THE NEUMANN PROBLEM FOR THE FRACTIONAL LAPLACIAN: BOUNDARY REGULARITY

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joint work with A. Audrito (ETH Zurich) and X. Ros-Oton (ICREA-UB)

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We consider the energy functional

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Regularity for the local case

- Interior regularity: $-\Delta u = f$ in B_1, with $f \in L^\infty(B_1)$

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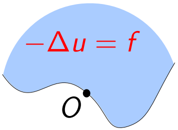

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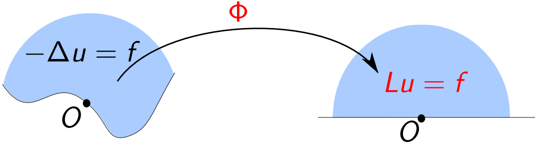

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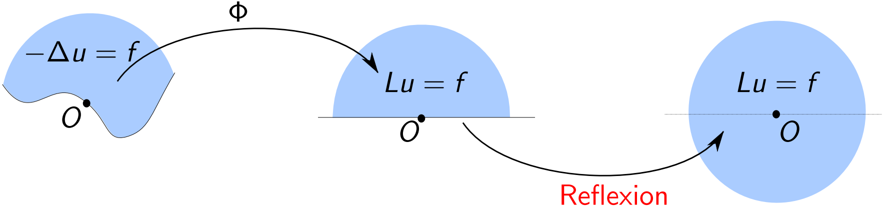

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- If Neumann boundary condition is imposed, the particle reflects inwards and the process continues

The fractional Laplacian

Defined for $s \in (0,1)$ by

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}} \, \mathrm{d}z.$$

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- It has an associated local extension problem.

Functional setting

When working with the fractional Laplacian in bounded domains, the natural functional space to work is

$$\mathcal{H}^s(\Omega) = \{u \in L^2(\Omega) \ \text{ such that } [u]_{\mathcal{H}^s(\Omega)} < +\infty\},$$

where

$$[u]_{\mathcal{H}^s(\Omega)} := \left(\iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}z \right)^{1/2},$$

and $\mathcal{H}^s(\Omega)$ is a Banach space with associated norm

$$\|u\|_{\mathcal{H}^s(\Omega)} = \|u\|_{L^2(\Omega)} + [u]_{\mathcal{H}^s(\Omega)}$$

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This problem has a nice probabilistic interpretation in terms of Levy flights.

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This regularity result is optimal since explicit solutions can be constructed:

$$\left.\begin{array}{rcl} (-\Delta)^s u &=& 1 \quad \text{in } B_1, \\ u &=& 0 \quad \text{in } B_1^c. \end{array}\right\} \Longrightarrow u(x) = c(1-|x|^2)_+^s$$

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where $\mathcal{N}_s$ is a "nonlocal Neumann derivative" defined as

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Existence and uniqueness of weak solutions can be found in [Dipierro, Ros-Oton and Valdinoci, 2017] ([Du-Gunzburger-Lehoucq-Zhou,'13])

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- Natural probabilistic interpretation
- Conservation of mass in the associated heat equation
- Convergence to the classical Neumann problem as $s \uparrow 1$

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Theorem (Audrito, F-N, Ros-Oton, '20)

Let $\Omega \subset \mathbb{R}^N$ be any bounded C^1 domain and $f \in L^\infty(\Omega)$ and $\int_\Omega f = 0$. Then, weak solutions are bounded, and

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C \left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)} \right),$$

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This is the first regularity result for the Neumann problem. Even the boundedness of solutions is new.

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- Classical methods do not work (flatten the boundary and even reflection)
- The values of u are not known neither inside nor outside Ω
- The methods from the Dirichlet case, based on barriers, do not work either

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On the other hand, it is also clear that

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This phenomenon is due to a missing regularity condition in the strong formulation that w is does not satisfy.

Equivalent formulation

It is equivalent to minimize the nonlocal energy

$$\mathcal{E}(u) := \frac{c_{n,s}}{4} \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z - \int_\Omega f\,u$$

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than minimizing the "localized" one

$$\overline{\mathcal{E}}(u) := \frac{c_{n,s}}{4} \iint_{\Omega \times \Omega} |u(x) - u(z)|^2 \, K_\Omega(x, z) \, \mathrm{d}x \, \mathrm{d}z - \int_\Omega f \, u$$

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$$\overline{\mathcal{E}}(u) := \frac{c_{n,s}}{4} \iint_{\Omega \times \Omega} |u(x) - u(z)|^2 \, K_\Omega(x, z) \, \mathrm{d}x \, \mathrm{d}z - \int_\Omega f \, u$$

where K_Ω is an explicit kernel satisfying

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Equivalent formulation

It is equivalent to minimize the nonlocal energy

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It was first noted in the strong form by [Abatangelo, '20].

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- In some of the computations it is sufficient to consider the flat case by doing a bi-Lipschitz transformation

The regional fractional Laplacian

Motivated by the equivalent formulation, we first study the minimization of the functional

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Theorem (Bogdan, Burdzy, Chen, 2003)

Assume $\partial\Omega \in C^2$ and $f \in L^\infty(\Omega)$. Then, $u \in C^{2s-1}(\overline{\Omega})$.

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In a more recent work, M. Fall proves $C^{2s-\varepsilon}$ regularity up to the boundary by using the extension problem.

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We use the ideas coming from the interior regularity theory for divergence form elliptic equations. That is, we do a delicate Moser iteration with logarithmic corrections.

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$$\Big\Downarrow \quad \varphi \approx u^{-p} \quad + \quad \text{J-N Ineq.}$$

$$u \in C^\alpha(\overline{\Omega}) \text{ for some small } \alpha$$

Higher regularity (when $s > 1/2$)

We want to prove that

$$|u(x) - u(y)| \leq C|x - y|^{\sigma} \quad \text{for all} \quad x, y \in \Omega.$$

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Thus, we arrive at a contradiction if we show that a weak solution with certain growth in a half-space is constant.

From dimension n to dimension 1

$$\left.\begin{array}{rcll} (-\Delta)^s v & = & 0 & \text{in } \mathbb{R}^n_+, \\ \mathcal{N}_s v & = & 0 & \text{in } \mathbb{R}^n_-, \\ \|v\|_{L^\infty(B_R)} & \leq & R^\sigma, & \text{with } \sigma < 1. \end{array}\right\} \Longrightarrow v \equiv w(x_n)$$

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- We construct the auxiliary function

$$v_1(x) = \frac{v(x+he) - v(x)}{|h|^\alpha}$$ for any given $e = (\tilde{e}, 0) \in S^{n-1}$ and $h > 0$.

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- Since it grows less than linear, it should be constant in that directions

1D Liouville theorem

We want to prove

$$\left.\begin{array}{rcll} (-\Delta)^s w & = & 0 & \text{in } (0,\infty), \\ \mathcal{N}_s w & = & 0 & \text{in } (-\infty,0), \\ |w(x)| & \leq & 1+|x|^{\sigma} & \text{in } \mathbb{R}. \end{array}\right\} \Longrightarrow w \equiv \text{ctt}$$

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- Letting $R \to \infty$ we arrive at $w(x) = C|x|^{2s-1}$.

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- In particular the solution satisfies the Dirichlet problem for a Regional type operator.
- We prove a boundary Harnack inequality in B_R by using the "fake solution" as barrier.
- Letting $R \to \infty$ we arrive at $w(x) = C|x|^{2s-1}$.
- We conclude that $w \equiv 0$.

Thank You