THE NEUMANN PROBLEM FOR THE FRACTIONAL LAPLACIAN: BOUNDARY REGULARITY

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joint work with A. Audrito (ETH Zurich) and X. Ros-Oton (ICREA-UB)

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- Interior regularity: $-\Delta u = f$ in B_1, with $f \in L^\infty(B_1)$

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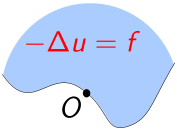

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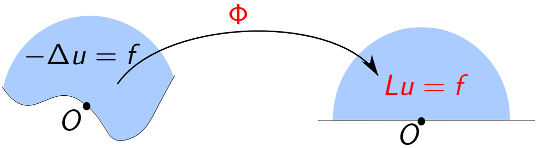

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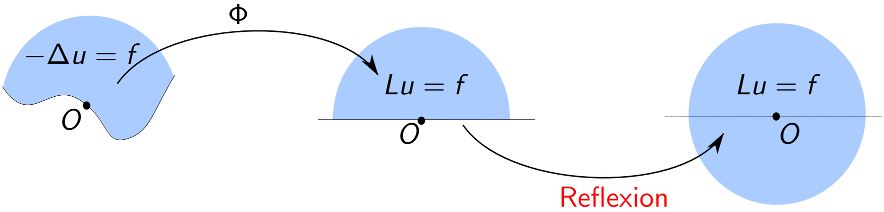

The fractional Laplacian

Defined for $s \in (0,1)$ by

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- It has an associated local extension problem.

Functional setting

When working with the fractional Laplacian in bounded domains, the natural functional space to work is

$$\mathcal{H}^s(\Omega) = \{u \in L^2(\Omega) \ \text{ such that } [u]_{\mathcal{H}^s(\Omega)} < +\infty\},$$

where

$$[u]_{\mathcal{H}^s(\Omega)} := \left(\int_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}z \right)^{1/2},$$

and $\mathcal{H}^s(\Omega)$ is a Banach space with associated norm

$$||u||_{\mathcal{H}^s(\Omega)} = ||u||_{L^2(\Omega)} + [u]_{\mathcal{H}^s(\Omega)}$$

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This problem has a nice probabilistic interpretation in terms of Levy flights.

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Assume $\partial\Omega \in C^2$ and $f \in L^\infty(\Omega)$. Then, $u \in C^s(\overline{\Omega})$.

This regularity result is optimal since explicit solutions can be constructed:

$$\left.\begin{array}{rcl} (-\Delta)^s u &=& 1 \quad \text{in } B_1, \\ u &=& 0 \quad \text{in } B_1^c. \end{array}\right\} \Longrightarrow u(x) = c(1-|x|^2)_+^s$$

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where $\mathcal{N}_s$ is a "nonlocal Neumann derivative" defined as

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Existence and uniqueness of weak solutions can be found in [Dipierro, Ros-Oton and Valdinoci, 2017] ([Du-Gunzburger-Lehoucq-Zhou,'13])

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- Convergence to the classical Neumann problem as $s \uparrow 1$

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Theorem (Audrito, F-N, Ros-Oton, '20)

Let $\Omega \subset \mathbb{R}^N$ be any bounded C^1 domain and $f \in L^\infty(\Omega)$ and $\int_\Omega f = 0$. Then, weak solutions are bounded, and

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This is the first regularity result for the Neumann problem. Even the boundedness of solutions is new.

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- Classical methods do not work (flatten the boundary and even reflection)
- The values of u are not known neither inside nor outside Ω
- The methods from the Dirichlet case, based on barriers, do not work either

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The trick is that w is not regular enough to pass from strong to weak sense.

Equivalent formulation

It is equivalent to minimize the nonlocal energy

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than minimizing the "localized" one

$$\overline{\mathcal{E}}(u) := \frac{c_{n,s}}{4} \iint_{\Omega \times \Omega} |u(x) - u(z)|^2 \, K_\Omega(x, z) \, \mathrm{d}x \, \mathrm{d}z - \int_\Omega f \, u$$

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where K_Ω is an explicit kernel satisfying

$$K_\Omega(x,z) \asymp \frac{1+\log^-\left(\frac{\min\{d(x),d(z)\}}{|x-z|}\right)}{|x-z|^{n+2s}} \qquad \text{for all} \quad x,z\in\Omega,$$

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It was first noted in the strong form by [Abatangelo, '20].

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$$u(z) = \left(\int_\Omega \frac{u(y)}{|z-y|^{n+2s}}\,\mathrm{d}y\right)\left(\int_\Omega \frac{\mathrm{d}y}{|z-y|^{n+2s}}\right)^{-1} \text{ for all } z \in \Omega^c$$

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The regional fractional Laplacian

Motivated by the equivalent formulation, we first study the minimization of the functional

$$\widetilde{\mathcal{E}}(u) := \frac{c_{n,s}}{4} \iint_{\Omega\times\Omega} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}} \, \mathrm{d}x \, \mathrm{d}z - \int_\Omega f\, u$$

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In a more recent work, M. Fall proves $C^{2s-\varepsilon}$ regularity up to the boundary by using the extension problem.

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$$\Big\Downarrow \quad \varphi \approx u^{-p} \quad + \quad \text{J-N Ineq.}$$

$$u \in C^\alpha(\overline{\Omega}) \text{ for some small } \alpha$$

Higher regularity (when $s > 1/2$)

We want to prove that

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Thus, we arrive at a contradiction if we show that a weak solution with certain growth in a half-space is constant.

From dimension n to dimension 1

$$\left.\begin{array}{rcll} (-\Delta)^s u & = & 0 & \text{in } \mathbb{R}^n_+, \\ \mathcal{N}_s u & = & 0 & \text{in } \mathbb{R}^n_-, \\ ||u||_{L^\infty(B_R)} & \leq & R^\sigma. & \end{array}\right\} \Longrightarrow u \equiv w(x_n)$$

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- We construct the auxiliary function

$$u_1(x) = \frac{u(x+he)-u(x)}{|h|^\alpha} \text{ for any given } e = (\tilde{e}, 0) \in S^{n-1} \text{ and } h > 0.$$

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- Since it grows less than linear, it should be constant in that directions

1D Liouville theorem

We want to prove

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- We conclude that $w \equiv 0$.

Thank You