ADVANCED COURSE IN PARTIAL DIFFERENTIAL EQUATIONS

Problems, solutions, and exams

Master in Advanced Mathematics
and Mathematical Engineering
Facultat de Matemàtiques i Estadistica - UPC

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CONTENTS:

TRANSPORT EQUATION

Exercise 1. Find the solution of

$$\begin{cases} u_t + cu_x = -\gamma u & x > 0,\ t > 0 \\ u(0,t) = 1 & t > 0 \\ u(x,0) = 0 & x > 0. \end{cases}$$

Is it classical?

Exercise 2. Consider the transport equation

$$\begin{cases} u_t + cu_x = f(x,t) & x \in (0,R),\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(x,0) = 0 & x \in (0,R), \end{cases}$$

where $c, R > 0$ are given constants. Prove the stability estimate:

$$\int_0^R u^2(x,t)\,dx \le e^t \int_0^t \left(\int_0^R f^2(x,s)\,dx \right) ds, \quad \text{for any } t > 0.$$

Hint: Multiply the equation by u, use $c > 0$ and $2fu \le f^2 + u^2$, to arrive at

$$\frac{d}{dt}\int_0^R u^2(x,t)\,dx \le \int_0^R f^2(x,t)\,dx + \int_0^R u^2(x,t)\,dx.$$

Exercise 3. We say that $u \in L^1_{\text{loc}}(\mathbb{R} \times [0,\infty))$, is a weak solution of

$$\text{(1)} \qquad \begin{cases} u_t + cu_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g & x \in \mathbb{R} \end{cases}$$

if

$$\int_0^\infty dt \int_{\mathbb{R}} u(v_t + cv_x)\,dx = -\int_{\mathbb{R}} g(x)v(x,0)\,dx$$

is satisfied for all functions $v \in C_c^\infty\big(\mathbb{R} \times [0,\infty)\big)$.

(a) Prove that if $g \in L^1_{\text{loc}}(\mathbb{R})$, then $u(x,t) = g(x - ct)$ is a weak solution of (1).

(b) Given $g \in C^0(\mathbb{R})$, prove that if u is a continuous weak solution of (1) then $u(x,t) = g(x - ct)$.

WAVE EQUATION

Exercise 4. Consider the problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in (0,\pi), t \in \mathbb{R} \\ u(0,t) = u(\pi,t) = 1 & t \in \mathbb{R} \\ u(x,0) = g(x) & x \in (0,\pi) \\ u_t(x,0) = h(x) & x \in (0,\pi). \end{cases}$$

(a) Give necessary conditions on g and h making the solution u belong to $C^2(\{x \geq 0, t \geq 0\})$.
(b) Apply d'Alembert's formula to compute $u(\pi/4, \pi/4)$.
(c) Solve the problem by using separation of variables when $g \equiv 1$ and $h = \sin^2 x$.
(d) Use the previous parts to compute

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2\{4-(2k+1)^2\}}$$

Exercise 5. Let Ω be a C^∞ bounded domain in $\mathbb{R}^n$. Consider the problem

$$(2) \qquad \begin{cases} u_{tt} - \Delta u + m^2 u = 0 & x \in \Omega,\ t > 0 \\ u(x,t) = 0 & x \in \partial\Omega,\ t > 0 \\ u(x,0) = f(x) & x \in \Omega \\ u_t(x,0) = g(x) & x \in \Omega. \end{cases}$$

(a) Which quantity is preserved during time? Specify it in terms of f and g.
(b) Set $\Omega = (0, L)$, $f(x) \equiv 0$ and $g \in C^1([0, L])$. Using separation of variables, show that the solution of (2) tends, when $m \to 0$, to the solution of the wave equation in $(0, L)$ with homogeneous Dirichlet boundary conditions and the same initial conditions as in (2).

Exercise 6. We have a solution of the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in (0, L),\ t > 0 \\ u(0,t) = u(L,t) = 0 & t > 0 \\ u(x,0) = g(x) & x \in (0, L) \\ u_t(x,0) = 0 & x \in (0, L). \end{cases}$$

(a) Is it true that, if $g \geq 0$ then $u \geq 0$?
(b) Same question, with zero Neumann boundary conditions instead of Dirichlet conditions.

DIFFUSION EQUATION

Exercise 7. Let u be a solution of the problem

$$\begin{cases} u_t - u_{xx} = 0 & x \in (0, \pi),\ t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = g(x) & x \in (0, \pi), \end{cases}$$

where $g : [0, \pi] \to \mathbb{R}$ is C^1 and satisfies $g'(0) = g'(\pi) = 0$. Moreover we know that $\max_{[0,\pi]} g = 2$.

(a) Study the sign of $u(x,t) - 2$ and describe when it is $> 0, = 0, < 0$. Justify your answer.
(b) Compute $\int_0^\pi u(x,t)\,dx$ as a function of t.
(c) Given $x \in (0, \pi)$, discuss the existence and the value of $\lim_{t\to\infty} u(x,t)$. Describe how fast it tends to its limit, i.e. its rate of convergence.

Exercise 8. Let u be a solution for the problem

$$\begin{cases} u_t - u_{xx} = 0 & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = x(1-x) & x \in (0,1). \end{cases}$$

After showing that u is non-negative, find two positive constants α, β such that

$$u(x,t) \le \alpha x(1-x)e^{-\beta t}.$$

Deduce that $u(x,t) \to 0$ uniformly in $[0,1]$ when $t \to +\infty$.

Exercise 9. Let u be the solution of the homogeneous diffusion equation in the interval $(0,\pi)$ with vanishing Dirichlet boundary conditions and with initial condition $g \in L^\infty\big((0,\pi)\big)$. Prove that, for all $t \ge 1$,

$$\|u(\cdot,t)\|_{L^\infty((0,\pi))} \le 4e^{-t}\|g\|_{L^\infty((0,\pi))}.$$

LAPLACE OPERATOR

Exercise 10. Let u be a harmonic function in an open set $\Omega \subseteq \mathbb{R}^n$. Show that:
(a) u^2 is subharmonic in Ω.
(b) If $\Omega = \mathbb{R}^n$ and $\int_{\mathbb{R}^n} u^2(x)\,dx < +\infty$, then $u \equiv 0$.

Exercise 11. Let u be a harmonic function in the whole of $\mathbb{R}^n$, such that

$$\|\nabla u\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx\right)^{1/2} < \infty.$$

Show that u is constant.

Exercise 12. Let u be a non-negative harmonic function in the ball $B_R \subset \mathbb{R}^n$ centered at the origin. Prove the validity of the following *Harnack inequality*:

$$\text{(HI)} \qquad R^{n-2}\frac{R-|x|}{(R+|x|)^{n-1}}u(0) \le u(x) \le R^{n-2}\frac{R+|x|}{(R-|x|)^{n-1}}u(0),$$

for any $x \in B_R$. Deduce that

$$\sup_{B_{R/2}} u \le 3^n \inf_{B_{R/2}} u.$$

Is the constant appearing in the last inequality optimal? [Hint: check carefully your proof of (HI). Where did you throw away the most?]

Exercise 13. Let $H \subset \mathbb{R}^n$ be a bounded, C^1 domain such that $\overline{B_\varepsilon} \subset H$, where B_ε denotes the open ball of radius ε, centered at the origin. Consider the set $\Omega = H \backslash \overline{B}_\varepsilon(0)$ and a solution $u \in C^2(\overline{\Omega})$ of

$$\begin{cases} \Delta u + a(x)u_{x_1} = 1 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -1 & \text{on } \partial H, \\ u = x_2 & \text{on } \partial B_\varepsilon(0), \end{cases}$$

where a is a continuous function in $\overline{\Omega}$ and ν is the exterior normal vector to Ω. Find the value of $\max_{\overline{\Omega}} u$.

Exercise 14. Let B_1 be the open unit ball in $\mathbb{R}^3$ and $A = \{v \in C^2(\overline{B_1}) : v_{|\partial B_1} \equiv 0\}$. Compute the value of

$$\min_{v \in A} \int_{B_1} \left\{ \frac{1}{2} |\nabla v(x)|^2 - 12|x| v(x) \right\} dx.$$

Exercise 15. Let $f(x, y) = x^2(x^2 + y^2)^{3/2}$.

(a) Compute a solution u to $-\Delta u = f$ in $\mathbb{R}^2$.

(b) Using (a), find the limit

$$\lim_{R \to +\infty} \frac{1}{R^7 \log R} \int_{B_R} \log\left(\sqrt{(x-1)^2 + y^2}\right) f(x, y)\, dx\, dy.$$

Exercise 16. Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\}$ and u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \{0\} \times (0, +\infty) \\ u = 1 & \text{on } (0, +\infty) \times \{0\}. \end{cases}$$

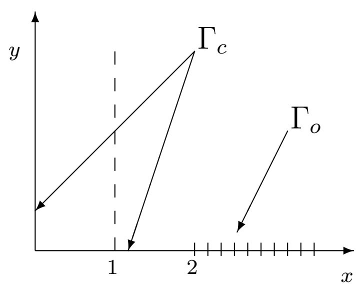
(a) Give the probabilistic interpretation of this problem. Deduce the value of u on the ray $\{x_1 = x_2 > 0\}$.

(b) Prove analytically the last assertion about the values of u on $\{x_1 = x_2 > 0\}$.

(c) Compute the Green function for the domain Ω. Deduce the expression of the Poisson kernel $P(x, y)$ of Ω at the points $x \in \Omega$ and $y \in \partial\Omega$ with $y_2 = 0$.

(d) Give an explicit expression for u in Ω.

Exercise 17. Consider the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$. Suppose that $\Gamma_o = \{(x, 0) \in \mathbb{R}^2 : x > 2\}$ is the open part on of the boundary of Ω and that $\Gamma_c = \{(x, 0) \in \mathbb{R}^2 : 0 \le x \le 2\} \cup \{(0, y) \in \mathbb{R}^2 : y > 0\}$ is the closed part of the boundary (look at the figure below). Calculate the y-coordinate so that the probability of exiting the domain starting a random walk at the point $(1, y)$ is maximum.



ISOPERIMETRIC INEQUALITY

Exercise 18. Let $u : [a, b] \longrightarrow \mathbb{R}$ be a C^1 function with $u(a) = u(b)$. Prove the so-called *Wirtinger inequality*

$$\int_a^b |u - c_u|^2 dt \le \left(\frac{b-a}{2\pi}\right)^2 \int_a^b |u'|^2 dt,$$

where $c_u = \frac{1}{b-a} \int_a^b u$. Characterize the functions for which the equality is attained.

Exercise 19. Let $\Omega \subset \mathbb{R}^2$ be any smooth bounded domain, and let $\Gamma = \partial\Omega$. The aim of this problem is to show the validity of the *isoperimetric inequality*

$$4\pi|\Omega| \leq |\partial\Omega|^2, \tag{3}$$

with equality if and only if Ω is a ball.

(a) Let $L = |\partial\Omega|$ and $\gamma : [0, L] \longrightarrow \mathbb{R}^2$ be the parametrization for Γ by arclength, that is $|\gamma'(t)| = 1$ for each $t \in [0, L]$. Let $x(t)$ and $y(t)$ denote the coordinate functions for γ, that is $\gamma(t) = (x(t), y(t))$. Show that

$$|\partial\Omega| = \int_0^L \left\{[x'(t)]^2 + [y'(t)]^2\right\} dt$$

and

$$|\Omega| = \left|\int_0^L x(t)y'(t)dt\right|.$$

(b) Using the Wirtinger inequality, show (3).

BANACH FIXED POINT THEOREM

Exercise 20. Consider the wave equation $u_{tt} - u_{xx} = f(x, t)$, for $x \in \mathbb{R}$ and $t > 0$.

(i) Compute the solution u with initial conditions $u(x, 0) = u_t(x, 0) = 0$, $x \in \mathbb{R}$.

(ii) Given $T > 0$, let $Q_T := \mathbb{R} \times (0, T)$. Prove that $\|u\|_{L^\infty(Q_T)} \leq \frac{T^2}{2}\|f\|_{L^\infty(Q_T)}$.

Consider now the nonlinear problem

$$\begin{cases} u_{tt} - u_{xx} = u^2(x, t) + g(x, t) & (x, t) \in Q_T, \\ u(x, 0) = u_t(x, 0) = 0 & x \in \mathbb{R}. \end{cases} \tag{4}$$

From (i) it follows that u solves this problem if and only if $u = N(u)$, where

$$N(u)(x, t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} (u^2 + g)(y, s)\, dy\, ds.$$

Let

$$X := \left\{u \in L^\infty(Q_T) \, : \, \|u\|_{L^\infty(Q_T)} \leq 1\right\}.$$

(iii) Show that, if $T < 1$ and $\|g\|_{L^\infty(Q_T)} \leq 1$, then N is a contraction from X into itself.

(iv) Prove that the Banach Fixed Point Theorem holds true in closed sets of complete metric spaces. Using this and the fact that $L^\infty(Q_T)$ is a complete metric space, deduce the existence of a soltuion $u \in L^\infty(Q_T)$ of the nonlinear problem (4)

Exercise 21. Consider the homogeneous heat equation $u_t - u_{xx} = 0$ in $(0, \pi)$ with vanishing Dirichlet conditions. Given an initial datum g, let $T_t g$ be the solution at time $t > 0$.

(a) Prove that

$$\|T_t g\|_{L^2(0,\pi)} \leq \|g\|_{L^2(0,\pi)}.$$

We now want to solve nonlinear problem

$$(5) \qquad \begin{cases} u_t - u_{xx} = 1 + f(u) & x \in (0,\pi),\ t \in (0,T), \\ u(0,t) = u(\pi,t) = 0 & t \in (0,T), \\ u(x,0) = 0 & x \in (0,\pi), \end{cases}$$

for some fixed $T > 0$ and with $f(u) = \frac{u}{1+u^2}$. (Notice that the function $u = 0$ is not a solution.)

(*b*) Show that

$$\|f(v) - f(w)\|_{L^2(0,\pi)} \leq \|v - w\|_{L^2(0,\pi)},$$

for any two functions $v, w \in L^2(0,\pi)$.

(*c*) Use Duhamel's formula to rewrite the nonlinear problem as a fixed-point problem. Work in the closed unit ball of the complete normed space

$$X = \left\{ u : (0,\pi) \times (0,T) \to \mathbb{R} \ : \ \sup_{t \in (0,T)} \|u(\cdot,t)\|_{L^2(0,\pi)} < \infty \right\}.$$

Use Banach Fixed Point Theorem together with (*a*) and (*b*) to prove that, for T sufficiently small, there exists a unique solution u to problem (5).

BANACH SPACES

Exercise 1.

(a) Show that the norm
$$\|f\|_\infty = \max_{t\in[-1,1]} |f(t)|$$
in the space $\mathcal{C}([-1,1])$ does not come from any scalar product, thus $\mathcal{C}([-1,1])$ is not a Hilbert space. Hint: use the parallelogram law.

(b) Let $\mathcal{C}^*([-1,1])$ be the space of real, continuous functions in the interval $[-1,1]$, with the norm
$$\|f\|_{L^2} = \left(\int_{-1}^{1} |f(t)|^2\, dt\right)^{1/2}.$$
Show that the norm comes from a scalar product, but that the space is not complete with respect to this norm. Hint: consider
$$f_n(t) = \begin{cases} 0, & -1 \le t \le 0, \\ nt, & 0 \le t \le 1/n, \\ 1, & 1/n \le t \le 1. \end{cases}$$

(c) Prove that the $\|\cdot\|_{L^p}$ does not satisfy the parallelogram law for any $1 \le p \le \infty$, $p \ne 2$. This shows, in particular, that L^p is not a Hilbert space for $p \ne 2$.

Solution. (a) In order to show it we only need to find two continuous functions f and g such that the parallelogram rule fails, i.e.
$$||f+g||_\infty^2 + ||f-g||_\infty^2 \ne 2||f||_\infty^2 + 2||g||_\infty^2.$$
The idea is to take functions such that the maximum is not attained at the same point. In fact, let us take $f(x) = 1 + x$ and $g(x) = 1 - x$, then we can compute the different norms and check that the equalities is not satisfied. That is,
$$||f+g||_\infty^2 + ||f-g||_\infty^2 = ||2||_\infty^2 + ||2x||_\infty^2 = 2^2 + 2^2 = 8,$$
and
$$2||f||_\infty^2 + 2||g||_\infty^2 = 2||1-x||_\infty^2 + 2||1+x||_\infty^2 = 2\cdot 2^2 + 2\cdot 2^2 = 16.$$

(b) We know that a given norm comes from a scalar product if and only if the parallelogram rule holds. Moreover, in case it is satisfied we can recover the scalar product in the following way
$$\langle f, g\rangle = \frac{1}{4}||f+g||^2 - \frac{1}{4}||f-g||^2.$$
In this particular case, it is easy to verify that the norm comes from the scalar product
$$\langle f, g\rangle_{L^2} = \int_{-1}^{1} f(t)\, g(t)\, dt.$$

Now, in order to see that this space is not complete we are proving that the given sequence f_n is Cauchy but it is not convergent in the space. That is, given $m, n \geq N$

$$||f_m - f_n||_{L^2}^2 = \int_0^{1/N} |f_m(t) - f_n(t)|^2 \, dt \leq \frac{4}{N} < \varepsilon^2$$

if N is big enough. On the other hand, let us suppose that $\mathcal{C}^*$ is a complete space. Then, there exists the limit $f = \lim_{k\to\infty} f_k$, i.e., there is a continuous function f such that $||f - f_k||_{L^2} \to 0$ when $k \to \infty$. In particular, this means that $f(x) = \lim_{k\to\infty} f_k(x)$ for almost every $x \in [-1, 1]$. Nevertheless, we can see that the pointwise limit is

$$f_k(x) \to \begin{cases} 0 & \text{if } \; -1 \leq x \leq 0, \\ 1 & \text{if } \; 0 < x \leq 1, \end{cases}$$

that is not continuous, so we arrive at a contradiction by uniqueness of the limit. Hence, the space $\mathcal{C}^*$ is not complete.

(c) We already know that L^2 is a Hilbert space with the scalar product

$$\langle f, g\rangle_{L^2} = \int_{-1}^{1} f(t)\, g(t)\, dt.$$

Now, we are showing that it is the unique L^p-space with this property. Given any $p \in [1, \infty] \setminus \{2\}$ we take the functions $f(t) = \chi_{(-1,0)}(t)$ and $g(t) = \chi_{(0,1)}(t)$. Then, we can check that they do not satisfy the parallelogram rule. That is,

$$||f + g||_{L^p}^2 + ||f - g||_{L^p}^2 = 2^{2/p} + 2^{2/p} = 2^{1+2/p},$$

and

$$2||f||_\infty^2 + 2||g||_\infty^2 = 2 + 2 = 4.$$

Note that we only have equality if and only if $1+2/p = 2$, which is equivalent to $p = 2$.

Exercise 2. Let Ω be an open set in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be a measurable function. Suppose that f satisfies the property $(\mathcal{B})$:

$$(\mathcal{B}) \quad \text{There exists } C > 0 \text{ such that } \; |f(x)| \leq C \; \text{ for a.e. } x \in \Omega.$$

We define

$$\|f\|_\infty = \inf\{\alpha : |f(x)| \leq \alpha \text{ a.e. in } \Omega\}.$$

(a) Show that $|f(x)| \leq \|f\|_\infty$ a.e. in Ω.

(b) Let $L^\infty(\Omega)$ the set of functions on Ω with values in $\mathbb{R}$ that satisfy $(\mathcal{B})$. Show that $\|\cdot\|_\infty$ is a norm in $L^\infty(\Omega)$.

(c) Let $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^2$ and let g be equal to f except at $x = 0$, where $g(0) = 2$, and at $x = \pm 1/2$, where $g(\pm 1/2) = 4$. Compare $\sup_{x\in[-1,1]} |f(x)|$, $\sup_{x\in[-1,1]} |g(x)|$, $\|f\|_\infty$ and $\|g\|_\infty$.

Solution. (a) We define the set

$$E = \{x \in \Omega \text{ such that } \; |f(x)| > ||f||_\infty\} \subset \Omega.$$

Let us assume by contradiction that $|E| > 0$. Then, we claim that there exists a subset $E_n \subset E$ with $|E_n| > 0$ such that $|f(x)| > ||f||_\infty + 1/n$ for every $x \in E_n$ and some

$n \in \mathbb{N}$. That is, it is clear that $E = \cup_{n\in\mathbb{N}} E_n$. Hence $0 < |E| \leq \sum_{n\in\mathbb{N}} |E_n|$, so it is needed $|E_n| > 0$ for some $n \in \mathbb{N}$. Finally, we arrive at a contradiction since

$$\begin{aligned}\|f\|_\infty &= \inf\{\alpha : |f(x)| \leq \alpha \text{ a.e. in } \Omega\} \geq \inf\{\alpha : |f(x)| \leq \alpha \text{ a.e. in } E_n\} \\ &\geq ||f||_\infty + 1/n > ||f||_\infty.\end{aligned}$$

The contradiction comes from assuming $|E| > 0$, so we can conclude that $|E| = 0$, and the desired result is proven.

(b) Let us show that $|| \cdot ||_\infty$ satisfies the three properties a norm must have.

- Homogeneity of degree one:

$$\begin{aligned}||\lambda f||_\infty &= \inf\{\alpha : |\lambda f(x)| \leq \alpha \text{ a.e. in } \Omega\} = \inf\{\alpha : |f(x)| \leq \frac{\alpha}{|\lambda|} \text{ a.e. in } \Omega\} \\ &= \inf\{|\lambda|\alpha : |f(x)| \leq \alpha \text{ a.e. in } \Omega\} = |\lambda| \inf\{\alpha : |f(x)| \leq \alpha \text{ a.e. in } \Omega\} \\ &= |\lambda|||f||_\infty\end{aligned}$$

- Triangular inequality:

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq ||f||_\infty + ||g||_\infty \quad \text{a.e. in } \Omega.$$

  Note the we have used here the result from part (a). Thus,

$$\begin{aligned}||f + g||_\infty &= \inf\{\alpha : |f(x) + g(x)| \leq \alpha \text{ a.e. in } \Omega\} \\ &\leq \inf\{\alpha : ||f||_\infty + ||g||_\infty \leq \alpha \text{ a.e. in } \Omega\} \\ &= ||f||_\infty + ||g||_\infty\end{aligned}$$

- Positive definition: by part (a) it is clear that $\|f\|_\infty \geq |f(x)| \geq 0$ a.e. in Ω. In paticular $\|f\|_\infty \geq 0$. Moreover,

$$||f||_\infty = 0 \Leftrightarrow |f(x)| = 0 \text{ a.e. in } \Omega \Leftrightarrow f \equiv 0 \text{ in } L^\infty.$$

  Note that functions in L^∞ are defined through a class of equivalence.

(c) On the one hand we have, since f is continuous it satisfies

$$||f||_\infty = \sup_{x\in[-1,1]} |f(x)| = 1.$$

On the other hand, g satisfies that $g(x) = f(x)$ a.e. in $(0, 1)$, so

$$||g||_\infty = ||f||_\infty = 1,$$

while it is clear that

$$\sup_{x\in[-1,1]} |g(x)| = 4 \neq ||g||_\infty.$$

Exercise 3.

(*a*) Let $\Omega \subset \mathbb{R}^n$, and let $f, g \in L^p(\Omega)$, $1 \leq p \leq \infty$. Show that $\|f - g\|_{L^p} = 0$ if and only if $f = g$ a.e. in Ω.

(*b*) Let $1 < p < \infty$. Find a function in $L^p(0, 1)$ that does not belong to $L^q(0, 1)$ for any $q > p$.

(*c*) Let $f \in L^\infty(\Omega)$ with $|\Omega| < \infty$. Prove that $f \in L^p(\Omega)$ for every $p \in [1, \infty]$. Moreover, it satisfies

$$\lim_{p\to+\infty} ||f||_{L^p(\Omega)} = ||f||_{L^\infty(\Omega)}$$

(d) Let $f \in L^p(\Omega)$ for every $p \in [1, \infty)$. Prove that if there exists a constant $C \geq 0$ such that $||f||_{L^p(\Omega)} \leq C$ for every $p \in [1, \infty)$, then $f \in L^\infty(\Omega)$ and it satisfies $||f||_{L^\infty(\Omega)} \leq C$. Is it true that $f \in L^\infty(\Omega)$ if $f \in L^p(\Omega)$ for every $p \in [1, \infty)$ but $||f||_{L^p(\Omega)} \to +\infty$ when $p \to +\infty$?

Solution. (a) It is clear that $\|f - g\|_{L^p} = 0$ if and only if

$$\int_\Omega |f(x) - f(x)|^p \, dx = 0.$$

Thus, $\|f - g\|_{L^p} = 0$ if and only if $|f(x) - f(x)|^p = 0$ for almost every x in Ω. Clearly this is equivalent to the result in the statement.

(b) We are showing that the function

$$f(x) = \frac{1}{x(1 - \log(x))^2}$$

belongs to $L^1(0, 1)$ but does not belong to $L^p(0, 1)$ for any $p > 1$. Knowing that, it is easy to construct a function satisfying the statement by taking powers of f. Then, let us first prove that $f \in L^1$. It is easy since the primitive is explicit. That is,

$$\int_0^1 |f| = \int_0^1 \frac{1}{x(1 - \log(x))^2} = [(1 - \log(x))^{-1}]_0^1 = 1.$$

On the other hand, we know that the logarithm grows less than any positive power at infinity, i.e.

$$\log(y) \leq \frac{e}{k} \, y^k \quad \text{in } \ (1, \infty) \quad \text{if} \quad k > 0.$$

From this inequality, given any $p > 1$ we take $k = \frac{p-1}{2p} > 0$. Then, we obtain that

$$(1 - \log(x))^{-2p} \geq C_p \, x^{p-1} \quad \text{in } \ (0, 1),$$

for some constant $C_p > 0$ depending only on $p > 1$. Thus,

$$\int_0^1 |f|^p = \int_0^1 \frac{1}{x^p(1 - \log(x))^{2p}} \geq C_p \int_0^1 x^{-1} = \infty.$$

(c) Given any $p \geq 1$ we have

$$\int_\Omega |f|^p \leq ||f||^p_{L^\infty(\Omega)} |\Omega| < \infty,$$

so $f \in L^p(\Omega)$. Note that here it is crucial the fact that $|\Omega|$ is finite. Otherwise the result is not true (take the constant function). Now, let us find the limit. From the previous computation we obtain

$$\limsup_{p \to \infty} ||f||_{L^p(\Omega)} \leq ||f||_{L^\infty(\Omega)} \ \limsup_{p \to \infty} |\Omega|^{1/p} = ||f||_{L^\infty(\Omega)}.$$

On the other hand, given any $\varepsilon > 0$ we define the set

$$\Omega_\varepsilon = \{x \in \Omega \ : \ |f(x)| > ||f||_{L^\infty(\Omega)} - \varepsilon\},$$

which satisfies $|\Omega_\varepsilon| > 0$ for every $\varepsilon > 0$. Then, it is clear that

$$\int_\Omega |f|^p \geq \int_{\Omega_\varepsilon} |f|^p > \int_{\Omega_\varepsilon} (||f||_{L^\infty(\Omega)} - \varepsilon)^p = (||f||_{L^\infty(\Omega)} - \varepsilon)^p \ |\Omega|,$$

so taking the lim inf we get

$$\liminf_{p\to\infty} ||f||_{L^p(\Omega)} \geq (||f||_{L^\infty(\Omega)} - \varepsilon) \ \liminf_{p\to\infty} |\Omega|^{1/p} = ||f||_{L^\infty(\Omega)} - \varepsilon.$$

Since this inequality holds true for any given $\varepsilon > 0$ we can take the limit and conclude that

$$\liminf_{p\to\infty} ||f||_{L^p(\Omega)} \geq ||f||_{L^\infty(\Omega)} \geq \limsup_{p\to\infty} ||f||_{L^p(\Omega)}.$$

Thus, the limit exits and it is equal to the L^∞-norm. Note that we need to work with lim inf and lim sup since a priori we don not know if the limit exists.

(d) Let us suppose that $f \notin L^\infty(\Omega)$. If we define the set

$$\Omega_M = \{x \in \Omega \quad : \quad |f(x)| > M\},$$

then, it satisfies $|\Omega_M| > 0$ for any given $M > 0$ (by definition of L^∞). Thus,

$$C \geq ||f||_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{1/p} \geq \left(\int_{\Omega_{2C}} |f|^p\right)^{1/p} \geq 2C\,|\Omega_{2C}|^{1/p},$$

for every $p \geq 1$. If we take p going to infinity we arrive at the contradiction $C \geq 2C$. Here, it is crucial the fact that $||\Omega_{2C}|| \neq 0$. Thus, $f \in L^\infty$. Now, let us take $M = ||f||_{L^\infty(\Omega)} - \varepsilon$. Then, doing the same computation, we obtain

$$C \geq ||f||_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{1/p} \geq \left(\int_{\Omega_M} |f|^p\right)^{1/p} \geq (||f||_{L^\infty(\Omega)} - \varepsilon)\,|\Omega_M|^{1/p}.$$

Finally, taking p going to infinity and ε to zero we arrive at

$$||f||_{L^\infty(\Omega)} \leq C.$$

Exercise 4. Let (M, d) be a metric space and let $0 < \alpha \leq 1$. We consider the space of Hölder continuous functions with exponent α:

$$\mathcal{C}^{0,\alpha}(M) = \{u : M \to \mathbb{R} \,|\, u \text{ bounded and } \exists\, C > 0 \text{ s.t. } \frac{|u(x) - u(y)|}{d(x,y)^\alpha} \leq C \text{ for } x \neq y\}.$$

If $0 < \alpha < 1$, $\mathcal{C}^{0,\alpha}(M)$ is also denoted by $\mathcal{C}^\alpha(M)$. If $\alpha = 1$, $\mathcal{C}^{0,1}(M) = Lip(M)$ is the space of bounded Lipschitz functions on M. It is well known that, for $0 < \alpha \leq 1$, $\mathcal{C}^{0,\alpha}(M)$ is a Banach space with the norm

$$\|u\|_{\mathcal{C}^{0,\alpha}} = \|u\|_\infty + \sup_{x\neq y} \frac{|u(x) - u(y)|}{d(x,y)^\alpha}.$$

Moreover, for $0 < \alpha < \beta < 1$, the following inclusions are true

$$Lip(M) \subset \mathcal{C}^{0,\beta}(M) \subset \mathcal{C}^{0,\alpha}(M) \subset \mathcal{C}_u(M),$$

where $\mathcal{C}_u(M) = \{u \in \mathcal{C}(M) \,|\, u \text{ uniformly continuous and bounded}\}$.

(a) Show that if $(u_k)_{k\geq 1}$ is a uniformly bounded sequence in $\mathcal{C}^{0,\alpha}(M)$ that converges pointwise to a function $u : M \to \mathbb{R}$, then $u \in \mathcal{C}^{0,\alpha}(M)$.

(b) Let $0 < \beta < \gamma < \alpha \leq 1$. Show the interpolation inequality

$$||u||_{\mathcal{C}^{0,\gamma}(M)} \leq ||u||^{\frac{\alpha-\gamma}{\alpha-\beta}}_{\mathcal{C}^{0,\beta}(M)} ||u||^{\frac{\gamma-\beta}{\alpha-\beta}}_{\mathcal{C}^{0,\alpha}(M)}.$$

(c) Let (M, d) be a path-connected metric space. Show that if $u \in C^{0,\alpha}(M)$ with $\alpha > 1$, then $u \equiv ctt$.

Solution. (a) Let us take $x, y \in M$. Given any $\varepsilon > 0$, by pointwise convergence we know that there exists k big enough (depending on x and y) such that $|u(x), u_k(x)| < \varepsilon$ and $|u(x), u_k(x)| < \varepsilon$. Then, taking this k we obtain

$$|u(x), u(y)| \leq |u(x), u_k(x)| + |u_k(x) - u_k(y)| + |u_k(y) - u(y)| \leq 2\varepsilon + Cd(x, y)^\alpha.$$

Since this is true for any $\varepsilon > 0$ and C is independent of it, we can take the limit and obtain

$$|u(x) - u(y)| \leq C\, d(x, y)^\alpha,$$

so $u \in C^{0,\alpha}(M)$.

(b) Since $\beta < \gamma < \alpha$, there exists $t \in (0, 1)$ such that $\gamma = (1-t)\beta + t\alpha$. Then,

$$\begin{aligned}
[u]_{C^{0,\gamma}} &= \sup_{x\neq y} \frac{|u(x) - u(y)|}{d(x, y)^\gamma} = \sup_{x\neq y} \frac{|u(x) - u(y)|^{1-t}}{d(x, y)^{(1-t)\beta}} \frac{|u(x) - u(y)|^t}{d(x, y)^{t\alpha}} \\
&\leq \left(\sup_{x\neq y} \frac{|u(x) - u(y)|}{d(x, y)^\beta} \right)^{1-t} \left(\sup_{x\neq y} \frac{|u(x) - u(y)|}{d(x, y)^\alpha} \right)^t \\
&= [u]^{1-t}_{C^{0,\beta}}\, [u]^t_{C^{0,\alpha}}.
\end{aligned}$$

Now, let us define $a = ||u||_{L^\infty}$, $b = [u]_{C^{0,\beta}}$ and $c = [u]^t_{C^{0,\alpha}}$. Hence,

$$\begin{aligned}
||u||_{C^{0,\gamma}} &= ||u||_{L^\infty} + [u]_{C^{0,\gamma}} \leq a + b^{1-t}\, c^t \\
&= (a + b)^{1-t} \left[\frac{a}{(a + b)^{1-t}} + \frac{b^{1-t}\, c^t}{(a + b)^{1-t}} \right] \\
&= (a + b)^{1-t} \left[\frac{a}{a + b}(a + b)^t + \frac{b}{a + b} \left(\frac{(a + b)c}{b} \right)^t \right] \\
&\leq (a + b)^{1-t} \left[\frac{a}{a + b}(a + b) + \frac{b}{a + b} \left(\frac{(a + b)c}{b} \right) \right]^t \\
&= (a + b)^{1-t}\, (a + c)^t = ||u||^{1-t}_{C^{0,\beta}} ||u||^t_{C^{0,\alpha}}
\end{aligned}$$

(c) Let us take any given $x, y \in M$. Since the space is path-connected, this means that there exists a continuous map $\gamma : M \to [0, 1]$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Moreover, we can take γ such that $d(\gamma(t), \gamma(s)) = (s - t)dist(x, y)$. Now, for any $k \geq 1$

we have

$$
\begin{aligned}
|u(x)-u(y)| = |u(\gamma(0))-u(\gamma(1))| &= \left|\sum_{j=0}^{k-1}\left\{u\left(\gamma\left(\frac{j}{k}\right)\right)-u\left(\gamma\left(\frac{j+1}{k}\right)\right)\right\}\right| \\
&\leq \sum_{j=0}^{k-1}\left|u\left(\gamma\left(\frac{j}{k}\right)\right)-u\left(\gamma\left(\frac{j+1}{k}\right)\right)\right| \\
&\leq C\sum_{j=0}^{k-1} d\left(\gamma\left(\frac{j}{k}\right)-\gamma\left(\frac{j+1}{k}\right)\right)^{\alpha} = C\sum_{j=0}^{k-1} d\left(\gamma\left(\frac{j}{k}\right)-\gamma\left(\frac{j+1}{k}\right)\right)^{\alpha} \\
&= Ck^{1-\alpha}.
\end{aligned}
$$

Thus, since $\alpha > 1$ and the previous inequality holds true for any given $k \geq 1$ we can take $k \to \infty$ and obtain $|u(x)-u(y)| = 0$, so $u \equiv ctt$.

HILBERT SPACE TECHNIQUES AND SOBOLEV SPACES

Exercise 5. Let $H = L^2(-1,1)$. We consider the subspace V of odd functions in H,

$$V = \{u \in H \ : \ u(-t) = -u(t) \text{ a.e. } t \in (-1,1)\}.$$

(a) Verify that it is possible to apply the projection theorem.
(b) After determining $V^{\perp}$, write the expression of $P_V f$, $Q_V f$ for a generic $f \in H$.

Solution. (a) In order to apply the projection theorem we only need to show that V is a closed subspace of L^2, which is a Hilbert space. First, it is clear that a linear combination of odd functions is also odd. Now, we are checking that it is closed. That is, let us take a sequence of odd functions $u_n \to \overline{u}$. Then, there is a subsequence u_{n_k} converging almost everywhere to $\overline{u}$. This means

$$0 = u_{n_k}(t) + u_{n_k}(t) \to \overline{u}(t) + \overline{u}(-t) \qquad \text{for almost every } \ t \in (-1,1).$$

Thus, $\overline{u}(-t) = -\overline{u}(t)$ for almost every $t \in (-1,1)$, and we conclude that the limit $\overline{u} \in V$, so it is a closed set.

An alternative way of proving that the set V is closed comes by rewriting it in the following way

$$V = \left\{u \in H \ : \ \int_{-1}^{1} |u(t)|^2 + u(t)u(-t)\,dt = 0\right\},$$

ans showing that the function $F : H \to \mathbb{R}$ defined as $F(u) = \int_{-1}^{1} |u(t)|^2 + u(t)u(-t)\,dt$ is continuous.

(b) Let us take $w \in V^{\perp}$. Then, it satisfies

$$
\begin{aligned}
0 = \int_{-1}^{1} w(t)u(t)\,dt &= \int_{-1}^{0} w(t)u(t)\,dt + \int_{0}^{1} w(t)u(t)\,dt \\
&= -\int_{0}^{1} w(-t)u(t)\,dt + \int_{0}^{1} w(t)u(t)\,dt = \int_{0}^{1} [w(t)-w(-t)]u(t)\,dt
\end{aligned}
$$

for every $u \in V$. Thus, we deduce that w must satisfy $w(t) = w(-t)$ for almost every $t \in (-1,1)$. So $V^\perp$ is the subspace of even functions.

In order to find the projection we are using the very well known decomposition of a function in the sum of an odd and an even function. That is,

$$f(x) = \frac{f(x) - f(-x)}{2} + \frac{f(x) + f(-x)}{2}.$$

Therefore, it is clear that

$$P_V f(t) = \frac{f(x) - f(-x)}{2} \quad \text{and} \quad Q_V f(t) = \frac{f(x) + f(-x)}{2}.$$

Exercise 6. Let $Q = (0,1) \times (0,1)$ and $H = L^2(Q)$. We consider the subspace

$$V = \{u \in H \, : \, u(x,y) = v(x), \text{ with } v \in L^2(0,1)\}.$$

Verify that $H = V \oplus V^\perp$ and determine P_V and $P_{V^\perp}$. Decompose the function $f(x,y) = xy$ as a sum of an element in V and an element in $V^\perp$.

Solution. We know that we can decompose $H = V \oplus V^\perp$ if V is closed. Then, we need to prove it. Let us take a sequence $u_k \in V$ such that $u_k \to \overline{u}$. Since $u_k(x,y) = v_k(x)$ and u_k is a Cauchy sequence in $L^2(Q)$ we deduce that v_k is a Cauchy sequence in $L^2(-1,1)$. That is,

$$||v_i - v_j||_{L^2(-1,1)} = ||u_i - u_j||_{L^1(Q)}.$$

Hence, there exists $\overline{v} \in L^2(-1,1)$ such that $v_k \to \overline{v}$ in $L^2(-1,1)$. Then, by definition of v_k it is clear that moreover $u_k \to \overline{v}$ in $L^2(Q)$. Finally, by uniqueness of the limit we deduce that $\overline{u}(x,y) = \overline{v}(x)$, so $\overline{u} \in V$ and V is a closed set.

Now, let us find $V^\perp$. If we take $w \in V^\perp$, it satisfies

$$\int_Q w(x,y)u(x,y)\,dxdy = 0$$

for every $u \in V$. This is equivalent to

$$0 = \int_Q w(x,y)v(x)\,dxdy = \int_0^1 v(x) \left(\int_0^1 w(x,y)\,dy \right) dx$$

for every $v \in L^2(0,1)$. Then, w must satisfy

$$\int_0^1 w(x,y)\,dy = 0 \text{ a.e. } x \in (0,1),$$

so we deduce that

$$V^\perp = \{u \in H \, : \, \int_0^1 u(x,y)\,dy = 0 \text{ a.e. } x \in (0,1)\}.$$

Finally, given any $u \in L^2(Q)$, let us write it as $u(x,y) = v(x,y) + w(x,y)$ with $v \in V$ and $w \in V^\perp$. Then, integrating with respect to y and using the definition of the sets we find

$$\int_0^1 u(x,y)\,dy = \int_0^1 v(x)\,dy + \int_0^1 w(x,y)\,dy = v(x).$$

Thus, we obtain that $v(x) = \int_0^1 u(x,y)\,dy$ and $w(x,y) = u(x,y) - \int_0^1 u(x,y)\,dy$. Since we know that this decomposition is unique we get

$$P_V u(x,y) = \int_0^1 u(x,y)\,dy \quad \text{and} \quad P_{V^\perp} u(x,y) = u(x,y) - \int_0^1 u(x,y)\,dy.$$

In the particular case $u(x,y) = xy$ we can apply the general result to obtain

$$P_V(xy) = \frac{x}{2} \quad \text{and} \quad P_{V^\perp}(xy) = xy - \frac{x}{2}.$$

Exercise 7.

(*a*) For which choices of exponents $\alpha > 0$ and $p \geq 1$ does the function $u(x) = |x|^{-\alpha}$ belong to the spaces $W^{1,p}(B_1)$ and $W^{1,p}(\mathbb{R}^n \backslash B_1)$?

(*b*) For which exponents α does the function $u(x) = (-\log|x|)^\alpha$ belong to $H^1(B_{1/2})$?

Solution. (a) As a first step, we are showing that $u \in L^p(B_1)$ if and only if $n > p\alpha$ and $u \in L^p(\mathbb{R}^n \setminus B_1)$ if and only if $n < p\alpha$. That is,

$$\int_{B_1} |u|^p = \int_{B_1} |x|^{-p\alpha} = |\partial B_1| \int_0^1 r^{-p\alpha}\, r^{n-1}\,dr = |\partial B_1| \int_0^1 r^{-p\alpha+n-1}\,dr < \infty$$

if and only if $-p\alpha + n - 1 > -1$. Here, we have use spherical coordinates and the fact that a power function in dimension 1 is integrable at zero if and only if the exponent is greater that -1. In a completely analogous way we obtain

$$\int_{\mathbb{R}^n \backslash B_1} |u|^p = \int_{\mathbb{R}^n \backslash B_1} |x|^{-p\alpha} = |\partial B_1| \int_1^\infty r^{-p\alpha}\, r^{n-1}\,dr = |\partial B_1| \int_1^\infty r^{-p\alpha+n-1}\,dr < \infty$$

if and only if $-p\alpha + n - 1 < -1$.

Now, we are using that $\nabla u(x) = |x|^{-\alpha-2}x$ in the weak sense (we will prove it latter) in order to show that $\nabla u \in L^p(B_1)$ if and only if $\alpha < \frac{n-p}{p} \leq n-1$ and $\nabla u \in L^p(\mathbb{R}^n \backslash B_1)$ if and only if $\alpha > \frac{n-p}{p}$. Then,

$$\int_{B_1} |\nabla u|^p = \int_{B_1} |x|^{(-\alpha-1)p} = |\partial B_1| \int_0^1 r^{-(\alpha+1)p+n-1}\,dr < \infty$$

if and only if $-(\alpha+1)p + n - 1 > -1$. For the other domain,

$$\int_{\mathbb{R}^n \backslash B_1} |\nabla u|^p = \int_{\mathbb{R}^n \backslash B_1} |x|^{(-\alpha-1)p} = |\partial B_1| \int_1^\infty r^{-(\alpha+1)p+n-1}\,dr < \infty$$

if and only if $-(\alpha+1)p + n - 1 < -1$.

Summarizing, we have that $u \in W^{1,p}(B_1)$ if and only if $\alpha < \frac{n-p}{p}$, while $u \in W^{1,p}(\mathbb{R}^n \backslash B_1)$ if and only if $\alpha > \frac{n-p}{p}$.

Finally let us prove that the weak derivative coincides with the classical derivative outside of the origin. We are using that $u \in C^1$ outside of the origin, and we will note u_i the classical derivatives of u. Then, given $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u\varphi_i = \lim_{\varepsilon \to 0} \int_{\Omega \backslash B_\varepsilon} u\varphi_i = \lim_{\varepsilon \to 0} \left\{ \int_{\partial B_\varepsilon} u\varphi\nu_i - \int_{\Omega \backslash B_\varepsilon} u_i\varphi \right\} = -\int_\Omega u_i\varphi,$$

so the weak derivative coincides with the classical one outside of the origin. Note that we have used that

$$\left| \int_{\partial B_\varepsilon} u\varphi\nu_i \right| \le C\,\varepsilon^{-\alpha+n-1} \to 0,$$

since $\alpha \le n-1$.

(b) Note that u is not continuous at the origin when $\alpha > 0$, then we can directly conclude that $u \notin H^1(B_{1/2})$ when $n = 1$ and $\alpha > 0$, since H^1 functions are continuous in that dimension. Now, we are showing that $u \in L^2(B_{1/2})$ for any given α. It is clear that it is true if $\alpha \le 0$ since in that case the function is bounded. Now, if $\alpha > 0$ we are using that

$$|-\log(r)| \le C_k r^{-k} \quad \text{in} \quad (0, 1/2)$$

for any $k > 0$. Then, taking $k = \frac{n}{4\alpha}$ we arrive at

$$\int_{B_{1/2}} |u|^2 = \int_{B_{1/2}} |-\log|x||^{2\alpha} \le C\int_0^{1/2} r^{-n/2+n-1} < +\infty.$$

Next, we are using that $\nabla u = -\alpha\frac{(-\log|x|)^{\alpha-1}}{|x|^2}\,x_i$ in the weak sense. We will prove it latter. Then we need to compute

$$||\nabla u||_{L^2}^2 = \alpha^2\int_0^1 (-\log|x|)^{2\alpha-2}\,|x|^{n-3}.$$

In order to do it we distinguish three cases. If $n \ge 3$,

$$||\nabla u||_{L^2}^2 = \alpha^2\int_0^1 (-\log|x|)^{2\alpha-2}\,|x|^{n-3} \le \alpha^2\int_0^1 (-\log|x|)^{2\alpha-2} < \infty,$$

for any $\alpha \in \mathbb{R}$. Here we have used the previous computations in this part. The second case, when $n = 2$

$$||\nabla u||_{L^2}^2 = \alpha^2\int_0^1 (-\log(r))^{2\alpha-2}\,r^{-1} = \alpha^2\left[\frac{(-\log(r))^{2\alpha-1}}{2\alpha-1}\right]_0^{1/2} < \infty,$$

if and only if $\alpha < 1/2$. Finally, when $n = 1$ and $\alpha < 0$

$$||\nabla u||_{L^2}^2 = \alpha^2\int_0^1 (-\log(r))^{2\alpha-2}\,r^{-2} \ge \alpha^2\int_0^1 r^{-1} = \infty.$$

Here, we have used the estimate $(-\log(r))^{2\alpha-2} \ge Cr$ for any $\alpha \in \mathbb{R}$.

At this point, we are proving that the weak derivative is the function we have said. We are using that $u \in C^1$ outside of the origin, and we will note u_i the classical derivatives of u. Then, given $\varphi \in C_c^\infty(B_{1/2})$ we have

$$\int_{B_{1/2}} u\varphi_i = \lim_{\varepsilon\to 0}\int_{B_{1/2}\setminus B_\varepsilon} u\varphi_i = \lim_{\varepsilon\to 0}\left\{\int_{\partial B_\varepsilon} u\varphi\nu_i - \int_{B_{1/2}\setminus B_\varepsilon} u_i\varphi\right\} = -\int_{B_{1/2}} u_i\varphi,$$

so the weak derivative coincides with the classical one outside of the origin. Note that we have used that

$$\left| \int_{\partial B_\varepsilon} u\varphi\nu_i \right| \le C(-\log(\varepsilon))^{\alpha}\,\varepsilon^{n-1} \to 0,$$

since $n > 1$ or $n = 1$ and $\alpha < 0$.

Exercise 8. Show that the functional

$$Lu = \int_0^1 u(t)\,dt$$

belongs to the space $H^{-1}(0,1) := (H_0^1(0,1))^*$. Find the element in $H_0^1(0,1)$ that represents it.

Exercise 9. Let $H = H^1(-1,1)$ and

$$V = \{u \in H : \, u(0) = 0\}.$$

After showing that V is a closed subspace of H, compute the orthogonal projection on V of $f(t) = 1$ for $t \in (-1,1)$.

Solution. Recall that H^1 functions in dimension 1 are continuous, so pointwise values make sense. Now, let us prove that in fact evaluating at a point is a continuous functional. That is, let $T : H^1(-1,1) \to \mathbb{R}$ such that $Tu = u(0)$, then by integrating and using Cauchy-Schwartz

$$|u(0)| = \left| u(x) + \int_x^0 u'(t)\,dt \right| \leq |u(x)| + \int_{-1}^1 |u'(t)|\,dt \leq |u(x)| + ||u'||_{L^2(0,1)}.$$

Next, taking the square, integrating and using that $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\begin{aligned} 2|u(0)|^2 = \int_{-1}^1 |u(0)|^2\,dt &\leq \int_{-1}^1 \left(|u(x)| + ||u'||_{L^2(0,1)}\right)^2\,dt \\ &\leq \int_{-1}^1 |u(x)|^2 + 2||u'||^2_{L^2(0,1)}\,dt = ||u||^2_{L^2(0,1)} + 4||u'||^2_{L^2(0,1)} \\ &\leq 4||u||^2_{H^1(0,1)}. \end{aligned}$$

Thus, $|Tu| = |u(0)| \leq \sqrt{2}\,||u||_{H^1(0,1)}$ and T is continuous. Note that the fact that evaluation is taken at 0 does not have any relevance in the computation.

Now, since $V = T^{-1}(\{0\})$, it is clear that it is a cosed set. Moreover, it is clearly a subspace. Therefore, the projection theorem guarantees the existence and uniqueness of the orthogonal projection.

Finally, let us characterize the projection. We know that the projection $w = P_V f$ satisfies

$$\int_{-1}^1 (f(t) - w(t))\,v(t)\,dt + \int_{-1}^1 (\nabla f(t) - \nabla w(t))\,\nabla v(t)\,dt = 0 \quad \forall v \in H.$$

Assuming the projection is regular enough and integrating by parts we obtain

$$\int_{-1}^1 (f(t) - w(t) - f''(t) + w''(t))\,v(t)\,dt + (f'(t) - w'(t))v(t)|_{-1}^1 = 0 \quad \forall v \in H.$$

In particular, the equality must hold for every $v \in H_0^1(-1,1)$, so the boundary term disappears and we get

$$\int_{-1}^1 (f(t) - w(t) - f''(t) + w''(t))\,v(t)\,dt = 0 \quad \forall v \in H_0^1(-1,1),$$

which is equivalent to

$$w'' - w = f'' - f \quad \text{a.e. in} \quad (-1,1).$$

But now the boundary term in the previous equality have to be zero too. This means that

$$(f'(1) - w'(1))v(1) - (f'(-1) - w'(-1))v(-1) = 0 \quad \forall v \in H,$$

and thus

$$w'(-1) = f'(-1) \qquad \text{and} \qquad w'(1) = f'(1).$$

Hence, the projection of f satisfies

$$\begin{cases} w'' - w = f'' - f \quad \text{a.e. in} \quad (-1,1) \\ w(0) = 0, \\ w'(-1) = f'(-1), \\ w'(1) = f'(1). \end{cases}$$

In the particular case $f \equiv 1$ it is easy to find that

$$w(t) = \begin{cases} 1 - \frac{1}{e^2+1}(e^{2+t} + e^{-t}), & \text{if } \; -1 \leq t \leq 0 \\ 1 - \frac{1}{e^2+1}(e^{t} + e^{2-t}), & \text{if } \quad 0 \leq t \leq 1. \end{cases}$$

is a solution and moreover it satisfies the projection condition. Since we know the projection is unique, it follows that it is in fact the projection.

Exercise 10. Let $p \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded, open, convex set. Denote with $(u)_\Omega$ the integral mean over Ω of any given function $u \in L^1(\Omega)$, that is

$$(u)_\Omega := \fint_\Omega u(x)\, dx.$$

Show that there exists a constant $C > 0$, depending on n, p and Ω, such that the *Poincaré-Wirtinger* inequality

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

holds for any $u \in W^{1,p}(\Omega)$.

Exercise 11. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.

(a) Show that

$$(u,v)_{\partial\Omega} = \int_{\partial\Omega} u|_{\partial\Omega} v|_{\partial\Omega}\, d\sigma + \int_\Omega \nabla u \cdot \nabla v\, dx$$

is an inner product in $H^1(\Omega)$.

(b) Show that the norm

$$\|u\|_{\partial\Omega} = \left(\int_{\partial\Omega} u|^2_{\partial\Omega}\, d\sigma + \int_\Omega |\nabla u|^2\, dx \right)^{1/2}$$

is equivalent to $\|u\|_{H^1}$.

Solution. (a) It is easy to see that it is bilinear and symmetric. Moreover, it is also clear by using the scalar product in L^2 that $(u,u)_{\partial\Omega} = ||u||^2_{L^2(\partial\Omega)} + ||\nabla u||^2_{L^2(\Omega)} \geq 0$. Finally, if $(w,w)_{\partial\Omega}$ then $||\nabla w||_{L^2(\Omega)} = 0$ and $||w||_{L^2(\partial\Omega)} = 0$. From the first condition we obtain that $\nabla w = 0$, or equivalently that w is constant. But the second conditions means that $w \equiv 0$ on $\partial\Omega$. Hence we conclude that $w \equiv 0$ in Ω.

(b) By density density arguments, we are only proving the equivalence of the norms for functions in $C^\infty(\Omega)$. Since Ω is bounded then $\Omega \subset B_R$ for some $R > 0$. Let us take $u \in C^\infty(\Omega)$, then

$$\int_\Omega u^2 = \int_\Omega u^2(x)\,\mathrm{div}\left(\frac{x}{n}\right)\,dx = \frac{1}{n}\int_{\partial\Omega} u^2(x)\,x\cdot\nu\,dx - \frac{1}{n}\int_\Omega \nabla(u^2(x))\cdot x\,dx.$$

Now, if we estimate each term separately we obtain

$$\left|\int_{\partial\Omega} u^2(x)\,x\cdot\nu\,dx\right| \leq \int_{\partial\Omega} |u|^2\,|x|\,|\nu|\,dx \leq R\int_{\partial\Omega}|u|^2\,dx,$$

and

$$\left|\int_\Omega \nabla(u^2(x))\cdot x\,dx\right| \leq \int_\Omega 2R|\nabla u||u|\,dx \leq \frac{1}{4}\int_\Omega |u|^2\,dx + 4R^2\int_\Omega |\nabla u|^2\,dx.$$

In the last estimate we have used that $2ab \leq a^2 + b^2$ with $a = |u|/2$ and $b = 2R|\nabla u|$. Joining this inequalities we get

$$\int_\Omega |u|^2 \leq \frac{1}{4n}\int_\Omega |u|^2 + \frac{R}{n}\int_{\partial\Omega}|u|^2\,dx + \frac{4R^2}{n}\int_\Omega |\nabla u|^2\,dx.$$

Thus

$$\begin{aligned} ||u||^2_{H^1(\Omega)} &= \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \leq \frac{4R}{4n-1}\int_{\partial\Omega}|u|^2\,dx + \frac{16R^2+4n-1}{4n-1}\int_\Omega |\nabla u|^2 \\ &\leq C_1^2||u||^2_{\partial\Omega}, \end{aligned}$$

where $C_1 > 0$ is a constant depending only on n and Ω.

On the other hand, by the trace inequlity we know that there is a positive constant C such that

$$||u||^2_{L^2(\partial\Omega)} \leq C||u||^2_{H^1(\Omega)}.$$

Thus,

$$||u||^2_{\partial\Omega} \leq (1+C^2)\,||u||_{H^1(\Omega)}.$$

Summarizing, there are two positive constants C_1 and C_2 such that

$$C_2\,||u||_{\partial\Omega} \leq ||u||_{H^1(\Omega)} \leq C_1\,||u||_{\partial\Omega},$$

so both norms are equivalent.

Note that if $u \in H^1_0(\Omega)$, then $||u||_{\partial\Omega} = ||\nabla u||_{L^2(\Omega)}$, so we recover Poincaré inequality and the fact that $||\nabla u||_{L^2}$ is a norm in that space.

WEAK FORMULATION

Exercise 12. We consider the following problem in the interval $(-1,1)$:

$$-u'' = f, \quad u(-1) = u(1) = 0$$

with $f = \chi_{(-1,0)} + 2\chi_{(0,1)}$.

(a) There exists a $\mathcal{C}^2$ solution? There exists a continuous solution? Find the solution in $H_0^1(-1,1)$.

(b) Same questions but for

$$-u'' - \lambda u = f, \quad u(-1) = u(1) = 0,$$

under suitable conditions on λ.

Solution. (a) First, note that there is no $\mathcal{C}^2$ solution. If there would be one, then u'' would be continuous. Nevertheless, since $u'' = -f$ and $-f$ is not continuous we would arrive at a contradiction. In fact, we are showing that there is a unique weak solution in $H_0^1(-1,1)$, and moreover it is C^1. It is clear that the weak formulation of the problem is to find $u \in H_0^1(-1,1)$ such that

$$\int_{-1}^{1} u'v' = \int_{-1}^{1} fv \quad \forall v \in H_0^1(-1,1).$$

Then, we only need to show that $F(v) = \int_{-1}^{1} fv$ is continuous in order to apply Riesz Theorem. That is, since $f \in L^2$ we have

$$|F(v)| \leq \int_{-1}^{1} |f||v| \leq ||f||_{L^2}||v||_{L^2} \leq C_P||f||_{L^2}||v||_{H_0^1}.$$

Thus, there is a unique weak solution $u \in H_0^1(-1,1)$. In particular $w = u' \in L^2$. Furthermore, it satisfies (from the weak formulation) that

$$\int_{-1}^{1} w\varphi' = \int_{-1}^{1} f\varphi \quad \forall \varphi \in C_c^\infty(-1,1),$$

so $-f \in L^2$ is the weak derivative of $w = u'$. Hence, u is more regular than H_0^1, indeed $u \in H^2 \subset \mathcal{C}^1$.

Finally, we want to find such solution. Since $f \in C^\infty((-1,0) \cap (0,1))$ we claim that u satisfies

$$\begin{cases} u'' = -1, & -1 < x < 0, \\ u'' = -2, & 0 < x < 1, \\ u(-1) = u(1) = 0, \\ u(0^-) = u(0^+) \\ u'(0^-) = u'(0^+). \end{cases}$$

Then, it is easy to see that the solution of the previous ODE is

$$u(x) = \begin{cases} -x^2/2 - x/4 + 3/4, & \text{if } \ -1 \leq x \leq 0 \\ -x^2 - x/4 + 3/4, & \text{if } \quad 0 \leq x \leq 1. \end{cases}$$

(b) Using the same arguments that in the previous part, we can deduce that there is no $\mathcal{C}^2$ solution. Here, we are proving by using Hilbert spaces techniques that there is existence and uniqueness of solution whenever $\lambda < C_P^{-2} = (\pi/2)^2$ (you can find the exact value of the Poincaré constant by following the same idea of the proof of Wirtinger inequality in Exercise 18 from Review List). Nevertheless it is possible to prove by using Fourier series that there is existence and uniqueness of solution when $4\lambda \neq (k\,\pi)^2$ for every $k = 1, 2, 3, 4...$ On the other cases, there is also existence (but not uniqueness) if f satisfies certain orthogonality condition.

It is clear that the weak formulation of the problem is to find $u \in H_0^1(-1,1)$ such that

$$\int_{-1}^{1} u'v' - \lambda \int_{-1}^{1} uv = \int_{-1}^{1} fv \quad \forall v \in H_0^1(-1,1).$$

Now, we only need to show that $B(u,v) = \int_{-1}^{1} u'v' - \lambda \int_{-1}^{1} uv$ is continuous and coercive in order to apply Lax-Milgram Theorem. On the one hand,

$$|B(u,v)| \leq \int_{-1}^{1} |u'||v'| + |\lambda| \int_{-1}^{1} |u||v| \leq C||u||_{H_0^1}\,||v||_{H_0^1}.$$

On the other hand, since $\lambda > C_P^{-2}$, there is a constant ε such that $0 < \varepsilon < 1 - C_P^2\,\lambda$. Then, taking such a constant we obtain

$$\begin{aligned} B(u,u) &= \varepsilon \int_{-1}^{1} |u'|^2 + (1-\varepsilon) \int_{-1}^{1} |u'|^2 - \lambda \int_{-1}^{1} |u|^2 \\ &\geq \varepsilon \int_{-1}^{1} |u'|^2 + \left(\frac{1-\varepsilon}{C_P^2}\right) \int_{-1}^{1} |u|^2 \geq \varepsilon ||u||_{H_0^1}^2. \end{aligned}$$

Thus, there is a unique weak solution $u \in H_0^1(-1,1)$. In particular $w = u' \in L^2$. Furthermore, it satisfies (from the weak formulation) that

$$\int_{-1}^{1} w\varphi' = \int_{-1}^{1} (f + \lambda u)\varphi \quad \forall \varphi \in C_c^\infty(-1,1),$$

so $-f - \lambda u \in L^2$ is the weak derivative of $w = u'$. Hence, u is more regular than H_0^1, indeed $u \in H^2 \subset \mathcal{C}^1$.

As in the previous part, we want to find such solution. Since $f \in C^\infty((-1,0) \cap (0,1))$ we claim that u satisfies

$$\begin{cases} u'' = -1 + \lambda u, & -1 < x < 0, \\ u'' = -2 + \lambda u, & 0 < x < 1, \\ u(-1) = u(1) = 0, \\ u(0^-) = u(0^+) \\ u'(0^-) = u'(0^+). \end{cases}$$

Then, it is easy to see that the solution of the previous ODE is

$$u(x) = \begin{cases} \frac{1-\cos(\sqrt{\lambda})}{2\lambda\cos(\sqrt{\lambda})}\cos(\sqrt{\lambda}x) + \frac{3-\cos(\sqrt{\lambda})}{2\lambda\sin(\sqrt{\lambda})}\sin(\sqrt{\lambda}x) - \frac{1}{\lambda}, & \text{if} \quad -1 \leq x \leq 0, \\ \frac{1+\cos(\sqrt{\lambda})}{2\lambda\cos(\sqrt{\lambda})}\cos(\sqrt{\lambda}x) + \frac{3-\cos(\sqrt{\lambda})}{2\lambda\sin(\sqrt{\lambda})}\sin(\sqrt{\lambda}x) - \frac{2}{\lambda}, & \text{if} \quad 0 \leq x \leq 1. \end{cases}$$

Exercise 13. Write the weak formulation of the problem

$$\begin{cases} (x^2+1)u'' - xu' = \sin 2\pi x, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

Show that there exists a unique solution $u \in H_0^1(0,1)$ and determine an explicit constant C for which

$$\|u'\|_{L^2(0,1)} \leq C.$$

Does this solution minimize an energy functional?

Solution. Since we have homogeneous Dirichlet conditions we multiply the equation by any $v \in H_0^1(0,1)$ and integrate by parts in order to obtain the weak formulation. That is,

$$\int_0^1 (x^2+1)u''(x)\,v(x) - xu'(x)\,v(x)\,dx = \int_0^1 \sin(2\pi x)\,v(x)\,dx \qquad \forall v \in H_0^1(0,1),$$

is equivalent to

$$\int_0^1 (x^2+1)u'(x)v'(x) + 3xu'(x)v(x)\,dx = -\int_0^1 \sin(2\pi x)\,v(x)\,dx \qquad \forall v \in H_0^1(0,1).$$

Then, if we define

$$B(u,v) = \int_0^1 (x^2+1)u'(x)v'(x) + 3xu'(x)v(x)\,dx,$$

and

$$F(v) = -\int_0^1 \sin(2\pi x)\,v(x)\,dx,$$

we obtain the weak formulation: find $u \in H_0^1(0,1)$ such that $B(u,v) = F(v)$ for every $v \in H_0^1(0,1)$. Now, we want to apply Lax-Milgram theorem in order to prove existence and uniqueness of solution. First, it is clear that B is bilinear. By using Cauchy-Schwartz and Poincaré inequalities we get that it is continuous. That is,

$$\begin{aligned} |B(u,v)| &\leq \int_0^1 (x^2+1)|u'(x)||v'(x)| + 3x|u'(x)||v(x)|\,dx \\ &\leq 2||u'||_{L^2}||v'||_{L^2} + 3||u'||_{L^2}||v||_{L^2} \\ &\leq (2+3\,C_P)||u||_{H_0^1}||v||_{H_0^1}. \end{aligned}$$

Moreover,

$$\begin{aligned} B(u,u) &= \int_0^1 (x^2+1)|u'(x)|^2 + 3xu'(x)u(x)\,dx = \int_0^1 (x^2+1)|u'(x)|^2 - \frac{3}{2}|u(x)|^2\,dx \\ &\geq ||u'||_{L^2}^2 - \frac{3}{2}||u||_{L^2}^2 \geq \left(1 - \frac{3}{2}C_P^2\right)||u'||_{L^2}^2 = \left(1 - \frac{3}{2}C_P^2\right)||u||_{H_0^1}^2. \end{aligned}$$

Here, we need $2 - 3C_P^2 > 0$. This holds true because $C_P = 1/\pi$ (you can prove it by following the same idea that in the proof of Wirtinger inequality in Exercise 18 from Review List). Thus, B is coercive. On the other hand let us show that F is continuous. That is,

$$|F(v)| \leq \int_0^1 |\sin(2\pi x)||v(x)|\,dx \leq ||\sin(2\pi x)||_{L^2}||v||_{L^2} = \frac{\sqrt{2}}{2}||v||_{L^2} \leq \frac{C_P\sqrt{2}}{2}||v||_{H_0^1}.$$

Since B is bilinear, continuous and coercive, and F is bilinear, we can apply Lax-Milgram and obtain existence and uniqueness of weak solution. Finally, we use the previous computations and take $v = u$ in the weak formulation in order to get

$$\begin{aligned} ||u'||_{L^2}^2 = ||u||_{H_0^1}^2 &\leq \left(1 - \frac{3}{2}C_P^2\right)^{-1} B(u,u) = \left(1 - \frac{3}{2}C_P^2\right)^{-1} F(u) \\ &\leq \frac{C_P\sqrt{2}}{2}\left(1 - \frac{3}{2}C_P^2\right)^{-1} ||u'||_{L^2}, \end{aligned}$$

which means that

$$||u'||_{L^2} \leq \frac{C_P\sqrt{2}}{2 - 3C_P^2} = \frac{\pi\sqrt{2}}{2\pi^2 - 3} \approx 0.27.$$

Note that since the bilinear form B is not symmetric the solution is not the minimizer of an energy functional.

Exercise 14. Write the variational formulation of the problem

$$\begin{cases} \cos x u'' - \sin x u' - xu = 1, \quad 0 < x < \pi/6, \\ u'(0) = -u(0),\ u(\pi/6) = 0. \end{cases}$$

Discuss existence and uniqueness.

Solution. Given a Dirichlet condition at $\pi/6$ we take the functional space

$$V = \{v \in H^1(0,\pi/6) \quad \text{such that} \quad v(\pi/6) = 0\}.$$

Then, in order to find the weak formulation we multiply the equation by $v \in V$ and integrate by parts. That is, the weak formulation consists on finding $u \in V$ such that

$$\int_0^{\pi/6} \cos(x)\,u'(x)\,v'(x)\,dx + \int_0^{\pi/6} x\,u(x)\,v(x)\,dx - u(0)\,v(0) = -\int_0^{\pi/6} v(x)\,dx \quad \forall v \in V.$$

Note that in case the weak solution is regular enough we can recover the strong formulation from the weak one just by integrating by parts (first using a test function in H_0^1 to recover the differential equation and then a general test function to obtain the boundary conditions).

Then, if we define

$$B(u,v) = \int_0^{\pi/6} \cos(x)\,u'(x)\,v'(x)\,dx + \int_0^{\pi/6} x\,u(x)\,v(x)\,dx - u(0)\,v(0),$$

and

$$F(v) = -\int_0^{\pi/6} v(x)\,dx$$

we can prove existence and uniqueness of weak solution by applying Lax-Milgram theorem if we show that B is bilinear, continuous and coercive and F is linear and continuous. Before proving that let us do some preliminary computations. Given any $v \in V$ we have

$$|v(x)| = \left| \int_x^{\pi/6} v'(t)\,dt \right| \leq \int_0^{\pi/6} |v'(t)|\,dt \leq \sqrt{\pi/6}\,||v'||_{L^2} \quad \forall x \in [0, \pi/6].$$

In particular we have a Poincaré inequality in V, i.e., there is a positive constant such that $||v||_{L^2} \leq C_P||v'||_{L^2}$ for every $V \in V$, so we can define the scalar product in V as

$$(u, v)_V = \int_0^{\pi/6} u'(x)\, v'(x)\, dx.$$

Now, we can check the hypothesis of Lax-Milgram theorem are satisfied. Linearity (and bilinearity) comes trivially. Let us show continuity. On the one hand

$$\begin{aligned}|B(u,v)| &\leq \int_0^{\pi/6} |\cos(x)|\,|u'(x)|\,|v'(x)|\,dx + \int_0^{\pi/6} |x|\,|u(x)|\,|v(x)|\,dx + |u(0)|\,|v(0)| \\ &\leq ||u'||_{L^2}\,||v'||_{L^2} + \frac{\pi}{6}||u||_{L^2}\,||v||_{L^2} + \frac{\pi}{6}||u'||_{L^2}\,||v'||_{L^2} \\ &\leq C||u||_V\,||v||_V.\end{aligned}$$

Here we have used the preliminary computation (at $x = 0$). On the other hand,

$$|F(v)| \leq \int_0^{\pi/6} |v(x)|\,dx \leq \sqrt{\pi/6}||v||_{L^2} \leq C||v||_V.$$

Finally, concerning the coercivity we get

$$\begin{aligned}B(u,u) &= \int_0^{\pi/6} \cos(x)\,|u'(x)|^2\,dx + \int_0^{\pi/6} x\,|u(x)|^2\,dx - |u(0)|^2 \\ &\geq \frac{\sqrt{3}}{2}||u'||_{L^2}^2 - |u(0)|^2 \geq \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)||u'||_{L^2}^2 \\ &\geq C||u||_V^2.\end{aligned}$$

Here, we have crucially used again the preliminary estimates.

In conclusion we can apply Lax-Milgram theorem and we obtain existence and uniqueness of weak solution to the problem.

Exercise 15. Let $Q = (0,1)\times(0,1) \subset \mathbb{R}^2$. We minimize, for $v \in H_0^1(Q)$, the functional

$$E(v) = \int_Q \left\{\tfrac{1}{2}|\nabla v|^2 - xv\right\}\,dxdy.$$

Write the Euler-Lagrange equation and show that there exists a unique minimizer $u \in H_0^1(Q)$. Find an explicit formula for u.

Solution. If we define

$$F(v) = \int_Q x\,v(x,y)\,dx\,dy,$$

then, we know that $u \in H_0^1$ is a minimizer of E if and only if

$$(u, v)_{H_0^1} = F(v) \quad \forall v \in H_0^1.$$

Note that this is the weak form of the problem

$$\begin{cases} -\Delta u = x & \text{in } Q, \\ u = 0 & \text{on } \partial Q. \end{cases}$$

Then, if we prove that F is linear and continuous we obtain existence and uniqueness of minimizer by using Riesz theorem. On the one hand, linearity comes trivial. On the other hand

$$|F(v)| \leq \int_Q x\,|v(x,y)|\,dx\,dy \leq ||v||_{L^1(Q)} \leq ||v||_{L^2(Q)} \leq C_P ||v||_{H_0^1(Q)}.$$

Thus, the existence and uniqueness of solution in $H_0^1(Q)$ is guaranteed.

In order to obtain an explicit solution let us use Fourier series. We know that $\{\sin(\pi\,i\,x)\,\sin(\pi\,j\,y)\}_{i,j>0}$ is a basis in $H_0^1(Q)$. Then, we can write the solution as

$$u(x,y) = \sum_{i,j=1}^{\infty} a_{ij}\,\sin(\pi\,i\,x)\,\sin(\pi\,j\,y).$$

The condition $u = 0$ on ∂Q is automatically satisfied. We only need to check that $-\Delta u = x$. That is, if we write x in Fourier serie as

$$x = \sum_{i,j=1}^{\infty} c_{ij}\,\sin(\pi\,i\,x)\,\sin(\pi\,j\,y),$$

we obtain that

$$u(x,y) = \sum_{i,j=1}^{\infty} \frac{c_{i,j}}{\pi^2(i^2+j^2)}\,\sin(\pi\,i\,x)\,\sin(\pi\,j\,y).$$

Now, we compute the explicit value of $c_{i,j}$,

$$c_{i,j} = 4\int_Q x\,\sin(\pi\,i\,x)\,\sin(\pi\,j\,y)\,dx\,dy = 4\frac{(-1)^{i+1}+(-1)^{i+j}}{\pi^2\,i\,j}.$$

Thus, taking into account that $c_{i,j} = 0$ when j is even we finally obtain

$$u(x,y) = -\frac{8}{\pi^4}\sum_{l,m=1}^{\infty} \frac{(-1)^l}{l\,(l^2+(2m-1)^2)\,(2m-1)}\,\sin(\pi\,l\,x)\,\sin(\pi\,(2m-1)\,y).$$

Exercise 16. Let Ω be a bounded open subset of $\mathbb{R}^n$. Consider the linear second order differential operator L, defined as

$$Lu := -\mathrm{div}(A\nabla u) + b\cdot\nabla u + cu,$$

where $A = (a_{ij}(x))_{i,j=1,\ldots,n}$ is a symmetric matrix ($a_{ij} = a_{ji}$), and $a_{ij}, b_i, c \in L^\infty(\Omega)$. The operator L is said to be *uniformly elliptic* if there exists a constant $\lambda > 0$ such that

$$A\xi\cdot\xi \geq \lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n \text{ and a.a. } x \in \Omega.$$

(a) Write the weak formulation of the problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that the bilinear form is continuous.

(b) Prove that if L is uniformly elliptic with constant $\lambda > 0$, then the so-called *Gårding inequality*

$$\frac{\lambda}{2}\,\|u\|^2_{H^1_0(\Omega)} \leq B_L(u,u) + \gamma\|u\|^2_{L^2(\Omega)} \quad \text{for any } u \in H^1_0(\Omega),$$

holds true for some $\gamma > 0$ depending only on n, λ, $\|b_i\|_{L^\infty(\Omega)}$ and $\|c_-\|_{L^\infty(\Omega)}$. Here B_L is the bilinear form found in the previous part.

(c) Prove by using parts (a) and (b) that if L is uniformly elliptic with constant $\lambda > 0$, there exists a constant $c_\star \geq 0$, depending only on n, λ and $\|b_i\|_{L^\infty(\Omega)}$, such that if $c \geq c_\star$ a.e. in Ω, then for any $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H^1_0(\Omega)$ of the problem. Furthermore, u satisfies

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

for some constant $C > 0$ depending only on Ω and λ.

Solution. (a) It is easy to see by multiplying the equation by $v \in H^1_0$ and integrating by parts that the weak formulation is: find $u \in H^1_0$ such that

$$\int_\Omega A\nabla u\cdot\nabla v + \int_\Omega b\cdot\nabla u\, v + \int_\Omega c\, u\, v = \int_\Omega f\, v \qquad \forall v \in H^1_0.$$

If we define

$$B_L(u,v) = \int_\Omega A\nabla u\cdot\nabla v + \int_\Omega b\cdot\nabla u\, v + \int_\Omega c\, u\, v,$$

and

$$F(v) = \int_\Omega f\, v,$$

we can rewrite the weak formulation as: find $u \in H^1_0$ such that $B_L(u,v) = F(v)$ for every $v \in H^1_0(\Omega)$. Here, it is clear that B_L is bilinear. Moreover, we can see that it is continuous. That is,

$$\begin{aligned}
|B_L(u,v)| &\leq \int_\Omega |A\nabla u|\,|\nabla v| + ||b||_{L^\infty}\int_\Omega |\nabla u|\,|v| + ||c||_{L^\infty}\int_\Omega |u|\,|v| \\
&\leq ||A||_{L^\infty}\,||\nabla u||_{L^2}\,||\nabla v||_{L^2} + ||b||_{L^\infty}\,||\nabla u||_{L^2}\,||v||_{L^2} + ||c||_{L^\infty}\,||u||_{L^2}\,||v||_{L^2} \\
&\leq \left(||A||_{L^\infty} + C_P\,||b||_{L^\infty} + C_P^2\,||c||_{L^\infty}\right)||u||_{H^1_0}\,||v||_{H^1_0},
\end{aligned}$$

where we are using Poincaré inequality. Moreover, we are using that since $a_{ij}(x)$ are bounded, then there is a constant (that we call $||A||_{L^\infty}$) such that $|A(x)\,\xi| \leq ||A||_{L^\infty}|\xi|$.

(b) Let us proved the proposed inequality. Given any $u \in H_0^1(\Omega)$ we get

$$\begin{aligned} B_L(u,u) &= \int_\Omega A\nabla u \cdot \nabla u + \int_\Omega b(x) \cdot \nabla u \, u + \int_\Omega c(x)|u|^2 \\ &\geq \lambda \int_\Omega |\nabla u|^2 - \int_\Omega |b(x)| \, |\nabla u| \, |u| - \int_\Omega |c_-(x)| \, |u|^2 \\ &\geq \lambda ||\nabla u||_{L^2}^2 - ||b||_{L^\infty} \, ||\nabla u||_{L^2} ||u||_{L^2} - ||c_-||_{L^\infty} \, ||u||_{L^2}^2 \\ &\geq \lambda ||\nabla u||_{L^2}^2 - \frac{\lambda}{2} ||\nabla u||_{L^2}^2 - \frac{||b||_{L^\infty}^2}{2\lambda} ||\nabla u||_{L^2}^2 - ||c_-||_{L^\infty} \, ||u||_{L^2}^2 \\ &= \frac{\lambda}{2} ||u||_{H_0^1}^2 - \left(\frac{||b||_{L^\infty}^2}{2\lambda} + ||c_-||_{L^\infty} \right) ||u||_{L^2}^2. \end{aligned}$$

Note that in the last inequality we have used that $-2st \geq -s^2 - t^2$ with $s = \sqrt{\lambda/2} ||\nabla u||_{L^2}$ and $t = (2\lambda)^{-1/2} ||u||_{L^2} ||b||_{L^\infty}$. Thus, we have proved the desired inequality with $\gamma = (2\lambda)^{-1} ||b||_{L^\infty}^2 + ||c_-||_{L^\infty}$.

(c) In order to prove existence and uniqueness of solution we are showing that B_L is coercive and F is linear and continuous. The fact that F is linear holds trivially. Moreover,

$$|F(v)| \leq \int_\Omega |f| \, |v| \leq ||f||_{L^2} \, ||v||_{L^2} \leq C ||v||_{H_0^1}.$$

Now, let us prove that B_L is coercive when $c \geq c_* = (2\lambda)^{-1} ||b||_{L^\infty}^2 \geq 0$. That is, let us define the auxiliary operator

$$\tilde{L} = -\mathrm{div}(A\nabla u) + b \cdot \nabla u + (c - c_*)u.$$

Then, we have

$$B_{\tilde{L}}(u,u) = B_L(u,u) - c_* ||u||_{L^2}^2.$$

Finally, taking into account that this auxiliary operator satisfies the conditions of the previous part, and moreover $(c - c_*)_- = 0$ we arrive at

$$B_L(u,u) - c_* ||u||_{L^2}^2 = B_{\tilde{L}}(u,u) \geq \frac{\lambda}{2} ||u||_{H_0^1}^2 - \frac{||b||_{L^\infty}^2}{2\lambda} ||u||_{L^2}^2 = \frac{\lambda}{2} ||u||_{H_0^1}^2 - c_* ||u||_{L^2}^2.$$

Thus, B_L is coercive and we can apply Lax-Milgram theorem to prove existence and uniqueness of solution. Now, we prove an estimate of the norm of the solution using the coercivity of B, the continuity of F and the weak formulation of the problem. That is,

$$||\nabla u||_{L^2}^2 \leq \frac{2}{\lambda} B_L(u,u) = \frac{2}{\lambda} F(u) \leq \frac{2}{\lambda} ||f||_{L^2} ||u||_{L^2} \leq \frac{2C_P}{\lambda} ||f||_{L^2} ||\nabla u||_{L^2},$$

so

$$||u||_{H_0^1}^2 = ||u||_{L^2}^2 + ||\nabla u||_{L^2}^2 \leq (C_P^2 + 1) ||\nabla u||_{L^2}^2 \leq (C_P^2 + 1) \frac{4C_P^2}{\lambda^2} ||f||_{L^2}^2,$$

where C_P is the constant from Poincaré inequality, and depends only on Ω.

Exercise 17. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Given $f \in L^2(\Omega)$, a function $u \in H_0^2(\Omega)$ is said to be a weak solution of the following boundary-value for the *bilaplacian operator*

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

if it satisfies

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \quad \text{for any } v \in H_0^2(\Omega).$$

Prove that there exists a unique weak solution of the bilaplacian equation.

Solution. First, let us recall that

$$H_0^2 = \{u \in H^2(\Omega) \quad \text{such that} \;\; Tr(u) \equiv 0 \;\; \text{and} \;\; Tr(\nabla u) \equiv 0\}.$$

Indeed, it is defined as the closure of $C_c^\infty(\Omega)$ with the H^2 norm. In particular, both the function and the gradient "vanish" on the boundary. Then, it is easy to see that the weak formulation comes from multiplying the equation (here, Δ^2 means $\Delta \circ \Delta$) by any test function $v \in H_0^2$ and integrating by parts twice. Furthermore, note that due to the continuity of the trace operator, H_0^2 is a closed subspace.

Now, let us show that we can equipped H_0^2 with the scalar product

$$(u, v)_{H_0^2} = \int_\Omega \Delta u \, \Delta v.$$

It is clear that it is bilinear and symmetric. We only need to check that it is positive definite. That is, on the one hand, we have integrating by parts and using the boundary conditions that

$$\begin{aligned} \int_\Omega |D^2 u|^2 &= \int_\Omega \sum_{i,j=1}^n u_{ij}^2 = \sum_{i,j=1}^n \int_\Omega u_{ij}^2 = -\sum_{i,j=1}^n \int_\Omega u_{iij}\, u_j = \sum_{i,j=1}^n \int_\Omega u_{ii}\, u_{jj} \\ &= \int_\Omega \left(\sum_{i=1}^n u_{ii}\right)\left(\sum_{j=1}^n u_{jj}\right) = \int_\Omega |\Delta u|^2. \end{aligned}$$

On the other hand, since both u and ∇u belong to H_0^1 we can use Poincaré inequality to obtain

$$||u||_{L^2}^2 + ||\nabla u||_{L^2}^2 \le (1 + C_P^2)||\nabla u||_{L^2}^2 \le (1 + C_P^2)\, C_P^2 ||D^2 u||_{L^2}^2.$$

Then, we get $0 \le ||u||_{H^2}^2 \le (1 + C_P^2 + C_P^4)||\Delta u||_{L^2}^2 = (1 + C_P^2 + C_P^4)\,(u, u)_{H_0^2}$, and moreover $u = 0$ if and only if $(u, u)_{H_0^2} = 0$. In fact, with these computation we have seen that this new scalar product in H_0^2 is equivalent to the usual one in H^2. Finally, it also clear that

$$F(v) = \int_\Omega f\, v$$

is continuous from H_0^2 to $\mathbb{R}$. Indeed,

$$|F(v)| \le \int_\Omega |f|\,|v| \le ||f||_{L^2}\,||v||_{L^2} \le ||f||_{L^2}\,||v||_{H^2} \le C\,||v||_{H_0^2}.$$

Hence, we can apply Riesz theorem to deduce the existence and uniqueness of weak solution of the problem.

Let us comment that one could ask why we use a Banach space (H_0^2) where both the function and the gradient vanish while we only need the function and the normal derivative to be zero on the boundary. The reason is that they are in fact the same. That is, if we take a regular enough function such that it is zero on the boundary, then it is clear that any tangential derivative is zero. Therefore, if we also impose the normal derivative to be zero we finally obtain that the whole gradient is zero on the boundary.

Exercise 18. Let $\Omega \subset \mathbb{R}^n$ be a bounded, regular domain, and let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Consider the following problem:

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, d\sigma, \quad \text{for all } v \in H^1(\Omega).$$

(*a*) To which classical problem does it correspond the previous weak formulation? Can you solve it when $\lambda > 0$?

(*b*) Let us take $\lambda = 0$ and $g \equiv 0$. Prove that such a weak solution exists if and only if

$$\int_\Omega f \, dx = 0.$$

Furthermore, under this assumption the solution is unique (in $H^1(\Omega)$) up to an additive constant.

(*c*) Let us take $\lambda = 0$. In spirit of the previous part, find a necessary and sufficient condition on f and g for the problem to be uniquely solvable up to an additive constant.

Solution. (a) If we assume all the functions are regular enough and we integrate by parts we obtain

$$-\int_\Omega \Delta u \, v \, dx \int_{\partial\Omega} \nabla u \cdot \nu \, v + \lambda \int_\Omega uv \, dx = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, d\sigma, \quad \text{for all } v \in C^\infty.$$

Thus, if we first take $v = 0$ on $\partial\Omega$ we arrive at $-\Delta u + \lambda u = f$ in Ω. Then, we can replace this pointwise identity in the previous characterization and get

$$\int_{\partial\Omega} \nabla u \cdot \nu \, v = \int_{\partial\Omega} gv \, d\sigma, \quad \text{for all } v \in C^\infty,$$

which means that $u_\nu = g$ on $\partial\Omega$. Thus, summarizing, the weal formulation corresponds to the problem

$$\begin{cases} -\Delta u + \lambda u = f, & \text{in } \in \Omega, \\ \partial_\nu u = g, & \text{on } \partial\Omega. \end{cases}$$

In order to prove the existence and uniqueness of weak solution let us define

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \lambda \int_\Omega uv \, dx$$

and

$$F(v) = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma.$$

We only need to prove that B is bilinear, continuous and coercive, while F is linear and continuous in order to apply Lax-Milgram theorem. The linearity (and bilinearity) is trivial. Now, on the one hand

$$\begin{aligned}
|B(u,v)| &\leq \int_\Omega |\nabla u|\,|\nabla v| + \lambda \int_\Omega |u|\,|v| \leq ||\nabla u||_{L^2(\Omega)}\,||\nabla v||_{L^2(\Omega)} + \lambda ||u||_{L^2(\Omega)}\,||v||_{L^2(\Omega)} \\
&\leq ||u||_{H^1}||v||_{H^1} + \lambda ||u||_{H^1}||v||_{H^1} = (1+\lambda)\,||u||_{H^1}||v||_{H^1},
\end{aligned}$$

so B is continuous (in fact for every $\lambda \geq 0$). On the other hand,

$$\begin{aligned}
B(u,u) &= \int_\Omega |\nabla u|^2 + \lambda \int_\Omega |u|^2 \\
&\geq \min(1,\lambda)\left(||\nabla u||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}\right) = \min(1,\lambda)||u||^2_{H^1}.
\end{aligned}$$

At this point it is crucial $\lambda > 0$ in order to obtain coercivity. Finally, we show the continuity of F. That is,

$$\begin{aligned}
|F(v)| &\leq \int_\Omega |f|\,|v|\,dx + \int_{\partial\Omega} |g|\,|v|\,d\sigma \leq ||f||_{L^2(\Omega)}\,||v||_{L^2(\Omega)} + ||g||_{L^2(\partial\Omega)}\,||v||_{L^2(\partial\Omega)} \\
&\leq ||f||_{L^2(\Omega)}\,||v||_{L^2(\Omega)} + C||g||_{L^2(\partial\Omega)}\,||v||_{H^1} \\
&\leq \left(||f||_{L^2(\Omega)} + C||g||_{L^2(\partial\Omega)}\right)||v||_{H^1}.
\end{aligned}$$

Here, we have used the fact that the trace operator is a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$. Finally, we can apply Lax-Milgram to obtain existence and uniqueness of weak solution.

(b) Let us begin by proving that the condition is necessary. Just by taking $v \equiv 1$ as test function in the weak formulation we obtain that the integral of f must be zero. Note that $v \equiv 1$ is an admissible test function since it clearly belongs to $H^1(\Omega)$.

Now, we are proving that under this condition there is existence of solution and uniqueness up to additive constant. This means that if we have a solution $u \in H^1$, then $u+C$ is also a solution for any given $C \in \mathbb{R}$. It is easy to check this last property. In order to prove existence let us define the set

$$V = \{w \in H^1(\Omega) \text{ such that } \int_\Omega w(x)\,dx = 0\}.$$

We know by Exercise 4 in Homework 2 that V is a Hilbert space with the scalar product

$$(u,v)_V = \int_\Omega \nabla u(x)\,\nabla v(x)\,dx.$$

Then, let us study the following auxiliary problem: find $u_0 \in V$ such that

$$(u_0, v)_V = \int_\Omega f\,v = F(v) \quad \text{for all } v \in V.$$

Since F is linear and continuous we can apply Riesz theorem to deduce the existence and uniqueness of a weak solution $u_0 \in V$ to this auxiliary problem. Now, we can use the condition on f to easily check that $u_0 \in V$ also satisfies

$$(u_0, v)_V = \int_\Omega f\, v \quad \text{for all } v \in H^1.$$

The idea is to write any given $v \in H^1$ as $v = \tilde{v} + C$ with $\tilde{v} \in V$ and C certain constant (in fact it is the mean of v in Ω). Thus, we have shown the existence of a solution to the original problem.

Finally, let us prove the uniqueness up to additive constant. On the one hand, we already know that we can construct different solutions to the original problem by adding constants to the unique solution $u_0 \in V$ of the auxiliary problem. On the other hand, let us suppose there is a solution $u \in H^1$ that is not of this form. Then, it is clear that $\tilde{u} = u - |\Omega|^{-1} \int_\Omega u \in V$ must be a solution of the auxiliary problem. Nevertheless, by the uniqueness of solution to this problem we have that $\tilde{u} = u_0$. So we arrive at a contradiction since $u = u_0 + C$.

(c) In order to find a similar necessary condition we use the constant function as test function, as we have done in the previous part. In this case, we obtain that

$$\int_\Omega f = -\int_{\partial\Omega} g.$$

Once we have this necessary condition we can proceed exactly as in the previous part and prove the existence and uniqueness (up to additive constant) of solution via working in the Hilbert space V.

Exercise 19. Find the mistake in the following argument and discuss if the conclusion is correct although the reasoning is not. For $\Omega \subset \mathbb{R}^n$ bounded and regular, consider the Neumann problem

$$\begin{cases} -\Delta u + \mathbf{c} \cdot \nabla u = f, & \text{in } \in \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\mathbf{c} \in \mathcal{C}^1(\overline{\Omega})^n$ and $f \in L^2(\Omega)$. Let $V = H^1(\Omega)$ and

$$B(u, v) = \int_\Omega \{\nabla u \cdot \nabla v + (\mathbf{c} \cdot \nabla u) v\}.$$

If div$\mathbf{c} = 0$, we may write

$$\int_\Omega (\mathbf{c} \cdot \nabla u) u \, dx = \frac{1}{2} \int_\Omega \mathbf{c} \cdot \nabla(u^2) \, dx = \int_{\partial\Omega} u^2 \mathbf{c} \cdot \nu \, d\sigma.$$

Thus if $\mathbf{c} \cdot \nu \geq c_0 > 0$ then,

$$B(u, u) \geq \|\nabla u\|^2_{L^2(\Omega)} + c_0 \|u\|^2_{L^2(\partial\Omega)} \geq C \|u\|^2_{H^1(\Omega)}$$

so that B is coercive and the problem has a unique solution.

Solution. First, note that the conclusion of the exercise is not true. In fact, if you have a solution u, it is easy to show that $u + C$ is also a solution for any given constant $C \in \mathbb{R}$. Thus, there is no uniqueness of solution. The mistake comes from assuming both div$\mathbf{c} = 0$ and and $\mathbf{c} \cdot \nu \geq c_0 > 0$. This two conditions are incompatible. That is, by using the divergence theorem we arrive at a contradiction

$$0 = \int_\Omega \text{div}\mathbf{c} = \int_{\partial\Omega} c \cdot \nu \geq \int_{\partial\Omega} c_0 = c_0 \, |\partial\Omega| > 0.$$

Exercise 20. (The obstacle problem) Let Ω be a bounded, convex domain in $\mathbb{R}^n$ and let ψ be strictly concave in Ω and such that $\max \psi > 0$ and $\psi < 0$ on $\partial\Omega$. Let K be the set

$$K = \{v \in H_0^1 \ : \ v \geq \psi \text{ a.e. in } \Omega\}.$$

(a) Verify that K is a convex, closed subset of $H_0^1(\Omega)$.
(b) Show that there exists a unique function $u \in K$ that minimizes in K the functional

$$J(v) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx,$$

and that it is characterized by the following expression:

$$\int_\Omega (\nabla v - \nabla u) \cdot \nabla u \, dx \geq 0 \text{ for all } v \in K.$$

(c) Conclude that if $u \in H_0^2(\Omega)$, u solves the obstacle problem (b) if and only if

$$-\Delta u \geq 0, \quad u - \psi \geq 0 \quad \text{and} \quad \Delta u(u - \psi) = 0 \quad \text{a.e. in } \Omega.$$

In particular, u is a harmonic function on each open subset where $u > \psi$.

Interpretation of the problem: for $n = 2$ we interpret the graph of u as an elastic membrane fixed at the boundary of Ω. $J(u)$ is proportional to the potential energy of deformations of the membrane. The problem consists of finding the configuration of minimal energy (equilibrium) under the condition that the membrane cannot go below ψ, which is interpreted as an obstacle.

Solution. (a) First, it is easy to check that K is convex. That is, if we take $u, w \in K$ and $t \in [0, 1]$, it is clear by definition that $u(x) \geq \psi(x)$ and $w(x) \geq \psi(x)$ almost everywhere in Ω. Then, we obtain that

$$t \, u(x) + (1 - t) \, w(x) \geq t \, \psi(x) + (1 - t) \, \psi(x) = \psi(x) \text{ a.e. in } \Omega.$$

Hence, $t \, u + (1 - t) \, w \in K$.

Now, let us take a sequence $u_k \in K$ converging in H_0^1 to certain u. By the completeness of H_0^1 we know that $u \in H_0^1$. Moreover, we know that there exists a subsequence $u_{k_j} \in K$ converging to u pointwise almost everywhere. Then, since $u_{k_j}(x) \geq \psi(x)$ a.e. we can deduce that $u(x) \geq \psi(x)$ almost everywhere. Hence, $u \in K$, and we have shown that K is a closed set.

(b) Let us take $w \equiv 0 \in H_0^1$. It is clear that $u \notin K$ since $\max \psi > 0$. Then, we can apply the projection on convex theorem to ensure the existence and uniqueness of the projection $u = P_K w$ of $w \equiv 0$. Indeed, by the definition of the projection it satisfies

$$J(u) = ||u||_{H_0^1}^2 = ||P_K w - w||_{H_0^1}^2 = \min_{v \in K} ||v - w||_{H_0^1}^2 = \min_{v \in K} ||v||_{H_0^1}^2 = \min_{v \in K} J(v),$$

so the projection of zero is the unique minimizer of the functional. Moreover, the projection on convex theorem gives us a characterization of the projection. That is, u is the unique element in K satisfying

$$-(u, v-u)_{H_0^1} = (w - P_K w, v - P_K w)_{H_0^1} \leq 0 \text{ for all } v \in K.$$

Note that this is exactly the characterization of the minimizer we are asked to show.

(c) Assume first that $u \in H_0^2(\Omega) \cap K$ solves the obstacle problem. We are proving that it satisfies the three properties in the statement. The second one is trivial since $u \in K$. Now, let $\Phi \in C_c^\infty(\Omega)$ be such that $\Phi \geq 0$. In that case $u + \Phi \in K$ and from the variational characterization of u we obtain

$$0 \leq \int_\Omega \nabla u \cdot \nabla \Phi = -\int_\Omega \Delta u \, \Phi \quad \text{ for all } \Phi \in C_c^\infty(\Omega) \text{ such that } \Phi \geq 0.$$

This implies that $-\Delta u \geq 0$ a.e. in Ω.

For the last condition we consider two subsets of Ω

$$\Omega_+ = \{x \in \Omega : \psi(x) \geq 0\} \quad \text{ and } \quad \Omega_- = \{x \in \Omega : \psi(x) < 0\}.$$

By the maximimum principle, using that $-\Delta u \geq 0$ in Ω and that $u = 0$ on $\partial\Omega$, we obtain that

$$u \geq 0 \quad \text{ in } \Omega.$$

Define

$$\psi_+ = \max\{0, \psi\} \in H_0^1(\Omega)$$

From the fact tha $u \geq 0$ in Ω and $u \geq \psi$ we obtain that $u - \psi_+ \geq 0 \in \Omega$ and therefore

$$(u - \psi_+)\Delta u \leq 0 \quad \text{ a.e. in } \Omega.$$

As $\psi_+ \geq \psi$ in Ω, we can use the variational characterization with $v = \psi_+$ to obtain, after an integration by parts, that

$$\int_\Omega (u - \psi_+)\Delta u \geq 0.$$

Combining this with the previous pointwise estimate yields that $(u - \psi_+)\Delta u = 0$ in Ω. Using now that $\psi_+ = \psi$ in Ω_+ we obtain that

$$(u - \psi)\Delta u = 0 \quad \text{ in } \Omega_+ .$$

Since $u \geq 0$ and $\psi < 0$ on Ω_- we have that for any $\Phi \in C_c^\infty(\Omega_-)$ (note that Ω_- is open, because ψ is continuous)

$$u + t\Phi \in K \quad \text{ for small } t \text{ (depending on } \Phi).$$

Now, suing that u is a minimizer we get that

$$\left.\frac{d}{dt}\right|_{t=0} J(u + t\Phi) = 0,$$

which is equivalent to

$$-\int_{\Omega_-} \Delta u \Phi = 0 \quad \text{ for all } \Phi \in C_c^\infty(\Omega_-).$$

This means that $\Delta u = 0$ in Ω_- and we obtain

$$(u - \psi)\Delta u = 0 \quad \text{ in } \Omega_- .$$

Finally, the converse direction is much easier. Let us assume the three conditions are satisfied. Let $v \in K$ be given. Then $v \geq \psi$ and $v - u \geq \psi - u$ almost everywhere in Ω. Then, using that $-\Delta u \geq 0$ and $-\Delta u(\psi - u) = 0$ we arrive at

$$-\Delta u(v-u) \geq -\Delta u(\psi - u) = 0 \quad \text{a.e. in } \Omega.$$

If we integrate the previous inequality in Ω and use $v - u \in H_0^1(\Omega)$ in the integration by parts we obtain the variational characterization of the solution to the obstacle problem.

Exercise 21. (From Sobolev to Isoperimetric) We define the *perimeter* of a bounded, measurable set $\Omega \subset \mathbb{R}^n$ as the quantity

$$\mathrm{Per}(\Omega) := \sup\left\{ \int_\Omega \mathrm{div} X \, dx : X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ |X| \leq 1 \text{ in } \mathbb{R}^n \right\}.$$

(a) Show that if Ω has smooth boundary, its perimeter coincides with the usual measure of $\partial\Omega$.

(b) Prove that, for any $v \in C_c^\infty(\mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n} |\nabla w| \, dx = \sup\left\{ -\int_{\mathbb{R}^n} \nabla w \cdot X \, dx : X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ |X| \leq 1 \text{ in } \mathbb{R}^n \right\}.$$

(c) Let $\{\rho_\varepsilon\}$ be a family of smooth and radially symmetric mollifiers. Given a set Ω of finite perimeter, we define $u_\varepsilon := \chi_\Omega * \rho_\varepsilon$. Take advantage of ($ii$) to prove that for any $\varepsilon > 0$

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \, dx \leq \mathrm{Per}(\Omega).$$

(d) Using (c), verify that the Sobolev inequality (with exponent $p = 1$) implies the Isoperimetric inequality, with the same constant, i.e, given any bounded measurable set Ω,

$$|\Omega|^{\frac{n-1}{n}} \leq C\, \mathrm{Per}(\Omega),$$

for some positive constant C depending only on n.

Solution. (a) On the one hand, given any $X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ such that $|X| \leq 1$ we can obtain from divergence theorem that

$$\int_\Omega \mathrm{div} X \, dx = \int_{\partial\Omega} X \cdot \nu \leq \int_{\partial\Omega} |X| \leq \int_{\partial\Omega} 1 = |\partial\Omega|.$$

Thus, we clearly obtain $\mathrm{Per}(\Omega) \leq |\partial\Omega|$. Moreover, we have equality if we construct a vector field X such $|X| \leq 1$ and it coincides with ν on the boundary of Ω. Let us do it. If we take any $\eta \in C_c^\infty((-1,1))$ such that $0 = \eta(0) \leq \eta \leq 1$. Then, since Ω is regular we can take $\varepsilon > 0$ small enough such that given any $x \in \partial\Omega$, the vector field X defined as

$$X(x - \lambda\nu(x)) = \eta(\lambda/\varepsilon)\nu(x)$$

satisfies the desired properties.

(b) Let us denote by

$$V(w) = \sup\left\{ -\int_{\mathbb{R}^n} \nabla w \cdot X \, dx : X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ |X| \leq 1 \text{ in } \mathbb{R}^n \right\}.$$

On the one hand, it is clear as in part (a) that $V(w) \leq \int_{\mathbb{R}^n} |\nabla w| \, dx$. Now, let us define the vector field

$$X(x) = \begin{cases} -\dfrac{\nabla w(x)}{|\nabla w(x)|} & \text{if } \nabla w(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is clear that $X \notin C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, but $X \in L^1(\mathbb{R}^n;\mathbb{R}^n)$. Thus, by density we can find a sequence of vector fields $X_\varepsilon \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $X_\varepsilon \to X$ in L^1 as ε goes to zero. Hence

$$\begin{aligned} -\int_{\mathbb{R}^n} \nabla w \cdot X_\varepsilon &= -\int_{\mathbb{R}^n} \nabla w \cdot X + \int_{\mathbb{R}^n} \nabla w \cdot (X - X_\varepsilon) \\ &= \int_{\mathbb{R}^n} |\nabla w|^2 + \int_{\mathbb{R}^n} \nabla w \cdot (X - X_\varepsilon) \\ &\leq \int_{\mathbb{R}^n} |\nabla w|^2 + \varepsilon ||\nabla w||_{L^\infty}, \end{aligned}$$

and we obtain the equality.

(c) Let us take any $X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $|X| \leq 1$ in $\mathbb{R}^n$. By using the properties of the convolution we have

$$-\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot X = -\int_{\mathbb{R}^n} (\chi_\Omega * \nabla \rho_\varepsilon) \cdot X = -\int_{\mathbb{R}^n} \chi_\Omega (\nabla \rho_\varepsilon * X) = \int_\Omega \operatorname{div}(-\rho_\varepsilon * X).$$

Now, if we define $X_\varepsilon = -\rho_\varepsilon * X$, it is clear that $X_\varepsilon \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$. Moreover it satisfies

$$|X_\epsilon(x)| = \left| \int_{\mathbb{R}^n} \rho_\varepsilon(y) \, X(x-y) \, dy \right| \leq \int_{\mathbb{R}^n} \rho_\varepsilon(y) \; |X(x-y)| \; dy \leq \int_{\mathbb{R}^n} \rho_\varepsilon(y) \, dy = 1.$$

Therefore, we can use X_ε as an admissible vector field in the definition of the perimeter and we obtain

$$-\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot X \leq \int_\Omega \operatorname{div} X_\varepsilon \leq \operatorname{Per}(\Omega)$$

for every $X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, which means that

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \, dx \leq \operatorname{Per}(\Omega).$$

(d) On the one hand, if we apply now Sobolev inequality with $p = 1$, which means $p^* = n/(n-1)$ we obtain

$$||u_\varepsilon||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C ||\nabla u_\varepsilon||_{L^1(\mathbb{R}^n)} \leq C \operatorname{Per}(\Omega).$$

On the other hand, taking into account that $u_\varepsilon \to \chi_E$ in L^p for every p as $\varepsilon \to 0$ we conclude that

$$|E|^{\frac{n-1}{n}} = ||\chi_E||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C \operatorname{Per}(\Omega).$$

Exercise 22. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in H^1_0(\Omega) \cap L^3(\Omega)$ be a weak solution of the nonlinear equation

$$\begin{cases} -\Delta u = u - u^3 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e. that it satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega (u - u^3)\varphi \, dx \quad \text{for all } \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Prove that $|u| \leq 1$ a.e. in Ω.

Solution. First, note that if u is a weak solution, then $-u$ is a weak solution too. Hence, it is sufficient to prove that $u \leq 1$. In order to do it, the idea it to take $(u-k)^+$ as test functions and check that this function is indeed equal to 0 for every $k > 1$. Nevertheless, we need to take a bit more care since test functions for this problem are bounded. That is, given any $k > 1$ we define the function

$$\varphi_k = \min(1, \max(u - k, 0)),$$

and the sets

$$\Omega_k = \{x \in \Omega : k < u(x) < k + 1\} \quad \text{and} \quad \widetilde{\Omega}_k = \{x \in \Omega : k < u(x)\}.$$

Note that by the boundary conditions on u it is clear that $\varphi_k = 0$ on $\partial\Omega$. Hence, taking into account that $\nabla \varphi_k = \chi_{\Omega_k} \nabla u$ and $u - u^3 < 0$ in $\widetilde{\Omega}_k$ we obtain

$$0 \leq \int_{\Omega_k} |\nabla u|^2 = \int_{\widetilde{\Omega}_k} (u - u^3)(u - k)^+ \leq 0,$$

which means that $(u - k)^+ \equiv 0$ a.e. for every $k > 1$. Thus, we conclude that $u \leq 1$ almost everywhere.

Note that the arguments are simpler when u is a classical solution. That is, let us suppose by contradiction that there are points where the function is above 1. Since u is bounded and it is zero on the boundary, this means that the maximum (that is grater than 1) is achieved at an interior point $x_0 \in \Omega$. Therefore, if we evaluate the equation at this point we arrive at a contradiction since

$$0 \leq -\Delta u(x_0) = u(x_0) - u^3(x_0) < 0.$$

The first inequality comes from assuming x_0 to be the point where the maximum is achieved and the last one from being $u(x_0) > 1$.

Advanced Course in PDEs.
MIDTERM EXAM **May 7th, 2020.**

- *You have 2 hours for doing this exam.*
- *You must upload your answers via the corresponding task in Atenea (you don't have to upload the calculations, just your final choice in each question).*
- *There is only one correct answer for each question.*
- *A correct answer scores* +1 *(over* 8*), a wrong answer scores* −1/3 *(over* 8*), and an empty answer scores* 0.
- *You must mark at most one answer per question.*

Question 1. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, $\alpha \in \mathbb{R}$, and

$$K = \left\{ u \in H^1(\Omega) : \int_\Omega (\partial_{x_1} u - \alpha u) = 1 \right\}.$$

Then, $\inf_{u\in K} \|u\|_{H^1(\Omega)}$

(a) is always attained.
(b) is never attained.
(c) is attained if and only if $\alpha > 0$.
(d) is only attained for $\alpha = |\Omega|$.

Question 2. Let $u \in C^2(\Omega) \cap C(\overline{\Omega}) \subset H^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, and set $f = -\Delta u + u$. Let $v = P_{H^1_0(\Omega)} u$, where $P_{H^1_0(\Omega)}$ is the orthogonal projection of $H^1(\Omega)$ onto $H^1_0(\Omega)$. Then,

(a) v is the unique minimizer of $\inf_{w\in H^1_0(\Omega)} (\|w\|^2_{H^1(\Omega)} - \int_\Omega fw)$.
(b) v may not exist depending on u.
(c) v solves $-\Delta v + v = f$ in the distributional sense in Ω.
(d) none of the previous answers is correct.

Question 3. Let $\varphi(x) = (1 - |x|)\chi_{(-1,1)}$ for $x \in \mathbb{R}$. Given $\Omega \subset \mathbb{R}$, set

$$F_\Omega = \{\varphi(\cdot - t) : t \in \Omega\} \subset L^1(\mathbb{R}).$$

Then, $\overline{F_\Omega} \subset\subset L^1(\mathbb{R})$ (i.e., the closure of F_Ω with respect to the $\|\cdot\|_{L^1(\mathbb{R})}$ norm is compact in $L^1(\mathbb{R})$)

(a) if and only if Ω is finite.
(b) if and only if Ω is bounded.
(c) if and only if $|\Omega| < +\infty$.
(d) always.

Question 4. Let $\delta_0 \in \mathcal{D}'(\mathbb{R})$ be the Dirac delta measure in $\mathbb{R}$, that is, $\delta_0(\varphi) = \varphi(0)$ for all $\varphi \in C^\infty_c(\mathbb{R})$. Then,

(a) $\sup_{\varphi\in C^\infty_c(\mathbb{R}),\, \varphi\not\equiv 0} \frac{|\delta_0(\varphi)|}{\int_\mathbb{R} |\varphi'|} = +\infty$.
(b) there exists a unique $u \in H^1((-1,1))$ such that $\delta_0(\varphi) = \int_{-1}^1 u'\varphi'$ for all $\varphi \in C^\infty_c((-1,1))$.
(c) there exist several $u \in H^1((-1,1))$ such that $\delta_0(\varphi) = \int_{-1}^1 u'\varphi'$ for all $\varphi \in C^\infty_c((-1,1))$.
(d) none of the previous answers is correct.

Question 5. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Assume that $u \in H^1(\Omega)$ weakly solves the Neumann problem $\Delta u = \pi + u\, e^{\sin(u)}$ in Ω and $\nabla u \cdot \nu = 0$ on $\partial\Omega$, where ν denotes the outward unit normal vector on $\partial\Omega$. Then,

(a) $v = u + \pi$ weakly solves $\Delta v = -\pi + v\, e^{-\sin(v)}$ in Ω.
(b) $u \geq \pi$ in Ω.
(c) $u \leq 0$ in Ω.
(d) none of the previous answers is correct.

Question 6. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with outward unit normal vector ν, A a non-symmetric $n \times n$ matrix (and A^* its adjoint), $a \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. If u solves

$$\begin{cases} u \in H^1(\Omega), \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v + \int_\Omega auv = \int_\Omega fv \quad \text{for all } v \in H^1(\Omega) \end{cases}$$

then u is a weak solution to

(a) $\operatorname{div}(A\nabla u) + au = f$ in Ω and $\nabla u \cdot A\nu = 0$ on $\partial\Omega$.
(b) $\operatorname{div}(A\nabla u) + au = f$ in Ω and $\nabla u \cdot A^*\nu = 0$ on $\partial\Omega$.
(c) $-\operatorname{div}(A\nabla u) + au = f$ in Ω and $\nabla u \cdot A\nu = 0$ on $\partial\Omega$.
(d) $-\operatorname{div}(A\nabla u) + au = f$ in Ω and $\nabla u \cdot A^*\nu = 0$ on $\partial\Omega$.

Question 7. Let us consider the function $u : \mathbb{R}^n \to \mathbb{R}$ defined as

$$u(x) = \cos\left(\frac{3\pi}{2}|x|^2\right).$$

(a) It is harmonic in $B_1 := \{x \in \mathbb{R}^n : |x| < 1\}$.
(b) It is not harmonic in B_1, but for any given $\varepsilon > 0$ there is a harmonic function w_ϵ such that $||u - w_\varepsilon||_{L^\infty(B_1)} < \varepsilon$.
(c) There is no harmonic function $w_{1/2}$ such that $||u - w_{1/2}||_{L^\infty(B_1)} < 1/2$ but there is one, w_2, such that $||u - w_2||_{L^\infty(B_1)} < 2$.
(d) There is no harmonic function w_2 such that $||u - w_2||_{L^\infty(B_1)} < 2$.

Question 8. Let $f \in L^2(B_1)$, where $B_1 := \{x \in \mathbb{R}^n : |x| < 1\}$. Consider the functional

$$E(w) = \int_{B_1} \left\{\frac{1}{2}|\nabla w(x)|^2 + \frac{1}{2}|w(x)|^2 - f(x)w(x)\right\} dx.$$

Let u and v weakly satisfy the following problems

$$\begin{cases} -\Delta u + u = f & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

and

$$\begin{cases} -\Delta v + v = f & \text{in } B_1, \\ \partial_\nu v = 0 & \text{on } \partial B_1. \end{cases}$$

Then,

(a) $E(u) \leq E(v)$ independently of f.
(b) $E(u) \geq E(v)$ independently of f.
(c) $E(u) \leq E(v)$ or $E(u) \geq E(v)$ depending on f.
(d) $E(u) \leq E(v)$ or $E(u) \geq E(v)$ depending on f and v (since the is no uniqueness of the Neumann problem).

Advanced Course in PDEs.
FINAL EXAM **June 2nd, 2020.**

- *You have 3 hours for doing this exam.*
- *You must upload your scanned answers via the corresponding task in Atenea.*
- *You must give a complete explanation of the development of your answers. You can use all the results from the theory/exercises lessons (without giving their proofs), but you must cite them properly verifying all the corresponding hypothesis.*
- *In each exercise, you can use a stage to solve the others, even if you have not solved it (say, you can use* (a) *to prove* (b), *etc.).*

Problem 1. *(5 points)* Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an open and bounded set. Let $\alpha \geq 0$ and set

$$a_\alpha(x) = \alpha \log(1 + |x|^2)$$

for all $x \in \mathbb{R}^n$. Consider the problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\Delta u + a_\alpha u = 1 \quad \text{in } \Omega. \end{cases} \tag{1}$$

(a) *(0.5 points)* Show that there exists a unique weak solution to (1). We call it u_α.
(b) *(1 point)* Prove that u_α is of class C^∞ in Ω.
(c) *(1 point)* Show that $0 \leq u_\alpha \leq u_0$ in Ω.
(d) *(0.5 points)* Let $\lambda_1(\Omega)$ be the first eigenvalue of the Dirichlet Laplacian in Ω. Prove that

$$\|u_\alpha\|_{L^2(\Omega)} \leq \frac{\sqrt{|\Omega|}}{\lambda_1(\Omega)}.$$

(e) *(1 point)* Show that there exist $\alpha_k > 0$ with $\alpha_k \to 0$ as $k \to +\infty$, for which

$$\lim_{k \to +\infty} \|u_{\alpha_k} - u_0\|_{L^2(\Omega)} = 0.$$

(f) *(1 point)* Compute u_0 for $n = 2$ and $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$.

Solution.

(a) We say that u is a weak solution to (1) iff

$$\begin{cases} u \in H_0^1(\Omega), \\ \int_\Omega \nabla u \cdot \nabla v + \int_\Omega a_\alpha uv = \int_\Omega v \quad \text{for all } v \in H_0^1(\Omega). \end{cases} \tag{2}$$

Since Ω is bounded, we have $a_\alpha \in L^\infty(\Omega)$. Also, $a_\alpha \geq 0$ in Ω. With this at hand, one can simply apply the theorem in [Chapter 3, page 3] of the handwritten notes (derived from Lax-Milgram theorem) to get the existence and uniqueness of the weak solution u_α.

(b) Observe that $a_\alpha \in C^\infty(\Omega)$ (because $1 + |x|^2 > 0$ for all $x \in \Omega$, and $x \mapsto |x|^2$ is a C^∞ function) and that $D^\beta a_\alpha \in L^\infty(\Omega)$ for all $\beta \geq 0$ (because Ω is bounded). Then, the fact that $-\Delta u_\alpha = 1 - a_\alpha u_\alpha$ weakly in $H_0^1(\Omega)$ suggests that $u_\alpha \in C^\infty(\Omega)$ by a bootstrap argument. We now prove this rigorously.

Since $u_\alpha \in H_0^1(\Omega) \subset L^2(\Omega)$, $a_\alpha \in L^\infty(\Omega)$, and Ω is bounded, we have $f := 1 - a_\alpha u_\alpha \in L^2(\Omega)$. Also, from (2) we deduce that

$$\begin{cases} u \in H_0^1(\Omega) \subset H^1(\Omega), \\ \int_\Omega \nabla u_\alpha \cdot \nabla v = \int_\Omega fv \quad \text{for all } v \in H_0^1(\Omega). \end{cases} \tag{3}$$

Therefore, using the theorem on "H^2–regularity" in [Chapter 4, page 9] of the handwritten notes, we deduce that $u_\alpha \in H^2(\Omega')$ for all $\Omega' \subset\subset \Omega$. Given one such Ω', we see that $f = 1 - a_\alpha u_\alpha \in H^2(\Omega')$ because $D^\beta a_\alpha \in L^\infty(\Omega)$ for all $\beta \leq 2$. Using now (3) and the theorem on "improving regularity" in [Chapter 4, page 12] of the handwritten notes, but applied on Ω', we deduce that $u_\alpha \in H^4(\Omega'')$ for all $\Omega'' \subset\subset \Omega'$. Since Ω' was arbitrary, we conclude that $u_\alpha \in H^2(\Omega')$ for all $\Omega' \subset\subset \Omega$. Now, we can iterate this argument to deduce that indeed $u_\alpha \in H^k(\Omega')$ for all $\Omega' \subset\subset \Omega$ and all $k \geq 1$.

Given $x_0 \in \Omega$, let $\epsilon > 0$ such that $B_\epsilon(x_0) \subset \Omega$. Take $\eta \in \mathcal{D}(\mathbb{R}^n)$ such that

$$\chi_{B_{\epsilon/2}(x_0)} \leq \eta \leq \chi_{B_\epsilon(x_0)}.$$

Since $u_\alpha \in H^k(\Omega')$ for all $\Omega' \subset\subset \Omega$ and all $k \geq 1$, we have $\eta u_\alpha \in H_0^k(B_\epsilon(x_0))$ for all $k \geq 1$. Finally, using that $k - n/2 \to +\infty$ as $k \to +\infty$ and the theorem on "regularity of Sobolev spaces" in [Chapter 4, page 5] of the handwritten notes, we obtain $\eta u_\alpha \in C^q(\overline{B_\epsilon(x_0)})$ for all integer $q \geq 0$. But $\eta u_\alpha = u_\alpha$ in $B_{\epsilon/2}(x_0)$, which yields that u_α is of class C^∞ in $B_{\epsilon/2}(x_0)$. Since $x_0 \in \Omega$ was arbitrary, we conclude that u_α is of class C^∞ in Ω.

(c) We first prove that $u_\alpha \geq 0$ in Ω. Set $Lu := -\Delta u + a_\alpha u$. By (1) we have $Lu = 1 \geq 0$ in $\mathcal{D}'(\Omega)$. Also, $u_\alpha \in H_0^1(\Omega)$ yields $u_\alpha \geq 0$ on $\partial\Omega$. Therefore, since $a_\alpha \geq 0$, the weak maximum principle for the operator L with Dirichlet boundary data (the theorem in [Chapter 3, page 5]), we deduce that $u \geq 0$ in Ω.

We now address the inequality $u_\alpha \leq u_0$ in Ω. From (1) we have $-\Delta u_\alpha = 1 - a_\alpha u_\alpha$ and $-\Delta u_0 = 1$ (since $a_0 = 0$). Therefore, $-\Delta(u_\alpha - u_0) = 1 - a_\alpha u_\alpha - 1 = -a_\alpha u_\alpha \leq 0$ in $\mathcal{D}'(\Omega)$, since $a_\alpha, u_\alpha \geq 0$. Also, $u_\alpha - u_0 \in H_0^1(\Omega)$, which yields $u_\alpha - u_0 \leq 0$ in $\partial\Omega$. Using again the weak maximum principle with Dirichlet boundary data, but this time for the operator $-\Delta$, we get $u_\alpha - u_0 \leq 0$ in Ω.

(d) Combining the Rayleigh quotient (the theorem in [Chapter 5, page 5]), $a_\alpha \geq 0$, (2) with $v = u_\alpha$, and Cauchy-Schwarz inequality, we see that

$$\begin{aligned}\lambda_1(\Omega)\|u_\alpha\|^2_{L^2(\Omega)} &= \lambda_1(\Omega)\int_\Omega |u_\alpha|^2 \leq \int_\Omega |\nabla u_\alpha|^2 \\ &\leq \int_\Omega |\nabla u_\alpha|^2 + \int_\Omega a_\alpha |u_\alpha|^2 = \int_\Omega u_\alpha \leq \sqrt{|\Omega|}\|u_\alpha\|_{L^2(\Omega)}.\end{aligned}$$

Dividing this by $\lambda_1(\Omega)\|u_\alpha\|_{L^2(\Omega)}$ we get the desired result.

(e) Combining the developments in (d) with (c) we see that

$$\int_\Omega |\nabla u_\alpha|^2 \leq \sqrt{|\Omega|}\|u_\alpha\|_{L^2(\Omega)} \leq \sqrt{|\Omega|}\|u_0\|_{L^2(\Omega)} < +\infty.$$

Therefore, $\int_\Omega |\nabla u_\alpha|^2$ is uniformly bounded for all $\alpha \geq 0$. In particular, there exists $C > 0$ such that $\int_\Omega |\nabla u_{1/j}|^2 < C$ for all integer $j \geq 1$. Since Ω is bounded, by the Remark in [Chapter 2, page 13] (derived from Fréchet-Kolmogorov theorem on $H_0^1(\Omega) \subset\subset L^2(\Omega)$), we see that there exist $u_* \in H_0^1(\Omega)$ and a subsequence $\{\alpha_k\}_{k\geq 1}$ of $\{1/j\}_{j\geq 1}$ such that

$$\begin{cases} \|u_{\alpha_k} - u_*\|_{L^2(\Omega)} \to 0 \quad \text{as } k \to +\infty, \\ u_{\alpha_k} \rightharpoonup u_* \quad \text{weakly in } H_0^1(\Omega) \text{ as } k \to +\infty. \end{cases}$$

Hence, to solve (e), we only need to show that $u_* = u_0$. By taking in $H_0^1(\Omega)$ the scalar product $(u, v)_{H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v$, that $u_{\alpha_k} \rightharpoonup u_*$ as $k \to +\infty$ means that

$$\int_\Omega \nabla u_{\alpha_k} \cdot \nabla v \to \int_\Omega \nabla u_* \cdot \nabla v$$

as $k \to +\infty$ for all $v \in H_0^1(\Omega)$. But (2) and $\|u_{\alpha_k} - u_*\|_{L^2(\Omega)} \to 0$ yield

$$\int_\Omega \nabla u_{\alpha_k} \cdot \nabla v = \int_\Omega (1 - a_{\alpha_k} u_{\alpha_k})v \to \int_\Omega (1 - a_0 u_*)v = \int_\Omega v.$$

Therefore, $u_* \in H_0^1(\Omega)$ satisfies $\int_\Omega \nabla u_* \cdot \nabla v = \int_\Omega v$ for all $v \in H_0^1(\Omega)$. Since the solution to (2) is unique by (a), we conclude that $u_* = u_0$.

(f) Set $n = 2$, $\alpha = 0$, and $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$. We will find the classical solution to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

which will be the weak solution u_0 to (1) for $\alpha = 0$.

We know that Δ is invariant under rotations, that is, if O is an orthogonal matrix then $\Delta(u(Ox)) = (\Delta u)(Ox)$. Therefore, if u solves (4) and we set $v(x) = u(Ox)$, then v also solves (4) because Ω, the nonlinear term, and the boundary condition are invariant under the rotation O. This shows that $u - v$ is a harmonic function in Ω whose boundary values vanish identically on $\partial\Omega$. By the maximum (and minimum) principle, we get that $u - v \equiv 0$ in Ω, which means that $u(x) = u(Ox)$. Since this holds for all orthogonal matrix O, we conclude that u is a radial function in Ω. Therefore, setting $r = |x|$, we can write $u(x) =: U(|x|) = U(r)$, where $1 \le r \le 2$.

Now, we can use the expression of the Laplacian in $\mathbb{R}^2$ for radial functions to deduce that

$$-1 = \Delta u = \frac{\partial^2}{\partial r^2} u + \frac{1}{r}\frac{\partial}{\partial r} u = U'' + \frac{1}{r} U'. \tag{5}$$

Multiplying both sides by r we get

$$-r = rU''(r) + U'(r) = (rU'(r))'$$

and, thus, $U'(r) = -\frac{r}{2} + \frac{c_1}{r}$ for some $c_1 \in \mathbb{R}$. Integrating once more in r, we obtain

$$U(r) = -\frac{r^2}{4} + c_1 \log(r) + c_2$$

for some c_1, $c_2 \in \mathbb{R}$. Imposing now the boundary condition $u = 0$ on $\partial\Omega$, which corresponds to $U(1) = U(2) = 0$, we can determine c_1 and c_2. Putting everything together and using that $u(x) = U(|x|)$, we conclude that

$$u_0(x) = -\frac{|x|^2}{4} + \frac{3\log(|x|)}{4\log(2)} + \frac{1}{4}.$$

Problem 2. *(5 points)* Let $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ be an open, bounded, and smooth domain. Let $u = u(x,t)$ be a smooth function satisfying

$$\begin{cases} u_t - \Delta u + u = 1 & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = ctt & \text{on } \partial\Omega \times (0,\infty), \\ \int_{\partial\Omega} u = 0 & \text{for every } t \in (0,\infty), \\ u(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{6}$$

(a) *(1 point)* Show that $\partial_\nu u(x,t) \le 0$ for every $x \in \partial\Omega$ and $t \in (0,+\infty)$.

(b) *(1 point)* Prove that there exists at most one smooth solution to (6).

(c) *(2 points)* Assume that $u^s : \Omega \to \mathbb{R}$ is smooth, such that $\int_{\partial\Omega} u^s = 0$, and it is a local minimizer of the functional

$$E(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 + \frac{1}{2}\int_\Omega |w|^2 - \int_\Omega w$$

among smooth functions $w : \Omega \to \mathbb{R}$ such that $\int_{\partial\Omega} w = 0$. Prove that it is the unique global minimizer and, moreover, it is a stationary solution of problem (6).

(d) *(1 point)* Prove that $u(\cdot,t)$ converges to u^s in $L^2(\Omega)$-norm as t goes to infinity. Moreover, show that

$$\|u(\cdot,t) - u^s\|_{L^2(\Omega)} \le e^{-t}\,\|u^s\|_{L^2(\Omega)} \quad \text{for every } t \in [0,\infty).$$

Solution.

(a) Let us take any $T > 0$. Since u is continuous and Ω is bounded, there is a point $(x_0, t_0) \in \overline{\Omega} \times [0, T]$ such that

$$u(x_0, t_0) = \inf_{(x,t) \in \overline{\Omega} \times [0,T]} u(x, t).$$

By the mean property on the boundary, it is clear that $u(x_0, t_0) \leq 0$. It is easy to check that $(x_0, t_0) \notin \Omega \times (0, T]$. If $(x_0, t_0) \in \Omega \times (0, T]$, then $u_t(x_0, t_0) \leq 0$ and $-\Delta u(x_0, t_0) \leq 0$. Thus, we arrive at a contradiction with the equation

$$0 \geq u_t(x_0, t_0) - \Delta u(x_0, t_0) + u(x_0, t_0) = 1.$$

Now, we distinguish two cases: either the infimum is strictly negative or zero. We will show that in both cases we can take (x_0, t_0) on the lateral boundary. In the first case, it is obvious from the initial condition. In the second case, the infimum of u is zero and the average of $u(\cdot, t)$ on $\partial\Omega$ is also zero for every t, therefore $u \equiv 0$ on the lateral boundary and we can take (x_0, t_0) there. Once we know the infimum is attained on the lateral boundary it is clear that $u_\nu(x_0, t_0) \leq 0$. Moreover, since the normal derivative is constant along the lateral boundary we conclude that

$$\partial_\nu u(x, t) \leq 0 \text{ for every } x \in \partial\Omega \text{ and } t \in (0, \infty).$$

(b) Let us assumme that there are two solutions u_1 and u_2 to (6). We will prove that the difference $w := u_1 - u_2$ is zero. That is, it is clear that w satisfies

$$\begin{cases} w_t - \Delta w + w = 0 & \text{in } \Omega \times (0, \infty), \\ \partial_\nu w = ctt & \text{on } \partial\Omega \times (0, \infty), \\ \int_{\partial\Omega} w = 0 & \text{for every } t \in (0, \infty), \\ w(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

Now, by using the equation and the boundary conditions (when integrating by parts) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega w^2(x, t)\, dx &= \int_\Omega w_t(x, t)\, w(x, t)\, dx \\ &= \int_\Omega \Delta w(x, t)\, w(x, t)\, dx - \int_\Omega w^2(x, t)\, dx \\ &= \int_{\partial\Omega} w_\nu(x, t)\, w(x, t) - \int_\Omega |\nabla w(x, t)|^2\, dx - \int_\Omega w^2(x, t)\, dx \\ &= ctt \int_{\partial\Omega} w(x, t) - \int_\Omega |\nabla w(x, t)|^2\, dx - \int_\Omega w^2(x, t)\, dx \\ &= -\int_\Omega |\nabla w(x, t)|^2\, dx - \int_\Omega w^2(x, t)\, dx \\ &\leq 0. \end{aligned}$$

This means that $G(t) := \int_\Omega w^2(x, t)\, dx$ is a nonincreasing function in t. Since by definition $G(t)$ is a nonnegative function and $G(0) = 0$ by the initial condition, we conclude that $G(t) = 0$ for every $t \geq 0$. It is clear that $G \equiv 0$ is equivalent to $w \equiv 0$.

(c) Let us take $\eta \in C^\infty(\Omega)$ such that $\int_{\partial\Omega} \eta = 0$. Then, it is clear that $u^s + \varepsilon\, \eta$ is an admissible competitor for every $\varepsilon \in \mathbb{R}$. By using the minimality properties of u^s it is clear that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E(u^s + \varepsilon\eta) = 0.$$

Now, if we use the explicit form of the functional and integration by parts we obtain

$$\begin{aligned} 0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E(u^s + \varepsilon\eta) &= \int_\Omega \nabla u^s \cdot \nabla\eta + \int_\Omega u^s\, \eta - \int_\Omega \eta \\ &= \int_\Omega (-\Delta u^s + u^s - 1)\, \eta + \int_{\partial\Omega} \partial_\nu u^s\, \eta \end{aligned} \tag{7}$$

for every $\eta \in C^\infty(\Omega)$ such that $\int_{\partial\Omega} \eta = 0$. Note that in particular we can take any $\eta \in C_c^\infty(\Omega)$ and obtain that

$$-\Delta u^s + u^s = 1 \qquad \text{in} \;\; \Omega,$$

since the boundary term vanishes in that case. Once we have this condition we can return to (7), which turns to be now

$$\int_{\partial\Omega} \partial_\nu u^s \, \eta = 0 \quad \forall \eta \in C^\infty(\Omega) \quad \text{such that} \int_{\partial\Omega} \eta = 0.$$

If we take $\eta = \partial_\nu u^s - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \partial_\nu u^s$ as a test function, we obtain that

$$0 = \int_{\partial\Omega} |\partial_\nu u^s|^2 - \frac{1}{|\partial\Omega|} \left| \int_{\partial\Omega} \partial_\nu u^s \right|^2 \geq 0.$$

Here, we have used Cauchy-Schwartz inequality with the functions $\partial_\nu u^s$ and 1. Since equality is only achieved in Cauchy-Schwartz inequality when the functions are proportional, we conclude that

$$\partial_\nu u^s = ctt \qquad \text{on} \;\; \partial\Omega.$$

Therefore, u^s is a solution of the stationary problem. Now, let us check that it is in fact the unique global minimizer. That is, given any v smooth such that $\int_{\partial\Omega} v = 0$ we have

$$\begin{aligned} E(v) &= E(u^s) + \frac{1}{2} \int_\Omega \nabla(v - u^s) \cdot \nabla(v + u^s) + \frac{1}{2} \int_\Omega (v - u^s)\,(v + u^s) - \int_\Omega (v - u^s) \\ &= E(u^s) + \frac{1}{2} \int_\Omega |\nabla(v - u^s)|^2 + \frac{1}{2} \int_\Omega |(v - u^s)|^2 + \int_\Omega (-\Delta u^s + u^s - 1)\,(v - u^s) \\ &= E(u^s) + \frac{1}{2} \int_\Omega |\nabla(v - u^s)|^2 + \frac{1}{2} \int_\Omega |(v - u^s)|^2 \\ &\geq E(u^s), \end{aligned}$$

with equality if and only if $v = u^s$. Note that this global minimality property comes from the convexity of the functional E.

(d) Let us call $\widetilde{w}(x,t) = u^s(x) - u(x,t)$. Then, it is clear that it satisfies

$$\begin{cases} \widetilde{w}_t - \Delta \widetilde{w} + \widetilde{w} = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_\nu \widetilde{w} = ctt & \text{on } \partial\Omega \times (0,\infty), \\ \int_{\partial\Omega} \widetilde{w} = 0 & \text{for every } t \in (0,\infty), \\ \widetilde{w}(x,0) = u^s(x) & \text{in } \Omega. \end{cases}$$

Then, by repeating the kind of computation from part (b) and denoting $\widetilde{G}(t) := \int_\Omega \widetilde{w}^2(x,t)\,dx$, we find that $\widetilde{G}$ satisfies

$$\frac{1}{2} \widetilde{G}'(t) \leq -\widetilde{G}(t) \qquad \text{for every} \quad t > 0,$$

which is equivalent to

$$\frac{d}{dt} \left(e^{2t} \widetilde{G}(t) \right) \leq 0 \qquad \text{for every} \quad t > 0.$$

This means that

$$e^{2t} \widetilde{G}(t) \leq \widetilde{G}(0) = \|u^s\|^2_{L^2(\Omega)} \quad \text{for every } t \in [0,\infty).$$

Finally, from here we directly deduce

$$\|u(\cdot,t) - u^s\|_{L^2(\Omega)} \leq e^{-t}\, \|u^s\|_{L^2(\Omega)} \quad \text{for every } t \in [0,\infty).$$

This clearly means that u converges to u^s in $L^2(\Omega)$-norm as t goes to infinity.

Let us remark that indeed we can obtain an estimate where the right-hand side depends only on Ω (and not on u^s). We only need to multiply the stationary equation by u^s, integrate by parts and use Cauchy-Schwartz to obtain

$$\|u^s\|_{L^2(\Omega)}^2 \leq \int_\Omega u^s(x)\,dx \leq \|u^s\|_{L^2(\Omega)}\,|\Omega|^{1/2}.$$

Advanced Course in PDEs.
MIDTERM EXAM **April 7th, 2021.**

- *You have 2 hours for doing this exam. Then, you must upload your scanned answers via the corresponding task in Atenea.*
- *You must give a complete explanation of the development of your answers. You can use all the results from the theory/exercises lessons (without giving their proofs), but you must cite them properly verifying all the corresponding hypothesis.*
- *In each exercise, you can use a stage to solve the others, even if you have not solved it (say, you can use* (a) *to prove* (b)*, etc.).*

Problem 1. *(2.5 points)* Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an open set, and $g \in L^2(\Omega)$. Consider the set

$$K = \left\{ v \in L^2(\Omega) : \, v(x) \leq g(x) \text{ for a.e. } x \in \Omega \right\}.$$

(a) *(1.25 points)* Show that K is a non-empty, closed, and convex subset of $L^2(\Omega)$.

(b) *(1.25 points)* Prove that, for every $u \in L^2(\Omega)$, the orthogonal projection of u on K is the function $(P_K u)(x) = \min\{u(x), g(x)\}$ for a.e. $x \in \Omega$.

Problem 2. *(2.5 points)* Let $\Omega = (-1, 1) \subset \mathbb{R}$.

(a) *(1.25 points)* Show that it does not exist $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega f\varphi = \varphi(0) \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

(b) *(1.25 points)* Let $u(x) = |x|$ for $x \in \Omega$. Prove that $u \in H^1(\Omega)$ but $u \notin H^2(\Omega)$.

Problem 3. *(5 points)* Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an open and bounded set with smooth boundary.

(a) *(0.75 points)* Let $u \in C^4(\Omega) \cap C(\overline{\Omega})$ be such that

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where Δ^2 is the bilaplacian operator, defined as $\Delta^2 u = \Delta(\Delta u)$. Show that $u \equiv 0$, or give an example of a nontrivial (not identically zero) function u and a set Ω such that (1) is satisfied.

(b) *(1.5 points)* Assuming the existence, prove uniqueness of $C^4(\Omega) \cap C^3(\overline{\Omega})$ solution for the problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases} \tag{2}$$

(c) *(0.75 points)* Let $B_1 \subset \mathbb{R}^n$ be the unit ball, and let $v, w \in C^4(B_1) \cap C^3(\overline{B_1})$ be such that

$$\begin{cases} \Delta w = 0 & \text{in } B_1, \\ w = f & \text{on } \partial B_1, \end{cases} \quad \text{and} \quad \begin{cases} \Delta v = 0 & \text{in } B_1, \\ v = \frac{g - \partial_\nu w}{2} & \text{on } \partial B_1. \end{cases}$$

Prove that $u = (|x|^2 - 1)v + w$ satisfies (2) with $\Omega = B_1$.

(d) *(2 points)* Using separation of variables, find the solution to (2) for $n = 2$, $\Omega = B_1 \subset \mathbb{R}^2$, $f(x) = x_1$, and $g(x) = x_2$, where we denoted $x = (x_1, x_2) \in \mathbb{R}^2$.

Solution 1.

(a) Obviously $g \in K$, thus $K \neq \emptyset$. That K is a subset of $L^2(\Omega)$ is also clear by definition. Let us first see that K is convex. If $u, v \in K$ and $0 \le t \le 1$, then $tu + (1-t)v \le tg + (1-t)g = g$ a.e. in Ω. This yields $tu + (1-t)v \in K$, which gives the convexity of K.

We now check that K is closed. Let $v_j \in K$ such that $\lim_{j\uparrow+\infty} \|v_j - v\|_{L^2(\Omega)} = 0$ for some $v \in L^2(\Omega)$. We want to show that $v \in K$. To prove this, we only need to check that $v \le g$ a.e. in Ω. One way to do it is to use a result which states that convergence in $L^2(\Omega)$ yields convergence almost everywhere in Ω of a subsequence (this result is indeed used to prove that $L^2(\Omega)$ is complete). We can also proceed without using this result, as follows. Assume by contradiction that there exists $U \subset \Omega$ measurable, with $0 < |U| < +\infty$, and such that $v(x) > g(x)$ for all $x \in U$. Since $U = \bigcup_{k\ge1} U_k$, where

$$U_k := \{x \in U : \, v(x) - g(x) > 1/k\},$$

we deduce that there exists some $k \ge 1$ for which $0 < |U_k| < +\infty$. For this chosen k, using Hölder inequality and that $v_j \in K$, we have

$$\begin{aligned} 0 < |U_k| &= \int_{U_k} 1 \le \int_{U_k} k(v-g) = \int_{U_k} k(v - v_j) + \int_{U_k} k(v_j - g) \\ &\le k\|v - v_j\|_{L^2(\Omega)}|U_k|^{1/2} + \int_{U_k} k(v_j - g) \le k\|v - v_j\|_{L^2(\Omega)}|U_k|^{1/2}. \end{aligned}$$

Taking now $j \to +\infty$ and using that $\lim_{j\uparrow+\infty} \|v_j - v\|_{L^2(\Omega)} = 0$ we get $0 < |U_k| \le 0$, which is a contradiction. Therefore, K is closed.

(b) We only need to check that if $w = \min\{u, g\}$ then $\|u-w\|^2_{L^2(\Omega)} \le \|u-v\|^2_{L^2(\Omega)}$ for all $v \in K$, since we obviously have $w \in K$. Set $\Omega_+ = \{x \in \Omega : \, u(x) > g(x)\}$ and $\Omega_- = \Omega \setminus \Omega_+$. Then, $w = g$ a.e. in Ω_+ and $w = u$ a.e. in Ω_-. Hence,

$$\begin{aligned} \|u - w\|^2_{L^2(\Omega)} &= \int_\Omega |u-w|^2 = \int_{\Omega_+} |u-w|^2 + \int_{\Omega_-} |u-w|^2 \\ &= \int_{\Omega_+} |u-g|^2 \le \int_{\Omega_+} |u-v|^2 \le \|u-v\|^2_{L^2(\Omega)}, \end{aligned}$$

where we also used that, for every $v \in K$, $u > g \ge v$ a.e. in Ω_+, which yields $|u-g| \le |u-v|$ a.e. in Ω_+.

Solution 2.

(a) We argue by contradiction. Assume that there exists $f \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega f\varphi = \varphi(0) \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{3}$$

Take $0 < \delta < 1$ and $\Omega' = \{x \in \mathbb{R} : \, \delta < |x| < 1 - \delta\}$. Then, $\Omega' \subset\subset \Omega$, and $f \in L^1(\Omega')$.

Let $0 < \epsilon < \delta$ and, given $x \in \Omega'$, let $\varphi_\epsilon(y) = \rho_\epsilon(x - y)$ for all $y \in \Omega$, where ρ_ϵ is a compactly supported approximate unit ($\rho_\epsilon \in C^\infty(\mathbb{R})$, $\int \rho_\epsilon = 1$, and $\operatorname{supp}\rho_\epsilon = B_\epsilon(0)$). Then $\varphi_\epsilon \in C_c^\infty(\Omega)$. Hence, by (3) we have

$$\rho_\epsilon(x) = \varphi_\epsilon(0) = \int_\Omega f\varphi_\epsilon = \int_{B_\epsilon(x)} f(y)\rho_\epsilon(x-y)\,dy =: f_\epsilon(x) \quad \text{for all } x \in \Omega'. \tag{4}$$

On one hand, $\rho_\epsilon = 0$ in Ω' since $\operatorname{supp}\rho_\epsilon = B_\epsilon(0)$ and $\epsilon < \delta$. On the other hand, by the theorem on approximation by mollifiers, $\lim_{\epsilon\downarrow0} \|f_\epsilon - f\|_{L^1(\Omega')} = 0$. Therefore, taking the limit $\epsilon \to 0$ in (4) we get $f = 0$ a.e. in Ω'. Since this holds for every δ, taking $\delta \to 0$ we deduce that $f = 0$ a.e. in Ω. Then, (3) gives $0 = \int_\Omega f\varphi = \varphi(0)$ for all $\varphi \in C_c^\infty(\Omega)$. This is a contradiction because there exists $\varphi \in C_c^\infty(\Omega)$ with $\varphi(0) \neq 0$.

(b) Clearly, $u \in L^2(\Omega)$. Let us compute the distributional (weak) derivatives of order 1 and 2 of u. If $\varphi \in C_c^\infty(\Omega)$ then, by integration by parts and using that $\varphi(\pm 1) = 0$,

$$\begin{aligned} -\int_\Omega u\varphi' &= \int_{-1}^0 x\varphi'(x)\,dx - \int_0^1 x\varphi'(x)\,dx \\ &= x\varphi(x)\Big|_{x=-1}^{x=0} - \int_{-1}^0 \varphi(x)\,dx - x\varphi(x)\Big|_{x=0}^{x=1} + \int_0^1 \varphi(x)\,dx \\ &= -\int_{-1}^0 \varphi(x)\,dx + \int_0^1 \varphi(x)\,dx = \int_\Omega \frac{x}{|x|}\varphi(x)\,dx. \end{aligned}$$

Therefore, $u'(x) = x/|x|$ in the sense of distributions on Ω. This gives $u' \in L^2(\Omega)$, and $u \in H^1(\Omega)$.

Let $v(x) = u'(x) = x/|x|$ for $x \in \Omega$. Then, since $\varphi(\pm 1) = 0$,

$$-\int_\Omega v\varphi' = \int_{-1}^0 \varphi'(x)\,dx - \int_0^1 \varphi'(x)\,dx = 2\varphi(0).$$

Therefore, $v' = u''$ is the distribution on Ω given by $\varphi \mapsto 2\varphi(0)$ for all $\varphi \in C_c^\infty(\Omega)$. By (a) it does not exist $f \in L^2(\Omega)$ such that $2\varphi(0) = \int_\Omega f\varphi$ for all $\varphi \in C_c^\infty(\Omega)$. This means that $u'' \notin L^2(\Omega)$ and, hence, $u \notin H^2(\Omega)$.

Solution 3.

(a) We are giving a nontrivial function satisfying the equation in the simplest case $\Omega = B_1$. For instance, it is clear that the unique solution of the problem

$$\begin{cases} \Delta\overline{u} = 1 & \text{in } B_1, \\ \overline{u} = 0 & \text{on } \partial B_1, \end{cases}$$

satisfies (1) and is not identically zero. In fact, it can be written explicitly as

$$\overline{u}(x) = \frac{|x|^2 - 1}{2n}.$$

(b) Given u_1 and u_2 two solutions of the problem (2), the difference $u_d := u_1 - u_2$ satisfies the homogeneous problem

$$\begin{cases} \Delta^2 u_d = 0 & \text{in } \Omega, \\ u_d = 0 & \text{on } \partial\Omega, \\ \partial_\nu u_d = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the PDE by u_d and integrating in Ω we obtain

$$\begin{aligned} 0 = \int_\Omega u_d\,\Delta^2 u_d &= -\int_\Omega \nabla u_d \cdot \nabla(\Delta u_d) + \int_{\partial\Omega} u_d \frac{\partial \Delta u_d}{\partial\nu} \\ &= \int_\Omega |\Delta u_d|^2 - \int_{\partial\Omega} \left(\Delta u_d \frac{\partial u_d}{\partial\nu} - u_d \frac{\partial\Delta u_d}{\partial\nu} \right) \\ &= \int_\Omega |\Delta u_d|^2 \end{aligned}$$

where we have integrated by parts twice and used the boundary conditions. Then, we deduce that

$$\Delta u_d = 0 \quad \text{in } \Omega.$$

From this and the boundary condition $u_d = 0$ on $\partial\Omega$ we conclude (by uniqueness of solution for the Laplace equation) that $u_d \equiv 0$.

(c) First, it is clear that $(|x|^2 - 1)v + w$ satisfies the Dirichlet condition on the boundary. Next, let us check that it is biharmonic. On the one hand,

$$\begin{aligned} \Delta\left((|x|^2 - 1)v + w\right) &= \Delta(|x|^2 - 1)v + 2\,\nabla(|x|^2 - 1)\cdot\nabla v + (|x|^2 - 1)\Delta v + \Delta w \\ &= 2n\,v + 4\,x\cdot\nabla v \end{aligned}$$

Then,

$$\begin{aligned}\Delta^2\big((|x|^2-1)v+w\big) &= \Delta\,(2n\,v+4\,x\cdot\nabla v)\\ &= 2n\,\Delta v+4\Delta x\cdot\nabla v+4x\cdot\nabla(\Delta v)+8\,\Delta v\\ &= 0.\end{aligned}$$

Finally, let us check that $(|x|^2-1)v+w$ satisfies the Neumann condition. That is, since

$$\nabla\big((|x|^2-1)v+w\big)=2v\,x+(|x|^2-1)\,\nabla v+\nabla w,$$

we obtain

$$\partial_\nu\big((|x|^2-1)v+w\big)=\frac{x}{|x|}\cdot\nabla\big((|x|^2-1)v+w\big)\Big|_{|x|=1}=2\,v+\partial_\nu w=g.$$

(d) In part (c) we deduced how to construct a solution. Since we know from part (b) that if it exists then it is unique, it should be of that form. Hence, let us find w and v satisfying

$$\begin{cases}\Delta w=0 & \text{in } B_1,\\ w=x_1 & \text{on } \partial B_1,\end{cases}\qquad\text{and}\qquad\begin{cases}\Delta v=0 & \text{in } B_1,\\ v=\frac{x_2-\partial_\nu w}{2} & \text{on } \partial B_1.\end{cases}$$

In order to find w let us write it and the PDE in polar coordinates: $w\equiv w(r,\theta)$ and

$$\begin{cases}r^{-1}(r\,w_r)_r+r^{-2}\,w_{\theta\theta}=0 & \text{in } (0,1)\times\mathbb{R},\\ w(1,\theta)=\cos\theta & \text{in } \mathbb{R},\\ w(r,\theta)=w(r,\theta+2\pi) & \text{in } (0,1)\times\mathbb{R}.\end{cases}$$

In the method of separation of variables we first look for solutions to $r^{-1}(r\,w_r)_r+r^{-2}\,w_{\theta\theta}=0$ of the form $w(r,\theta)=R(r)\Theta(\theta)$. The equation on w yields $r^{-1}(r\,R')'\Theta+r^{-2}\Theta''R=0$. Then, using that R only depends on r, and that Θ only depends on θ, we get $\Theta''=-\lambda\Theta$ and $r(rR')'=\lambda R$ for some $\lambda\in\mathbb{R}$. Now, the fact that θ is an angle means that Θ is 2π-periodic and satisfies the ODE

$$\Theta''=-\lambda\Theta.$$

Under these conditions, we know that $\lambda=k^2$ for some $k\in\mathbb{N}$ and

$$\Theta(\theta)=A\sin(k\,x)+B\cos(k\,x),\qquad A,\,B\in\mathbb{R}.$$

Taking into account these admissible values for λ and looking at the ODE on R, we must have $r(rR')'=k^2R$ for some $k\in\mathbb{N}$. Therefore, R must be of the form

$$R(r)=Cr^k+Dr^{-k},\qquad C,\,D\in\mathbb{R},\quad k=0,1,2,\ldots$$

Then, imposing that R must be bounded at $r=0$ because w must be regular at the origin, we must have $D=0$.

Up to now, we have seen that, for every $k=0,1,2,\ldots$ and every $a_k,\,b_k\in\mathbb{R}$,

$$w(r,\theta)=\big(a_k\sin(k\,\theta)+b_k\cos(k\,\theta)\big)r^k$$

satisfies the PDE and the periodic conditions. By the linearity of the equation, it is natural to consider solutions of the form

$$w(r,\theta)=\sum_{k=0}^{\infty}\big(a_k\sin(k\,\theta)+b_k\cos(k\,\theta)\big)\,r^k,$$

whenever the series converge in some sense. Looking now at the boundary conditions, we must have

$$\cos\theta=w(1,\theta)=\sum_{k=0}^{\infty}\big(a_k\sin(k\,\theta)+b_k\cos(k\,\theta)\big),$$

which means that the unique nonzero coefficient is $b_1=1$. Hence, the solution is

$$w(r,\theta)=r\cos\theta.$$

Once we know the function w, we can proceed in a similar way to find v, which satisfies

$$\begin{cases} r^{-1}(r\, v_r)_r + r^{-2}\, v_{\theta\theta} = 0 & \text{in } (0,1)\times\mathbb{R}, \\ v(1,\theta) = \frac{\sin\theta - \cos\theta}{2} & \text{in } \mathbb{R}, \\ v(r,\theta) = v(r,\theta+2\pi) & \text{in } (0,1)\times\mathbb{R}. \end{cases}$$

That is,

$$v(r,\theta) = \sum_{k=0}^{\infty} \big(c_k \sin(k\,\theta) + d_k \cos(k\,\theta)\big)\, r^k,$$

and by imposing the boundary condition we obtain

$$v(r,\theta) = r\,\frac{\sin\theta - \cos\theta}{2}.$$

Finally, we can write the solution of the original problem

$$u(r,\theta) = (r^2-1)v(r,\theta) + w(r,\theta) = \frac{r^3 - r}{2}\,\sin\theta + \frac{3r - r^3}{2}\,\cos\theta,$$

that written in Cartesian coordinates is

$$u(x_1,x_2) = \frac{-x_1^3 - x_1x_2^2 + 3x_1 + x_2x_1^2 + x_2^3 - x_2}{2}.$$

Advanced Course in PDEs **FME-UPC**
FINAL EXAM **June 7th, 2021**
Authors: Juan-Carlos Felipe-Navarro, Albert Mas

- *You have 3 hours for doing this exam. Then, you must upload your scanned answers via the corresponding task in Atenea.*
- *You must give a complete explanation of the development of your answers. You can use all the results from the theory/exercises lessons (without giving their proofs), but you must cite them properly verifying all the corresponding hypothesis.*
- *In each exercise, you can use any part to solve the others, even if you have not solved it (say, you can use* (a) *to prove* (b)*, etc.).*

Problem 1. Let $\Omega \subset \mathbb{R}^3$ ($n = 3$) be an open and bounded set with smooth boundary. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Assume that $u \in H^1(\Omega)$ is a weak solution to $-\Delta u + u = f(u)$ in Ω, that is,

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv = \int_\Omega f(u)v \quad \text{for all } v \in H_0^1(\Omega).$$

Prove the following (independent) statements:

(a) *(1 point)* If f is of class C^1 on $\mathbb{R}$ and $\|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})} < +\infty$, then u is of class C^1 on Ω.

(b) *(1 point)* If $u \in H_0^1(\Omega)$, f is odd, and $f(t) < (1 + \lambda_1)t$ for all $t > 0$, where $\lambda_1 > 0$ denotes the first eigenvalue of the Dirichlet Laplacian on Ω, then $u = 0$ in Ω and $f(0) = 0$.
[Hint: Show that $u^\pm = 0$ in Ω.]

(c) (1 point) *There exists $C > 0$ only depending on Ω such that $\|u\|_{L^6(\Omega)} \le C$ if $u \in H_0^1(\Omega)$ and $|f(t)| \le |t|^{1/2}$ for all $t \in \mathbb{R}$.*
[Hint: Note that $6 = \frac{3\cdot 2}{3-2} = \frac{np}{n-p}$ with $n = 3$ and $p = 2$.]

Problem 2. Let $\Omega \subset \mathbb{R}^n$ ($n \ge 1$) be an open, bounded, and smooth domain. Let $u = u(x,t)$ be a smooth function satisfying

$$\begin{cases} u_t - \Delta u + 2\,x \cdot \nabla u = 1 & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u + \gamma u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

for some $\gamma > 0$.

(a) *(1 point)* Write the weak formulation of the stationary problem associated to (1).

(b) *(1 point)* Prove that

$$(v,w)_{exp} := \int_\Omega e^{-|x|^2}\,\nabla v(x) \cdot \nabla w(x)\,dx + \gamma \int_{\partial\Omega} e^{-|x|^2} v(x)\,w(x)\,dS(x)$$

is a scalar product in $H^1(\Omega)$. Show also that the associated norm is equivalent to the usual one.

(c) *(1 point)* Show that u^s is a weak solution to the stationary problem if and only if

$$(u^s, w)_{exp} = \int_\Omega e^{-|x|^2} w(x)\,dx \quad \text{for all } w \in H^1(\Omega).$$

(d) *(1 point)* Prove that there exists a unique weak solution u^s to the stationary problem. Furthermore, show that u^s is the unique minimizer of an energy functional.

(e) *(1 point)* Show that $u^s \ge 0$ a.e. in Ω.

Assume now that the stationary solution $u^s : \Omega \to \mathbb{R}$ is smooth.

(f) *(1.5 points)* Prove that $0 \le u \le u^s$.

(g) *(0.5 points)* Show that there exists a set $E \subset \Omega$ of positive measure such that

$$u^s(y) \ge \frac{1}{\gamma\,|\partial\Omega|} \int_\Omega e^{-|x|^2}\,dx \quad \text{for all } y \in E.$$

Solution 1.

(a) Let us first show that $f(u) \in H^1(\Omega)$. On one hand,

$$\int_\Omega |f(u)|^2 \leq 4\left[\int_\Omega |f(u) - f(0)|^2 + \int_\Omega |f(0)|^2\right] \leq 4\left[\|f'\|^2_{L^\infty(\mathbb{R})}\int_\Omega |u|^2 + \|f\|^2_{L^\infty(\mathbb{R})}|\Omega|\right] < +\infty.$$

On the other hand, $\nabla\big(f(u)\big) = f'(u)\nabla u$ in the sense of distributions. Since $\|f'\|_{L^\infty(\mathbb{R})} < +\infty$ and $\nabla u \in L^2(\Omega)^n$, we get $\nabla\big(f(u)\big) \in L^2(\Omega)^n$. Therefore, $f(u) \in H^1(\Omega)$.

From the weak formulation we have

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega (f(u) - u)v \quad \text{for all } v \in H^1_0(\Omega).$$

Since $f(u) - u \in H^1(\Omega)$, by regularity theory we deduce that $u \in H^3(\Omega'')$ for all $\Omega'' \subset\subset \Omega$. Take now $\Omega' \subset\subset \Omega$ and $\eta \in C^\infty_c(\Omega)$ such that $\eta = 1$ on Ω'. Then,

$$\eta u \in H^3_0(\Omega) \equiv W^{3,2}_0(\Omega) \hookrightarrow C^{1,1/2}(\overline{\Omega}) \subset C^1(\overline{\Omega})$$

by Morrey's inequality. Finally, $\eta u \in C^1(\overline{\Omega})$ and $\eta = 1$ on Ω' yield that u is of class C^1 on Ω'. Since Ω' is arbitrary, u is of class C^1 on Ω.

(b) Since $u \in H^1_0(\Omega)$, we have $u^\pm \in H^1_0(\Omega)$. Assume that $u^+ > 0$ in a set of positive measure in Ω. From the weak formulation and the estimate on f, we see that

$$\begin{aligned}\int_\Omega |\nabla(u^+)|^2 + \int_\Omega |u^+|^2 &= \int_\Omega \nabla u \cdot \nabla(u^+) + \int_\Omega uu^+ = \int_\Omega f(u)u^+ \\ &= \int_\Omega f(u^+)u^+ < (1+\lambda_1)\int_\Omega |u^+|^2,\end{aligned}$$

which leads to $\int_\Omega |\nabla(u^+)|^2 < \lambda_1 \int_\Omega |u^+|^2$, but this contradicts the characterization of λ_1 as the infimum value of a Rayleigh quotient. Hence, u^+ vanishes identically in Ω.

Assume now that $u^- > 0$ in a set of positive measure in Ω. From the weak formulation, the fact that f is odd, and the estimate on f, we see that

$$\begin{aligned}\int_\Omega |\nabla(u^-)|^2 + \int_\Omega |u^-|^2 &= -\int_\Omega \nabla u \cdot \nabla(u^-) - \int_\Omega uu^- = -\int_\Omega f(u)u^- \\ &= -\int_\Omega f(-u^-)u^- = \int_\Omega f(u^-)u^- < (1+\lambda_1)\int_\Omega |u^-|^2,\end{aligned}$$

which leads to $\int_\Omega |\nabla(u^-)|^2 < \lambda_1 \int_\Omega |u^-|^2$. As before, we deduce that u^- vanishes identically in Ω.

In conclusion, $u^\pm = 0$ in Ω, thus $u = 0$ in Ω. In particular, $0 = -\Delta u + u = f(u) = f(0)$.

(c) We know that $H^1_0(\Omega) \equiv W^{1,2}_0(\Omega) \hookrightarrow L^6(\Omega)$ by Sobolev-Gagliardo-Nirenberg inequality. Therefore, there exists $C_\Omega > 0$ only depending on Ω such that $\|u\|_{L^6(\Omega)} \leq C_\Omega \|u\|_{H^1(\Omega)}$. Then, using also the weak formulation, the estimate on f, and Hölder inequality, we get

$$\begin{aligned}\|u\|_{L^6(\Omega)} &\leq C_\Omega\|u\|_{H^1(\Omega)} = C_\Omega\left[\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2\right]^{1/2} = C_\Omega\left[\int_\Omega f(u)u\right]^{1/2} \leq C_\Omega\left[\int_\Omega |u|^{3/2}\right]^{1/2} \\ &\leq C_\Omega\left[\left(\int_\Omega |u|^{\frac{3}{2}4}\right)^{1/4}|\Omega|^{3/4}\right]^{1/2} = C_\Omega|\Omega|^{3/8}\left[\int_\Omega |u|^6\right]^{1/8} = C_\Omega|\Omega|^{3/8}\|u\|^{3/4}_{L^6(\Omega)}.\end{aligned}$$

This yields $\|u\|_{L^6(\Omega)} \leq C^4_\Omega|\Omega|^{3/2}$.

Solution 2.

(a) Multiplying the stationary equation by v, integrating in Ω and applying integration by parts we obtain

$$\begin{aligned}\int_\Omega v &= \int_\Omega -\Delta u^s\, v + 2\int_\Omega x \cdot \nabla u^s\, v = \int_\Omega \nabla u^s \cdot \nabla v - \int_{\partial\Omega} \partial_\nu u^s\, v + 2\int_\Omega x \cdot \nabla u^s\, v \\ &= \int_\Omega \nabla u^s \cdot \nabla v + \gamma\int_{\partial\Omega} u^s\, v + 2\int_\Omega x \cdot \nabla u^s\, v.\end{aligned}$$

Here, we have used the boundary condition $\partial_\nu u^s + \gamma u^s = 0$. Then, the weak formulation is finding $u^s \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u^s \cdot \nabla v + \gamma \int_{\partial\Omega} u^s\, v + 2\int_\Omega x \cdot \nabla u^s\, v = \int_\Omega v \quad \text{for all } \ v \in H^1(\Omega).$$

(b) It is clear that $(\cdot,\cdot)_{exp}$ is symmetric and bilinear. Moreover, since $e^{-|x|^2} > 0$ and $\gamma > 0$, it satisfies $(w,w)_{exp} \geq 0$, with equality if and only if $w \equiv 0$.

In order to show that the induced norm is equivalent to the usual one we will show that it is equivalent to the norm

$$\|w\|_{\partial\Omega} := \Big(\int_\Omega |\nabla w(x)|^2\, dx + \int_{\partial\Omega} |w(x)|^2\, dS(x) \Big)^{1/2}$$

from Exercise 11 in the Advanced List. Since Ω is bounded, we use that

$$0 < e^{-R^2} \leq e^{-|x|^2} \leq 1 \qquad \text{for all } \ x \in \overline{\Omega},$$

where $R > 0$ is such that $\Omega \subset B_R(0)$. Then, we conclude

$$e^{-R^2} \min\{1,\gamma\}\|w\|^2_{\partial\Omega} \leq (w,w)_{exp} \leq \max\{1,\gamma\}\|w\|^2_{\partial\Omega}.$$

(c) Given $w \in H^1(\Omega)$, it is clear that $v = e^{-|x|^2} w \in H^1(\Omega)$, and moreover $\nabla v = e^{-|x|^2}\nabla w - 2\,x\,e^{-|x|^2}\, w$. Hence, using v as a test function in the weak formulation found in (a) we obtain

$$\begin{aligned}\int_\Omega e^{-|x|^2}\, \nabla u^s(x) \cdot \nabla w(x)\, dx - 2\int_\Omega e^{-|x|^2}\, x \cdot \nabla u^s(x)\, w(x)\, dx + \gamma \int_{\partial\Omega} e^{-|x|^2}\, u^s(x)\, w(x)\, dS(x) \\ + 2\int_\Omega e^{-|x|^2}\, x \cdot \nabla u^s(x)\, w(x)\, dx = \int_\Omega e^{-|x|^2}\, w(x)\, dx.\end{aligned}$$

Since this can be done for every $w \in H^1(\Omega)$, we conclude that if u^s is a weak solution to the stationary problem then

$$(u^s, w)_{exp} = \int_\Omega e^{-|x|^2}\, w(x)\, dx \qquad \text{for all } \ w \in H^1(\Omega).$$

The converse result is analogous.

(d) By Riesz-Frechet theorem and parts (b) and (c) it is enough to check that

$$F(w) := \int_\Omega e^{-|x|^2}\, w(x)\, dx$$

is a continuous linear functional on $H^1(\Omega)$. The linearity is clear. Regarding the continuity, we see that

$$|F(w)| \leq \int_\Omega e^{-|x|^2}\, |w(x)|\, dx \leq \int_\Omega |w| \leq |\Omega|^{1/2}\|w\|_{L^2(\Omega)} \leq C\|w\|_{H^1(\Omega)} \leq C\|w\|_{exp}$$

by (b). In addition, the solution is the unique minimizer of

$$\begin{aligned} E(w) :&= \frac{1}{2}(w,w)_{exp} - F(w) \\ &= \frac{1}{2}\int_\Omega e^{-|x|^2}\, |\nabla w(x)|^2\, dx + \frac{\gamma}{2}\int_{\partial\Omega} e^{-|x|^2}\, |w(x)|^2\, dS(x) - \int_\Omega e^{-|x|^2}\, w(x)\, dx\end{aligned}$$

among $H^1(\Omega)$ functions.

(e) Let us take $w = (u^s)^+$. It is well known that $w \in H^1(\Omega)$ and $\nabla w = \nabla u^s\, \chi_{\{u^s>0\}}$ a.e. in Ω. Using the energy functional found in (d), we see that

$$\begin{aligned} E(w) &= \frac{1}{2}\int_{\Omega\cap\{u^s>0\}} e^{-|x|^2}\, |\nabla u^s(x)|^2\, dx + \frac{\gamma}{2}\int_{\partial\Omega\cap\{u^s>0\}} e^{-|x|^2}\, |u^s(x)|^2\, dS(x) - \int_{\Omega\cap\{u^s>0\}} e^{-|x|^2}\, u^s(x)\, dx \\ &\leq \frac{1}{2}\int_\Omega e^{-|x|^2}\, |\nabla u^s(x)|^2\, dx + \frac{\gamma}{2}\int_{\partial\Omega} e^{-|x|^2}\, |u^s(x)|^2\, dS(x) - \int_\Omega e^{-|x|^2}\, u^s(x)\, dx \\ &= E(u^s).\end{aligned}$$

Since u^s is the unique minimizer of E, this means that $u^s = w = u^s\, \chi_{\{u^s>0\}}$. Thus, $u^s \geq 0$.

(f) Let us suppose by contradiction that $u < 0$ in a certain region of $\overline{\Omega} \times [0,T]$ for some $T > 0$. Then, the minimum of u in $\overline{\Omega} \times [0,T]$ is attained at some point (x_0, t_0) such that $u(x_0, t_0) < 0$. From the equation we discard that $(x_0, t_0) \in \Omega \times (0, T]$. Hence, (x_0, t_0) must belong to the parabolic boundary, but since $u(\cdot, 0) = 0$, the only possibility is being on the lateral boundary. Nevertheless, in that case,

$$\partial_\nu u(x_0, t_0) = -\gamma\, u(x_0, t_0) > 0,$$

which contradicts the fact of being a minimum. Here, it is crucial the positivity of γ.

The upper bound follows similarly once we know $u^s \geq 0$ from (e). That is, writing the equation satisfied by $u^s - (1-\varepsilon)u$, we deduce that $u^s \geq (1-\varepsilon)u$ for every $\varepsilon > 0$. Finally, taking the limit ε to zero we obtain the desired result.

(g) By taking $w \equiv 1$ in the weak formulation with the exponential weight from part (c), and since we know that $u^s \geq 0$ from (e), we deduce

$$\gamma \int_{\partial\Omega} u^s \geq \gamma \int_{\partial\Omega} e^{-|x|^2} u^s = \int_\Omega e^{-|x|^2},$$

which is equivalent to

$$K := \frac{1}{\gamma|\partial\Omega|} \int_\Omega e^{-|x|^2} \leq \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u^s.$$

Now, we can distinguish two cases:

- $u^s \equiv K$ on $\partial\Omega$. By repeating the arguments from part (f) to the stationary problem we deduce that u^s attains the minimum on $\partial\Omega$. Hence, we conclude that $u^s \geq K$ in $\overline{\Omega}$ and the result follows in this scenario.
- $u^s(x_0) > K$ for some $x_0 \in \partial\Omega$. By continuity of u^s, it is clear that $u^s \geq K$ in $B_\varepsilon(x_0) \cap \overline{\Omega}$ for some $\varepsilon > 0$ small enough. Thus, the result also follows in this second case.