LIOUVILLE TYPE RESULTS FOR FRACTIONAL SCHRÖDINGER OPERATORS IN 1D

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Integro-differential operators

Definition 1 (Integro-differential operator)

We say that the operator L is integro-differential if it is of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(u(x+y) - u(x) - y \cdot \nabla u(x)\, \chi_{B_1}(y)\right) K(x,y)\, dy,$$

with $K \geq 0$.

- Translation invariant: $K(x,y) \equiv K(y)$
- Symmetric: $K(x,-y) = K(x,y)$

$$Lu(x) = P.V. \int_{\mathbb{R}^n} \left(u(x+y) - u(x)\right) K(y)\, dy.$$

The fractional Laplacian

Fractional Laplacian: canonical example of integro-differential operator.

$$K(y) = |y|^{-n-2s}$$

It corresponds to radially symmetric Lévy process of order $2s$.

Proposition 1

$$(-\Delta)^s u = \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)\right).$$

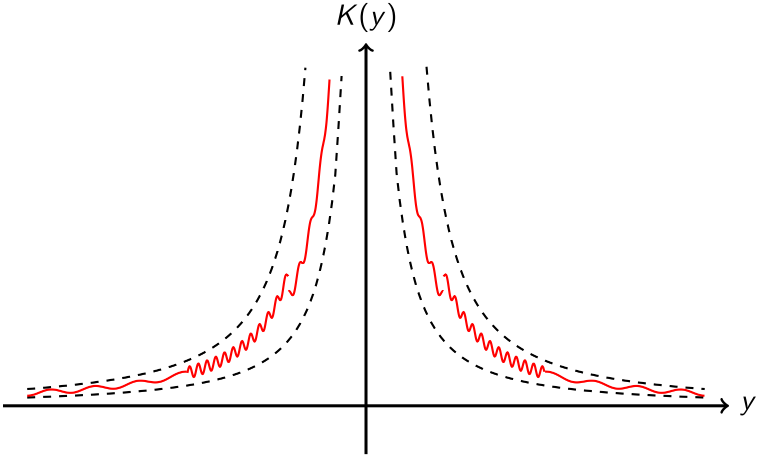
Definition 2 (Elipticity class $\mathcal{L}_0$)

We say that an integro-differential operator belongs to the ellipticity class $\mathcal{L}_0(s)$ if

$$\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}},$$

with $\Lambda \geq \lambda > 0$.

Elipticity class $\mathcal{L}_0$



An extension problem for the fractional Laplacian

Definition 3 (a-harmonic extension)

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and $a \in (-1,1)$ a real parameter. We define its *a-harmonic extension* U as the solution of the problem

$$\begin{cases} div(y^a \nabla U) = 0, & \text{in } \mathbb{R}^n \times (0,+\infty), \\ U(x,0) = u(x), & \text{in } \mathbb{R}^n. \end{cases}$$

Theorem 1 (Caffarelli & Silvestre 2007)

Let be $u : \mathbb{R}^n \to \mathbb{R}$ a smooth function, $a \in (-1,1)$ a real parameter, and U its a-harmonic extension, then we have

$$C_{n,s}\,(-\Delta)^s\,u = -\lim_{y\to 0^+} y^a\,U_y(x,y),$$

with $a = 1-2s$, and $C_{n,s} > 0$ only depending on n and s.

Schrödinger operators

Definition 4 (Classical Schrödinger operator)

We say that $\mathcal{L}$ is a Schrödinger operator if it is of the form

$$\mathcal{L}u(x) = (-\Delta)u(x) - V(x)\,u(x),$$

for any function $V(x)$, which is called the potential of the operator.

Schrödinger equation for quantum mechanical particles:

$$i\frac{\partial}{\partial t}\psi = \mathcal{L}\psi,$$

with $F(x) = \nabla V(X)$ the magnetic field.

Fractional Schrödinger operator: $(-\Delta) \longrightarrow (-\Delta)^s$

The problem

A Liouville type result is of the form:

Given a linear operator $\mathcal{L}$ and two bounded solutions $w, \tilde{w}$ of the equation

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^n,$$

with $w > 0$. Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

We focus on the case $\mathcal{L} = L - V(x)$ with

$$L\, u(x) = \int_{\mathbb{R}^n} \{u(x+y) - u(x)\}\; K(y)\, dy.$$

Motivation I

Conjecture 1 [E. De Giorgi 1978]

Let $u \in C^2(\mathbb{R}^n)$ be a solution of the Allen-Cahn equation

$$-\Delta u = u - u^3 \qquad \text{in } \mathbb{R}^n$$

such that

$$|u| \leq 1 \text{ and } \partial_{x_n} u > 0$$

in the whole $\mathbb{R}^n$. Then, all level sets $\{u = \lambda\}$ of u are hyperplanes, at least if $n \leq 8$. Equivalently, u is a $1D$ solution, that is, a function depending only on one Euclidean variable.

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- $n = 2$ [Ghoussoub & Gui, 1998]
- $n = 3$ [Ambrosio & Cabré, 2000]
- $4 \leq n \leq 8$ [Savin, 2009] with $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$
- $n \geq 9$ Counterexample [del Pino, Kowalczyk & Wei, 2011]

Motivation I

- The conjecture is equivalent to proving that there exist a vector $c \in \mathbb{R}^n$ and a scalar function $v : \mathbb{R}^n \to \mathbb{R}$ such that

$$\nabla u = c\, v(x) \quad \text{for all } \; x \in \mathbb{R}^n.$$

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- If we call $f(u) = u - u^3$ and differentiate the Allen-Cahn equation

$$-\Delta u_i - f'(u)\, u_i = 0 \quad \text{for all } i = 1, ..., n.$$

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- Let be

$$\mathcal{L} := -\Delta - f'(u(x)),$$

therefore

$$\boxed{\text{A Liouville type result for } \mathcal{L} \Longrightarrow \text{The conjecture is true}}$$

Motivation II

Definition 5 (Nondegeneracy)

We say that u is a nondegenerate solution of the semilinear equation

$$Lu = f(u) \ \text{ in } \mathbb{R}^n$$

if it is a solution, and the linearized operator

$$\mathcal{L} = L - f'(u)$$

is such that

$$\ker(\mathcal{L}) = \mathrm{span}(u_{x_1}, \ldots, u_{x_n}).$$

- Stability of stationary solutions in time dependent problems
- Application of implicit function arguments

Some Liouville type results for Schrödinger operators

$$\boxed{\mathcal{L} = L - V(x)}$$

- Ambrosio & Cabré, 2000

$$L = -\Delta \quad \text{and} \quad n \leq 2$$

$$\sigma = \frac{\tilde{w}}{w} \implies \operatorname{div}(w^2 \nabla \sigma) = 0 \text{ in } \mathbb{R}^n.$$

- Cabré & Sire, 2014

$$L = (-\Delta)^s \text{ with } s \in [1/2, 1) \quad \text{and} \quad n = 1$$

$$\sigma = \frac{\tilde{w}}{w} \implies \begin{cases} \operatorname{div}(w^2 y^a \nabla \sigma) = 0 \text{ in } \mathbb{R}^n \times (0, +\infty), \\ \lim_{y \to 0^+} y^a U_y(x, y) = 0 \text{ in } \mathbb{R}^n. \end{cases}$$

Some Liouville type results for Schrödinger operators

$$\boxed{\mathcal{L} = L - V(x)}$$

- Hamel, Ros-Oton, Sire & Valdinoci, 2016

L integro-differential operator in $\mathbb{R}^2$ s.t. $\begin{cases} \text{- Harnack Inequality} \\ \text{- } K \text{ compact support} \end{cases}$

$$\sigma = \frac{\tilde{w}}{w}$$

$$\Downarrow$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [\sigma(x) - \sigma(y)]^2 \left[\tau^2(x) + \tau^2(y)\right] \, w(x)\, w(y)\, K(x-y)\, dx\, dy =$$

$$= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left[\sigma^2(x) - \sigma^2(y)\right] \left[\tau^2(x) - \tau^2(y)\right] w(x)\, w(y)\, K(x-y)\, dx\, dy.$$

New results

Theorem 2 (Cabré & F-N.)

Let $L \in \mathcal{L}_0(s)$, with $s \in [1/2, 1)$. Assume that the potential function $V \in C^\beta(\mathbb{R})$ and satisfies

$$V \leq -b < 0 \quad \text{in} \quad \mathbb{R} \setminus [-M, M]$$

for some positive constants b, M and β.
Let $w, \widetilde{w}$ be two solutions of the linear equation

$$Lu - V(x)u = 0 \quad \text{in } \mathbb{R},$$

with $w, \widetilde{w} \in L^\infty(\mathbb{R})$ and $w > 0$. Then

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with $w, \widetilde{w} \in L^{\infty}(\mathbb{R})$ and $w > 0$. Then

$$\frac{\widetilde{w}}{w} \equiv ctt.$$

Analogous result if w and $\tilde{w}$ are odd with $w > 0$ in $(0, +\infty)$.

Ideas of the proof

- A bound for σ via a Maximum Principle:

$$\begin{cases} \mathcal{L}\varphi & \geq & 0 \text{ in } \mathbb{R}\setminus[-M,M] \\ \varphi & \geq & 0 \text{ in } [-M,M] \end{cases} \Longrightarrow \varphi \geq 0 \text{ in } \mathbb{R}.$$

Then,

$$\varphi = Cw \pm \tilde{w} \Longrightarrow |\sigma| \leq C \text{ in } \mathbb{R}.$$

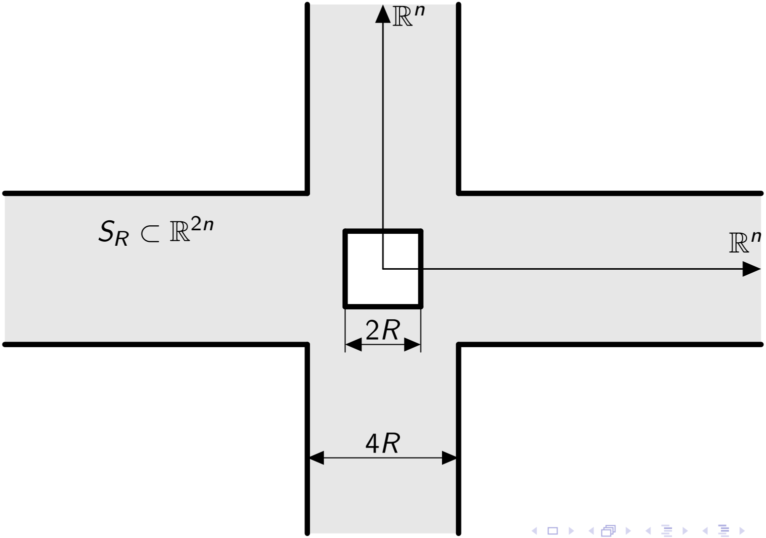
- Integrability result for the kernel:

$$\iint_{S_R} \min\left\{\frac{|x-y|^2}{R^2}, 1\right\} K(x-y)\,dx\,dy \leq C,$$

with C a positive constant independent of R.

Ideas of the proof

$$S_R = [(B_{2R} \times \mathbb{R}^n) \cup (\mathbb{R}^n \times B_{2R})] \setminus (B_R \times B_R)$$



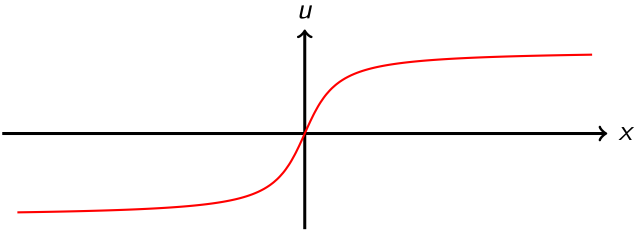
New results

Corollary 1

Let L be an integral operator in the ellipticity class $\mathcal{L}_0(s)$, with $s \in [1/2, 1)$, and u be a layer solution of

$$Lu - f(u) = 0 \quad \text{in } \mathbb{R},$$

with f of bistable type. Then, u is nondegenerate.



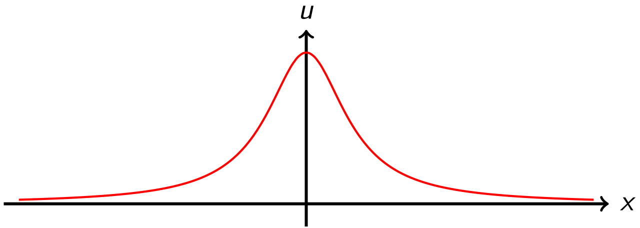
New results

Corollary 2

Let L be an integral operator in the ellipticity class $\mathcal{L}_0(s)$, with $s \in [1/2, 1)$, K decreasing for almost every point in $(0, +\infty)$ and u be a ground state solution of

$$Lu - f(u) = 0 \quad \text{in } \mathbb{R},$$

with f such that $f(0) = 0$ and $f'(0) < 0$. Then, u is nondegenerate.



Thank You