EQUACIONS EN DERIVADES PARCIALS

Exercicis, solucions i exàmens

Grau en matemàtiques

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CONTINGUTS:

Problema 1. Resoleu

$$\begin{cases} u_x + u_y + u = e^{x+2y} & (x,y) \in \mathbb{R}^2 \\ u(x,0) = 0 & x \in \mathbb{R}. \end{cases}$$

Problema 2. Trobeu la solució del problema

$$\begin{cases} u_x + 3y^{2/3} u_y = 2 \\ \qquad u(x,1) = 1 + x. \end{cases}$$

Problema 3.* (a) Digueu per quines funcions $d(t)$ i $g(x)$ el següent problema té solució clàssica:

$$\begin{cases} u_t + u_x = -u & x > 0,\ t > 0 \\ u(0,t) = d(t) & t > 0 \\ u(x,0) = g(x) & x > 0. \end{cases}$$

(b) Si $d \equiv 1$ i $g \equiv 0$, demostreu que la solució trobada pel mètode de les característiques és solució generalitzada (és a dir, en sentit integral).

Problema 4. Considereu l'equació de transport següent amb $c > 0$

$$\begin{cases} u_t + c u_x = f(x,t) & 0 < x < R,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < R. \end{cases}$$

Discutiu l'existència i unicitat de solució (no cal trobar-ne l'expressió explícita). Demostreu la desigualtat (estimació d'estabilitat)

$$\int_0^R u^2(x,t)\,dx \le e^t \int_0^t \int_0^R f^2(x,s)\,dx\,ds, \quad \text{per a tot } t > 0.$$

[Indicació: Multipliqueu l'equació per u, useu $c > 0$ i $2fu \le f^2 + u^2$, per arribar a

$$\frac{d}{dt} \int_0^R u^2(x,t)\,dx \le \int_0^R f^2(x,t)\,dx + \int_0^R u^2(x,t)\,dx.]$$

Problema 5.* Al pla $\mathbb{R}^2$, considerem l'EDP quasi-lineal de primer ordre

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{1}$$

Per resoldre-la, definim el camp vectorial a $\mathbb{R}^3$ donat per

$$X(x,y,z) = (a(x,y,z), b(x,y,z), c(x,y,z)).$$

Com la direcció normal a la gràfica de u és $\nu(x,y,z) = (-u_x, -u_y, 1)$, llavors la EDP és equivalent a $\nu \cdot X = (-u_x, -u_y, 1) \cdot (a,b,c) = 0$. Donat un domini $\Omega \subset \mathbb{R}^2$ i donada u a una part de la frontera $\partial\Omega$, parametritzada per una corba $(\alpha(s), \beta(s))$, volem trobar u a tot (o a una part de) Ω. Les corbes característiques

$$\begin{cases} dx/dt = a(x,y,z) \\ dy/dt = b(x,y,z) \\ dz/dt = c(x,y,z) \end{cases}$$

defineix un sistema de EDOs amb condicions inicials, fixat un s, $x(0) = \alpha(s)$, $y(0) = \beta(s)$ and $z(0) = \gamma(s) := u(\alpha(s), \beta(s))$. Resolent el sistema, obtenim $x = x(s,t), y = y(s,t), z = z(s,t)$. Si, pel teorema de la funció inversa, podem invertir $x = x(s,t), y = y(s,t)$ i obtenim $s = s(x,y), t = t(x,y)$, demostreu que llavors $u = z(s(x,y), t(x,y))$ és la solució del problema. Demostreu també que la invertibilitat anterior requereix que

$$\begin{vmatrix} \dot\alpha & a \\ \dot\beta & b \end{vmatrix} \neq 0.$$

Problema 6.* Trobeu la solució $u = u(x,y)$ del problema

$$\begin{cases} uu_x + yu_y = x \\ \quad u(x,1) = 2x. \end{cases}$$

Problema 7.* (a) En una autopista rectilínia infinita, $v = v(x,t)$ representa la densitat de cotxes, essent $v = 1$ la densitat màxima (en certes unitats; llavors tots els cotxes estan tocant-se i parats) mentre $v = 0$ és l'absència de cotxes. La velocitat d'un cotxe (el cotxe és infinitesimal i es representa per un punt x) depèn de la densitat de cotxes en aquell punt i ve donada per $120(1-v)$, en km/h. Els cotxes estan, per tant, parats en cas de densitat màxima i circulen a 120 km/h en absència d'altres. Escriviu la llei de conservació de massa en qualsevol interval d'espai a un cert temps t i deduïu l'equació del trànsit.

(b) Feu un canvi de la variable incògnita v per escriure l'equació com la de Burgers: $u_t + uu_x = 0$. Sigui $g(x) = u(x,0)$ la condició inicial. Useu el mètode de les característiques per demostrar que u satisfà l'equació funcional $u - g(x - ut) = 0$. Useu el teorema de la funció implícita per demostrar que, si g té derivada acotada a tot $\mathbb{R}$, llavors existeix una solució clàssica del problema de Cauchy en un interval $[0, t_0]$ amb $t_0 > 0$, i estimeu aquest temps t_0.

Demostreu que si g és creixent llavors podem prendre $t_0 = \infty$. Interpreteu el fet "g és creixent" traduït a l'apartat (a) sobre l'equació del trànsit (després de fer el canvi de u a v) i comenteu el sentit que no hi hagi xocs i la solució clàssica existeixi per tot temps positiu per aquestes condicions inicials. Es podria resoldre l'equació per temps negatius?

Problema 8.* (i) Ordeneu per ordre de feblesa les topologies provinents de les normes estàndard als espais $C([a,b])$, $C^1([a,b])$, $L^1(a,b)$ i $L^2(a,b)$. Discutiu l'existència de desigualtats relacionant aquestes normes (és a dir, que una norma sigui més petita o igual que una altra, llevat d'una constant multiplicativa), i trobeu les millors constants en aquestes desigualtats.

(ii) Digueu si les successions $\{x^k\}$, $\{(x/2)^k\}$ i $\{\sin(k\pi x)\}$ tenen parcials convergents a $L^1\big((0,1)\big)$, $L^2\big((0,1)\big)$, $C\big([0,1]\big)$, $C^2\big([0,1]\big)$.

Problema 9. Sigui E un espai de Banach. Per $g \in E$ i $u : [-\varepsilon, \varepsilon] \subset \mathbb{R} \to E$, considereu l'EDO

$$\begin{cases} u_t = F(u), & \text{per } t \in [-\varepsilon, \varepsilon], \\ u(0) = g, \end{cases}$$

on $F : E \to E$ és una funció localment Lipschitz (és a dir, és Lipschitz quan restringida sobre cada conjunt fitat de E). Escrivint l'EDO de manera integral, useu el teorema del punt fix per contraccions en espais de Banach per demostrar l'existència i unicitat de solució $u \in C^1([-\varepsilon, \varepsilon], E)$ de l'EDO si ε és prou petit.

Problema 10. Useu el mètode de les característiques per resoldre

$$\begin{cases} u_t + cu_x = e^{-t} \sin x & x \in \mathbb{R},\ t > 0 \\ u(x, 0) = 0. \end{cases}$$

Comproveu que obteniu el mateix resultat usant la fórmula de Duhamel.

Problema 11. (a) Resoleu

$$\begin{cases} u_t + b \cdot \nabla_x u = -\gamma u & x \in \mathbb{R}^n,\ t > 0 \\ u(x, 0) = g(x) & x \in \mathbb{R}^n. \end{cases}$$

Observeu la diferència en el comportament de la solució u segons el signe de γ. En dimensió $n = 1$ i per $\gamma > 0$, aquesta equació modelitza el transport en un tub infinit amb corrent constant d'un contaminant que es degrada (per exemple degut a descomposició biològica o a que és absorbit pel medi). Aquest tipus de solució s'anomena ona viatgera esmorteïda (damped traveling wave).

(b) La solució u de l'apartat anterior es pot escriure com $u = T_t g$ on

$$T_t : C^k(\mathbb{R}^n) \longrightarrow C^k(\mathbb{R}^n)$$

és un semigrup d'operadors lineals. Trobeu T_t i comproveu la identitat de semigrup: $T_{t+s} = T_t T_s$.

Problema 12.* Un contaminant en un tub infinit amb un corrent de velocitat constant de c m/s cap a la dreta, inicialment amb una concentració de $g(x)$ g/m, s'absorbeix (en g/m/s) proporcionalment al quadrat de la seva concentració. Trobeu la concentració de contaminant a temps $t > 0$ i el semigrup no lineal associat. Comproveu que la llei de semigrup es satisfà.

Els problemes amb asterisc $*$ es resoldran a classe de problemes

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Problema 1. Considerem l'operador
$$(Av)(x) := \frac{1}{x}\int_0^x v(y)\,dy, \quad x \in (0,1],$$
actuant sobre funcions $v \in C^0([0,1])$. Demostreu que, per tota $g \in C^0([0,1])$, el problema
$$\begin{cases} u_t = Au + \sin u, \\ u(x,0) = g(x), \quad x \in [0,1], \end{cases}$$
admet una única solució $u = u(x,t)$ que és contínua a tot $[0,1]$ per a cada temps t en un cert interval. Quin és l'interval màxim de temps t en el que la solució u existeix?

Problema 2.* Considereu l'operador A definit per
$$(Av)(x) = \int_0^1 \big(v(y) - v(x)\big)\,dy, \quad x \in [0,1].$$

(a) Comproveu que $A : C^0([0,1]) \to C^0([0,1])$ és lineal i continu.

(b) Demostreu l'existència i unicitat de solució (per temps prou petits) del problema no lineal $u_t - Au = u^3$, $u(x,0) = g(x)$.

(c) Trobeu la solució de $u_t - Au = 0$, $x \in (0,1)$, $t > 0$, amb condició inicial $u(x,0) = g(x)$. Calculeu $\lim_{t\to+\infty} u(x,t)$ (fins i tot si no heu resolt la qüestió anterior d'aquest apartat c).

(d) Quan val Av en els punts on s'assoleix el màxim (i el mínim) de $v = v(x)$? En aquesta línia, comenteu breument les diferències i similituds entre l'operador A i el Laplacià.

Problema 3. Escrivim l'equació del transport lineal $u_t + cu_x = 0$ com $u' = Au$ on, per cada t, $u = u(t)$ pertany a un espai de funcions de x, $u(t)(x) = u(x,t)$, u' denota la derivada respecte al temps t, i $A = -c\partial_x$ és un operador diferencial lineal. Si bé aquest operador no és continu entre l'espai de Banach C^k i ell mateix, amb k enter, insistim a usar, almenys formalment, la fórmula per $e^{tA}g$ donada per una sèrie de potències (on $g = g(x)$ és la condició inicial). Feu-ho i sumeu la sèrie per arribar a una fórmula útil.

[Nota: el càlcul formal que heu fet requereix i només s'aplicaria a condicions inicials $g \in C^\infty$ (i encara més, necessitaríem g analítica). De totes formes, inclús així el càlcul segueix sent formal, doncs es pot demostrar que l'espai vectorial de funcions C^∞ no es pot dotar d'una norma tal que: el faci de Banach, doni la topologia habitual de convergència uniforme de la funció i de totes les seves derivades i, finalment, faci que l'operador A sigui continu.]

Problema 4.* Considerem l'equació d'evolució no lineal

$$\begin{cases} u_t = -cu_x + u(x,t)\int_0^1 |x-y|^{-1/3}u(y,t)\,dy, & x\in(0,1), t\in(0,t_m), \\ u(0,t)=0, & t\in(0,t_m), \\ u(x,0)=g(x), & x\in(0,1), \end{cases}$$

amb $c>0$ una constant i $g\in L^2(0,1)$ donada. Demostreu l'existència i unicitat de solució en sentit integral (en un cert espai de Banach a trobar) per temps t_m prou petit. Doneu una cota inferior explícita pel temps t_m.

Noteu i discutiu el fet següent: en aquest problema per l'equació diferencial del transport (que involucra l'operador A del problema 3) hem hagut de posar una condició de vora o de contorn a $x=0$. En canvi, en els problemes que involucren operadors A lineals integrals continus (problemes 1, 2 i 5) ja hi ha existència i unicitat de solució sense imposar condicions de vora (i, per tant, no es poden imposar tals condicions).

Problema 5.* Considerem l'operador

$$(Bv)(x) := \int_0^x v(y)\,dy, \quad x\in[0,1],$$

actuant sobre funcions $v\in C^0([0,1])$. Considerem el problema d'evolució, per $u=u(x,t)$,

$$\begin{cases} u_t = Bu, \\ u(x,0)=g(x), \quad x\in[0,1], \end{cases}$$

per una $g\in C^0([0,1])$ donada.

(i) Usant la sèrie de potències per a l'exponencial d'un operador, trobeu una fórmula explícita per $u=u(x,t)$ (la fórmula pot involucrar una sèrie de potències que no cal sumar). Justifiqueu que es pot usar la fórmula de l'exponencial.

(ii) [Aquest apartat es pot fer independentment d'haver fet o no l'apartat anterior] Demostreu que hi ha una única solució del problema d'evolució i que aquesta és de la forma

$$u(x,t) = g(x) + \int_0^x tK(t(x-z))g(z)\,dz,$$

per una certa funció $K:\mathbb{R}\to\mathbb{R}$. Trobeu l'EDO i les condicions inicials que satisfà aquesta funció K, justificant que determinen la funció K de manera única. Trobeu el nom i propietats de la funció K (a "WolframAlpha" per exemple).

Problema 1.* (Equipartició de l'energia) Sigui $u \in C^2(\mathbb{R} \times [0,\infty))$ una solució del problema de la corda vibrant

$$\begin{cases} \rho u_{tt} - \tau u_{xx} = 0 & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \\ u_t(x,0) = h(x) & x \in \mathbb{R}. \end{cases}$$

Suposem que g i h són prou regulars i tenen suport en un interval afitat $[a,b]$. L'*energia cinètica* i l'*energia interna* (o potencial) de la corda venen donades per

$$K(t) = \frac{1}{2}\int_{\mathbb{R}} \rho u_t^2(x,t)\,dx \qquad \text{i} \qquad P(t) = \frac{1}{2}\int_{\mathbb{R}} \tau u_x^2(x,t)\,dx.$$

Demostreu:

(a) $K(t) + P(t)$ és constant en t.

(b) $K(t) = P(t)$ per a $t > (b-a)/(2c)$, on $c = \sqrt{\tau/\rho}$ és la velocitat de l'ona al llarg de la corda.

Problema 2.* Considerem el problema

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x > 0,\ t > 0 \\ u(0,t) = d(t) & t > 0 \\ u(x,0) = 0 & x > 0 \\ u_t(x,0) = 0 & x > 0, \end{cases}$$

on $d \in C^2(\mathbb{R})$ i $d(0) = d'(0) = 0$.

(a) Resoleu el problema restant $d(t)$ a la solució i fent reflexió senar.

(b) Alternativament, resoleu el problema factoritzant l'equació d'ones i usant el mètode de les característiques.

(c) Per un impuls d d'un segon donat per

$$d(t) = \begin{cases} 1 - \cos(2\pi t) & t \in (0,1) \\ 0 & t \geq 1 \end{cases}$$

i amb $c = 10\,\mathrm{m/s}$, determineu la longitud, amplitud i posició de l'ona a temps $t = 4\,\mathrm{s}$. És solució clàssica per aquesta d?

Problema 3. Resoleu

$$\begin{cases} u_{tt} - u_{xx} = 1 & x > 0,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(x,0) = 0 & x > 0 \\ u_t(x,0) = 0 & x > 0. \end{cases}$$

És u solució clàssica?

Problema 4. Una ona plana amb suport compacte i alçada h rebota en la vora d'una piscina. Quina es l'alçada màxima que assol.leix l'ona durant la reflexió?

Problema 5.* Considerem l'equació d'ones no lineal

$$u_{tt} - u_{xx} = u^2 + f(x,t) \quad \text{a } Q_T := \mathbb{R} \times (0,T),$$

amb condicions inicials $u(x,0) = u_t(x,0) = 0$, on f és una funció contínua i fitada a $\overline{Q_T}$ amb $\|f\|_{L^\infty(Q_T)} \leq 1$. Treballant en l'espai de funcions contínues i afitades a $\overline{Q_T}$, demostreu l'existència i unicitat de solució (en sentit integral) per T prou petit. Doneu una cota inferior explícita pel temps T.

Problema 1.* (Equipartició de l'energia) Sigui $u \in C^2(\mathbb{R} \times [0, \infty))$ una solució del problema de la corda vibrant

$$\begin{cases} \rho u_{tt} - \tau u_{xx} = 0 & x \in \mathbb{R},\ t \geq 0 \\ u(x, 0) = g(x) & x \in \mathbb{R} \\ u_t(x, 0) = h(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

Suposem que g i h són prou regulars i tenen suport en un interval fitat $[a, b]$. L'*energia cinètica* i l'*energia interna* (o potencial) de la corda venen donades per

$$K(t) = \frac{1}{2} \int_{\mathbb{R}} \rho u_t^2(x, t)\, dx \qquad \text{i} \qquad P(t) = \frac{1}{2} \int_{\mathbb{R}} \tau u_x^2(x, t)\, dx.$$

Demostreu:

(a) $K(t) + P(t)$ és constant en t.
(b) $K(t) = P(t)$ per a $t > (b - a)/(2c)$, on $c = \sqrt{\tau/\rho}$ és la velocitat de l'ona al llarg de la corda.

Solution.

Motivation for definitions: K comes from the formula $\frac{1}{2}mv^2$ for the kinetic energy, and is obtained by integrating $\frac{1}{2}mv^2$ over the string, m considered as infinitesimal mass (or better density ρ) and $v = u_t$ the velocity of the vertical displacement. For the potential energy use that if one stretches an elastic string then the energy is proportional to the new length minus the unstretched length, so in our case, parametrizing the string by $x \mapsto (x, u(x))$ on some interval $I \subset \mathbb{R}$:

$$\text{P for stretched string} = \int_I \tau \sqrt{1 + u_x^2} dx, \qquad \text{P for string resting} = \int_I \tau\, dx.$$

We get that

$$\text{Potential energy in } I = \int_I \tau(\sqrt{1 + u_x^2} - 1) dx.$$

Now assume that u_x is small (under which assumption wave equation was derived!) and use first order Taylor expansion $\sqrt{1 + y} = 1 + y/2 + O(y^2)$ to get that, approximately,

$$\text{Potential enery in } I = \int_I \tau \frac{u_x^2}{2}\, dx.$$

(a) By differentiation and using the wave equation, and then integration by parts (recall that if g and h have support in $[a, b]$, then by d'Alembert formula $u(\cdot, t)$ has support in $[a-ct, b+ct]$) we get

$$\begin{aligned}\frac{d}{dt}K(t) &= \int_{\mathbb{R}} \rho u_t(x,t)\, u_{tt}(x,t)\, dx = \int_{a-ct}^{b+t} \rho u_t(x,t)\, u_{tt}(x,t)\, dx \\ &= \int_{a-ct}^{b+t} \tau u_t(x,t)\, u_{xx}(x,t)\, dx \\ &= [\tau u_t(x,t) u_x(x,t)]_{x=a-ct}^{x=b+t} - \int_{a-ct}^{b+t} \tau u_{tx}(x,t) u_x(x,t)\, dx \\ &= -\int_{\mathbb{R}} \tau u_{tx}(x,t) u_x(x,t)\, dx\,.\end{aligned}$$

On the other hand,

$$\frac{d}{dt}P(t) = \int_{\mathbb{R}} \tau u_x(x,t)\, u_{tx}(x,t).$$

Then,

$$\frac{d}{dt}\left(K(t)+P(t)\right) = 0,$$

and therefore $K(t)+P(t)$ is constant in t.

(b) In this part we are going to prove that if t is big enough, then $u_t^2(\cdot,t) = c^2\, u_x^2(\cdot,t)$, and therefore $K(t) = P(t)$ (recall that $c = \sqrt{\tau/\rho}$). From d'Alembert formula we know that the solution u to (1) is given by

$$\begin{aligned}u(x,t) =& \frac{1}{2}\left(g(x-ct)+g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy \\ =& F(x+ct) + G(x-ct),\end{aligned}$$

where

$$F(s) = \frac{1}{2}g(s) + \frac{1}{2c}\int_0^s h(y)dy, \qquad G(s) = \frac{1}{2}g(s) - \frac{1}{2c}\int_0^s h(y)dy.$$

This gives

$$u_t = cF'(x+ct) - cG'(x-ct), \qquad u_x = F'(x+ct) + G'(x-ct),$$

and therefore

$$\begin{aligned}u_t^2 =& c^2\left(F'(x+ct)^2 + G'(x-ct)^2 - 2F'(x+ct)G'(x-ct)\right), \\ u_x^2 =& \left(F'(x+ct)^2 + G'(x-ct)^2 + 2F'(x+ct)G'(x-ct)\right).\end{aligned}$$

Hence, in order to show $u_t^2(\cdot,t) = c^2\, u_x^2(\cdot,t)$ it is sufficient to prove that

$$F'(x+ct)G'(x-ct) = 0$$

for all x if t is big enough. Indeed, note that

$$F'(s) = \frac{1}{2}g'(s) + \frac{1}{2c}h(s), \quad G'(s) = \frac{1}{2}g'(s) - \frac{1}{2c}h(s),$$

and both are supported in $[a,b]$. From this we see that

$$\begin{aligned} (i) \quad & F'(x+ct) = 0 \quad \text{if } x+ct \notin [a,b], \\ (ii) \quad & G'(x-ct) = 0 \quad \text{if } x-ct \notin [a,b]. \end{aligned}$$

Now, let us verify that if $2ct > b-a$, then either $F'(x+ct) = 0$ or $G'(x-ct) = 0$ for every $x \in \mathbb{R}$. Note that if $2ct > b-a$, then $(x+ct)-(x-ct) = 2ct > b-a$. Therefore, at least one of the two points $x-ct$ and $x+ct$ is outside of the interval $[a,b]$. Thus, using (i) and (ii) we see that $F'(x+ct)G'(x-ct) = 0$ for all x if $t > (b-a)/(2c)$, and we are done.

Problema 2.* Considerem

$$\begin{cases} u_{tt} - u_{xx} = 0 & x > 0,\ t > 0 \\ u(0,t) = d(t) & t > 0 \\ u(x,0) = 0 & x > 0 \\ u_t(x,0) = 0 & x > 0, \end{cases}$$

on $d \in C^2(\mathbb{R})$ i $d(0) = d'(0) = 0$.

(a) Resoleu el problema restant $d(t)$ a la solució i fent reflexió senar.

(b) Alternativament, resoleu el problema factoritzant l'equació d'ones i usant el mètode de les característiques.

(c) Per un impuls d d'un segon donat per

$$d(t) = \begin{cases} 1 - \cos(2\pi t) & t \in (0,1) \\ 0 & t \geq 1 \end{cases}$$

i amb $c = 10\,\mathrm{m/s}$, determineu la longitud, amplitud i posició de l'ona a temps $t = 4\,\mathrm{s}$. És solució clàssica per aquesta d?

Solution.

First, let us see that the conditions $d(0) = d'(0) = 0$ imposed in the statement are necessary in order to have $u \in C^1([0,+\infty) \times [0,+\infty))$, i.e., that u is C^1 up to the boundary/initial conditions. On the one hand, if we want the solution u to be continuous at $(0,0)$ we must have

$$d(0) = \lim_{t \to 0^+} u(0,t) = \lim_{x \to 0^+} u(x,0) = 0.$$

On the other hand, if we want u_t to be also continuous at the origin, we must have

$$d'(0) = \lim_{t \to 0^+} d'(t) = \lim_{t \to 0^+} u_t(0,t) = \lim_{x \to 0^+} u_t(x,0) = 0.$$

(a) If we set $v(x,t) = u(x,t) - d(t)$, then v must satisfy

$$\begin{cases} v_{tt} - v_{xx} = -d'' & \text{in } x,t > 0 \\ v(0,t) = 0 & t > 0 \\ v(x,0) = 0 & x > 0 \\ v_t(x,0) = 0 & x > 0. \end{cases}$$

Since we have Dirichlet boundary conditions in the half-line, we can solve the problem by doing an odd reflection and using d'Alembert-Duhamel formula for the whole line, as we have seen in the theory lessons. In fact, if $f(x,t)$ is odd with respect to the variable x, i.e. $f(x,t) = -f(-x,t)$ for all x in $\mathbb{R}$, and w solves the inhomogeneous equation in the whole $\mathbb{R}$

$$w_{tt} - w_{xx} = f(x,t) \quad x \in \mathbb{R},\, t > 0, \quad w(x,0) = w_t(x,0) = 0,$$

then, by uniqueness of solution to the wave equation in the whole line, the odd symmetry of f is inherited by w, and in particular one gets $w(0,t) = 0$ for all $t > 0$.

Thus, in order to obtain v satisfying $v(0,t) = 0$, we construct it by doing an odd reflection (in the x variable) of $-d''$ in $\mathbb{R} \times (0,\infty)$ and using d'Alembert-Duhamel formula. That is, we define

$$f(x,t) = \begin{cases} -d''(t) & \text{if } x > 0, t > 0 \\ d''(t) & \text{if } x < 0, t > 0. \end{cases}$$

With this f, the solution is given by

$$v(x,t) = \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+t-s} f(y,s)dy \right) ds = \frac{1}{2}\int_\Delta f,$$

which is the integral of f over the triangle Δ (in the y,s plane) of basis $(x-t, x+t)$ and height t at x. Let us calculate this integral for $x, t > 0$.

Case 1. If we are in the region $x > t$, then the triangle Δ lies entirely where $f(y,s)$ is equal to $-d''(s)$ –see Figure below, so that using integration by parts and the fact that $d(0) = d'(0) = 0$ we get

$$\begin{aligned} v(x,t) &= -\frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+t-s} d''(s)dy \right) ds = -\frac{1}{2}\int_0^t d''(s) \left(\int_{x-(t-s)}^{x+t-s} dy \right) ds \\ &= \int_0^t d''(s)(s-t)ds = [(t-s)d'(s))]_{s=0}^{s=t} - \int_0^t d'(s)ds \\ &= td'(0) - d(t) + d(0) = -d(t). \end{aligned}$$

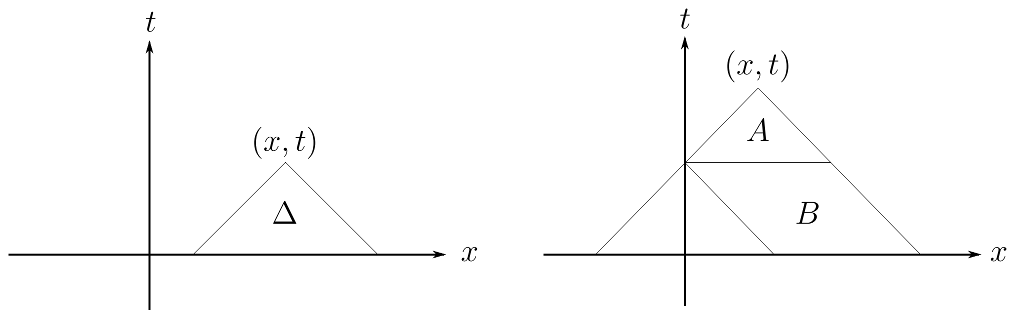
Case 2. We now assume that the triangle Δ intersects the region $x < 0$ –see Figure below, which happens when $x < t$. As f is odd in y, in the integral on Δ the part intersecting the negative y-region cancels with the a part in the positive y-region (reflected on s-axis), and one easily sees that

$$\int_\Delta f = \int_A f + \int_B f,$$

where (A is the upper triangle, B is the lower parallelogram),

$$A = \{(y,s) : \, s \in (t-x, t), \quad y \in (s-(t-x), -s+t+x)\},$$

$$B = \{(y,s) : \, s \in (0, t-x), \quad y \in ((t-x)-s, t+x-s)\}.$$



In these regions $f(y,s)$ is equal to $-d''(s)$. We see that

$$-\frac{1}{2}\int_B d''(s)ds = -\frac{1}{2}\int_0^{t-x} d''(s)2x\,ds = -xd'(t-x)$$

and

$$-\frac{1}{2}\int_A d''(s)ds = -\frac{1}{2}\int_{t-x}^{t} d''(s)(2t-2s) = \int_{t-x}^{t} d''(s)s\,ds - t\int_{t-x}^{t} d''(s)ds.$$

After some integration by parts we also get

$$-\frac{1}{2}\int_A d''(s)ds = -d(t) + d(t-x) + xd'(t-x).$$

Therefore, for $x < t$ we have

$$v(x,t) = \frac{1}{2}\int_\Delta f = -d(t) + d(t-x).$$

Summing up Cases 1 and 2, and using that $u = v + d$, we conclude that

$$u(x,t) = \begin{cases} d(t-x) & \text{if } t > x \\ 0 & \text{if } t \le x. \end{cases}$$

(b) We factorize the equation as (composition of differential operators)

$$(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0. \tag{2}$$

The order of factorization is important for the following computations. One can also try with the factorization $(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0$ but, as we will see below, the right order for our boundary conditions and domain is $(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$.

Set $w = u_t + u_x$ so that, from (2), w must satisfy (transport equation)

$$w_t - w_x = 0 \quad \text{ in } x, t > 0.$$

Given $a > 0$ and the curve $\gamma_a(s) = (a - s, s)$, thanks to the equation on w and the boundary conditions on u we have

$$\frac{d}{ds} w(\gamma_a(s)) = 0, \qquad w(\gamma_a(0)) = u_t(a,0) + u_x(a,0) = 0.$$

This implies that $w \equiv 0$ in $x, t > 0$. Taking into account the definition of w, this yields

$$u_t + u_x = 0 \quad \text{in } x, t > 0.$$

Using once more the method of characteristics but now for the equation $u_t + u_x = 0$ and the boundary conditions $u(0,t) = d(t)$ if $t > 0$ and $u(x,0) = 0$ if $x > 0$, we obtain

$$u(x,t) = \begin{cases} d(t-x) & \text{if } t > x \\ 0 & \text{if } t \leq x. \end{cases}$$

Note that, as it must be due to the uniqueness of solution, this u coincides with the one previously obtained in part (a) using d'Alembert-Duhamel formula.

In case we would have tried with the factorization $(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0$ and $w = u_t - u_x$, the characteristic curves for w would be $\gamma_{a,1}(s) = (a+s, s)$ for $a > 0$ (corresponding to the boundary data of u on $(0, +\infty) \times \{0\}$), and $\gamma_{b,2}(s) = (s, b+s)$ for $b > 0$ (arising form the boundary data of u on $\{0\} \times (0, +\infty)$). Then, in the region $x > t$ we would get, as before, $w \equiv 0$ because we still have $w(\gamma_{a,1}(0)) = u_t(a,0) - u_x(a,0) = 0$. However, in the region $x < t$ we would be led to the ODE

$$\frac{d}{ds} w(\gamma_{b,2}(s)) = 0, \qquad w(\gamma_{b,2}(0)) = d'(b) - u_x(0,b).$$

This yields $u_t(x,t) - u_x(x,t) = w(x,t) = d'(t-x) - u_x(0, t-x)$ on the region $x < t$. The problem is that in principle we do not have any information on the boundary data $u_x(0, \cdot)$, which appears in the non-homogeneous term on the right hand side of this last transport equation. The interested reader may try to solve $u_t(x,t) - u_x(x,t) = d'(t-x) - u_x(0, t-x)$ using Duhamel's formula for the transport equation and see what he/she gets in the end...

(c) Here we take a wave speed c different than $c = 1$ from the previous parts (a) and (b). This means that we are now considering the problem

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 & x > 0,\ t > 0 \\ w(0,t) = d(t) & t > 0 \\ w(x,0) = 0 & x > 0 \\ w_t(x,0) = 0 & x > 0, \end{cases} \tag{3}$$

where $c = 10$ and

$$d(t) = \begin{cases} 1 - \cos(2\pi t) & t \in (0,1), \\ 0 & t \geq 1. \end{cases}$$

However, by a simple rescaling we can get the solution to this new problem from the solution found in (a) or (b). More precisely, given $u(x,t)$ set $w(x,t) = u(x/c, t)$. If u solves $u_{tt} - u_{xx} = 0$ then it is straightforward to check that w solves $w_{tt} - c^2 w_{xx} = 0$. Moreover, if $u(0,t) = d(t)$ for all $t > 0$, and $u(x,0) = 0$ and $u_t(x,0) = 0$ for all $x > 0$, then the same holds true for w. Therefore, if

$$u(x,t) = \begin{cases} d(t-x) & \text{if } t > x \\ 0 & \text{if } t \leq x \end{cases}$$

is the solution found in (a) or (b), then

$$w(x,t) = \begin{cases} d\big(t - \frac{x}{c}\big) & \text{if } ct > x \\ 0 & \text{if } ct \le x \end{cases} \tag{4}$$

solves (3), where here

$$d\big(t - \tfrac{x}{c}\big) = \begin{cases} 1 - \cos\big(2\pi\big(t - \frac{x}{c}\big)\big) & \text{if } 0 < t - \frac{x}{c} < 1 \\ 0 & \text{if } t - \frac{x}{c} \ge 1. \end{cases}$$

We will use (4) to se that w is not a classical solution since w_{xx} is not continuous on the lines $x = c(t-1)$ and $x = ct$. However, that v is not a classical solution was expected without using the explicit formula (4): note that by one of the compatibility relations we should have $d''(0) = 0$ (using that $w_{tt} = c^2 w_{xx}$ at $(0,0)$) but our d does not satisfy $d''(0) = 0$. Besides d'' is not well defined at $t = 1$.

Let us now see that, in fact, these two singularities for d'' propagate or travel and make w_{xx} discontinuous on the lines $x = ct$ and $x = c(t-1)$ passing through the points $(0,0)$ and $(0,1)$. First, observe that

$$\big(d\big(t - \tfrac{x}{c}\big)\big)_{xx} = \tfrac{1}{c^2} d''\big(t - \tfrac{x}{c}\big).$$

Then, using the definition of w given in (4),

$$\lim_{x\to(ct)^+} w_{xx}(x,t) = \lim_{x\to(ct)^+} 0 = 0, \quad \text{and}$$
$$\lim_{x\to(ct)^-} w_{xx}(x,t) = \lim_{x\to(ct)^-} \big(d\big(t - \tfrac{x}{c}\big)\big)_{xx} = \lim_{y\to 0^+} \tfrac{1}{c^2} d''(y) = 4\pi^2/c^2.$$

Therefore, w_{xx} is not continuous on the line $x = ct$. Similarly,

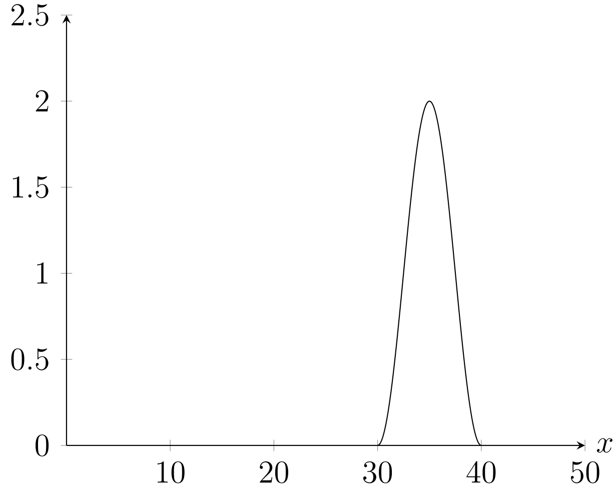
$$\lim_{x\to(c(t-1))^+} w_{xx}(x,t) = \lim_{x\to(c(t-1))^+} \big(d\big(t - \tfrac{x}{c}\big)\big)_{xx} = \lim_{y\to 1^-} \tfrac{1}{c^2} d''(y) = 4\pi^2/c^2, \quad \text{and}$$
$$\lim_{x\to(c(t-1))^-} w_{xx}(x,t) = \lim_{x\to(c(t-1))^-} \big(d\big(t - \tfrac{x}{c}\big)\big)_{xx} = \lim_{y\to 1^+} \tfrac{1}{c^2} d''(y) = 0,$$

which shows that w_{xx} is not continuous on the line $x = c(t-1)$.

Finally, in order to address the question in (c) about the description of $w(\cdot, 4)$, let us take $c = 10$ and $t = 4$ in (4). We distinguish 3 cases:

- *The case of $x \ge ct$ in definition of w in (4):* We have $t - x/c \le 0$, which is equivalent to $4 - x/10 \le 0$ or, equivalently, $x \ge 40$. In this case $w(x,4) = 0$.
- *The case of $0 < t - x/c < 1$:* We have $0 < 4 - x/10 < 1$ and hence $30 < x < 40$. In this case $w(x,4) = 1 - \cos\big(2\pi\big(4 - \frac{x}{10}\big)\big)$.
- *The case of $t - x/c \ge 1$:* We have $x \le 30$. In this case $w(x,4) = 0$.

Thus, $w(x,4)$ is a “bell-shaped” function (see Figure below), with wave length of 10m (between 30 and 40), 2m of amplitude (height at the maximum, in this case at $x = 35$), and it is zero outside of the interval $(30, 40)$.



2.5
2
1.5
1
0.5
0
10
20
30
40
50
x

Problema 5.* Considerem l'equació d'ones no lineal

$$u_{tt} - u_{xx} = u^2 + f(x,t) \quad \text{a } Q_T := \mathbb{R} \times (0,T),$$

amb condicions inicials $u(x,0) = u_t(x,0) = 0$, on f és una funció contínua i afitada a $\overline{Q_T}$ amb $\|f\|_{L^\infty(Q_T)} \leq 1$. Treballant en l'espai de funcions contínues i afitades a $\overline{Q_T}$, demostreu l'existència i unicitat de solució (en sentit integral) per T prou petit. Doneu una cota inferior explícita pel temps T.

Solution.

In order to solve this nonlinear wave equation we will use Banach fixed point theorem on a contraction based on d'Alembert-Duhamel formula for the non-homogeneous linear wave equation $v_{tt} - c^2 v_{xx} = \tilde{f}$.

Recall that, if we consider the problem

$$v_{tt} - c^2 v_{xx} = \tilde{f}(x,t) \tag{1}$$

with initial conditions $v(x,0) = g(x)$ and $v_t(x,0) = h(x)$, where $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and $\tilde{f} \in C^1(\mathbb{R} \times \mathbb{R})$, we know that the unique classical solution is given by

$$v(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy + \frac{1}{2c}\int_{\Delta_{x,t}} \tilde{f}(y,s)\,dy\,ds, \tag{2}$$

where $\Delta_{x,t} = \{(y,s) \in \mathbb{R}^2 : 0 < s < t,\ x - c(t-s) < y < x + c(t-s)\}$. However, the right hand side of (2) also makes sense when g, h, and $\tilde{f}$ are only assumed to be continuous (and even less). A solution v to the problem (1) in the integral sense means that v is of the form given by (2) (which may not be a classical solution depending on the regularity of g, h, and $\tilde{f}$).

If we use (2) with $c = 1$, $v = u$, $g = h = 0$, and $\tilde{f} = u^2 + f$, we see that u is a solution (in the integral sense) to $u_{tt} - u_{xx} = u^2 + f(x,t)$ on Q_T with $u(x,0) = u_t(x,0) = 0$ if and only if u satisfies

$$u(x,t) = \frac{1}{2}\int_{\Delta_{x,t}} (u^2 + f) = \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} \left[u(y,s)^2 + f(y,s)\right] dy \right) ds \tag{3}$$

for all $(x,t) \in Q_T$. The right hand side of (3) suggests what contraction should be taken in order to use Banach fixed point theorem (if we take T small enough).

Let $C_b(\overline{Q_T})$ be the vector space of continuous and bounded functions in $\overline{Q_T}$. We define $\|\cdot\|_\infty := \|\cdot\|_{C_b(\overline{Q_T})}$ and $N : C_b(\overline{Q_T}) \to C_b(\overline{Q_T})$ by

$$(Nw)(x,t) := \frac{1}{2}\int_{\Delta_{x,t}} (w^2 + f).$$

Note that the operator is well defined from $C_b(\overline{Q_T})$ to $C_b(\overline{Q_T})$. That is, given w continuous, since f is also continuous and integration preserves continuity we get that Nw is also continuous. Regarding the boundedness, observe that if $w \in C_b(\overline{Q_T})$, since the area of the triangle $\Delta_{x,t}$ is $|\Delta_{x,t}| = t^2 \le T^2$, we have that

$$\|Nw\|_\infty \le \frac{T^2}{2}\|w^2 + f\|_\infty. \tag{4}$$

We will now show that, if T is small enough, N has a fixed point when restricted to the set $X_M = \{u \in C_b(\overline{Q_T}) : \|u\|_\infty \le M\}$. Note that X_M is complete, being a closed subspace of the Banach space $C_b(\overline{Q_T})$. In order to apply Banach fixed point theorem, it is enough to show the following two facts:

- *N maps X_M into itself:* By (4), if $w \in X_M$ then

$$\|Nw\|_\infty \le \frac{T^2}{2}\|w^2 + f\|_\infty \le \frac{T^2}{2}(M^2 + \|f\|_\infty) \le M$$

  if

$$T^2 \le \frac{2M}{M^2 + \|f\|_{L^\infty}}.$$

- *N is a contraction on X_M:* Given $(x,t) \in \overline{Q_T}$ and $v, w \in X_M$,

$$\begin{aligned}
|Nv(x,t) - Nw(x,t)| &= \left| \frac{1}{2}\int_{\Delta_{x,t}} (v^2 - w^2) \right| \\
&\le \frac{1}{2}\int_{\Delta_{x,t}} |v^2 - w^2| \le \frac{T^2}{2}\|v - w\|_\infty \|v + w\|_\infty \\
&\le \frac{T^2}{2}\|v - w\|_\infty \left(\|v\|_\infty + \|w\|_\infty\right) \\
&\le T^2 M \, \|v - w\|_\infty.
\end{aligned}$$

  Taking the supremum in $(x,t) \in \overline{Q_T}$ we obtain

$$\|Nv - Nw\|_\infty \le T^2 M \, \|v - w\|_\infty.$$

  Then, N is a contraction if

$$T^2 < \frac{1}{M}.$$

Therefore, the Banach fixed point theorem applies to N if we take

$$T^2 < \min\left\{\frac{2M}{M^2 + \|f\|_{L^\infty}}, \frac{1}{M}\right\},$$

showing that N has a unique fixed point $u \in X_M$, which is a solution to (3) and, therefore, a solution in the integral sense to our initial problem.

To get a big value for T one could maximize the right hand side of the previous inequality among all $M > 0$. One certainly has a solution for all $t \in [0,T)$ and $T < 1$, by taking $M = 1$ and using $\|f\|_{L^\infty} \le 1$.

A final remark is in order. The reader may verify that indeed

$$C_b(\overline{Q_T}) = C([0,T]; C_b(\mathbb{R})),$$

hence the approach we followed in the resolution of this exercise is the same that we have been using in previous exercises about evolution equations on Banach space valued functions of time, i. e., $u_t = Au$ being A an operator from a Banach space to itself.

Problema 1.* (Cada apartat es pot resoldre independentment d'haver contestat als apartats previs) Considereu el problema de Neumann següent:

$$\begin{cases} u_{tt} - u_{xx} = & -\alpha u + f(x,t), & x \geq 0,\ t \geq 0, \\ u_x(0,t) = & 0, & t \geq 0, \\ u(x,0) = & g(x), & x \geq 0, \\ u_t(x,0) = & 0, & x \geq 0. \end{cases}$$

(1) Suposem $\alpha = 0$ i $f(x,t) = \sin(\pi(x-1)/2)$. Quant val u a $(x,t) = (1,2)$?
(2) Suposem ara $\alpha \in \mathbb{R}$. Doneu condicions necessàries en α, f i g per tal que
  (a) $u \in C^2([0,+\infty) \times [0,+\infty))$.
  (b) $u \in C^3([0,+\infty) \times [0,+\infty))$.
(3) També per a $\alpha \in \mathbb{R}$, suposant que $g \in C([0,+\infty))$, $f \in C([0,+\infty)\times[0,+\infty))$, i que ambdues són fitades, demostreu existència i unicitat de solució en sentit integral del problema anterior (per a temps positius prou petits). Existeix solució per a tot temps $t \geq 0$?

Problema 1. (a) Considereu el domini del pla

$$\Omega = \{(x,y) : a < x < b, \varphi(x) < y < \psi(x)\}$$

on $\varphi, \psi \in C^1\big([a,b]\big)$ son funcions que satisfan $\varphi(x) < \psi(x)$ per a tot $x \in (a,b)$. Demostreu que per tota $u \in C^1(\overline{\Omega})$ es té

$$\int_\Omega \partial_y u \; dxdy = \int_{\partial\Omega} u\nu^y \, dl,$$

on ν^y es la segona component de la normal unitària exterior a $\partial\Omega$.

(b) Deduïu, en el domini Ω anterior i per $u, v \in C^1(\overline{\Omega})$, la fórmula d'integració per parts:

$$\int_\Omega (\partial_y u) v \, dxdy = -\int_\Omega u \partial_y v \, dxdy + \int_{\partial\Omega} uv\nu^y \, dl.$$

Nota IMPORTANT: La formula d'integració per parts a $\mathbb{R}^n$ també és vàlida amb una demostració similar. Enuncia:

Sigui $\Omega \subset \mathbb{R}^n$ un domini fitat Lipschitz, i siguin $u, v \in C^1(\overline{\Omega})$. Per a tot $i = 1, 2, \ldots, n$ es té

$$\int_\Omega (\partial_i u) v \, dx = -\int_\Omega u \partial_i v \, dx + \int_{\partial\Omega} uv\nu^i \, dS,$$

on ν^i és la i-èssima component de la normal unitària exterior a $\partial\Omega$.

Problema 2. El *teorema de la divergència* diu que per a tot domini fitat Lipschitz $\Omega \subset \mathbb{R}^n$ i per a tot camp $X \in C^1(\overline{\Omega}; \mathbb{R}^n)$ es té

$$\int_\Omega \operatorname{div} X \, dx = \int_{\partial\Omega} X \cdot \nu \, dS,$$

on ν és la normal exterior a $\partial\Omega$. Demostreu que el teorema de la divergència i la fórmula d'integració per parts de la Nota del problema anterior són equivalents.

Problema 3. (a) Donat el domini $\Omega = \{(x,y) : x^2/a^2 + y^2 < 1\}$, considereu les el·lipses $E_\lambda = \{\sqrt{x^2/a^2 + y^2} < \lambda\}$ $(0 < \lambda < 1)$. Compareu $\int_\Omega u \, dxdy$ i $\int_0^1 \int_{\partial E_\lambda} u \, dl d\lambda$ fent un canvi de variables. Demostreu que les dues integrals anteriors són iguals per tota $u \in L^1(\Omega)$, si i només si, $a = 1$.

(b) En dimensió $n \geq 2$, demostreu el següent "Teorema de Fubini esfèric". Per a tota funció $u \in L^1(\mathbb{R}^n)$, es té

$$\int_{\mathbb{R}^n} u \, dx = \int_0^\infty \int_{\partial B_r} u \, dS dr.$$

[Indicació: Primer feu el cas $n = 2$ i 3 usant coordenades polars o esfèriques. En dimensió general $n \geq 2$, per densitat podem suposar que $u \in C^1(\mathbb{R}^n)$ i, usant el teorema de la divergència, demostrar que

$$\frac{d}{dr}\int_{B_r} u\,dx = \frac{d}{dr}\int_{B_1} u(ry)\,r^n\,dy = \int_{B_1} ... = \int_{B_r} ... = \int_{\partial B_r} u\,dS.]$$

Problema 4. Demostreu les següents propietats importants del Laplacià:

(a) La definició de Laplacià no depèn de la base ortonormal de $\mathbb{R}^n$ triada. És a dir

$$\sum_i \partial_{e_i e_i} u(x) = \sum_i \partial_{e'_i e'_i} u(x)$$

per cada parella de bases ortonormals $\{e_i\}$, $\{e'_i\}$ de $\mathbb{R}^n$.

(b) El Laplacià és invariant per rotacions. És a dir, per a tota matriu ortogonal O i $u \in C^2$, si es defineix $u^*(x) := u(Ox)$ i $x^* = Ox$ es té

$$\Delta u^*(x) = \Delta u(Ox) = \Delta u(x^*).$$

(c) El Laplacià és invariant per translacions.

(d) El Laplacià és invariant per isometries de $\mathbb{R}^n$.

Problema 5.* (i) Proveu que si u és de classe C^2 en un entorn de $x \in \mathbb{R}^n$ llavors

$$u(x+h) = u(x) + \nabla u(x)\cdot h + \frac{1}{2}h^T D^2u(x)h + o(|h|^2) \quad \text{quan } h \to 0,$$

on $D^2u(x)$ és la matriu Hessiana, i.e., $D^2u(x) = \big(\partial_{ij}u(x)\big)_{1\leq i,j\leq n}$. [Indicació: Fixeu $x \in \mathbb{R}^n$, considereu la funció d'una variable $g(t) = u(x+th)$ i apliqueu la fòrmula de Taylor a $\mathbb{R}$.]

(ii) Donat un obert $\Omega \subset \mathbb{R}^n$, $u \in C^2(\Omega)$ i $x \in \Omega$, demostreu que

$$\begin{aligned}\Delta u(x) &= \lim_{r\downarrow 0}\frac{2n}{r^2}\fint_{\partial B_r(x)}\big(u(y)-u(x)\big)\,dS(y)\\ &= \lim_{r\downarrow 0}\frac{2n}{r^2}\left\{\left(\fint_{\partial B_r(x)} u(y)\,dS(y)\right) - u(x)\right\},\end{aligned} \tag{1}$$

a on $\fint_E := \frac{1}{|E|}\int_E$ denota fer la mitjana. [Indicació: Feu servir l'aproximació de Taylor de u al voltant del punt x.]

(iii) Useu l'apartat (ii) per donar una prova alternativa de les questions (a) i (b) del problema anterior.

(iv) Useu l'apartat (ii) per determinar el signe del Laplacià en un punt de màxim (o mínim) local de u.

Problema 6.* [Desigualtat de Wirtinger] Sigui $u : \mathbb{R} \longrightarrow \mathbb{R}$ una funció C^1 i 2π-periòdica. Proveu que

$$\int_0^{2\pi} |u - c_u|^2 dt \leq \int_0^{2\pi} |u'|^2 dt,$$

on $c_u = \frac{1}{2\pi}\int_0^{2\pi} u$.

[Indicació: Escriviu la desigualtat en termes de la sèrie de Fourier de u.]

Deduïu que si $v : \mathbb{R} \longrightarrow \mathbb{R}$ és una funció C^1 i $(b-a)$-periòdica, llavors

$$\int_a^b |v - c_v|^2 dt \leq \left(\frac{b-a}{2\pi}\right)^2 \int_a^b |v'|^2 dt,$$

on $c_v = \frac{1}{b-a}\int_a^b v$.

Problema 7.* [Desigualtat isoperimètrica] Sigui $\Omega \subset \mathbb{R}^2$ un domini (i.e., un obert connex) regular i fitat. La desigualtat isoperimètrica enuncia que

$$4\pi|\Omega| \leq |\partial\Omega|^2, \tag{2}$$

amb igualtat si, i només si, Ω és un cercle. Per tant, (2) ens diu que d'entre tots els dominis amb mateix perimètre (dominis isoperimètrics), els cercles són els que tenen màxima àrea.

L'objectiu del problema és demostrar aquest resultat quan la vora $\Gamma = \partial\Omega$ de Ω és una corba simple (tancada i prou regular) de Jordan.

(a) Siguii $L = |\partial\Omega|$ i $\gamma : [0, L] \longrightarrow \mathbb{R}^2$ una parametrització de Γ amb el paràmetre arc, i.e., $|\gamma'(t)| = 1$ per a tot $t \in [0, L]$. Siguin $x(t)$ i $y(t)$ les funcions coordenades de γ, i.e., $\gamma(t) = (x(t), y(t))$. Proveu que

$$|\partial\Omega| = \int_0^L dt = \int_0^L \left\{|x'(t)|^2 + |y'(t)|^2\right\} dt$$

i

$$|\Omega| = \left|\int_0^L x(t)y'(t)dt\right|.$$

(b) Usant la desigualtat de Wirtinger del problema anterior, proveu (2).

Problema 1. [Desigualtat de Wirtinger] Sigui $u : \mathbb{R} \longrightarrow \mathbb{R}$ una funció C^1 i 2π-periòdica. Proveu que

$$\int_0^{2\pi} |u - c_u|^2 dt \le \int_0^{2\pi} |u'|^2 dt,$$

on $c_u = \frac{1}{2\pi}\int_0^{2\pi} u$.

[Indicació: Escriviu la desigualtat en termes de la sèrie de Fourier de u.]

Deduïu que si $v : \mathbb{R} \longrightarrow \mathbb{R}$ és una funció C^1 i $(b-a)$-periòdica, llavors

$$\int_a^b |v - c_v|^2 dt \le \left(\frac{b-a}{2\pi}\right)^2 \int_a^b |v'|^2 dt,$$

on $c_v = \frac{1}{b-a}\int_a^b v$.

Solution.

Recall that every function $f \in L^2([0, 2\pi])$ can be expressed in Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \big(a_k \cos(kx) + b_k \sin(kx)\big), \tag{1}$$

where, for $k \ge 0$,

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx, \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx. \tag{2}$$

The equality (1) holds in $L^2([0, 2\pi])$.

In the sequel we will use the following results:

- *Cauchy-Schwarz inequality* (this will only be used in Problem 2): if f, $g \in L^2([a, b])$ then

$$\left|\int_a^b fg\right| \le \|f\|_{L^2([a,b])}\|g\|_{L^2([a,b])}.$$

  Moreover, the equality holds if and only if $f \equiv 0$ or $g = \lambda f$ for some $\lambda \in \mathbb{R}$. Cauchy-Schwarz inequality follows easily from the fact that $\int_a^b fg$ defines a scalar product in $L^2([a, b])$ for which the associated norm is

$$\|f\|_{L^2([a,b])} = \left(\int_a^b |f|^2\right)^{1/2}.$$

- *Parseval's identity:* For every $f \in L^2([0, 2\pi])$ expressed as in (1) we have

$$\|f\|^2_{L^2([0,2\pi])} = \frac{\pi}{2}a_0^2 + \pi\sum_{k=1}^{\infty}(a_k^2 + b_k^2).$$

Parseval's identity is a consequence of the orthogonality of the $L^2([0,2\pi])$ Fourier basis $\{1, \cos(kx), \sin(kx)\}_{k\geq 1}$ (the Pythagorean Theorem in an infinite dimensional space).

- *The Fourier series of f'*: If $f : \mathbb{R} \to \mathbb{R}$ is piecewise C^1, 2π-periodic, and continuous in $\mathbb{R}$, then the Fourier series of f' is obtained by differentiating the Fourier series of f term by term. That is,

$$f'(x) = \sum_{k=1}^{\infty} k\big(b_k \cos(kx) - a_k \sin(kx)\big),$$

where a_k and b_k are as in (2). In particular, from Parseval's identity we deduce that

$$\|f'\|^2_{L^2([0,2\pi])} = \pi \sum_{k=1}^{\infty} k^2(a_k^2 + b_k^2).$$

With these results at hand, let us now address Wirtinger's inequality. We express u and u' in Fourier series in $L^2([0,2\pi])$:

$$u = c_u + \sum_{k=1}^{\infty} \big(a_k \cos(kx) + b_k \sin(kx)\big),$$

$$u' = \sum_{k=1}^{\infty} k\big(b_k \cos(kx) - a_k \sin(kx)\big),$$

where, for $k \geq 1$, $a_k = \frac{1}{\pi}\int_0^{2\pi} u(x)\cos(kx)\,dx$ and $b_k = \frac{1}{\pi}\int_0^{2\pi} u(x)\sin(kx)\,dx$. Note that c_u here corresponds to $a_0/2$ in (1). Therefore, from Parseval's identity applied to $u - c_u$ and u', we deduce that

$$\int_0^{2\pi} |u - c_u|^2 = \pi \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \pi \sum_{k=1}^{\infty} k^2(a_k^2 + b_k^2) = \int_0^{2\pi} |u'|^2,$$

which is Wirtinger's inequality. Here we have used $k \geq 1$. From these computations we also see that we have an equality in Wirtinger's inequality if and only if $a_k = b_k = 0$ for all $k \geq 2$, that is, if and only if

$$u = c_u + a_1 \cos(x) + b_1 \sin(x).$$

Now, the general case of a $(b-a)$-periodic function will follow easily. Given v as in the statement, set

$$u(x) = v\left(\frac{b-a}{2\pi}x + a\right),$$

which is C^1 and 2π-periodic. Then, by the change of variables $y = \frac{b-a}{2\pi}x + a$,

$$c_u = \frac{1}{2\pi}\int_0^{2\pi} v\left(\frac{b-a}{2\pi}x + a\right)dx = \frac{1}{2\pi}\frac{2\pi}{b-a}\int_a^b v(y)\,dy = c_v.$$

In the same way one can check that

$$\int_0^{2\pi} |u - c_u|^2 = \frac{2\pi}{b-a} \int_a^b |v - c_v|^2$$

and

$$\int_0^{2\pi} |u'|^2 = \frac{b-a}{2\pi} \int_a^b |v'|^2.$$

Thus, the general case follows directly from the one in $[0, 2\pi]$.

Finally, as before, we have an equality in the general Wirtinger inequality if and only if

$$v(y) = c_x + a_1 \cos\left(\frac{2\pi}{b-a} y\right) + b_1 \sin\left(\frac{2\pi}{b-a} y\right),$$

where a_1 and b_1 are the corresponding Fourier coefficients of the $(b-a)$-periodic function v.

Note that an alternative proof of Wirtinger Inequality (by Hardy, Littlewood and Polya) can be done without using Fourier series. It is based on integrating in $[0, 2\pi]$ the following identity

$$\left[u'(t)^2 - u(t)^2\right] = \left[u'(t) - \frac{\cos(t)}{\sin(t)} u(t)\right]^2 + \left[u(t)^2 \frac{\cos(t)}{\sin(t)}\right]'$$

for any given u such that $c_u = 0$ and $u(0) = u(2\pi) = 0$.

Problema 2. [Desigualtat isoperimètrica] Sigui $\Omega \subset \mathbb{R}^2$ un domini (i.e., un obert connex) regular i fitat. La desigualtat isoperimètrica enuncia que

$$4\pi|\Omega| \leq |\partial\Omega|^2, \tag{3}$$

amb igualtat si, i només si, Ω és un cercle. Per tant, (3) ens diu que d'entre tots els dominis amb mateix perímetre (dominis isoperimètrics), els cercles són els que tenen màxima àrea.

L'objectiu del problema és demostrar aquest resultat quan la vora $\Gamma = \partial\Omega$ de Ω és una corba simple (tancada i prou regular) de Jordan.

(a) Sigui $L = |\partial\Omega|$ i $\gamma : [0, L] \longrightarrow \mathbb{R}^2$ una parametrització de Γ amb el paràmetre arc, i.e., $|\gamma'(t)| = 1$ per a tot $t \in [0, L]$. Siguin $x(t)$ i $y(t)$ les funcions coordenades de γ, i.e., $\gamma(t) = (x(t), y(t))$. Proveu que

$$|\partial\Omega| = \int_0^L dt = \int_0^L \left\{|x'(t)|^2 + |y'(t)|^2\right\} dt$$

i

$$|\Omega| = \left|\int_0^L x(t) y'(t) dt\right|.$$

(b) Usant la desigualtat de Wirtinger del problema anterior, proveu (3).

Solution.

(a) The formula for $|\partial\Omega|$ is a consequence of the fact that we are taking the parametrization of $\partial\Omega$ by the arc length, hence $1 = |\gamma'(t)| = \sqrt{|x'(t)|^2 + |y'(t)|^2}$.

The formula for $|\Omega|$ will follow by the divergence theorem. Setting

$$\nu(t) = (-y'(t), x'(t))$$

we clearly have $\gamma \cdot \nu \equiv 0$. That is, ν is the unit normal vector field on $\partial\Omega$ (pointing inside or outside of Ω depending on wether the parametrization $\gamma(t)$ is clowckwise or anticlockwise). Also, if we set $F(x, y) = (x, 0)$ then $\operatorname{div} F = 1$. Therefore, the divergence theorem yields

$$|\Omega| = \int_\Omega 1 = \int_\Omega \operatorname{div} F = \left| \int_{\partial\Omega} F \cdot \nu \right| = \left| \int_0^L x(t)(-y'(t))dt \right|.$$

(b) Recall that $|\partial\Omega| = L$. Since $\gamma(t) = (x(t), y(t))$ is the parametrization of $\partial\Omega$ by the arc length, we deduce that both x and y are L-periodic functions. In particular,

$$\int_0^L y' = y(L) - y(0) = 0. \tag{4}$$

Therefore, if we set $c_x = \frac{1}{L}\int_0^L x(t)\,dt$, using (a), (4), and Cauchy-Schwarz inequality (see Problem 1) we deduce that

$$|\Omega| = \left| \int_0^L xy' \right| = \left| \int_0^L (x - c_x)y' \right| \le \left(\int_0^L |x - c_x|^2 \right)^{1/2} \left(\int_0^L |y'|^2 \right)^{1/2}. \tag{5}$$

Note that we have used

$$\int_0^L c_x\, y' = c_x \int_0^L y' = 0.$$

Moreover, the equality in (5) is attained if and only if $x - c_x = \lambda y'$ for some $\lambda \neq 0$ (recall that Γ is a Jordan curve and, thus, $x \not\equiv$ and $y \not\equiv 0$). If we now apply Wirtinger inequality (Problem 1) to the right hand side of (5) we obtain

$$|\Omega| \le \frac{L}{2\pi} \left(\int_0^L |x'|^2 \right)^{1/2} \left(\int_0^L |y'|^2 \right)^{1/2}. \tag{6}$$

Recall also that Wirtinger inequality is an equality if and only if x is of the form

$$x(t) = c_x + a_1 \cos\left(\frac{2\pi}{L}t\right) + b_1 \sin\left(\frac{2\pi}{L}t\right).$$

Let us now estimate the right hand side of (6). Using that $ab \le (a^2 + b^2)/2$ for all a, $b \in \mathbb{R}$ (with equality if and only if $a = b$), and taking

$$a = \left(\int_0^L |x'|^2 \right)^{1/2}, \qquad b = \left(\int_0^L |y'|^2 \right)^{1/2},$$

from (6) and (a) we conclude that

$$|\Omega| \leq \frac{L}{2\pi}\, ab \leq \frac{L}{2\pi}\frac{a^2+b^2}{2} = \frac{L}{2\pi}\frac{|\partial\Omega|}{2} = \frac{1}{4\pi}L^2 = \frac{1}{4\pi}|\partial\Omega|^2, \tag{7}$$

and the isoperimetric inequality follows.

It only remains to consider the case of equality in the isoperimetric inequality. Note that $4\pi|\Omega| = |\partial\Omega|^2$ if and only if the equality holds in all the intermediate steps towards the proof of $4\pi|\Omega| \leq |\partial\Omega|^2$. This means that the inequalities in (5), (6), and (7) must be indeed equalities. Recall that

(i) the equality in (5) holds if and only if $x - c_x = \lambda y'$ for some $\lambda \neq 0$. In the case of equality in (5),

(ii) the equality in (6) holds if and only if $x(t) = c_x + a_1 \cos\left(\frac{2\pi}{L}t\right) + b_1 \sin\left(\frac{2\pi}{L}t\right)$. In the case of equality in (6),

(iii) the equality in (7) holds if and only if $\int_0^L |x'|^2 = \int_0^L |y'|^2$.

From (i) and (ii) we deduce that

$$x(t) = c_x + a_1 \cos\left(\frac{2\pi}{L}t\right) + b_1 \sin\left(\frac{2\pi}{L}t\right)$$

and

$$y(t) = \frac{1}{\lambda}\left[\frac{L}{2\pi}a_1 \sin\left(\frac{2\pi}{L}t\right) - \frac{L}{2\pi}b_1 \cos\left(\frac{2\pi}{L}t\right)\right] + c_y$$

(the integration constant on the right hand side of the formula for $y(t)$ is precisely c_y because the integrals of $\sin(\frac{2\pi}{L}t)$ and $\cos(\frac{2\pi}{L}t)$ in $[0, L]$ vanish). Finally, using (iii) with the formulas for $x(t)$ and $y(t)$, and orthogonality, we have

$$\frac{1}{|\lambda|}\frac{L}{2\pi} = 1,$$

that is, $\lambda = \pm\frac{L}{2\pi}$. This yields

$$y(t) = c_y \pm \left(a_1 \sin\left(\frac{2\pi}{L}t\right) - b_1 \cos\left(\frac{2\pi}{L}t\right)\right).$$

Therefore, we conclude that $4\pi|\Omega| = |\partial\Omega|^2$ if and only if

$$\gamma(t) = (c_x, c_y) + \Big(a_1 \cos\left(\tfrac{2\pi}{L}t\right) + b_1 \sin\left(\tfrac{2\pi}{L}t\right), \pm\left(a_1 \sin\left(\tfrac{2\pi}{L}t\right) - b_1 \cos\left(\tfrac{2\pi}{L}t\right)\right)\Big).$$

This is a parametrization of a circle centered at (c_x, c_y) and of radius $\sqrt{a_1^2 + b_1^2}$ (observe that $(x(t) - c_x)^2 + (y(t) - c_y)^2 = a_1^2 + b_1^2$). Depending on the choice of the sign $\pm$ in the expression for $y(t)$, γ runs the circle clockwise or anticlockwise.

Finally, let us comment that in fact Wirtinger and Isoperimetric inequalities are equivalent. Therefore, it is also possible to prove Wirtinger by assuming Isoperimetric inequality holds.

Problema 1.* Considerem l'operador
$$(Av)(x) = \int_0^1 \frac{x^2\, v(y)}{y^{1/4}}\, dy$$
i els espais de Banach $L^1((0,1))$, $L^2((0,1))$ i $C([0,1])$.

(a) Discutiu l'existència i unicitat de solució $u = u(x,t)$ pel problema
$$\begin{cases} u_t = Au \\ u(x,0) = g(x) \end{cases}$$
quan la funció g pertany a cadascun dels tres espais de Banach anteriors.

(b) Podeu trobar la suposada solució de manera explícita?

(c) Discutiu l'existència i unicitat de solució $u = u(x,t)$ pel problema
$$\begin{cases} u_t = Au + u^2 \\ u(x,0) = x^{-1/3} \end{cases}$$
en cadascun dels tres espais de Banach anteriors.

Problema 2.* Sigui E un espai de Banach de funcions definides a $[0,1]$. Per a $g \in E$ i $u = u(x,t)$, considereu el problema
$$\begin{cases} u_t = & Au + \cos(u), \quad t \in I := [-T,T], \\ u(0) = & g, \end{cases}$$
on A és l'operador integral definit per
$$(Av)(x) := \int_0^1 \frac{v(y) - v(x)}{|y-x|^{1/4}}\, dy\,, \qquad \text{per a } x \in [0,1].$$
Discutiu l'existència i unicitat de solució a l'espai $C^0(I;E)$ (en sentit integral) en els casos següents:

(1) $E = L^2((0,1))$.
(2) $E = C^0([0,1])$.

Problema 3.* Considerem l'operador
$$(Av)(x) := \int_0^x \frac{v(y)}{\sqrt{y}}\, dy, \qquad 0 \le x \le 1.$$

(a) Demostreu que A envia l'espai $C([0,1])$ en si mateix de manera contínua. Calculeu $\|A\|$.

(b) Donada $g \in C([0,1])$, volem resoldre l'equació d'evolució

$$u_t = Au + u^2 \qquad \text{per } t \in I \subset \mathbb{R},$$

amb la condició inicial $u(\cdot, 0) = g$. Considerem dos casos: (i) resoldre el problema anterior sense imposar condicions de vora; (ii) imposar $u(0,t) = 0$ per $t \in I$. Justifiqueu si (i) i/o (ii) admeten existència i unicitat de solució dient, si cal, per quines condicions inicials. Pels problemes que ho fan, demostreu en detall l'existència i unicitat de solució (en sentit integral).

Problema 1.* (a) Resoleu per separació de variables el problema

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in (0,\pi), t > 0 \\ u(0,t) = u(\pi,t) = 1 & t > 0 \\ u(x,0) = 1 & x \in (0,\pi) \\ u_t(x,0) = \sin^2 x & x \in (0,\pi). \end{cases}$$

(b) Exteneu adequadament les condicions inicials i apliqueu la fórmula de d'Alembert per calcular $u(\pi/4, 3)$.

(c) Calculeu

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2\{4-(2k+1)^2\}}.$$

Problema 2.* Determineu la temperatura d'una barra de longitut π, si inicialment la temperatura és $x(\pi - x)$, en l'extrem de l'esquerra la temperatura sempre és nul·la i en el de la dreta no hi ha fluxe de calor. Preneu el coeficient de difusió $D = 1$.

Problema 3. En un instrument de vent recte (modelitzat per un interval per fer un primer estudi), $u(x,t)$ representa, per $x \in (0, L)$ i $t \geq 0$, la pressió de l'aire dins l'instrument. Es satisfà l'equació d'ones $u_{tt} - c^2 u_{xx} = 0$. Quan una solució és periòdica en el temps, amb període T, la seva freqüència és $(2\pi)/T$.

En un instrument obert pels dos costats, com la flauta, la pressió u als dos extrems és igual a la pressió atmosfèrica, diem 0. Tenim, per tant, condicions de Dirichlet. En un instrument, com el clarinet, obert per un costat però no per l'altre (a on el músic exergeix pressió), tenim condicions de vora mixtes (una de Dirichlet, l'altre de Neumann; ambdues nul·les). Veure el llibre (free online) "Music: A Mathematical Offering", by Dave Benson.

Resoleu el problema de valors inicials per ambdos instruments usant sèries de Fourier. Per cada instrument, calculeu totes les freqüències que apareixen en la base de Fourier corresponent a l'instrument. S'anomenen harmònics de l'instrument. La freqüència més baixa, o to més greu, s'anomena to fonamental; els altres harmònics són sobre-tons del to fonamental (comprobeu que són múltiples enters del to fonamental).

Compareu els harmònics de la flauta i del clarinet. Perquè es fan les flautes més llargues que els clarinets en relació als seus diàmetres? Compareu també els sobretons que apareixen en cadascun dels dos instruments (això defineix el "timbre" de l'instrument).

Problema 4. Donat el problema

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = x(x-\pi) & x \in (0,\pi) \\ u_t(x,0) = \sin x & x \in (0,\pi), \end{cases}$$

calculeu $u(\pi/2, 2)$.

Problema 5.* (Una corda en un camp magnètic). Una corda metal·lica està fixada als punts $(0,0,0)$ i $(\pi,0,0)$ de l'eix x de $\mathbb{R}^3$. Per parametritzar el desplaçament en tres dimensions de la corda fem servir una funció que pren valors complexos $u(x) = u^z(x) + iu^y(x)$. La corda està carregada elèctricament i està immersa en un camp magnètic constant en la direcció x. Degut a les forces de Lorentz, l'equació per a petites vibracions d'aquesta corda és

$$\begin{cases} u_{tt} - c^2 u_{xx} = i\kappa u_t & x \in (0,\pi), t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = g(x) & x \in (0,\pi) \\ u_t(x,0) = h(x) & x \in (0,\pi), \end{cases}$$

on κ serà una certa constant (la densitat lineal de càrrega de la corda multiplicada per la intensitat del camp magnètic i dividida per la densitat lineal de la corda).

Useu separació de variables per calcular $u(x,t)$ per a $t > 0$ per $\kappa = 1$. Les forces de Lorentz provoquen amortiment de les solucions?

Problema 6. (Finances: l'equació de Black-Scholes). Els preus $u(s,t)$ dels principals derivats financers (com les *European call options*) es modelitzen amb l'equació de Black-Scholes:

$$u_t + \frac{1}{2}\sigma^2 s^2 u_{ss} + rsu_s - ru = 0 \qquad s > 0,\ t \in (0,T),$$

on la volatilitat σ i el tipus d'interès r son constants donades. Una "European call option" dóna dret a comprar un determinal actiu (*asset*) a un temps futur $T > 0$ i a un preu P fixats a priori. El preu real de la "call" a temps $t < T$ és igual a $u(s,t)$, i depèn per tant del valor s que té l'actiu a temps t.

(a) Trobeu $a > 0$ de forma que el canvi de variables del $s = e^{x/a}$ transformi l'equació de Black-Scholes en

$$v_t + v_{xx} + cv_x - rv = 0, \qquad x \in \mathbb{R},\ t > 0$$

on $u(s,t) = v(a \log s, t)$.

(b) Comproveu que la funció $\overline{v}(x,t) = v(x+ct,t)$ satisfà

$$\overline{v}_t + \overline{v}_{xx} - r\overline{v} = 0, \qquad x \in \mathbb{R},\ t \in (0,T).$$

(c) Feu el canvi $w(x,t) = e^{bt}\,\overline{v}(x, T-t)$ per a b apropiada per transformar l'equació anterior en la de la calor:

$$w_t - w_{xx} = 0, \qquad x \in \mathbb{R},\ t \in (0,T).$$

(d) Raoneu que el preu a temps $t = T$ d'una "European call option" és $u(s,T) = (s-P)^+$, on $(\,\cdot\,)^+$ denota la part positiva. Com podrieu calcular explícitament el valor de la opció $u(s,t)$ per a temps $t < T$?

Problema 7.* Es té l'equació de la calor amb condicions de frontera de Neumann:

$$\begin{cases} u_t - Du_{xx} = 0, & 0 < x < \pi,\ t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0, & t > 0 \\ u(x,0) = g(x), & 0 < x < \pi, \end{cases}$$

amb $D > 0$ i $g \in L^2\big((0,\pi)\big)$.

(a) Calculeu u per separació de variables i demostreu la convergència uniforme de u cap a la mitjana de les temperatures inicials $m = \frac{1}{\pi}\int_0^\pi g(x)\,dx$ quan $t \to +\infty$.

(b) Proveu que si $g \in C^1\big([0,\pi]\big)$, llavors $u \in C^0\big([0,\pi] \times [0,\infty)\big)$.

(c) Proveu que si $g \in C^2\big([0,\pi]\big)$ i $g'(0) = g'(\pi) = 0$ llavors $u_x(\cdot,t)$ convergeix uniformement a g' en $[0,\pi]$ quan $t \to 0$. Discutiu la necessitat de la condició $g'(0) = g'(\pi) = 0$.

On Fourier series expansions.

If $f \in L^2((0,T))$, we can write f in Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2k\pi}{T}x\right) + b_k \sin\left(\frac{2k\pi}{T}x\right)\right), \tag{1}$$

where the equality holds in the $L^2((0,T))$ sense, and where

$$a_k := \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2k\pi}{T}x\right)dx, \quad b_k := \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2k\pi}{T}x\right)dx. \tag{2}$$

From this Fourier expansion it is easy to derive the sinus and cosinus series for any given $f \in L^2((0,L))$. If we extend f on $(0,2L)$ in an even or odd way with respect to $x = L$, we obviously have $f \in L^2((0,2L))$. Also, by taking $T = 2L$ in (2), we see that $b_k = 0$ if we made the even extension, and $a_k = 0$ if we made the odd one. Therefore, (1) yields

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}x\right), \quad a_k := \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi}{L}x\right)dx \tag{3}$$

and

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right), \qquad b_k := \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx, \tag{4}$$

both equalities holding in the $L^2((0,L))$ sense.

Problema 1.* Es té l'equació de la calor amb condicions de frontera de Neumann:

$$\begin{cases} u_t - Du_{xx} = 0, & 0 < x < \pi,\ t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0, & t > 0 \\ u(x,0) = g(x), & 0 < x < \pi, \end{cases}$$

amb $D > 0$ i $g \in L^2\big((0,\pi)\big)$.

(a) Calculeu u per separació de variables i demostreu la convergència uniforme de u cap a la mitjana de les temperatures inicials $m = \frac{1}{\pi}\int_0^\pi g(x)\,dx$ quan $t \to +\infty$.

(b) Proveu que si $g \in C^1\big([0,\pi]\big)$, llavors $u \in C^0\big([0,\pi]\times[0,\infty)\big)$.

(c) Proveu que si $g \in C^2\big([0,\pi]\big)$ i $g'(0) = g'(\pi) = 0$ llavors $u_x(\cdot,t)$ convergeix uniformement a g' en $[0,\pi]$ quan $t \to 0$. Discutiu la necessitat de la condició $g'(0) = g'(\pi) = 0$.

Solution.

(a) In the method of separation of variables, we first look for solutions to $u_t - Du_{xx} = 0$ of the form $u(x,t) = X(x)T(t)$. The equation on u yields $T'X - DTX'' = 0$. Then, using that X only depends on x and T only depends on t, we get $X'' =$

$-\lambda X$ and $T' = -D\lambda T$ for some $\lambda \in \mathbb{R}$. Now, the Neumann boundary condition $u_x(0,t) = u_x(\pi,t) = 0$ for all $t > 0$ means that X must satisfy the ODE

$$X'' = -\lambda X, \quad X(0) = 0, \quad X(\pi) = 0. \tag{5}$$

Under these conditions, we claim that $\lambda \geq 0$. To see this, we multiply the two sides of the equality $X'' = -\lambda X$ by X and integrate by parts using the boundary conditions $X'(0) = 0$ and $X'(\pi) = 0$ to get

$$\lambda \int_0^\pi |X|^2 = \int_0^\pi |X'|^2 \geq 0.$$

Therefore, $\lambda \geq 0$ and the general solution to $X'' = -\lambda X$ is

$$X(x) = B\sin(\sqrt{\lambda}x) + C\cos(\sqrt{\lambda}x), \qquad B,\, C \in \mathbb{R}.$$

Then, imposing the boundary conditions, we must have $B = 0$ and $\sin(\sqrt{\lambda}\pi) = 0$, which is equivalent to $\lambda = k^2$ for some $k \in \mathbb{Z}$. Thus, the general solution to (5) is

$$X(x) = C\cos(kx), \qquad C \in \mathbb{R}, \quad k = 0, 1, 2, \dots$$

Taking into account these admissible values for λ and looking at the ODE on T, we must have $T' = -Dk^2T$ for some $k \in \mathbb{Z}$. Therefore, T must be of the form

$$T(t) = e^{-Dk^2t}, \quad k = 0, 1, 2, \dots$$

Up to now, we have seen that, for every $k = 0, 1, 2, \dots$ and every $C_k \in \mathbb{R}$,

$$u(x,t) = C_k \cos(kx)e^{-Dk^2t}$$

satisfies $u_t - Du_{xx} = 0$ and $u_x(0,t) = u_x(\pi,t) = 0$. By the linearity of the equation and the boundary conditions, it is natural to consider solutions of the form

$$u(x,t) = \sum_{k=0}^{\infty} C_k \cos(kx)e^{-Dk^2t},$$

whenever the series converge in some sense. Looking now at the initial condition $u(x,0) = g(x)$, we must formally have

$$g(x) = u(x,0) = \sum_{k=0}^{\infty} C_k \cos(kx)$$

for certain $C_k \in \mathbb{R}$. By (3), we conclude that a candidate to solution of our initial problem is

$$u(x,t) = \sum_{k=0}^{\infty} a_k \cos(kx)e^{-Dk^2t}, \tag{6}$$

where

$$a_0 = \frac{1}{\pi}\int_0^\pi g(x)\,dx = m \quad \text{and} \quad a_k = \frac{2}{\pi}\int_0^\pi g(x)\cos(kx)\,dx \quad \text{if } k \geq 1.$$

For every $t > 0$, the series defining $u(x,t)$ (and all the derivatives of the series with respect to the x variable) converge uniformly in $x \in [0,\pi]$ because $g \in L^2((0,\pi))$

(hence $\sum_{k=1}^{\infty}|a_k|^2 < +\infty$ by Parseval identity and, in particular, $|a_k| \to 0$ as $k \to +\infty$). Therefore, if u is as in (6), then $u \in C^{\infty}([0,\pi] \times (0,+\infty))$ and satisfies $u_t - Du_{xx} = 0$, and $u_x(0,t) = u_x(\pi,t) = 0$ for all $t > 0$. Finally, regarding the convergence to the initial data, by Parseval identity we have

$$\begin{aligned}\|u(\cdot,t) - g\|^2_{L^2((0,\pi))} &= \frac{\pi}{2}\sum_{k=1}^{\infty}|a_k|^2|e^{-Dk^2t} - 1|^2 \\ &\le \frac{\pi}{2}\sum_{k=1}^{N}|a_k|^2|e^{-Dk^2t} - 1|^2 + 2\pi\sum_{k=N+1}^{\infty}|a_k|^2\end{aligned}$$

for all $N \geq 1$. Since $\sum_{k=1}^{\infty}|a_k|^2 < +\infty$, for every $\epsilon > 0$ there exists N such that

$$2\pi\sum_{k=N+1}^{\infty}|a_k|^2 < \frac{\epsilon}{2}.$$

Once this N is chosen, there exists $t_0 > 0$ such that

$$\frac{\pi}{2}\sum_{k=1}^{N}|a_k|^2|e^{-Dk^2t} - 1|^2 < \frac{\epsilon}{2}$$

for all $0 \le t < t_0$, since it is the finite sum of continuous functions. Therefore, for every $\epsilon > 0$ there exists $t_0 > 0$ such that $\|u(\cdot,t) - g\|^2_{L^2((0,\pi))} < \epsilon$ for all $0 \le t < t_0$. This means that $u(\cdot,t)$ tends to g in the $L^2((0,\pi))$ sense when $t \to 0$.

It only remains to be shown the uniform convergence of $u(x,t)$ to the mean temperature $m = \frac{1}{\pi}\int_0^{\pi} g(x)\,dx$ as $t \to +\infty$. Note that, for all $t \geq 1$,

$$\begin{aligned}|u(x,t) - m| = |u(x,t) - a_0| &\le \sum_{k=1}^{\infty}|a_k|e^{-Dk^2t} \\ &= e^{-Dt}\sum_{k=1}^{\infty}|a_k|e^{-D(k^2-1)t} \le e^{-Dt}\sum_{k=1}^{\infty}|a_k|e^{-D(k^2-1)}.\end{aligned}$$

Thus, to check that $\|u(\cdot,t) - m\|_{L^{\infty}((0,\pi))} \to 0$ when $t \to +\infty$, it suffices to show that

$$\sum_{k=1}^{\infty}|a_k|e^{-D(k^2-1)} < +\infty. \tag{7}$$

We will prove (7) in two different ways. The first one is as follows. As we already mentioned, since $g \in L^2((0,\pi))$, we have $\sum_{k=1}^{\infty}|a_k|^2 < +\infty$ by Parseval identity. Therefore, $|a_k|$ are uniformly bounded in k. Moreover, it is clear that

$$\lim_{k\to\infty} k^2 e^{-D(k^2-1)} = 0,$$

which easily yields $e^{-D(k^2-1)} \leq C/k^2$ for some $C > 0$ and all $k \geq 1$. Then, we deduce that there exists $C > 0$ such that

$$\sum_{k=1}^{\infty} |a_k| e^{-D(k^2-1)} \leq C \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty, \tag{8}$$

which proves (7).

For the second proof of (7), we simply use Cauchy-Schwarz to get

$$\sum_{k=1}^{\infty} |a_k| e^{-D(k^2-1)} \leq \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} e^{-2D(k^2-1)} \right)^{1/2} < +\infty.$$

(b) Given $N \in \mathbb{N}$ we define

$$u_N(x,t) = \sum_{k=0}^{N} a_k \cos(kx) e^{-Dk^2 t}.$$

Clearly, $u_N \in C^0([0,\pi] \times [0,+\infty))$. To solve (b) it is enough to show that, for every $T > 0$, u_N converges uniformly to u in $[0,\pi] \times [0,T]$ when $N \to +\infty$. To see this, recall that if $K \subset \mathbb{R}^2$ is a compact set, and a sequence of functions f_N continuous in K converge uniformly in K to some function f when $N \to +\infty$, then f is also continuous in K.

Since $|\cos(kx) e^{-Dk^2 t}| \leq 1$ for all $x \in [0,\pi]$ and all $t \geq 0$, we can easily estimate

$$|u_N(x,t) - u(x,t)| \leq \sum_{k=N+1}^{\infty} |a_k|. \tag{9}$$

Thus, the uniform convergence of u_N to u will follow if we show that $\sum_{k=1}^{\infty} |a_k|$ is convergent. Using that $g \in C^1([0,\pi])$ and integration by parts, for $k \geq 1$ we have

$$a_k = -\frac{2}{\pi k} \int_0^{\pi} g'(x) \sin(kx)\, dx, \text{ that is, } -ka_k = \frac{2}{\pi} \int_0^{\pi} g'(x) \sin(kx)\, dx. \tag{10}$$

This means that $-ka_k$ is the Fourier coefficient of g' in sinus series, see (4). Since $g' \in L^2((0,\pi))$, from Parseval identity applied to g' we deduce that

$$\sum_{k=1}^{\infty} k^2 a_k^2 < \infty.$$

Therefore, using Cauchy-Schwarz inequality we get

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} k|a_k| \frac{1}{k} \leq \Big(\sum_{k=1}^{\infty} k^2 a_k^2 \Big)^{1/2} \Big(\sum_{k=1}^{\infty} \frac{1}{k^2} \Big)^{1/2} < +\infty,$$

which shows that the right hand side of (9) is as small as we want if we take N big enough.

(c) Assume that we know that $u_x \in C^0([0,\pi] \times [0,T])$ for some $T > 0$. Then u_x will be uniformly continuous in the compact set $[0,\pi] \times [0,T]$, and this yields the uniform convergence of $u_x(\cdot,t)$ to $u_x(\cdot,0) = g'$ as $t \to 0$.

Therefore, it is enough to show that $u_x \in C^0([0,\pi] \times [0,T])$ for some $T > 0$. To prove this statement, we will proceed in essentially the same way as in (b), checking that $(u_N)_x$ is uniformly convergent in $[0,\pi] \times [0,T]$ when $N \to +\infty$. Note that this also shows that u is actually differentiable (recall that if a sequence of functions f_N are such that $(f_N)' \to h$ uniformly and $f_N \to f$ pointwise as $N \to +\infty$, then f is differentiable and $f' = h$).

We have

$$(u_N)_x(x,t) = -\sum_{k=1}^{N} a_k k \sin(kx) e^{-Dk^2 t}.$$

Therefore, to show that $(u_N)_x$ converges uniformly when $N \to +\infty$ it is enough to check that, for every $\epsilon > 0$,

$$\sum_{k=N+1}^{\infty} k|a_k||\sin(kx)|e^{-Dk^2t} \leq \sum_{k=N+1}^{\infty} k|a_k| < \epsilon \tag{11}$$

if N is big enough. Following the integration by parts on the definition of a_k done in (b) (see (10)), and integrating by parts once more (using this time that $g'(0) = g'(\pi) = 0$), we get

$$a_k = -\frac{2}{\pi k^2} \int_0^\pi g''(x) \cos(kx)\, dx.$$

Since $g'' \in L^2((0,\pi))$, we can proceed exactly as in (b) to get that $\sum (k^2|a_k|)^2 < \infty$. Then, using Cauchy-Schwarz inequality as in (b) we obtain

$$\sum_{k=1}^{\infty} k|a_k| < \infty,$$

which yields (11).

We have seen that the condition $g'(0) = g'(\pi) = 0$ is sufficient to get that $u_x(\cdot,t)$ converges uniformly to g' in $[0,\pi]$ when $t \to 0$. It is also necessary because $u_x(0,t) = u_x(\pi,t) = 0$ for all $t > 0$ and, if $u_x(0,t)$ converges to $g'(0)$, we must have $g'(0) = 0$. The same argument applies to $g'(\pi)$.

Problema 2.* Determineu la temperatura d'una barra de longitut π, si inicialment la temperatura és $x(\pi - x)$, en l'extrem de l'esquerra la temperatura sempre és nul·la i en el de la dreta no hi ha fluxe de calor. Preneu el coeficient de difusió $D = 1$.

Solution.

We want to solve the problem

$$\begin{cases} u_t - u_{xx} = 0, & x \in (0,\pi),\ t > 0 \\ u(0,t) = 0, & t > 0, \\ u_x(\pi,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in [0,\pi], \end{cases}$$

where $g(x) = x(\pi - x)$. Using the method of separation of variables and the notation from Problem 1, for solutions of the form $u(x,t) = X(x)T(t)$ we must have

$$X(x) = B\sin(\sqrt{\lambda}x) + C\cos(\sqrt{\lambda}x)$$

with B, $C \in \mathbb{R}$ and $\lambda \geq 0$. Now, imposing that $X(0) = 0$ (which corresponds to $u(0,t) = 0$), we get that $C = 0$, hence

$$X(x) = B\sin(\sqrt{\lambda}x).$$

Using then that $X'(\pi) = 0$ (which corresponds to $u_x(\pi,t) = 0$), it must hold $\cos(\sqrt{\lambda}\pi) = 0$ (or $B = 0$, which would yield the trivial solution $X \equiv 0$), thus

$$\sqrt{\lambda} = \frac{2k+1}{2}, \quad k = 0,1,2,\ldots$$

Arguing as in stage (a) of Problem 1 we conclude that a formal solution, satisfying the boundary conditions at $x = 0$ and $x = \pi$, and satisfying the equation $u_t - u_{xx} = 0$ is given by

$$u(x,t) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{2k+1}{2}x\right) e^{-\left(\frac{2k+1}{2}\right)^2 t}.$$

It remains to choose the coefficients a_k in agreement with the initial condition, that is, such that

$$u(x,0) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{2k+1}{2}x\right) = g(x). \tag{12}$$

Let us check that we can write g as a series of the form given in (12), and then obtain the corresponding coefficients. If we extend g in an even way with respect to $x = \pi$ (motivated by the Neumann condition at this point) and apply identity (4) with $L = 2\pi$, we obtain that g can be written in the $L^2((0,2\pi))$ sense as

$$g(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m}{2}x\right)$$

with

$$b_m := \frac{1}{\pi}\int_0^{2\pi} g(x)\sin\left(\frac{m}{2}x\right)dx, \qquad m \geq 1.$$

Now, we use that g is even with respect to $x = \pi$ in order to simplify the expression of the coefficients b_m. On the one hand, since $\sin(mx/2)$ is odd (and g is even) with respect to $x = \pi$ when m is even, we obtain that $b_{2k} = 0$ for all $k \in \mathbb{N}$. On the other hand, $\sin(mx/2)$ is even with respect to $x = \pi$ when m is odd, hence we get

$$b_{2k+1} = \frac{1}{\pi} \int_0^{2\pi} g(x) \sin\left(\frac{2k+1}{2} x\right) dx = \frac{2}{\pi} \int_0^{\pi} g(x) \sin\left(\frac{2k+1}{2} x\right) dx. \qquad (13)$$

Therefore, we conclude that $a_k = b_{2k+1}$ and

$$g(x) = \sum_{k=0}^{\infty} b_{2k+1} \sin\left(\frac{2k+1}{2} x\right).$$

Taking $g(x) = x(\pi - x)$ and using integration by parts twice in (13), one can show that

$$b_{2k+1} = 8 \frac{4 - (-1)^k (2k+1)\pi}{(2k+1)^3 \pi}.$$

Here we used that $\sin((2k+1)\pi/2) = (-1)^k$. We leave the details for the reader.

Then, the solution to our initial problem we finally obtained is

$$u(x,t) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{4 - (-1)^k (2k+1)\pi}{(2k+1)^3} \sin\left(\frac{2k+1}{2} x\right) e^{-\left(\frac{2k+1}{2}\right)^2 t}.$$

Problema 3.* (a) Resoleu per separació de variables el problema

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 1 & t > 0, \\ u(x,0) = 1 & x \in (0,\pi), \\ u_t(x,0) = \sin^2(x) & x \in (0,\pi). \end{cases}$$

(b) Exteneu adequadament les condicions inicials i apliqueu la fórmula de d'Alembert per calcular $u(\pi/4,3)$.

(c) Calculeu

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2\{4-(2k+1)^2\}}.$$

Solution.

(a) In order to have homogeneous boundary conditions at $x = 0$ and $x = \pi$ we define the auxiliary function $v(x,t) = u(x,t) - 1$. Then, it is easy to check that v satisfies the problem

$$\begin{cases} v_{tt} - v_{xx} = 0 & x \in (0,\pi),\ t > 0, \\ v(0,t) = v(\pi,t) = 0 & t > 0, \\ v(x,0) = 0 & x \in (0,\pi), \\ v_t(x,0) = \sin^2(x) & x \in (0,\pi). \end{cases}$$

In the method of separation of variables, we first look for solutions to $v_{tt} - v_{xx} = 0$ of the form $v(x,t) = X(x)T(t)$. The equation on v yields $XT'' - X''T = 0$. Then, using that X only depends on x and T only depends on t, we get $X'' = -\lambda X$ and $T'' = -\lambda T$ for some $\lambda \in \mathbb{R}$. Now, the homogeneous Dirichlet boundary conditions $v(0,t) = v(\pi,t) = 0$ for all $t > 0$ means that X must satisfy the ODE

$$X'' = -\lambda X, \quad X(0) = 0, \quad X(\pi) = 0. \tag{1}$$

Under these conditions, we claim that $\lambda \geq 0$. To see this, we multiply the two sides of the equality $X'' = -\lambda X$ by X and integrate by parts using the boundary conditions $X(0) = 0$ and $X(\pi) = 0$ to get

$$\lambda \int_0^\pi |X|^2 = \int_0^\pi |X'|^2 \geq 0.$$

Therefore, $\lambda \geq 0$, and the general solution to $X'' = -\lambda X$ is

$$X(x) = C\sin(\sqrt{\lambda}x) + D\cos(\sqrt{\lambda}x), \qquad C,\ D \in \mathbb{R}.$$

Then, imposing the boundary conditions, we must have $D = 0$ and $\sin(\sqrt{\lambda}\pi) = 0$, which is equivalent to $\lambda = k^2$ for some $k \in \mathbb{Z}$. Thus, the general solution to (1) is

$$X(x) = C\sin(kx), \qquad C \in \mathbb{R}, \quad k = 1, 2, \ldots$$

Taking into account these admissible values for λ and looking at the ODE on T, we must have $T'' = -k^2 T$ for some $k \in \mathbb{N}$. Therefore, T must be of the form

$$T(t) = A\sin(kt) + B\cos(kt), \qquad A,\ B \in \mathbb{R}, \quad k = 1, 2, \ldots$$

Up to now, we have seen that, for every $k = 1, 2, \ldots$ and every $a_k,\ b_k \in \mathbb{R}$,

$$v(x,t) = \big(a_k \sin(kt) + b_k \cos(kt)\big)\sin(kx)$$

satisfies $v_{tt} - v_{xx} = 0$ and $v(0,t) = v(\pi,t) = 0$. By the linearity of the equation and the boundary conditions, it is natural to consider solutions of the form

$$v(x,t) = \sum_{k=1}^{\infty} \big(a_k \sin(kt) + b_k \cos(kt)\big)\sin(kx),$$

whenever the series converge in some sense. Looking now at the initial conditions $v(x,0) = 0$ and $v_t(x,0) = \sin^2(x)$, we must formally have

$$0 = v(x,0) = \sum_{k=1}^{\infty} b_k \sin(kx)$$

and

$$\sin^2(x) = v_t(x,0) = \sum_{k=1}^{\infty} a_k\, k\, \sin(kx)$$

for certain $a_k,\ b_k \in \mathbb{R}$. By using the Fourier expansion in sinus series, we conclude that a candidate to solution is

$$v(x,t) = \sum_{k=1}^{\infty} \big(a_k \sin(kt) + b_k \cos(kt)\big)\sin(kx),$$

where

$$a_k\, k = \frac{2}{\pi}\int_0^{\pi} \sin^2(x)\, \sin(kx)\, dx \quad \text{ and } \quad b_k = 0 \quad \text{for } k = 1, 2, \ldots$$

Let us compute a_k. Using that

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \quad \text{ and } \quad \int_0^{\pi} \sin(kx)\, dx = \frac{1}{k}(1-(-1)^k)$$

we get

$$\int_0^{\pi} \sin^2(x)\sin(kx)\, dx = \frac{1-(-1)^k}{2k} - \frac{1}{2}\int_0^{\pi} \cos(2x)\sin(kx)\, dx.$$

Furthermore, since

$$\cos(2x)\sin(kx) = \frac{1}{2}\sin\big((2+k)x\big) - \frac{1}{2}\sin\big((2-k)x\big),$$

we arrive at

$$a_k\, k = \frac{2}{\pi}\int_0^{\pi} \sin^2(x)\sin(kx)\, dx = \frac{1-(-1)^k}{\pi}\left(\frac{1}{k} - \frac{1}{2(2+k)} + \frac{1}{2(2-k)}\right),$$

which yields

$$a_k = \frac{4(1-(-1)^k)}{\pi k^2(4-k^2)} = \begin{cases} \frac{8}{\pi k^2(4-k^2)} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Finally, taking into account that $u = v + 1$, we conclude that

$$u(x,t) = 1 + \sum_{k=0}^{\infty} \frac{8}{\pi(2k+1)^2\{4-(2k+1)^2\}} \sin\big((2k+1)t\big) \sin\big((2k+1)x\big).$$

(b) Recall that d'Alembert formula

$$w(x,t) = \frac{1}{2}g(x-ct) + \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy \tag{2}$$

gives the solution to the problem

$$\begin{cases} w_{tt} - c\,w_{xx} = 0 & x \in \mathbb{R},\, t > 0, \\ w(x,0) = g(x) & x \in \mathbb{R}, \\ w_t(x,0) = h(x) & x \in \mathbb{R}. \end{cases}$$

The function v found in (a) solves this problem for $c = 1$ and $(x,t) \in (0,\pi) \times [0,+\infty)$, with $g(x) = 0$ and $h(x) = \sin^2(x)$ for $x \in (0,\pi)$, and has vanishing Dirichlet boundary conditions at $x = 0$ and $x = \pi$. Hence, we can also find v using d'Alembert formula by doing an odd reflection with respect to $x = 0$ and then a 2π-periodic extension in the whole real line of $g(x) = 0$ and $h(x) = \sin^2(x)$, originally given in $(0,\pi)$. That is, if for the function $\sin^2(x)$ in $(0,\pi)$ we denote by h its odd reflection with respect to $x = 0$ and 2π-periodic extension in $\mathbb{R}$, (2) yields

$$v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(y)dy. \tag{3}$$

Then, using that $-\pi < \pi/4 - 3 < 0$ and $\pi < \pi/4 + 3 < 2\pi$, together with the fact that h is odd with respect to $x = 0$ and $x = \pi$, we get

$$v(\pi/4, 3) = \frac{1}{2}\int_{\pi/4-3}^{\pi/4+3} h(y)\,dy = \frac{1}{2}\int_{3-\pi/4}^{7\pi/4-3} \sin^2(y)\,dy = \frac{1}{2}(\pi - 3).$$

Finally, recalling that $u = v + 1$, we conclude that

$$u(\pi/4, 3) = \frac{1}{2}(\pi - 1).$$

(c) From (a) we know that

$$v(x,t) = \sum_{k=0}^{\infty} \frac{8}{\pi(2k+1)^2\{4-(2k+1)^2\}} \sin\big((2k+1)t\big) \sin\big((2k+1)x\big).$$

In order to relate this expression for v to the sum in the statement of the exercise, we want to choose a point (x,t) such that $\sin((2k+1)t)\sin((2k+1)x) = 1$ for all

$k \in \mathbb{N}$. This is satisfied, for example, by $(x,t) = (\pi/2, \pi/2)$. Thus,

$$v(\pi/2, \pi/2) = \sum_{k=0}^{\infty} \frac{8}{\pi(2k+1)^2\{4-(2k+1)^2\}}.$$

On the other hand, using (3) from (b) we get

$$v(\pi/2, \pi/2) = \frac{1}{2}\int_0^{\pi} h(x)\,dx = \frac{1}{2}\int_0^{\pi} \sin^2(x)\,dx = \frac{\pi}{4}.$$

Combining both identities we conclude

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2\{4-(2k+1)^2\}} = \frac{\pi^2}{32}.$$

Problema 6.* (Una corda en un camp magnètic). Una corda metal·lica està fixada als punts $(0,0,0)$ i $(\pi,0,0)$ de l'eix x de $\mathbb{R}^3$. Per parametritzar el desplaçament en tres dimensions de la corda fem servir una funció que pren valors complexos $u(x) = u^z(x) + iu^y(x)$. La corda està carregada elèctricament i està immersa en un camp magnètic constant en la direcció x. Degut a les forces de Lorentz, l'equació per a petites vibracions d'aquesta corda és

$$\begin{cases} u_{tt} - c^2 u_{xx} = i\kappa u_t & x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0 & t > 0, \\ u(x,0) = g(x) & x \in (0,\pi), \\ u_t(x,0) = h(x) & x \in (0,\pi), \end{cases}$$

on κ serà una certa constant (la densitat lineal de càrrega de la corda multiplicada per la intensitat del camp magnètic i dividida per la densitat lineal de la corda).

Useu separació de variables per calcular $u(x,t)$ per a $t > 0$ per $\kappa = 1$. Les forces de Lorentz provoquen amortiment de les solucions?

Physical interpretation: the Lorentz force.

A particle of charge q moving with a velocity v in a magnetic field B experiences a force $F = (F_1, F_2, F_3)$ given by

$$F = qv \times B.$$

In our case $B = (b,0,0)$ with $b \in \mathbb{R}$, and we are supposing that the metallic string is parametrized as $x \mapsto (x, u(x)) = (x, u^z(x), u^y(x)) \in \mathbb{R}^3$.

On one hand, if the string has linear density ρ, by the wave equation with an external force term, we have

$$\rho u^z_{tt} - \tau u^z_{xx} = F_2\,, \qquad \rho u^y_{tt} - \tau u^y_{xx} = F_3\,.$$

On the other hand, by the Lorentz force

$$F = q(0, u^z_t, u^y_t) \times (b,0,0) = qb(0, u^y_t, -u^z_t).$$

Therefore, setting $c^2 = \tau/\rho$, we obtain the two equations

$$u^z_{tt} - c^2 u^z_{xx} = \tfrac{qb}{\rho}\, u^y_t\,, \qquad u^y_{tt} - c^2 u^y_{xx} = -\tfrac{qb}{\rho}\, u^z_t\,.$$

Using the complex notation $u = u^z + iu^y$ and setting $\kappa = -qb/\rho$, both equations can be written as one: $u_{tt} - c^2 u_{xx} = i\kappa u_t\,$.

Solution.

We assume $\kappa = 1$. In the method of separation of variables we first look for solutions of the form $u(x,t) = X(x)T(t)$, now complex valued. Then, the equation $u_{tt} - c^2 u_{xx} = iu_t$ yields the existence of a constant $\lambda \in \mathbb{C}$ such that

$$X'' = -\lambda X, \qquad T'' - iT' = -\lambda c^2 T.$$

From the boundary conditions $X(0) = X(\pi) = 0$ we get that $\lambda = k^2$ and

$$X(x) = B\sin(kx), \qquad B \in \mathbb{R}, \quad k = 1, 2, \ldots$$

Therefore, T satisfies

$$T'' - iT' + c^2k^2T = 0.$$

For every k, this is a second order linear ODE with constant coefficients, hence the general solution is given by a linear combination of $T_j(t) = e^{\beta_j t}$ $(j = 1, 2)$ with $\beta_j \in \mathbb{C}$ satisfying the equation

$$\beta^2 - i\beta + c^2k^2 = 0. \tag{4}$$

If we set $\gamma_k = \sqrt{1 + 4c^2k^2} > 1$ for $k = 1, 2, \ldots$, then the solutions to (4) are $\beta = \frac{i}{2}(1 \pm \gamma_k)$. Therefore, a formal solution to our initial problem is given by

$$u(x,t) = \sum_{k=1}^{\infty} \sin(kx)\left(A_k e^{\frac{i}{2}(1+\gamma_k)t} + C_k e^{\frac{i}{2}(1-\gamma_k)t}\right), \tag{5}$$

where A_k and C_k are complex numbers that must be chosen in order to satisfy the initial conditions $u(\cdot,0) = g$ and $u_t(\cdot,0) = h$ in $(0,\pi)$. Writing the real and imaginary parts of the coefficients

$$A_k = a_k + ib_k, \quad C_k = c_k + id_k,$$

and using that $e^{i\varphi} = \cos\varphi + i\sin\varphi$ for all $\varphi \in \mathbb{R}$, from (5) we get $u = u^z + iu^y$ with

$$u^z(x,t) = \sum_{k=1}^{\infty} \sin(kx)\Big(a_k \cos\left(\tfrac{1+\gamma_k}{2}t\right) + c_k \cos\left(\tfrac{1-\gamma_k}{2}t\right) \\ -b_k \sin\left(\tfrac{1+\gamma_k}{2}t\right) - d_k \sin\left(\tfrac{1-\gamma_k}{2}t\right)\Big)$$

and

$$u^y(x,t) = \sum_{k=1}^{\infty} \sin(kx)\Big(b_k \cos\left(\tfrac{1+\gamma_k}{2}t\right) + d_k \cos\left(\tfrac{1-\gamma_k}{2}t\right) \\ +a_k \sin\left(\tfrac{1+\gamma_k}{2}t\right) + c_k \sin\left(\tfrac{1-\gamma_k}{2}t\right)\Big).$$

On the one hand, if we now impose the initial condition on the position

$$u^z(x,0) + iu^y(x,0) = u(x,0) = g(x) = g_1(x) + ig_2(x),$$

where g_1 and g_2 are real valued, we obtain

$$\sum_{k=1}^{\infty} \sin(kx)(a_k + c_k) = g_1(x), \quad \sum_{k=1}^{\infty} \sin(kx)(b_k + d_k) = g_2(x).$$

Hence, taking the Fourier expansion in sinus series of g_1 and g_2, we must have

$$\begin{aligned} a_k + c_k &= \frac{2}{\pi}\int_0^{\pi} g_1(x)\sin(kx)\,dx, \\ b_k + d_k &= \frac{2}{\pi}\int_0^{\pi} g_2(x)\sin(kx)\,dx. \end{aligned} \tag{6}$$

On the other hand, imposing the initial velocity,

$$u_t^z(x,0) + iu_t^y(x,0) = u_t(x,0) = h(x) = h_1(x) + ih_2(x),$$

where h_1 and h_2 are real valued, we get that

$$\begin{aligned} h_1(x) = u_t^z(x,0) &= \sum_{k=1}^{\infty} \sin(kx)\left(-b_k\frac{1+\gamma_k}{2} - d_k\frac{1-\gamma_k}{2}\right), \\ h_2(x) = u_t^y(x,0) &= \sum_{k=1}^{\infty} \sin(kx)\left(a_k\frac{1+\gamma_k}{2} + c_k\frac{1-\gamma_k}{2}\right). \end{aligned}$$

Taking the Fourier expansion in sinus series of h_1 and h_2, we obtain

$$\begin{aligned} -b_k\frac{1+\gamma_k}{2} - d_k\frac{1-\gamma_k}{2} &= \frac{2}{\pi}\int_0^{\pi} h_1(x)\sin(kx)\,dx, \\ a_k\frac{1+\gamma_k}{2} + c_k\frac{1-\gamma_k}{2} &= \frac{2}{\pi}\int_0^{\pi} h_2(x)\sin(kx)\,dx. \end{aligned} \tag{7}$$

From (6) and (7) we get a system of four linear equations whose solution is

$$\begin{aligned} a_k &= \frac{1}{\pi\gamma_k}\int_0^{\pi} \big(g_1(x)(\gamma_k - 1) + 2h_2(x)\big)\sin(kx)\,dx, \\ c_k &= \frac{1}{\pi\gamma_k}\int_0^{\pi} \big(g_1(x)(\gamma_k + 1) + 2h_2(x)\big)\sin(kx)\,dx, \\ b_k &= \frac{1}{\pi\gamma_k}\int_0^{\pi} \big(g_2(x)(\gamma_k - 1) - 2h_1(x)\big)\sin(kx)\,dx, \\ d_k &= \frac{1}{\pi\gamma_k}\int_0^{\pi} \big(g_2(x)(\gamma_k + 1) - 2h_1(x)\big)\sin(kx)\,dx. \end{aligned}$$

Finally, using that $A_k = a_k + ib_k$, $C_k = c_k + id_k$, and $ih = i(h_1 + ih_2) = -h_2 + ih_1$, we obtain that

$$A_k = \frac{1}{\pi\gamma_k}\int_0^\pi \big(g(x)(\gamma_k - 1) - 2ih(x)\big)\sin(kx)\,dx,$$
$$C_k = \frac{1}{\pi\gamma_k}\int_0^\pi \big(g(x)(\gamma_k + 1) - 2ih(x)\big)\sin(kx)\,dx.$$

Plugging this into (5) gives the final form of the solution.

The solution u oscillates in time because $i(1 \pm \gamma_k)$ are purely imaginary numbers (and recall (5)), hence it is clear that there is no damping. We now show that indeed the energy is preserved along time. The total energy E of u at time t is given by (see Problem 1 in List 3)

$$E(t) = \frac{1}{2}\int_0^\pi \left(|u_t|^2 + c^2|u_x|^2\right)dx = \frac{1}{2}\int_0^\pi \Big((u_t^z)^2 + (u_t^y)^2 + c^2\big((u_x^z)^2 + (u_x^y)^2\big)\Big)\,dx.$$

Then,

$$\frac{d}{dt}E(t) = \int_0^\pi \left(u_t^z u_{tt}^z + u_t^y u_{tt}^y + c^2 u_x^z u_{xt}^z + c^2 u_x^y u_{xt}^y\right)dx.$$

Now, using the equation $u_{tt} - c^2 u_{xx} = iu_t$, the boundary conditions $u(0,t) = u(\pi,t) = 0$ which yield $u_t(0,t) = u_t(\pi,t) = 0$, and integration by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_0^\pi \left(u_t^z u_{tt}^z + c^2 u_x^z u_{xt}^z + u_t^y u_{tt}^y + c^2 u_x^y u_{xt}^y\right)dx \\
&= \int_0^\pi \left(c^2 u_t^z u_{xx}^z - u_t^z u_t^y + c^2 u_x^z u_{xt}^z + c^2 u_t^y u_{xx}^y + u_t^y u_t^z + c^2 u_x^y u_{xt}^y\right)dx \\
&= c^2\int_0^\pi (u_t^z u_{xx}^z + u_x^z u_{xt}^z)\,dx + c^2\int_0^\pi (u_t^y u_{xx}^y + u_x^y u_{xt}^y)\,dx \\
&= c^2\int_0^\pi (u_t^z u_{xx}^z - u_{xx}^z u_t^z)\,dx + c^2\int_0^\pi (u_t^y u_{xx}^y - u_{xx}^y u_t^y)\,dx = 0.
\end{aligned}$$

Therefore E is constant in time, and there is no damping. Otherwise the energy would decrease as t increases.

Problema 1. Demostreu les següents propietats importants del Laplacià:

(a) La definició de Laplacià no depèn de la base ortonormal de $\mathbb{R}^n$ triada. És a dir

$$\sum_i \partial_{e_i e_i} u(x) = \sum_i \partial_{e'_i e'_i} u(x)$$

per cada parella de bases ortonormals $\{e_i\}$, $\{e'_i\}$ de $\mathbb{R}^n$.

(b) El Laplacià és invariant per rotacions. És a dir, per a tota matriu ortogonal O i $u \in C^2$, si es defineix $u^*(x) := u(Ox)$ i $x^* = Ox$ es té

$$\Delta u^*(x) = \Delta u(Ox) = \Delta u(x^*).$$

(c) El Laplacià és invariant per translacions.

(d) El Laplacià és invariant per isometries de $\mathbb{R}^n$.

Problema 2. Donada una funció f a $[0, L]$, considerem el problema

$$\begin{cases} -u'' = f(x) & \text{a } (0, L) \\ u(0) = u(L) = 0. \end{cases}$$

Resoleu el problema per simple integració de l'EDO. Demostreu que la solució ve donada per l'expressió

$$u(x) = \int_0^L G(x, y) f(y) dy, \tag{1}$$

on $G(x, y)$ es una funció explícita que no depén de f. S'anomena la funció de Green del problema. Dibuixeu la gràfica de G com a funció de y per un x donat. Comproveu que G satisfà les seguents propietats: $G(x, y) \geq 0$, $G(x, y) = G(y, x)$ i $G(0, y) = G(L, y) = 0$, per tot x i y.

Problema 3. Considerem el problema de Dirichlet pel Laplacià a la bola o disc unitat B_1 de $\mathbb{R}^2$:

$$\begin{cases} -\Delta u = 0 & \text{a } B_1 \\ u = g & \text{a } \partial B_1, \end{cases}$$

on $g : \partial B_1 \to \mathbb{R}$ és una funció contínua donada. Volem trobar un mètode alternatiu al de separació de variables (polars) per demostrar que la solució ve donada per l'expressió

$$u(x) = \int_{\partial B_1} P(x, y) g(y) dy, \qquad \text{per } x \in B_1, \tag{2}$$

on P s'anomena el nucli de Poisson i ve donat (en coordenades polars) per

$$P(x, y) = P(re^{i\alpha}, e^{i\beta}) = \frac{1 - r^2}{2\pi} \frac{1}{1 + r^2 - 2r\cos(\alpha - \beta)}. \tag{3}$$

Per fer-ho, sigui u la part real d'una funció holomorfa φ. Fixem $z = re^{i\alpha} \in B_1$ i considerem les funcions

$$\zeta \to \frac{\varphi(\zeta)}{\zeta - z} \qquad \text{i} \qquad \zeta \to \frac{\varphi(\zeta)\bar{z}}{1 - \zeta\bar{z}}.$$

Sumeu les seves integrals a ∂B_1 i, usant la fòrmula de Cauchy per funcions holomorfes, deduïu (2)-(3).

[Noteu la similitud de (1) i (2). Fórmules explícites com aquestes per resoldre problemes per EDPs només existeixen, en general, en dimensió $n = 1$ o bé, en dimensió $n \geq 2$, per dominis amb moltes simetries com una bola, un rectangle, cilindres, semi-espais, etc. Per dominis generals, les fórmules encara són vàlides i útils, però els nuclis G i P no són explícits (si bé, com veurem, es poden calcular numèricament discretitzant).

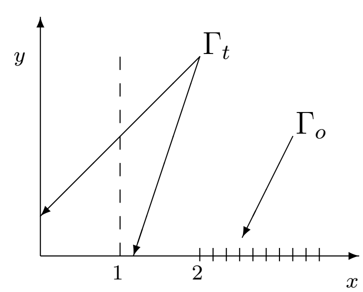
Problema 4. Quina regularitat tenen les solucions de $u_{xx} - 2u_{xy} + 2u_{yy} = 0$ en un obert de $\mathbb{R}^2$? I les de $u_{xx} - 3u_{xy} + 2u_{yy} = 0$? (Indicació: considereu de fer canvis lineals de les variables x i y).

Problema 5. Donat un domini $\Omega \subset \mathbb{R}^n$, trobeu la interpretació probabilística (en termes de costos o "pagaments" durant el passeig aleatori discret) per la solució del problema

$$\begin{cases} -\Delta u = 1 & \text{a} \quad \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{a} \quad \partial\Omega. \end{cases}$$

Quan $\Omega = B_2 \setminus \overline{B}_1 \subset \mathbb{R}^n$, trobeu els punts de Ω a on el cost és màxim.

Problema 6.* Considerem el domini $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$. Suposem que $\Gamma_o = \{(x, 0) \in \mathbb{R}^2 : x > 2\}$ és la part oberta de la frontera de Ω i que $\Gamma_t = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 2\} \cup \{(0, y) \in \mathbb{R}^2 : y > 0\}$ és la part tancada de la frontera (veure la fígura adjunta). Calculeu la coordenada y tal que la probabilitat de sortir del domini començant camins aleatoris des del punt $(1, y)$ sigui màxima.



Problema 7.* Una imatge en blanc i negre és una funció $u : \Omega \subset \mathbb{R}^2 \to [0, 255] \subset \mathbb{R}$ a valors reals definida en un domini Ω del pla. El valor $u(x, y)$ representa el nivell de gris de la imatge en el punt (x, y), on 0 representa negre i 255 blanc. Si la imatge

és digital (o discreta) i rectangular, llavors el seu domini de definició ve donat per la discretització d'un rectangle:

$$\{(x_i, y_j) = (ih, jh) : i = 1, \ldots, I; j = 1, \ldots, J\},$$

on $h > 0$. El punt (x_i, y_j) és el centre d'un pixel quadrat de mida $h \times h$ ("in 'pixel', 'pix' is slang for 'picture' and 'el' stands for 'element' "). Direm que la imatge és de mida $I \times J$.

Transmitint imatges, a vegades es perd un pixel. Si es tracta d'un pixel interior (és a dir $1 < i < I$ i $1 < j < J$), una manera natural de reassignar-li un valor és per la mitjana

$$u(x_i, y_j) = \frac{1}{4}\left\{u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1})\right\}. \tag{4}$$

Els pixels no interiors els anomenem de vora (són els que toquen a algun dels quatre costats de la imatge rectangular).

Una imatge digital es diu que és *ideal* si satisfà (4) en cadascun dels seus pixels interiors. Equivalentment, $\Delta_h u(x_i, y_j) = 0$ en cada pixel interior (x_i, y_j), a on

$$\begin{aligned}&\Delta_h u(x_i, y_j) := &(5)\\ &:= \frac{4}{h^2}\left\{\frac{1}{4}\big(u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1})\big) - u(x_i, y_j)\right\},\end{aligned}$$

s'anomena el *Laplacià discret* de u.

Demostreu que, coneguts els pixels de vora d'una imatge digital, existeix una única imatge digital *ideal* que els té com valors de vora.

Problema 8.* Les partícules d'una substància es mouen a $\mathbb{R}$ amb pas $h > 0$ i passeig aleatori d'anar a la dreta amb probabilitat $p = \frac{1}{2} + ah$ ($a > 0$ constant) i probabilitat d'anar a l'esquerra $p = \frac{1}{2} - ah$. El pas temporal és $\Delta t = h^2$.

(a) Fent el desenvolupament de Taylor de la concentració de partícules $u(x, t)$ en x a temps t, demostreu que u és solució de

$$u_t - \frac{1}{2}u_{xx} + 2au_x = 0, \quad x \in \mathbb{R}, \quad t > 0.$$

Suposeu que resolem aquesta equació amb condició inicial $u(x, 0) = g(x)$ amb g senar, estrictament creixent i fitada a $\mathbb{R}$, i amb g' fitada.

(b) Vista l'essència del moviment aleatori a (a), decidiu el signe de $u(0, t)$ per $t > 0$.

(c) Sigui $v(y, t) := u(y + 2at, t)$. Demostreu que

$$\begin{cases} v_t - \frac{1}{2}v_{yy} = 0 & \text{a} \quad \mathbb{R} \times (0, +\infty) \\ v(\cdot, 0) = g & \text{a} \quad \mathbb{R}. \end{cases}$$

Deduïu que v és senar en y. Considerant l'equació satisfeta per la funció v_y, demostreu que $v_y > 0$ per tot $y \in \mathbb{R}$ i $t > 0$. Per tant, $u_x > 0$ i u és subcalòrica ($u_t - \frac{1}{2}u_{xx} < 0$). Trobeu ara rigurosament el signe de $u(0, t)$ per tot $t > 0$.

Problema 9.* Donat el problema

$$\begin{cases} u_t - u_{xx} = 1 & 0 < x < 1 \quad t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 \leq x \leq 1 \end{cases}$$

(a) Determineu la solució estacionaria $u^s(x)$ que satisfà les condicions de contorn.
(b) Demostreu que $u(x,t) \leq u^s(x)$ per tot $t > 0$.
(c) Determineu $\beta > 0$ tal que $u(x,t) \geq (1 - e^{-\beta t})u^s(x)$.
(d) Deduïu que $u(x,t) \to u^s(x)$ uniformement en $[0,1]$ quan $t \to +\infty$.
(e) Resoleu el problema usant separació de variables.

Problema 10.* Considereu el domini $\Omega = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$. Sigui u la solució del problema

$$\begin{cases} u_t - \Delta u = 0 & \text{a} & \Omega \times (0,\infty), \\ u_\nu(x,y,t) = 0 & \text{a} & \{x^2 + y^2 = 1\} \times (0,\infty), \\ u_\nu(x,y,t) = \frac{15}{2}xy & \text{a} & \{x^2 + y^2 = 4\} \times (0,\infty), \\ u(x,y,0) = g(x,y) & \text{a} & \Omega \times \{0\}, \end{cases}$$

on

$$g(x,y) = 15xy\left(1 - \frac{2}{\sqrt{x^2+y^2}}\right).$$

a) És u parella o senar en x? I en y? Justifiqueu les respostes. Quina relació hi ha entre $u(3/4, 3\sqrt{3}/4, 100)$ i $u(-3/4, -3\sqrt{3}/4, 100)$?
b) Es conserva, en el temps, la temperatura mitjana?
c) Calculeu els equilibris tèrmics del problema.
d) Quan val $\lim_{t\to+\infty} u(x,y,t)$?

Problema 11. Sigui $g : [0,\infty) \to \mathbb{R}$ contínua i fitada. Usant el nucli de Gauss a tot $\mathbb{R}$, trobeu la solució del problema

$$\begin{cases} u_t - u_{xx} = 0, & x > 0, \quad t > 0 \\ u(0,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \geq 0. \end{cases}$$

(Indicació: considereu de fer alguna reflexió).

Problema 5.* Donat un domini $\Omega \subset \mathbb{R}^n$, trobeu la interpretació probabilística (en termes de costos o "pagaments" durant el passeig aleatori discret) per la solució del problema

$$\begin{cases} -\Delta u = 1 & \text{a} \quad \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{a} \quad \partial\Omega. \end{cases}$$

Quan $\Omega = B_2 \setminus \overline{B}_1 \subset \mathbb{R}^n$, trobeu els punts de Ω a on el cost és màxim.

Solution.

To simplify the exposition, let us reduce ourselves to the case $n = 2$. We leave as an exercise that the same computations and arguments work in higher dimensions.

First, let us discretize the problem. Assume that a particle initially inside Ω moves as time goes on doing steps of length $h > 0$ in any of the four cardinal directions (north, east, south, west) with uniform probability. Assume also that to take a step has a cost or payoff given by $\tau(h)$. We denote by $u^h(x, y)$ the expected total cost that a particle initially located at $(x, y) \in \Omega$ will have to pay before leaving Ω.

By the conditional probability formula, we have

$$u^h(x,y) = \tau(h) + \frac{1}{4}\Big(u^h(x+h,y) + u^h(x-h,y) + u^h(x,y+h) + u^h(x,y-h)\Big).$$

This is equivalent to

$$\begin{aligned} 0 = \frac{4\tau(h)}{h^2} &+ \frac{u^h(x+h,y) + u^h(x-h,y) - 2u^h(x,y)}{h^2} \\ &+ \frac{u^h(x,y+h) + u^h(x,y-h) - 2u^h(x,y)}{h^2}. \end{aligned} \tag{1}$$

Let us set $\tau(h) = h^2/4$. If we assume that $u^h \to u$ when $h \to 0$, taking the formal limit in (1) for $h \to 0$, we obtain the equation

$$0 = 1 + u_{xx}(x,y) + u_{yy}(x,y),$$

that is, $-\Delta u = 1$. This equation holds in Ω.

Note that in the limit h to 0, the movement of the particle is expected to be continuous. Moreover, it is also natural to expect that the probability that the particle hits the boundary of the domain is 1 since it cannot go out without touching the boundary. Thus, $u(x, y)$ is the expected total cost that a particle initially located at $(x, y) \in \Omega$ will have to pay before touching $\partial\Omega$ for the first time. Then, it is clear that the expected cost of a particle initially located at the boundary of the domain is zero since no step is needed.

We now address the second part of the exercise. Set $\Omega = B_2 \setminus \overline{B}_1 \subset \mathbb{R}^n$. By Problem 1(b) we know that Δ is invariant under rotations, that is, if O is an orthogonal matrix then $\Delta(u(Ox)) = (\Delta u)(Ox)$. Therefore, if we set $v(x) = u(Ox)$, then v also solves

$$\begin{cases} -\Delta v = 1 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

because Ω, the nonlinear term, and the boundary condition are invariant under the rotation O. This shows that $u-v$ is a harmonic function in Ω whose boundary values vanish identically on $\partial\Omega$. By the maximum (and minimum) principle, we get that $u - v \equiv 0$ in Ω, which means that $u(x) = u(Ox)$. Since this holds for all orthogonal matrix O, we conclude that u is a radially symmetric function in Ω, i.e., it depends only on $|x|$. Therefore, setting $r = |x|$, we can write $u(x) =: U(|x|) = U(r)$, where $1 \leq r \leq 2$.

Now, we can use the expression of the Laplacian for radially symmetric functions to deduce that

$$-1 = \Delta u = \frac{\partial^2}{\partial r^2} u + \frac{n-1}{r}\frac{\partial}{\partial r} u = U'' + \frac{n-1}{r} U'. \tag{2}$$

Multiplying both sides by r^{n-1} we get

$$-r^{n-1} = r^{n-1}U''(r) + (n-1)r^{n-2}U'(r) = (r^{n-1}U'(r))'$$

and, thus, $U'(r) = -\frac{r}{n} + \frac{c_1}{r^{n-1}}$ for some $c_1 \in \mathbb{R}$. Integrating once more in r, we obtain

$$U(r) = \begin{cases} -\frac{r^2}{2n} - \frac{c_1}{(n-2)r^{n-2}} + c_2 & \text{if } n \neq 2, \\ -\frac{r^2}{4} + c_1 \log(r) + c_2 & \text{if } n = 2, \end{cases}$$

for some c_1, $c_2 \in \mathbb{R}$. Imposing now the boundary condition $u = 0$ on $\partial\Omega$, which corresponds to $U(1) = U(2) = 0$, we can determine c_1 and c_2. One gets,

$$c_1 = \begin{cases} \frac{3(n-2)2^{n-1}}{(2^n-4)n} & \text{if } n \neq 2, \\ \frac{3}{4\log(2)} & \text{if } n = 2, \end{cases} \tag{3}$$

and

$$c_2 = \begin{cases} \frac{2(2^n-1)}{(2^n-4)n} & \text{if } n \neq 2, \\ \frac{1}{4} & \text{if } n = 2. \end{cases}$$

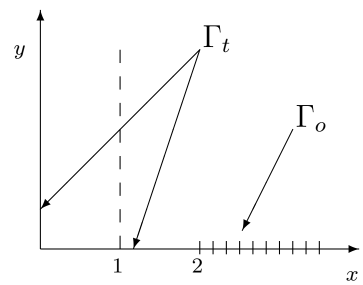
Putting everything together and using that $u(x) = U(|x|)$, we find the explicit form of the solution u, which is

$$u(x) = \begin{cases} -\frac{|x|^2}{2n} - 3\frac{2^{n-1}}{(2^n-4)n|x|^{n-2}} + \frac{2(2^n-1)}{(2^n-4)n} & \text{if } n \neq 2, \\ -\frac{|x|^2}{4} + \frac{3}{4\log(2)}\log(|x|) + \frac{1}{4} & \text{if } n = 2. \end{cases}$$

Finally, to find the points in Ω which have maximum cost, we only have to find the maximum of U in $[1, 2]$. Recall that $U'(r) = -\frac{r}{n} + \frac{c_1}{r^{n-1}}$, hence $U'(r) = 0$ if and only if $r^n = nc_1$. Moreover, it is easy to check that $U'(r) > 0$ if $r < \sqrt[n]{nc_1}$ and $U'(r) < 0$ if $r > \sqrt[n]{nc_1}$. Since $U(1) = U(2) = 0$, this shows that indeed the unique critical point $r = \sqrt[n]{nc_1}$ of U is in $(1, 2)$ and is the unique global maximum of U in

$(0,+\infty)$. In conclusion, the points $x \in \Omega$ which have maximum cost $u(x)$ are the ones with $|x| = \sqrt[n]{nc_1}$, where c_1 is as in (3).

Problema 6.* Considerem el domini $\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$. Suposem que $\Gamma_o = \{(x,0) \in \mathbb{R}^2 : x > 2\}$ és la part oberta de la frontera de Ω i que $\Gamma_t = \{(x,0) \in \mathbb{R}^2 : 0 \leq x \leq 2\} \cup \{(0,y) \in \mathbb{R}^2 : y > 0\}$ és la part tancada de la frontera (veure la figura adjunta). Calculeu la coordenada y tal que la probabilitat de sortir del domini començant camins aleatoris des del punt $(1,y)$ sigui màxima.



Solution.

In order to solve the problem we will use some facts from complex analysis. In particular, we will use that

- the real and imaginary parts of a holomorphic function are harmonic,
- if v is harmonic in $V \subset \mathbb{R}^2$ and $f : U \subset \mathbb{C} \to V$ is holomorphic in U, then $v \circ f$ is harmonic in U.

Both facts can be checked simply by computing the Laplacian and using the Cauchy-Riemann equations that the real and imaginary parts of a holomorphic function satisfy. We leave the details for the reader.

Let u be the solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \Gamma_o, \\ u = 0 & \text{on } \Gamma_t. \end{cases} \tag{4}$$

Given $(x,y) \in \Omega$, the value $u(x,y)$ represents the probability that a random path starting at (x,y) exits Ω through Γ_o. In the exercise, we are asked to find the coordinate $y \geq 0$ for which $u(1,y)$ is maximum. The existence of such a maximum is guaranteed by the fact that the probability $u(1,y)$ is expected to go to zero both when y goes to zero and when y goes to infinity. Indeed, if y is small enough, then the particle is really close to Γ_t, and thus it is expected to hit the boundary in that part with very high probability. Similarly, if y is big enough, the particle is much closer to Γ_t than to Γ_o, and therefore the same happens.

Our strategy to find the point where the maximum is attained will be to transform the region Ω, via a holomorphic mapping f, to a flat region (that is, to "open" Ω) in such a way that Γ_o and Γ_t go to the positive and negative real axes. Then, in this new region, we will find a simple function v inspired by the *angle function* that solves the new problem, and finally we will take $u = v \circ f$. Once u is found, we will maximize $u(1, y)$ for $y \geq 0$.

Consider the holomorphic map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2 - 4$. Then,

$$\begin{cases} f(\Omega) = \{(x, y) \in \mathbb{R}^2 : \ y > 0\}, \\ f(\Gamma_o) = \{(x, 0) \in \mathbb{R}^2 : \ x > 0\}, \\ f(\Gamma_t) = \{(x, 0) \in \mathbb{R}^2 : \ x \leq 0\}. \end{cases}$$

Here, note that the function z^2 is the responsible of opening the quarter of plane to a half-plane (recall that if $z = re^{i\theta}$, then $z^2 = re^{i2\theta}$), while subtracting 4 is needed in order to center the problem at 0.

Hence, if we find the solution v to

$$\begin{cases} \Delta v = 0 & \text{in } \{(x, y) \in \mathbb{R}^2 : \ y > 0\}, \\ v = 1 & \text{on } \{(x, 0) \in \mathbb{R}^2 : \ x > 0\}, \\ v = 0 & \text{on } \{(x, 0) \in \mathbb{R}^2 : \ x \leq 0\}, \end{cases} \tag{5}$$

and we set $u = v \circ f$, then u will be the solution to (4).

The solution to (5) can be easily found once one recalls the *angle function* $\Theta(z)$, usually defined on $\mathbb{C} \setminus \{(x, 0) : x \leq 0\}$ by

$$\Theta(x, y) \equiv \Theta(z) = \theta \quad \text{ if } x + iy = z = re^{i\theta} \text{ with } \theta \in (-\pi, \pi).$$

Note that $\Theta(x, 0) = 0$ for all $x > 0$, and one can obviously extend Θ to $\{(x, 0) : x \leq 0\}$ by setting $\Theta(x, 0) = \pi$ for all $x \leq 0$, being then a continuous function in $\{(x, y) : \ y \geq 0\} \setminus \{(0, 0)\}$. Moreover, it satisfies

$$\Delta\Theta = 0 \quad \text{ in } \{(x, y) : \ y > 0\}$$

because it is the imaginary part of $\log(z) = \log(|z|) + i\Theta(z)$, a holomorphic function in $\mathbb{C} \setminus \{(x, 0) : x \leq 0\}$.

With these considerations at hand, it is easy to check that the solution to (5) is

$$v(x, y) = 1 - \frac{\Theta(x, y)}{\pi},$$

and then, abusing of the complex notation $(x, y) \equiv x + iy$, the solution to (4) is

$$\begin{aligned} u(x, y) &= (v \circ f)(x, y) = v(f(x + iy)) = v((x + iy)^2 - 4) \\ &= v(x^2 - y^2 - 4, 2xy) = 1 - \frac{1}{\pi}\Theta(x^2 - y^2 - 4, 2xy). \end{aligned}$$

Now, we want to maximize $u(1, y)$ for $y \geq 0$. Note that

$$\lim_{y \to 0^+} \Theta(-3 - y^2, 2y) = \Theta(-3, 0) = \pi, \qquad \lim_{y \to +\infty} \Theta\left(-3 - y^2, 2y\right) = \pi,$$

which yields $u(1, 0) = \lim_{y \to 0^+} u(1, y) = \lim_{y \to +\infty} u(1, y) = 0$. By the minimum principle $u \geq 0$ in $\overline{\Omega}$. Hence, the maximum of $u(1, y)$ must be attained at some

$y>0$, being a critical point of $u(1,y)$. Since $-3-y^2<0$ for all $y\in\mathbb{R}$, we have

$$u(1,y) = 1-\frac{1}{\pi}\Theta(-3-y^2,2y) = 1-\frac{1}{\pi}\Big(\arctan\Big(\frac{-2y}{3+y^2}\Big)+\pi\Big)$$
$$= -\frac{1}{\pi}\arctan\Big(\frac{-2y}{3+y^2}\Big),$$

where here $\arctan(\cdot)$ is the usual inverse of the tangent, $\arctan:\mathbb{R}\to(-\pi/2,\pi/2)$. Therefore,

$$\frac{d}{dy}u(1,y) = -\frac{1}{\pi}\frac{d}{dy}\arctan\Big(\frac{-2y}{3+y^2}\Big) = -\frac{2(y^2-3)}{\pi(y^4+10y^2+9)},$$

and the unique critical point of $u(1,y)$ for $y>0$ is $y=\sqrt{3}$, which is an absolute maximum for all $y\geq 0$. Also,

$$u(1,\sqrt{3}) = -\frac{1}{\pi}\arctan\Big(-\frac{\sqrt{3}}{3}\Big) = \frac{1}{6}.$$

Problema 7.* Una imatge en blanc i negre és una funció $u:\Omega\subset\mathbb{R}^2\to[0,255]\subset\mathbb{R}$ a valors reals definida en un domini Ω del pla. El valor $u(x,y)$ representa el nivell de gris de la imatge en el punt (x,y), on 0 representa negre i 255 blanc. Si la imatge és digital (o discreta) i rectangular, llavors el seu domini de definició ve donat per la discretització d'un rectangle:

$$\{(x_i,y_j)=(ih,jh): i=1,\dots,I; j=1,\dots,J\},$$

on $h>0$. El punt (x_i,y_j) és el centre d'un pixel quadrat de mida $h\times h$ (*" in 'pixel', 'pix' is slang for 'picture' and 'el' stands for 'element' "*). Direm que la imatge és de mida $I\times J$.

Transmitint imatges, a vegades es perd un pixel. Si es tracta d'un pixel interior (és a dir $1<i<I$ i $1<j<J$), una manera natural de reassignar-li un valor és per la mitjana

$$u(x_i,y_j) = \frac{1}{4}\left\{u(x_{i+1},y_j)+u(x_{i-1},y_j)+u(x_i,y_{j+1})+u(x_i,y_{j-1})\right\}. \tag{6}$$

Els pixels no interiors els anomenem de vora (són els que toquen a algun dels quatre costats de la imatge rectangular).

Una imatge digital es diu que és *ideal* si satisfà (6) en cadascun dels seus pixels interiors. Equivalentment, $\Delta_h u(x_i,y_j)=0$ en cada pixel interior (x_i,y_j), a on

$$\Delta_h u(x_i,y_j) \tag{7}$$
$$:= \frac{4}{h^2}\left\{\frac{1}{4}\big(u(x_{i+1},y_j)+u(x_{i-1},y_j)+u(x_i,y_{j+1})+u(x_i,y_{j-1})\big)-u(x_i,y_j)\right\},$$

s'anomena el *Laplacià discret* de u.

Demostreu que, coneguts els pixels de vora d'una imatge digital, existeix una única imatge digital *ideal* que els té com valors de vora.

Solution.

First, note that once the boundary pixels of a bounded picture are given, we can find an *ideal* picture with those boundary values by solving a finite linear system with the same number of equations as unknowns. That is, we have as many unknowns (the gray levels of the pixels) as interior pixels, and moreover we also have a linear equation of the form (6) for each of the interior pixels. Thus, the problem can be described as $Az = b$, where A is a square matrix, z denotes the vector of gray level of interior pixels, and b is a vector containing the information associated to the boundary data.

Now, let us address the uniqueness of solution. Note that this is equivalent to show that the determinant of the square matrix A is not zero. This is a nontrivial problem from linear algebra that surely can be solved by manipulating the matrix in a clever way. Nevertheless, we are using a completely different argument. Assume that u_1 and u_2 are two *ideal* pictures with the same boundary values. We want to show that then they must be the same picture, that is, $u_1 = u_2$. By the linearity of the discrete Laplacian Δ_h, we see that $u = u_1 - u_2$ is an *ideal* picture with zero boundary values. Thus, u satisfies (6) at all interior points, and $u(x_i, y_j) = 0$ if (x_i, y_j) is a pixel on the boundary. Our target is to prove that indeed $u \equiv 0$, and this will follow by a discrete version of the strong maximum principle to be proven below. More precisely, we will show that either the maximum and the minimum of an *ideal* picture v are only attained at the boundary of the picture or, otherwise, v is constant. But then, since u vanishes on the boundary, in both cases we get $u \equiv 0$ and, therefore, $u_1 = u_2$.

Proof (of the discrete strong maximum principle)

Assume that v is an *ideal* picture such that

$$v(x_p, y_q) = \sup_{(x_i,y_j) \text{ pixel}} v(x_i, y_j) =: M$$

for some interior pixel (x_p, y_q). Then, v is smaller than or equal to M at all other pixels of the picture –in particular, in the four neighboring pixels of (x_p, x_q). On the other hand, by (6) we know that $v(x_p, y_q)$ is equal to the mean ot the neighboring pixels. Thus, we deduce that the maximum M is not only attained at (x_p, y_q) but also in the four neighboring pixels. Now, we continue repeating this argument for each of the neighboring points (where we already know that v also attains its maximum value) and we finally deduce that $v = M$ everywhere.

Similarly, we can prove that if v is an *ideal* picture such that

$$v(x_p, y_q) = \inf_{(x_i,y_j) \text{ pixel}} v(x_i, y_j) =: m$$

for some interior pixel (x_p, y_q), then $v = m$ everywhere.

In conclusion, either

$$\inf_{(x_i,y_j) \text{ pixel}} v(x_i, y_j) < v(x_p, y_q) < \sup_{(x_i,y_j) \text{ pixel}} v(x_i, y_j)$$

for all interior pixel (x_p, y_q), or v is a constant function. $\square$

Observe that the discrete strong maximum principle that we have proven applies to every connected image Ω, not only to rectangular images.

Finally, let us now show existence of solution. Since we already know uniqueness of solution to the system $Az = b$, we deduce that $\det A \neq 0$, hence A is invertible and, therefore, $z = A^{-1}b$ is the unique solution.

Problema

Considerem l'operador

$$(Av)(x) = \int_0^1 x\, v(y)\, dy, \qquad \text{per } x \in [0,1].$$

(a) Discutiu l'existència i unicitat de solució $u = u(x,t)$ pel problema

$$\begin{cases} u_t = Au, \\ u(x,0) = g(x), \quad x \in (0,1). \end{cases} \tag{1}$$

(b) Podeu trobar la suposada solució de manera explícita?

(c) Digueu el nom dels tres matemàtics les contribucions dels quals permeten demostrar l'existència de solució per

$$\begin{cases} u_t = Au + u^2, \\ u(x,0) = \begin{cases} 0 & \text{si } 0 < x < 1/2, \\ 1 & \text{si } 1/2 < x < 1. \end{cases} \end{cases}$$

Justifiqueu la vostra resposta explicant la demostració d'existència de solució, breument però dient amb precisió els seus passos i punts claus.

Solució

(a) En primer lugar, vamos a ver que el operador $A : L^2(0,1) \to L^2(0,1)$ es lineal y continuo. La linealidad del operador se deduce de forma trivial a partir de la linealidad de la integral. En cuanto a la continuidad, tenemos

$$\begin{aligned} \|Av\|_{L^2}^2 &= \int_0^1 |(Av)(x)|^2\, dx = \int_0^1 x^2 \left(\int_0^1 v(y)\, dy\right)^2 dx \\ &= \left(\int_0^1 x^2\, dx\right)\left(\int_0^1 v(y)\, dy\right)^2 = \frac{1}{3}\left(\int_0^1 v(y)\, dy\right)^2 \\ &\le \frac{1}{3}\int_0^1 |v(y)|^2\, dy = \frac{1}{3}\|v\|_{L^2}^2, \end{aligned} \tag{2}$$

donde hemos usado la desigualdad de Cauchy-Schwarz en la última línea del cálculo anterior. Esta estimación nos dice al mismo tiempo que el operador está bien definido de $L^2(0,1)$ en sí mismo, así como que es acotado en $L^2(0,1)$ ya que $\|Av\|_{L^2} \le C\|v\|_{L^2}$ para toda función $v \in L^2(0,1)$. Debido a la equivalencia entre acotado y continuo para operadores lineales, concluimos que A es continuo.

Al ser A lineal y continuo, en particular es globalmente Lipschitz y, por tanto, sabemos que existe una única $u \in C^0(\mathbb{R}, L^2(0,1))$ solución en sentido integral del problema inicial, es decir, tal que

$$u(x,t) = g(x) + \int_0^t (Au)(x,s)\,ds \quad \text{ para todo } \; t \in \mathbb{R}.$$

Observad que hemos escogido $L^2(0,1)$ como espacio de Banach en el que trabajar teniendo en mente el apartado (c), en el que la condición inicial es una función característica (no es continua). En ese sentido, otra elección también válida hubiera sido $L^p(0,1)$ para cualquier $1 \leq p \leq +\infty$, ya que es un espacio de Banach que contiene la condición inicial del apartado (c). Respecto al argumento de existencia y unicidad, para $1 < p < +\infty$ solo sería necesario reemplazar en (2) el uso de la desigualdad de Cauchy-Schwarz por la de Hölder, y para $p = 1$ o $p = +\infty$ la estimación análoga a (2) es trivial.

Mencionamos también que este apartado se podría haber resuelto de forma similar (como hemos hecho habitualmente en clase) trabajando en el espacio de Banach $C^0([0,1])$ si suponemos $g \in C^0([0,1])$.

(b) Ya que el operador A es lineal y continuo sabemos que la única solución de (1) viene dada por la aplicación exponencial, $u(t) = e^{At}g$. Por tanto vamos a calcular la aplicación exponencial. Recordemos que se define del siguiente modo:

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k\, A^k}{k!},$$

donde $A^k = A \circ \overset{(k)}{\ldots} \circ A$. Es por ello que necesitamos calcular las potencias del operador. Es sencillo demostrar por inducción que, si $k > 0$,

$$(A^k v)(x) = \frac{1}{2^{k-1}}(Av)(x) = \frac{x}{2^{k-1}} \int_0^1 v(y)dy. \tag{3}$$

El caso base $k = 1$ claramente se cumple. Suponemos que (3) es cierto para $k-1$ y vamos a ver que también se cumple para k. Así es,

$$\begin{aligned}(A^k v)(x) = A(A^{k-1}v)(x) &= \int_0^1 x\,(A^{k-1}v)(y)\,dy = x\int_0^1 \frac{y}{2^{k-2}}\left(\int_0^1 v(z)\,dz\right)dy \\ &= \frac{x}{2^{k-2}}\left(\int_0^1 v(z)\,dz\right)\left(\int_0^1 y\,dy\right) = \frac{x}{2^{k-1}}\int_0^1 v(z)\,dz.\end{aligned}$$

Por tanto,

$$\begin{aligned} u(x,t) = e^{At} g(x) = \sum_{k=0}^{\infty} \frac{t^k\,(A^k g)(x)}{k!} &= g(x) + \sum_{k=1}^{\infty} \left(x \frac{t^k}{2^{k-1} k!} \int_0^1 g(y)\,dy \right) \\ &= g(x) + 2x \left(\sum_{k=1}^{\infty} \frac{(t/2)^k}{k!} \right) \int_0^1 g(y)\,dy \\ &= g(x) + 2x(e^{t/2} - 1) \int_0^1 g(y)\,dy. \end{aligned}$$

A continuación se explica una construcción alternativa de la solución $u(x,t)$ sin usar la aplicación exponencial. Si definimos $w(t) = \int_0^1 u(x,t)\,dx$, es facil ver que ésta satisface la EDO

$$\left\{ \begin{array}{l} w' = w/2, \\ w(0) = \int_0^1 g(x)\,dx. \end{array} \right.$$

Por tanto,

$$\int_0^1 u(x,t)\,dx = w(t) = e^{t/2} \int_0^1 g(x)\,dx.$$

De este modo, podemos sustituir esta expresión en la ecuación original (1) (observad que $A(u(\cdot,t))(x) = x \int_0^1 u(y,t)\,dy$), reduciéndola a la ecuación

$$\left\{ \begin{array}{ll} u_t(x,t) = x e^{t/2} \int_0^1 g(y)\,dy, & \\ u(x,0) = g(x), & x \in (0,1), \end{array} \right.$$

que es una EDO lineal para cada x fijada, y cuya solución se calcula fácilmente y coincide con la encontrada anteriormente.

(c) Los matemáticos son Lebesgue (por construir y demostrar la completitud de los espacios L^p -que por tanto son espacios de Banach- usados en el apartado (a) y que también usaremos a continuación), Picard (por el método iterativo que permite obtener la solución de una EDO) y Banach (por el teorema del punto fijo que aplicaremos).

Para demostrar la existencia de solución trabajaremos en el espacio de Banach

$$F = C^0([0,T], L^2(0,1))$$

para un $T > 0$ fijado y aplicaremos el método de Picard. En F estamos considerando la norma

$$\|u\|_F := \sup_{t \in [0,T]} \|u(\cdot,t)\|_{L^2(0,1)}.$$

Igual que en el apartado (a), otro espacio válido para los argumentos que vienen a continuación es $C^0([0,T], L^p(0,1))$ para cualquier $1 \le p \le +\infty$.

Por tanto, si tomamos

$$g(x) = \left\{ \begin{array}{ll} 0 & \text{si } 0 < x < 1/2, \\ 1 & \text{si } 1/2 < x < 1. \end{array} \right. ,$$

queremos encontrar $u \in F$ tal que

$$u(x,t) = g(x) + \int_0^t (Au(x,s) + u^2(x,s))\, ds =: Nu(x,t),$$

o lo que es lo mismo, un punto fijo para la aplicación no lineal N. Para ello consideramos la bola cerrada de F centrada en 0 y de radio R, y queremos ver que

$$N : \overline{B_R(0)} \subset F \to \overline{B_R(0)}$$

y es una contracción. Dejamos al lector comprobar que $N(F) \subset F$.

Por un lado, dado $u \in \overline{B_R(0)}$ (es decir, $\|u(s)\|_{L^2} \le R$ para todo $s \in [0,T]$) tenemos

$$\|(Nu)(\cdot,t)\|_{L^2} \le \|g\|_{L^2} + \int_0^T (\|(Au)(\cdot,s)\|_{L^2} + \|u(\cdot,s)\|_{L^2}^2)\, ds \le \|g\|_{L^2} + T\,(1+R)\,R.$$

Aquí hemos usado que la norma del operador A calculada en el apartado (a) es menor que 1 (véase (2)). Si elegimos

$$R = 2\|g\|_{L^2},$$

entonces $\|(Nu)(\cdot,t)\|_{L^2} \le R$ para todo $t \in [0,T]$ si

$$T < \frac{1}{2(1+2\|g\|_{L^2})}. \tag{4}$$

Tomando el supremo en $t \in [0,T]$, deducimos que $\|Nu\|_F \le R$ si T cumple (4). Esto significa que $N : \overline{B_R(0)} \to \overline{B_R(0)}$, como queríamos. Observad que si definimos $\tilde{g}(x,t) = g(x)$ para todo x y t, entonces $\tilde{g} \in \overline{B_R(0)}$.

Veamos ahora que N es una contracción sobre $\overline{B_R(0)}$. Esto es, dados $u, \tilde{u} \in \overline{B_R(0)}$ y $t \in [0,T]$, tenemos

$$\begin{aligned}
\|(Nu)(\cdot,t) - (N\tilde{u})(\cdot,t)\|_{L^2} &= \left\| \int_0^t \Big(A(u-\tilde{u})(\cdot,s) + u^2(\cdot,s) - \tilde{u}^2(\cdot,s) \Big) ds \right\|_{L^2} \\
&\le \int_0^T \Big(\|A(u-\tilde{u})(\cdot,s)\|_{L^2} + \|u^2(\cdot,s) - \tilde{u}^2(\cdot,s)\|_{L^2} \Big) ds \\
&\le \int_0^T \Big(\|(u-\tilde{u})(\cdot,s)\|_{L^2} + \|(u-\tilde{u})(\cdot,s)\|_{L^2}\, \|(u+\tilde{u})(\cdot,s)\|_{L^2} \Big) ds \\
&\le T(1+2R)\|u-\tilde{u}\|_F.
\end{aligned}$$

Tomando el supremo en $t \in [0,T]$, deducimos que $\|Nu - N\tilde{u}\|_F \le T(1+2R)\|u-\tilde{u}\|_F$ y que, por tanto, N es una contracción si $T(1+2R) < 1$. Recordando la elección de R hecha anteriormente, N es una contracción si

$$T < \frac{1}{1+4\|g\|_{L^2}}. \tag{5}$$

En conclusión, teniendo en cuenta las cotas en T encontradas en (4) y (5), tendríamos que $N : \overline{B_R(0)} \subset F \to \overline{B_R(0)}$ es una contracción si

$$T < \frac{1}{2 + 4\|g\|_{L^2}}.$$

Aplicando el teorema del punto fijo de Banach, obtenemos que la aplicación N tiene un único punto fijo en $\overline{B_R(0)}$. Por lo tanto, hemos demostrado existencia y unicidad de solución para el problema no lineal para todo tiempo en $[0, T]$.

Finalmente, mencionamos que de forma análoga uno puede mostrar existencia y unicidad de solución para el problema no lineal para todo tiempo en $[-T, T]$ si $T > 0$ es suficientemente pequeño. Para ello, basta empezar considerando el espacio de Banach $F = C^0([-T, T], L^2(0, 1))$ y repetir las estimaciones anteriores para $t \in [-T, T]$.

Equacions en Derivades Parcials FME-UPC

14 de maig de 2020 Test-2 per a l'avaluació contínua *Temps: 1 hora. 11:15-12:15*

Problema

Donada una funció g, considerem el problema

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u_x(1,t) = 1, & t > 0, \\ u(x,0) = g(x), & 0 < x < 1. \end{cases}$$

(a) (4 *punts*) Calculeu la solució expressant-la com una sèrie de Fourier. Digues, amb paraules, quina propietat tenen les funcions que has fet servir en la sèrie de Fourier i que fa que aquestes siguin les funcions apropiades per aquest problema.

(b) (3 *punts*) La solució, té límit quan $t \to +\infty$ i, en tal cas, quan val el límit? Quin problema resol el límit?

(c) (3 *punts*) Doneu condicions suficients (sobre la funció g) per tal que la solució sigui contínua a $[0, \pi] \times [0, +\infty)$, o sigui, fins a $t = 0$ i fins la vora de $(0, \pi)$. Són necessàries aquestes condicions? En la mesura que sigui possible, justifiqueu i demostreu les vostres respostes.

Solució

(a) In order to have homogeneous boundary conditions at $x = 0$ and $x = 1$ we define the auxiliary function $v(x,t) = u(x,t) - x$. Then, it is easy to check that v satisfies the problem

$$\begin{cases} v_t - v_{xx} = 0, & 0 < x < 1, \quad t > 0, \\ v(0,t) = 0, \quad v_x(1,t) = 0, & t > 0, \\ v(x,0) = \tilde{g}(x) := g(x) - x, & 0 < x < 1. \end{cases}$$

Using the method of separation of variables and the notation from problems in List 5 for solutions of the form $v(x,t) = X(x)T(t)$ we obtain that

$$X(x) = B \sin(\sqrt{\lambda} x) + C \cos(\sqrt{\lambda} x), \qquad B, C \in \mathbb{R}, \ \lambda \in [0, +\infty).$$

Now, imposing that $X(0) = 0$ (which corresponds to $u(x,0) = 0$), we get that $C = 0$, and therefore

$$X(x) = B \sin(\sqrt{\lambda} x), \qquad B \in \mathbb{R}.$$

Using then that $X'(1) = 0$ (which corresponds to $v_x(1,t) = 0$), it must hold $\cos(\sqrt{\lambda}) = 0$ (or $B = 0$, which would yield the trivial solution $X \equiv 0$), thus

$$\sqrt{\lambda} = \frac{2k+1}{2}\pi, \quad k = 0, 1, 2, \ldots$$

Taking into account these admissible values for λ and looking at the ODE on T, it must be of the form

$$T(t) = Ae^{-\left(\frac{2k+1}{2}\pi\right)^2 t}, \qquad A \in \mathbb{R}, \quad k = 1, 2, \ldots$$

Hence, we conclude that a formal solution, satisfying the boundary conditions at $x = 0$ and $x = 1$, and satisfying the equation $v_t - v_{xx} = 0$ is given by

$$v(x,t) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{2k+1}{2}\pi x\right) e^{-\left(\frac{2k+1}{2}\pi\right)^2 t}.$$

It remains to choose the coefficients a_k in agreement with the initial condition, that is, such that

$$v(x,0) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{2k+1}{2}\pi x\right) = \tilde{g}(x). \tag{1}$$

Let us check that if $g \in L^2((0,1))$ we can write $\tilde{g}$ as a series of the form given in (1), and then obtain the corresponding coefficients. If we extend $\tilde{g}$ in an even way with respect to $x = 1$ (motivated by the Neumann condition at this point) and we use the Fourier expansion in sinus series with $L = 2$, we obtain that $\tilde{g}$ can be written in the $L^2((0,2))$ sense as

$$\tilde{g}(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m}{2}\pi x\right)$$

with

$$b_m := \frac{1}{\pi}\int_0^{2\pi} \tilde{g}(x) \sin\left(\frac{m}{2}\pi x\right) dx, \qquad m \geq 1.$$

Now, if we use that $\tilde{g}$ is even with respect to $x = 1$ we obtain that $b_{2k} = 0$ for all $k \in \mathbb{N}$, while

$$b_{2k+1} = \int_0^2 \tilde{g}(x) \sin\left(\frac{2k+1}{2}\pi x\right) dx = 2\int_0^1 \tilde{g}(x) \sin\left(\frac{2k+1}{2}\pi x\right) dx. \tag{2}$$

Therefore, we conclude that $a_k = b_{2k+1}$ and

$$\tilde{g}(x) = g(x) - x = \sum_{k=0}^{\infty} b_{2k+1} \sin\left(\frac{2k+1}{2}\pi x\right).$$

Finally, summarizing and taking into account the definition of v in terms of u we obtain

$$u(x,t) = x + \sum_{k=0}^{\infty} a_k \sin\left(\frac{2k+1}{2}\pi x\right) e^{-\left(\frac{2k+1}{2}\pi\right)^2 t}$$

with
$$a_k = 2\int_0^1 (g(x) - x)\sin\left(\frac{2k+1}{2}\pi x\right)dx.$$
Let us remark that the functions we are using in the Fourier series
$$\sin\left(\frac{2k+1}{2}\pi x\right)$$
are such that they satisfy the homogeneous boundary conditions of the problem. That is, they vanish at $x = 0$ and their derivatives at $x = 1$.

(b) In this part we are proving that $u(x,t) \to x$ uniformly when t goes to infinity. That is, note that, for all $t \geq 1$,
$$\begin{aligned}|u(x,t) - x| &\leq \sum_{k=0}^{\infty} |a_k| e^{-\left(\frac{2k+1}{2}\pi\right)^2 t} \\ &= e^{-\frac{\pi^2}{4}t}\sum_{k=0}^{\infty} |a_k| e^{-\left\{\left(\frac{2k+1}{2}\pi\right)^2 - \frac{\pi^2}{4}\right\}t} \\ &\leq e^{-\frac{\pi^2}{4}t}\sum_{k=0}^{\infty} |a_k| e^{-\left\{\left(\frac{2k+1}{2}\pi\right)^2 - \frac{\pi^2}{4}\right\}}.\end{aligned}$$
Thus, to check that $\|u(\cdot,t) - Id\|_{L^\infty((0,\pi))} \to 0$ when $t \to +\infty$, it suffices to show that
$$\sum_{k=0}^{\infty} |a_k| e^{-\left\{\left(\frac{2k+1}{2}\pi\right)^2 - \frac{\pi^2}{4}\right\}} < +\infty.$$
Given $g \in L^2((0,1))$, we have $\sum_{k=0}^{\infty} |a_k|^2 < +\infty$ by Parseval identity. Therefore, if we simply use Cauchy-Schwarz we obtain
$$\sum_{k=0}^{\infty} |a_k| e^{-\left\{\left(\frac{2k+1}{2}\pi\right)^2 - \frac{\pi^2}{4}\right\}} \leq \left(\sum_{k=0}^{\infty} |a_k|^2\right)^{1/2}\left(\sum_{k=0}^{\infty} e^{-2\left\{\left(\frac{2k+1}{2}\pi\right)^2 - \frac{\pi^2}{4}\right\}}\right)^{1/2} < +\infty.$$
Note that, as it is natural when it exists, the limit function x satisfies the stationary problem
$$\begin{cases} -u^s_{xx} = 0, \quad 0 < x < 1, \\ u^s(0) = 0, \\ u^s_x(1) = 1. \end{cases}$$

(c) We are showing that u is continuous in $[0,\pi] \times [0,+\infty)$ if $g \in C^1([0,1])$ and $g(0) = 0$. First, note that in fact proving the continuity of u is the same as proving the continuity of v. Thus, we are showing the last one. We proceed as in Problem 1 from List 5. Given $N \in \mathbb{N}$ we define
$$v_N(x,t) = \sum_{k=0}^{N} a_k \sin\left(\frac{2k+1}{2}\pi x\right) e^{-\left(\frac{2k+1}{2}\pi\right)^2 t}.$$

Clearly, $v_N \in C^0([0,\pi] \times [0,+\infty)$ from being a finite linear combination of continuous functions. Thus, in order to obtain the continuity of u it is enough to show that, for every $T > 0$, v_N converges uniformly to v in $[0,\pi] \times [0,T]$ when $N \to +\infty$.

Since $\left| \sin\left(\frac{2k+1}{2}\pi x\right) e^{-\left(\frac{2k+1}{2}\pi\right)^2 t} \right| \leq 1$ for all $x \in [0,\pi]$ and all $t \geq 0$, we can easily estimate

$$|v_N(x,t) - v(x,t)| \leq \sum_{k=N+1}^{\infty} |a_k|. \tag{3}$$

Thus, the uniform convergence of v_N to v will follow if we show that $\sum_{k=0}^{\infty} |a_k|$ is convergent. Using that $g \in C^1([0,\pi])$, $g(0) = 0$ and integration by parts, for $k \geq 0$ we have

$$a_k = \frac{4}{(2k+1)\pi} \int_0^1 (g'(x) - 1) \cos\left(\frac{2k+1}{2}\pi x\right) dx.$$

This means that $(2k+1)\pi a_k/2$ is the Fourier coefficient of $g'(x) - 1$ in cosinus series. Since $g' \in C^0([0,1]) \subset L^2((0,1))$, from Parseval identity applied to $g'(x) - 1$ we deduce that

$$\sum_{k=1}^{\infty} (2k+1)^2 a_k^2 < \infty.$$

Therefore, using Cauchy-Schwarz inequality we get

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} (2k+1)|a_k| \frac{1}{2k+1} \leq \left(\sum_{k=1}^{\infty} (2k+1)^2 a_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}\right)^{1/2} < +\infty,$$

which shows that the right hand side of (3) is as small as we want if we take N big enough.

We have seen that the condition $g(0) = 0$ is sufficient to get that $u(\cdot, t)$ converges uniformly to g in $[0,\pi]$ when $t \to 0$. It is also necessary because $u(0,t) = 0$ for all $t > 0$ and, if $u(0,t)$ converges to $g(0)$, we must have $g(0) = 0$.

Regarding the regularity of g we know that the continuity is a necessary condition. Otherwise, it is clear that u cannot be continuous up to the boundary. On the other hand, concerning the differentiability of g we are only using that $\sum_{k=0}^{\infty} |a_k|$ is convergent, which is a weaker condition.

Equacions en Derivades Parcials

FME-UPC

3 de juny de 2021 Examen Final *Temps: 9:00–12:00*

Autoria: Xavier Cabré, Juan-Carlos Felipe-Navarro, Albert Mas

Donat $c \in \mathbb{R}$, considereu el problema

$$\begin{cases} u_t = e^{-x}(e^x u_x)_x + c, & x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in (0,\pi). \end{cases} \tag{1}$$

Problema 1. En tot aquest problema considereu (1) amb $c = 1$. Cada apartat es pot resoldre independentment dels altres.

(a) (1.5 *punts*) Doneu la interpretació probabilística del problema estacionari associat a (1) (a nivell discret) i deduïu (sense demostrar-ho rigorosament) si el màxim de la solució u per temps prou grans se situarà a l'esquerra o a la dreta de $x = \pi/2$.

(b) (1.5 *punts*) Calculeu el valor exacte del límit quan $t \to +\infty$ del punt on s'assoleix el màxim de $u(\cdot, t)$.

(c) (1 *punt*) Demostreu la propietat essencial de la qual resulta que l'operador estacionari associat a (1) diagonalitza en una base de funcions pròpies. Demostreu també que tots els seus valors propis són reals.

Problema 2. (3 *punts*) Trobeu la solució de (1) per $c = 0$ i $g(x) = e^{-x/2}$. [Recordeu que les funcions exponencials poden resoldre les EDOs lineals homogènies amb coeficients constants.]

Problema 3. Donat $a \in \mathbb{R}$, considereu el problema

$$\begin{cases} u_t = u_{xx} + a|u|^{3/2}, & x \in (0,\pi),\ t \in I, \\ u(0,t) = u(\pi,t) = 0, & t \in I, \\ u(x,0) = g(x), & x \in (0,\pi). \end{cases} \tag{2}$$

Part de l'apartat (ii) es pot resoldre independentment de (i).

(i) (1.2 *punts*) Recordem que

$$(S_t h)(x) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} h(y)\, dy, \quad t > 0,$$

és el semigrup associat a l'equació de difusió $v_t - v_{xx} = 0$ a tota la recta real en l'espai de funcions contínues i fitades a $\mathbb{R}$. Useu aquest fet per resoldre (2) quan $a = 0$ i $g \in C_0([0,\pi]) := \{g \in C([0,\pi]) \, : \, g(0) = g(\pi) = 0\}$ de manera alternativa al mètode de sèries de Fourier.

(ii) (1.8 *punts*) Amb $a = 1$, discutiu la possibilitat de donar un resultat d'existència i unicitat de solució de (2) per certs intervals I depenent de l'espai de Banach $L^2(0,\pi)$ o $C_0([0,\pi])$ al que pertanyi la funció g. Doneu els principals detalls de la demostració en els casos afirmatius.

Equacions en Derivades Parcials FME-UPC

2 de juliol de 2021 Examen Extraordinari *Temps: 09:00–11:30*

Autoria: Xavier Cabré, Juan-Carlos Felipe-Navarro, Albert Mas

Problema 1. (7 *punts*) Considereu el quadrat $\Omega = (-1,1) \times (0,2) \subset \mathbb{R}^2$ i la solució $u = u(x,y)$ de

$$\begin{cases} \Delta u = 0, & (x,y) \in \Omega, \\ u = 0, & (x,y) \in \partial\Omega,\ y < 2, \\ u(x,2) = g(x^2), & x \in [-1,1], \end{cases} \tag{1}$$

on $g \in C^0([0,1])$ és una funció donada amb $g(1) = 0$. Noteu que a (1) escrivim $g(x^2)$, no pas $g(x)$.

[*Cada apartat es pot resoldre independentment dels altres.*]

(*a*) (0,4 *punts*) Doneu la interpretació probabilística del problema (1).

(*b*) (0,5 *punts*) On se situa el punt de màxim de u? Per què? Juga $g(1) = 0$ un paper aquí?

(*c*) (1 *punt*) Quant val $u_x(0, \frac{1}{2})$? Demostreu rigorosament la vostra resposta.

(*d*) (2 *punts*) Calculeu la solució de (1) quan $g(x^2) = 1 - \sqrt{x^2} = 1 - |x|$.

(*e*) (1,3 *punts*) Tornant a una funció $g \in C^0([0,1])$ general amb $g(1) = 0$, però que a més satisfà $g \leq 1$, expliqueu com trobarieu una fita superior a per $u(0, \frac{1}{2})$, amb $0 < a < 1$.

(*f*) (1 *punt*) Tornant a una funció $g \in C^0([0,1])$ general amb $g(1) = 0$, demostreu o doneu un contraexemple de la següent afirmació:

$$\tfrac{1}{4} \min_{[0,1]} g \leq u(0,1) \leq \tfrac{1}{4} \max_{[0,1]} g.$$

(*g*) (0,8 *punts*) Expliqueu breument com trobarieu la solució del problema $\Delta v = 0$ a Ω i $v = h$ a $\partial\Omega$, on $h \in L^2(\partial\Omega)$ és una funció donada.

Problema 2. (3 *punts*) Donat $a \in \mathbb{R}$, considereu el problema

$$\begin{cases} u_{tt} = u_{xx} + f(x) + a\,\chi_{(0,1)}(x)\,u^2, & x > 0,\ t \in I, \\ u_x(0,t) = 0, & t \in I, \\ u(x,0) = g(x), & x > 0, \\ u_t(x,0) = 0, & x > 0, \end{cases} \tag{2}$$

on $\chi_{(0,1)}$ és la funció característica de l'interval $(0,1)$ i $I \subset \mathbb{R}$ és un interval contenint 0.

(*i*) (0,6 *punts*) Prenent $a = 0$ a (2), trobeu les condicions necessàries en f i g per tal que $u \in C^2([0,+\infty) \times \mathbb{R})$.

(*ii*) (0,4 *punts*) Prenent $a = 0$ a (2), calculeu la solució (generalitzada) u assumint $f, g \in C^0([0,+\infty))$.

(*iii*) (2 *punts*) Prenent $a = 1$ en (2), discutiu la possibilitat de donar un resultat d'existència i unicitat de solució de (2) (en algun sentit) si $f, g \in L^2((0,+\infty))$ i f té suport a $(0,1)$. En cas afirmatiu dieu quina és l'aplicació per la que trobareu un punt fix en una certa bola tancada, l'espai de Banach on viurà aquesta bola (i per què) i comproveu que l'aplicació envia l'espai de Banach en si mateix. [No cal discutir que l'aplicació és contractiva.]