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El momento magnético anómalo del muón: tensión entre teoría y experimento

The anomalous magnetic moment of the muon: tension between theory and experiment

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The anomalous magnetic moment of the muon: tension between theory and experiment

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In this master thesis I describe the state of the art of the theoretical and experimental determinations of the anomalous magnetic moment of the muon. Specifically, I have computed the tree and one-loop level QED contributions and I have studied the remaining corrections to the theoretical result. Regarding the experimental determination, I have studied and reviewed the experimental setup, and I have also rederived and clarified the theoretical formalism needed to extract the final result. In addition, I have estimated the effect of the synchrotron radiation, which has been omitted in the literature.

I. INTRODUCTION AND MOTIVATION

The anomalous magnetic moment of the muon, a_μ, has a huge relevance nowadays due to the fact that there exists a significant deviation between the Standard Model prediction and the experimental value (see [1]),

$$\begin{aligned} a_\mu^{\text{th}} &= 116\,591\,810(43)\,10^{-11}, \\ a_\mu^{\text{exp}} &= 116\,592\,061(41)\,10^{-11}, \\ \Rightarrow &\boxed{\Delta a_\mu(\text{exp-th}) = (251 \pm 59)10^{-11}}. \end{aligned} \tag{1}$$

The discrepancy is of 4.2 standard deviations, and as a consequence, it could hint to physics beyond the Standard Model. The aim of this work is to understand both the theoretical and the experimental determinations.

In order to explain what the anomalous magnetic moment means, let me first consider a classical particle with charge q and mass m orbiting in a circular loop of radius r in the XY plane. Its angular momentum is

$$\vec{L} = m\vec{r} \times \vec{v} = mrv\vec{e_z}. \tag{2}$$

In this situation, the charge, while moving, generates a current $I = q/T = qv/(2\pi r)$, where $T = 2\pi r/v$ is the movement period. Then, the particle carries a magnetic moment

$$\vec{\mu} = IA\vec{n} = I\pi r^2 \vec{e_z} = \frac{qvr}{2}\vec{e_z} = \frac{q}{2m}\vec{L}, \tag{3}$$

where I is the current generated, A the surface within the trajectory and $\vec{n}$ the unitary vector normal to it. Due to this magnetic moment the particle interacts with an external magnetic field $\vec{B}$ in the following way

$$H_M = -\vec{\mu} \cdot \vec{B}. \tag{4}$$

Similarly, when considering particles with spin $\vec{S}$, which can be thought of as an intrinsic angular momentum, they satisfy, in the presence of a magnetic field, an equation similar to (3)

$$\vec{\mu} = g\frac{q}{2m}\vec{S}, \tag{5}$$

where the g-factor is an adimensional constant.

From the non-relativistic limit of Dirac equation, we will obtain $g = 2$ for all the charged leptons. That is the Dirac equation prediction, but it is also the tree level QED calculation. For this reason, the "anomalous" magnetic moment is any deviation from this Dirac simple prediction. We will see in the next section that QED at one loop introduces the first correction to this value, and that there are many other contributions to the theoretical determination of the g-factor. The anomalous magnetic moment is usually parametrized as

$$a_\ell = \frac{g_\ell - 2}{2}, \tag{6}$$

where the $\ell = e, \mu, \tau$ subindex runs over the three leptons in the Standard Model.

Why are we so interested in the muon's magnetic moment? What makes it different from a_e or a_τ? On the one hand, it can be shown that the contributions to a lepton anomalous magnetic moment from new physics at an energy scale Λ verify (see [2] or [3] for more details)

$$\delta a_\ell \propto \frac{m_\ell^2}{\Lambda^2}. \tag{7}$$

Due to (7), a_μ is more sensitive to new physics than a_e. Moreover, a_e is dominated by QED contributions up to a high precision, but it is barely sensitive to hadronic, weak and new physics effects.

On the other hand, although from (7) it follows that a_τ is more sensitive than a_μ to new physics, τ has a very short lifetime ($\tau_\tau = 2.906 \cdot 10^{-13}$ s, to be compared with $\tau_\mu = 2.197$ μs) that does not allow for high precision studies. This fact is exhibited by the most stringent bonds [4] for a_τ at 95% confidence level

$$-0.052 < a_\tau < 0.013. \tag{8}$$

Thus in practice, a_μ is the most interesting anomalous magnetic moment for New Physics searches.

This master thesis is organized as follows. In Section II, I will obtain $g = 2$ from the non-relativistic limit of the Dirac equation. Then, I will perform the theoretical calculation of a_μ at tree and one-loop level in QED, and next I will briefly comment the remaining Standard

Model contributions together with some computations. In Section III, I will describe the experimental determination of the anomalous magnetic moment of the muon, with the theoretical analysis of the dynamical equations of the particle and its spin polarization. Then, I will show the experimental results, as well as their comparison with the theoretical calculation. Finally, in Section IV, I will discuss further contributions from the synchrotron radiation in the experiment.

II. THEORETICAL CALCULATION OF a_μ

A. g from the Dirac equation

Now, I will focus on the "non-anomalous" part of the magnetic moment of the muon (of a lepton in general). Let me consider Dirac equation in presence of an electromagnetic field, which describes wavefunction the evolution of fermions of spin $\frac{1}{2}$, mass m and charge e

$$i\frac{\partial \Psi}{\partial t} = (\vec{\alpha}(\vec{p} - e\vec{A}) + eA^0 + \beta m)\Psi, \tag{9}$$

where Ψ is the fermion wavefunction, $A^\mu = (A^0, \vec{A})$ is the electromagnetic four-vector, and, in Dirac representation, $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ and β are

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{10}$$

with σ_x, σ_y and σ_z the Pauli matrices. Observe that we are considering $c = \hbar = 1$. In addition, the mass dependence of the wavefunction can be expressed, in the non-relativistic limit, as

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-imt}, \tag{11}$$

where φ and χ do not depend on m. Introducing (10) and (11) in (9), we obtain the following equations

$$\begin{cases} i\frac{\partial}{\partial t}\varphi + m\varphi = \vec{\sigma}(\vec{p} - e\vec{A})\chi + (eA^0 + m)\varphi, \\ i\frac{\partial}{\partial t}\chi + m\chi = \vec{\sigma}(\vec{p} - e\vec{A})\varphi + (eA^0 - m)\chi, \end{cases}$$

or, reordering terms

$$\begin{cases} i\frac{\partial}{\partial t}\varphi = \vec{\sigma}(\vec{p} - e\vec{A})\chi + eA^0\varphi, \\ i\frac{\partial}{\partial t}\chi = \vec{\sigma}(\vec{p} - e\vec{A})\varphi + (eA^0 - 2m)\chi. \end{cases} \tag{12}$$

The non-relativistic limit also implies that both the kinetic and potential energies are negligible when compared with the rest energy, i.e., $|i\frac{\partial}{\partial t}\chi|, |eA^0\chi| \ll |m\chi|$, and therefore the second equation gives us in this limit

$$\chi \approx \frac{\vec{\sigma}(\vec{p} - e\vec{A})}{2m}\varphi. \tag{13}$$

If we put together (12) and (13), we get

$$i\frac{\partial}{\partial t}\varphi \approx \frac{(\vec{\sigma}(\vec{p} - e\vec{A}))(\vec{\sigma}(\vec{p} - e\vec{A}))}{2m}\varphi + eA^0\varphi. \tag{14}$$

Using the following Pauli's matrices property

$$(\vec{\sigma}\vec{b})(\vec{\sigma}\vec{c}) = \vec{b}\vec{c}I + i\vec{\sigma}(\vec{b} \times \vec{c}), \tag{15}$$

with $\vec{b}$ and $\vec{c}$ arbitrary vectors, we have

$$\begin{gathered} (\vec{\sigma}(\vec{p} - e\vec{A}))(\vec{\sigma}(\vec{p} - e\vec{A})) = \\ (\vec{p} - e\vec{A})^2 + i\vec{\sigma}((-i\vec{\nabla} - e\vec{A}) \times (-i\vec{\nabla} - e\vec{A})) = \\ (\vec{p} - e\vec{A})^2 - e\vec{\sigma}(\vec{\nabla} \times \vec{A} + \vec{A} \times \vec{\nabla}) = \\ (\vec{p} - e\vec{A})^2 - e\vec{\sigma}(\vec{\nabla} \times \vec{A}) = (\vec{p} - e\vec{A})^2 - e\vec{\sigma}\vec{B}, \end{gathered} \tag{16}$$

where, in the third step, I have used that the operators are acting on φ and $\vec{\nabla} \times (\vec{A}\varphi) = (\vec{\nabla} \times \vec{A})\varphi - \vec{A} \times (\vec{\nabla}\varphi)$. Therefore, substituting (16) in (14), we obtain

$$i\frac{\partial}{\partial t}\varphi \approx \left[\frac{(\vec{p} - e\vec{A})^2}{2m} - \frac{e}{2m}\vec{\sigma}\vec{B} + eA^0\right]\varphi, \tag{17}$$

also known as the Pauli equation. In this non-relativistic equation, it is verified that $\vec{S} = \frac{\vec{\sigma}}{2}$. Taking this fact into account, when comparing with (4) and (5), we find

$$\vec{\mu} = \frac{e}{m}\vec{S} \quad \Rightarrow \quad g = 2. \tag{18}$$

As promised, we have obtained the Dirac equation's prediction $g = 2$.

B. a_μ at tree level in QED

Following mainly [5], I will prove that QED at tree level predicts $a_\mu = 0$, according to Dirac equation's prediction. The idea is to apply the non-relativistic limit to the interaction term between a muon and a magnetic field at tree level in order to obtain a term of the form $-\vec{\mu}\vec{B}$ and identify the g-factor (same outline as in Dirac equation's case).

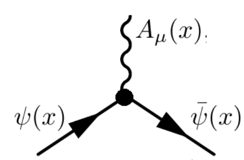

FIG. 1. Interaction between a muon and a magnetic field in QED. At tree level, the interaction vertex is given in (19).

Therefore, we begin with the interaction term in QED

$$\mathcal{H}_I = -e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x), \tag{19}$$

where ψ is the muonic field, $\bar{\psi} = \psi^\dagger\gamma^0$ is its Dirac adjoint, $A_\mu(x)$ is the four-vector of the electromagnetic field, and γ^μ are the Dirac matrices, that we conveniently choose in the Weyl representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (I, \vec{\sigma}), \quad \bar{\sigma}^\mu = (I, -\vec{\sigma}). \tag{20}$$

The free-particle solutions for the field $\psi(x)$ in this representation are

$$\psi(x) = u(p)e^{-ipx}, \quad u(p) = \frac{1}{\sqrt{2E_p}} \begin{pmatrix} \sqrt{p\sigma}\chi \\ \sqrt{p\bar{\sigma}}\chi \end{pmatrix}, \tag{21}$$

where χ is a normalized 2-component spinor $\chi^\dagger\chi = 1$, and $E_p = (|\vec{p}|^2 + m_\mu^2)^{1/2}$. In the non-relativistic limit, since $|\vec{p}| \ll m_\mu \approx E_p$, the solutions are simplified

$$\begin{aligned} u(p) =& \frac{1}{\sqrt{2E_p}} \begin{pmatrix} \sqrt{p\sigma}\chi \\ \sqrt{p\bar{\sigma}}\chi \end{pmatrix} = \frac{1}{\sqrt{2E_p}} \begin{pmatrix} E_p\sqrt{I - \frac{\vec{p}\vec{\sigma}}{E_p}}\chi \\ E_p\sqrt{I + \frac{\vec{p}\vec{\sigma}}{E_p}}\chi \end{pmatrix} \\ =& \frac{1}{\sqrt{2m_\mu}} \begin{pmatrix} m_\mu \left(I - \frac{\vec{p}\vec{\sigma}}{2m_\mu} + O(|\frac{\vec{p}}{m_\mu}|^2)\right)\chi \\ m_\mu \left(I + \frac{\vec{p}\vec{\sigma}}{2m_\mu} + O(|\frac{\vec{p}}{m_\mu}|^2)\right)\chi \end{pmatrix} = \\ =& \frac{1}{\sqrt{2}} \begin{pmatrix} (I - \frac{\vec{p}\vec{\sigma}}{2m_\mu})\chi \\ (I + \frac{\vec{p}\vec{\sigma}}{2m_\mu})\chi \end{pmatrix} + O\left(\left|\frac{\vec{p}}{m_\mu}\right|^2\right). \end{aligned} \tag{22}$$

And, due to the fact that we are interested in the interaction of the muonic field with the magnetic field, we can consider $A_\mu = (0, -\vec{A})$. Together with (19) and (22), this leads to

$$\begin{aligned} \mathcal{H}' :=& \mathcal{H}_I e^{-i(p'-p)x} = e\bar{u}(p')\gamma^i u(p) A^i(x) = \\ & \frac{e}{2}\chi^\dagger \left(I + \frac{\vec{p}'\vec{\sigma}}{2m_\mu}\right)\sigma^i A^i \left(I + \frac{\vec{p}\vec{\sigma}}{2m_\mu}\right)\chi \\ & -\frac{e}{2}\chi^\dagger \left(I - \frac{\vec{p}'\vec{\sigma}}{2m_\mu}\right)\sigma^i A^i \left(I - \frac{\vec{p}\vec{\sigma}}{2m_\mu}\right)\chi + O\left(\left|\frac{\vec{p}}{m_\mu}\right|^2\right) \\ & \approx eA^i\chi^\dagger \left(\frac{\vec{p}'\vec{\sigma}}{2m_\mu}\sigma^i + \sigma^i\frac{\vec{p}\vec{\sigma}}{2m_\mu}\right)\chi. \end{aligned} \tag{23}$$

Using the property of the Pauli matrices $\sigma^i\sigma^j = \delta^{ij}I + i\epsilon^{ijk}\sigma^k$, equation (23) gets simplified

$$\begin{aligned} \mathcal{H}' \approx& eA^i\chi^\dagger \left(\frac{p'^i + p^i}{2m_\mu} + i\epsilon^{ijk}\sigma^j\frac{p'^k}{2m_\mu} + i\epsilon^{ijk}\sigma^k\frac{p^j}{2m_\mu}\right)\chi \\ =& eA^i\chi^\dagger\frac{(p'+p)^i}{2m_\mu}\chi + ieA^i\epsilon^{ijk}\frac{(p-p')^j}{2m_\mu}\chi^\dagger\sigma^k\chi. \end{aligned} \tag{24}$$

The second term in (24) will provide us with the g-factor. Recall we want to recast the Hamiltonian into a $-\vec{\mu}\vec{B}$ term. Let us then integrate the second term in (24)

$$\begin{aligned} & \int d^4x ieA^i(x)\epsilon^{ijk}\frac{(p-p')^j}{2m_\mu}\chi^\dagger\sigma^k\chi e^{i(p'-p)x} = \\ & \int d^4x \frac{e}{2m_\mu}\epsilon^{ijk}A^i(x)\frac{\partial}{\partial x^j}\left((\chi e^{-ip'x})^\dagger\sigma^k\chi e^{-ipx}\right) = \\ & -\int d^4x \frac{e}{2m_\mu}\epsilon^{ijk}\frac{\partial A^i}{\partial x^j}(x)(\chi e^{-ip'x})^\dagger\sigma^k\chi e^{-ipx} = \\ & \int d^4x \frac{e}{m_\mu}B^k(x)s^k(x) = \int d^4x \frac{e}{m_\mu}\vec{s}(x)\vec{B}(x), \end{aligned} \tag{25}$$

where $s^k(x) = (\chi e^{-ip'x})^\dagger\frac{\sigma^k}{2}\chi e^{-ipx}$ is the density of spin, i.e., $S^k = \int d^4 s^k(x)$. Comparing equation (25) with $H_I = -g\frac{e}{2m_\mu}\vec{S}\vec{B}$ (and taking into account that we are dealing with a charge $-e$) we finally obtain

$$g_\mu = 2 \quad \Rightarrow \quad a_\mu = 0, \tag{26}$$

as I had advanced. Nevertheless, as we will see in the next section, at one loop in QED we obtain the first correction to a_μ, i.e., the first theoretical prediction of an anomalous magnetic moment of the muon (of a lepton in general).

C. a_μ at one loop in QED

In this section I will calculate the first correction to a_μ. For this aim, we need to do a general analysis (valid for any quantum field theory) of the most general form of the vertex function and the terms which contribute to a_μ. Any correction to the diagram from figure 1 will have an amplitude given by

$$\mathcal{M} = -ie\bar{u}(p')\Gamma^\mu(p',p)u(p)\epsilon_\mu(p'-p), \tag{27}$$

where Γ^μ is the vertex function and ϵ_μ is the polarization vector of the electromagnetic field. As the diagram only depends on the moments p', p and the spin polarizations of the muons, the most general vertex function is given by[1]

$$\begin{aligned} \Gamma^\mu(p',p) = A\gamma^\mu + B(p'+p)^\mu + C(p'-p)^\mu \\ +iD\sigma^{\mu\nu}(p'+p)_\nu + iE\sigma^{\mu\nu}q_\nu, \end{aligned} \tag{28}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $q = p' - p$, and A, B, ..., E are scalar functions[2] of q^2.

However, we can further simplify (28). First, if we use the Ward identity $q_\mu\bar{u}(p')\Gamma^\mu u(p) = 0$ (the conservation of the electromagnetic current), we obtain[3]

$$\begin{aligned} \bar{u}(p')\left[Cq^2 + iDq_\mu\sigma^{\mu\nu}(p'+p)_\nu\right]u(p) = 0, \ \forall p', p, \\ \Rightarrow C(q^2) = D(q^2) = 0. \end{aligned} \tag{29}$$

Additionally, making use of the Gordon identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m_\mu} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m_\mu}\right]u(p), \tag{30}$$

we can absorb the B term into the A and E terms in (28), leading to the next general expression for the vertex function

$$\Gamma^\mu(p',p) = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m_\mu}, \tag{31}$$

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[1] Notice that terms with γ_5 are not allowed due to the parity conservation of the electromagnetic interaction, and terms with more than one γ matrix can be related to $\sigma^{\mu\nu}$ and $g^{\mu\nu}$ using $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I$.

[2] Because we have only two independent four-vectors p'^μ and p^μ, and $pp = p'p' = m_\mu^2$, $pp' = -\frac{q^2}{2} + m_\mu^2$.

[3] Notice that, from the Dirac equation $(\not{p}+m_\mu)u(p) = 0$, $\bar{u}(p')(\not{p}'+m_\mu) = 0$, which, together with the antisymmetry of $\sigma^{\mu\nu}$ guarantee the conservation of the terms with A, B and E of (28).

where $F_1(q^2)$ and $F_2(q^2)$ are the so-called Dirac and Pauli form factors, respectively.

From the non-relativistic limit, we can interpret these form factors. In the static limit, $eF_1(0)$ is the physical charge, so the normalization condition $F_1(0) = 1$ is taken in order to maintain the charge e. In addition, $F_2(0)$ is related with a_μ, as we will see next. Within the non-relativistic limit, we have[4]

$$u(p) = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi\\ \chi\end{pmatrix} + O\left(\left|\frac{\vec{p}}{m_\mu}\right|\right). \tag{32}$$

For our purposes, as we did at tree level, we can consider $A_\mu = (0, -\vec{A})$. Remember that in this limit $m_\mu \approx E_p$, so we can approximate $q^\mu = (0, \vec{q})$. As a consequence, we only have to consider the components σ^{ij} that are multiplied by the γ^0 from $\bar{u}(p')$

$$\begin{aligned}\gamma^0\sigma^{ij} &= \frac{i}{2}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}[\sigma^j,\sigma^i] & 0\\ 0 & [\sigma^j,\sigma^i]\end{pmatrix}\\ &= \frac{i}{2}2i\epsilon^{jik}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} = \epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}.\end{aligned} \tag{33}$$

Consequently, the amplitude (27) is

$$\begin{aligned}\mathcal{M} = \mathcal{M}_1 - ie\bar{u}(p')F_2(q^2)\frac{i\sigma^{ij}(p'-p)_j}{2m_\mu}u(p)A_i(x) \approx\\ \mathcal{M}_1 + eF_2(0)A^i(x)\epsilon^{ijk}\frac{(p'-p)^j}{2m_\mu}\chi^\dagger\sigma^k\chi,\end{aligned} \tag{34}$$

where $\mathcal{M}_1$ is the amplitude term proportional to $F_1(0)$ which, after fixing $F_1(0) = 1$, provides a g-factor $g'_\mu = 2$ (same calculation as at tree level). Comparing the term with $F_2(0)$ in (34) with the second term of (24) (and remembering that the amplitude has an extra i factor with respect to the Hamiltonian density) we see that the only difference is $e \to eF_2(0)$. As a consequence, we obtain an additional Hamiltonian term that adds an "extra g-factor" $g''_\mu = 2F_2(0)$. The two terms in (34) lead to

$$g_\mu = g'_\mu + g''_\mu = 2 + 2F_2(0) \quad \Rightarrow \quad \boxed{a_\mu = F_2(0)}, \tag{35}$$

which is the relation that justifies the standard definition (6) of a_μ.

Notice that if we apply the Feynman rules to get a structure similar to (31) for an arbitrary the vertex function, we only need to calculate the terms with $(p'-p)_\nu$ to identify $F_2(q^2)$. In the $q^2 \to 0$ limit it will provide us with a_μ. After these considerations, and taking into account that there was no term with $(p'-p)_\nu$ in (19), we find again that $a_\mu = 0$ at tree level in QED.

From now on, I will focus on the one-loop calculation[5], with the strategy I have just explained: reach the structure (31) and identify $F_2(q^2)$, which will provide a_μ.

[4] Since the term with $F_2(q^2)$ in (31) contains $\frac{(p'-p)_\nu}{2m_\mu}$, we are thus approximating that term up to first order in $\left|\frac{\vec{p}}{m_\mu}\right|$.

[5] A similar analysis can be found in reference [6].

At one loop in QED, there is only one diagram (figure 2) that contributes to a_μ (i.e., with a non-vanishing F_2 term).

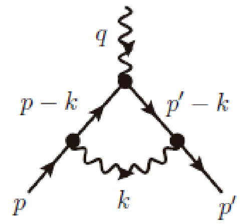

FIG. 2. The only Feynman diagram that contributes to a_μ at one loop in QED .

Using the Feynman rules, the vertex for the diagram in figure 2 is

$$\Gamma^\mu = -i\frac{e^2}{(2\pi)^4}\int d^4k\frac{N^\mu(k^\nu)}{D(k^\nu)}, \tag{36}$$

with

$$\begin{aligned}N^\mu(k^\nu) &= \gamma^\nu(\not{p}' - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m_\mu)\gamma_\nu,\\ D(k^\nu) &= ((p'-k)^2 - m^2 + i0)\\ &\quad\cdot((p-k)^2 - m^2 + i0)(k^2 + i0),\end{aligned} \tag{37}$$

where $m = m_\mu$. Let us now recast our expression in the form (31). For this aim, we use the standard Feynman trick and we work out the Dirac algebra. Taking into account that the vertex always appears in the structure $\bar{u}(p')\Gamma^\mu u(p)$, we arrive to

$$\Gamma^\mu = -i\frac{2e^2}{(2\pi)^4}\int d^4k\int_0^1 dx\int_0^{1-x} dy\frac{N'^\mu(k^\nu,x,y)}{D'(k^\nu,x,y)^3}, \tag{38}$$

where in this case

$$\begin{aligned}N'^\mu &= N_1^\mu + N_2^\mu,\\ N_1^\mu &= -2\not{k}\gamma^\mu\not{k} - \gamma^\mu 2m^2(2x + 2y - x^2 - y^2)\\ &\quad-\gamma^\mu 4pp'(1 - x - y - xy),\\ N_2^\mu &= 4mp^\mu(x - xy - y^2) + 4mp'^\mu(y - xy - x^2),\\ D' &= k^2 + p'^2(x - x^2) + p^2(y - y^2) - 2pp'xy\\ &\quad-m^2(x+y) + i0.\end{aligned} \tag{39}$$

Notice that from (31) and the Gordon identity (30), the N_2^μ term corresponds to $-F_2(q^2)(p'+p)^\mu/(2m)$. Additionally, only N_1^μ is divergent, so the renormalization scheme only changes that term, while the term with N_2^μ remains unaffected. As we are only interested in $F_2(0)$, performing the integration (38) only for N_2^μ in the $q^2 \to 0$ limit, we can obtain $F_2(0)$ and a_μ

$$\begin{aligned}\frac{(p+p')^\mu F_2(0)}{2m} &= \lim_{q^2\to 0}\frac{ie^2}{8\pi^4}\int d^4k\int_0^1 dx\int_0^{1-x} dy\frac{N_2^\mu}{D'^3}\\ &= \frac{(p+p')^\mu}{2m}\frac{e^2}{8\pi^2} \quad\Rightarrow\quad \boxed{a_\mu = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}},\end{aligned} \tag{40}$$

that is the anomalous magnetic moment of a lepton up to one loop in QED.

D. Other Standard Model contributions to a_μ

The theoretical prediction of the anomalous magnetic moment of the muon is decomposed in QED, Electroweak and hadronic contributions

$$a_\mu^{\rm th} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm had}. \tag{41}$$

The QED contribution is given by a perturbative expansion, due to the smallness of its coupling constant α

$$a_\mu^{\rm QED} = \sum_n A^{(2n)} \left(\frac{\alpha}{\pi}\right)^n, \tag{42}$$

where $a_\mu^{(2n)}$ is the QED contribution at n-loop level.

In the previous section we obtained the one-loop order anomalous magnetic moment in QED. Making use of the value of the fine-structure constant $\alpha^{-1} = (e^2/(4\pi))^{-1} = 137.035\,999\,084(51)$, obtained from the precise experimental value of a_e [7], we can evaluate (40)

$$\boxed{a_\mu^{(2),\rm QED}} = \frac{\alpha}{2\pi} = \boxed{116\,140\,973.301(81)10^{-11}}. \tag{43}$$

This contribution is shared by the three charged leptons, and it is the predominant one as can be observed in the experimental values of a_e [7] and a_μ [1]

$$\begin{aligned} a_e^{\rm exp} &= 115\,965\,218.073(28)10^{-11}, \\ a_\mu^{\rm exp} &= 116\,592\,061(41)10^{-11}. \end{aligned} \tag{44}$$

At two loops in QED, many other diagrams that contribute to a_μ must be calculated (see figure 3): 7 diagrams involving only muonic lines (the so-called universal contribution) and 2 mass-dependent diagrams (where electron and tau loops appear). Their contribution is

$$\begin{aligned} &\boxed{a_\mu^{(4),\rm QED}} = a_{\mu,\rm uni}^{(4)} + a_{\mu,\rm mass}^{(4)} = \Bigg(\frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 \\ &+ \frac{3}{4}\zeta(3)\Bigg)\left(\frac{\alpha}{\pi}\right)^2 + \left(c_e\left(\frac{m_\mu}{m_e}\right) + c_\tau\left(\frac{m_\mu}{m_\tau}\right)\right)\left(\frac{\alpha}{\pi}\right)^2 = \\ &0.765857410(27)\left(\frac{\alpha}{\pi}\right)^2 = \boxed{413\,217.620(14)10^{-11}}, \end{aligned} \tag{45}$$

where the mass-dependent contributions c_e y c_τ can be found in [8]. Comparing (43) and (45), we can observe that the two-loop contribution is three orders of magnitude smaller than the one-loop one, as we could expect from (42) due to the small value of α.

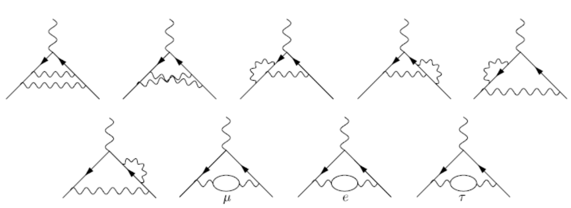

FIG. 3. Two-loop contributions in QED to a_μ. The first 7 diagrams correspond to the universal contribution, while the last two diagrams are mass dependent.

Higher-order results in QED are much more involved: at three loops there are 72 diagrams, the result for the universal diagrams was given first by Remiddi and Laporta in 1996 [9] (for the rest, see references in [8]); at four loops the number of diagrams is 891, which were first computed by Kinoshita, Laporta and their collaborators during several decades, making use of numerical methods (and even nowadays improvements and recalculations are being made [10]). Furthermore, the five-loop contribution has been recently computed also by Kinoshita and his collaborators [11].

Electroweak (EW) contributions are obtained similarly to QED ones. At one loop, there are three diagrams (see figure 4), involving the gauge bosons $W^\pm$ and Z, and also the Higgs boson. Their contribution (detailed in [12]) is

$$\boxed{a_\mu^{(2),\rm EW} = 194.82(2)10^{-11}}, \tag{46}$$

which is very small in comparison with the QED contribution at one loop (43).

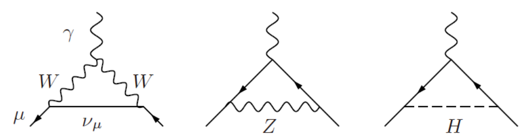

FIG. 4. Weak contributions to a_μ at one loop.

At two loops in the EW sector, there are many more diagrams containing quark loops, electromagnetic corrections or hadronic effects. The first full two-loop computation without approximations was obtained by Czarnecki, Krause and Marciano [13]. Furthermore, it has been estimated that the three-loop correction is negligible [14].

Finally, hadronic contributions must be taken into account. They are the most problematic calculations since QCD is non perturbative at low energy. The leading correction (see figure 5(a)) is the hadronic vacuum polarization (HVP), which can be determined through causality and unitarity (optical theorem), in terms of the experimental $\sigma(e^+e^- \to \text{hadrons})$ cross-section (different values depending on the experiment), which is known as a data-driven determination.

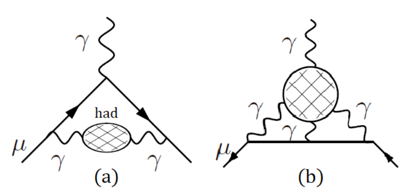

FIG. 5. (a) Hadronic vacuum polarization contribution. (b) Hadronic light-by-light scattering contribution.

Nevertheless, the most complex hadronic contributions arise from the so-called "light-by-light scattering" (HLbL, figure 5(b)), because for long they were only calculated using phenomenological models. Nevertheless, in the last years, an approach similar to the one employed for the hadronic vacuum polarization has been developed, leading to a data-driven determination of this contribution with few model dependencies (see [8]).

All the contributions I have mentioned in this section (see [2, 8, 12, 15] for further details) are necessary for the complete theoretical calculation, and appear in the White Paper of the "Muon $g-2$ Theory Initiative" [8], as can be seen in figure 6. It is interesting to compare the orders of magnitude and uncertainties of the different contributions: the main correction comes from QED while the main contribution to the error comes from hadronic vacuum polarization at lowest order (HPV(LO)) (it is limited by the experimental precision of $\sigma(e^+e^- \to \text{hadrons})$).

Contribution	Value $\times 10^{11}$
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)

FIG. 6. SM contributions to a_μ taken from the White Paper of the "Muon $g-2$ Theory Initiative" [8].

III. EXPERIMENTAL DETERMINATION OF a_μ

After having examined the theoretical calculation, in this section we want to understand how the experimental value for a_μ is obtained, i.e., our aim is to comprehend the "Muon $g-2$" experiment. I have mostly made use of references [2, 12, 16].

A. Sketch of the experiment Muon $g-2$

The $g-2$ experiment consists in a muon storage ring where highly polarized muons travel in an approximately circular trajectory in the presence of a uniform magnetic field. A precession of its magnetic moment is observed, and from it a_μ can be determined.

To begin with, I will focus on how the muons are produced. First, an accelerated beam of protons hits a target and charged pions are produced. Then, they decay into a muon and a muonic neutrino (with branching ratio $\sim$1). As in the experiment, we will consider the case $\pi^+ \to \mu^+ + \nu_\mu$. Since pions are spinless and due to the left-handed chirality of neutrinos[6], we obtain a beam of muons longitudinally polarized.

Then, muons are injected in a 14-meter-diameter storage ring, which contains a uniform magnetic field that forces them to travel in a approximate cyclotron trajectory of angular frequency ω_c (the motion is more complex as we will see). In addition, the spin (or equivalently, the magnetic moment) of the muon experiments a Larmor precession [17], due to the magnetic field, of angular frequency ω_s. Both dynamics lead to an effective precession of the spin around the flight axis, with angular frequency $\omega_a = \omega_s - \omega_c$ (see figure 7), given by

$$\vec{\omega}_a = -\frac{e}{m_\mu}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{v}\times\vec{E}}{c^2} - a_\mu \frac{\gamma}{\gamma+1}\frac{(\vec{B}\cdot\vec{v})\vec{v}}{c^2}\right], \tag{47}$$

where the electric field, normal to the circular orbit, is necessary to maintain the beam focused (since electromagnetic repulsions tend to scatter muons in the beam).

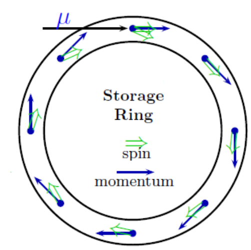

FIG. 7. Muons and their spin dynamics in the storage ring.

As the last term of (47) is approximately zero (although it introduces a correction), if we fix $\gamma^2 = (1 + a_\mu)/a_\mu$, known as "magic γ", then (47) gets simplified to

$$\vec{\omega}_a = -a_\mu \frac{e}{m_\mu}\vec{B}. \tag{48}$$

The magic γ has a value $\gamma_{\text{mag}} \approx 29.3$, which leads to an energy of $E_{\text{mag}} \approx 3.1$ GeV. Thus, if the muon beam is accelerated to this energy, then (48) holds, and provides a_μ when ω_a and B are measured. Moreover, the lifetime of muons is greatly extended because of the Lorentz dilation (to 64.4 μs, while its rest lifetime is $\tau_\mu = 2.2$ μs), which facilitates the measurements (in contrast to the a_τ case).

But, how do we measure ω_a and B with high precision? As muons are unstable, they decay with probability almost 1 to an electron and the two associated neutrinos

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu. \tag{49}$$

In order to observe ω_a, it is necessary a highly anisotropic distribution of the decay with respect to the muon spin direction in the rest frame. In this case, parity is maximally violated in (49), and as a consequence there exists a big correlation between the direction of the emitted positron and the muon spin direction in the rest frame (see figure 8).

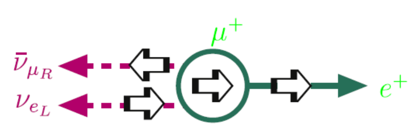

FIG. 8. Muon decay in the zero mass limit taken from [2]. White arrows represent the spin direction, while coloured arrows are the motion directions of each particle.

[6] Indeed, the property required is the helicity, which is not exactly conserved, since neutrinos have a small but non-vanishing mass, and therefore chirality and helicity are not identical.

Additionally, as the beam is boosted at almost the speed of light, the correlation in the laboratory frame is even stronger (see figure 9). In any case, the angular distribution of the emission is strongly peaked in the direction of the muon spin, and this gives a clear reference of the spin precession. In the experiment, positrons from the muon decays are detected by electromagnetic calorimeters inside the ring, and a global signal is reconstructed. In figure 9 we can see this measurement, where we observe an oscillatory signal of angular frequency ω_a together with a negative exponential $e^{-t/(\gamma\tau_\mu)}$ from the decay of the muons.

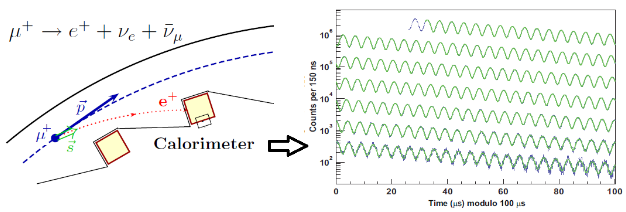

FIG. 9. Schematic drawing of the muon decay and representation of the measured signal.

Nevertheless, for the precise determination of the magnetic field, a complex procedure is needed. Making use of Magnetic Nuclear Resonance and a probe of water, it is found

$$B = \frac{\hbar\omega_p}{2\mu_p}, \tag{50}$$

where ω_p is the Larmor angular frequency of a proton in water and μ_p the modulus of the magnetic moment of the proton. But (48) is not directly used since it would require a highly accurate measure of e/m_μ. Observe that

$$\begin{cases} \dfrac{\omega_a}{\omega_p} = a_\mu \dfrac{e}{m_\mu}\dfrac{B}{\omega_p} = a_\mu \dfrac{e\hbar}{2\mu_p m_\mu}, \\ \mu_\mu = (1+a_\mu)\dfrac{e\hbar}{2m_\mu} \Rightarrow \dfrac{1}{m_\mu} = \dfrac{2\mu_\mu}{(1+a_\mu)e\hbar}, \end{cases} \tag{51}$$

where I have used $S = \hbar/2$. From these two relations, and defining $\mathcal{R} = \omega_a/\omega_p$ and $\lambda = \mu_\mu/\mu_p$:

$$\begin{aligned} \mathcal{R} = a_\mu \frac{e\hbar}{2\mu_p m_\mu} = a_\mu \frac{2e\hbar\mu_\mu}{2\mu_p(1+a_\mu)e\hbar} = \frac{a_\mu}{1+a_\mu}\lambda, \\ \Rightarrow \boxed{a_\mu = \frac{\mathcal{R}}{\lambda - \mathcal{R}}}, \end{aligned} \tag{52}$$

which is the formula actually used, since ω_a, ω_p and λ (obtained from the hyperfine structure of the muonium atom μ^+e^-) are accurately measured, leading to a high-precision experimental determination of a_μ.

B. Dynamical equations of the muon

Due to the fact that forces associated with the anomalous magnetic moment are very weak in comparison with the forces governing the "orbital" motion, we can study separately both dynamics [12]. In this section we will focus on the movement of the muon in the storage ring. I have tried to formalize the dynamical analysis presented in [12] or [16], which are somewhat confusing or incomplete.

First, we will consider the motion in the presence of an electromagnetic field. The effect on the muon μ^+ is given by the Lorentz force

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}), \tag{53}$$

where e is the charge of the muon, $\vec{v}$ its velocity, and $\vec{E}$ and $\vec{B}$ the electric and magnetic fields, respectively. We will consider relativistic dynamics, since $v = |\vec{v}| \approx c$, and we will work in cylindrical coordinates (ρ, θ, z), taking the axis of the ring as the Z direction. In the absence of an electric field, and for a uniform magnetic field $\vec{B} = -B_0\vec{e}_z$, the equations of motion (53) take the form

$$\begin{aligned} \frac{d(m(v)\dot{\rho})}{dt} &= m(v)\rho\dot{\theta}^2 - e\rho\dot{\theta}B_0, \\ \frac{d(m(v)\rho^2\dot{\theta})}{dt} &= 0, \\ \frac{d(m(v)\dot{z})}{dt} &= 0, \end{aligned} \tag{54}$$

where $m(v) = m_\mu/\sqrt{1 - v^2/c^2}$ and the second equation is the conservation of the z component of the angular moment (it holds if $\dot{z}(t=0) = 0$) instead of the θ-component of the Lorentz force (less simple but equivalent in this case). Observe that the energy is conserved

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v} = 0 \Rightarrow E = \frac{m_\mu c^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \text{const}, \tag{55}$$

and therefore, v is also constant, i.e., the muon describes a circular motion. Then, $d\vec{p} = \vec{p}d\theta$ and

$$evB_0 = |\vec{F}| = \left|\frac{d\vec{p}}{dt}\right| = p\left|\frac{d\theta}{dt}\right| = p\omega_c \Rightarrow \omega_c = \frac{eB_0}{m(v)}, \tag{56}$$

where $p = m(v)v$ has been used. ω_c is known as the relativistic cyclotron frequency, the one appearing in the definition of ω_a. The radius of the orbit is

$$r_0 = \frac{v}{\omega_c} = \frac{m(v)v}{eB_0}. \tag{57}$$

Actually, the Lorentz force has more terms, since due to the electromagnetic repulsion of the muons of the beam, a focusing system is required in the Z axis. For this aim, an electric field is applied, which provides a restoring force in the vertical direction, and a repulsive one in the radial direction in the following way

$$\vec{E} = (E_\rho, E_\theta, Ez) = (k(\rho - r_0), 0, -kz), \tag{58}$$

where k is a positive constant. Note that $|\vec{E}|$ must be small enough such that the attractive radial force produced by the magnetic field would be stronger than the radial repulsive electric force. This allows us to treat the electric forces as perturbations following (54). Therefore, the solution is $\rho(t) = r_0 + \delta\rho(t)$, $z(t) = \delta z(t)$, with $\delta\rho(t) \ll r_0$. We will neglect quadratic "δ" terms. Since $\vec{v}$ is approximately in the θ direction and has an approximately constant value close to c, $\delta\theta$ will be negligible, and we can consider that the perturbations only affect to ρ and z (as if corrections to v and θ where of a higher order). With all this considerations, our equations are

$$\begin{aligned} \frac{d(m(v)\dot{\rho})}{dt} &= m(v)\rho\dot{\theta}^2 - e\rho\dot{\theta}B_0 + ek\delta\rho, \\ \frac{d(m(v)\dot{z})}{dt} &= -ekz. \end{aligned} \tag{59}$$

As v can be considered constant under our approximations, $m(v)$ and r_0 will be also constant, and we can use the circular motion formula $v = \rho\dot{\theta}$. If we apply these approximations to (59), it is obtained

$$\begin{aligned} m(v)\ddot{\delta\rho} &= m(v)\frac{v^2}{\rho} - evB_0 + ek\delta\rho, \\ m(v)\ddot{z} &= -ekz. \end{aligned} \tag{60}$$

For the first equation, if we consider $\rho = r_0$, then the first two terms in the right side cancel within our approximations (it is just the right side term of the first equation in (54)). Therefore, we have that $evB_0 = m(v)v^2/r_0$. Using

$$m(v)\frac{v^2}{\rho} = m(v)\frac{v^2}{r_0 + \delta\rho} \approx m(v)\frac{v^2}{r_0}\left(1 - \frac{\delta\rho}{r_0}\right), \tag{61}$$

we get

$$m(v)\ddot{\delta\rho} = -m(v)\frac{v^2}{r_0^2}\delta\rho + ek\delta\rho. \tag{62}$$

Defining $n = kr_0/vB_0$ ($n < 1$ since the magnetic force is stronger than the electric one), we obtain the equations[7]

$$\begin{aligned} \ddot{\delta\rho} &= -\frac{v^2}{r_0^2}(1-n)\delta\rho = -\omega_c^2(1-n)\delta\rho, \\ \ddot{z} &= -\frac{v^2}{r_0^2}nz = -\omega_c^2 nz, \end{aligned} \tag{63}$$

which have the oscillatory solutions

$$\rho(t) = r_0 + A\cos(\sqrt{1-n}\,\omega_c t),\ z(t) = B\cos(\sqrt{n}\,\omega_c t), \tag{64}$$

with $|A|, |B| \ll r_0$. Therefore, we have obtained that muons describe a circular motion, but when a focusing electric field is introduced, the particles also perform simple harmonic motions in radial and vertical directions known as "betatron oscillation". This correction to the circular motion will suppose a contribution to the precession frequency given in (47). In the Fermi National Accelerator Laboratory (Fermilab) experiment, the geometry of the ring is more complex, but the dynamics follow the same behaviour.

[7] It has been used that $ekr_0^2/m(v)v^2 = kr_0/vB_0 = n$.

C. Dynamics of the muon spin polarization

Now, I am interested in the dynamics concerning the muon spin polarization $\vec{P}$ in order to obtain the formula (47). The analysis will be based on references [12] and [18]. In the presence of a magnetic field $\vec{B}$, the muon polarization changes according to

$$\frac{d\vec{P}}{dt} = g_\mu \frac{e}{2m_\mu}\vec{P}\times\vec{B}, \tag{65}$$

and as a consequence, the component of $\vec{P}$ perpendicular to $\vec{B}$ rotates with angular frequency $g_\mu \frac{e}{2m_\mu}B$, while the parallel one remains unchanged. Nevertheless, this result is only valid in the rest frame of the muon.

Observe that there are infinite rest frames for a particle, all of them related by a rotation. In the laboratory frame, the polarization $\vec{P}$ is defined as the polarization measured in the rest frame obtained from the laboratory system by a Lorentz boost. Since we are dealing with accelerated particles, we need to work in "momentary"/"instantaneous" rest frames, and therefore relate the dynamics between them and with the laboratory frame O.

For that aim, we need to consider a Lorentz boost in an arbitrary direction (given by $\vec{v}$)

$$t' = \gamma(v)\left(t - \frac{\vec{v}\cdot\vec{r}}{c^2}\right),\ \vec{r'} = \vec{r} + \alpha(v)(\vec{v}\cdot\vec{r})\vec{v} - \gamma(v)\vec{v}t, \tag{66}$$

with $\gamma(v) = (1-v^2/c^2)^{-1/2}$ and $\alpha(v) = (\gamma(v)-1)/v^2$. At a given time, the muon has a velocity $\vec{v}$, and right after it changes to $\vec{v}+\delta\vec{v}$. Let O' and O'' be the corresponding instantaneous rest frames. Making use of (66), it can be shown [18] that their coordinates are related by

$$t'' = t' - \delta\vec{u}'\cdot r', \quad \vec{r''} = \vec{r'} + \delta\vec{\theta'}\times r' - \delta\vec{u'}t', \tag{67}$$

at linear order in $\delta\vec{v}$, where

$$\delta\vec{u}' = \gamma\left(1 + \frac{\gamma-1}{v^2}\vec{v}\vec{v}\cdot\right)\delta\vec{v}, \quad \delta\vec{\theta'} = \frac{\gamma-1}{v^2}(\delta\vec{v}\times\vec{v}). \tag{68}$$

If we compare (66) at first order in $\vec{v}$ and (67), we reach the conclusion that O' and O'' are related by an infinitesimal Lorentz boost of velocity $\delta\vec{u}'$ and an infinitesimal rotation given by $\delta\vec{\theta'}$. So in general, the transformation between two consecutive rest systems is not just a Lorentz boost (except if $\delta\vec{v} \parallel \vec{v}$).

Our objective is to obtain the dynamical equation for $\vec{P}$ in the laboratory frame. As muons do not have electric dipole moment, when O' is the instantaneous rest frame the electric field only (linearly) accelerates the particle, and therefore (65) holds, so we have that, until O'' would be the momentary rest frame, the changes in $\vec{P'}$ and in the velocity of the particle $\vec{u}'$ in O' verify

$$\delta\vec{P'} = g_\mu\frac{e}{2m_\mu}\vec{P}\times\vec{B}'\delta t', \quad \delta\vec{u}' = \frac{e}{m_\mu}\vec{E}'\delta t'. \tag{69}$$

The crucial point here is that, because of our definition of $\vec{P}$ in the laboratory frame, $\vec{P} = \vec{P}'$ when O' is the rest frame, but a instant later when the rest frame is O'', $\vec{P} + \delta\vec{P} \neq \vec{P}' + \delta\vec{P}'$, because in that moment $\vec{P}$ is given by O'', and we must transform $\delta\vec{P}'$ accordingly to (67) (rotate $\vec{P}$ with $\delta\vec{\theta}'$ since by (69) the boost is already taken into account). Hence

$$\delta\vec{P} = \delta\vec{P}' + \delta\vec{\theta}' \times \vec{P} = \frac{eg_\mu}{2m_\mu}\vec{P} \times \vec{B}'\delta t' + \frac{\gamma - 1}{v^2}(\delta\vec{v} \times \vec{v}) \times \vec{P}. \quad (70)$$

Finally, in order to obtain a dynamical equation in O, we need to transform $\delta t'$ and $\vec{B}'$ into quantities of the laboratory frame. Making use of (66) and that $\vec{B}' = \gamma(\vec{B} - \vec{v} \times \vec{E}/c^2) + (1-\gamma)(\vec{v} \cdot \vec{B})\vec{v}/v^2$, one obtains

$$\frac{d\vec{P}}{dt} = \vec{\omega}_s \times \vec{P}, \quad (71)$$

$$\vec{\omega}_s = \frac{\gamma-1}{v^2}\frac{d\vec{v}}{dt} \times \vec{v} - \frac{eg_\mu}{2m_\mu}\left(\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2} + \frac{1-\gamma}{\gamma}\frac{\vec{v}\cdot\vec{B}}{v^2}\vec{v}\right), \quad (72)$$

where the first term in (72) is the so-called "Thomas precession" (see [17] for more details), which we would like to manipulate to obtain a more useful expression. Deriving the left term in the Lorentz force formula $d(m_\mu\gamma\vec{v})/dt = e(\vec{E} + \vec{v} \times \vec{B})$, it is straightforward to obtain[8]

$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{e}{m_\mu\gamma}(\vec{E} + \vec{v} \times \vec{B}) - \frac{\gamma^2}{c^2}\left(\vec{v}\frac{d\vec{v}}{dt}\right)\vec{v} = \\ &\frac{e}{m_\mu\gamma}(\vec{E} + \vec{v} \times \vec{B}) - \frac{e}{c^2\gamma m_\mu}\left(\vec{E}\cdot\vec{v}\right)\vec{v},\end{aligned} \quad (73)$$

which, if introduced in (72), leads to

$$\begin{aligned}\vec{\omega}_s = -\frac{e}{m_\mu\gamma}\Bigg((1+\gamma a_\mu)\vec{B} + a_\mu\frac{1-\gamma}{v^2}(\vec{B}\cdot\vec{v})\vec{v} \\ -\gamma\left(a_\mu + \frac{1}{1+\gamma}\right)\frac{\vec{v}\times\vec{E}}{c^2}\Bigg).\end{aligned} \quad (74)$$

Then, since $\vec{v} \cdot \vec{E} = 0$ holds first order, equation (73) can be rewritten as

$$\frac{d\vec{v}}{dt} = \vec{\omega}_c \times \vec{v}, \quad \vec{\omega}_c = -\frac{e}{m_\mu\gamma}\left(\vec{B} + \frac{\vec{E}\times\vec{v}}{v^2}\right), \quad (75)$$

which together with (74), and remembering that $\vec{\omega}_a = \vec{\omega}_s - \vec{\omega}_c$, leads, as promised, to equation (47)

$$\vec{\omega}_a = -\frac{e}{m_\mu}\left[a_\mu\vec{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{v}\times\vec{E}}{c^2} - a_\mu\frac{\gamma}{\gamma+1}\frac{(\vec{B}\cdot\vec{v})\vec{v}}{c^2}\right]. \quad (76)$$

As I commented in Section III A, with the "magic γ" and the approximation $\vec{v} \perp \vec{B}$, equation (76) gets simplified to $\vec{\omega}_a = -ea_\mu\vec{B}/m_\mu$, which could be used to obtain a_μ. Nevertheless, as we obtained in the previous section, that condition is not verified since $\vec{v}$ has a non-vanishing component parallel to $\vec{B}$. This introduces the so-called *Vertical Pitch Correction*, denoted by C_P. Similarly, the radial electric field employed in the focusing of the beam leads to a vertical component of $\vec{v} \times \vec{E}$, and as a consequence the second term in the right side of (76) introduces another correction, C_E, known as *Radial Electric Field Correction.* This corrections are, for the Fermilab experiment

$$C_P \approx 0.27 \text{ ppm}, \quad C_E \approx 0.47 \text{ ppm}. \quad (77)$$

Although they are very small, they become really relevant due to the high precision of the experiment. More details can be found in references [12] and [16].

D. Experimental results. Comparison with theory

After the analysis of the experiment and the dynamics of the muons in it, we are now in position to quote the experimental result and compare it to the theoretical one. The following values have been extracted from [1]. On April the 7^{th}, 2021, the result of the Fermilab experiment was published

$$a_\mu = 116\,592\,040(54)10^{-11}. \quad (78)$$

If (78) is combined with the results of the previous experiment, carried out in the Brookhaven National Laboratory (BNL), it gives the following average

$$\boxed{a_\mu^{\text{exp}} = 116\,592\,061(41)10^{-11}}. \quad (79)$$

Although there used to be different theoretical results from model calculations of the hadronic contributions, nowadays the "consensus" theoretical value provided by the "Muon $g-2$ Theory Initiative" is [8]

$$\boxed{a_\mu^{\text{th}} = 116\,591\,810(43)10^{-11}}. \quad (80)$$

If we compare both values, we obtain

$$\boxed{\Delta a_\mu(\text{exp-th}) = (251 \pm 59)10^{-11}}, \quad (81)$$

a 4.2σ discrepancy, which may hint to possible new physics effects.

IV. INFLUENCE OF RADIATION

During this master thesis, I have wondered whether contributions from the synchrotron radiation had been taken into account in the $g-2$ experiments. As I have not found any account of such a calculation in the literature, I have performed a simple estimate that I will detail next.

Considering the simple case of an ultrarrelativistic particle (good approximation in the experiment since $v \approx c$ and $E_{\text{mag}} \approx 3.1$ GeV $\gg m_\mu c^2 \approx 105.6$ MeV) with charge

[8] $\vec{v}\frac{d\vec{v}}{dt} = e\vec{E}\cdot\vec{v}/(m_\mu\gamma^3)$ is obtained from the first equality in (73), after multiplying by $\vec{v}$.

e describing a circular motion under the conditions of the experiment, its instantaneous power radiated is [17]

$$P = \frac{dE}{dt} = \frac{e^2 v^4 \gamma^4(v)}{6\pi\epsilon_0 c^3 R^2} \approx \frac{e^2 c \gamma^4(v)}{6\pi\epsilon_0 R^2}. \tag{82}$$

Then, we can estimate the total energy loss in a cycle, with a period $\sim (\omega_c/2\pi)^{-1}$, assuming a constant radius and speed, which is a good approximation in the experimental setup. In our case, $\gamma \approx 29.3$, $R = 7.112$ m and $\omega_c/2\pi \approx 6.71$ MHz [19], and therefore

$$\begin{aligned} \Delta E = \frac{dE}{dt}\frac{2\pi}{\omega_c} \approx 1.001 \cdot 10^{-22} J = 6.25 \cdot 10^{-4} \text{ eV}, \\ \Rightarrow \frac{\Delta E}{E} \approx 2.02 \cdot 10^{-13}. \end{aligned} \tag{83}$$

The cyclotron frequency ω_c, defined in (56), can be related with the energy E in the following way

$$\omega_c = \frac{eBc^2}{E}, \tag{84}$$

and hence, the radiated power induces a change in ω_c

$$\frac{\Delta\omega_c}{\omega_c} = \frac{\Delta E}{E} \approx 2.02 \cdot 10^{-13}. \tag{85}$$

Finally, as a_μ is determined as

$$a_\mu = \frac{\mathcal{R}}{\lambda - \mathcal{R}} = \frac{\frac{\omega_s - \omega_c}{\omega_p}}{\lambda - \frac{\omega_s - \omega_c}{\omega_p}}, \tag{86}$$

the change in ω_c introduces a correction

$$\boxed{\Delta a_\mu} = \left|\frac{\partial a_\mu}{\partial \omega_c}\right| \Delta\omega_c = \frac{a_\mu(1+a_\mu)}{\omega_s - \omega_c}\Delta\omega_c \approx \boxed{6.9 \cdot 10^{-15}}, \tag{87}$$

where I have used $(\omega_s - \omega_c)/2\pi \approx 229.08$ kHz [19]. Comparing with (80), it is clear that this estimate is beyond the experimental precision since it is 5 orders of magnitude lower that the experimental uncertainty.

CONCLUSIONS

In this work, I have been able to understand the theoretical and experimental calculations of the anomalous magnetic moment of the muon. On the one hand, I have performed the calculations of a_μ at tree and one-loop level in QED, as well as understood the general picture of the remaining SM contributions. On the other hand, I have presented a clarifying exposition of the dynamical analysis of particle and spin motions, whose presentation is very confusing in the literature, that leads to the "master formula" (48) and the corrections (76) and (77).

In addition, I have estimated the effect of the synchrotron radiation in the determination of a_μ for the Fermilab experiment, obtaining that its effect is negligible for the current experimental precision. A more rigorous analysis would be required to determine the importance of this contribution for future high-precision experiments.

In conclusion, I have performed a detailed study of the tension between theoretical and experimental determinations of a_μ which, at the current level of accuracy, implies a discrepancy of 4.2 standard deviations and hence, is one of the most promising observables to identify signs of physics beyond the Standard Model.

[1] B. Abi *et al.* [Muon g-2], Phys. Rev. Lett. **126**, no.14, 141801 (2021).
[2] F. Jegerlehner and A. Nyffeler, Phys. Rept. **477**, 1-110 (2009).
[3] F.J.M. Farley and Y.K. Semertzidis, Prog. Part. Nucl. Phys. **52**, 1-83 (2004).
[4] S. Eidelman, M. Giacomini, F.V. Ignatov and M. Passera, Nucl. Phys. B Proc.Suppl. **169**, 226-231 (2007)
[5] S. Li, Y. Xiao and J.M. Yang, Physics **4**, 40-47 (2021).
[6] S. Mandl and G. Shaw, "Quantum Field Theory", John Wiley & Sons, Inc., Great Britain (1986).
[7] D. Hanneke, S. Fogwell and G. Gabrielse, Phys. Rev. Lett. **100**, 120801 (2008).
[8] T. Aoyama, N. Asmussen, M. Benayoun, J. Bijnens, T. Blum, M. Bruno, I. Caprini, C.M. Carloni Calame, M. Cè and G. Colangelo, *et al.* Phys. Rept. **887**, 1-166 (2020).
[9] S. Laporta and E. Remiddi, Phys. Lett. B **379**, 283-291 (1996).
[10] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. Lett. **99**, 110406 (2007).
[11] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. Lett. **109**, 111808 (2012).
[12] F. Jegerlehner, Springer Tracts Mod. Phys. **274**, pp.1-693 (2017).
[13] A. Czarnecki, B. Krause and W.J. Marciano, Phys. Rev. D **52**, R2619-R2623 (1995).
[14] A. Czarnecki, W.J. Marciano and A. Vainshtein, Phys. Rev. D **67**, 073006 (2003).
[15] F. Jegerlehner, Acta Phys. Polon. B **38**, 3021 (2007).
[16] J. Grange *et al.* [Muon g-2], "Muon (g-2) Technical Design Report". [arXiv:1501.06858 [physics.ins-det]].
[17] Jackson, J.D.: *Classical Electrodynamics.* John Wiley & Sons, Inc., United States of America, 1962.
[18] J. S. Bell, "Polarized Particles for Accelerator Physicists," doi:10.5170/CERN-1975-011
[19] T. Albahri *et al.* [Muon g-2], Phys. Rev. D **103**, no.7, 072002 (2021).