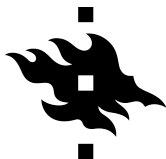


How to know we have the minimizer?

From Bernoulli to Weierstrass

Juan Carlos Felipe-Navarro



**HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI**

4th BYMAT Conference

November 9-11, 2022

Introduction

Motivation

Minimization/maximization has always been a main interest for all civilizations

Motivation

Minimization/maximization has always been a main interest for all civilizations

Given $\mathcal{E} : \mathcal{A} \rightarrow \mathbb{R}$ we want to find $a_0 \in \mathcal{A}$ such that

$$\mathcal{E}(a_0) \leq \mathcal{E}(a) \quad \text{for all } a \in \mathcal{A}.$$

Minimization/maximization has always been a main interest for all civilizations

Given $\mathcal{E} : \mathcal{A} \rightarrow \mathbb{R}$ we want to find $a_0 \in \mathcal{A}$ such that

$$\mathcal{E}(a_0) \leq \mathcal{E}(a) \quad \text{for all } a \in \mathcal{A}.$$

- Sometimes the problem can be modeled as optimization of a function in **finite variables**, i.e.,

Minimization/maximization has always been a main interest for all civilizations

Given $\mathcal{E} : \mathcal{A} \rightarrow \mathbb{R}$ we want to find $a_0 \in \mathcal{A}$ such that

$$\mathcal{E}(a_0) \leq \mathcal{E}(a) \quad \text{for all } a \in \mathcal{A}.$$

- Sometimes the problem can be modeled as optimization of a function in finite variables, i.e.,

$$\mathcal{A} = \Omega \subset \mathbb{R}^n$$

Minimization/maximization has always been a main interest for all civilizations

Given $\mathcal{E} : \mathcal{A} \rightarrow \mathbb{R}$ we want to find $a_0 \in \mathcal{A}$ such that

$$\mathcal{E}(a_0) \leq \mathcal{E}(a) \quad \text{for all } a \in \mathcal{A}.$$

- Sometimes the problem can be modeled as optimization of a function in finite variables, i.e.,

$$\mathcal{A} = \Omega \subset \mathbb{R}^n$$

- In other situations the admissible set is a collection of **paths**, **trajectories**, **regions**...

Minimization/maximization has always been a main interest for all civilizations

Given $\mathcal{E} : \mathcal{A} \rightarrow \mathbb{R}$ we want to find $a_0 \in \mathcal{A}$ such that

$$\mathcal{E}(a_0) \leq \mathcal{E}(a) \quad \text{for all } a \in \mathcal{A}.$$

- Sometimes the problem can be modeled as optimization of a function in finite variables, i.e.,

$$\mathcal{A} = \Omega \subset \mathbb{R}^n$$

- In other situations the admissible set is a collection of paths, trajectories, regions...

$\mathcal{A} =$ “set of real-valued functions”

Motivation

Minimization/maximization has always been a main interest for all civilizations

Given $\mathcal{E} : \mathcal{A} \rightarrow \mathbb{R}$ we want to find $a_0 \in \mathcal{A}$ such that

$$\mathcal{E}(a_0) \leq \mathcal{E}(a) \quad \text{for all } a \in \mathcal{A}.$$

- Sometimes the problem can be modeled as optimization of a function in finite variables, i.e.,

$$\mathcal{A} = \Omega \subset \mathbb{R}^n$$

- In other situations the admissible set is a collection of paths, trajectories, regions...

$$\mathcal{A} = \text{“set of real-valued functions”}$$

Calculus of Variations treats this scenario

Examples

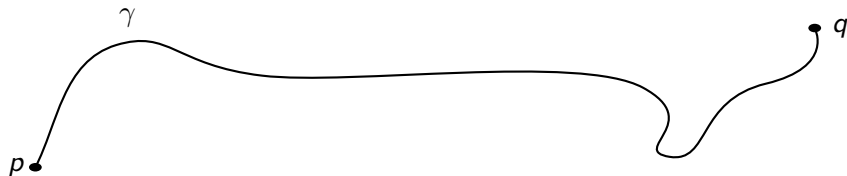
- 1 Shortest path between two points

p •

• q

Examples

1 Shortest path between two points

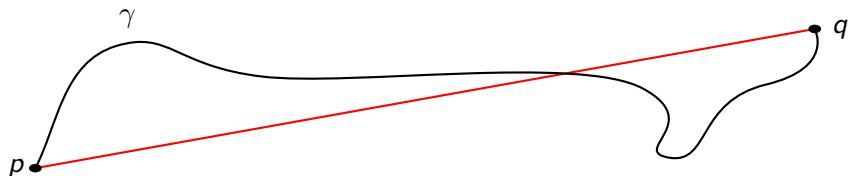


$$\mathcal{A} = \{ \gamma : [0, 1] \rightarrow \mathbb{R}^n \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q \}$$

$$\mathcal{E}(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt$$

Examples

1 Shortest path between two points

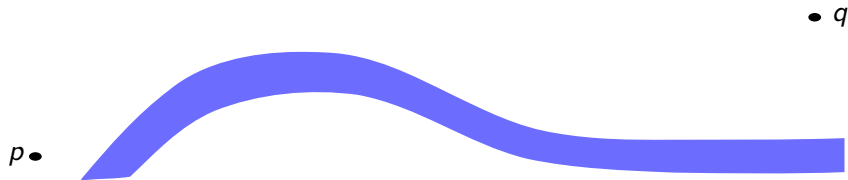


$$\mathcal{A} = \{ \gamma : [0, 1] \rightarrow \mathbb{R}^n \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q \}$$

$$\mathcal{E}(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt$$

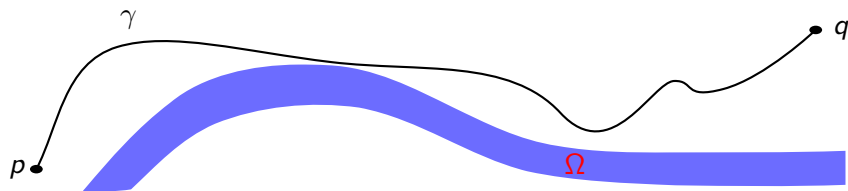
Examples

- 1 Shortest path between two points



Examples

① Shortest path between two points

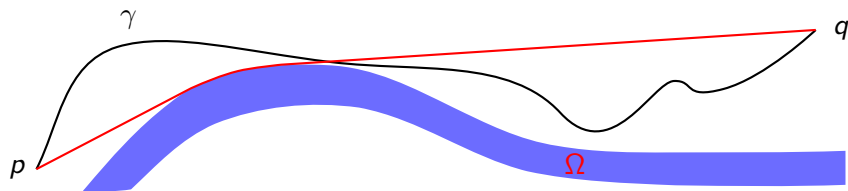


$$\mathcal{A} = \{ \gamma : [0, 1] \rightarrow \mathbb{R}^n \setminus \Omega \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q \}$$

$$\mathcal{E}(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt$$

Examples

1 Shortest path between two points

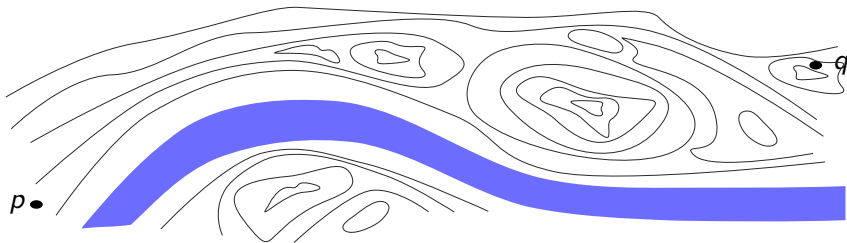


$$\mathcal{A} = \{ \gamma : [0, 1] \rightarrow \mathbb{R}^n \setminus \Omega \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q \}$$

$$\mathcal{E}(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt$$

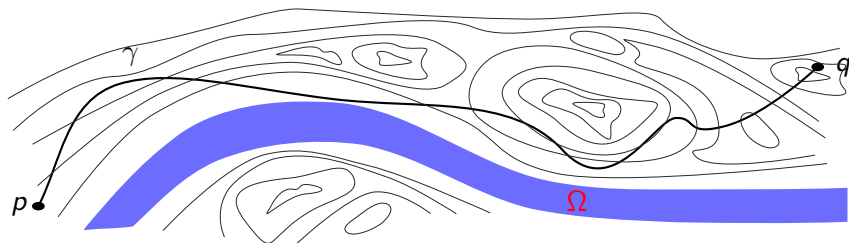
Examples

① Shortest path between two points



Examples

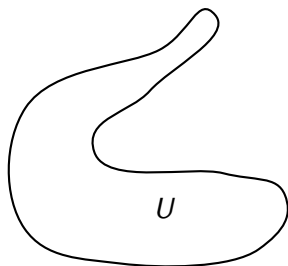
1 Shortest path between two points



$$\mathcal{A} = \{ \gamma : [0, 1] \rightarrow \mathbb{R}^n \setminus \Omega \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q \}$$

$$\mathcal{E}(\gamma) = \int_0^1 c(\gamma(t)) |\dot{\gamma}(t)| dt$$

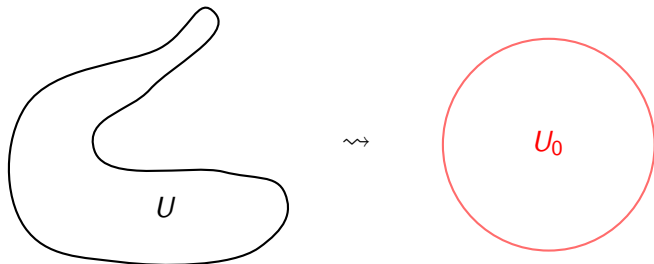
2 Isoperimetric problem



$$\mathcal{A} = \{U \subset \mathbb{R}^n \text{ such that } |U| = 1\}$$

$$\mathcal{E}(U) = |\partial U|$$

2 Isoperimetric problem

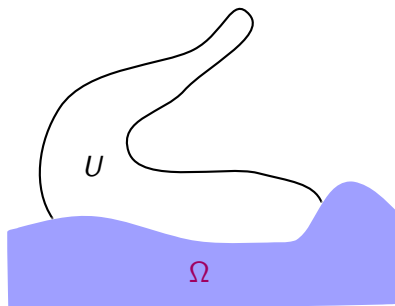


$$\mathcal{A} = \{U \subset \mathbb{R}^n \text{ such that } |U| = 1\}$$

$$\mathcal{E}(U) = |\partial U|$$

Examples

② Isoperimetric problem



$$\mathcal{A} = \{U \subset \mathbb{R}^n \setminus \Omega \text{ such that } |U| = 1\}$$

$$\mathcal{E}(U) = |\partial U \setminus \bar{\Omega}|$$

Examples

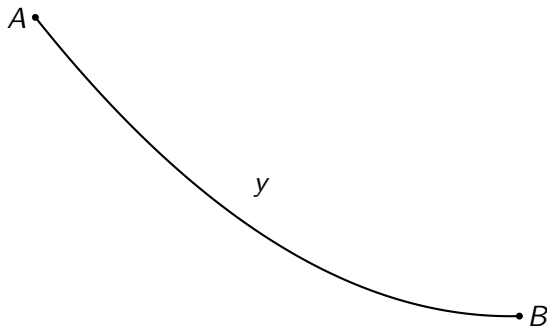
③ Brachistochrone

$A \bullet$

$\bullet B$

Examples

③ Brachistochrone

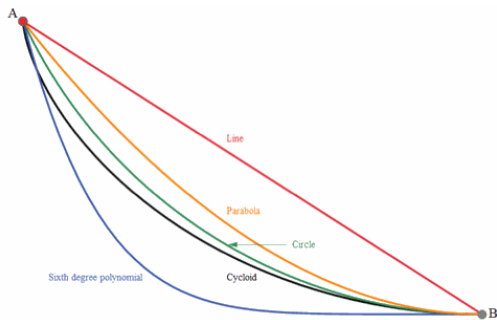


$\mathcal{A} = \{y : [0, 1] \rightarrow \mathbb{R} \text{ such that } y(0) = a \text{ and } y(1) = b\}$

$$\mathcal{E}(y) = \int_0^1 \sqrt{\frac{1 + (y')^2}{y}} dx$$

Examples

③ Brachistochrone



$$\mathcal{A} = \{y : [0, 1] \rightarrow \mathbb{R} \text{ such that } y(0) = a \text{ and } y(1) = b\}$$

$$\mathcal{E}(y) = \int_0^1 \sqrt{\frac{1 + (y')^2}{y}} dx$$

Examples

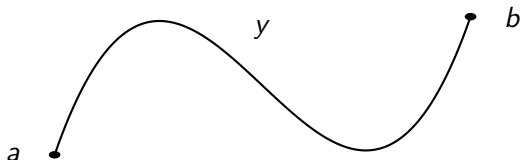
④ Catenary problem

• b

a •

Examples

4 Catenary problem

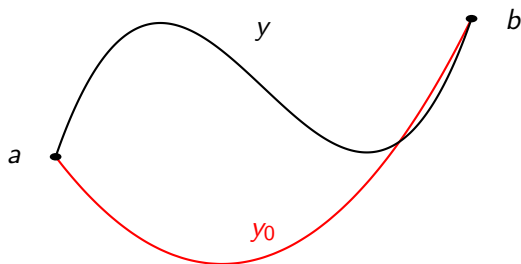


$$\mathcal{A} = \left\{ y : [0, 1] \rightarrow \mathbb{R} \text{ such that } y(0) = a, \ y(1) = b \right. \\ \left. \text{and } \int_0^1 \sqrt{1 + (y')^2} = L \right\}$$

$$\mathcal{E}(y) = \int_0^1 y \sqrt{1 + (y')^2} \, dx$$

Examples

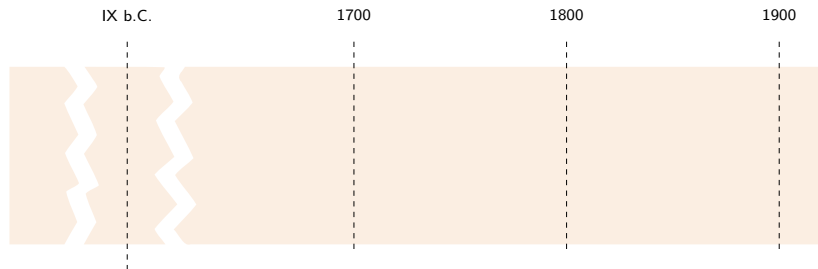
④ Catenary problem



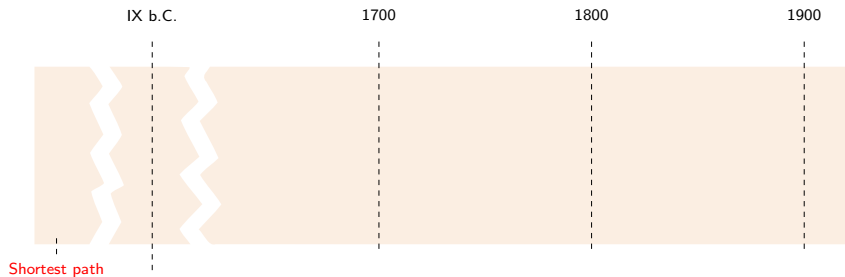
$$\mathcal{A} = \left\{ y : [0, 1] \rightarrow \mathbb{R} \text{ such that } y(0) = a, \ y(1) = b \right. \\ \left. \text{and } \int_0^1 \sqrt{1 + (y')^2} = L \right\}$$

$$\mathcal{E}(y) = \int_0^1 y \sqrt{1 + (y')^2} dx$$

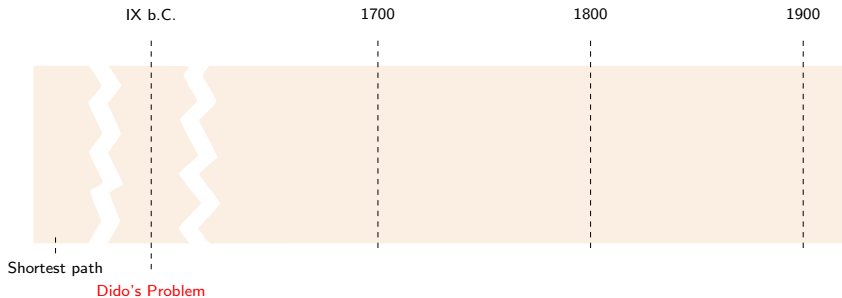
History



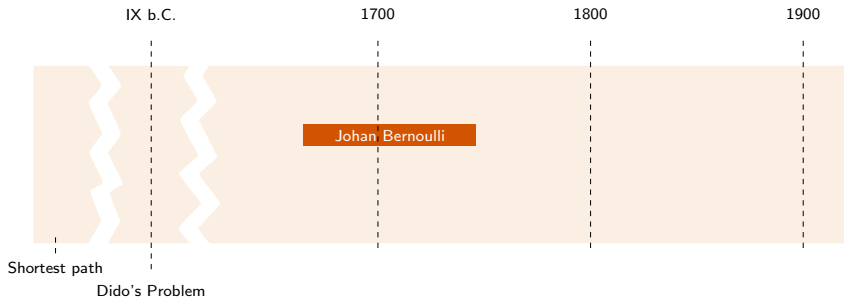
History



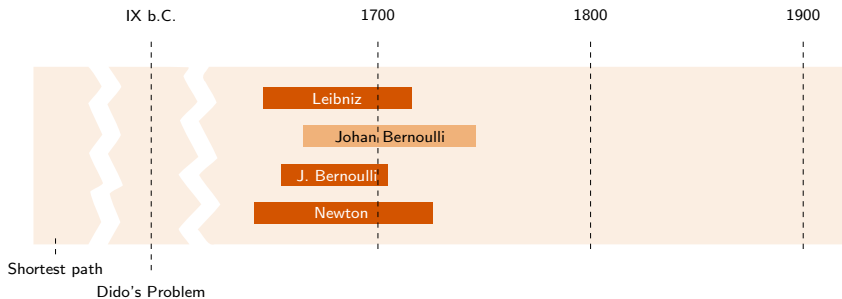
History



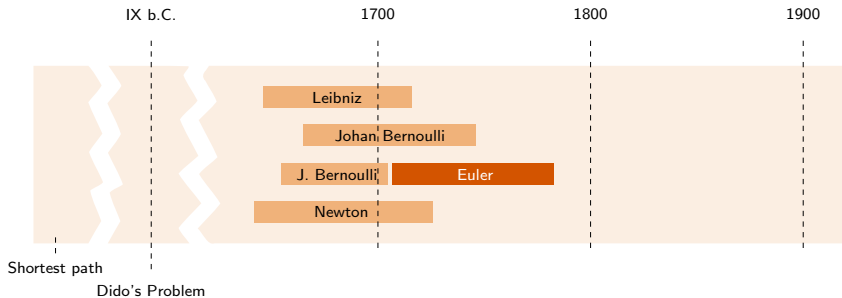
History



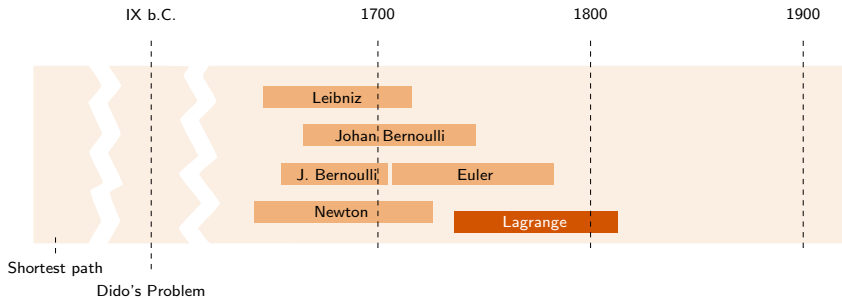
History



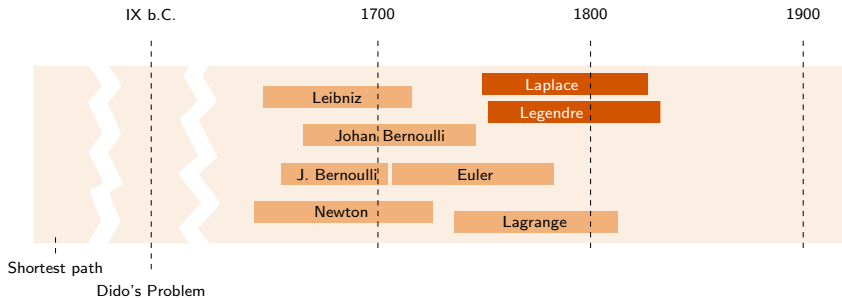
History



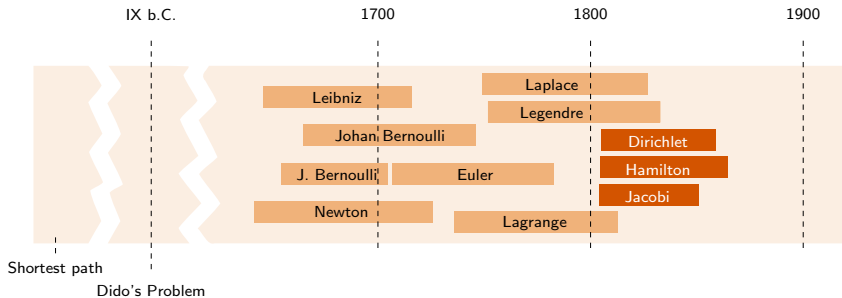
History



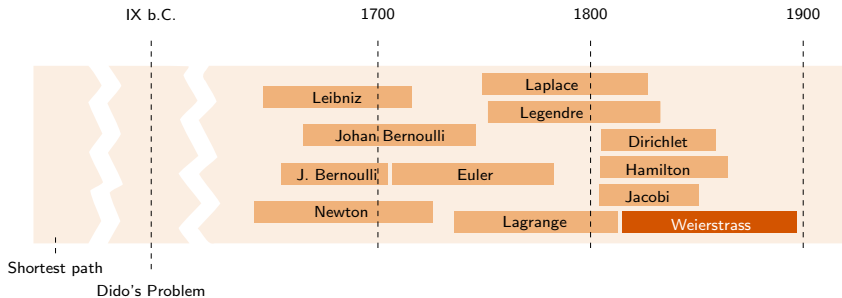
History



History



History



Classical Theory of Calculus of Variations

Setting

Given a **bounded domain** $\Omega \subset \mathbb{R}^n$ and a function $g : \partial\Omega \rightarrow \mathbb{R}$ we define

$$\mathcal{A}_g = \{w : \bar{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } w = g \text{ on } \partial\Omega\}$$

Setting

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $g : \partial\Omega \rightarrow \mathbb{R}$ we define

$$\mathcal{A}_g = \{w : \bar{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } w = g \text{ on } \partial\Omega\}$$

Setting

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $g : \partial\Omega \rightarrow \mathbb{R}$ we define

$$\mathcal{A}_g = \{w : \bar{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } w = g \text{ on } \partial\Omega\}$$

and

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Setting

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $g : \partial\Omega \rightarrow \mathbb{R}$ we define

$$\mathcal{A}_g = \{w : \bar{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } w = g \text{ on } \partial\Omega\}$$

and

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

where

$$\begin{aligned} G : \Omega \times \mathbb{R} \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (x, \lambda, q) &\mapsto G(x, \lambda, q) \end{aligned}$$

Setting

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $g : \partial\Omega \rightarrow \mathbb{R}$ we define

$$\mathcal{A}_g = \{w : \bar{\Omega} \subset \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } w = g \text{ on } \partial\Omega\}$$

and

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

where

$$\begin{aligned} G : \Omega \times \mathbb{R} \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (x, \lambda, q) &\mapsto G(x, \lambda, q) \end{aligned}$$

The canonical example is

$$\mathcal{E}_1(w) = \frac{1}{2} \int_{\Omega} |\nabla w(x)|^2 \, dx - \int_{\Omega} F(w(x)) \, dx$$

Necessary conditions

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Necessary conditions

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

$$\mathcal{E}(u) \leq \mathcal{E}(w) \text{ for any } w = u \text{ on } \partial\Omega$$

Necessary conditions

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

$$\mathcal{E}(u) \leq \mathcal{E}(w) \text{ for any } w = u \text{ on } \partial\Omega$$



$$\mathcal{E}(u) \leq \mathcal{E}(u + t\eta) \text{ for any } \eta = 0 \text{ on } \partial\Omega \text{ and } t \in \mathbb{R}$$

Necessary conditions

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

$$\mathcal{E}(u) \leq \mathcal{E}(w) \text{ for any } w = u \text{ on } \partial\Omega$$



$$\mathcal{E}(u) \leq \mathcal{E}(u + t\eta) \text{ for any } \eta = 0 \text{ on } \partial\Omega \text{ and } t \in \mathbb{R}$$



$$f(t) = \mathcal{E}(u + t\eta)$$

f has a minimum at $t = 0$ for any $\eta = 0$ on $\partial\Omega$

Necessary conditions

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

$$\mathcal{E}(u) \leq \mathcal{E}(w) \text{ for any } w = u \text{ on } \partial\Omega$$



$$\mathcal{E}(u) \leq \mathcal{E}(u + t\eta) \text{ for any } \eta = 0 \text{ on } \partial\Omega \text{ and } t \in \mathbb{R}$$



$$f(t) = \mathcal{E}(u + t\eta)$$

f has a minimum at $t = 0$ for any $\eta = 0$ on $\partial\Omega$



$$f'(0) = 0 \text{ and } f''(0) \geq 0 \text{ for any } \eta = 0 \text{ on } \partial\Omega$$

Necessary conditions

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

$$\mathcal{E}(u) \leq \mathcal{E}(w) \text{ for any } w = u \text{ on } \partial\Omega$$

\Downarrow

$$\mathcal{E}(u) \leq \mathcal{E}(u + t\eta) \text{ for any } \eta = 0 \text{ on } \partial\Omega \text{ and } t \in \mathbb{R}$$

\Downarrow

$$f(t) = \mathcal{E}(u + t\eta)$$

f has a minimum at $t = 0$ for any $\eta = 0$ on $\partial\Omega$

\Downarrow

$$f'(0) = 0 \text{ and } f''(0) \geq 0 \text{ for any } \eta = 0 \text{ on } \partial\Omega$$

$$\delta_{\eta}\mathcal{E}(u) = f'(0) \quad \text{and} \quad \delta_{\eta}^2\mathcal{E}(u) = f''(0)$$

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\delta_{\eta} \mathcal{E}(u) = \int_{\Omega} \partial_{\lambda} G(x, w(x), \nabla w(x)) \eta(x) + \partial_q G(x, w(x), \nabla w(x)) \cdot \nabla \eta(x) \, dx$$

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\delta_{\eta} \mathcal{E}(u) = \int_{\Omega} \partial_{\lambda} G(x, w(x), \nabla w(x)) \eta(x) + \partial_q G(x, w(x), \nabla w(x)) \cdot \nabla \eta(x) \, dx$$

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\begin{aligned} \delta_{\eta} \mathcal{E}(u) &= \int_{\Omega} \partial_{\lambda} G(x, w(x), \nabla w(x)) \eta(x) + \partial_q G(x, w(x), \nabla w(x)) \cdot \nabla \eta(x) \, dx \\ &= \int_{\Omega} \left\{ -\operatorname{div}(\partial_q G(x, w(x), \nabla w(x))) + \partial_{\lambda} G(x, w(x), \nabla w(x)) \right\} \eta(x) \, dx \\ &\quad + \int_{\partial\Omega} \partial_q G(x, w(x), \nabla w(x)) \cdot \nu(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\begin{aligned} \delta_{\eta} \mathcal{E}(u) &= \int_{\Omega} \partial_{\lambda} G(x, w(x), \nabla w(x)) \eta(x) + \partial_q G(x, w(x), \nabla w(x)) \cdot \nabla \eta(x) \, dx \\ &= \int_{\Omega} \left\{ -\operatorname{div}(\partial_q G(x, w(x), \nabla w(x))) + \partial_{\lambda} G(x, w(x), \nabla w(x)) \right\} \eta(x) \, dx \\ &\quad + \int_{\partial\Omega} \partial_q G(x, w(x), \nabla w(x)) \cdot \nu(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \\ &= \int_{\Omega} \mathcal{L}w(x) \eta(x) \, dx + \int_{\partial\Omega} \mathcal{N}w(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\begin{aligned} \delta_{\eta} \mathcal{E}(u) &= \int_{\Omega} \partial_{\lambda} G(x, w(x), \nabla w(x)) \eta(x) + \partial_q G(x, w(x), \nabla w(x)) \cdot \nabla \eta(x) \, dx \\ &= \int_{\Omega} \left\{ -\operatorname{div}(\partial_q G(x, w(x), \nabla w(x))) + \partial_{\lambda} G(x, w(x), \nabla w(x)) \right\} \eta(x) \, dx \\ &\quad + \int_{\partial\Omega} \partial_q G(x, w(x), \nabla w(x)) \cdot \nu(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \\ &= \int_{\Omega} \mathcal{L}u(x) \eta(x) \, dx + \int_{\partial\Omega} \mathcal{N}u(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\begin{aligned} \delta_{\eta} \mathcal{E}(u) &= \int_{\Omega} \partial_{\lambda} G(x, u(x), \nabla u(x)) \eta(x) + \partial_q G(x, u(x), \nabla u(x)) \cdot \nabla \eta(x) \, dx \\ &= \int_{\Omega} \left\{ -\operatorname{div}(\partial_q G(x, u(x), \nabla u(x))) + \partial_{\lambda} G(x, u(x), \nabla u(x)) \right\} \eta(x) \, dx \\ &\quad + \int_{\partial\Omega} \partial_q G(x, u(x), \nabla u(x)) \cdot \nu(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \\ &= \int_{\Omega} \mathcal{L}u(x) \eta(x) \, dx + \int_{\partial\Omega} \mathcal{N}u(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

u is **minimizer** of \mathcal{E}

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\begin{aligned} \delta_{\eta} \mathcal{E}(u) &= \int_{\Omega} \partial_{\lambda} G(x, u(x), \nabla u(x)) \eta(x) + \partial_q G(x, u(x), \nabla u(x)) \cdot \nabla \eta(x) \, dx \\ &= \int_{\Omega} \left\{ -\operatorname{div}(\partial_q G(x, u(x), \nabla u(x))) + \partial_{\lambda} G(x, u(x), \nabla u(x)) \right\} \eta(x) \, dx \\ &\quad + \int_{\partial\Omega} \partial_q G(x, u(x), \nabla u(x)) \cdot \nu(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \\ &= \int_{\Omega} \mathcal{L}u(x) \eta(x) \, dx + \int_{\partial\Omega} \mathcal{N}u(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

u is **minimizer** of \mathcal{E}

Euler-Lagrange equation

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

First variation:

$$\delta_{\eta} \mathcal{E}(u) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\begin{aligned} \delta_{\eta} \mathcal{E}(u) &= \int_{\Omega} \partial_{\lambda} G(x, u(x), \nabla u(x)) \eta(x) + \partial_q G(x, u(x), \nabla u(x)) \cdot \nabla \eta(x) \, dx \\ &= \int_{\Omega} \left\{ -\operatorname{div}(\partial_q G(x, u(x), \nabla u(x))) + \partial_{\lambda} G(x, u(x), \nabla u(x)) \right\} \eta(x) \, dx \\ &\quad + \int_{\partial\Omega} \partial_q G(x, u(x), \nabla u(x)) \cdot \nu(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \\ &= \int_{\Omega} \mathcal{L}u(x) \eta(x) \, dx + \int_{\partial\Omega} \mathcal{N}u(x) \eta(x) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

u is **minimizer** of $\mathcal{E} \implies \mathcal{L}u = 0$ in Ω

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Second variation:

$$\delta_{\eta}^2 \mathcal{E}(u) = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Second variation:

$$\delta_{\eta}^2 \mathcal{E}(u) = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\begin{aligned} \delta_{\eta}^2 \mathcal{E}(u) = \int_{\Omega} \{ & \partial_{\lambda\lambda}^2 G(x, u(x), \nabla u(x)) \eta^2(x) + 2\partial_{\lambda q}^2 G(x, u(x), \nabla u(x)) \cdot \eta(x) \nabla \eta(x) \\ & + \nabla \eta(x) \cdot \partial_{qq}^2 G(x, u(x), \nabla u(x)) \nabla \eta(x) \} \end{aligned}$$

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Second variation:

$$\delta_{\eta}^2 \mathcal{E}(u) = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\begin{aligned} \delta_{\eta}^2 \mathcal{E}(u) = \int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, u(x), \nabla u(x)) \eta^2(x) + 2\partial_{\lambda q}^2 G(x, u(x), \nabla u(x)) \cdot \eta(x) \nabla \eta(x) \right. \\ \left. + \nabla \eta(x) \cdot \partial_{qq}^2 G(x, u(x), \nabla u(x)) \nabla \eta(x) \right\} \end{aligned}$$

u is **minimizer** of $\mathcal{E} \implies \mathcal{L}u=0$ in Ω
 $\delta_{\eta}^2 \mathcal{E}(u) \geq 0 \quad \forall \eta=0$ on $\partial\Omega$

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

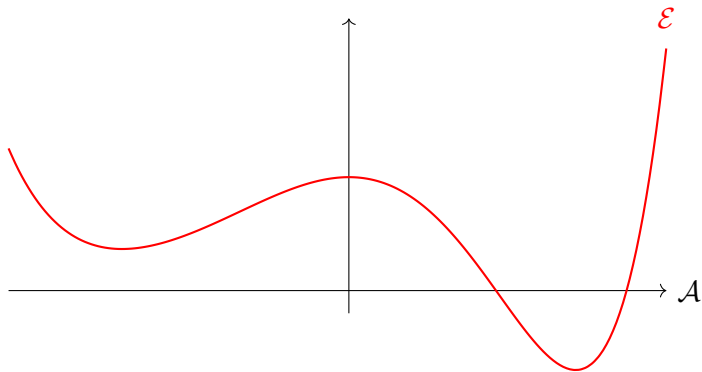
Second variation:

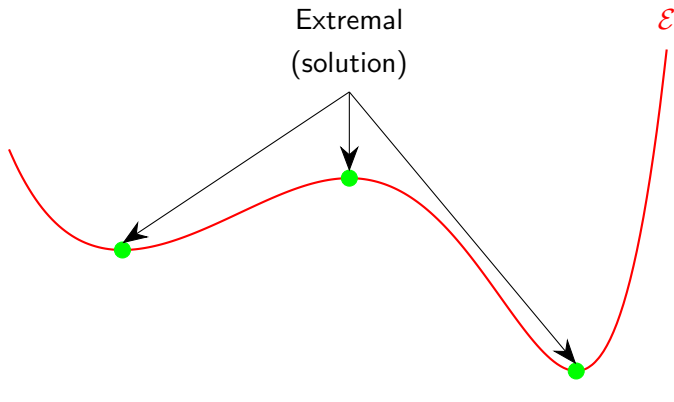
$$\delta_{\eta}^2 \mathcal{E}(u) = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} \mathcal{E}(w + t\eta)$$

$$\delta_{\eta}^2 \mathcal{E}(u) = \int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, u(x), \nabla u(x)) \eta^2(x) + 2\partial_{\lambda q}^2 G(x, u(x), \nabla u(x)) \cdot \eta(x) \nabla \eta(x) + \nabla \eta(x) \cdot \partial_{qq}^2 G(x, u(x), \nabla u(x)) \nabla \eta(x) \right\}$$

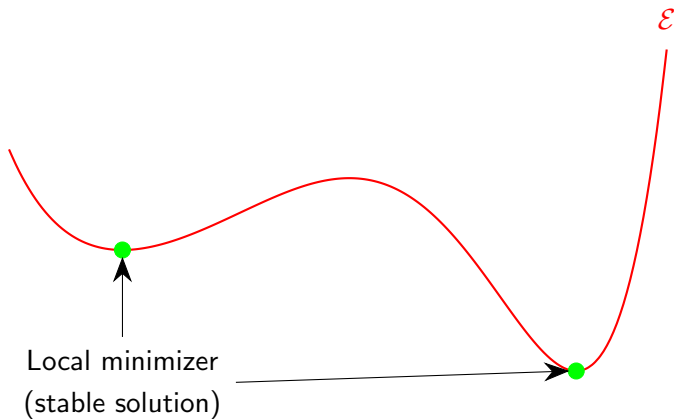
u is minimizer of $\mathcal{E} \implies \mathcal{L}u=0$ in Ω
 $\delta_{\eta}^2 \mathcal{E}(u) \geq 0 \, \forall \eta=0$ on $\partial\Omega \iff u$ is a **stable solution**

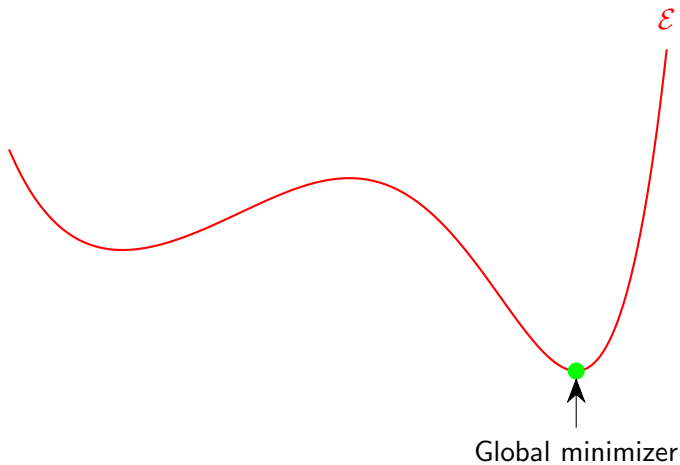
1D scheme





1D scheme





Necessary conditions: Legendre

Stability condition:

$$\int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, w(x), \nabla w(x)) \eta^2(x) + 2\partial_{\lambda q}^2 G(x, w(x), \nabla w(x)) \cdot \eta(x) \nabla \eta(x) + \nabla \eta(x) \cdot \partial_{qq}^2 G(x, w(x), \nabla w(x)) \nabla \eta(x) \right\} \geq 0 \quad \text{for all } \eta = 0 \text{ on } \partial\Omega$$

Necessary conditions: Legendre

Stability condition:

$$\int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, w(x), \nabla w(x)) \eta^2(x) + 2\partial_{\lambda q}^2 G(x, w(x), \nabla w(x)) \cdot \eta(x) \nabla \eta(x) + \nabla \eta(x) \cdot \partial_{qq}^2 G(x, w(x), \nabla w(x)) \nabla \eta(x) \right\} \geq 0 \quad \text{for all } \eta = 0 \text{ on } \partial\Omega$$

Theorem (Legendre Condition)

Let u be a *stable solution* of the energy functional \mathcal{E} . Then, it satisfies

$$\partial_{qq}^2 G(\cdot, u, \nabla u) \geq 0 \text{ in } \Omega.$$

Necessary conditions: Legendre

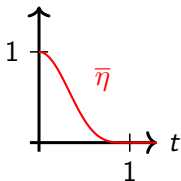
Stability condition:

$$\int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, w(x), \nabla w(x)) \eta^2(x) + 2\partial_{\lambda q}^2 G(x, w(x), \nabla w(x)) \cdot \eta(x) \nabla \eta(x) + \nabla \eta(x) \cdot \partial_{qq}^2 G(x, w(x), \nabla w(x)) \nabla \eta(x) \right\} \geq 0 \quad \text{for all } \eta = 0 \text{ on } \partial\Omega$$

Theorem (Legendre Condition)

Let u be a stable solution of the energy functional \mathcal{E} . Then, it satisfies

$$\partial_{qq}^2 G(\cdot, u, \nabla u) \geq 0 \text{ in } \Omega.$$



$$\rightsquigarrow \eta_{\varepsilon}(x) = \varepsilon \bar{\eta} \left(\frac{|x-x_0|}{\varepsilon} \right)$$

Necessary conditions: Legendre

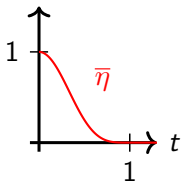
Stability condition:

$$\int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, w(x), \nabla w(x)) \eta_{\varepsilon}^2(x) + 2\partial_{\lambda q}^2 G(x, w(x), \nabla w(x)) \cdot \eta_{\varepsilon}(x) \nabla \eta_{\varepsilon}(x) + \nabla \eta_{\varepsilon}(x) \cdot \partial_{qq}^2 G(x, w(x), \nabla w(x)) \nabla \eta_{\varepsilon}(x) \right\} \geq 0$$

Theorem (Legendre Condition)

Let u be a stable solution of the energy functional \mathcal{E} . Then, it satisfies

$$\partial_{qq}^2 G(\cdot, u, \nabla u) \geq 0 \text{ in } \Omega.$$



$$\rightsquigarrow \eta_{\varepsilon}(x) = \varepsilon \bar{\eta} \left(\frac{|x-x_0|}{\varepsilon} \right)$$

Necessary conditions: Legendre

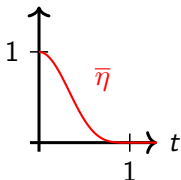
Stability condition:

$$\int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, w(x), \nabla w(x)) \eta_{\varepsilon}^2(x) + 2\partial_{\lambda q}^2 G(x, w(x), \nabla w(x)) \cdot \eta_{\varepsilon}(x) \nabla \eta_{\varepsilon}(x) + \nabla \eta_{\varepsilon}(x) \cdot \partial_{qq}^2 G(x, w(x), \nabla w(x)) \nabla \eta_{\varepsilon}(x) \right\} \geq 0$$

Theorem (Legendre Condition)

Let u be a stable solution of the energy functional \mathcal{E} . Then, it satisfies

$$\partial_{qq}^2 G(\cdot, u, \nabla u) \geq 0 \text{ in } \Omega.$$



$$\rightsquigarrow \eta_{\varepsilon}(x) = \varepsilon \bar{\eta} \left(\frac{|x-x_0|}{\varepsilon} \right)$$

Necessary conditions: Legendre

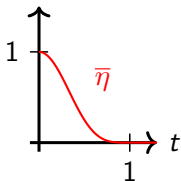
Stability condition:

$$\int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, w(x), \nabla w(x)) \eta_\varepsilon^2(x) + 2\partial_{\lambda q}^2 G(x, w(x), \nabla w(x)) \cdot \eta_\varepsilon(x) \nabla \eta_\varepsilon(x) + \nabla \eta_\varepsilon(x) \cdot \partial_{qq}^2 G(x, w(x), \nabla w(x)) \nabla \eta_\varepsilon(x) \right\} \geq 0$$

Theorem (Legendre Condition)

Let u be a stable solution of the energy functional \mathcal{E} . Then, it satisfies

$$\partial_{qq}^2 G(\cdot, u, \nabla u) \geq 0 \text{ in } \Omega.$$



$$\rightsquigarrow \eta_\varepsilon(x) = \varepsilon \bar{\eta} \left(\frac{|x-x_0|}{\varepsilon} \right)$$

Necessary conditions: Legendre

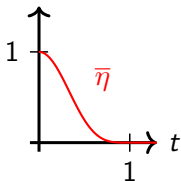
Stability condition:

$$\int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, w(x), \nabla w(x)) \eta_{\varepsilon}^2(x) + 2\partial_{\lambda q}^2 G(x, w(x), \nabla w(x)) \cdot \eta_{\varepsilon}(x) \nabla \eta_{\varepsilon}(x) + \nabla \eta_{\varepsilon}(x) \cdot \partial_{qq}^2 G(x, w(x), \nabla w(x)) \nabla \eta_{\varepsilon}(x) \right\} \geq 0$$

Theorem (Legendre Condition)

Let u be a stable solution of the energy functional \mathcal{E} . Then, it satisfies

$$\partial_{qq}^2 G(\cdot, u, \nabla u) \geq 0 \text{ in } \Omega.$$



$$\rightsquigarrow \eta_{\varepsilon}(x) = \varepsilon \bar{\eta} \left(\frac{|x-x_0|}{\varepsilon} \right)$$

Necessary conditions: Legendre

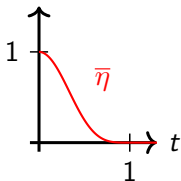
Stability condition:

$$\int_{\Omega} \left\{ \partial_{\lambda\lambda}^2 G(x, w(x), \nabla w(x)) \eta^2(x) + 2\partial_{\lambda q}^2 G(x, w(x), \nabla w(x)) \cdot \eta(x) \nabla \eta(x) + \nabla \eta(x) \cdot \partial_{qq}^2 G(x, w(x), \nabla w(x)) \nabla \eta(x) \right\} \geq 0$$

Theorem (Legendre Condition)

Let u be a stable solution of the energy functional \mathcal{E} . Then, it satisfies

$$\partial_{qq}^2 G(\cdot, u, \nabla u) \geq 0 \text{ in } \Omega.$$



$$\rightsquigarrow \eta_{\varepsilon}(x) = \varepsilon \bar{\eta} \left(\frac{|x-x_0|}{\varepsilon} \right)$$

Necessary conditions: Weierstrass

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Weierstrass excess function:

$$E(x, \lambda, q, \tilde{q}) = G(x, \lambda, \tilde{q}) - G(x, \lambda, q) - \partial_q G(x, \lambda, q) \cdot (\tilde{q} - q)$$

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Weierstrass excess function:

$$E(x, \lambda, q, \tilde{q}) = G(x, \lambda, \tilde{q}) - G(x, \lambda, q) - \partial_q G(x, \lambda, q) \cdot (\tilde{q} - q)$$

Theorem (Weierstrass Necessary Condition)

Let u be a *minimizer* of the energy functional \mathcal{E} among functions with the same boundary data. Then, it satisfies

$$E(x, u(x), \nabla u(x), \xi) \geq 0 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n.$$

Calibrations

The main question

Given an **extremal** of the functional \mathcal{E} , when is it a **minimizer** among functions with the same boundary data?

The main question

Given an extremal of the functional \mathcal{E} , when is it a minimizer among functions with the same boundary data?

- If \mathcal{E} is (strictly) **convex**, then there is a unique extremal that turns out to be a minimizer

The main question

Given an extremal of the functional \mathcal{E} , when is it a minimizer among functions with the same boundary data?

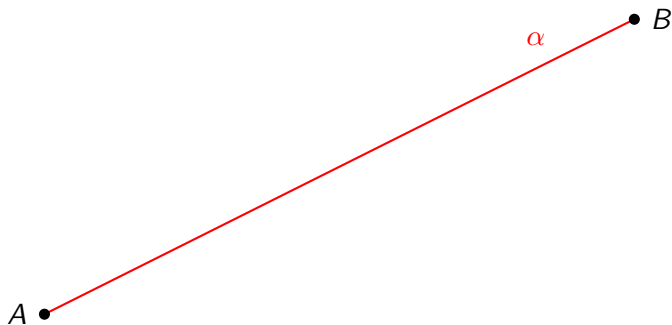
- If \mathcal{E} is (strictly) convex, then there is a unique extremal that turns out to be a minimizer
- Important models with **nonconvex** functional:
 - Allen-Cahn energy
 - Bernoulli free boundary problem
 - Perimeter

Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?

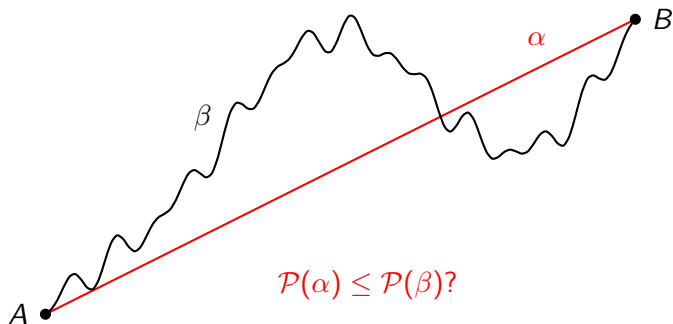
Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



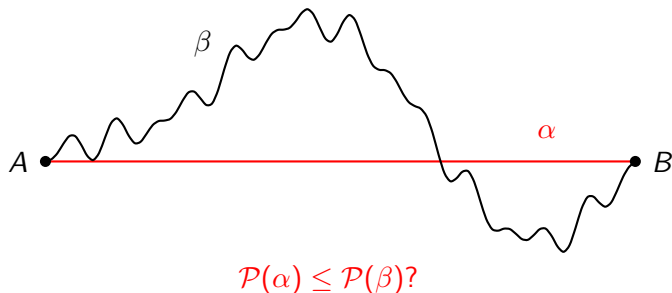
Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



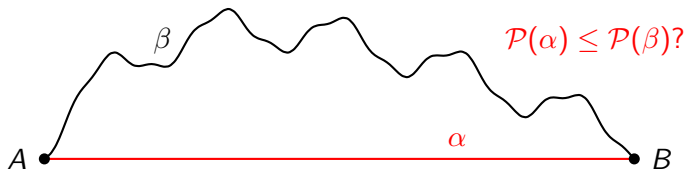
Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



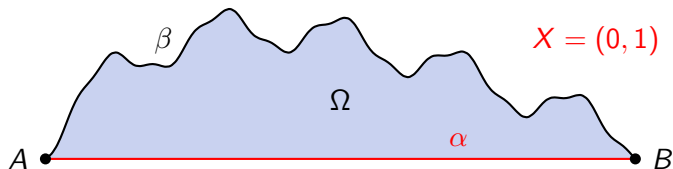
Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



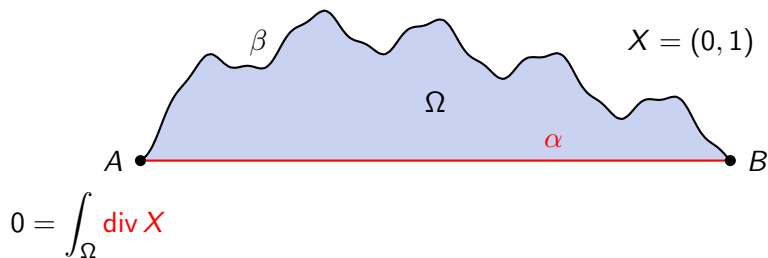
Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



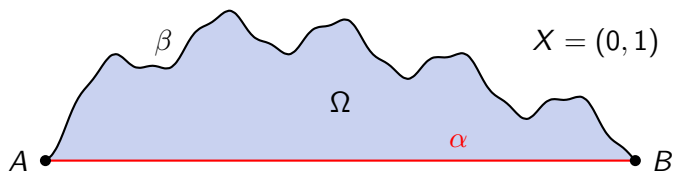
Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



Toy example

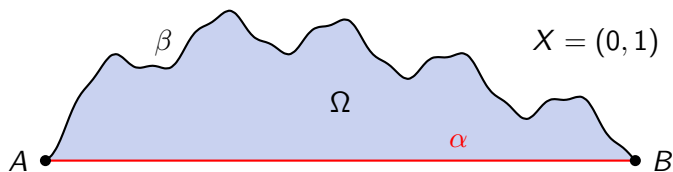
How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



$$0 = \int_{\Omega} \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu = \int_{\beta} X \cdot \nu_{\beta} - \int_{\alpha} X \cdot \nu_{\alpha}$$

Toy example

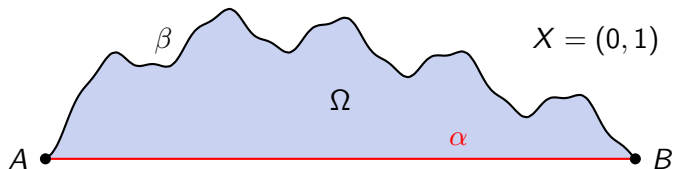
How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



$$0 = \int_{\Omega} \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu = \int_{\beta} X \cdot \nu_{\beta} - \int_{\alpha} X \cdot \nu_{\alpha}$$

Toy example

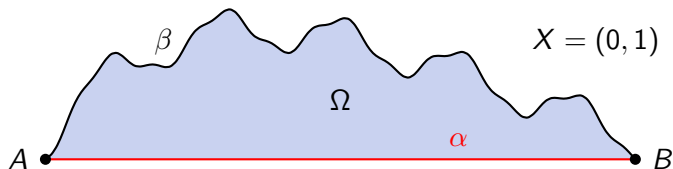
How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



$$0 = \int_{\Omega} \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu = \int_{\beta} X \cdot \nu_{\beta} - \int_{\alpha} \mathbf{1}$$

Toy example

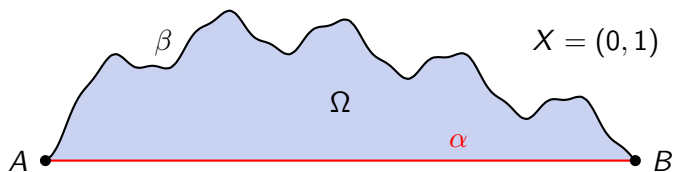
How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



$$0 = \int_{\Omega} \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu = \int_{\beta} X \cdot \nu_{\beta} - \int_{\alpha} 1 = \int_{\beta} X \cdot \nu_{\beta} - \mathcal{P}(\alpha)$$

Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?

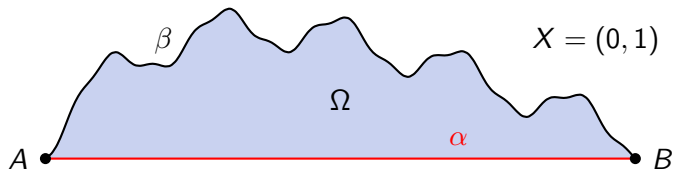


$$0 = \int_{\Omega} \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu = \int_{\beta} X \cdot \nu_{\beta} - \int_{\alpha} X \cdot \nu_{\alpha} = \int_{\beta} X \cdot \nu_{\beta} - \mathcal{P}(\alpha)$$

$$\mathcal{P}(\alpha) = \int_{\beta} X \cdot \nu_{\beta}$$

Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?

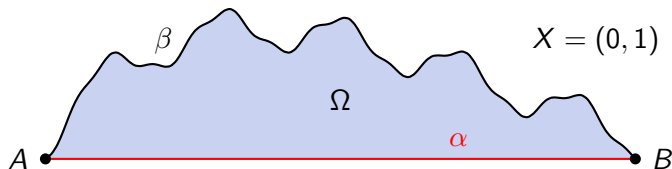


$$0 = \int_{\Omega} \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu = \int_{\beta} X \cdot \nu_{\beta} - \int_{\alpha} X \cdot \nu_{\alpha} = \int_{\beta} X \cdot \nu_{\beta} - \mathcal{P}(\alpha)$$

$$\mathcal{P}(\alpha) = \int_{\beta} X \cdot \nu_{\beta} \leq \int_{\beta} 1$$

Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?

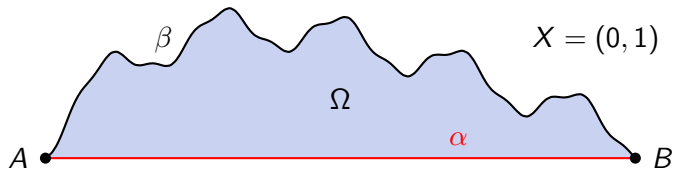


$$0 = \int_{\Omega} \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu = \int_{\beta} X \cdot \nu_{\beta} - \int_{\alpha} X \cdot \nu_{\alpha} = \int_{\beta} X \cdot \nu_{\beta} - \mathcal{P}(\alpha)$$

$$\mathcal{P}(\alpha) = \int_{\beta} X \cdot \nu_{\beta} \leq \int_{\beta} 1 = \mathcal{P}(\beta) \quad \checkmark$$

Toy example

How to prove that a line in \mathbb{R}^2 is a minimizer of the perimeter functional?



$$0 = \int_{\Omega} \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu = \int_{\beta} X \cdot \nu_{\beta} - \int_{\alpha} X \cdot \nu_{\alpha} = \int_{\beta} X \cdot \nu_{\beta} - \mathcal{P}(\alpha)$$

$$\mathcal{P}(\alpha) = \int_{\beta} X \cdot \nu_{\beta} \leq \mathcal{P}(\beta) \quad \checkmark$$

$$\mathcal{C}(\gamma) = \int_{\gamma} X \cdot \nu_{\gamma} \, dl, \quad \text{where } |X| = 1 \text{ and } \operatorname{div} X = 0$$

Definition

A functional $\mathcal{C}: \mathcal{A} \rightarrow \mathbb{R}$ is a *calibration* for \mathcal{E} and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$.
- $\mathcal{C}(w) = \mathcal{C}(\tilde{w})$ for all $w, \tilde{w} \in \mathcal{A}$.

Definition

A functional $\mathcal{C}: \mathcal{A} \rightarrow \mathbb{R}$ is a *calibration* for \mathcal{E} and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$.
- $\mathcal{C}(w) = \mathcal{C}(\tilde{w})$ for all $w, \tilde{w} \in \mathcal{A}$.

The calibration functional

Definition

A functional $\mathcal{C}: \mathcal{A} \rightarrow \mathbb{R}$ is a *calibration* for \mathcal{E} and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$.
- $\mathcal{C}(w) = \mathcal{C}(\tilde{w})$ for all $w, \tilde{w} \in \mathcal{A}$.

Given $w \in \mathcal{A}$

$$\mathcal{E}(u) = \mathcal{C}(u) = \mathcal{C}(w) \leq \mathcal{E}(w).$$

The calibration functional

Definition

A functional $\mathcal{C}: \mathcal{A} \rightarrow \mathbb{R}$ is a *calibration* for \mathcal{E} and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$.
- $\mathcal{C}(w) = \mathcal{C}(\tilde{w})$ for all $w, \tilde{w} \in \mathcal{A}$.

Given $w \in \mathcal{A}$

$$\mathcal{E}(u) = \mathcal{C}(u) = \mathcal{C}(w) \leq \mathcal{E}(w).$$

Then, u is a minimizer.

The calibration functional

Definition

A functional $\mathcal{C}: \mathcal{A} \rightarrow \mathbb{R}$ is a *calibration* for \mathcal{E} and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$.
- $\mathcal{C}(u) \leq \mathcal{C}(w)$ for all $w \in \mathcal{A}$.

Given $w \in \mathcal{A}$

$$\mathcal{E}(u) = \mathcal{C}(u) \leq \mathcal{C}(w) \leq \mathcal{E}(w)$$

Then, u is a minimizer.

The calibration functional

Definition

A functional $\mathcal{C}: \mathcal{A} \rightarrow \mathbb{R}$ is a *calibration* for \mathcal{E} and $u \in \mathcal{A}$ if the following conditions hold:

- $\mathcal{C}(u) = \mathcal{E}(u)$.
- $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$.
- $\mathcal{C}(u) \leq \mathcal{C}(w)$ for all $w \in \mathcal{A}$.

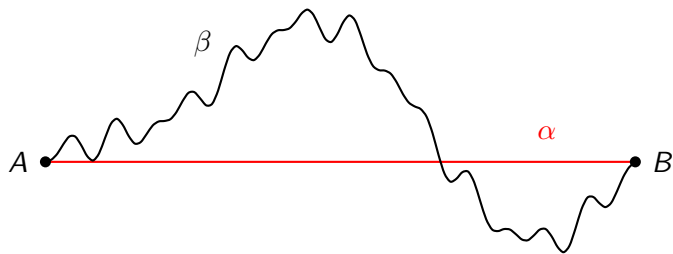
Given $w \in \mathcal{A}$

$$\mathcal{E}(u) = \mathcal{C}(u) \leq \mathcal{C}(w) \leq \mathcal{E}(w)$$

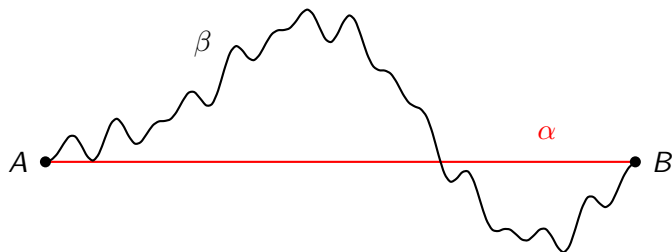
Then, u is a minimizer.

When is it possible to find such a functional?

Construction of Calibration: Perimeter functional



Construction of Calibration: Perimeter functional



The functional

$$\mathcal{C}(\gamma) = \int_{\gamma} X \cdot \nu_{\gamma} dl, \quad \text{where } X = (0, 1)$$

provides a **Calibration** for the line α and the perimeter \mathcal{P} .

Construction of Calibration: Convex framework

$G(x, \lambda, q)$ convex in $(\lambda, q) \implies \mathcal{E}$ is convex

Construction of Calibration: Convex framework

$G(x, \lambda, q)$ convex in $(\lambda, q) \implies \mathcal{E}$ is convex

Lemma (Comparison of energies)

Assume that $G(x, \lambda, q)$ is convex in (λ, q) . Then, given $u, w \in H^1(\Omega)$, they satisfy

$$\begin{aligned} \mathcal{E}(w) \geq \mathcal{E}(u) &+ \int_{\Omega} \mathcal{L}u(x) (w(x) - u(x)) \, dx \\ &+ \int_{\partial\Omega} \mathcal{N}u(x) (w(x) - u(x)) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

Construction of Calibration: Convex framework

$G(x, \lambda, q)$ convex in $(\lambda, q) \implies \mathcal{E}$ is convex

Lemma (Comparison of energies)

Assume that $G(x, \lambda, q)$ is convex in (λ, q) . Then, given $u, w \in H^1(\Omega)$, they satisfy

$$\begin{aligned} \mathcal{E}(w) \geq \mathcal{E}(u) &+ \int_{\Omega} \mathcal{L}u(x) (w(x) - u(x)) \, dx \\ &+ \int_{\partial\Omega} \mathcal{N}u(x) (w(x) - u(x)) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

It provides a Calibration functional for u and \mathcal{E}

$$\begin{aligned} \mathcal{C}(w) = \mathcal{E}(u) &+ \int_{\Omega} \mathcal{L}u(x) (w(x) - u(x)) \, dx \\ &+ \int_{\partial\Omega} \mathcal{N}u(x) (w(x) - u(x)) \, d\mathcal{H}^{n-1}(x) \end{aligned}$$

Weierstrass Extremal Field Theory

Weierstrass Sufficient Condition

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Theorem (Weierstrass Sufficient Condition)

Assume that $G(x, \lambda, q)$ is **convex in q** . If u is **embedded** in an **extremal field**, then it is a minimizer of \mathcal{E} .

Weierstrass Sufficient Condition

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Theorem (Weierstrass Sufficient Condition)

Assume that $G(x, \lambda, q)$ is convex in q . If u is embedded in an extremal field, then it is a minimizer of \mathcal{E} .

We say that a family of functions $\{u^t\}_{t \in \mathbb{R}}$ is a *field* if

- the functions $t \mapsto u^t(x)$ are **increasing** for each x
- the map $(x, t) \mapsto u^t(x)$ is **continuous**

Weierstrass Sufficient Condition

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Theorem (Weierstrass Sufficient Condition)

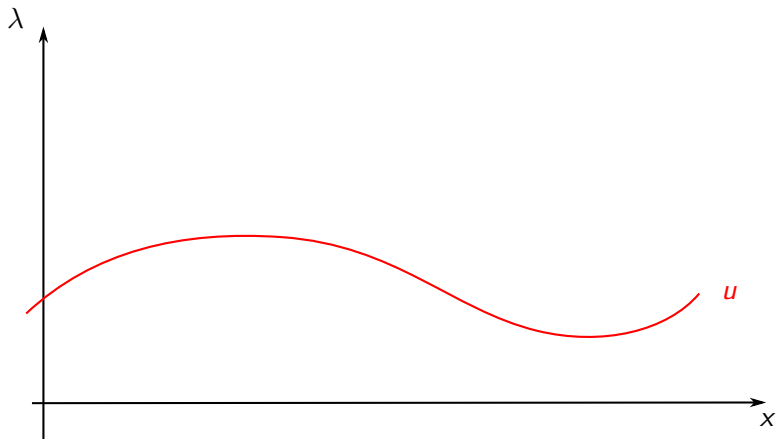
Assume that $G(x, \lambda, q)$ is convex in q . If u is embedded in an extremal field, then it is a minimizer of \mathcal{E} .

We say that a family of functions $\{u^t\}_{t \in \mathbb{R}}$ is a *field* if

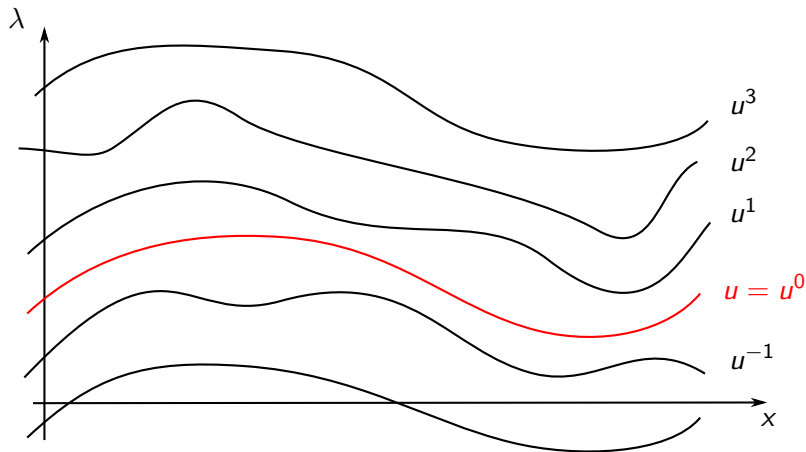
- the functions $t \mapsto u^t(x)$ are increasing for each x
- the map $(x, t) \mapsto u^t(x)$ is continuous

Moreover, it is an **extremal field** if each leaf u^t satisfies the E-L equation.

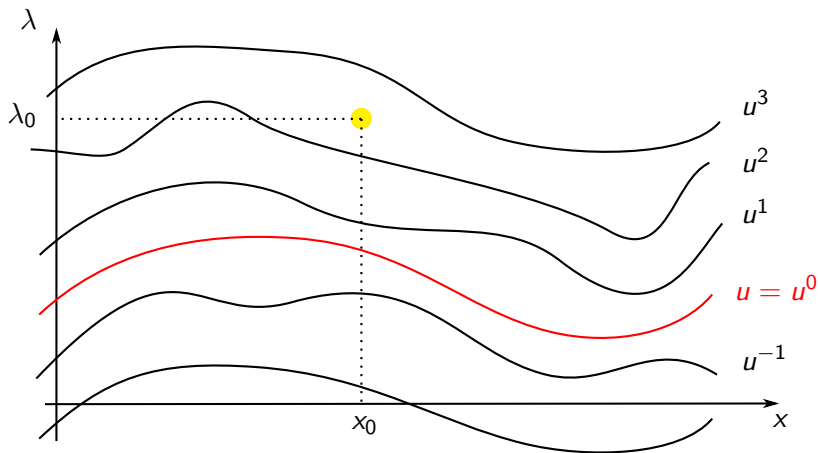
The leaf-parameter function



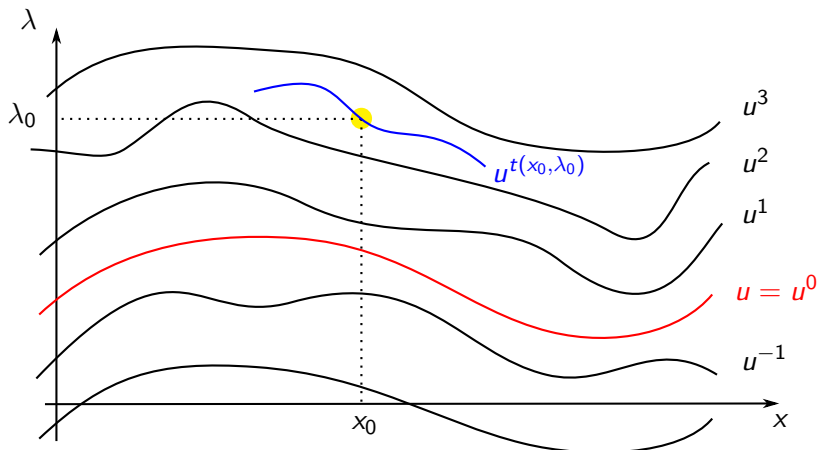
The leaf-parameter function



The leaf-parameter function



The leaf-parameter function



$t(x_0, \lambda_0)$ is the unique $\tau \in \mathbb{R}$ such that $u^\tau(x_0) = \lambda_0$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

1 $\mathcal{C}(u) = \mathcal{E}(u)$?

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

① $\mathcal{C}(u) = \mathcal{E}(u)$?

$$\begin{aligned} \mathcal{C}(u) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla u(x) - \nabla u^t(x)) \Big|_{t=t(x, u(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, u(x))} \, dx \end{aligned}$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

① $\mathcal{C}(u) = \mathcal{E}(u)$?

$$\begin{aligned} \mathcal{C}(u) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla u^0(x) - \nabla u^t(x)) \Big|_{t=t(x, u^0(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, u^0(x))} \, dx \end{aligned}$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

① $\mathcal{C}(u) = \mathcal{E}(u)$?

$$\begin{aligned} \mathcal{C}(u) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla u^0(x) - \nabla u^t(x)) \Big|_{t=t(x, u^0(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, u^0(x))} \, dx \end{aligned}$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

① $\mathcal{C}(u) = \mathcal{E}(u)$?

$$\begin{aligned} \mathcal{C}(u) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla u^0(x) - \nabla u^t(x)) \Big|_{t=0} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=0} \, dx \end{aligned}$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

① $\mathcal{C}(u) = \mathcal{E}(u)$?

$$\begin{aligned} \mathcal{C}(u) = & \int_{\Omega} \partial_q G(x, u^0(x), \nabla u^0(x)) (\nabla u^0(x) - \nabla u^0(x)) \, dx \\ & + \int_{\Omega} G(x, u^0(x), \nabla u^0(x)) \, dx \end{aligned}$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

① $\mathcal{C}(u) = \mathcal{E}(u)$?

$$\begin{aligned} \mathcal{C}(u) = & \int_{\Omega} \partial_q G(x, u^0(x), \nabla u^0(x)) (\nabla u^0(x) - \nabla u^0(x)) \, dx \\ & + \int_{\Omega} G(x, u^0(x), \nabla u^0(x)) \, dx \end{aligned}$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

① $\mathcal{C}(u) = \mathcal{E}(u)$?

$$\mathcal{C}(u) = \int_{\Omega} G(x, u^0(x), \nabla u^0(x)) \, dx = \mathcal{E}(u)$$

Proof via the construction of a calibration (1/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

① $\mathcal{C}(u) = \mathcal{E}(u)$? ✓

$$\mathcal{C}(u) = \int_{\Omega} G(x, u(x), \nabla u(x)) \, dx = \mathcal{E}(u)$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u ?

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) &= \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ &\quad + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u ?

$$\mathcal{E}(w) - \mathcal{C}(w)$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u ?

$$\begin{aligned} \mathcal{E}(w) - \mathcal{C}(w) = & \int_{\Omega} \left\{ G(x, w(x), \nabla w(x)) - G(x, u^t(x), \nabla u^t(x)) \right. \\ & \left. - \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \right\} \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) &= \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ &\quad + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u ?

$$\begin{aligned} \mathcal{E}(w) - \mathcal{C}(w) &= \int_{\Omega} \left\{ G(x, u^t(x), \nabla w(x)) - G(x, u^t(x), \nabla u^t(x)) \right. \\ &\quad \left. - \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \right\} \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u ?

$$\begin{aligned} \mathcal{E}(w) - \mathcal{C}(w) = & \int_{\Omega} \left\{ G(x, u^t(x), \nabla w(x)) - G(x, u^t(x), \nabla u^t(x)) \right. \\ & \left. - \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \right\} \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u

$$\mathcal{E}(w) - \mathcal{C}(w) = \int_{\Omega} E(x, u^t(x), \nabla u^t(x), \nabla w(x)) \Big|_{t=t(x, w(x))} \, dx$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) &= \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ &\quad + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u ?

$$\mathcal{E}(w) - \mathcal{C}(w) = \int_{\Omega} E(x, u^t(x), \nabla u^t(x), \nabla w(x)) \Big|_{t=t(x, w(x))} \, dx$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) &= \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ &\quad + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u

$$\mathcal{E}(w) - \mathcal{C}(w) = \int_{\Omega} E(x, u^t(x), \nabla u^t(x), \nabla w(x)) \Big|_{t=t(x, w(x))} \, dx$$

Proof via the construction of a calibration (2/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 2 $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all w with the same Dirichlet condition as u ? ✓

$$\mathcal{E}(w) - \mathcal{C}(w) = \int_{\Omega} E(x, u^t(x), \nabla u^t(x), \nabla w(x)) \Big|_{t=t(x, w(x))} \, dx$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- 3 $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ?

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- ③ $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ?

$$\mathcal{C}(w) = \int_{\Omega} \begin{pmatrix} X^1(x, t) \\ X^2(x, t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x, w(x))} \, dx$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = \int_{\Omega} & \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- ③ $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ?

$$\mathcal{C}(w) = \int_{\Omega} \begin{pmatrix} X^1(x, t) \\ X^2(x, t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x, w(x))} \, dx$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- ③ $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ?

$$\mathcal{C}(w) = \int_{\Omega} \begin{pmatrix} X^1(x, t) \\ X^2(x, t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x, w(x))} \, dx$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- ③ $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ?

$$\mathcal{C}(w) = \int_{\Omega} \begin{pmatrix} X^1(x, t) \\ X^2(x, t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x, w(x))} \, dx$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- ③ $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ?

$$\begin{aligned} \mathcal{C}(w) &= \int_{\Omega} \begin{pmatrix} X^1(x, t) \\ X^2(x, t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x, w(x))} \, dx \\ &= \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n(x, \lambda) \end{aligned}$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) &= \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ &\quad + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

- ③ $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ?

$$\begin{aligned} \mathcal{C}(w) &= \int_{\Omega} \begin{pmatrix} X^1(x, t) \\ X^2(x, t) \end{pmatrix} \cdot \begin{pmatrix} -\nabla w(x) \\ 1 \end{pmatrix} \Big|_{t=t(x, w(x))} \, dx \\ &= \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n(x, \lambda), \quad \text{with} \quad \text{div } X = \mathcal{L}(u^t)(x) \Big|_{t=t(x, \lambda)} \end{aligned}$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$



$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

③ $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ?

$$\mathcal{C}(w) = \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n, \quad \text{where } \operatorname{div} X = 0$$

Proof via the construction of a calibration (3/3)

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

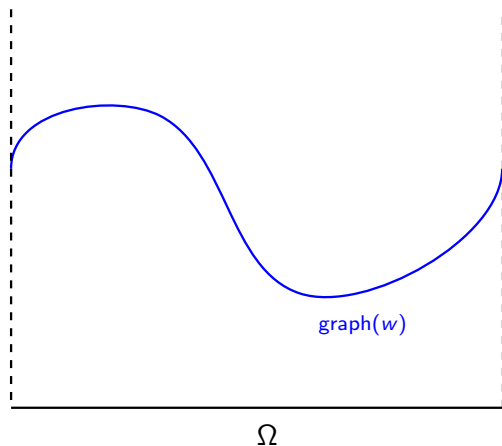


$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

③ $\mathcal{C}(u) = \mathcal{C}(w)$ for all w with the same Dirichlet condition as u ? ✓

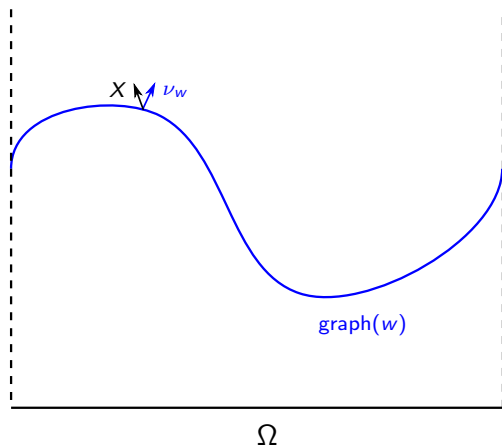
$$\mathcal{C}(w) = \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n, \quad \text{where} \quad \text{div} X = 0$$

Geometric interpretation



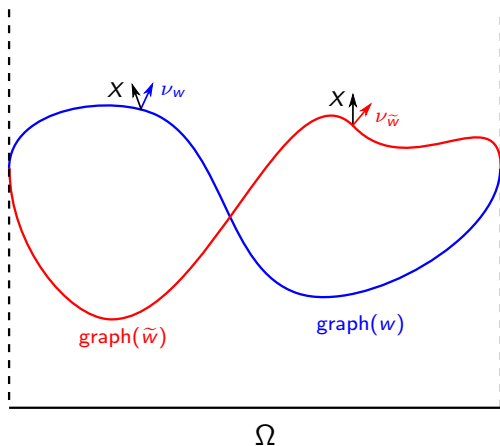
$$C(w) = \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n, \quad \text{where} \quad \text{div} X = 0$$

Geometric interpretation



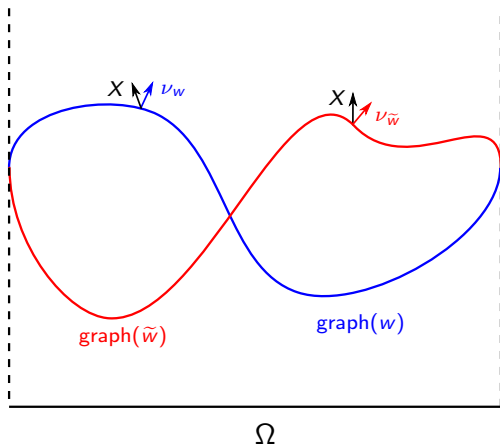
$$C(w) = \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n, \quad \text{where } \operatorname{div} X = 0$$

Geometric interpretation



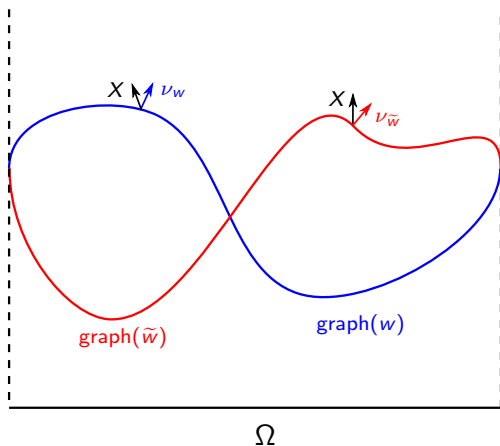
$$C(w) = \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n, \quad \text{where} \quad \text{div} X = 0$$

Geometric interpretation



$$C(w) = \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n, \quad \text{where } \text{div } X = 0$$

Geometric interpretation



$$C(w) = \int_{\text{graph}(w)} X \cdot \nu_w \, d\mathcal{H}^n, \quad \text{where } \text{div } X = 0$$

This geometric strategy fails when applying to other problems

Is there a **structure** behind the calibration that we can understand from a PDE point of view?

Is there a structure behind the calibration that we can understand from a PDE point of view?

We found it in the calibration for the fractional perimeter by [Cabr , '19]

Is there a structure behind the calibration that we can understand from a PDE point of view?

We found it in the calibration for the fractional perimeter by [Cabr , '19]

$$\text{Calibration} = \boxed{\phantom{\text{Calibration}}} + \boxed{\phantom{\text{Calibration}}}$$

Is there a structure behind the calibration that we can understand from a PDE point of view?

We found it in the calibration for the fractional perimeter by [Cabr e, '19]

$$\text{Calibration} = \boxed{\begin{array}{c} \text{Euler-Lagrange on} \\ \text{the leafs in } \Omega \end{array}} + \boxed{\phantom{\text{Euler-Lagrange on the leafs in } \Omega}}$$

Is there a structure behind the calibration that we can understand from a PDE point of view?

We found it in the calibration for the fractional perimeter by [Cabr , '19]

$$\text{Calibration} = \boxed{\begin{array}{c} \text{Euler-Lagrange on} \\ \text{the leafs in } \Omega \end{array}} + \boxed{\text{Boundary terms}}$$

Is there a structure behind the calibration that we can understand from a PDE point of view?

We found it in the calibration for the fractional perimeter by [Cabr e, '19]

$$\text{Calibration} = \boxed{\begin{array}{c} \text{Euler-Lagrange on} \\ \text{the leafs in } \Omega \end{array}} + \boxed{\text{Boundary terms}}$$

Can we find the same **structure** in the **local theory**?

Can we find the same structure in the local theory?

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Can we find the same structure in the local theory?

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

\Downarrow

$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Can we find the same structure in the local theory? **YES** ✓

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

↓

$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

New identity

Can we find the same structure in the local theory? **YES** ✓

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

⇓

$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Proposition [Cabr , Erneta, F-N, '22]

$$\begin{aligned} \mathcal{C}(w) = & \mathcal{E}(u) + \int_{\Omega} \int_{u(x)}^{w(x)} L u^t(x) \Big|_{t=t(x, \lambda)} \, d\lambda \, dx \\ & + \int_{\partial\Omega} \int_{u(x)}^{w(x)} \mathcal{N} u^t(x) \Big|_{t=t(x, \lambda)} \, d\lambda \, d\mathcal{H}^{n-1}(x). \end{aligned}$$

Can we find the same structure in the local theory? **YES** ✓

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

⇓

$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Proposition [Cabr , Erneta, F-N, '22]

$$\begin{aligned} \mathcal{C}(w) = & \mathcal{E}(u) + \int_{\Omega} \int_{u(x)}^{w(x)} Lu^t(x) \Big|_{t=t(x, \lambda)} \, d\lambda \, dx \\ & + \int_{\partial\Omega} \int_{u(x)}^{w(x)} \mathcal{N}u^t(x) \Big|_{t=t(x, \lambda)} \, d\lambda \, d\mathcal{H}^{n-1}(x). \end{aligned}$$

Can we find the same structure in the local theory? **YES** ✓

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

⇓

$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Proposition [Cabr , Erneta, F-N, '22]

$$\begin{aligned} \mathcal{C}(w) = & \mathcal{E}(u) + \int_{\Omega} \int_{u(x)}^{w(x)} L u^t(x) \Big|_{t=t(x, \lambda)} \, d\lambda \, dx \\ & + \int_{\partial\Omega} \int_{u(x)}^{w(x)} \mathcal{N} u^t(x) \Big|_{t=t(x, \lambda)} \, d\lambda \, d\mathcal{H}^{n-1}(x). \end{aligned}$$

Can we find the same structure in the local theory? **YES** ✓

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

⇓

$$\begin{aligned} \mathcal{C}(w) = & \int_{\Omega} \partial_q G(x, u^t(x), \nabla u^t(x)) (\nabla w(x) - \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \\ & + \int_{\Omega} G(x, u^t(x), \nabla u^t(x)) \Big|_{t=t(x, w(x))} \, dx \end{aligned}$$

Proposition [Cabr , Erneta, F-N, '22]

$$\begin{aligned} \mathcal{C}(w) = & \mathcal{E}(u) + \int_{\Omega} \int_{u(x)}^{w(x)} \mathcal{L}u^t(x) \Big|_{t=t(x, \lambda)} \, d\lambda \, dx \\ & + \int_{\partial\Omega} \int_{u(x)}^{w(x)} \mathcal{N}u^t(x) \Big|_{t=t(x, \lambda)} \, d\lambda \, d\mathcal{H}^{n-1}(x). \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\mathcal{C}(w) - \mathcal{E}(u) = \mathcal{C}(w) - \mathcal{C}(u)$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ G(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q G(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ G(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q G(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ G(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q G(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \\ &= \dots \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned}\mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ G(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q G(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \\ &= \dots \\ &= \int_0^1 \int_\Omega \mathcal{L}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) dx d\theta + \int_0^1 \int_{\partial\Omega} \mathcal{N}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\mathcal{H}^{n-1} d\theta\end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ G(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q G(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \\ &= \dots \\ &= \int_\Omega \int_0^1 \mathcal{L}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\theta dx + \int_{\partial\Omega} \int_0^1 \mathcal{N}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\theta d\mathcal{H}^{n-1} \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ G(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q G(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \\ &= \dots \\ &= \int_\Omega \int_0^1 \mathcal{L}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\theta dx + \int_{\partial\Omega} \int_0^1 \mathcal{N}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\theta d\mathcal{H}^{n-1} \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ G(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q G(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \\ &= \dots \\ &= \int_\Omega \int_0^1 \mathcal{L}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\theta dx + \int_{\partial\Omega} \int_0^1 \mathcal{N}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\theta d\mathcal{H}^{n-1} \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ \mathcal{G}(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q \mathcal{G}(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \\ &= \dots \\ &= \int_0^1 \int_\Omega \mathcal{L}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) dx d\theta + \int_0^1 \int_{\partial\Omega} \mathcal{N}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\mathcal{H}^{n-1} d\theta \\ &= \int_\Omega \int_{u(x)}^{w(x)} \mathcal{L}(u^t) \Big|_{t=t(x, \lambda)} d\lambda dx + \int_{\partial\Omega} \int_{u(x)}^{w(x)} \mathcal{N}(u^t) \Big|_{t=t(x, \lambda)} d\lambda d\mathcal{H}^{n-1} \end{aligned}$$

Ideas of the Proof

Given any w we define $\zeta = u - w$ and $w_\theta = (1 - \theta)w + \theta u$

$$\begin{aligned} \mathcal{C}(w) - \mathcal{E}(u) &= \mathcal{C}(w) - \mathcal{C}(u) \\ &= \int_0^1 \frac{d}{d\theta} \mathcal{C}(w_\theta) d\theta \\ &= \int_0^1 \int_\Omega \frac{d}{d\theta} \left\{ \mathcal{G}(x, u^t, \nabla u^t) \Big|_{t=t(x, w_\theta(x))} \right\} dx d\theta \\ &\quad + \int_0^1 \int_\Omega \frac{d}{d\theta} \left(\left\{ \partial_q \mathcal{G}(x, u^t, \nabla u^t) (\nabla w_\theta - \nabla u^t) \right\} \Big|_{t=t(x, w_\theta(x))} \right) dx d\theta \\ &= \dots \\ &= \int_0^1 \int_\Omega \mathcal{L}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) dx d\theta + \int_0^1 \int_{\partial\Omega} \mathcal{N}(u^t) \Big|_{t=t(x, w_\theta(x))} \zeta(x) d\mathcal{H}^{n-1} d\theta \\ &= \int_\Omega \int_{u(x)}^{w(x)} \mathcal{L}(u^t) \Big|_{t=t(x, \lambda)} d\lambda dx + \int_{\partial\Omega} \int_{u(x)}^{w(x)} \mathcal{N}(u^t) \Big|_{t=t(x, \lambda)} d\lambda d\mathcal{H}^{n-1} \end{aligned}$$

$$u^{t(x, \lambda)}(x) = \lambda \implies \nabla u^{t(x, \lambda)}(x) \partial_\lambda t(x, \lambda) + \nabla t(x, \lambda) = 0$$

Summary of questions

Given an **extremal** u of the functional

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

Summary of questions

Given an extremal u of the functional

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

When is u a **minimizer** of \mathcal{E} among functions with the same boundary data?

Summary of questions

Given an extremal u of the functional

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

When is u a minimizer of \mathcal{E} among functions with the same boundary data?



When is it possible to find a **calibration** functional \mathcal{C} for u and \mathcal{E} ?

Summary of questions

Given an extremal u of the functional

$$\mathcal{E}(w) = \int_{\Omega} G(x, w(x), \nabla w(x)) \, dx$$

When is u a **minimizer** of \mathcal{E} among functions with the same boundary data?



When is it possible to find a calibration functional \mathcal{C} for u and \mathcal{E} ?



When can u be **embedded** into an **extremal field** for \mathcal{E} ?

$$\mathcal{E}(w) = \int_{\Omega} G(x, \nabla w(x)) \, dx$$

Vertical invariance

$$\mathcal{E}(w) = \int_{\Omega} G(x, \nabla w(x)) \, dx$$



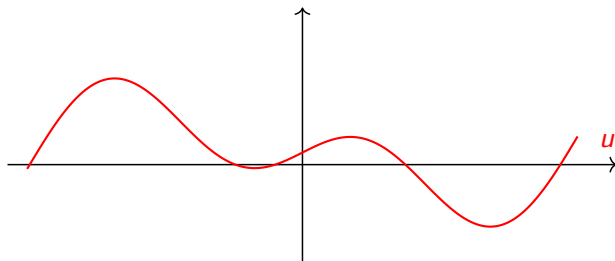
$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u + t) = 0$$

Vertical invariance

$$\mathcal{E}(w) = \int_{\Omega} G(x, \nabla w(x)) \, dx$$

⇓

$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u + t) = 0$$

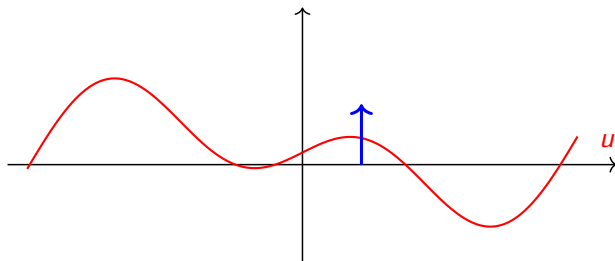


Vertical invariance

$$\mathcal{E}(w) = \int_{\Omega} G(x, \nabla w(x)) dx$$

⇓

$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u + t) = 0$$

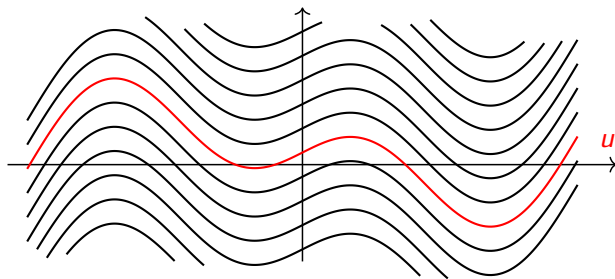


Vertical invariance

$$\mathcal{E}(w) = \int_{\Omega} G(x, \nabla w(x)) dx$$

⇓

$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u + t) = 0$$

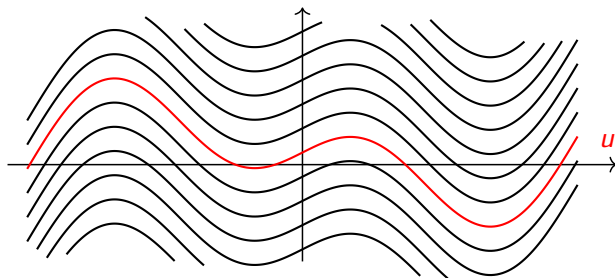


Vertical invariance

$$\mathcal{E}(w) = \int_{\Omega} G(x, \nabla w(x)) dx$$

⇓

$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u + t) = 0$$



$$u^t(x) := u(x) + t$$

$$\mathcal{E}(w) = \int_{\Omega} G(w(x), \nabla w(x)) \, dx$$

Monotonicity and horizontal invariance

$$\mathcal{E}(w) = \int_{\Omega} G(w(x), \nabla w(x)) \, dx$$

↓

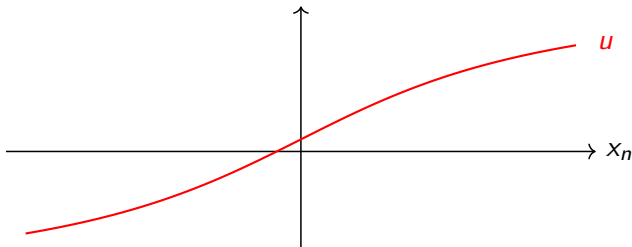
$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u(\cdot + te)) = 0$$

Monotonicity and horizontal invariance

$$\mathcal{E}(w) = \int_{\Omega} G(w(x), \nabla w(x)) \, dx$$

⇓

$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u(\cdot + te)) = 0$$

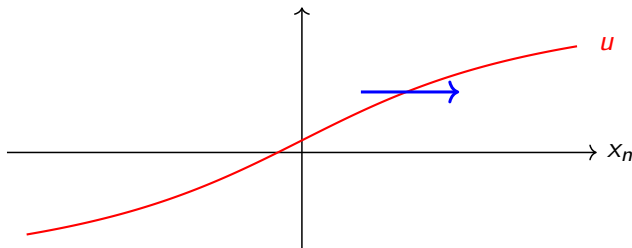


Monotonicity and horizontal invariance

$$\mathcal{E}(w) = \int_{\Omega} G(w(x), \nabla w(x)) \, dx$$

⇓

$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u(\cdot + te)) = 0$$

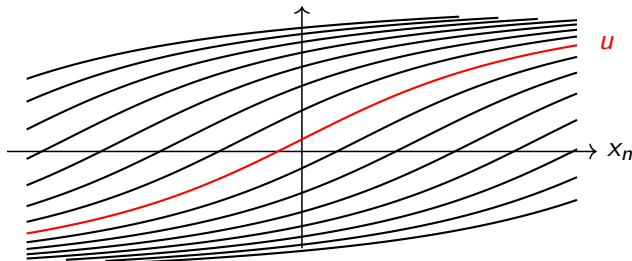


Monotonicity and horizontal invariance

$$\mathcal{E}(w) = \int_{\Omega} G(w(x), \nabla w(x)) \, dx$$

⇓

$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u(\cdot + te)) = 0$$

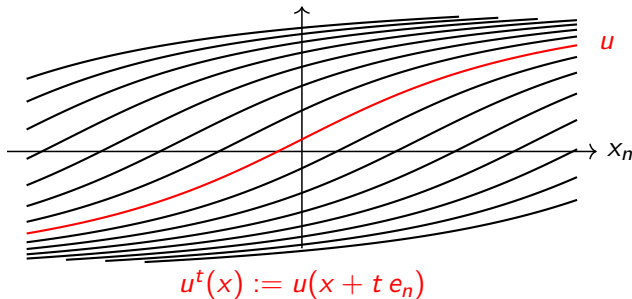


Monotonicity and horizontal invariance

$$\mathcal{E}(w) = \int_{\Omega} G(w(x), \nabla w(x)) \, dx$$

⇓

$$\mathcal{L}(u) = 0 \implies \mathcal{L}(u(\cdot + te)) = 0$$



When can u be embedded into an extremal field for \mathcal{E} ?

- If there is translation invariance ✓

When can u be embedded into an extremal field for \mathcal{E} ?

- If there is translation invariance
- In problems with many **symmetries**

When can u be embedded into an extremal field for \mathcal{E} ?

- If there is translation invariance
- In problems with many symmetries \rightsquigarrow **Simons Cone**

When can u be **embedded** into an **extremal field** for \mathcal{E} ?

- If there is translation invariance
- In problems with many symmetries \rightsquigarrow Simons Cone
- In many cases if we only want a **local embedding**

When can u be embedded into an extremal field for \mathcal{E} ?

- If there is translation invariance
- In problems with many symmetries \rightsquigarrow Simons Cone
- In many cases if we only want a local embedding \rightsquigarrow Viscosity theory

Nonlocal theory of Calibrations

$$\mathcal{E}_N(w) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} G_N(x, y, w(x), w(y)) \, dx \, dy$$

Nonlocal calibration

$$\mathcal{E}_N(w) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} G_N(x, y, w(x), w(y)) \, dx \, dy$$

Theorem [Cabr , Erneta & F-N '22]

Let u be embedded in an **extremal field** $\{u^t\}_{t \in \mathbb{R}}$ such that $(x, t) \rightarrow u^t(x)$ is a bounded C^2 function. Assume that G_N is a function of (x, y, a, b) satisfying $\partial_{ab}^2 G_N \leq 0$ and let \mathcal{C}_N be the functional

$$\begin{aligned} \mathcal{C}_N(w) = & \iint_{(\Omega^c \times \Omega^c)^c} \int_{u(x)}^{w(x)} \partial_a G_N(x, y, u^t(x), u^t(y)) \Big|_{t=t(x,\lambda)} \, d\lambda \, dx \, dy \\ & + \mathcal{E}_N(u) \end{aligned}$$

Then, it follows that \mathcal{C}_N is a **calibration** for the functional \mathcal{E}_N and u .

Nonlocal calibration

$$\mathcal{E}_N(w) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} G_N(x, y, w(x), w(y)) \, dx \, dy$$

Theorem [Cabré, Erneta & F-N '22]

Let u be embedded in an **extremal field** $\{u^t\}_{t \in \mathbb{R}}$ such that $(x, t) \rightarrow u^t(x)$ is a bounded C^2 function. Assume that G_N is a function of (x, y, a, b) satisfying $\partial_{ab}^2 G_N \leq 0$ and let \mathcal{C}_N be the functional

$$\begin{aligned} \mathcal{C}_N(w) = & \iint_{(\Omega^c \times \Omega^c)^c} \int_{u(x)}^{w(x)} \partial_a G_N(x, y, u^t(x), u^t(y)) \Big|_{t=t(x, \lambda)} \, d\lambda \, dx \, dy \\ & + \mathcal{E}_N(u) \end{aligned}$$

Then, it follows that \mathcal{C}_N is a calibration for the functional \mathcal{E}_N and u .

The condition $\partial_{ab}^2 G_N \leq 0$ turns to be the natural ellipticity condition for the problem.

Examples of nonlocal Lagrangians

- The case

$$G_N(x, y, a, b) = \frac{|a - b|^p}{2p|x - y|^{n+ps}} - \frac{1}{2|\Omega|} \mathbb{1}_{\Omega \times \Omega}(x, y)(F(a) + F(b))$$

comes from equations driven by **fractional p -Laplacian**.

Examples of nonlocal Lagrangians

- The case

$$G_N(x, y, a, b) = \frac{|a - b|^p}{2p|x - y|^{n+ps}} - \frac{1}{2|\Omega|} \mathbb{1}_{\Omega \times \Omega}(x, y)(F(a) + F(b))$$

comes from equations driven by fractional p -Laplacian.

- The Lagrangian

$$G_N(x, y, a, b) = \frac{G\left(\frac{a-b}{|x-y|}\right)}{|x-y|^{n+s-1}},$$

where $s \in (0, 1)$,

$$G''(\tau) = \frac{1}{(1 + \tau^2)^{\frac{n+s+1}{2}}}, \quad \text{and} \quad G'(0) = G(0) = 0,$$

gives rise to the **fractional perimeter for subgraphs**.

Examples of nonlocal Lagrangians

- The case

$$G_N(x, y, a, b) = \frac{|a - b|^p}{2p|x - y|^{n+ps}} - \frac{1}{2|\Omega|} \mathbb{1}_{\Omega \times \Omega}(x, y)(F(a) + F(b))$$

comes from equations driven by fractional p -Laplacian.

- The Lagrangian

$$G_N(x, y, a, b) = \frac{G\left(\frac{a-b}{|x-y|}\right)}{|x-y|^{n+s-1}},$$

where $s \in (0, 1)$,

$$G''(\tau) = \frac{1}{(1 + \tau^2)^{\frac{n+s+1}{2}}}, \quad \text{and} \quad G'(0) = G(0) = 0,$$

gives rise to the fractional perimeter for subgraphs.

- The case

$$G_N(x, y, a, b) = -\mathbb{1}_{\Omega \times \Omega}(x, y)K(x - y)ab$$

corresponds to **convolution-type energies**.

Thank You