# THE NEUMANN PROBLEM FOR THE FRACTIONAL LAPLACIAN: BOUNDARY REGULARITY


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joint work with A. Audrito (ETH Zurich) and X. Ros-Oton (ICREA-UB)

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PDEs Session

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- Neumann Problem: Minimize $\mathcal{E}(u)$ among all functions $u \in H^1(\Omega)$.

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# Regularity for the local case

- Interior regularity: $-\Delta u = f$ in $B_1$, with $f \in L^\infty(B_1)$

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- Boundary regularity: $-\Delta u = f$ in $\Omega$, with $f \in L^\infty(\Omega)$ and $\partial\Omega \in C^2$

  The main idea is to flatten the boundary and apply interior estimates to odd/even reflections (depending on the boundary conditions) of $u$ through the boundary.

# The fractional Laplacian

Defined for $s \in (0,1)$ by

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, \mathrm{d}z.$$

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- It has an associated local extension problem.

# The Dirichlet problem for the fractional Laplacian

We consider the nonlocal energy functional

$$\mathcal{E}(u) := \frac{c_{n,s}}{4} \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z - \int_\Omega f\,u$$

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This problem has a nice probabilistic interpretation in terms of Levy flights.

# Regularity of the fractional Dirichlet problem

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*Assume $\partial\Omega \in C^2$ and $f \in L^\infty(\Omega)$. Then, $u \in C^s(\overline{\Omega})$.*

This regularity result is optimal since explicit solutions can be constructed:

$$\left.\begin{array}{rcl} (-\Delta)^s u &=& 1 \quad \text{in } B_1, \\ u &=& 0 \quad \text{in } B_1^c. \end{array}\right\} \Longrightarrow u(x) = c(1-|x|^2)_+^s$$

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$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \Omega^c, \end{cases}$$

where $\mathcal{N}_s$ is a "nonlocal Neumann derivative" defined as

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Existence and uniqueness of weak solutions can be found in [Dipierro, Ros-Oton and Valdinoci, 2017] ([Du-Gunzburger-Lehoucq-Zhou,'13]).

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- Conservation of mass in the associated heat equation
- Convergence to the classical Neumann problem as $s \uparrow 1$

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$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \Omega^c. \end{cases}$$

**Theorem (Audrito, F-N, Ros-Oton, '20)**

*Let $\Omega \subset \mathbb{R}^N$ be any bounded $C^1$ domain and $f \in L^\infty(\Omega)$ and $\int_\Omega f = 0$. Then, weak solutions are bounded, and*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C\left(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

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## Theorem (Audrito, F-N, Ros-Oton, '20)

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This is the first regularity result for the Neumann problem. Even the boundedness of solutions is new.

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- Classical methods do not work (flatten the boundary and even reflection)
- The values of $u$ are not known neither inside nor outside $\Omega$
- The methods from the Dirichlet case, based on barriers, do not work either

# Ideas of the proof

- We find an equivalent "localized" formulation of the problem

$$\overline{\mathcal{E}}(u) := \frac{c_{n,s}}{4} \iint_{\Omega\times\Omega} |u(x)-u(z)|^2 \, K_\Omega(x,z) \, \mathrm{d}x \, \mathrm{d}z - \int_\Omega f\, u$$

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with $K_\Omega$ satisfying

$$K_\Omega(x,z) \asymp \frac{1 + \log^-\left(\frac{\min\{d(x),d(z)\}}{|x-z|}\right)}{|x-z|^{n+2s}} \qquad \text{for all} \quad x,z \in \Omega,$$

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$$(-\Delta)^s u(x) = c_{n,s} \int_{\color{red}\Omega} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\, \mathrm{d}z + c_{n,s} \int_{\color{red}\Omega^c} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\, \mathrm{d}z$$

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# Thank You