcal{L}_N\varphi(x_0) \geq 0$? Assume not



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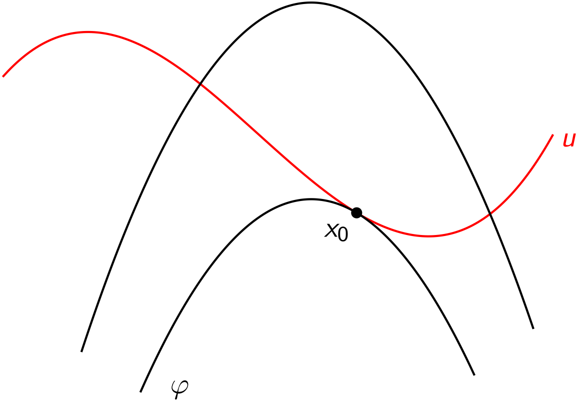

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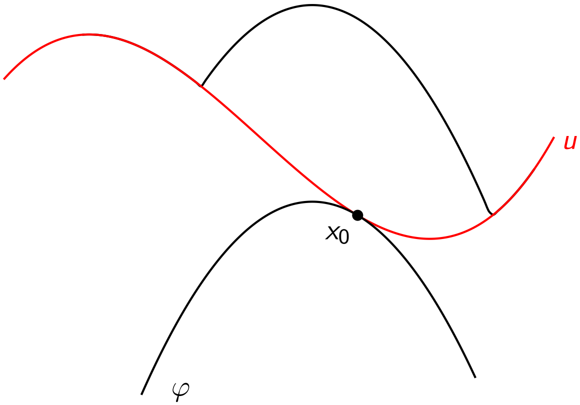

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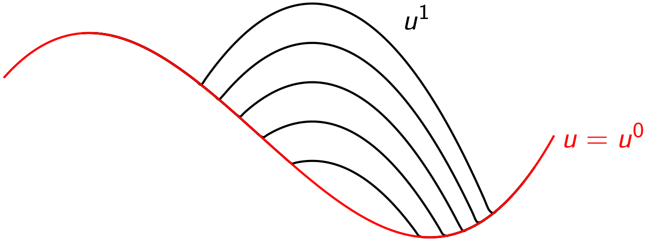

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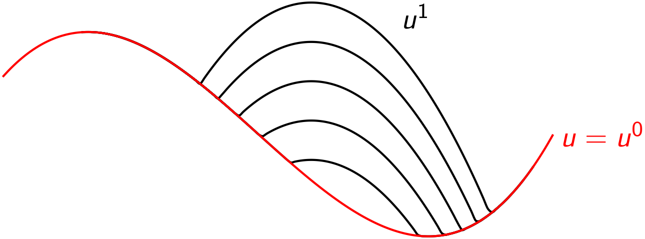

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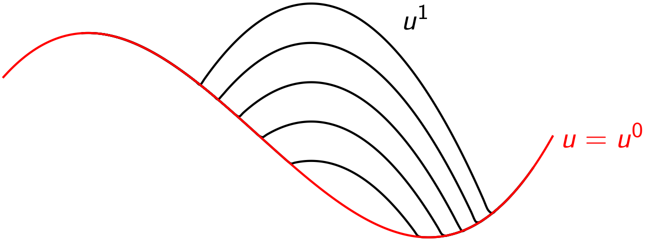

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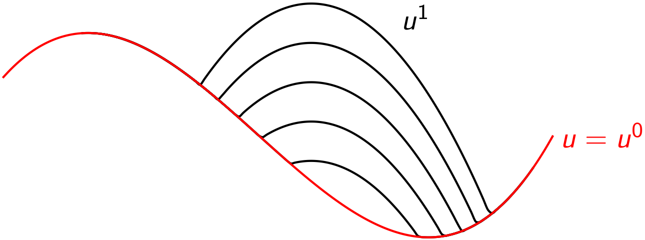

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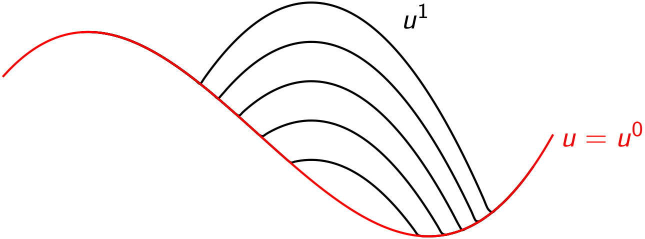

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Open questions

- Can we find nonlocal analogues to Legendre and Weierstrass Necessary conditions?

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- Can we find nonlocal analogues to Legendre and Weierstrass Necessary conditions?
- Can we develop a nonlocal Weierstrass extremal field theory for nonlocal functionals of the form:

$$\mathcal{E}(w) = \int_\Omega G(x, w(x), \nabla^s w(x))\,\mathrm{d}x \quad \text{or} \quad \mathcal{E}(w) = \int_\Omega G(x, w(x), (-\Delta)^{s/2} w(x))\,\mathrm{d}x?$$

Thank You

Towards the fractional calibration

$$\mathcal{C}_1(w) = \int_\Omega \left\{ \nabla u^t(x) \cdot \nabla w(x) - \frac{1}{2} |\nabla u^t(x)|^2 \right\} \Big|_{t=t(x,w(x))} \mathrm{d}x - \int_\Omega F(w(x))\, \mathrm{d}x$$

- Replace the gradient terms by differences and integrals by double integrals

$$\begin{aligned}
\mathcal{F}_s^2(w) = {} & \frac{c_{n,s}}{2} \iint_{(\Omega^c \times \Omega^c)^c} \frac{(u^\tau(x) - u^t(y))(w(x) - w(y))}{|x-y|^{n+2s}} \Bigg|_{\substack{t=t(x,w(x)) \\ \tau=t(y,w(y))}} \mathrm{d}x\, \mathrm{d}y \\
& - \frac{c_{n,s}}{4} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|u^\tau(x) - u^t(y)|^2}{|x-y|^{n+2s}} \Bigg|_{\substack{t=t(x,w(x)) \\ \tau=t(y,w(y))}} \mathrm{d}x\, \mathrm{d}y \\
& - \int_\Omega F(w(x))\, \mathrm{d}x
\end{aligned}$$

Nonlocal perimeter

$$\mathcal{P}(F) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} |\mathbb{1}_F(x) - \mathbb{1}_F(y)| K(x-y)\, \mathrm{d}x\, \mathrm{d}y$$

$$\Downarrow \quad \text{[Cabré '19]}$$

$$\begin{aligned} \mathcal{C}_{\mathcal{P}}(F) = &\int_{\Omega} \mathbb{1}_F(x)\, H_K[E^t](x)\Big|_{t=\phi(x)} \mathrm{d}x \\ &+ \int_{\Omega^c} \mathbb{1}_F(x) \left\{ \int_{\Omega} \big(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y)\big) K(x-y)\, \mathrm{d}y \right\} \Bigg|_{t=\phi(x)} \mathrm{d}x \end{aligned}$$

Here the Euler-Lagrange operator is given by

$$H_K[E](x) = \int_{\mathbb{R}^n} \big(\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)\big) K(x-y)\, \mathrm{d}y$$