False vacua in field theory models

Eduardo Garnacho Velasco*
Departamento de Física Teórica, Universidad Complutense de Madrid, 28040 Madrid, Spain
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An effective Hamiltonian in Coulomb gauge is used to model Quantum Chromodynamics (QCD) at the hadronic scale, including confinement through a linear potential plus an hyperfine correction due to a transverse gluon exchange. Using standard many-body techniques, quarks are represented as quasiparticles performing a Bogoliubov-Valatin rotation, diagonalizing the Hamiltonian. Minimizing the Hamiltonian expected value, the mass gap equation is obtained, whose solutions are defined as the vacua of the theory. This equation is solved numerically and an analysis of these vacua is carried out. Finally, a comparison to typical models appearing in cosmology is shown.

I. INTRODUCTION

The spontaneous symmetry breaking (SSB) of the chiral symmetry $SU(2)_L \times SU(2)_R$ in QCD (with two light quarks) is responsible for many physical consequences. One of them is the division of vacua (defined as extremal points of the expected value of the Hamiltonian) in two different classes: a chirally symmetric class and another class which breaks chiral invariance. The first type of vacua is the well known perturbative QCD vacuum $|0\rangle$. Due to the SSB of the chiral symmetry, it is not the ground state of the theory. The second type of vacua can be studied diagonalizing the Hamiltonian through a Bogoliubov-Valatin rotation, the same technique used in the BCS theory for superconductivity, leading to a coherent vacuum state of interacting quasiparticles. We will denote the quasiparticle vacuum (or BCS vacuum) as $|\Omega\rangle$ and this is precisely the ground state of the theory. The QCD vacuum space can then be visualized like a Mexican hat potential, with the perturbative vacuum at the top of the hat.

However, QCD may possess excited vacuum states called replicas or false vacua (see [1] for example) which also break chiral invariance. Here is where we have to understand a vacuum as an extremal point of $\langle H\rangle$. If these replicas are truly metastable, the hadronic spectrum built on them would also be metastable yielding new resonances, but the stability of the replicas will be a topic of discussion through the paper.

This work has the following structure. In Sec. II we develop the formalism of the model Hamiltonian used to represent QCD. In Sec. III, we explain the BCS transformation and how it leads to the mass gap equation, whose solutions are defined as the vacua of the theory and numerically found. In Sec. IV we analyse the stability of these solutions, the resulting vacuum picture and some other properties of the mass gap equation solutions, such as the quark condensate, the degree of symmetry breaking or at which temperature a gas of pions would populate the replicas. Finally, in Sec. V we try to look for an analogy of this model in cosmology, in particular in a scalar inflation theory.

* egarnach@ucm.es

II. EFFECTIVE HAMILTONIAN

As is well known, a perturbative approach to QCD is only possible at high energies, where the coupling constant is small compared to the scale of the theory. If we have to deal with low energies, we need to make use of other tools such as effective theories. In this work we represent QCD in the Coulomb gauge

$$H = H_q + H_g + H_{qg} + V_C, \tag{1}$$

with

$$H_q = \int \mathrm{d}^3x\, \Psi(\vec{x})^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m_q)\Psi(\vec{x}), \tag{2}$$

$$H_g = \mathrm{Tr}\int \mathrm{d}^3x\,[\vec{\Pi}^a(\vec{x})\cdot\vec{\Pi}^a(\vec{x}) + \vec{B}^a(\vec{x})\cdot\vec{B}^a(\vec{x})], \tag{3}$$

$$H_{qg} = g\int \mathrm{d}^3x\, \vec{J}^a(\vec{x})\cdot\vec{A}^a(\vec{x}), \tag{4}$$

$$V_C = -\frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\,\rho^a(\vec{x})V(|\vec{x}-\vec{y}\,|)\rho^a(\vec{y}), \tag{5}$$

by means of a model Hamiltonian. There Ψ and m_q are the (bare) quark field and mass, $\rho^a(\vec{x}) = \Psi_x^\dagger T^a\Psi_x$ and $\vec{J}^a(\vec{x}) = \Psi_x^\dagger T^a\vec{\alpha}\Psi_x$ are the density and current of colour respectively, with T^a the generators of SU(3), g is the QCD coupling, $\vec{A}^a$ are the gauge fields, $\vec{\Pi}^a$ are the conjugate fields and $\vec{B}^a$ are the non-abelian magnetic fields defined by

$$\vec{B}^a \equiv \vec{\nabla}\times\vec{A}^a + \frac{1}{2}f^{abc}\vec{A}^b\times\vec{A}^c \tag{6}$$

with f^{abc} the structure constants of SU(3). For a complete analysis of the Coulomb gauge Hamiltonian see [2].

Let us focus on the potential V_C of Eq. (5). Any model of the strong interaction should reflect the phenomenon of confinement, that is, the absence of isolated color charged particles (such as gluons or quarks) in the spectrum. In this work, we model confinement through the potential V_C. The V_C part accounts for a Cornell potential, i.e. a Coulomb potential due to the exchange of a gluon plus a linear part which is responsible for confinement:

$$V_{\rm Cornell}(k) = -4\pi\frac{\alpha_s}{k^2} - 8\pi\frac{\sigma}{k^4} \tag{7}$$

where α_s is the coupling in QCD and σ is a string tension constant, which can be inferred from experimental data (for example, of charmonium spectrum) in Lattice QCD calculations.

In particular, we use a modified Cornell potential numerically fitted to pure Yang-Mills variational computations (see [3])

$$V(p) = \begin{cases} C(p) \equiv -\dfrac{8.07}{p^2}\dfrac{\log^{-0.62}\left(\frac{p^2}{m_g^2}+0.82\right)}{\log^{0.8}\left(\frac{p^2}{m_g^2}+1.41\right)} & \text{if } p > m_g \\ L(p) \equiv -\dfrac{12.25\, m_g^{1.93}}{p^{3.93}} & \text{if } p < m_g \end{cases} \tag{8}$$

where the low momentum component is (numerically) close to a pure linear potential and the other term represents a renormalized high energy Coulomb tail.

Because chiral symmetry is a feature of quarks, not gluons, H_g is omitted and we substitute H_{qg} by a generic transverse hyperfine interaction V_T due to the exchange of a transverse gluon:

$$V_T = \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\, \vec{J}_i^a(\vec{x})U_{ij}(\vec{x},\vec{y})\vec{J}_j^a(\vec{y}), \tag{9}$$

$$U_{ij}(\vec{x},\vec{y}) = \left(\delta_{ij} - \frac{\nabla_i\nabla_j}{\nabla^2}\right)_{\vec{x}} U(|\vec{x}-\vec{y}\,|), \tag{10}$$

$$U(p) = \begin{cases} C(p) & \text{if } \ p > m_g \\ -\dfrac{C_h}{p^2+m_g^2} & \text{if } p < m_g \end{cases} \tag{11}$$

with a Yukawa type interaction at low momentum and C_h a constant determined by matching the high and low momentum regions. This interaction is sensible for transferred momenta not much larger than the dynamical mass of the gluon m_g, which we are using as a scale of the theory.

III. BCS TRANSFORMATION AND GAP EQUATION

As we stated in Sec. I, when chiral symmetry is broken, a new type of vacua arises. Because $|0\rangle$ is no longer the ground state of the theory but $|\Omega\rangle$ instead, the theory has a non vanishing quark condensate $\langle\Omega|\bar{\Psi}\Psi|\Omega\rangle \neq 0$. The formation of these pairs can be seen in analogy to the formation of Cooper pairs in superconductors. So, following with the analogy, it is standard to use the BCS many-body techniques to analyse the non-chiral ground state (or simply BCS vacuum) $|\Omega\rangle$.

We begin by writing the plane wave expansion of the quark field

$$\Psi(\vec{x}) = \sum_{c\lambda}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[u_{c\lambda}(\vec{k})b_{c\lambda}(\vec{k}) + v_{c\lambda}(-\vec{k})d^\dagger_{c\lambda}(-\vec{k})\right]e^{i\vec{k}\cdot\vec{x}} \tag{12}$$

with $u_{c\lambda}$, $v_{c\lambda}$ the particle, antiparticle bare spinors, $b_{c\lambda}$, $d_{c\lambda}$ the particle, antiparticle bare annihilation operators, λ the spin state and $c = 1, 2, 3$ the color index which will be suppressed hereafter.

We can expand Ψ in terms of any complete basis, so we choose to expand it using a new quasiparticle basis

$$\Psi(\vec{x}) = \sum_{\lambda}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[U_{\lambda}(\vec{k})B_{\lambda}(\vec{k}) + V_{\lambda}(-\vec{k})D^\dagger_{\lambda}(-\vec{k})\right]e^{i\vec{k}\cdot\vec{x}}. \tag{13}$$

The two basis are related by a linear transformation called Bogoliubov-Valatin transformation (see [4], [5] for an in-depth treatment):

$$B_\lambda(\vec{k}) = \alpha_k b_\lambda(\vec{k}) - \beta_k d^\dagger_\lambda(\vec{k}), \tag{14}$$

$$D_\lambda(-\vec{k}) = \alpha_k d_\lambda(-\vec{k}) + \beta_k b^\dagger_\lambda(\vec{k}). \tag{15}$$

The coefficients α_k, β_k only depend on $|\vec{k}|$, are real and c-numbers. This transformation is canonical if and only if the new operators obey the same commutation laws as the original ones

$$\{B_k, B^\dagger_{k'}\} = \{D_k, D^\dagger_{k'}\} = \delta_{kk'}. \tag{16}$$

This implies $|\alpha_k|^2 + |\beta_k|^2 = 1$ and we can implement this transformation as a rotation, parametrized by an angle $\theta(k) \equiv \theta_k$ called the BCS angle, which is a function of k. This parametrization yields the next relation between the two basis

$$B_\lambda(\vec{k}) = \cos\frac{\theta_k}{2}b_\lambda(\vec{k}) - \lambda\sin\frac{\theta_k}{2}d^\dagger_\lambda(\vec{k}), \tag{17}$$

$$D_\lambda(-\vec{k}) = \cos\frac{\theta_k}{2}d_\lambda(-\vec{k}) + \lambda\sin\frac{\theta_k}{2}b^\dagger_\lambda(\vec{k}). \tag{18}$$

It is more convenient to work in terms of the gap angle $\phi_k \equiv \phi(k)$, which is related to the BCS angle by $\phi = \theta + \alpha$, where α is the perturbative mass angle which satisfies $\sin\alpha = m_q/E_k$, and $E_k = \sqrt{m_q^2 + k^2}$. Then the rotated quasiparticle spinors can be expressed in terms of the original spinors as follows

$$\begin{aligned} U_\lambda(\vec{k}) &= \cos\frac{\theta_k}{2}u_\lambda(\vec{k}) - \lambda\sin\frac{\theta}{2}v_\lambda(-\vec{k}) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+\sin\phi_k}\,\xi_\lambda \\ \sqrt{1-\sin\phi_k}\,\vec{\sigma}\cdot\hat{k}\,\xi_\lambda\end{pmatrix}, \end{aligned} \tag{19}$$

$$\begin{aligned} V_\lambda(-\vec{k}) &= \cos\frac{\theta_k}{2}v_\lambda(-\vec{k}) + \lambda\sin\frac{\theta}{2}u_\lambda(\vec{k}) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}-\sqrt{1-\sin\phi_k}\,\vec{\sigma}\cdot\hat{k}\,i\sigma_2\,\xi_\lambda \\ \sqrt{1+\sin\phi_k}\,i\sigma_2\,\xi_\lambda\end{pmatrix} \end{aligned} \tag{20}$$

with ξ_λ a two-dimensional Pauli spinor.

The trivial vacuum, defined by $b_\lambda|0\rangle = d_\lambda|0\rangle = 0$, is related to the BCS vacuum, defined by $B_\lambda|\Omega\rangle = D_\lambda|\Omega\rangle = 0$, by the transformation

$$|\Omega\rangle = \exp\Bigg(- \sum_{\lambda_1,\lambda_2} \int \frac{d^3k}{(2\pi)^3} (\vec{\sigma}\cdot\hat{k})_{\lambda_1\lambda_2} \tan\frac{\theta_k}{2} \times b^\dagger_{\lambda_1}(\vec{k}) d^\dagger_{\lambda_2}(-\vec{k}) \Bigg) |0\rangle. \tag{21}$$

If we expand the exponential, we see the operators $b^\dagger d^\dagger$ create a quark/antiquark pair current, showing the BCS vacuum as a coherent state of $q\bar{q}$ excitations representing a $^3P_0$ condensate. Notice that, in the infinite volume limit, the trivial vacuum and the BCS vacuum are orthogonal $\langle\Omega|0\rangle = 0$.

The main goal of this work is to find and analyse the vacua of this theory, that can be defined as extremal points of the Hamiltonian expected value, which in the new basis is

$$\begin{aligned}\rho \equiv& \frac{\langle\Omega|H|\Omega\rangle}{V} = \int \frac{d^3k}{(2\pi)^3} \big[-6(kc_k + m_q s_k) \\ &- 2\int \frac{d^3q}{(2\pi)^3} \hat{V}(|\vec{k}-\vec{q}|)(1 - s_k s_q - c_k c_q x) \\ &+ 4\int \frac{d^3q}{(2\pi)^3} \hat{U}(|\vec{k}-\vec{q}|)(1 + s_k s_q) + c_q c_k \hat{W}(|\vec{k}-\vec{q}|)\big]\end{aligned} \tag{22}$$

where ρ is the energy density of the system such that $\int d^3x\,\rho = H$, $s_k \equiv \sin\phi(\vec{k})$, $c_k \equiv \cos\phi(\vec{k})$ and

$$\hat{W}(|\vec{k}-\vec{q}|) \equiv \frac{x(|\vec{k}|^2 + |\vec{q}|^2) - |\vec{k}||\vec{q}|(1+x^2)}{|\vec{k}-\vec{q}|^2} \hat{U}(|\vec{k}-\vec{q}|) \tag{23}$$

with $x = \hat{k}\cdot\hat{q}$. To minimize this expected value, we use the gap angle ϕ_k as a variational parameter

$$\frac{\delta\langle\Omega|H|\Omega\rangle}{\delta\phi_k} = 0. \tag{24}$$

This leads to an equality known in the literature as the mass gap equation

$$\begin{aligned}ks_k - m_q c_k = &\frac{2}{3}\int \frac{d^3q}{(2\pi)^3} \hat{V}(|\vec{k}-\vec{q}|)[s_k c_q x - c_k s_q] \\ &- \frac{4}{3}\int \frac{d^3q}{(2\pi)^3} \big(c_k s_q \hat{U}(|\vec{k}-\vec{q}|) - c_q s_k \hat{W}(|\vec{k}-\vec{q}|)\big).\end{aligned} \tag{25}$$

The solutions $\phi^i(k) \equiv \phi^i_k$ of this equation, substituted with $\theta_k = \phi_k - \arctan(m_q/k)$ in Eq. (21), are interpreted as possible vacua of the theory, so we need to solve this nonlinear integral equation. It is precisely the nonlinear character of this equation what allows the existence of other solutions besides the perturbative and the BCS vacuum.

The angular integrals can be analytically evaluated

$$\begin{aligned}ks_k - m_q c_k = \frac{1}{6\pi^2}\int_0^\infty dq\, q^2 [&s_k c_q V_1 - c_k s_q V_0 \\ &- 2(c_k s_q U_0 - c_q s_k W_0)]\end{aligned} \tag{26}$$

where

$$I_n = \int_{-1}^{1} dx\, I(|\vec{k}-\vec{q}|) x^n \tag{27}$$

with $I = V, U, W$ from Eqs. (22) and (23). Let us first consider the chiral limit $m_q = 0$. In this limit, it is straightforward to check that we have the trivial solution $\sin\phi_k = 0$. This solution is clearly the trivial vacuum $|0\rangle$ (there is no rotation). The non-trivial solutions on the other hand need to be found numerically. In addition, in this limit $m_q \to 0$ the mass gap equation is symmetric under the exchange of the $\sin\phi_k$ sign, that is, if $\sin\phi_k$ is a solution, then $-\sin\phi_k$ is a solution as well.

For the case $m_q = 0$, we find three solutions which can be seen in Fig. 1. The first solution, ϕ^0_k, represents the BCS vacuum $|\Omega\rangle$. The next two solutions, ϕ^1_k and ϕ^2_k, represent the two first excited vacua or replicas.

Once the mass gap has been solved, we can calculate the vacuum quark-antiquark condensate given by

$$\langle\bar{q}q\rangle \equiv \langle\Omega|\bar{\Psi}(0)\Psi(0)|\Omega\rangle = -\frac{3}{\pi^2}\int_0^\infty dk\, k^2 \sin\phi_k. \tag{28}$$

This condensate is quadratically UV divergent for $m_q \neq 0$, so beyond the chiral limit we need to regulate this by subtracting the trivial vacuum contribution

$$\langle\bar{q}q\rangle_{reg} = -\frac{3}{\pi^2}\int_0^\infty dk\, k^2 \left(\sin\phi_k - \frac{m_q}{E_k}\right). \tag{29}$$

Although the bare quark mass is given by the parameter m_q, when we consider quasiparticles we can introduce a dressed or constituent mass $M(k)$ for the quasiparticle (see [6] for further details). This constituent mass is related with the gap angle by

$$\sin\phi(k) = \frac{M(k)}{E} \tag{30}$$

with $E = \sqrt{M^2(k) + k^2}$. So once we have found the solutions ϕ^0_k, ϕ^1_k, ϕ^2_k, it is straightforward to calculate the associated constituent masses. For the case $m_q = 0$, the dressed masses $M(k)$ can be seen in Fig. 2. We can notice the solutions are characterized by the nodes the mass plot has, with the BCS solution having no nodes and the first and second replica having one and two respectively.

We can try to add a small current mass of $m_q = 1$ MeV to the quark and analyse the solutions. In this case we have only found two solutions instead of three (Fig. 3), the BCS solution ϕ^0_k and the first replica ϕ^1_k. Although we thoroughly examined the solution space, we weren't able to reproduce the second replica we find in the chiral limit. This is probably caused by the addition of a small

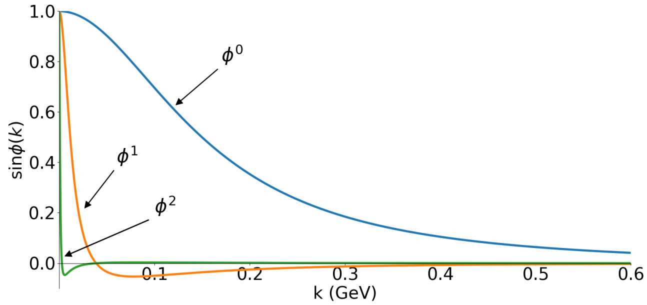

FIG. 1. The first three (non trivial) solutions of the mass gap equation in the chiral limit $m_q = 0$. We have analogous solutions under the exchange $\sin\phi_k \leftrightarrow -\sin\phi_k$, but we choose here to show only the first quadrant solutions.

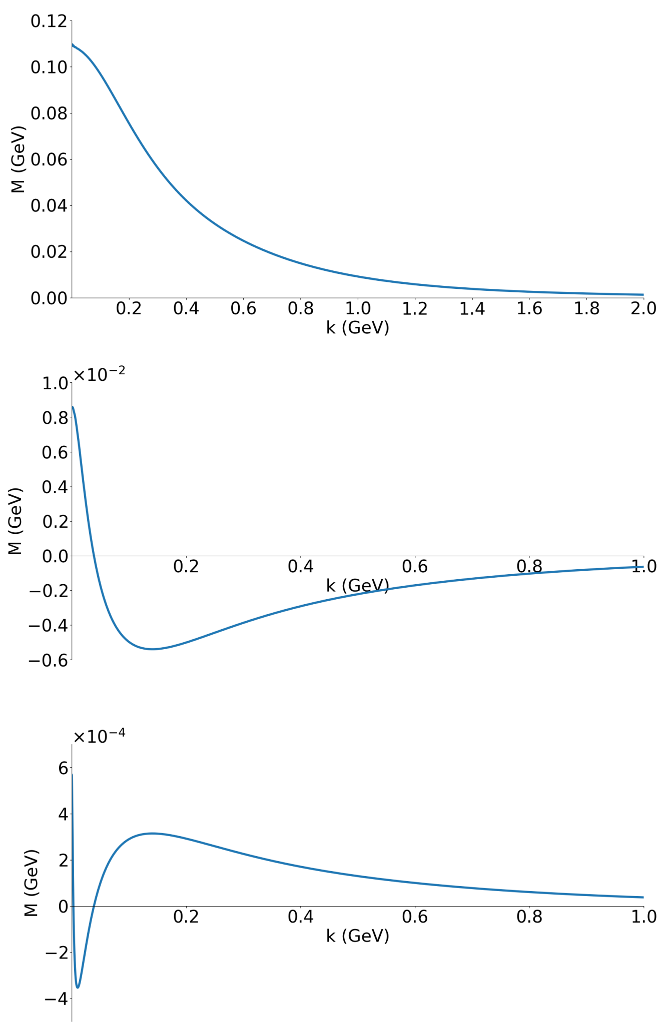

FIG. 2. Constituent mass function of the quark $M(k)$ in the chiral limit $m_q = 0$. From top to bottom, they correspond to the ϕ_k^0, ϕ_k^1 and ϕ_k^2 solutions.

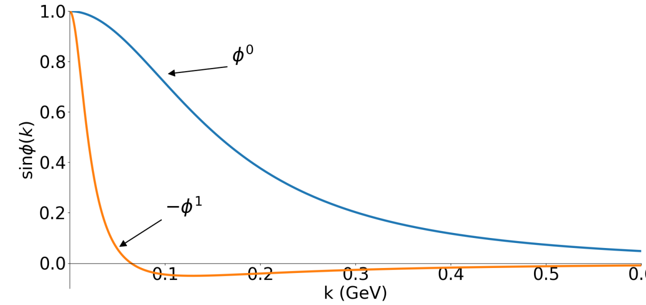

FIG. 3. The first two (non trivial) solutions of the mass gap equation with $m_q = 1$ MeV. We show ϕ_k^1 with a sign change to compare it with the chiral limit in Fig. 1.

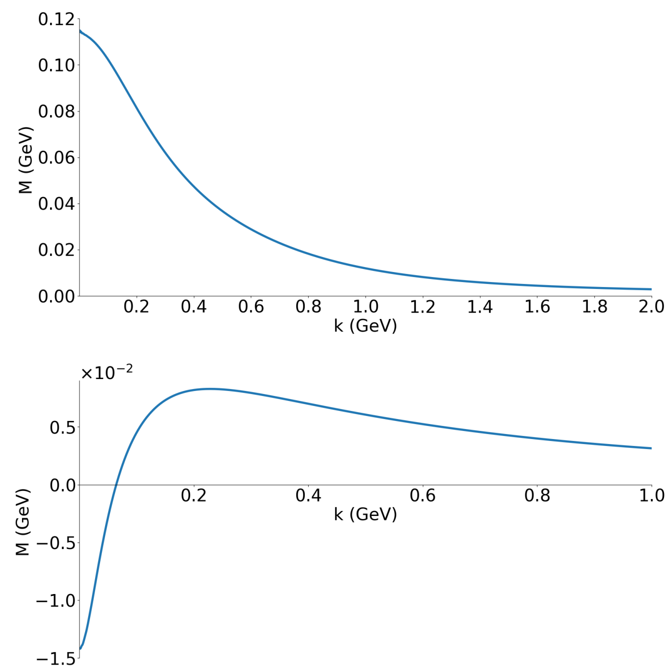

FIG. 4. Constituent mass function of the quark $M(k)$ with $m_q = 1$ MeV. From top to bottom, they correspond to the ϕ_k^0, and ϕ_k^1 solutions.

current mass, which provokes the second replica to be closer to the origin making it more numerically challenging. A more subtle study may be needed to find this solution and more excited vacua. The dressed masses of the solutions for the case $m_q = 1$ MeV can be seen in Fig. 4, showing the same behaviour as in the chiral limit case.

IV. EXTREMAL POINTS ANALYSIS

Now that we have found several solutions for the mass gap equation, we can analyse the properties of these replicas. In [7], the Lisbon investigators have suggested that in a similar model (with an harmonic potential) the vacuum replicas may be metastable. So the main objective of this section is to check if in this model the excited vacua we have found are metastable or unstable. If they are indeed metastable, we can build a hadronic spectrum in the replicas and study its properties. If they are not, the system can classically decay from the replicas to the BCS vacuum and no stable hadronic spectrum would be possible in the excited vacua. From now on, we will only study the solutions within the chiral limit $m_q = 0$. This is motivated by two main reasons. First, this is the limit used in [7], where it is discussed that it is enough to study this case, because the addition of a small current mass for the quark $m_q =$ 1-5 MeV can be seen as a small increase in the constituent mass of the quark $M(k)$, not changing the general picture of the model. Second, it seems logical to use the chiral limit in our case because it is where we have found more solutions, so we have a richer vacua structure. However, the analysis we will carry out is completely analogous for a small current quark mass.

Although the mass gap equation solutions are extremal points of the energy density (recall Eq. (24)), to study if the replicas are metastable we need to calculate the second functional derivative of the vacuum energy density

$$F(k,q) = \frac{\delta^2 \langle \Omega | H | \Omega \rangle}{\delta \phi_q \delta \phi_k}. \tag{31}$$

The fastest way to asses the positivity (or otherwise) of such a quadratic form is to look at the sign of the eigenvalues of its matrix evaluated at each solution ϕ_k^i of Eq. (25). If all the eigenvalues are strictly positive, i.e. the matrix is definite positive, the solution corresponds to a minimum. If the eigenvalues are all negative, it is a maximum. Finally, if the eigenvalues are mixed, the solution is a saddle point. In the first case and interpreting ϕ_k as a classical effective field, there would not be any classical trajectory to decay from that solution to another and the vacuum would be metastable. However, in the last two cases there would be classical trajectories to decay and the vacuum would be unstable. The eigenvalue equation for this matrix reads

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} F(k,q)\psi_i(q) = \lambda_i \psi_i(k) \tag{32}$$

which in the model of Eq. (22) leads to

$$\begin{aligned}\lambda_i\psi_i(k) =& 6(kc_k + m_q s_k)\psi_i(k) - 4\int \mathrm{d}^3 q \left[\hat{V}(|\vec{k}-\vec{q}|)(s_k s_q + c_q c_k x) + 2\hat{U}(|\vec{k}-\vec{q}|)s_k s_q + 2c_k c_q \hat{W}(|\vec{k}-\vec{q}|)\right]\psi_i(k) \\ &+ 4\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\left[\hat{V}(|\vec{k}-\vec{q}|)(c_k c_q + s_q s_k x) + 2\hat{U}(|\vec{k}-\vec{q}|)c_k c_q + 2 s_k s_q \hat{W}(|\vec{k}-\vec{q}|)\right]\psi_i(q).\end{aligned} \tag{33}$$

TABLE I. Summary of the nature of the mass gap equation solutions (vacua) for the model. Eq. (33) is discretized in a grid of 600 points to yield mostly positive eigenvalues, but we list here whether any of them is negative.

Vacuum	Critical point	# negative eigenvalues
Perturbative	Saddle point	3
BCS	Minimum	0
1st replica	Saddle point	1
2nd replica	Saddle point	2

We solve the eigenvalue problem numerically, finding the results summarised in Table I. As we expected, the BCS vacuum ϕ_k^0 is a minimum, so it is stable and there is, presumably, where the QCD hadronic spectrum lives. Regarding the perturbative vacuum, it is a saddle point and therefore it is unstable. Although from the mental picture of the Mexican hat we introduced at the beginning of the paper some might argue the perturbative vacuum has to be a maximum, the Mexican hat is just a graphical assistance to visualize the idea of the two classes of vacua. Actually, we are working in the space function L^2 for the gap function $\sin\phi(k)$, which is an infinite dimensional space and the vacuum picture is not as clear as a Mexican hat. However, the instability of the perturbative vacuum fits with the spontaneous symmetry breaking of the theory and its (classical) decay to the BCS vacuum. Finally, we find both replicas ϕ_k^1 and ϕ_k^2 are saddle points and hence they seem to be classically unstable.

As we mentioned before, the Lisbon investigators found in [7] that these replicas are metastable. This follows from the orthogonality of the Fock spaces of the vacua (in the infinite volume limit, although even for small volumes the Fock spaces are for all intents and purposes orthogonal). This orthogonality prevents the decay from one replica to another via the emission of particle-antiparticle pairs, because it builds an effective infinite potential between the replicas.

In that case, the stability is addressed at a second level, examining the creation of meson-like $q\bar{q}$ states over the replica. They obtained that

$$\langle\Omega_i|[H, \int \mathrm{d}^3k\, \Psi B_k^\dagger D_{-k}^\dagger]|\Omega_i\rangle \tag{34}$$

is a matrix with only positive eigenvalues, where $|\Omega_i\rangle$ $i = 1, 2$ is the first replica and second respectively. This means all the meson excitations they found in the replicas have positive mass squared (no tachyons) (see [8] for a confirmation of this result).

So it is not possible neither to decay from the replicas varying the gap angle (due to the orthogonality of the Fock spaces) nor in a trajectory with a constant gap angle emitting mesons (because the spectrum has no tachyons). Hence the replicas may seem stable.

However, the discussion is subtle. If for any gap angle the Fock spaces are orthogonal and hence it is not possible to decay, it would not be necessary for the spectrum to sit at a minimum of H. It could just live in a state with any gap angle, no matter how many descending classical trajectories existed because the effective infinite potential would prevent the particles from decaying. This is also true for the case of the trivial vacuum $|0\rangle$ and the BCS vacuum $|\Omega\rangle$, as it would not be possible to decay from the first to the second due to their orthogonality. Therefore it seems possible that there is some physical mechanism that allows the QCD vacuum to relax to the BCS vacuum, so decays within the replicas and the BCS vacuum would be possible. If this type of mechanism exists, from our previous analysis the replicas would seem to be unstable. However, further investigation and discussion is needed to clarify the nature of the replicas.

During all this work, we have assumed that the quark condensate is a spacetime-independent constant that fills all the space. However, there is another possibility. Some investigators (see for example [9]) argue that the quark condensate is only a matter of hadrons, and therefore it only exists within a bubble surrounding the hadron with enough radius for the gluons and quarks to propagate, while the rest of the space is "empty" in terms of the condensate (in analogy to the old quark model). This hypothesis, if correct, completely solves the QCD contribution to the cosmological constant problem. However, this hypothesis is far from being broadly accepted by the scientific community.

Although the space is infinite-dimensional, there is a visual check we can carry out to see the space of vacua. We take a slice of L^2 by choosing a curve $f(k)$ parametrized by $\alpha \in \mathbb{R}$. We force this curve to go through the four solutions we have found (including the trivial) and we evaluate $\langle\Omega|H|\Omega\rangle$ in $f(k)$. Then we can plot the energy density (which is just a number) as a function of the parameter α. We have chosen the following curve in the function space:

$$\begin{aligned} f(k) = &\left[\frac{4}{3}(\phi_k^2 - \phi_k^t) - 4(\phi_k^1 - \phi_k^t) + 4(\phi_k^0 - \phi_k^t)\right]\alpha^3 \\ &+ \left[-2(\phi_k^2 - \phi_k^t) + 8(\phi_k^1 - \phi_k^t) - 10(\phi_k^0 - \phi_k^t)\right]\alpha^2 \\ &+ \left[\frac{2}{3}(\phi_k^2 - \phi_k^t) - 3(\phi_k^1 - \phi_k^t) + 6(\phi_k^0 - \phi_k^t)\right]\alpha \\ &+\phi_k^t \end{aligned} \tag{35}$$

where ϕ_k^t is the trivial solution. It is easy to check that for $\alpha = 0$ we are in the trivial vacuum $f(k) = \phi_k^t$, for $\alpha = 1/2$ in the BCS vacuum $f(k) = \phi_k^0$, for $\alpha = 1$ in the first replica $f(k) = \phi_k^1$ and for $\alpha = 3/2$ in the second replica $f(k) = \phi_k^2$. We have already calculated the energy density as a function of ϕ_k in Eq. (22), so making use of the angular integrals already defined in Eq. (27), we have to numerically calculate

$$\begin{aligned} \rho = &-\frac{3}{\pi^2}\int_0^\infty \mathrm{d}k\,(k^3 c_k + m_q k^2 s_k) \\ &- \frac{1}{4\pi^4}\int_0^\infty \mathrm{d}k\,k^2 \int_0^\infty \mathrm{d}q\,q^2 \big[V_0(1 - s_k s_q) - V_1 c_k c_q \\ &- 2U_0(1 + s_k s_q) - 2c_k c_q W_0\big] \end{aligned} \tag{36}$$

in $\phi(k) = f(k)$ for $\alpha \in [0, 3/2]$. In fact, as in the case of the quark condensate, we have to subtract the trivial vacuum contribution $\rho_t = \langle 0|H|0\rangle/V$ to control the UV divergence:

$$\begin{aligned} \rho_{reg} = &-\frac{3}{\pi^2}\int_0^\infty \mathrm{d}k\,(k^3(c_k - 1) + m_q k^2 s_k) \\ &+ \frac{1}{4\pi^4}\int_0^\infty \mathrm{d}k\,k^2 \int_0^\infty \mathrm{d}q\,q^2 \big[V_0 s_k s_q + V_1(c_k c_q - 1) \\ &+ 2U_0 s_k s_q + 2W_0(c_k c_q - 1)\big]. \end{aligned} \tag{37}$$

We can see in Fig. 5 the results for representing this slice of the function space.

In this slice we recover the Mexican hat form for the trivial and BCS vacuum, with the excited vacua following. From the point of view of such slice, the replicas (that we found are saddle points), look like maxima. However, if we represent the same picture only with pairs of vacua, using the parametrization

$$f(k) = \alpha\phi_k^i + (1 - \alpha)\phi_k^j \tag{38}$$

as we can see in Figs. 6, 7 and 8, we recover the saddle point appearance. From the trivial vacuum, the trajectory is descending for both replicas. However, from the BCS vacuum to the replicas the trajectory is ascending in both cases. This is an illustration of the mixed eigenvalues we found for the replicas.

Turning to our numerical results, we found for the first and second replicas respectively the following energy

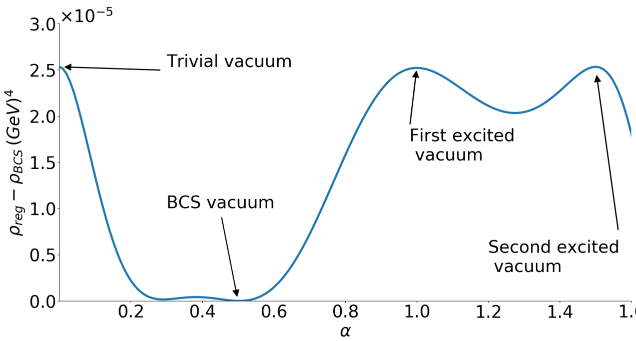

FIG. 5. Energy density along a slice of the function space for the vacua in the chiral limit $m_q = 0$. We plot $\rho_{\rm reg} - \rho_{\rm BCS}$, where$\rho_{\rm BCS}$ is the regularized energy density of the BCS vacuum ϕ_k^0, so the BCS vacuum is the 0 energy level.

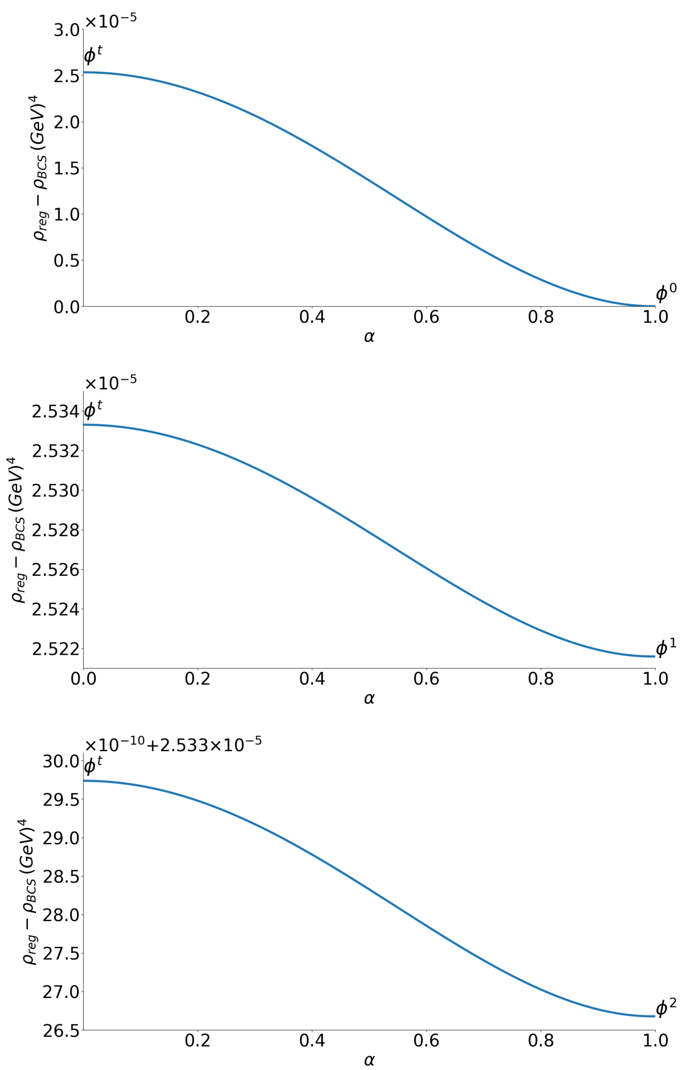

FIG. 6. Energy density along trajectories from the trivial vacuum to the BCS vacuum and the two replicas.

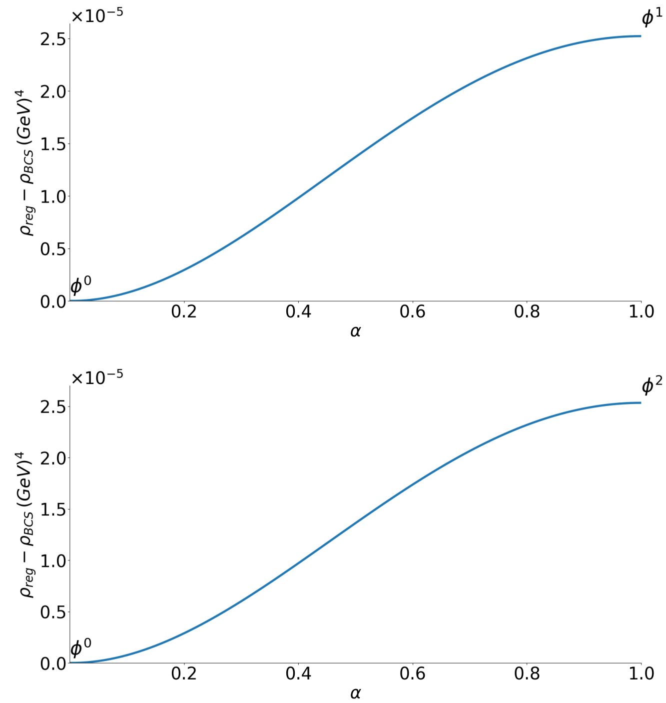

FIG. 7. Energy density along trajectories from the BCS vacuum to the two replicas.

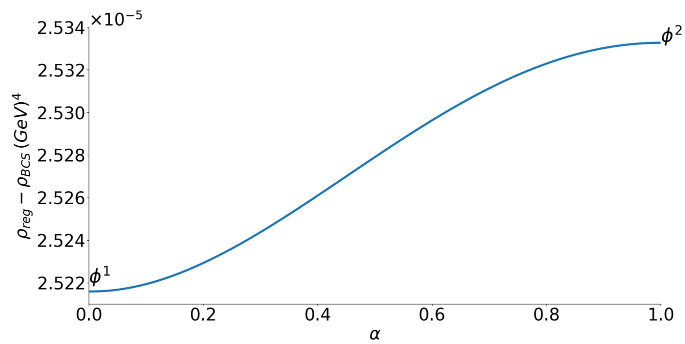

FIG. 8. Energy density along a trajectory from the first replica to the second replica.

densities:

$$\rho_1 = (0.1181 m_g)^4 = 3.30\ \mathrm{MeV/fm}^3, \tag{39}$$

$$\rho_2 = (0.1182 m_g)^4 = 3.31\ \mathrm{MeV/fm}^3, \tag{40}$$

where $m_g = 600$ MeV is the scale for the theory and they are measured from the BCS ground state ϕ_k^0.

Next we can calculate the quark condensate for each solution by means of Eq. (29). In the chiral limit $m_q = 0$ we have

$$\langle \bar{q}q \rangle_{\phi^t} = 0, \tag{41}$$

$$\langle \bar{q}q \rangle_{\phi^0} = -(178\ \mathrm{MeV})^3, \tag{42}$$

$$\langle \bar{q}q \rangle_{\phi^1} = (73\ \mathrm{MeV})^3, \tag{43}$$

$$\langle \bar{q}q \rangle_{\phi^2} = -(61\ \mathrm{MeV})^3. \tag{44}$$

Adding a small quark mass $m_q = 1$ MeV we find

$$\langle \bar{q}q \rangle_{\phi^0} = -(189\ \mathrm{MeV})^3, \tag{45}$$

$$\langle \bar{q}q \rangle_{\phi^1} = -(111\ \mathrm{MeV})^3. \tag{46}$$

For this calculation we have recovered the small quark mass case we found the solutions for in Sec. III because it will help us to understand the sign of the quark condensate. First, we can compare the value of the quark condensate in the BCS solution ϕ_k^0 with recent Lattice estimations of this value (see [10]). Our result is smaller than the latest Lattice calculations of $\langle \bar{q}q \rangle_{\phi^0} = -(272\ \mathrm{MeV})^3$. Turning off the hyperfine interaction of Eq. (9), the value of the condensate decrease to about $-(120\ \mathrm{MeV})^3$. We therefore conclude an improved model may be needed to reach the Lattice estimation for the condensate, including higher order terms to the Hamiltonian.

Let us turn now to the sign of the quark condensate and the dressed quark mass $M(k)$. First, consider the constituent masses $M(k)$ we have shown in Fig. 2 and 4. We can see in these figures the dressed mass has negative parts, what seems unphysical. However, $M(k)$ has to be understood as an auxiliary function for a confined quark, not like a physical mass. In fact, the masses of the physical mesons can be calculated in the BCS vacuum and in the replicas and they have positive mass (see [7], [8]). Now consider the quark condensate calculated in Eq. (29). We can see it has an explicit minus sign and the result is expected to be negative, which is what happens in all the solutions except for the first replica in the chiral limit, as we can see in Eq. (43). We can understand the change in the sign making use of the Gell-Mann-Oakes-Renner relation

$$-\langle \bar{q}q \rangle m_q(\mu) = m_\pi^2 f_\pi^2 \tag{47}$$

where m_π and f_π^2 are respectively the mass and decay constant of the pion and μ is the renormalization scale. This relation yields several interesting results. We now understand that what is a physical quantity is not the condensate or the quark mass, but the product of them (eventually, $m_q(\mu) \to M(k = \mu)$). In that case, the left side of the identity has to be positive, so if we have a positive condensate, the dressed mass has to be negative. And this is exactly what happens for the first replica in the chiral limit, as we can see in Fig. 2, the dressed mass has mainly negative values. Notice the opposite happens with $m_q = 1$ MeV, the quark condensate is negative (Eq. (46)) because the dressed quark mass is positive (Fig. 4).

The constituent quark mass $M(k)$ characterizes the degree of symmetry breaking. We can analyse how the increase of the current quark mass m_q affects the chiral symmetry breaking. We plot the dressed quark mass $M(k)$ in the BCS vacuum ϕ_k^0 for different values of m_q (Fig. (9)). We can see for a small value of $m_q = 5$ MeV, which is characteristic of the u and d quarks, the degree of chiral symmetry breaking remains close to the chiral limit $m_q = 0$. When we increase the current quark mass to $m_q = 30, 100$ MeV, characteristic of the s quark, the degree of chiral symmetry breaking has rapidly increased and when we arrive to masses of about the c quark, with $m_q = 1500$ MeV, the symmetry is completely broken.

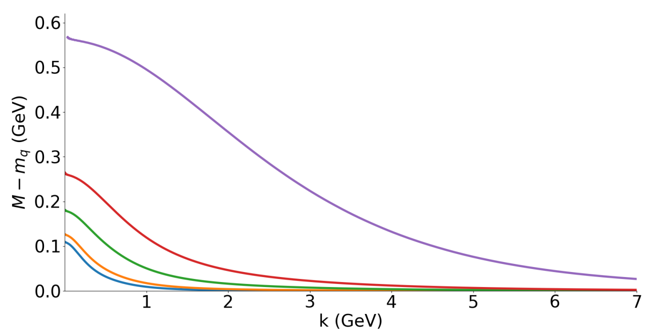

FIG. 9. Constituent quark mass of the BCS solution ϕ_k^0 for different current quark masses. From bottom to top, $m_q = 0, 5, 30, 100, 1500$ MeV. The subtraction of m_q is convenient to compare the relative intensity of vacuum chiral symmetry breaking.

To finish this section, we can calculate at what temperature a gas of pions (that are the first excitation appearing over $|\Omega_i\rangle$) populate the excited vacua. This will give us some insight in how much temperature would be necessary to reach the replicas and if the needed temperature is lower than the phase transition from gas to plasma, allowing the gas to occupy the replicas.

We use the energy density of the gas of pions calculated in the frame of reference where the gas is at rest (see [11] and references therein)

$$\rho_{\mathrm{gas}}(T) = g \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\sqrt{m_\pi^2 + k^2}}{e^{\sqrt{m_\pi^2+k^2}/T} - 1} \tag{48}$$

where $g = 3$ is the degeneration for the pion isospin triplet, m_π is the pion mass and T is the temperature. We can easily calculate this integral numerically for a given temperature and then represent this energy density as lines in the vacuum space we have shown in Fig. 5, so we can check at what temperature the gas is energetic enough to populate the replicas if there is a mechanism allowing it. The results are shown in Fig. 10. We can see both replicas are populated for temperatures near $T = 117 - 120$ MeV. This result is compatible with the gas occupying the replicas without reaching the quark-gluon plasma phase, since the transition has $T_c \approx 170$ MeV. Moreover, it shows these replicas could have been populated in the early stages of the universe, and then decay with the decrease of temperature to the BCS vacuum.

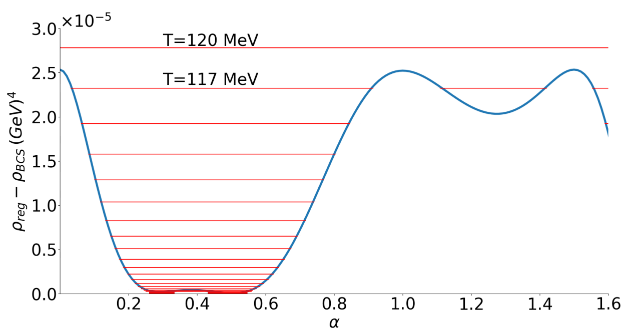

FIG. 10. Gas of pions populating the vacuum space for different values of the temperature. From top to bottom, each horizontal red line represents the energy density of the gas for a given temperature, with a 3 MeV decrease with each line.

V. COMPARISON TO COSMOLOGY

We can take advantage of the classical path to decay from the replicas to the BCS vacuum to perhaps model some early stage of the universe. So we are going to look for an analogy for the scalar inflation theory in our model (see [12] for an introduction to inflation).

Consider the inflaton scalar field $\phi(t, \vec{x})$ and an associated potential $V(\phi)$ (units of energy density). Models of inflation must fulfill a slow-roll condition to guarantee a large enough number of e-foldings. This slow-roll condition in the scalar inflaton model reads as

$$\left(\frac{E_P}{V}\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2 \ll 1 \tag{49}$$

with E_P the Planck energy $E_P = \sqrt{1/G} = 1.22 \cdot 10^{19}$ GeV. We are going to evaluate the left side of the condition in our QCD model and compare it with the slow-roll requirement for inflation.

In the model of Eq. (22), we use the quark condensate $\langle\bar{q}q\rangle^{1/3}$ as the scalar field ϕ (recall it is a ${}^3P_0$ condensate so it is a scalar and has energy cube units so we need to take the cubic root to match the inflaton field dimensions). Moreover, we identify the potential $V(\phi)$ with the energy density expected value ρ. In order to study the evolution of the system, we make use of the eigenvectors we calculated through Eq. (33).

We choose the most negative eigenvalue of the replica we are rolling from, which is the fastest path to decay, and we use the corresponding eigenvector $\psi(k)$. Then we parametrize the trajectory as

$$\varphi_g(k) \equiv \phi^r(k) + \alpha\psi(k) \tag{50}$$

where ϕ^r_k is one of the replicas. In particular, we explore the case $\phi^r_k = \phi^1_k$, i.e. the decay from the first replica to the BCS vacuum. As we can see in Table I there is only one negative eigenvalue for the first replica solution, so it is the eigenvector associated with this eigenvalue the one we will choose to make the gap angle evolve. So $\varphi_g(k)$ is the angle where we will evaluate the quark condensate and the energy density for different values of α, which will parametrize the evolution.

We calculate numerically the left side of Eq. (49), leading to

$$\left(\frac{E_P}{V}\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2 \sim 10^{38} \gg 1. \tag{51}$$

Clearly the slow-roll condition is not fulfilled, the rolling down the potential is much faster than what is needed for an exponential expansion and therefore this model would not be suitable for inflation. We can easily understand why this happens. The Planck constant is $E_P \sim 10^{19}$ GeV and from our numerical results we have

$$V = \rho \sim 10^{-5}\ \mathrm{GeV}^4 \approx (10^{-1}m_g)^4, \tag{52}$$

$$\frac{\mathrm{d}V}{\mathrm{d}\phi} = \frac{\mathrm{d}\rho}{\mathrm{d}\langle\bar{q}q\rangle^{1/3}} \sim 10^{-5}\ \mathrm{GeV}^3 \approx (10^{-2}m_g)^3. \tag{53}$$

We can see the problem is V is much smaller than E_P, so the term E_p/V is much bigger than the derivative term $\mathrm{d}V/\mathrm{d}\phi$, making the whole result too large for the condition to be satisfied. In terms of the energy scale m_g we have

$$\left(\frac{E_P}{V}\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2 \sim \left(\frac{10^{17}\ \mathrm{GeV}}{m_g}\right)^2. \tag{54}$$

So in order to the slow-roll condition to be fulfilled, we would need an energy scale of about $10^{18}m_g$, which is much higher than the QCD scale. So from our numerical results we can conclude the analogy with inflation fails for models at the QCD scale. A confining gauge theory at a higher scale such as $10^{18}m_g$ should however work.

VI. CONCLUSIONS

In Sec. III, we have analysed the vacua of a simple family of models of QCD in the Coulomb gauge. These vacua were found solving numerically the mass gap equation, a non-linear integral equation. In the chiral limit, we found four solutions, corresponding to the perturbative vacuum, the BCS vacuum and two excited replicas, confirming previous results in this type of models. For the case of a small quark mass, namely $m_q = 1$ MeV, we found analogous solutions but we were only able to identify one replica, probably due to numerical issues. This is also the case for more excited vacua in both $m_q = 0, 1$ MeV, as the solutions are closer to the y-axes in each excited vacuum, so a very careful numerical analysis needs to be done in order to find further replicas.

In Sec. IV, we have analysed the nature of the vacua in the chiral limit, confirming the metastability of the BCS vacuum and finding the two replicas we obtained are classically unstable. The Lisbon investigators have suggested in [7] that these replicas are in fact stable due to second quantization issues. However, the discussion is subtle and needs further consideration. We have also studied the quark condensate in all the solutions, with the BCS solution quark condensate smaller than the Lattice estimations, suggesting more refined models including higher order corrections would be needed to reach the Lattice value. Moreover, we have analysed the degree of symmetry breaking of the theory by means of the dressed quark mass $M(k)$, getting the expected result that for current masses m_q of about the s quark mass and beyond, the degree of chiral symmetry breaking is much higher than for smaller current masses of about the u and d quarks masses (explicit m_q chiral symmetry breaking induces additional spontaneous breaking). In addition, we have calculated that for a temperature of about 120 MeV, the replicas could be populated.

Finally, in Sec. V, due to the instability of the replicas, we have tried to look for an analogy in cosmology, comparing it with a scalar inflation model. The slow-roll conditions are trivially not fulfilled at the QCD scale, so an analogy with inflation is only possible for similar theories at a much higher scale.

As is stated in [7], a metastable phase would have important physical consequences. In a metastable replica bubble, it is possible to have a stable hadron spectrum, and therefore these excitations would be detectable in the experiments, confirming the existence of excited vacua in QCD. Even if these replicas are unstable, they could be indirectly detected (see [1]). If we consider a small bubble of about 5 fm for the replicas, the spectrum built there would contain tachyons and therefor be unstable. However, they may still live a finite amount of time comparable to the time of hadronic processes. Then this process can go through an intermediate process of formation of a local bubble and then a decay to the BCS vacuum, via the emission of pairs of particles such as pions. These could be detected experimentally, being a signal of the existence of replicas in QCD. In any of the cases, whether they are amenable to experimental detection is challenging, but the theoretical and experimental efforts in the direction of understanding this false vacua could be rewarded with a deeper understanding of QCD vacua and the Standard Model in general [13].

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[13] A poster partially based on the results of this work will be presented in July in the European Physical Society 2021 HEP Conference.