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Dispersión de leptones por núcleos

Lepton-nucleus scattering

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Abstract:

Precise measurements in modern neutrino oscillation experiments require an accurate description of neutrino–nucleus interactions. One of the main sources of uncertainty arises from multinucleon processes, in which the incoming lepton interacts with correlated nucleons inside the nucleus. These effects can significantly impact neutrino-energy reconstruction and therefore the extraction of oscillation parameters. Electron scattering provides an essential benchmark for testing nuclear models, since the electromagnetic interaction is well understood and high-precision experimental data are available. In particular, exclusive $(e, e'p)$ reactions offer direct access to nuclear spectral functions, missing-energy distributions, and nucleon–nucleon correlations. In this study, we will introduce the theoretical framework for lepton–nucleus scattering, including the impulse approximation, spectral functions, and short-range correlations (SRC). A Monte Carlo event generator developed to study one- and two-nucleon emission channels will be presented. This simulation explores the available phase space and studies different kinematics and relevant nuclei in neutrino experiments, with the aim of comparing the simulated results with the results from $(e, e'p)$ experiments. Once the simulations are validated with data from electron-nucleus scattering, the idea will be to apply this program to neutrino experiments, comparing it to recent data and searching for regions where the one-nucleon emission channel is predominant against other reaction channels. The ultimate goal is to contribute to the improvement of nuclear models for future neutrino oscillation experiments.

Resumen:

Los futuros experimentos de oscilaciones de neutrinos buscan una gran precisión en sus medidas; para ello necesitan una descripción adecuada de las interacciones del neutrino. Una de las principales fuentes de incertidumbre en las medidas viene de los procesos de más de un nucleón, en los que el lepton incidente interactúa con nucleones correlacionados dentro del núcleo. Estos efectos pueden impactar de forma significativa la reconstrucción de la energía del neutrino y, por lo tanto, la extracción de los parámetros de oscilación. Los experimentos de dispersión de electrones proporciona un marco para testear y validar los modelos nucleares, ya que la interacción electromagnética es bien conocida y existen experimentos capaces de proporcionar datos con gran precisión. En particular, las reacciones exclusivas $(e, e'p)$ ofrecen un acceso directo a las funciones espectrales nucleares, las distribuciones de energía perdida o “missing energy” y las correlaciones nucleón-nucleón. En este estudio se introducirá el marco teórico para dispersiones leptón-núcleo, incluyendo la aproximación de impulso, la función espectral y las correlaciones de corto alcance (SRC). Además, se ha desarrollado un generador de eventos Monte Carlo para estudiar los canales de emisión de uno y dos nucleones. La simulación explora el espacio de fases disponible, al igual que diferentes cinemáticas y núcleos relevantes para experimentos de neutrinos, con el objetivo de comparar las simulaciones con los datos experimentales disponibles de $(e, e'p)$. Una vez validadas las simulaciones con datos de dispersión de electrones, se quiere utilizar el programa para aplicarlo a experimentos de neutrinos, comparando con datos recientes y buscando las zonas donde el canal de emisión de un nucleón es predominante frente a la emisión de más de un nucleón. El objetivo de este estudio es contribuir a mejorar los modelos nucleares para así mejorar los resultados de los futuros experimentos de oscilaciones de neutrinos.

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1 Introduction

1.1 Motivation

The neutrino was postulated in 1931 by Wolfgang Pauli to solve the problem of the continuous energy distribution of the β decay. This particle carries the “missing” energy of the beta decay and must be electrically neutral for the conservation of the electrical charge. It is also a requirement for the neutrino to have spin $1/2$, in order to explain the change in angular momentum of the initial and final states in these β decays [1].

$${}^A_ZX \rightarrow {}^A_{Z+1}Y + e^- + \bar{\nu}_e \quad (1)$$

The name neutrino was later introduced by Enrico Fermi when he developed his theory of β decay and weak interaction. In this theory, Fermi explained that neutrinos interact solely through the weak interaction, making them very hard to detect. In fact, it wasn’t until 1956 when the existence of the neutrino was confirmed by Cowan and Reines through the inverse beta decay at a nuclear reactor [2].

$$\bar{\nu}_e + p \rightarrow n + e^+ \quad (2)$$

Initially, neutrinos were thought to be massless particles within the Standard Model framework. However, this assumption was refuted when the discovery of neutrino oscillations by Super-Kamiokande in 1998 and the Sudbury Neutrino Observatory in 2001 was published, granting them a Nobel Prize in 2015 [3], [4].

The Super-Kamiokande experiment focused on the atmospheric neutrino anomaly. Atmospheric neutrinos are produced in particle showers initiated when cosmic rays strike the Earth’s atmosphere, creating pions that decay into muons and muon neutrinos (ν_μ). The muons subsequently decay into electrons, electron neutrinos (ν_e), and additional muon neutrinos. In this experiment, they compared the number of muon neutrinos that came from above, traveling a short distance through the atmosphere, versus the number of them that came from below after going through the Earth; therefore, covering a lot more distance. They observed a significant, distance- and energy-dependent deficit in the ν_μ flux that had crossed the Earth, approximately half for certain trajectories. Crucially, Super-Kamiokande did not directly detect the appearance of new neutrino flavors; rather, they observed the disappearance of ν_μ . The data that they obtained was statistically consistent with the hypothesis of $\nu_\mu \rightarrow \nu_\tau$ oscillation.

This idea of oscillation of neutrinos was later confirmed by the SNO collaboration when they studied the longstanding Solar Neutrino Problem. In the Sun, nuclear fusion processes, particularly the decay of 8B , produce exclusively electron neutrinos. By using heavy water and studying different reaction mechanisms, they were able to measure the solar neutrino flux. The experiment was sensitive to both Charged Current (CC) processes, caused only by electron neutrinos, $\nu_e + d \rightarrow p + p + e^-$, and Neutral Current (NC) processes, that were equally caused by all three neutrino flavors and, therefore, represented the total flux. With this experiment, SNO observed that the flux of electron neutrinos was approximately one third of the total flux, while the total flux derived from the NC processes was perfectly matched the theoretical predictions of the Standard Solar Model. This proved unequivocally that the missing electron neutrinos had not simply vanished, but had oscillated into muon and tau neutrinos. Together, these complementary experiments demonstrated that the flavor of neutrinos is actively changing, a property that requires neutrinos to be massive particles.

However, their absolute masses, the neutrino mass hierarchy and the nature of the neutrino, whether

neutrinos are Dirac or Majorana particles, remain open questions in physics. This has motivated a worldwide effort to set-up experiments dedicated to precision neutrino physics. Among them, we can name some of the current experiments such as T2K [5], NO ν A [6], MINER ν A [7] and Micro-BooNE [8], as well as next-generation facilities such as Hyper-Kamiokande [9] and DUNE [10].

Despite the different specific setups and detector technologies, all these experiments share a common fundamental premise: they use intense wide-band neutrino beams directed at massive and complex nuclear targets to measure oscillation probabilities. Therefore, a good understanding of neutrino-nucleus interactions and how nuclear effects can influence the results obtained in the detectors is crucial for the success of these experiments.

To accurately measure neutrino oscillation parameters, experiments must reconstruct the energy of the incident neutrino (E_ν) on an event-by-event basis. Although monochromatic or quasi-monochromatic neutrino sources exist, such as those produced by pion decay-at-rest [11], they are generally constrained to fixed, low energies. While innovative solutions have been proposed to produce tunable, quasi-monochromatic beams at higher energies, they currently face significant beam-intensity and flux challenges. Consequently, contemporary and next-generation long-baseline experiments rely on broadband neutrino beams. Therefore, E_ν reconstruction relies heavily on the kinematics and calorimetry of the observed final-state particles.

Additionally, neutrinos interact exclusively via the weak force, so their cross sections are small, at least at the energies attainable in these current or planned facilities, which means that these experiments require huge amounts of material to obtain high statistics. The simplest targets, which would eliminate complex nuclear effects, such as electrons or free protons (hydrogen), are not feasible options due to their reduced cross sections and small density. Additionally, to study charged-current interactions for neutrinos, we need target materials with neutrons, which will be converted into protons in the final state, ejected from the nucleus and detected in the experiment, alongside the electron or muon in the leptonic vertex. Detecting neutrons after charged reactions on protons, on the other hand, is a much harder endeavour. Consequently, modern experiments are forced to construct detectors based on targets made of heavy, dense materials such as oils (hydrocarbons), water and argon.

These nuclei are not a passive collection of free nucleons. Depending on the energy transferred to the nucleus, many different reaction mechanisms can contribute to the observed signal. They can be more or less classified as follows: [12].

- Elastic / coherent. This is the case in which the nuclear target as a whole is what interacts with the neutrinos, and recoils also as a whole after the interaction. The elastic peak corresponds to the case where the target nucleus remains in the ground state, and it shows up as a sharp peak in the cross-section. The nucleus can also remain whole, but get enough energy to reach a discrete excited state, which would be seen as an additional collection of discrete peaks.
- Giant Resonances (GR). As the excitation energy given to the target nucleus increases, the interaction of the neutrino with the nucleus reaches collective modes of excitation, known as giant resonances, due to their relatively large cross-section and energy width.
- Quasi-Elastic (QE). In this case, the nuclear target is broken, so that a nucleon is knocked-out of the nucleus, and it may be detected in the final state (often a proton). The theoretical definition implies that, in this process, the neutrino interacts with a single nucleon, which is then ejected from the nucleus. However, the experiment can only determine that there is

one nucleon in the final state and thus different “estimations” of the QE signal are obtained depending on the experiment.

- Multi-nucleon contributions. Nuclear effects such as nuclear correlations or final state interactions can produce the emission of more than one nucleon, even if the neutrino interacts mainly with only one nucleon of the nuclear target.
- Single pion production (SPP). If the excitation energy given to the target nuclear system is high enough, and in the proper kinematical conditions, it is possible to excite baryon resonances, such as the Δ , which then decay producing pions in the final state.
- Deep inelastic regime (DIS). Finally, at even larger excitation energies, beyond the excitation of the lightest resonances, the neutrino interacts with the internal quark structure of the nucleon.

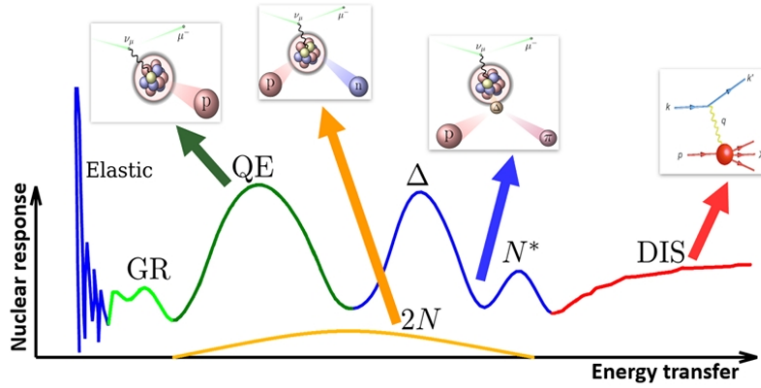


Figure 1: Sketch of nuclear response as a function of the energy transferred by a light lepton to the nucleus. From low to high energy transfer, the mechanisms are: elastic interaction, nuclear giant resonances (GR), quasielastic scattering (QE), multi-nucleon contributions (dominated by two-nucleon knock-out reactions, $2N$), pion production (Δ and N^*), and deep inelastic scattering (DIS). The importance and position of the different regions depends on the kinematical conditions beyond the energy transfer.

The goal of neutrino oscillation experiments is to measure the energy, momentum and flavor of the incident neutrinos. Therefore, the experimentalist should select the events that may produce the most accurate estimation of the energy of the initial neutrino. The flavor is identified by the outgoing lepton (muon or electron). To accurately reconstruct the neutrino’s energy, experimentalists would preferably use quasielastic (QE) events, which closely resemble scattering of a free nucleon, if they can be disentangled from the background of events produced by many other processes. By detecting the final-state lepton and proton, we can calculate the kinematics of the unobserved residual nucleus using energy-momentum conservation. In neutrino experiments, however, the incident energy is unknown. To derive it, we must first know the energy-momentum relationship of the residual nucleus. In light nucleus, such as ^{12}C or ^{16}O , we can try to put kinematic windows to maximize the probability that the knocked-out nucleon was in the neutron p-shell, for instance, and thus we would know with certainty the energy of the residual nucleus and, therefore, the energy of the incoming neutrino. The rest of the channels other than QE, contributing in the chosen dataset, represent a background that needs to be estimated to be removed from the measured signal, and minimized by means of the selection of kinematical cuts applied to the events.

1.2 Current Challenges

As neutrino oscillation experiments enter a high-precision era, the primary source of systematic uncertainties comes from the modeling of neutrino interactions with the target nuclei, or to say it in a compact way, the uncertainty stems from the lack of full knowledge of the correlations between energy and momentum of the residual nuclear system. In order to reduce uncertainties, theoretical models must be able to accurately describe all the reaction mechanisms that occur in neutrino-nucleus scattering. However, this task is deeply correlated to how the different neutrino experiments and detectors reconstruct the incident neutrino energy (E_ν).

Accelerator-based experiments traditionally rely on two main reconstruction philosophies, each tied to specific detector designs [13]:

- Kinematic (or QE-like) method: This method is used in experiments such as MiniBooNE and T2K's Super-Kamiokande that use primarily water and mineral-oil Cherenkov detectors. What they do is calculate E_ν using only the measured energy and emission angle of the outgoing charged lepton. They assume that the event is a true Charged-Current Quasi-Elastic (CCQE) interaction on a free nucleon at rest (neutrino + nucleon at rest $\rightarrow$ lepton + nucleon), corrected by an average nuclear binding energy. Because Cherenkov detectors cannot easily resolve low-energy hadrons in the final states, this method is the only possible experimental assumption in these experiments. However, because multi-nucleon processes, when a 2p2h (two-particles two-holes) final states stemming from short-range correlations (SRC), which also present as one of the reaction mechanisms. This causes a bias in the reconstructed energy downward by 100–200 MeV [14].
- Calorimetric method: used by segmented scintillator detectors in experiments such as NO ν A and MINER ν A, where E_ν is reconstructed by adding the measured energy of the outgoing lepton and the total visible hadronic energy deposited in the detector (from protons, pions, and gammas) without relying on angular assumptions or interaction hypotheses. It is less dependent on initial-state kinematic assumptions, but calorimetry results are dependent on detector thresholds, undetectable neutral particles (such as escaping neutrons), and binding energy losses. Consequently, its accuracy depends heavily on Monte Carlo generators to estimate the unobserved missing energy fraction [13][15].

To correct these significant systematic biases, theoretical models must move beyond simple Fermi-gas approximations (which assume a fixed binding energy) and implement realistic nuclear Spectral Functions. The spectral function provides the joint probability of finding a nucleon with a specific momentum and separation energy inside the nucleus [16]. This accounts for both the discrete energy levels of mean-field shell orbitals and the high-momentum tails generated by SRCs. Implementing shell-structure spectral functions has been shown to shift the mean reconstructed energy upward, drastically reducing the uncertainties. However, there are still challenges involving multinucleon emission from Short-range correlations and FSI.

To mitigate many problems that plague current neutrino experiments, next-generation facilities, particularly DUNE [17] and the Short-Baseline Neutrino (SBN) program (SBND, MicroBooNE, ICARUS) [18], introduce a paradigm shift by utilizing Liquid Argon Time Projection Chambers (LArTPCs). LArTPCs act as 3D imaging detectors with millimeter-level spatial resolution, enabling explicit particle identification and track counting down to low kinetic energy thresholds. This tracking capability allows experimentalists to select specific event topologies—such as exclusive $1\mu 1p$ (one muon, one proton) events—to filter out multi-nucleon backgrounds and isolate cleaner single-nucleon knockout processes, thereby significantly improving energy reconstruction.

This technological leap introduces a new challenge: the target nucleus is Argon (^{40}Ar), which, unlike lighter targets like Carbon (^{12}C) or Oxygen (^{16}O), is a dense, heavy nucleus with strong Final State Interactions (FSI), where outgoing nucleons frequently scatter or re-absorb before escaping. In addition, its spectral function is harder to measure, relying on titanium experiments to obtain the spectral function for neutrons. Validating theoretical nuclear models becomes essential to fully understand this nuclei, but this presents significant challenges. Since these experiments use non-monochromatic beams with broad energy distributions, the incoming neutrino energy is not known event by event. To address this limitation, we will draw our attention towards electron-nucleus scattering experiments, as a clean probe to test and validate the nuclear models that will be assume to analyze neutrino scattering.

Historically, electron scattering has been a very powerful tool for gaining insight into the nuclear structure. Electrons interact through electromagnetic interaction, which is a well understood interaction described by Quantum Electrodynamics (QED). The primary advantage of using electrons to study the nuclear structure is the fact that we know the energy and direction of the incident beam, coupled with an exact theoretical description of the leptonic vertex. Furthermore, we have a good amount of information about the interaction in the hadronic vertex, particularly for electron scattering on free nucleons. Therefore, we are able to control the kinematics of the process and use energy-momentum conservation to map the energy-momentum distribution of the target nucleus.

However, due to the strong nucleon-nucleon interaction at short distances, nucleons can form temporary correlated pairs characterized by large relative momentum and small center-of-mass momentum. If one member of the pair is struck with sufficient energy, its correlated partner may also be emitted. Experimentally, when the energy transferred to the nuclear system is above the 2-nucleon emission threshold, detecting just one proton in the final state does not guarantee that there was only one nucleon emitted, as the partner nucleon might escape undetected or fall below the detector's visibility threshold.

Therefore, beyond the two-nucleon emission threshold, multi-nucleon processes contribute and the analysis of the experiment is more complicated, unless we are able of measuring /detecting /disentangling every emitted particle, which is hardly ever the case. In both electron as in neutrino scattering, these two (and more) nucleon emission processes arise from correlated nucleon pairs inside the nucleus, or from the interaction of the knocked out nucleon with another nucleon while traveling out of the residual nucleus.

There are many different facilities that have done or are currently carrying out $(e, e'p)$ experiments. In the past, the $(e, e'p)$ reaction was studied mainly to understand the nuclear structure and the quasielastic region, while current experiments are performed at higher energies to apply the results to neutrino oscillation experiments. Among them, we can highlight NIKHEF-K in Amsterdam, which is a medium energy Linear Accelerator later coupled to a storage ring [19]. It uses magnets to perform matching dispersion and obtained a very good resolution in energy. There are other Linear Accelerators used for electron scattering experiments such as Saclay in France [20] and Bates in MIT [21]. In addition, MAMI in Germany and Jefferson lab (JLab) in the states, are also electron accelerators, but MAMI consists of a cascade of three (later four) racetrack microtrons, which are small coupled Linacs [22], and JLab is a recirculating superconducting Linac [23]. One last example of electron scattering facilities is RIKEN in Japan [24], which is fundamentally different in purpose from the other five. Instead of studying stable nuclei with fixed targets, it is built to perform electron scattering directly off short-lived, online-produced unstable nuclei using a racetrack microtron.

Electron experiments, both inclusive electron scattering (e, e') as well as $(e, e'p)$ will yield insights into the different processes that happen in lepton-nucleus scattering. Therefore, by modeling single- and multi-nucleon emission channels in electron scattering off nuclei like ^{12}C , ^{16}O , and ^{40}Ar , we will be able to better understand what happens in neutrino-nucleus interactions, test our nuclear models for the case of multinucleon knockout and find kinematic settings where the one-nucleon emission channel is predominant, thereby reducing the uncertainties of neutrino oscillation experiments.

2 Objectives

The overall goal of this Master's Thesis is to study coincidence electron-nucleus $(e, e'p)$ scattering and analyze the phase space and cross sections of single- and multi-nucleon emission channels, to extract implications for neutrino experiments. For this, the work was divided into the following specific objectives:

- Theoretical cross-section analysis of $(e, e'p)$ scattering.
- Experimental set-up and detector acceptance modeling.
- Development of a Monte Carlo Event Generator for the one nucleon emission and two nucleon emission channels.
- Study of different target nuclei and kinematics, comparing the simulated results with experimental data.

3 Theoretical framework

3.1 Electron-Nucleus scattering

In current neutrino experiments, the incoming neutrino energy is not known event by event. Therefore, we use electron scattering to validate nuclear models and its effects on neutrino scattering. Electron scattering experiments can be divided into different groups depending on what is detected in the final state [25]:

- Inclusive scattering (e, e') . In this type of experimental setup, we do not measure any particular reaction channel. Only the incident and final electrons are measured, so all possible processes compatible with the kinematical restrictions, from the full range of electron-nucleus interactions, contribute to the final signal. In inclusive scattering, it is difficult to evaluate the contributions of the different reaction mechanisms, unless the kinematics allow us to isolate only one channel. For instance, in elastic electron scattering of a nucleus, we know the energy and momentum transferred to the target, and we can determine that the nucleus has not been given any excitation energy. For a given momentum transfer, this represents the case where the least amount of energy is lost by the electron; that is the leftmost contribution in Figure 1. If the experiment's resolution can separate this elastic peak from all other processes, we are effectively doing inclusive electron scattering while capturing only the elastic channel. This is then an 'inclusive-exclusive' experiment: inclusive because we include all possible channels for a given kinematics, and exclusive because physics dictates that only one specific channel can occur.
- Exclusive scattering. We turn to exclusive scattering when it is necessary to isolate a specific reaction channel and fully constrain the final state. This is the case, for example, when we

have a single-nucleon emission and we know the energy is below the separation energy of two nucleons. Under these conditions, kinematics guarantee that no additional nucleons were emitted, fully isolating the one-nucleon emission channel.

- Coincidence scattering $(e, e'p)$, corresponds to the coincidence of electron and proton in the final state, but the rest of the final nuclear state is left unobserved. This can be done under exclusive conditions, if we know, by the kinematics, that there is only energy enough to knockout one nucleon. If the energy transfer remains below the two-nucleon emission threshold, we can guarantee that no additional nucleons were knocked out; thus we know that we have measured every contributing channel. Beyond the two-nucleon threshold, however, these coincidence experiments are no longer exclusive, and more particles than the actually detected proton are emitted.

Coincidence / Semi-Inclusive Scattering $(e, e'p)$ is the direct analogue to neutrino energy reconstruction channels, so we can use it to translate what we obtain to the quasi-elastic peak observed in neutrino-nucleus interactions.

3.2 Kinematics of Electron-nucleus scattering

In $(e, e'p)$ scattering experiments, we are interested in reactions where the incoming electron, with four-momentum $k_i^\mu = (\epsilon_i, \vec{k}_i)$, is scattered resulting in an electron and a proton with four momentum $k_f^\mu = (\epsilon_f, \vec{k}_f)$ and $P_N^\mu = (E_F, \vec{P}_F)$, respectively. The incoming and final electron are contained in the incident plane, while the proton is emitted with an angle ϕ_F with respect to the electron's scattering plane, as shown in Fig. 2 [26], [27].

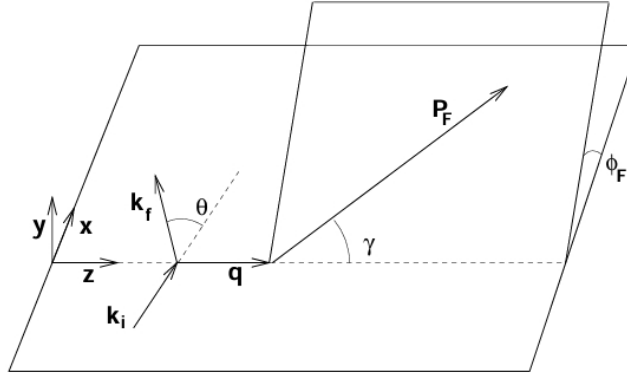


Figure 2: Kinematics of the exclusive process $(e, e'p)$ [26]

To describe the electromagnetic interaction between the electron and the target nucleus, we use the first-order Born approximation, where we assume the scattering process is mediated by the exchange of only one singular photon between the lepton and the hadronic system. this first-order Born approximation is justified in the fact that the electromagnet coupling constant is relatively weak, higher order contributions will be less than 1% contributions. This virtual photon carries the transferred four-momentum, q^μ . This momentum transfer can be obtained from energy-momentum conservation in the leptonic vertex:

$$q^\mu = k_i^\mu - k_f^\mu = (\omega, \vec{q}) \quad (3)$$

where ω is the energy transferred to the nuclear target and $\vec{q}$ is the three momentum transfer, whose direction can be used to define the z-axis. This leads to the invariant quantity $Q^2 = -q^2 =$

$$|\vec{q}|^2 - \omega^2 > 0.$$

On the other hand, looking at the hadronic vertex, the target nucleus has four-momentum $P_A^\mu = (E_A, \vec{P}_A)$, while the emitted nucleon and the residual nucleus have momenta: $P_F^\mu = (E_F, \vec{P}_F)$ and $P_{A-1}^\mu = (E_{A-1}, \vec{P}_{A-1})$, respectively. For the calculations, we will choose the reference frame so that the target is at rest: $P_A^\mu = (M_A, 0)$.

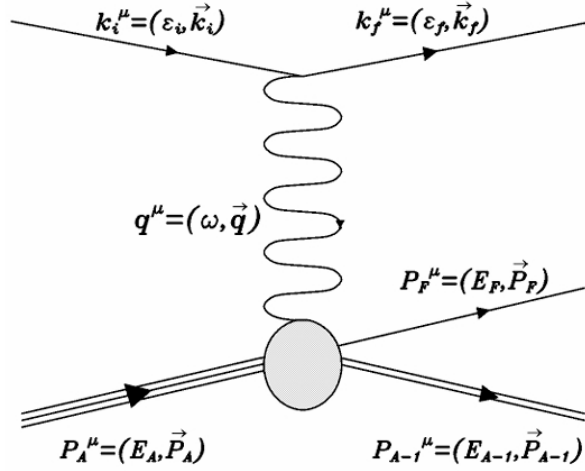


Figure 3: Feynman diagram for an electron-proton scattering with a photon exchange [27]

With these variables, we can define the missing energy and momentum, which will allow us to reconstruct the complete kinematics of the reaction. The **missing energy**, E_m , is the energy transformed into internal energy during the process [27]. It is computed from the energy imbalance between the measured final-state particles and the energy transferred by the electron:

$$E_m = \omega - T_F - T_B = M_B - M_A + M \quad (4)$$

where the kinetic energies are:

$$T_{A-1} = E_{A-1} - M_{A-1} \quad (5)$$

$$T_F = E_F - M \quad (6)$$

$$T_A = E_A - M_A \quad (7)$$

$M_B - M_A + M$ represents the difference in the binding energy between the final and initial nuclear states. Therefore, if the residual nucleus is in the ground state, E_m is simply the separation energy of the emitted nucleon.

The missing momentum can also be defined in a similar way to account for the momentum of the undetected nuclear part, or residual system.

$$\vec{p}_m = \vec{P}_F - \vec{q} \quad (8)$$

These two variables are the ones usually represented by experimentalists, since they allow us to reconstruct the kinematics. It is important to note that in some modern experiments at higher energies the recoil energy of the residual system is neglected, since at high energies the transversality

is negligible and the residual nucleus doesn't move considerably. Theoretical models must take this into account when comparing with experimental data. Examples of missing energy spectrum from $(e, e'p)$ scattering are the following:

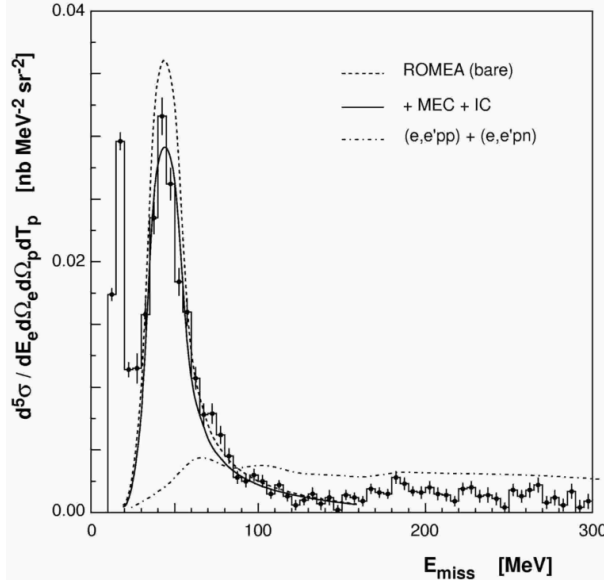


Figure 4: Example of missing energy spectrum for ^{16}O at $E_{\text{beam}} = 0.843$ GeV[28]

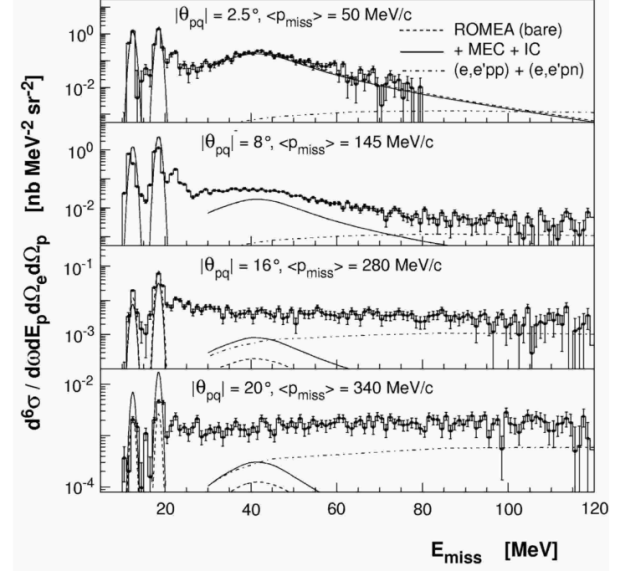


Figure 5: Cross-section as a function of the missing energy spectrum for ^{16}O at $E_{\text{beam}} = 2.442$ GeV [28]

3.3 Cross section

In scattering processes, one of the most important observables is the cross section, σ , which quantifies the probability of a specific process in a scattering reaction. Physically, it represents an effective area presented by the target particle for a specific reaction to occur [1], [29].

$$\sigma = \frac{\dot{N}}{\phi N_b} = \frac{\text{number of reactions}}{\text{incident flux} \times \text{number of target nuclei}} \quad (9)$$

For electron-nucleus scattering, this cross section can be calculated analytically, whose derivation can be seen in Appendix A, obtaining[27]:

$$d\sigma = \frac{m_e^2 M_p^2}{\epsilon_i \epsilon_f E_I E_F} \frac{e^4}{q^4} (2\pi)^4 \delta^{(4)}(P_I + k_i - P_F - k_f) \frac{d^3 \vec{k}_f}{(2\pi)^3} \frac{d^3 \vec{P}_F}{(2\pi)^3} \frac{1}{|\vec{v}_{rel}|} \frac{1}{4m_e^2} \cdot \left[|(k_i^\mu + k_f^\mu) \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right] \quad (10)$$

This expression can be simplified using Mott's cross section:

$$d\sigma = |\vec{P}_F| E_F \sigma_M \frac{M_p^2}{4\epsilon_f E_F \cos^2(\theta/2)} \delta^{(4)}(P_I + k_i - P_F - k_f) \frac{1}{|E_I \vec{k}_i - \epsilon_i \vec{P}_I|} \cdot \left[|(k_i^\mu + k_f^\mu) \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right] d\epsilon_f d\Omega_f dE_F d\Omega_F \quad (11)$$

The term in brackets in Eq. (11) represents the transition matrix element squared which can be written as the contraction of the leptonic tensor $L_{\mu\nu}$, which accounts for the QED electron vertex, and the hadronic tensor $H^{\mu\nu}$, which contains the nuclear currents $\langle J_\mu \rangle$ [26].

The same formalism can be applied to the neutrino-nucleus interaction, exchanging the electromagnetic leptonic tensor for the neutral leptonic current and adding an axial term to the nuclear current.

Furthermore, we can separate the responses into longitudinal, transversal and interference terms, which can be seen in the Appendix C.

3.4 Plane Wave Impulse Approximation (PWIA)

To calculate the cross section, there is a series of assumptions and approximations usually made, defining the so-called Impulse Approximation: We assume that the (single) virtual photon emitted by the electron interacts exclusively with single nucleon inside the target, and it is this nucleon the one that is ejected from the nucleus, while the rest of the nucleons remain as spectators. This approximation is valid in the region of quasielastic scattering where the momentum transfer is sufficiently large to resolve individual nucleons, allowing the primary interaction to be treated as a single-nucleon knockout process, while the rest of the nucleons remain bounded together forming the residual nucleus.

If, furthermore, we consider the emitted nucleon as a free particle whose wave functions are represented by plane wave functions, then we are talking about Plane Wave Impulse Approximation (PWIA). In this case, the cross section factorizes cleanly (at least in nonrelativistic formalisms), making it possible to link in an easy way the cross-section for this reaction to single-particle details of the nuclear structure, such as nucleon binding energies and single-particle momentum distributions. If, on the other hand, we consider the interactions of the emitted nucleon and the residual nucleus with the surrounding material, known as the interactions in the final states (FSI), we would have a Distorted Wave Impulse Approximation (DWIA). In DWIA, the wave function of the outgoing nucleon is distorted by a nuclear potential to account for FSI [25], [27], [30].

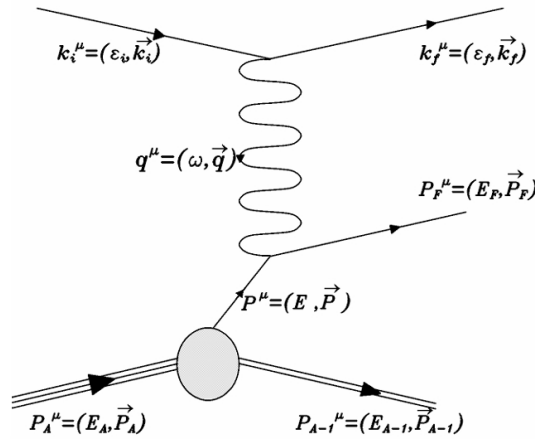


Figure 6: Feynman diagram for an electron-proton scattering with a photon exchange within the Impulse Approximation [27]

To analyze the reaction kinematics of an exclusive $(e, e'p)$ process, we start from momentum conservation in the laboratory frame, where the target nucleus A is initially at rest ($\vec{P}_A = \vec{0}$):

$$\vec{q} + \vec{P}_A = \vec{P}_F + \vec{P}_{A-1} \implies \vec{q} = \vec{P}_F + \vec{P}_{A-1} \quad (12)$$

Note that this conservation law holds generally, independently of the specific reaction mechanism.

If we apply this in Eq. 8, we can see that the missing momentum is related to the momentum of the residual nucleus:

$$\vec{p}_m \equiv \vec{P}_F - \vec{q} = -\vec{P}_{A-1} \quad (13)$$

Therefore, by definition, the missing momentum is equal and opposite to the recoil momentum of the residual $(A - 1)$ system.

When we introduce the Impulse Approximation (IA), it is assumed that the virtual photon interacts exclusively with a single nucleon, leaving the remaining $(A - 1)$ nucleons as passive spectators during the quick knockout process. Since the spectator system receives no direct momentum from the probe, its total momentum remains unchanged throughout the collision. Considering that the initial nucleus was at rest ($\vec{P}_I + \vec{P}_{A-1} = \vec{0}$), the initial momentum of the spectator system must balance the initial momentum $\vec{P}_I$ of the bound nucleon:

$$\vec{P}_{A-1} = -\vec{P}_I \quad (14)$$

Substituting this result into the definition of the missing momentum, we obtain the fundamental kinematic relation of the Impulse Approximation:

$$\vec{p}_m = \vec{P}_I \quad (15)$$

Thus, within the IA framework, by measuring the missing momentum $\vec{p}_m$, we measure directly the initial momentum $\vec{P}_I$ of the struck nucleon inside the target nucleus before the interaction.

3.5 Spectral Function

The ultimate goal of exclusive $(e, e'p)$ experiments is to understand the structure of the target nucleus prior to the interaction. This initial state is formally described by the Spectral Function, $S(E_m, p_m)$. The spectral function represents the probability density of finding a nucleon inside the nucleus with a specific binding energy E_α compatible with E_m and an initial momentum P_I , which in the Impulse Approximation corresponds to the missing momentum p_m [31]. Theoretical models usually build this spectral function adding two contributions [32]:

- Independent-Particle / Shell Model: Contribution stemming from a single nucleon, not belonging to a correlated pair. This is usually computed from a mean field model. This accounts for approximately 70 – 80% of the total spectral function, and it is the main contribution at low missing energy and momentum.
- Short-Range Correlations (SRC): Contribution to the spectral function when the interacting nucleon belongs to a correlated nucleon pair. The interaction breaks the pair, often resulting in a 2N final state, being the main contribution at high energy and momentum.
- Other correlation effects, of long range (LRC).

To extract the spectral function from experiments, the **Factorization Approximation** is claimed [33], [34]. Within the factorization approach, the total measured six-fold differential cross section is expressed as a product of three terms: a kinematic phase space and recoil factor $K \propto E_F \cdot |\vec{P}_F|$, the elementary off-shell electron-nucleon cross section σ_{ep} (such as the de Forest σ_{cc1} or σ_{cc2} prescriptions), and the nuclear Spectral Function $S(\vec{p}_m, E_m)$ [27], [35]:

$$\frac{d^6\sigma}{d\epsilon_f d\Omega_f dE_F d\Omega_F} = K \sigma_{ep} S(\vec{p}_m, E_m) \quad (16)$$

While factorization is not an exact approximation, even within the IA, under QE kinematics, the general features of the cross-section are reproduced within this approach, which has an extraordinary appeal, because the interpretation is straightforward: the cross-section is simply proportional to the probability of finding an initial nucleon with the given missing energy and momentum, times the probability for the lepton to scatter a nucleon with said energy and momentum into the observed final nucleon state. Indeed, apart from the spectral function, we have the elementary cross section, $\sigma_{ep} \equiv \frac{d\sigma}{d\Omega_f}$. Expressions for these quantities and a brief justification of the factorization approach are presented in Appendix B.

Inspired in the factorization approach, many $(e, e'p)$ experiments present their results in terms of the “reduced cross-section” (σ_{red}) which is derived from the primary measured observable, the six-fold differential cross section, $\sigma_{exp} \equiv \frac{d^6\sigma}{d\epsilon_f d\Omega_f dE_F d\Omega_F}$:

$$\sigma_{red}(p_m, E_m) = \frac{\sigma_{exp}}{K \cdot \sigma_{ep}} \quad (17)$$

By dividing the cross section by the kinematic factor, K , and the elementary off-shell electron-proton cross section, σ_{ep} , we remove the dependency on the electron-nucleon electromagnetic variables from the observed cross section, if factorization holds.

What remains contains just the information about the state of the nucleon inside the target nucleus before the event. As mentioned before, under suitable approximations, the factorization approximation for the exclusive cross-section holds exactly. Such is the case for the non-relativistic Plane Wave Impulse Approximation (PWIA), where the outgoing nucleon does not interact with the surrounding medium. In this case, reduced cross-sections correspond to the Spectral Function, $S(E_m, p_m)$ [33], [30]:

$$\sigma_{red}(p_m, E_m) \approx S(E_m, p_m) \quad (18)$$

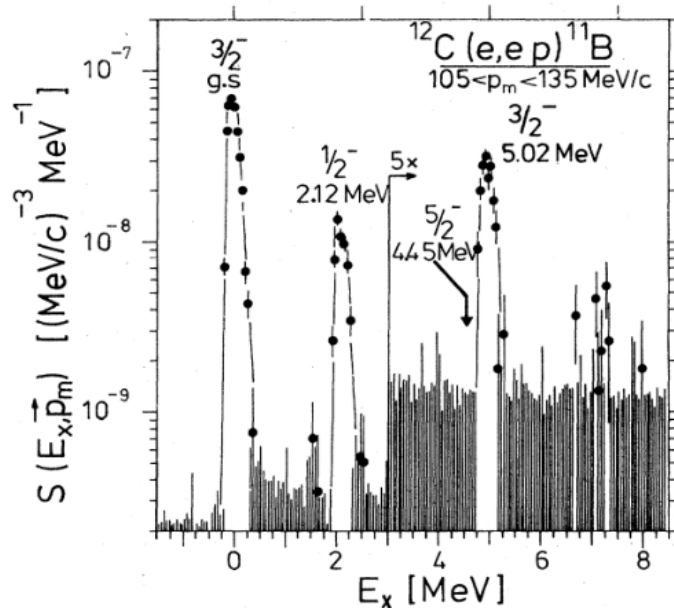


Figure 7: One-proton hole states excitation energy spectrum of ^{11}B observed in the $^{12}\text{C}(e, e'p)^{11}\text{B}$ reaction at approximately 400 MeV/c of momentum transfer and under quasielastic kinematics [36].

The spectral function depends on the missing energy and the missing momentum of the initial nucleon. Experimentally, reduced cross-sections can be integrated in the momentum variable in a certain range, to obtain a missing-energy spectrum. For instance, ^{12}C is illustrated in the Figure 7, where the characteristic peaks corresponding to one-hole states measured under exclusive conditions at low missing energy and momentum are clearly distinguishable.

4 Emission of one proton versus multinucleon emission

4.1 Multi-nucleon emission mechanisms (SRCs)

In the context of the Spectral Function, when we go to higher values of missing energies and momenta, instead of just getting the discrete peaks corresponding to one-hole states in the residual system, as seen in Figure 7, we start seeing a broad continuous tail like in Figure 5 and 4. This is due to the contribution of correlated pairs, due to the strong short-range nucleon-nucleon interaction [37].

The nucleons inside the nucleus are bound via the nuclear force, which, simplifying a bit, is repulsive at short distances, attractive at intermediate distances, and vanishes when the distance between the nucleons becomes large. The combination of the finite range of the nuclear interaction, the fact that proton and neutrons are fermions, and thus Pauli principle makes it less likely that two protons or two neutrons appear at short distances of each other, makes it possible that a model of the nucleons based upon nucleons that behave approximately as independent particles in a mean field created by their average interaction with the other nucleons, is able to describe to a large extent many properties of finite nuclei [38]. However, if two nucleons get close enough to each other, roughly $0.5 - 0.7$ fm, so that their wavefunctions can overlap, the effects beyond mean field approximation may be conspicuous. These correlations mentioned here are named, for obvious reasons, short range correlations (SRC), and loosely speaking, it is usually said that 20 to 30% of the total nucleons in medium/heavy nuclei correspond to SRC correlated pairs [37]. There are also long-range correlations, that we will not comment in this work.

Uncorrelated nucleons can have a momentum in the target nucleons of moderate maximum magnitude, of the order of the Fermi level, below 250 MeV/c, beyond where the mean field momentum distributions gets depleted. Thus, we can look for correlations in experiments focusing in nucleons with large momentum [39]. These arise from correlated pairs, and the high (relative) momentum is generated due to the short range repulsive core. The full pair of nucleons might have a relatively small momentum, but the nucleons inside the pair, may have a high momentum relative to each other, arranged so that the total momentum of two nucleon pair cancels to a large extent, or in simpler words, the two nucleons in a SRC pair have momenta largely canceling each other.

Some experiments have hinted that these correlated pairs are predominantly composed of proton-neutron (np) pairs, which would require a strong tensor component of the nuclear force at short distances [40]. One has to be aware, however, that short-range correlations, mean field, or final state interactions (FSI), which will be introduced later, are not direct observable quantities. They can be connected to experiments only in the framework of a model. Different models can produce more or less correlations, and high momentum components in the nuclear function can be produced in a model of relativistic FSI without explicit SRC.

The large relative momentum of correlated pairs arises from the repulsive character at short distances of the strong nuclear force, which prevents nucleons from getting very close to each other.

Similar to the repulsive potentials observed in atomic physics that we can observe in Figure 8, there exists a minimum allowed distance for two nucleons, typically around $0.5 - 0.7$ fm. This introduces a physical barrier that acts as a step function—in the two-nucleon spatial wave function. Consequently, the Fourier transform of this sharp spatial gradient generates heavily populated high-momentum tails ($p > 400$ MeV/c) in momentum space. There are also correlated pairs with smaller momentum, but these configurations are less likely, due to the strong short-range repulsion, and difficult to disentangle from mean field components [39].

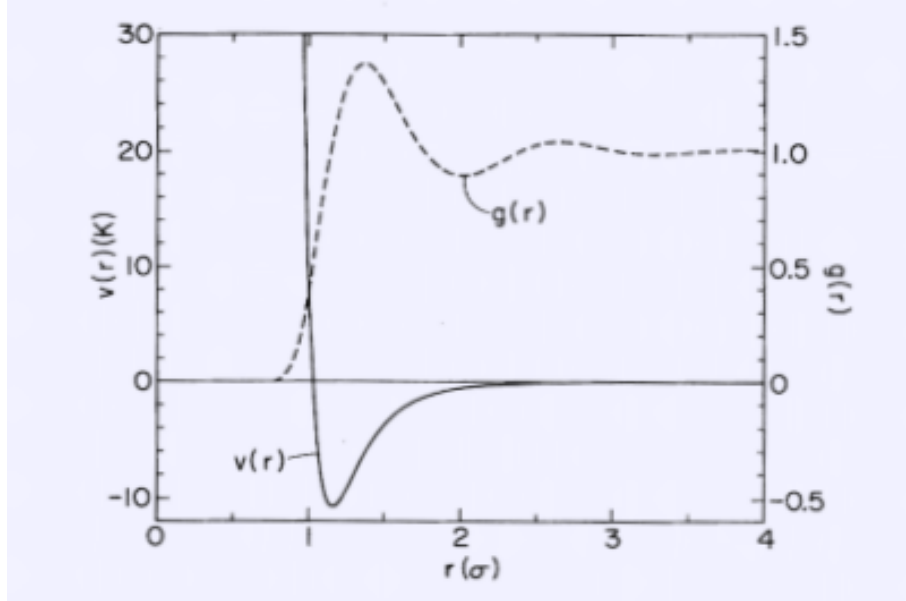


Figure 8: Inter-atomic potential $V(r)$ and correlation function $g(r)$ for 4He [39]

In $(e, e'p)$ scattering experiments, the virtual photon interacts with one of the protons, which is then ejected from the nucleus. If the kinematics allow for enough energy to be transferred to the nucleus to break a correlated pair and knock out one of the nucleons, then the other nucleon, without the additional binding provided by correlations, would also be unbound and knocked-out, even if the exchanged virtual boson did not interact with it.

On the other hand, if the experiment is performed so that the excitation energy provided to the nucleus is below the two-nucleon emission threshold, it will not be possible for the virtual boson to knock-out nucleons belonging to a correlated pair. This is why, in coincidence $(e, e'p)$ experiments performed under exclusive conditions, *i.e.* below the two nucleon emission threshold, we can be certain that we are exciting pure one-hole states of the nuclear target. And this is why we only 'find' about 70% of the nucleons while the remaining 30% of nucleons would correspond to correlated pairs.

In $(e, e'p)$ experiments, only the scattered electron and proton are detected, so if the kinematics is not exclusive, that is, we have reached the two-nucleon emission threshold, the additional nucleons that could be knocked-out due to interaction with a pair would go unnoticed. Therefore, the kinetic energy taken by this secondary (and further in energy, tertiary, etc), unobserved, nucleon would be seen as strength at high apparent missing energy.

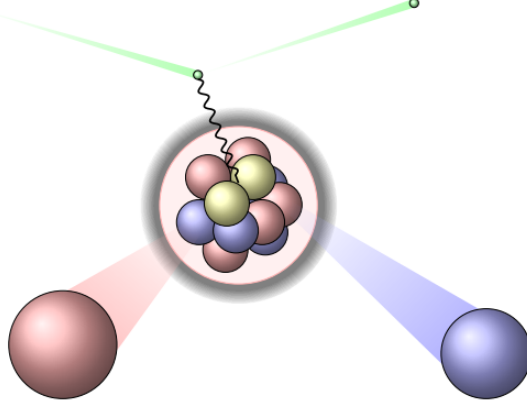


Figure 9: Scheme of the two nucleon knockout channel [38]

The distribution of available energy and momentum changes, since the energy transferred now has to be distributed among two (or more) nucleons and the residual $(A-2)$ nucleus. Comparing it to the kinematics for one-nucleon emission, we still have the energy and momentum of the detected proton $(E_F, \vec{P}_F)$, that we can call now $(E_1, \vec{P}_1)$, and the residual $A-1$ energy and momentum will be now distributed between the $A-2$ residual nucleus and the additional emitted nucleon:

$$(E_{A-1}, \vec{P}_{A-1}) = (E_2, \vec{P}_2) + (E_{A-2}, \vec{P}_{A-2}) \quad (19)$$

Thus, the missing momentum is distributed between the secondary nucleon and the $A-2$ core:

$$\vec{p}_m = -(\vec{P}_2 + \vec{P}_{A-2}) \quad (20)$$

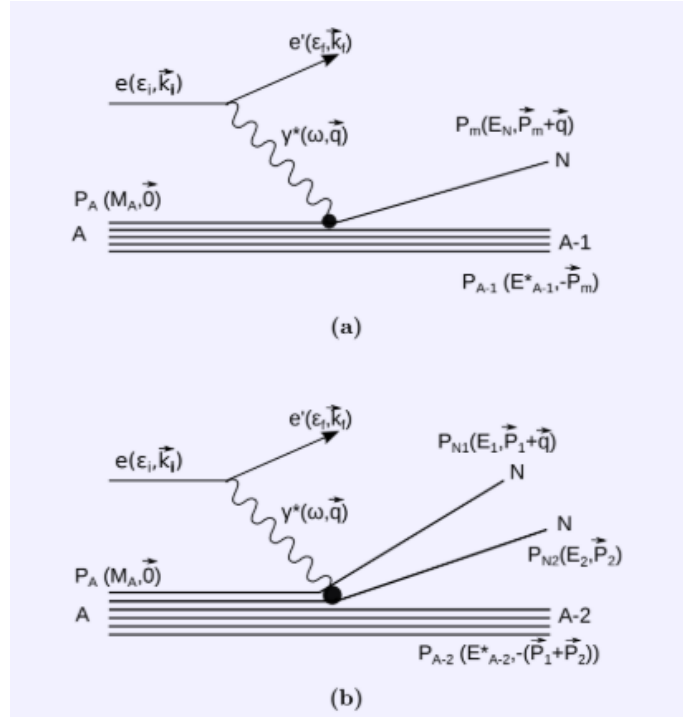


Figure 10: Feynman diagrams for quasi-elastic scattering off an uncorrelated nucleon with initial momentum p_m vs a quasi-elastic scattering off a nucleon involved in SRC [39]

We would assume that the nucleons in a correlated pair have a large relative momentum but a very small center-of-mass momentum relative to the rest of the nucleus and that, in the spirit of the Impulse Approximation, the $A - 2$ residual nucleus behaves as a spectator and is nearly at rest ($\vec{P}_C \approx 0$), and the initial momentum of the correlated nucleons is approximately equal, but in opposite directions.

$$\vec{P}_{I,1} \approx -\vec{P}_{I,2} \quad (21)$$

where, the secondary nucleon does not interact with the virtual photon, so its initial momentum is equal to the final momentum, $\vec{P}_{I,2} = \vec{P}_2$.

In a missing energy spectrum we can disentangle two regions. Below the two-nucleon emission, the experiment will produce pure 1-hole states into the final A-1 system, which will be either in its ground state, or in one of the discrete states below the particle emission threshold of the A-1 system. The natural width of these states is very small, negligible compared to the experimental resolution in missing energy. Thus, in this region, we will see very defined peaks, with a width due to experimental resolution, characteristics of the kinematics and the performance of the different facilities, from as little as 100 keV at NIKHEF-K to several MeV at JLAB [19].

However, when the missing energy is above the two nucleon threshold, then the virtual photon can knock-out a deeply bound nucleon, leaving the residual nucleus in a one-hole resonance configuration, above the particle emission level, which will almost instantaneously decay by emission of an additional particle. These resonances would have a natural width of several MeV or tens of MeV, as it is the case of the s - shell one-hole states seen in ^{11}B after $(e, e'p)$ on ^{12}C . But it can be the case that the virtual photon knocks-out a nucleon from a correlated pair, inducing then the emission of a second nucleon (thus, the two nucleons in the pair), and thus the total apparent missing energy must take into account the separation energy required to remove two nucleons from the nucleus (S_{2N}), plus the kinetic energies of the undetected A-2 and additional nucleon, resulting in:

$$E_m = S_{2N} + T_2 + T_{A-2} \quad (22)$$

Therefore, we can see that, for this process to occur, the missing energy must be larger than the kinematic threshold for the emission of two nucleons, which is the two-nucleon separation energy (S_{2N} or S_{2p} in the case of two protons). Above this threshold, the missing energy is no longer characterized by narrow discrete peaks, but rather a combination of wide peaks corresponding to deep one-hole resonances, superimposed with the missing energy tail due to emission of nucleons from correlated pairs. This second contribution grows slowly after the two-nucleon threshold and, if the kinematics of the experiment spans a very large range of energy transfers, will rise up to a constant value, forming a tail of strength up to large missing energies, as we add other processes such as higher order multinucleon emissions ($3N$ and $4N$, see Fig.4, left panel).

4.2 Final State Interactions

While correlations are mainly mentioned in the context of the interaction of the virtual boson with the nucleon in the initial (target) state, the (in-passing) knocked-out nucleon, during its path outwards the residual nucleus, interacts with the remaining nucleons, or more generally, it will experience nuclear interactions in the final state, or in short, 'Final State Interactions' or FSI. [16]. These FSI can lead these nucleons to knock-out further nucleons. One has to notice that, while there will always be FSI affecting the nucleon that has interacted with the virtual boson, these FSI may produce knock-out of further nucleons only in the case that the missing energy of the experiment is above the two-nucleon threshold. A detailed discussion of FSI is beyond the scope of this work.

4.3 Missing Energy Spectrum and Emission Thresholds

To assess the reaction mechanism contributing to an experimental data set, either just single-nucleon emission or single-nucleon combined with multinucleon emissions, we inspect the separation energy for one-nucleon removal, two-nucleons or more, which can be determined through the differences in binding energies of the target and possible final nuclear systems. It can be computed from the atomic mass excesses (Δ) of the initial and final nuclear states:

$$S_p = \Delta({}_{Z-1}^{A-1}X) + \Delta({}^1\text{H}) - \Delta({}_Z^AX) \quad (23)$$

$$S_n = \Delta({}_Z^{A-1}X) + \Delta(n) - \Delta({}_Z^AX) \quad (24)$$

$$S_{2p} = \Delta({}_{Z-2}^{A-2}X) + 2\Delta({}^1\text{H}) - \Delta({}_Z^AX) \quad (25)$$

$$S_{2n} = \Delta({}_Z^{A-2}X) + 2\Delta(n) - \Delta({}_Z^AX) \quad (26)$$

$$S_{pn} = \Delta({}_{Z-1}^{A-2}X) + \Delta({}^1\text{H}) + \Delta(n) - \Delta({}_Z^AX) \quad (27)$$

Taking the mass excess from IAEA *Live chart of nuclides* [41], we obtain the following separation energies for the nuclei of study:

Table 1: Separation energies for one- and two-nucleon knockout channels in ${}^{12}\text{C}$, ${}^{16}\text{O}$, ${}^{40}\text{Ar}$ and ${}^{48}\text{Ti}$, in MeV. Values calculated using atomic mass excesses.[41]

Target	Reaction Channel	Residual Nucleus	Threshold	S (MeV)
${}^{12}\text{C}$	${}^{12}\text{C}(e, e'p){}^{11}\text{B}$	${}^{11}\text{B}$	S_p	15.96
	${}^{12}\text{C}(e, e'n){}^{11}\text{C}$	${}^{11}\text{C}$	S_n	18.72
	${}^{12}\text{C}(e, e'2p){}^{10}\text{Be}$	${}^{10}\text{Be}$	S_{2p}	27.19
	${}^{12}\text{C}(e, e'2n){}^{10}\text{C}$	${}^{10}\text{C}$	S_{2n}	31.84
	${}^{12}\text{C}(e, e'pn){}^{10}\text{B}$	${}^{10}\text{B}$	S_{pn}	27.41
${}^{16}\text{O}$	${}^{16}\text{O}(e, e'p){}^{15}\text{N}$	${}^{15}\text{N}$	S_p	12.13
	${}^{16}\text{O}(e, e'n){}^{15}\text{O}$	${}^{15}\text{O}$	S_n	15.66
	${}^{16}\text{O}(e, e'2p){}^{14}\text{C}$	${}^{14}\text{C}$	S_{2p}	22.34
	${}^{16}\text{O}(e, e'2n){}^{14}\text{O}$	${}^{14}\text{O}$	S_{2n}	28.89
	${}^{16}\text{O}(e, e'pn){}^{14}\text{N}$	${}^{14}\text{N}$	S_{pn}	22.96
${}^{40}\text{Ar}$	${}^{40}\text{Ar}(e, e'p){}^{39}\text{Cl}$	${}^{39}\text{Cl}$	S_p	12.53
	${}^{40}\text{Ar}(e, e'n){}^{39}\text{Ar}$	${}^{39}\text{Ar}$	S_n	9.87
	${}^{40}\text{Ar}(e, e'2p){}^{38}\text{S}$	${}^{38}\text{S}$	S_{2p}	22.76
	${}^{40}\text{Ar}(e, e'2n){}^{38}\text{Ar}$	${}^{38}\text{Ar}$	S_{2n}	16.47
	${}^{40}\text{Ar}(e, e'pn){}^{38}\text{Cl}$	${}^{38}\text{Cl}$	S_{pn}	20.60
${}^{48}\text{Ti}$	${}^{48}\text{Ti}(e, e'p){}^{47}\text{Sc}$	${}^{47}\text{Sc}$	S_p	11.44
	${}^{48}\text{Ti}(e, e'n){}^{47}\text{Ti}$	${}^{47}\text{Ti}$	S_n	11.63
	${}^{48}\text{Ti}(e, e'2p){}^{46}\text{Ca}$	${}^{46}\text{Ca}$	S_{2p}	19.94
	${}^{48}\text{Ti}(e, e'2n){}^{46}\text{Ti}$	${}^{46}\text{Ti}$	S_{2n}	20.51
	${}^{48}\text{Ti}(e, e'pn){}^{46}\text{Sc}$	${}^{46}\text{Sc}$	S_{pn}	22.09

As mentioned before, these separation energies establish clear physical regimes in the missing energy spectrum. Below the lowest separation energy (that is, the two nucleon emission threshold) for two nucleons, we can guarantee that only one nucleon will be emitted, ($E_m < S_{2N}$) from an outer

shell, which will give rise to narrow discrete peaks in the missing energy spectrum. As mentioned before, for energies above the two-nucleon threshold ($E_m > S_{2N}$), the spectrum will show the two main contributions: the multinucleon emission with its long continuous tail and the deep one-hole broad resonances ($1s$ one-hole states in light nuclei). In either case, either from the decay of the deep one-hole resonance by emission of particles (nucleons or otherwise), or due to the breakdown of SRC pairs, we can generally assume that there will always be more than one nucleon emitted beyond the two-nucleon emission threshold [42].

We will delve further in the reason of the different width of the discrete outer-shell peaks compared to the resonance of the inner s -shell one-hole states (or rather, resonance). The reason behind it is simple to understand. In the emission of nucleons from outer shells, the residual nucleus is left in a single-hole state that is essentially stable against nucleon emission. Therefore, their natural width is very small. The width we see in experimental plots in $(e, e'p)$ is solely due to the energy resolution in experiments, not because the outer levels have a width of their own. For inner levels, however, the situation is different. When a hole is created deep within the nuclear core, the residual nucleus is left in a high excited energy configuration. This deep hole is filled almost instantaneously by a nucleon from a higher shell dropping down into it. This creates a short-lived state, a resonance. From Heisenberg's uncertainty principle ($\Delta E \Delta t \geq \hbar/2$) we can estimate the width in energy of the short live-time resonances, that follows a Breit-Wigner distribution. This difference between the narrow outer-shell states and the broad inner-shell strength is clearly observed experimentally in the ^{16}O missing energy spectra, as it is explained in detail in Appendix D. In particular, it can be seen that the s -hole strength is distributed over a broad excitation-energy region and exhibits several substructures [42].

When comparing with experimental data to obtain the distribution of the strength against missing energy, one would then introduce a phenomenological width in the contribution from every one-hole state. A relatively small phenomenological widths for the p -shell states, consistent with the experimental resolution in missing energy, and a substantially larger width for the $1s_{1/2}$ or other resonances opened up beyond the two nucleon emission threshold in the present model.

If we go to heavier nuclei, such as ^{40}Ar and ^{48}Ti , the number of shells increases considerably. Therefore, there are many more contributions of inner shells that will sit above the SRC distribution. These deep-hole states will, thus, contribute to the missing energy spectrum past the two-nucleon emission threshold.

Therefore, to properly model the missing energy spectrum, we will need to take into account not only the outer-most one-hole states the nucleus, but also the deeper one-hole states, since the emitted nucleon could belong to a deeply bounded shell, and the emission of nucleons from SRC pairs.

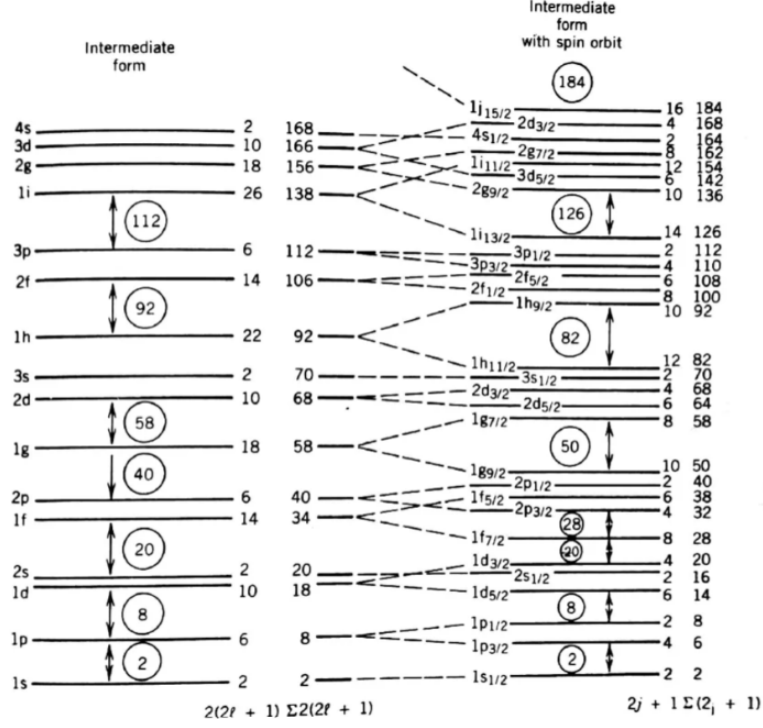


Figure 11: Nuclear shell model

As previously mentioned, apart from the natural theoretical widths of each energy level, it is necessary to take into account the resolution in missing energy from the experiments. This resolution will depend on the specific facility since the raw energy resolution of the incident beam, the momentum resolution of both spectrometers (electron and proton arms), and the effect of dispersion matching will contribute to the final resolution. There are experiments like the ones performed in NIKHEF that achieved a 100 keV resolution for a hundreds of MeV incident beam [19], thanks to the use of matching dispersion. This is a technique that consists of canceling out the variation in the initial electron energy. Since the initial beam is not perfectly monochromatic, its energy variation causes a blurring of the final resolution of the experiment. To counteract this, before the electron hits the target, two magnets are placed to deviate the motion of the electron. Because more energetic electrons are deviated differently to less energetic ones creating a relation between the transversal position of the electron and its energy. In this way, by knowing the position of impact of the electron, the energy of the incoming electron can be adjusted, improving the experimental resolution. On the other hand, other experiments such as JLab, can't go below a few MeV in resolution, so their results will present broader peaks for the p -shells, than the data obtained from NIKHEF. A summary of the experimental facilities and the resolution in missing energy that they achieved can be found in Appendix D.

Therefore, to try to model the width of each state and compare them to the experimental data, we will introduce phenomenological gaussian widths that will try to represent the natural width of each state and the experimental resolution. The low-lying p -shell states appear as relatively well-defined structures, and therefore we will establish relatively small Gaussian widths of $\sigma_\alpha \sim 1\text{--}3$ MeV. On the other hand, for deeply bounded s -shells the strength is considerably more spread in missing energy, so we will need to use a larger width of $\sigma_\alpha = 8$ MeV to describe this broader distribution. For heavier nuclei, we can see that in reference [43] they use a Gaussian width of $\sigma_\alpha = 4$ to represent intermediate levels.

Table 2: Phenomenological parameters for the proton shells of ^{12}C , ^{16}O , ^{40}Ar and ^{48}Ti used in $(e, e'p)$ spectral function models. Values for ^{48}Ti and ^{40}Ar are taken directly from Table IV from reference [43], while the shell binding energies for ^{12}C and ^{16}O are chosen consistently with the missing energy positions observed in NIKHEF $(e, e'p)$ measurements [44] [45]. The Gaussian widths σ_α are phenomenological parameters used to represent the corresponding energy distributions. Here E_α denotes the centroid binding energy, σ_α the Gaussian energy width, $2j + 1$ the theoretical maximum occupation of the shell in the IPM, and $[E_{\text{low}}, E_{\text{high}}]$ the integration range.

Nucleus	Shell (α)	E_α (MeV)	σ_α (MeV)	$(2j + 1)$	E_{low} (MeV)	E_{high} (MeV)
^{12}C	$1p_{3/2}$	15.50	2.50	4	15.0	22.0
	$1s_{1/2}$	36.90	8.00	2	30.0	50.0
^{16}O	$1p_{1/2}$	12.10	1.00	2	11.5	14.0
	$1p_{3/2}$	18.40	3.00	4	16.0	22.0
	$1s_{1/2}$	40.00	8.00	2	30.0	55.0
^{40}Ar	$1d_{3/2}$	12.53	2.00	4	8.0	14.0
	$2s_{1/2}$	12.93	2.00	2	8.0	14.0
	$1d_{5/2}$	18.23	4.00	6	14.0	20.0
	$1p_{1/2}$	28.00	8.00	2	20.0	45.0
	$1p_{3/2}$	33.00	8.00	4	20.0	45.0
	$1s_{1/2}$	52.00	8.00	2	45.0	70.0
^{48}Ti	$1f_{7/2}$	11.45	2.00	8	8.0	14.0
	$2s_{1/2}$	12.21	2.00	2	14.0	30.0
	$1d_{3/2}$	12.84	2.00	4	14.0	30.0
	$1d_{5/2}$	15.46	4.00	6	14.0	30.0
	$1p_{1/2}$	35.00	8.00	2	30.0	54.0
	$1p_{3/2}$	40.00	8.00	4	30.0	54.0
	$1s_{1/2}$	62.00	8.00	2	53.0	80.0

4.4 Phase Space

To model these Short-Range Correlations, we must consider how the ejection of a correlated nucleon pair fundamentally changes the observable kinematics of the scattering process. The transition from one-nucleon emission to two-nucleon emission alters the available phase space of the final hadronic state, which dictates the shape of the measured energy and momentum distributions. This is because when we have a one nucleon emission we are working with a two body final state: the emitted proton and the residual nucleus that recoils. Energy and momentum conservation tightly restrict the available phase space, producing discrete kinematic lines. In contrast, in the two-nucleon knockout we have an additional nucleon that is emitted. Because of this, the available energy will be distributed between the two emitted nucleons giving rise to a continuous distribution.

The physical origin of this difference is directly analogous to the distinction between beta decay and electron capture.

- Two-body final state (electron capture). In electron capture, a nucleus absorbs an atomic electron and emits a single neutrino, leaving a recoiling daughter nucleus. Because there are only two bodies in the final state, one of much larger mass than the other, energy and

momentum conservation fixes their kinematic variables, restricting the available phase space, and produces neutrinos that are mono-energetic and therefore have discrete peaks in the energy spectrum.

- **Three-body final state (beta decay).** In beta decay, the parent nucleus decays into a beta particle (electron or positron), their corresponding neutrino or antineutrino and a daughter nucleus. In this scenario, the energy is distributed among the three final particles (mainly the neutrino and the electron as the daughter nucleus simply recoils) in an infinite number of combinations, expanding the phase space and producing a continuous energy spectrum.

To formalize the phase space arguments discussed in the previous analogies, we evaluate the density of final states ρ_f for nucleon emission. In our non-covariant normalization convention ($\bar{u}u = 1$), the phase space volume element for a final state particle is purely given by $\frac{d^3\vec{p}}{(2\pi)^3}$.

1p Knockout: In the case of emission of a single proton, energy conservation fixes the value of the momentum for certain angle. The density of final states for the emitted proton is restricted by the energy conservation Dirac delta:

$$\rho_1 = \int \frac{d^3\vec{p}}{(2\pi)^3} \delta(E_{\text{avail}} - E_{\text{final}}) \quad (28)$$

where,

- $E_{\text{avail}} = E_m - S_{1p}$ is the “available” energy transferred by the lepton
- $E_{\text{final}} = E_p - E_{A-1}$ is the energy of the final hadronic system

Expressing the volume element in spherical coordinates ($d^3\vec{p}_f = p_f^2 dp_f d\Omega_f = p_f E_f dE_f d\Omega_f$) and integrating over all solid angles (4π), we obtain:

$$\rho_1 = \frac{4\pi}{(2\pi)^3} \int p_f E_f \delta(E_{\text{avail}} - E_f - E_{A-1}) dE_f \quad (29)$$

The delta function strictly forces the proton to possess a unique, discrete energy E_0 and momentum p_0 for a given scattering angle. Integrating over the energy trivially collapses the integral:

$$\rho_1 = \frac{p_0 E_0}{2\pi^2} d\Omega \quad (30)$$

where $E_0 = E_{\text{avail}} - E_{A-1}$ and $p_0 = \sqrt{E_0^2 - m_p^2}$.

This result mathematically confirms that the 1-proton final state yields a discrete value for the differential cross section at a fixed outgoing angle.

Near the emission threshold, where the kinetic energy is: $T = E_0 - m_p \ll m_p$, we can take the momentum as $p_0 \approx \sqrt{2m_p T}$. In this way we obtain the characteristic square-root dependence of single-particle phase space:

$$\rho_1(T) \propto \sqrt{T} \quad (31)$$

2p Knockout: In the case of a two-nucleon knockout process, the phase space now requires the integrating over the momenta of both emitted nucleons:

$$\rho_2 = \int \frac{d^3\vec{p}_1}{(2\pi)^3} \frac{d^3\vec{p}_2}{(2\pi)^3} \delta(E_{\text{avail}} - E_1 - E_2 - E_{A-2}) \quad (32)$$

Writing both differentials in spherical coordinates ($d^3\vec{p}_i = p_i E_i dE_i d\Omega_i$) and integrating over the full solid angles $d\Omega_1$ and $d\Omega_2$ we obtain an expression of the form:

$$\rho_2 = \frac{16\pi^2}{(2\pi)^6} \iint p_1 E_1 p_2 E_2 \delta(E_{\text{avail}} - E_1 - E_2 - E_{A-2}) dE_1 dE_2 \quad (33)$$

The energy conservation delta function can only constrain one of the two particles. If we integrate over the variables of the second proton, the single energy delta function constrains the energy of the second nucleon to $E_2(E_1) = E_{\text{avail}} - E_{A-2} - E_1$. Integrating over E_2 leaves an unconstrained integration over the full three-momentum of the first proton.

$$\rho_2 \propto \int (p_1 E_1)(p_2 E_2) dE_1 \quad (34)$$

$$\frac{d\rho_2}{dp_1} = \frac{1}{4\pi^4} p_1^2 \cdot p_2(p_1) E_2(p_1) \quad (35)$$

Because the available energy is continuously shared between the two emitted nucleons (E_1 and E_2), the remaining integral over $d^3\vec{p}_1$ does not collapse to a discrete peak and instead we get a continuous distribution.

The integration boundaries for p_2 are constrained by the physical emission threshold ($E_2 \geq m_p$) and the available missing energy range E_m . Integrating Eq. (34) over the total available kinetic energy $T_{\text{tot}} = E_{\text{avail}} - E_{A-2} - 2m_p$ gives the total $2p$ phase-space volume:

$$\rho_2(T_{\text{tot}}) \propto \int_0^{T_{\text{tot}}} \sqrt{T_1} \sqrt{T_{\text{tot}} - T_1} dT_1 \propto T_{\text{tot}}^2 \quad (36)$$

Near the threshold ($E_m \gtrsim S_{2p}$), this quadratic behavior ($\sim T_{\text{tot}}^2$) causes ρ_2 to grow very slowly at initial energies before increasing quadratically, contrasting sharply with the steep square-root rise ($\sim \sqrt{T}$) observed in single-nucleon knockout.

We can calculate these density of final states numerically from Eq. 30 and 34, and normalize them so that they can be compared, obtaining the following figure:

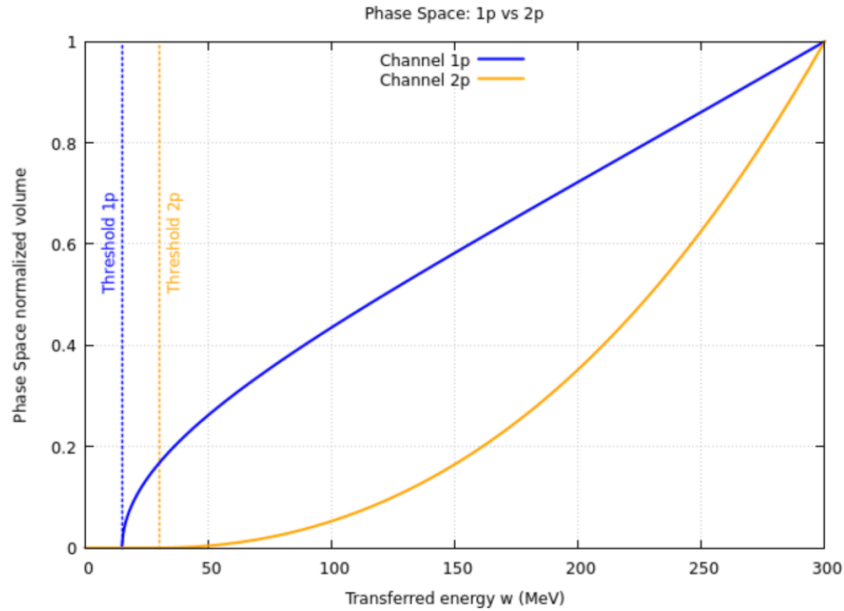


Figure 12: Normalized phase space volume for one- and two-proton emission

What we can observe is that for the one-nucleon emission case has a square root dependence; while the case of two-nucleon emission, right after crossing the threshold, it grows very slowly and then it has a quadratic behavior, as expected.

5 Results and Discussion

5.1 Simulation of the Kinematics

To connect theoretical predictions with actual experimental observables, we rely on simulations, where we can establish the physics of the experiments paired with physical limits and conservation laws to generate a certain number of events. The aim is to create models that can study the contributions of the phase space of different reaction mechanisms.

In our case, the code will model our two channels of interest: the emission of one nucleon from the electron-nucleus scattering, assuming Impulse Approximation, and for the cases where the missing energy is larger than the two-nucleon emission threshold, we will add this contribution and obtain, for every event of the 1N channel, a number N_2 of events for the 2N channel. Therefore, the idea is to obtain an simulator of the kinematics of emission of one or more nucleons (without taking FSI into consideration) to see where each region is predominant.

To achieve this, we will rely on Monte Carlo simulations where we use randomized variables to reconstruct the values of the missing energy and missing momentum using energy-momentum conservation laws. We will validate these models with known experimental data from different $(e, e'p)$ experiments, kinematics and nuclei with the aim of obtaining a program that can be used to predict the contribution of the two-nucleon emission channel. In this way, experimentalists can introduce it in their generators and find regions where the 2N channel is smaller, thereby obtaining a clearer signal for their desired QE region. We will apply this code to electron scattering, but with the aim of exchanging it to neutrinos so that it can be applied to neutrino oscillation experiments.

5.2 Acceptance of the experiment

Before trying to simulate the phase space of one-nucleon emission and two-nucleon emission, we must model and look into the experimental set-up. The detectors used will have their own kinematic window depending on the energy of the beam, the angle of the incoming electrons, etc; so they will only be able to measure a determined range of values of missing energy and momentum for each kinematic setting. This available phase space covered by the experimental set-up is what we know as experimental acceptance and it can be simulated using Monte Carlo generators.

For this study, a custom Monte Carlo Fortran code developed and provided by Prof. José Manuel Udías was used. The framework of this code is designed to map the accessible (E_m, p_m) phase space by simulating the geometric and operational constraints of the detectors used for $(e, e'p)$ scattering experiments and neutrino oscillation experiments. How the code works is by defining a set of central values for the electron part: $(\epsilon_i, \epsilon_f, \theta_e, \phi_e)$ and for the emitted nucleon or proton: $(|\vec{P}_N|, \theta_N, \phi_N)$. With this kinematic setting, we can generate thousands of randomized events. For each event, the variables are sampled using a uniform random distribution centered around the preset values of energy, momentum and angles. The limits of this sampling correspond to the experimental resolution, which is typically a relative width of around 6% for momentum and energy and of the order of milliradians ($\approx 3-6$ mrad) for angular dispersions.

For every accepted event, the code then applies energy-momentum conservation, considering the target nucleus at rest, to obtain the four-vectors of the initial and scattered electron, the initial and residual nucleus and the emitted proton. Applying the definitions of the transferred momentum $q^\mu = (\omega, \vec{q})$, the missing energy (E_m) and missing momentum p_m , Eqs. 3, 4 and 8, we reconstruct the fundamental experimental variables.

Furthermore, we can impose cuts to remove events that are not of our interest. For example, imposing that the cosine of the angle between $\vec{q}$ and $\vec{p}_N$ is close to unity ($\cos(\theta_{pq}) \approx \pm 1$), we keep only the parallel events.

Once all the events have been generated and the cuts have been applied, we are left with a raw acceptance map (E_m vs p_m) that represents the boundaries of our detectors. This first mapping is purely geometric, so to fully understand what is measured, we will later have to apply all the different physical constraints and introduce the nuclear target's behavior.

We can test this code with a set of different kinematics. To make it easier and not choose random settings, we relied on the kinematic settings from the Jefferson Lab E12-14-012 $^{48}\text{Ti}(e, e'p)$ experiment [46].

In this experiment, the main goal was to better understand the interactions of Argon with neutrinos for experiments like DUNE, with the aim of obtaining the spectral function for both protons and neutrons. The measurement of neutrons in electron scattering experiments is usually challenging due to low efficiency and difficulty in detection. Instead, ^{48}Ti is used in $(e, e'p)$ experiments to obtain the spectral function of the neutrons from ^{40}Ar . This is because ^{48}Ti has 22 protons which is the same number of neutrons in ^{40}Ar . Therefore, we study the energy and momentum distributions of ^{48}Ti as an approximation to neutrons in Argon. In this specific experiment, a beam of electrons of 2.222 GeV was used against a thin target of solid natural Titanium. The scattered electron and proton were detected in coincidence using High Resolution Spectrometers, obtaining the missing energy and missing momentum to reconstruct the initial state of the proton inside the nucleus.

We will use the kinematic settings of this experiment to model the experimental acceptance of the detectors.

Table 3: Central kinematic settings for the $^{48}\text{Ti}(e, e'p)$ experiment E12-14-012 with beam energy $E_{\text{beam}} = 2.222$ GeV, [46].

	E'_e (GeV)	θ_e (deg)	Q^2 (GeV^2/c^2)	$ P $ (MeV/c)	$T_{p'}$ (MeV)	$\theta_{p'}$ (deg)	$ q $ (MeV/c)	p_m (MeV/c)	E_m (MeV)
Kin 1	1.777	21.5	0.549	915	372	-50.0	865	50	73
Kin 2	1.716	20.0	0.460	1030	455	-44.0	846	184	50
Kin 3	1.799	17.5	0.370	915	372	-47.0	741	174	50
Kin 4	1.799	15.5	0.291	915	372	-44.5	685	230	50
Kin 5	1.716	15.5	0.277	1030	455	-39.0	730	300	50

It is important to note that it is possible to obtain negative missing momentum kinematics, since the missing momentum is defined from momentum conservation as in Eq. 8. The sign convention adopted for the missing momentum is determined by the relative orientation of the proton spectrometer with respect to the momentum-transfer direction (q). When the proton is detected at an angle smaller than the direction of q , $\theta_p < \theta_q$, the missing momentum is conventionally assigned a negative sign $p_m < 0$ (“minus” kinematics). In contrast, when the proton is detected at an angle larger than the direction of q , $\theta_p > \theta_q$, p_m is assigned a positive sign $p_m > 0$ (“plus” kinematics). When the proton momentum is parallel to q , then $p_m = 0$, corresponding to parallel kinematics.

This convention allows the experimental acceptance to be represented as a function of signed missing momentum, resulting in distributions that can contain both positive and negative p_m values [25].

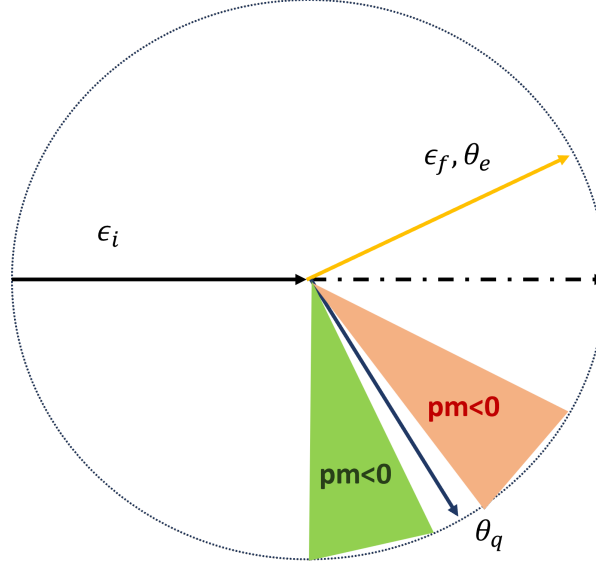


Figure 13: Kinematic scheme of the reaction. The red zone corresponds to the negative p_m kinematics while the green region corresponds to the positive p_m kinematics

Introducing the different kinematic settings from Table 3 into our Monte Carlo generator, we obtain the following results:

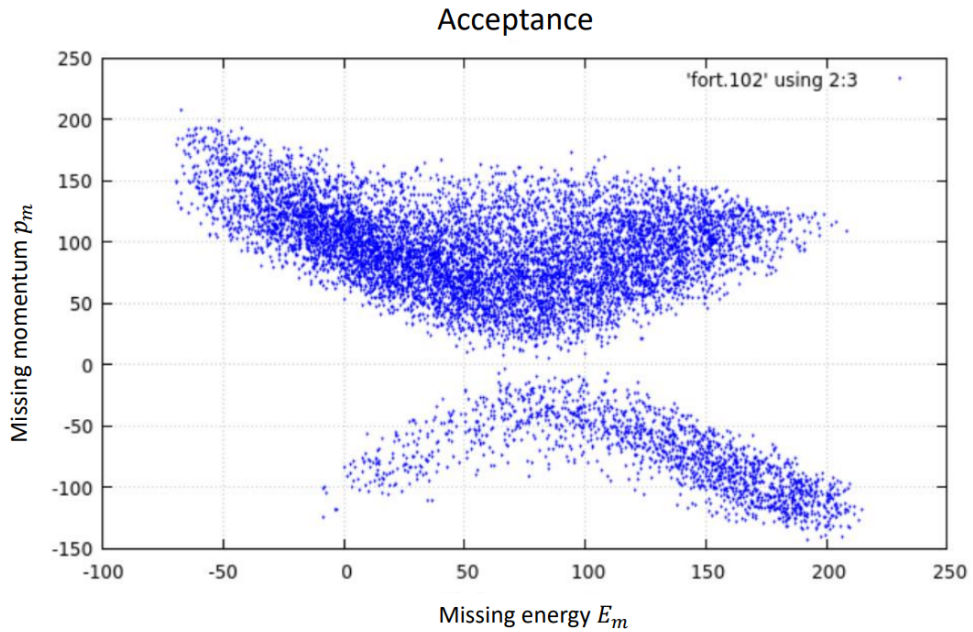


Figure 14: Experimental acceptance for kinematic 1

In this acceptance map, we can observe a dominant upper cloud and a lower cloud, which corresponds to the different (plus and minus) kinematic solutions. We also have empty spaces that

correspond to kinematic regions that are forbidden or outside the geometry of the detector.

If we change to one-dimensional plots, we get histograms of the acceptance only as a function of E_m or p_m , allowing us to study the efficiency of the detectors across the different variables.

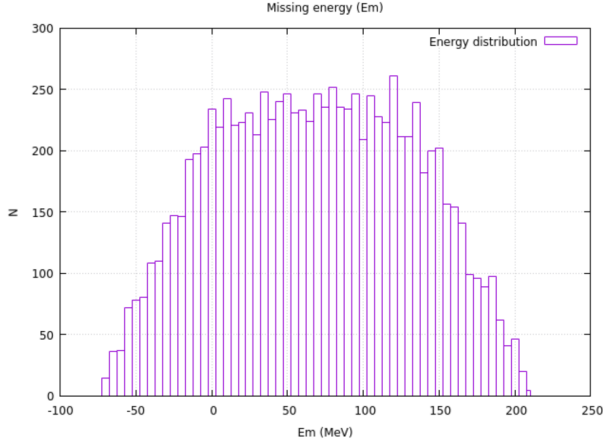


Figure 15: Missing energy spectrum from kinematic 1

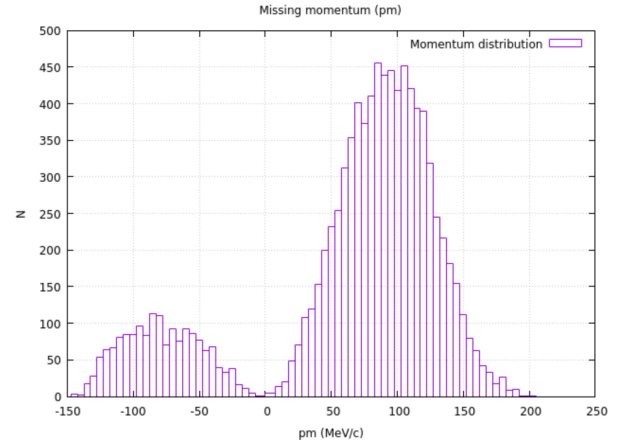


Figure 16: Missing momentum spectrum from kinematic 1

In Figure 15, we have a missing energy distribution that grows, then stays constant from approximately 0 to 150 MeV, and finally decreases. This central plane region is the uniform acceptance; which means our Monte Carlo simulation is generating homogeneous events in this region. The borders, where the curve falls, do not represent a lack of physics, but rather a geometric limit of the detector, the electrons or protons are not giving a signal near the edges or outside of the detector, normally, experimental data is only gathered in the regions of uniform acceptance, so that the edges of the detector need not to be simulate din detail.

It is important to note, we just produce randomly generated angles and energies from isotropic and uniform distributions in our generator, which is why we obtain events that would represent a negative energy.

On the other hand, Figure 16 represents the missing momentum acceptance. In this case, we have two regions: one larger of positive momentum and a smaller one of negative momentum separated by an empty space at zero momentum. This shows us the predominant direction allowed by the kinematics of the experiment. The reason why we get no events at zero momentum is because detectors usually have a blind spot at the point where the beam goes through. At 0 degrees, the emitted particles will collide with the beam pipe and will not reach the detector.

In both cases, for E_m and p_m , experimentalists usually apply cuts to clean the data and keep only the data of interest. We are interested in the uniform plane regions, as they will allow us to have a better resolution in energy. Therefore, experimentalists will have to decide on these cuts to keep the statistics high, while ensuring their detectors are measuring data uniformly. For example, in the case of Figure 15 it would be reasonable to keep only the region comprehended between 0 and 150 MeV.

We can do the same thing with the rest of the kinematics from Table 3:

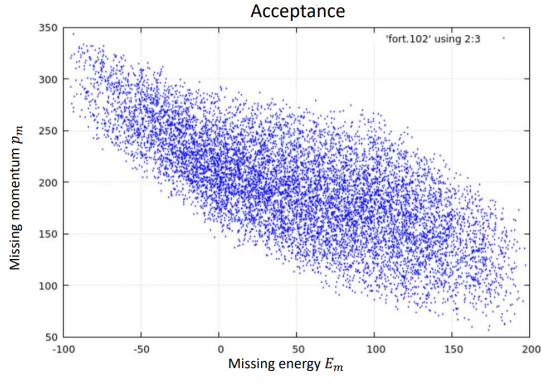


Figure 17: Experimental acceptance for kinematic 2

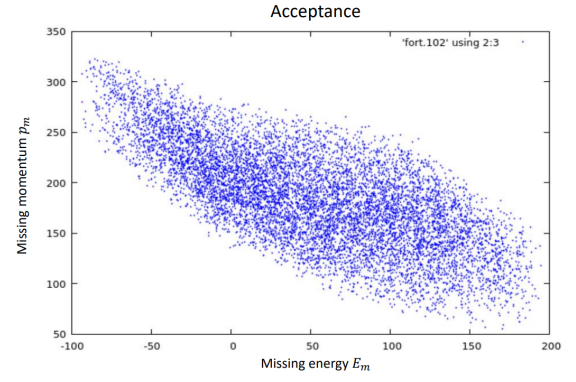


Figure 18: Experimental acceptance for kinematic 3

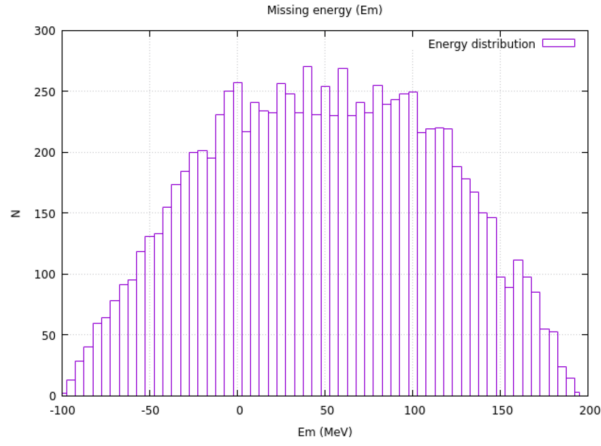


Figure 19: Acceptance (a.u.) as a function of missing energy from kinematic 2

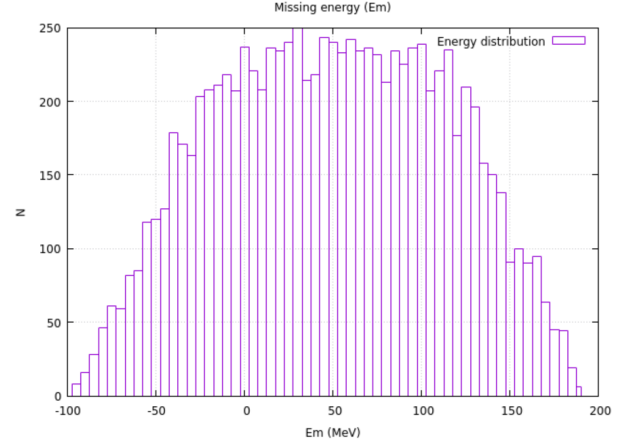


Figure 20: Acceptance (a.u.) as a function of missing energy from kinematic 3

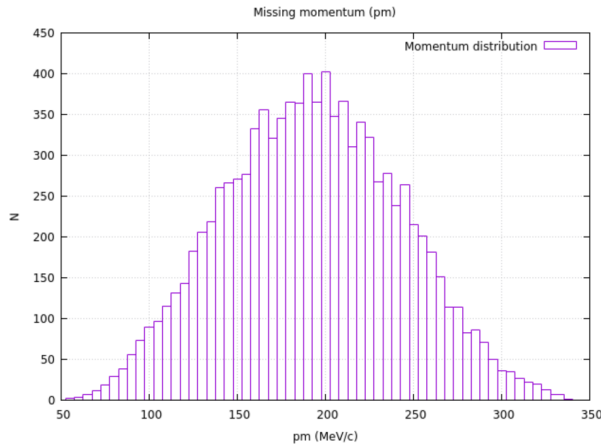


Figure 21: Acceptance (a.u.) as a function of missing momentum from kinematic 2



Figure 22: Acceptance (a.u.) as a function of missing momentum from kinematic 3

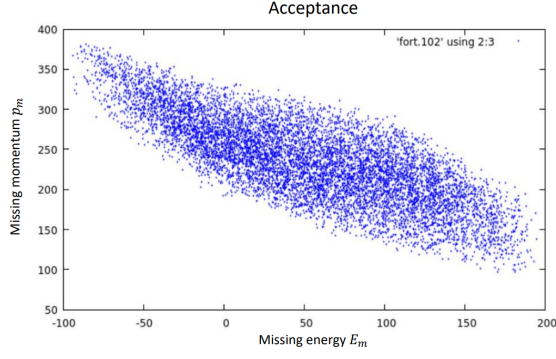


Figure 23: Experimental acceptance for kinematic 4

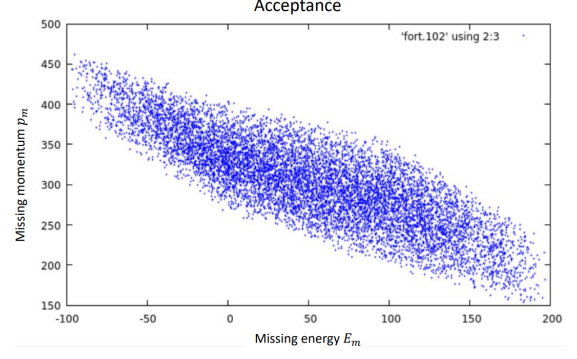


Figure 24: Experimental acceptance for kinematic 5

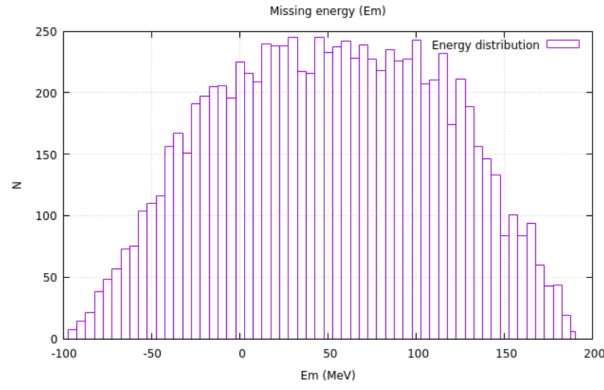


Figure 25: Acceptance (a.u.) as a function of missing energy from kinematic 4

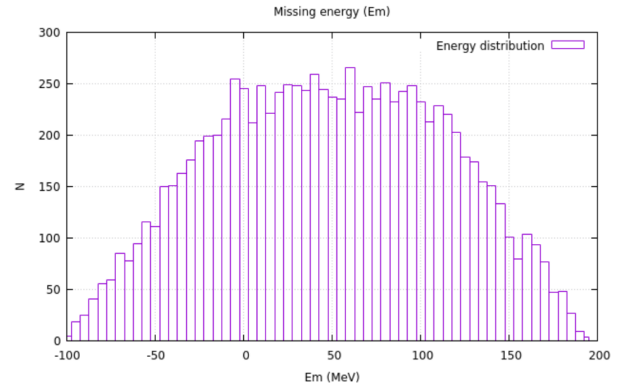


Figure 26: Acceptance (a.u.) as a function of missing energy from kinematic 5

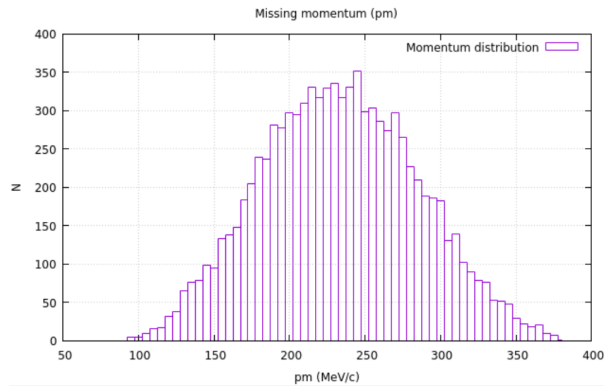


Figure 27: Acceptance (a.u.) as a function of missing momentum from kinematic 4

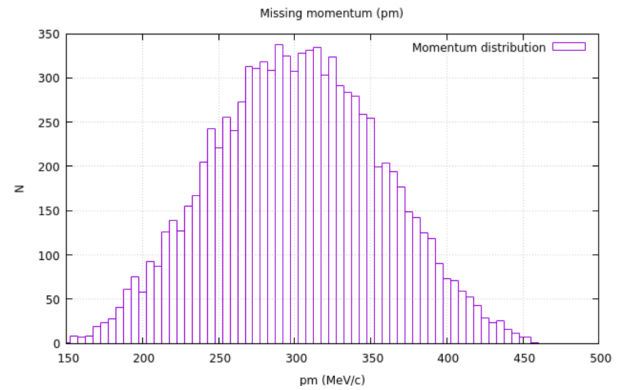


Figure 28: Acceptance (a.u.) as a function of missing momentum from kinematic 5

What we see is that for these kinematic settings we obtain similar results that are, in general, more uniform than the ones obtain with kinematic 1. Therefore, experimentalists might consider these configurations as more favorable when taking measurements.

5.3 Phase space comparison: 1N vs 2N channels

Using the main structure of this code, we can start to modify it to implement the physics of the scattering process and introduce the two-nucleon channel. The idea is to transition from a raw geometric phase space to physically weighted distributions.

We will first focus on the pure kinematic volume before implementing cross section calculations. For the one-nucleon emission channel, we will open an outer loop of $N_1 = 10000$, do as in the experimental acceptance code and generate a set of uniformly randomized variables around a set of central values for $\epsilon_f, \theta_e, \phi_e, p_N, \theta_N$ and ϕ_N . With energy-momentum conservation, we reconstruct the values for q^μ, E_m and p_m . For every accepted event with $E_m > S_{2p}$, we open a secondary loop for the two-nucleon emission channel of $N_2 = 300$, where we sample the second nucleon's momentum vector $\mathbf{p}_2$ over a sphere up to $p_{2,\max} = 1500$ MeV/c. Subtracting this value from the momentum of the residual system, we obtain the momentum for the $A - 2$ residual nucleus and its invariant mass M_{A-2} .

To prevent the generation of unphysical events, we must apply boundaries and constraints:

- First, we use a geometric constraint $\cos(\theta_{pq}) > 0.5$ to ensure the emitted proton's direction is approximately parallel to the transferred momentum.
- Then, to ensure that we obtain real physical states we establish the invariant mass squared of both the $A - 1$ and $A - 2$ residual nuclei are larger than zero, as well as the missing energy.
- For the inner loop, we must ensure that it is only open when there is a possibility of emitting a correlated pair. This happens when the missing energy is larger than the separation energy of two nucleons.
- Finally, for the 2N channel, we must ensure that the reconstructed invariant mass of the $A - 2$ core, M_{A-2} is greater or equal to its theoretical ground-state mass, M_{A-2}^0 .

To account for the integration over the undetected second nucleon: $d^3p_2 = p_2^2 dp_2 d(\cos \theta_2) d\phi_2$, we need to weigh the second channel by a differential volume factor:

$$w_{\text{PS}} = \frac{p_2^2 \cdot \text{vol}_{p_2}}{N_2} \quad (37)$$

With this we will see that the phase space for the 2N channel is overwhelmingly larger than the one for the 1N channel. This is because for every one-nucleon knockout event we are able to create a lot of different possible combinations for the distribution of the energy among the three final particles.

However, in reality, there are mechanisms that suppress the 2N channel contribution and instead we obtain figures like Figure 4. To model the actual behavior of the nuclear target, we can establish different types of weights and study the physically weighted distributions.

To implement this, we are going to use ^{48}Ti as the target nucleus and the kinematic configuration 1 from Table 3. And improve our code to make it closer to reality.

Just to see how the phase space would look like, we can establish that all the events are equally probable, by putting $w_{1N} = w_{2N} = 1$. However, by doing this, the contribution of the two-nucleon emission will be much larger than the one-nucleon channel, since our code is creating a lot of events for the 2N channel for every event of the 1N channel. Therefore, assigning equal weights to both

channels is not realistic. Instead, as a first approximation, we are going to assume the proton is emitted only from the outmost shell, in the case of ^{48}Ti this corresponds in the theoretical shell model to the $1f_{7/2}$ shell; and we are going to use gaussian distributions. For the one-nucleon emission channel, we model the ejected protons through a Gaussian distribution of the the reconstructed missing energy centered at $S_p = 11.45$ MeV with a width $\sigma_{E_m} = 2$ MeV. While for the two-nucleon emission, we weigh the events by using a Gaussian of the obtained residual mass M_{A-2} centered around its ground-state mass M_{A-2}^0 using a Gaussian width $\sigma_M = 50$ MeV. We can compare both cases: equal weights for both channels versus using the Gaussian weights.

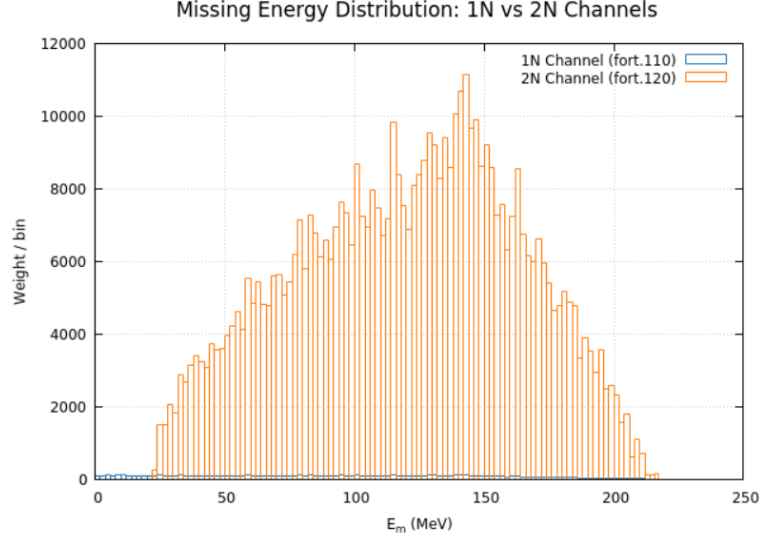


Figure 29: Volume of the phase space as a function of the missing energy for the one- and two-nucleon emission channels for ^{48}Ti using equal weights for both channels, as explained in Table 4, and the kinematic settings 1 from Table 3

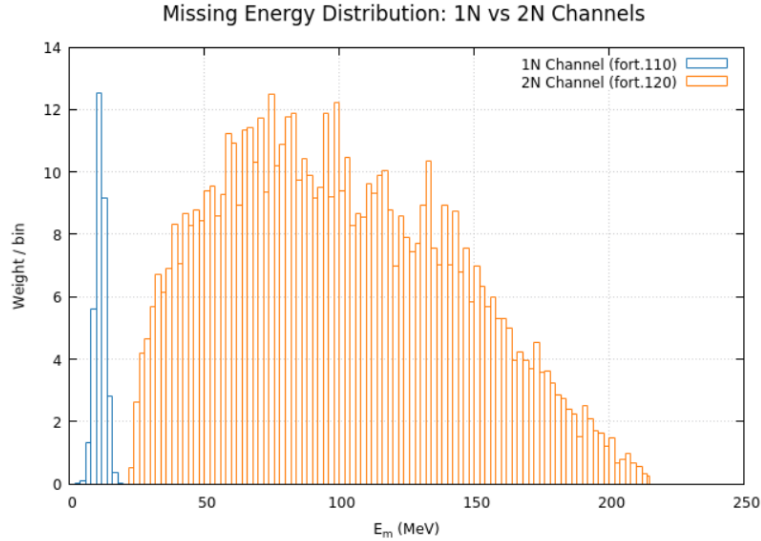


Figure 30: Volume of the phase space as a function of the missing energy for the one- and two-nucleon emission channels for ^{48}Ti using Gaussian distributions, as explained in Table 4, and the kinematic settings 1 from Table 3

For the case where the gaussian distributions weights were used, the channels are comparable, but the 2N contribution is still too high.

If we want to make this code more realistic, we can forget about using a gaussian approach for the 2N channel and apply the distribution of momentum of the correlated pair, $n(p_1, p_2)$, which is the probability of finding simultaneously the two protons with momentum p_1 and p_2 [31]. We describe the system using through the relative momentum of the pair, $\vec{k}_{rel}$ and the momentum of the center of mass, $\vec{K}_{CM}$:

$$\vec{k}_{rel} = \frac{\vec{p}_1 - \vec{p}_2}{2} \quad (38)$$

$$\vec{K}_{CM} = \vec{p}_1 + \vec{p}_2 \quad (39)$$

Factorizing,

$$n(p_1, p_2) \approx n_{rel}(\vec{k}_{rel})n_{CM}(\vec{K}_{CM}) \quad (40)$$

n_{CM} represents the movement of the correlated pair as a whole. It can be modeled through a gaussian distribution of the form:

$$w_{CM} = \exp\left(-\frac{K_{CM}^2}{2\sigma_{CM}^2}\right) \quad (41)$$

with σ_{CM} is usually between 140 and 170 MeV/c [47].

While n_{rel} shows the intensity of the interaction between the two correlated nucleons. Its shape it's not a Gaussian distribution, but rather a tail that decays slowly. To model this we can use a simple approach:

$$w_{rel} = \left(1 + \left(\frac{k_{rel}}{p_0}\right)^2\right)^{-2} \quad (42)$$

with a a cutoff factor of around $p_0 = 350$ MeV.

Or we can turn to Hulthén distribution, which decays more abruptly as $1/k^4$ and uses two scales of energy $\alpha_h = 200$ and $\beta_h = 1500$ to differentiate interactions at short and intermediate distances [37].

Finally, we also need the energy distribution ρ that represents the probability distribution of the residual nucleus. It represents how much energy was used to break up the correlated pair. For this, we introduce the variable $x_{Em} = E_m - B_{2N}$ that corresponds to the available energy after subtracting the binding energy of two nucleons to the missing energy. The form of this distribution is asymmetric with its peak fixed at $E_0 = 60$ MeV with a width of $\Gamma = 80$ MeV.

$$f_\rho \propto \frac{\sqrt{x_{Em}}}{(x_{Em} - E_0)^2 + (\Gamma/2)^2} \quad (43)$$

Therefore, the weight of the channel 2N turns out to be: $w_{2N} = occ_{2N} \cdot w_{rel} \cdot w_{CM} \cdot f_\rho$, where occ_{2N} is 30% to represent the percentage of nucleons that belong to correlated pairs [37].

If we try to represent this phase space, we obtain something of the form:

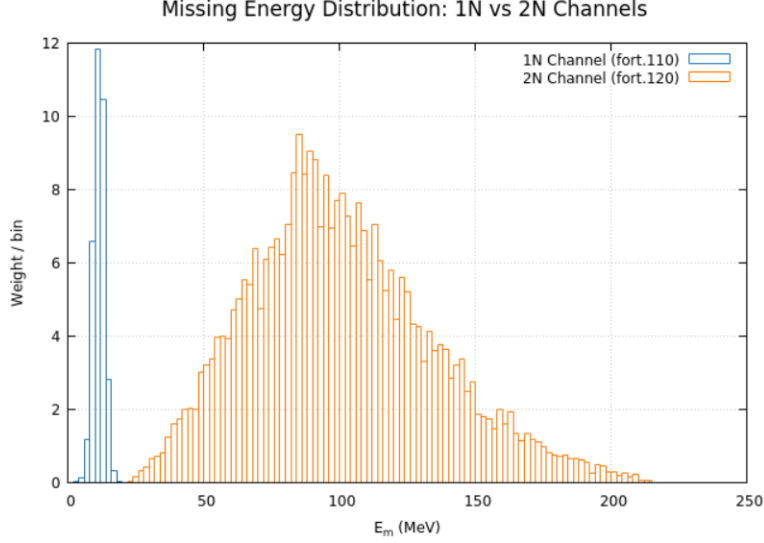


Figure 31: Volume of the phase space as a function of the missing energy for the one- and two-nucleon emission channels for ^{48}Ti using the momentum distribution of the correlated pair, as shown in Table 4, and the kinematic settings 1 from Table 3

However, this would not be useful if we want to compare it to real experimental data, since, in that case, we would need cross section values. To transform the pure phase space into a simulation of the real experiment, we need to introduce the electromagnetic cross section using the “de Forest CC1 cross section” [48], which corresponds to the σ_{ep} explained previously in the theoretical chapter, Eq. 16. We also need to add the kinematic factor, $K = p_N \cdot E_N$ to the weight and the spectral functions S_{1N} and S_{2N} , obtaining the following equations for the weights of each channel:

$$w_{1N} = occ_{1N} \cdot S_1 \cdot K \cdot \sigma_{cc1} \quad (44)$$

$$w_{2N} = (vol_{p2}/N_2) \cdot (p2 \cdot p2) \cdot S_2 \cdot K \cdot \sigma_{cc1} \quad (45)$$

where $occ_{1N} = 70\%$ and $occ_{2N} = 30\%$ are the probabilities of the nucleon being a free nucleon versus part of a correlated pair [37].

Spectral functions are usually complicated to compute, so instead we will use simplified versions. For the spectral function of the channel 1N, S_1 , we introduce the shell model, using a harmonic oscillator, making the spectral function the sum of the probabilities of each level:

$$S_1 = \sum_{i=1}^{n_{max}} (\text{Occupation}_i \times n_i(p_m) \times \rho_i(E_m)) \quad (46)$$

where we include the fractional occupation of each shell, the momentum distribution, $n_i(p_m)$, using the solutions from the harmonic oscillator, and the energy distribution, $\rho_i(E_m)$, modeled with the binding energy of each level.

For the spectral function of the channel 2N, S_2 , we simply use the momentum distribution of the correlated pair as previously explained, using Hulthén’s distribution to represent the tail at high momentum.

$$S_2 = occ_{2N} \cdot f n_{cm} \cdot f n_{rel} \cdot f \rho \quad (47)$$

To compare this with the previous images, we are going to assume that all the protons that are emitted belong to the outermost shell of ^{48}Ti , theoretically $1f_{7/2}$, so we will calculate S_1 assuming

that there is only one shell. With this program the contributions of the one-nucleon emission channel and the two-nucleon channel are no longer the phase space volume but represent actual differential cross sections, weighted by the de Forest CC1 off-shell prescription and realistic spectral functions.

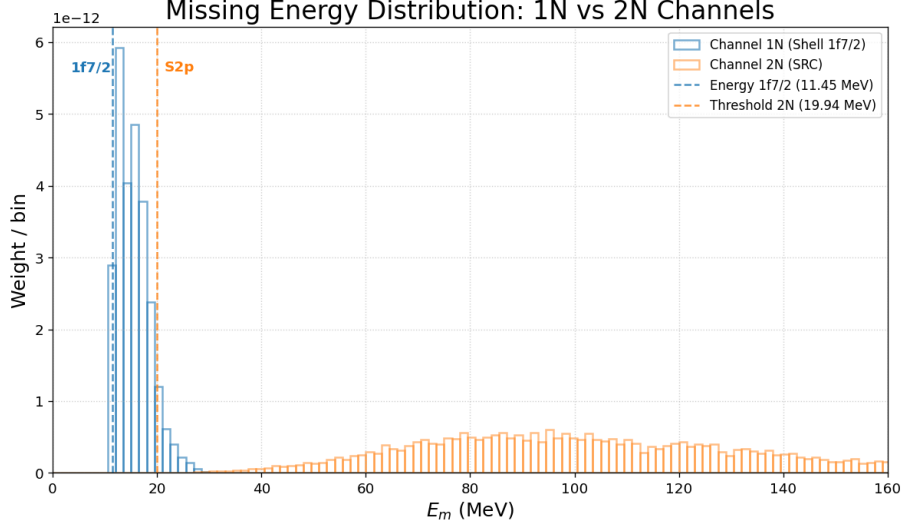


Figure 32: Differential cross section as a function of the missing energy for the one- and two-nucleon emission channels for the shell $1f_{7/2}$ of ^{48}Ti , calculated using the spectral functions and the de Forest CC1 off-shell prescription, Table 4. Using the kinematic settings 1 from Table 3

To make the progression clear, the physical inputs used in each figure of this section are summarized in Table 4. In this preliminary stage, we are not yet introducing a complete multi-shell nuclear structure model. Instead, the 1N knockout channel is parameterized solely by the single-proton separation energy $S_p = 11.45$ MeV (representing emission from the outermost shell in ^{48}Ti), while the 2N emission channel is activated strictly above the two-proton separation energy $S_{2p} = 11.94$ MeV.

Table 4: Summary of physical models and energy threshold parameters used for each figure in this section. No multi-shell structure is included at this stage; 1N emission ($ooc1N = 70\%$) is modeled considering only the single valence proton separation energy S_p , while 2N emission ($ooc2N = 30\%$) is controlled by the two-proton threshold S_{2p} .

Figure	Physical Approximation	1N Channel Model ($S_p = 11.45$ MeV)	2N Channel Model ($S_{2p} = 11.94$ MeV)
Fig. 29	Pure Unweighted Phase Space	Constant weight ($w_{1N} = 1$)	Constant weight ($w_{2N} = 1$, open for $E_m > S_{2p}$)
Fig. 30	Simple Gaussian Weighting	Gaussian in E_m centered at S_p ($\sigma = 2$ MeV)	Gaussian in M_{A-2} centered at M_{A-2}^0 ($\sigma = 50$ MeV)
Fig. 31	SRC Pair Correlations	Gaussian in E_m centered at S_p ($\sigma = 2$ MeV)	Hulthén $n_{rel} \cdot n_{CM} \cdot \rho(E_m - S_{2p})$ distribution
Fig. 32	Differential Cross Section	$\sigma_{CC1} \cdot K \cdot S_1$ (Single $1f_{7/2}$ state at S_p)	$\sigma_{CC1} \cdot K \cdot S_2$ (Hulthén + $\rho(E_m - S_{2p})$ continuum)

The next step is to incorporate both the p-shells and s-shells to the code, test it and compare it to different $(e, e'p)$ scattering experiments with different nuclei and kinematics, with the aim of seeing if the approximations taken are enough or if a more complex code is necessary to properly describe two-nucleon knockouts in electron-nucleus scattering.

5.4 Target analysis: $(e, e'p)$ in ^{12}C , ^{16}O and heavier nuclei

Once we have a program that simulates the emission of one and two nucleons from an electron-nucleus scattering we can start testing it against different nuclei and kinematics. For this, we will use data from $(e, e'p)$ experiments for some of the most common used targets in neutrino oscillation experiments, particularly from targets of ^{12}C , ^{16}O , ^{40}Ar and ^{48}Ti .

Firstly, ^{12}C was one of the targets that were mainly used in $(e, e'p)$ scattering experiments from the 80's and 90's to study the reaction mechanisms of the quasi-elastic region and its dependence on the momentum transfer. Therefore, there are many experimental data on this nucleus. However, the energies of the incident electron beam were lower than the ones used today. One of these experiments was the one conducted at the MIT Bates Linear Accelerator Center by Weinstein et al. in 1990, [21]. This experiment studied the missing energy spectra for the $^{12}\text{C}(e, e'p)$ reaction mechanisms in parallel kinematics ($p_f || q$) using incident beam energies of around 500 to 700 MeV. With this study, they were able to obtain the clear distributions of the 1p and 1s shells, as well as the broad continuum from the two-nucleon emission contribution. To do this, they used two spectrometers to detect the electron and proton in coincidence. We will compare their experimental results from Figure 1 b) of the article with our simulated results using the kinematic settings b) from Table I [21]. It is important to note, in the Figure we will try to recreate there are a series of dashed lines for the recoil momentum. This represents that the recoil momentum or missing momentum, p_m , doesn't stay constant, but instead varies as a function of the missing energy, E_m . Because the experiment was performed in parallel kinematics with fixed values of momentum transfer (q) and energy transfer (ω), the only way to scan across the different missing energies is by varying the final momentum of the detected proton. In our code, in order to span across all the desired values of the missing energy, we will perform the simulation six times with the central value of the momentum of the detected proton $|P|$ comprehended between 397.7–863.7 MeV/c in intervals of $\Delta p \approx 90$ MeV/c.

The next nucleus to study is the ^{16}O , which is an easy to model nucleus for its double closed shell. To validate our simulation we will use the experimental data from Figure 4 derived from the 1997 Jefferson Lab measurements published by Fissum et al. [28]. In this experiment, they used an unpolarized beam of electrons and a waterfall target that allowed for better calibration. Using High Resolution Spectrometers (HRS) they obtained values for the cross section for a wide range of missing momentum and missing energy and were able to extract the asymmetry (A_{LT}), as well as the structure functions (R_L , R_T , R_{LT} , y R_{L+TT}). Comparing the data to theoretical models, they pinpointed the need to include relativistic effects and two body currents (MEC). For our study, we will use the experimental data from Figure 25 from this article where they used an electron beam energy of 1.643 GeV and the central momentum of the proton varied from 828 MeV/c to 1190 MeV/c in five steps of $\Delta p_p \approx 70$ MeV/c per step.

Nowadays, neutrino oscillation experiments are studying heavier nuclei as possible targets to increase the statistics with more dense materials with higher number of neutrons. One of these suitable heavier nuclei is ^{40}Ar , which will be used in future generation neutrino oscillation facilities, such as DUNE. This nucleus presents a lot of benefits for detection of particles from lepton-nucleus scattering; however, it is a very asymmetric nucleus that complicates the theoretical modeling, due to short range correlations (SRC) and final state interactions (FSI). As mentioned before, to study the spectral function for the neutrons of ^{40}Ar we turn to the protons of ^{48}Ti . Therefore, a proper study of both nuclei is needed to use ^{40}Ar as target in neutrino oscillation experiments. To properly model this nuclei, there have been a series of $(e, e'p)$ experiments done in the past decade. One of these experiments is the E12-14-012 carried out in Jefferson Lab, whose objective

was obtaining the spectral function of the protons of both ^{40}Ar and ^{48}Ti , as explained in Chapter 5.2. For the data of the cross sections, we will use the results obtained by Gu et al. [43], where they used an electron beam of 2.222 GeV. Our goal will be to compare the missing energy and missing momentum distributions from our simulations, using the kinematics from Table 1 of this article, with Figures 3 and 4 obtained in the experiment for both nuclei.

For the kinematics, we will use the kinematics from Table 1 setting b) from [21] for ^{12}C , from Appendix B from [28] for ^{16}O and Table 1 from [43] for ^{40}Ar and ^{48}Ti .

Table 5: Central kinematic settings for $^{12}\text{C}(e, e'p)$, $^{16}\text{O}(e, e'p)$ and $^{48}\text{Ti}(e, e'p)$ with beam energy $E_{\text{beam}} = 0.698$ GeV, 1.643 GeV and 2.222 GeV, respectively.

	E'_e (GeV)	θ_e (deg)	Q^2 (GeV^2/c^2)	$ P $ (MeV/c)	$T_{p'}$ (MeV)	$\theta_{p'}$ (deg)	$ q $ (MeV/c)	p_m (MeV/c)	E_m (MeV)
^{12}C	0.3430	90	0.4788	397.7 - 863.7	80.0 - 337.0	26.2	775	-377.3 - 88.7	18 - 275
^{16}O	1.056	37.17	0.706	828 - 1190	313.1 - 577.1	38.45	1026	-198 - 164	11.9 - 275.9
^{40}Ar	1.777	21.5	0.549	915	372	-50.0	865	50	73
^{48}Ti	1.799	21.5	0.556	915	372	-50.0	857	58	51

We are now going to implement the different shells from the nuclei; therefore, we need the separation energies needed, as well as the widths and occupations introduced in Chapter 4.3 in Tables 1 and 2. These relevant parameters are summarized in the following table.

Table 6: Target-specific parameters for the spectral functions of ^{12}C , ^{16}O , ^{40}Ar , and ^{48}Ti . The table includes one- and two-proton separation energies (S_p , S_{2p}), and the parameters for the Gaussian energy distributions of the proton shells: centroid binding energy (E_α), intrinsic Gaussian width (σ_α), and the proton occupancy.

Nucleus	S_p (MeV)	S_{2p} (MeV)	Shell (α)	E_α (MeV)	σ_α (MeV)	Occupation
^{12}C	15.96	27.19	$1p_{3/2}$	15.50	2.50	4
			$1s_{1/2}$	36.90	8.00	2
^{16}O	12.13	22.34	$1p_{1/2}$	12.10	1.00	2
			$1p_{3/2}$	18.40	3.00	4
			$1s_{1/2}$	40.00	8.00	2
^{40}Ar	12.53	22.76	$1d_{3/2}$	12.53	2.00	4
			$2s_{1/2}$	12.93	2.00	2
			$1d_{5/2}$	18.23	4.00	6
			$1p_{1/2}$	28.00	8.00	2
			$1p_{3/2}$	33.00	8.00	4
			$1s_{1/2}$	52.00	8.00	2
^{48}Ti	11.44	19.94	$1f_{7/2}$	11.45	2.00	2
			$2s_{1/2}$	12.21	2.00	2
			$1d_{3/2}$	12.84	2.00	4
			$1d_{5/2}$	15.46	4.00	6
			$1p_{1/2}$	35.00	8.00	2
			$1p_{3/2}$	40.00	8.00	4
			$1s_{1/2}$	62.00	8.00	2

In addition, recalling Chapter 5.3. the following table shows the common parameters used to describe the momentum distributions implemented into our simulation code:

Table 7: Phenomenological parameters used across the simulation code to model the momentum distributions, the two-nucleon (2N) continuum background. The total two-body fraction is scaled to 30% ($\text{occ2N} = 0.3$) of the cross section.

Distribution Component	Parameter	Value	Physical Description
Energy distribution $f_p(E_m)$	E_0	60.0 MeV	Centroid of the Lorentzian profile
	Γ	80.0 MeV	Width (FWHM) of the Lorentzian profile
	B_{2N}	S_{2p}	2N emission threshold
C.M. Momentum $n_{cm}(P_{cm})$	σ_{cm}	140.0 MeV/c	Gaussian width of the pair C.M. momentum
Relative Momentum $n_{rel}(k_{rel})$	α_h	200.0 MeV/c	Soft scale (Hulthén-like inner SRC)
	β_h	1500.0 MeV/c	Hard scale ($1/k^4$ tail limit)

Inserting these parameters in our program, we can study the different nuclei of interest.

- ^{12}C

With our simulation we obtain weighted distributions of the missing energy and missing momentum.

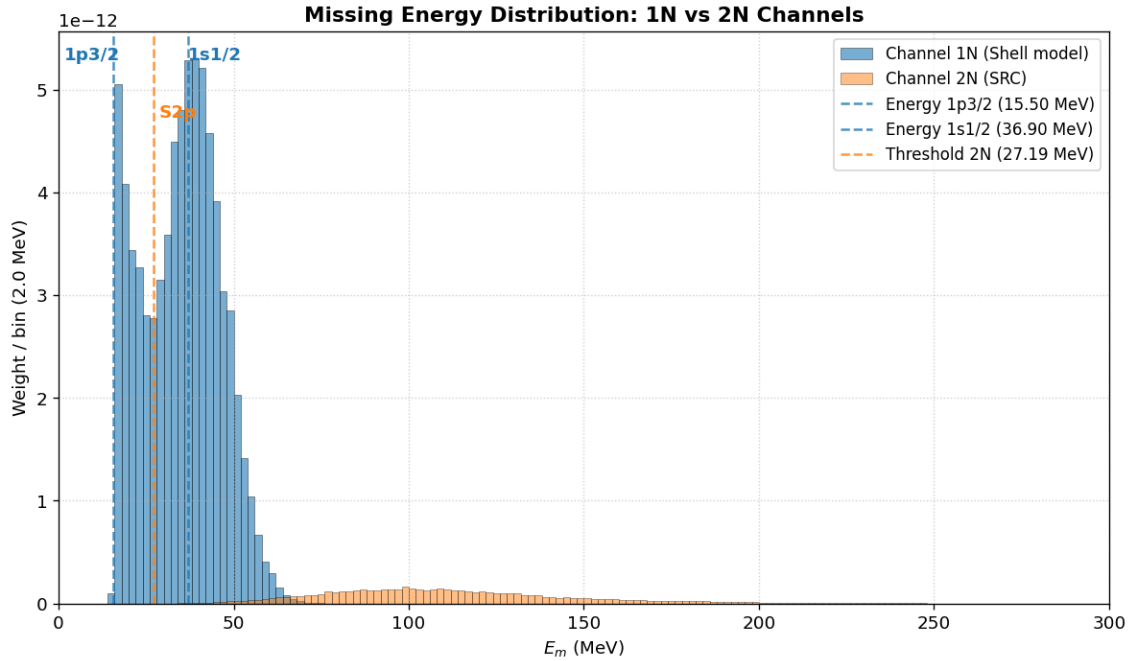


Figure 33: Simulated missing energy histogram for the $^{12}\text{C}(e, e'p)$ scattering using the kinematics settings from Table 5 and parameters from Tables 6 and 7, separated into the one-nucleon emission (blue) and two-nucleon emission (orange) contributions. The events have been scaled by their respective statistical weights w_{1N} and w_{2N} . The width of the bin is 2 MeV.

In the case of ^{12}C , we want to compare our simulated missing energy distribution to the cross section experimental six-fold differential cross section from reference [21]. To do this we need to rescale our simulation to be able to make the comparison obtaining the following figure:

$^{12}\text{C}(e, e'p)$ missing energy spectrum from L. B. Weinstein et al. (1990)
 $E_e=0.698$ GeV, $e_f=343$ MeV, $\theta_e=90^\circ$, $|\mathbf{p}_N|=397.7 - 863.7$ MeV/c, $\theta_N=26.2^\circ$ (MC normalized in shape to exp area)

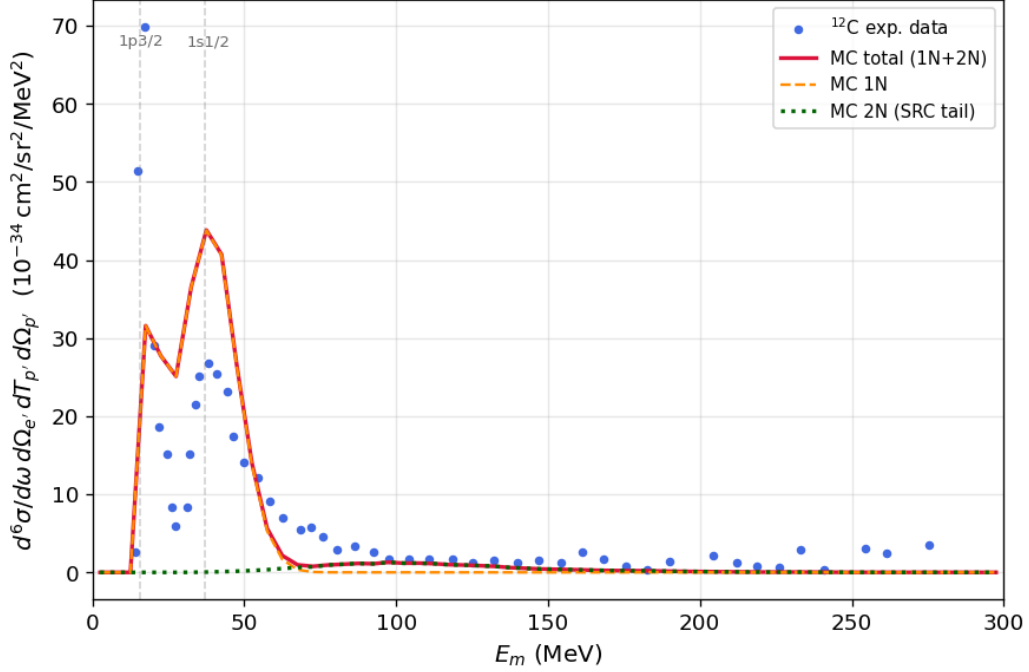


Figure 34: Differential cross section $d^6\sigma/d\omega d\Omega_e dT_{p'} d\Omega_{p'}$ as a function of the missing energy (E_m) for the reaction $^{12}\text{C}(e, e'p)$, with a beam energy of $E_{beam} = 0.698$ GeV. The blue dots correspond to the experimental data from Weinstein et al. (1990) [21], while the red line is the total Monte Carlo simulation which corresponds to the sum of the 1N and 2N channels. The MC simulation (performed with settings for ^{12}C from Tables 5, 6 and 7) has been normalized to the area of the experimental data to compare the structure.

In both Figure 33 and Figure 34, we can see the two discrete peaks from the p-shell and the s-shell, where the s-shell is wider than the p-shell. If we compare the simulation with the experimental data, we observe that, even though the overall behavior is very similar, the p-shell peak is lower than the expected result and the s-shell is slightly larger. This could be due to the nature of the simulations. If the proton's momentum chosen makes the intervals to have overlapping values of E_m this can make the distribution in the middle to have more events than it would happen in reality. To solve this computationally, it is possible to reduce the number of momentum intervals to reduce the overlapping between them. On the other hand, we can see the two-nucleon knockout channel is not very prominent for this kinematic setting. For E_m around 75 MeV we can see that the experimental data is slightly larger than the simulated values. this could be due to a not wide enough s-shell distribution or the need for a larger contribution from the 2N channel.

• ^{16}O

If we try to simulate ^{16}O we expect the peaks from the two p-shells, $1p_{1/2}$ and $1p_{3/2}$, as well as a clear and broader s-shell distribution. Since the $1p_{1/2}$ and the $1p_{3/2}$ levels are very close in energy it will be more difficult to differentiate from one another, compared to the s-shell. For this nuclei, we have experimental data for both the cross section as a function of the energy and as a function of the missing momentum, so we can study both:

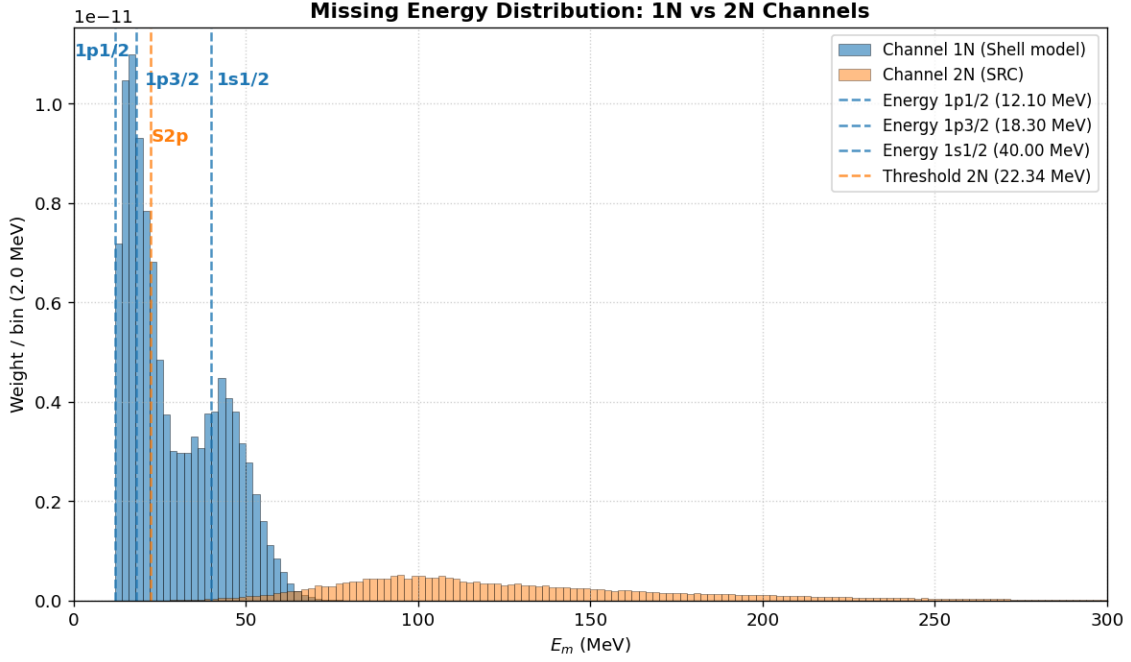


Figure 35: Simulated missing energy histogram for the $^{16}\text{O}(e, e'p)$ scattering using the kinematics settings from Table 5 and parameters from Tables 6 and 7, separated into the one-nucleon emission (blue) and two-nucleon emission (orange) contributions. The events have been scaled by their respective statistical weights w_{1N} and w_{1N} . The width of the bin is 2 MeV.

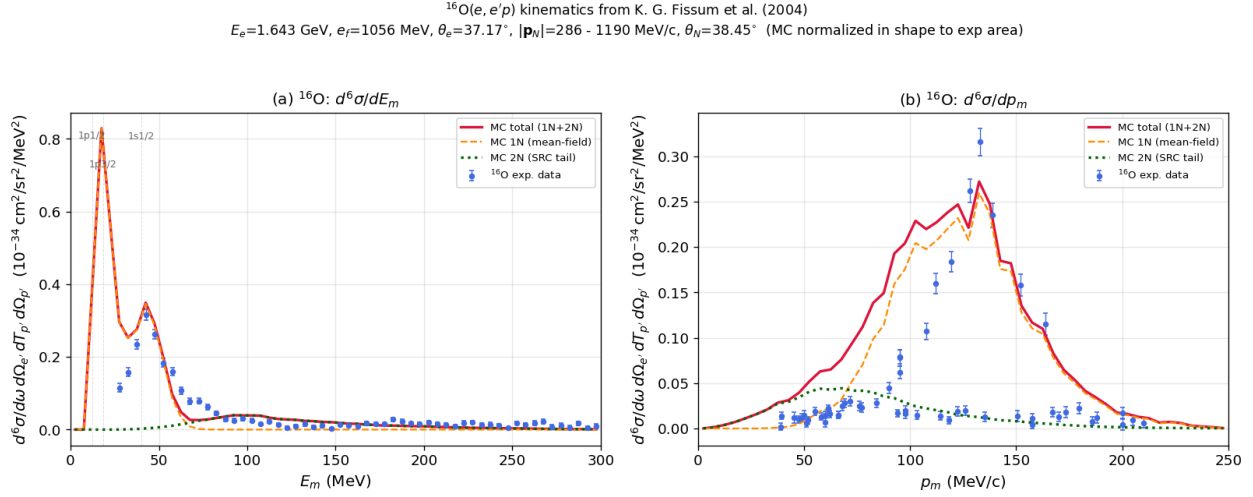


Figure 36: Differential cross section $d^6\sigma/d\omega d\Omega_{e'} dT_{p'} d\Omega_{p'}$ as a function of a) the missing energy (E_m) and b) the missing momentum (p_m) for the reaction $^{16}\text{O}(e, e'p)$, with a beam energy of $E_{beam} = 1.643$ GeV. The blue dots correspond to the experimental data from Fissum et al. (2004) [28], while the red line is the total Monte Carlo simulation which corresponds to the sum of the 1N and 2N channels. The MC simulation (performed with settings for ^{16}O from Tables 5, 6 and 7) has been normalized to the area of the experimental data to compare the structure.

For ^{16}O we obtain a simulated missing energy distribution that is very similar to the experimental data. As in the case of ^{12}C around $E_m = 75$ MeV the experimental data sits above our simulation.

This is the point where the one-nucleon emission stops being predominant and the two-nucleon emission starts becoming the main reaction mechanism. The possible reasons behind this discrepancy are the need for a wider distribution for the s-shell and the need for a slightly larger contribution from the two-nucleon emission channel, which will make the contribution from the SRC tail slightly higher at lower energies. If we see the theoretical model done by Ghent's group in Figure 4, we can see the s-shell distribution stops at a further value of E_m and the 2N contribution is slightly higher. For the missing momentum, the overall shape is very similar. We can see here that there are two sets of experimental data for the missing momentum, one that goes to higher values of the cross section and another that stays at lower values. If we compare them to the contributions from the 1N and 2N channels we can see that the high cross section results are fitted quite well by the 1N simulation while the 2N simulation follows the shape of the lower cross section results. Therefore, we can conclude the set of higher cross section as a function of the missing momentum belong to free nucleons inside the nucleus while the low cross sections results come from nucleons that belonged to correlated pairs inside the initial nucleus.

- ^{40}Ar and ^{48}Ti

Moving on to heavier nuclei, there are now many more shells. This is going to create an overlapping of the discrete peaks, and we are going to observe a continuous distribution from the one-nucleon emission contribution. The two-nucleon emission channel will start for both ^{40}Ar and ^{48}Ti at around the two-nucleon separation threshold ($E_m \approx 20$ MeV) and will slowly grow to form a broad, relatively flat continuum at higher missing energies.

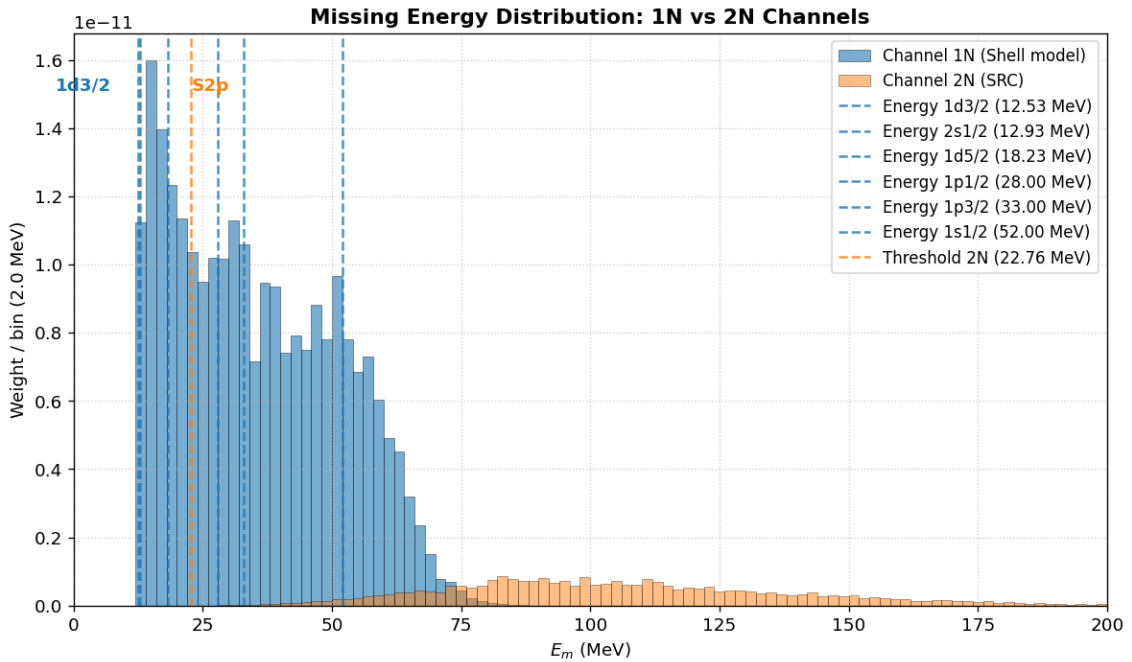


Figure 37: Simulated missing energy histogram for the $^{40}\text{Ar}(e, e'p)$ scattering using the kinematics settings from Table 5 and parameters from Tables 6 and 7, separated into the one-nucleon emission (blue) and two-nucleon emission (orange) contributions. The events have been scaled by their respective statistical weights w_{1N} and w_{2N} . The width of the bin is 2 MeV.

$^{40}\text{Ar}(e,e'p)$ kinematics from Gu et al., PRC 103, 034604 (2021)
 $E_e=2.222$ GeV, $e_f=1777$ MeV, $\theta_e=21.5^\circ$, $|\mathbf{p}_N|=915$ MeV/c, $\theta_N=50^\circ$ (MC normalized in shape to exp area)

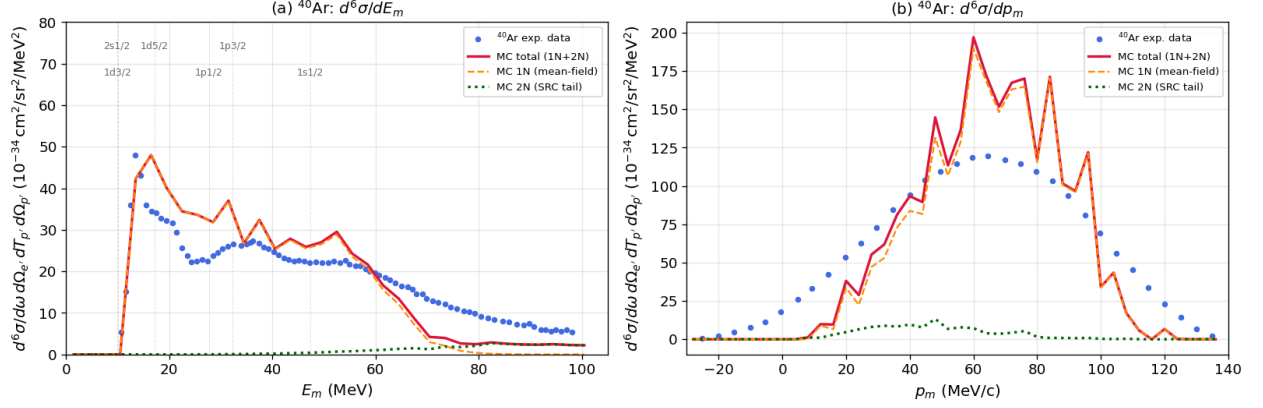


Figure 38: Differential cross section $d^6\sigma/d\omega d\Omega_e dT_p d\Omega_{p'}$ as a function of a) the missing energy (E_m) and b) missing momentum (p_m) for the reaction $^{40}\text{Ar}(e, e'p)$, with a beam energy of $E_{beam} = 2.222$ GeV. The blue dots correspond to the experimental data from Gu et al. (2021) [43], while the red line is the total Monte Carlo simulation which corresponds to the sum of the 1N and 2N channels. The MC simulation (performed with settings for ^{40}Ar from Tables 5, 6 and 7) has been normalized to the area of the experimental data to compare the structure.

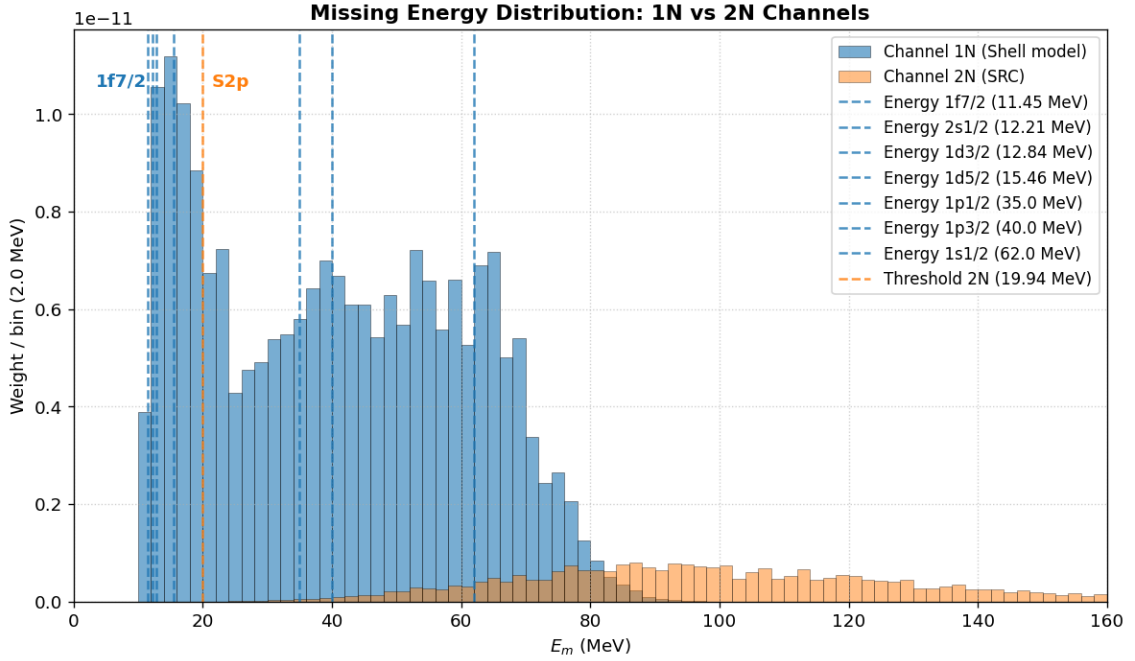


Figure 39: Simulated missing energy histogram for the $^{48}\text{Ti}(e, e'p)$ scattering using the kinematics settings from Table 5 and parameters from Tables 6 and 7, separated into the one-nucleon emission (blue) and two-nucleon emission (orange) contributions. The events have been scaled by their respective statistical weights w_{1N} and w_{2N} . The width of the bin is 2 MeV.

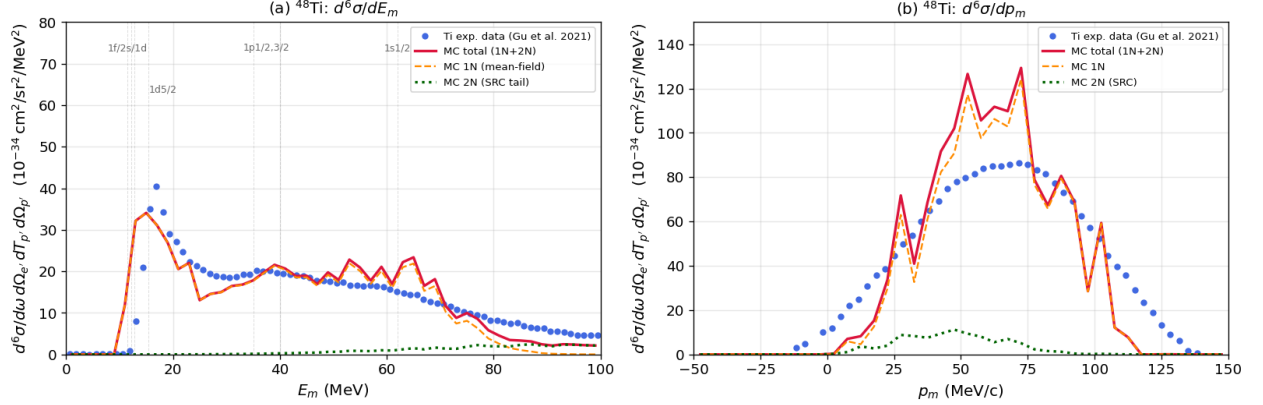


Figure 40: Differential cross section $d^6\sigma/d\omega d\Omega_e dT_p d\Omega_{p'}$ as a function of a) the missing energy (E_m) and b) missing momentum (p_m) for the reaction $^{48}\text{Ti}(e, e'p)$, with a beam energy of $E_{beam} = 2.222$ GeV. The blue dots correspond to the experimental data from Gu et al. (2021) [43], while the red line is the total Monte Carlo simulation which corresponds to the sum of the 1N and 2N channels. The MC simulation (performed with settings for ^{48}Ti from Tables 5, 6 and 7) has been normalized to the area of the experimental data to compare the structure.

In Figures 38 and 40, we can see the simulated results present sharp peaks and a jagged appearance. This is a result to the binning in missing energy required for the statistics of the Monte Carlo event generation. Bins with a low number of simulated events (N) suffer from large relative statistical fluctuations. The normalization used to compare the simulation with the experimental data magnifies this effects creating the zig-zag effect. Using thinner bins and increasing the number of simulated events will reduce this effect.

In spite of this, the simulations fit the overall shape of the experimental data. For higher missing energies the contribution from the 2N channel becomes insufficient to justify the experimental data. This could be just the fact that we haven't added final state interactions (FSI) and other multinucleon emissions that happen at a thresholds beyond three nucleon emission.

To illustrate the kinematic distribution of the simulated events in the phase space, we can do as in Chapter 5.2 and study the 2D map of E_m versus p_m to see the total density. In this plot, each event is weighed by the elementary electron-nucleon cross section (σ_{ep}), the phase-space kinematic factor, and the spectral function that models both the titanium shell structure and the high-momentum tail associated with short-range correlations (SRC).

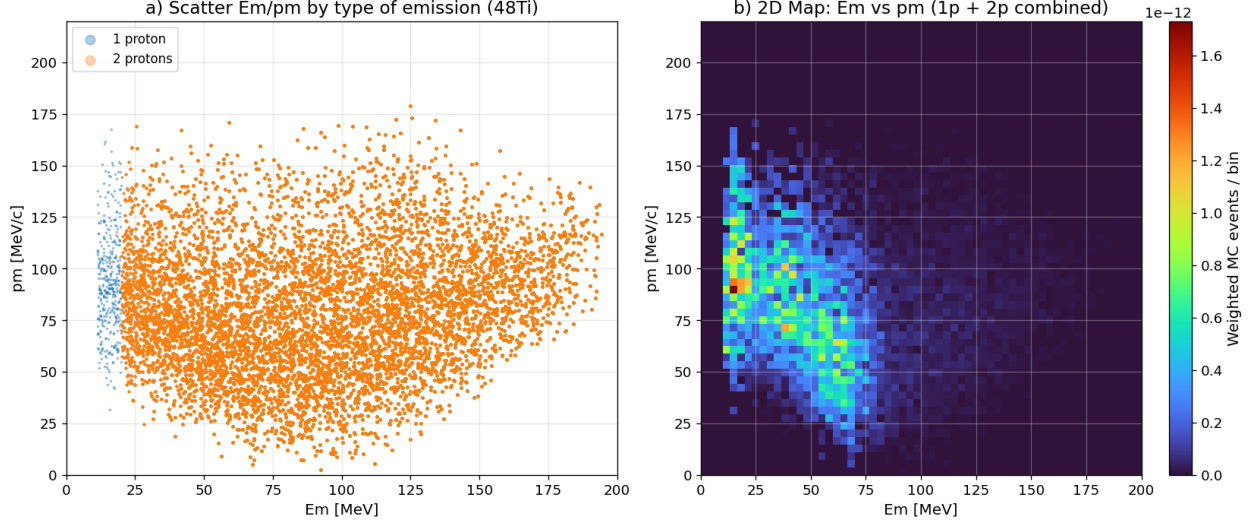


Figure 41: 2D acceptance for $(e, e'p)$ in ^{48}Ti . a) Scatter plot (E_m, p_m) . One-nucleon emission events are represented in blue while two-nucleon emission events are in orange. b) k-density plot of the six-fold cross-section versus E_m and p_m for ^{48}Ti . Both one-nucleon and two-nucleon emissions are present and weighted by w_{1N} and w_{2N} , respectively.

The color scale reflects the expected physical rate, allowing for visual identification of which regions of the E_m - p_m plane are populated under this particular configuration. Comparing Figure 41 a) with b) we can clearly see where do we expect more events, within the acceptance. This 2D map is a valuable tool for understanding how reaction mechanisms and kinematic constraints shape the event distribution across phase space in lepton-nucleus scattering.

5.4.1 Factorized Spectral function

Instead of modeling separately the 1N and 2N emission contributions, from kinematics relationship and simple models of the one-hole missing and momentum distributions, and the same for the 2N contributions, we can compare, within the same factorized image, with the results obtained with a semi-phenomenological spectral function which, in principle, will contain both 1N, 2N and multi-nucleon effects. As there are much better data in detailed spectral functions in ^{16}O , we will use it in this comparison.

We start with this ansatz: $S(p_m, E_m) = n(p_m)\rho(E_m)$ and use a parametrization similar to the one presented in reference [49]. The differential cross section evaluates the off-shell single-nucleon De Forest CC1 prescription [48], a phase-space kinematic factor ($K = p_p E_p$), a recoil correction (f_{rec}), and the effective nuclear spectral function (S_{eff}).

This nuclear spectral function contains the contribution from pure one-hole states, plus an explicit continuum background:

One-hole States ($p_{1/2}, p_{3/2}, p_{3/2}^*$): The single-particle energy distributions $\rho(E_m)$ are modeled via Gaussian profiles centered at $E_m = 12.1$ MeV, 18.4 MeV, and 23.0 MeV, respectively.

Deep Bound (resonance) one-hole ($s_{1/2}$): Features an asymmetric energy distribution starting at $E_m \geq 22.3$ MeV with mean parameters $\mu = 37.0$ MeV and $\sigma = 19.0$ MeV based on the Franco-Patiño model [50]. The momentum distributions $n(p_m)$ for all one-hole shells are numerically

interpolated from tabulated calculations from the relativistic mean field calculations for these nuclei [27], [51].

Continuum Background: To account for multi-nucleon emission (2N topologies) and short-range correlations (SRC), a continuous background is introduced for $E_m \geq 26.0$ MeV. It adopts a piecewise exponential decay in energy [50] and an analytical momentum dependence:

$$n_{\text{bg}}(p_m) \propto \frac{1}{\left(\alpha_{\text{bg}}^2 + (p_m/\hbar c)^2\right)^4} \quad (48)$$

where $\alpha_{\text{bg}} = 0.7 \text{ fm}^{-1}$.

The total effective spectral function S_{eff} combines each channel weighted by its respective spectroscopic factor (Z) and a shell-dependent Final State Interaction (FSI) transparency coefficient (T):

$$S_{\text{eff}}(p_m, E_m) = T_p \left(Z_{p_{1/2}} S_{p_{1/2}} + Z_{p_{3/2}} S_{p_{3/2}} + Z_{p_{3/2}}^* S_{p_{3/2}}^* \right) + T_s Z_{s_{1/2}} S_{s_{1/2}} + T_{\text{bg}} S_{\text{bg}} \quad (49)$$

The transparency factors are set to $T_p = 0.70$ for p -shell protons and $T_s = T_{\text{bg}} = 0.55$ for the s -shell and continuum background, providing a realistic absolute normalization when compared with experimental quasielastic cross sections. Occupancies and transparencies are taken from the adopted values extracted from many experiments, as reviewed in K.Fisumm et al. [28].

With this more realistic spectral function, we can simulate Figures 42, which corresponds to Figure 21 from reference [28], which reports all the results from experiment E89003 of JLab, in which the GFN-UCM contributed both to the data taken, analysis of data and comparison to theory. This Figure presents the six-fold differential cross section as a function of the missing energy for a specific kinematic of the lepton arm, but changing the angle θ_{pq} which allows us to cover several ranges of p_m . To obtain them, we have folded in the acceptance of the JLAB experiment as mentioned in [28].

Table 8: Central kinematic settings for the $^{16}\text{O}(e, e'p)$ reaction with beam energy $E_{\text{beam}} = 2.445 \text{ GeV}$ from Figure 21 ref. [28].

	E'_e (GeV)	θ_e (deg)	$ P $ (MeV/c)	$ \theta_{pq} $ (deg)	p_m (MeV/c)
^{16}O	2.000	23.36	1000.3	2.5, 8.0, 16.0, 20.0	50.0 - 340.0

To obtain the following figure, we have implemented in the code the following parameters to represent the different shells and channels to compute the factorized spectral function:

Table 9: Theoretical parameters for the spectral function adapted to ^{16}O

Shell/Channel	Energy E_m (MeV)	width σ (FWHM) (MeV)	Occupation (Z)	Transparency (T)
$p_{1/2}$	12.1	0.6370 (1.5)	0.60	0.70
$p_{3/2}$	18.4	0.6370 (1.5)	0.65	0.70
$p_{3/2}^*$ (doublet state)	23.0	0.8493 (2.0)	0.20	0.70
$s_{1/2}$	≥ 22.3 ($\mu = 37.0$)	19.0000	0.70	0.55
Background (2N)	≥ 26.0	$\alpha_{\text{bg}} = 0.7 \text{ fm}^{-1}$	—	0.55

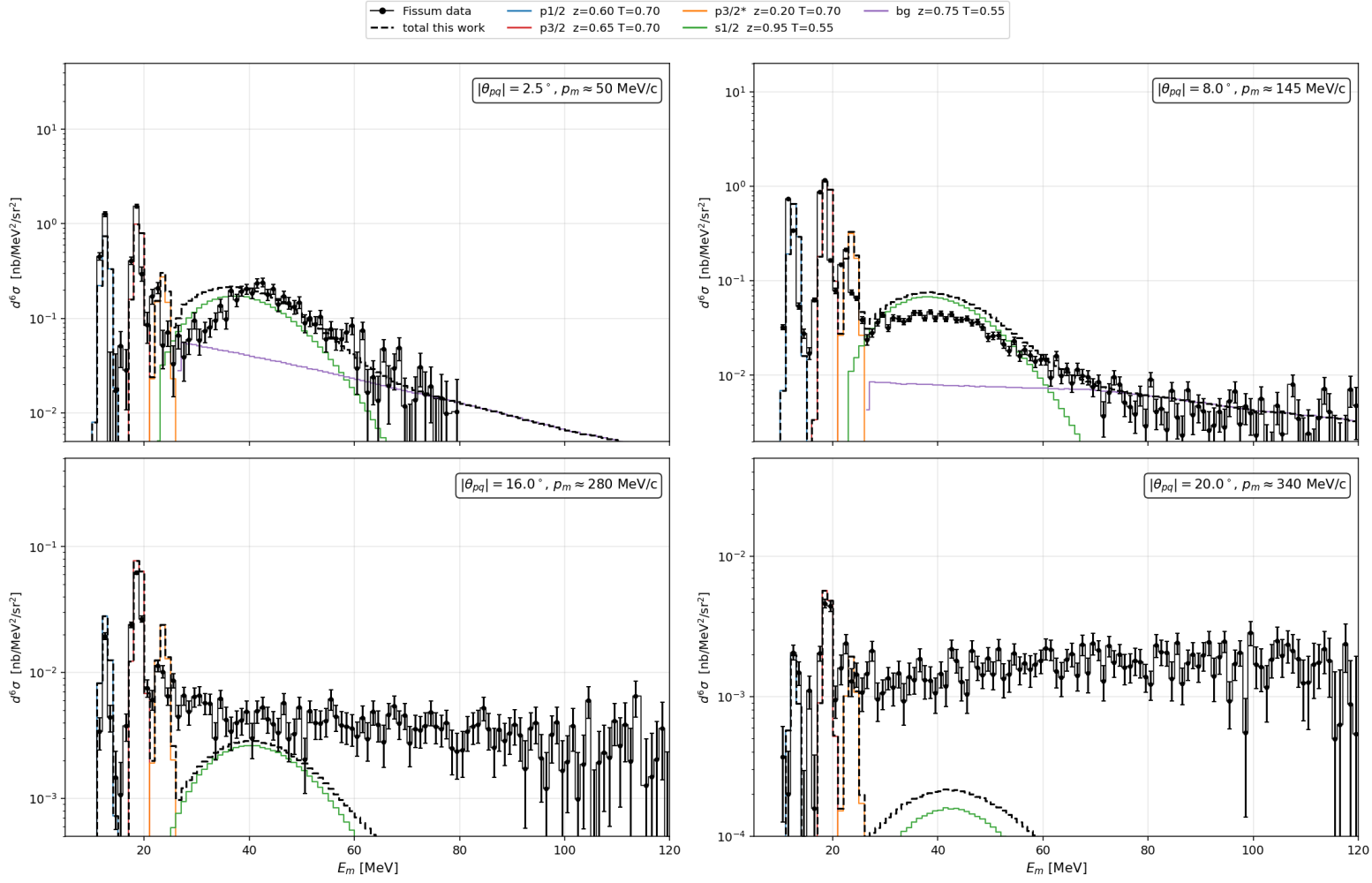


Figure 42: Figure 21 from reference [28] simulated with a factorized approach from a realistic spectral function and FSI as mentioned in the text. Green line: contribution of the one-hole s -state. Purple line, background contribution, orange line, contribution of the other one-hole states introduced in the theory calculation. Dashed line: total theoretical prediction for the cross-section. Histograms with error bars, experimental data.

We can see in Figure 42 that the cross-section below the two-nucleon threshold is very well reproduced in the whole range of missing momentum, indicating that the one-hole momentum distributions employed reproduces well the experiment in this region. At the kinematics corresponding to the lowest missing momentum, at high missing energy, the cross-section is dominated by the broad contribution of the one-hole s -wave, with some contribution of the background terms. The background dominates the results at high missing energies for the second kinematics at $p_m = 145$ MeV/c, and it reproduces the data well. At larger missing momentum and missing energies, however, the theoretical prediction is clearly below the data. Processes beyond the Impulse Approximation such as meson exchange currents, and multi-nucleon mechanisms have been involved to explain the data in this region.

6 Conclusion

In this work, we have employed the electron-nucleus scattering framework and experiments as a way to disentangle the nuclear response, that is, the energy and momentum that the nuclear system absorbs when it is scattered by a virtual photon. We focused on $(e, e'p)$ experiments, which allows to build the phenomenological spectral functions. We have built a MC simulation code to populate the experimental phases space (trajectories and momentum of the lepton and proton) which, coupled with the factorization approach, allows us to make predictions for the cross-sections measured in modern facilities. We advocate that the nuclear response derived in this way may be employed to help improve the description of the nuclear targets in neutrino oscillation experiments. In particular, we have studied the contribution of 1N and 2N channels to the experimental signals, modeling the different reaction mechanisms, specifically the channels where one proton is emitted and detected, and where one proton is detected but more nucleons are emitted because they were correlated in the initial nucleus through the strong nuclear force. The latter is one of the main sources of systematic uncertainties in neutrino oscillation experiments, thus its importance when generating theoretical models.

Our Monte Carlo simulator models the acceptance of the experiments within the kinematics of one-nucleon (1N) and two-nucleon (2N) emission channels. The initial implementation allowed for the detailed mapping of the experimental acceptance, replicating the geometric and operational constraints of high-resolution spectrometers from typical facilities (such as those of the Jefferson Lab E12-14-012 experiment). To study how the available phase space changes when going from a two body final state (single-nucleon knockout) to a three-body final state (two-nucleon knockout), we transitioned to weighted distributions, obtaining a code that allows us to map the phase space and reconstruct the values of the missing energy, E_m , and the missing momentum, p_m . By implementing the Plane Wave Impulse Approximation (PWIA), realistic Spectral Functions (combining the shell model and SRC momentum distributions), and the off-shell de Forest CC1 electron-nucleon cross-section, we were able to obtain realistic six-fold differential cross sections which can be later compared with experimental data from relevant nuclei in the field of neutrino experiments (^{12}C , ^{16}O , ^{40}Ar and ^{48}Ti).

The results from ^{40}Ar and ^{48}Ti are of special interest since they are complex nuclei, whose spectral function is difficult to model. Therefore, a good understanding of their nuclear structure is necessary, since ^{40}Ar will be the target nucleus used in next generation neutrino oscillation facilities, such as DUNE. Using $(e, e'p)$ experiments helps us validate our model to see if our program can correctly describe the phase space of this nuclei.

With our code we have been able to obtain results that fit the overall shape of the experimental data, showing the peaks of the p-shells and the broader distribution of the s-shell, as well as the continuous tail of the 2N channel, where it starts for $E_m > S_{2p}$, grows slowly and then becomes the predominant channel for higher missing energies. However, there are still disagreements with the experimental values. For smaller nuclei (^{12}C , ^{16}O), the simulated data in the region where the s-shell starts to decrease and the 2N channel becomes more important is lower than expected, due to an underestimation of the width of the s-shell or the strength of the 2N channel, or the lack of other processes present. While for heavier nuclei (^{40}Ar , ^{48}Ti), this underestimation of the simulated data happens at higher energies where FSI and other multinucleon channels become important, which makes the 2N channel no longer enough to describe the events. The MC developed here will help to identify the regions of the phase-space populated in neutrino experiments, for instance, and to look for experimental regions where the 1N contributions makes most of the events gathered, as

these are the ones that will provide the most accurate estimations of the neutrino energies.

We have also compared to data of one of the most comprehensive study of the nuclear response of ^{16}O available, obtained at JLAB. These data are a very demanding test for the validity of the extraction of the spectral function. In this case, we modeled the cross-section directly from a phenomenological spectral function.

The overall fair agreement with the data, in large region of missing energy and momentum, that we obtain with relatively simple models, makes us believe that the factorization approach works well to describe the bulk of the nuclear response. This factorization approach enables us to apply our findings in $(e, e'p)$ experiments directly into the neutrino nucleus arena, by simply substituting the σ_{ep} by the corresponding elementary neutrino-nucleon cross-section.

These codes and results provided a good start and solid foundation for our assumptions, but there are several things that can be done to improve it:

- First thing is to chose more realistic momentum distributions, and more detailed models of FSI, that here are just taken as transparency constant factors. The actual code uses a harmonic oscillator for Ti to describe the shell model which falls short when trying to describe in detail the experiment. For ^{16}O the momentum distributions are more realistic, and the agreement with data is also superior.
- Using a spectral function in titanium that correctly describes the nuclear structure across different energy and momentum and has been validated against simple nuclei, as we did for oxygen, instead of using an approximated constant value.
- We might also want to do a study on how to set the width of the missing energy distributions according to resolution of the different experiments, that is, we will need to introduce the energy resolution into the MC simulation of the experiment, so that we can properly use the natural widths of the resonances in the modeling, and obtain the experimental spectra of apparent missing energies, without fitting the width of the one-hole states.
- Finally, we need to add more than two nucleon emission contributions and more realistic final states interactions (FSI). The latter is usually done through a cascade model of with an optical potential, to describe how the emitted proton interacts with the residual nucleus, potentially emitting new particles.

The study and the program generated throughout this work has been done with the mindset of transitioning into neutrino-nucleus scattering in future projects. To do this, it is necessary to swap σ_{ep} by σ_ν , turning this model into a functional generator for the experimental community. In this way, we will be able to validate the theoretical models with recent data that uses neutrino-nucleus scattering and give experiments such as DUNE tools to optimize the kinematic cuts to obtain clearer signals for the desired $1p$ region, and improve energy reconstruction algorithms, ultimately contributing to a more precise extraction of neutrino oscillation parameters.

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A Formalism of Lepton-Nucleus Scattering

One of the most important observables in scattering processes is the cross section, σ , which quantifies the probability of a specific process in a scattering reaction. Physically, it represents an effective area presented by the target particle for a specific reaction to occur [1].

$$\sigma = \frac{\dot{N}}{\phi N_b} = \frac{\text{number of reactions}}{\text{incident flux} \times \text{number of target nuclei}} \quad (50)$$

Units of the cross section are usually $[\sigma] = \text{cm}^2$ or $[\sigma] = b = 10^{-28} \text{m}^2$.

Our detectors only measure scattered particles in a small solid angle $d\Omega$, and therefore do not observe all of the outgoing particles. Additionally, experiments are usually interested in a certain final energy range dE' . This gives us the so-called double differential cross section $d^2\sigma/dE'd\Omega$; so to obtain the total cross section, we must integrate over all the possible angles and energies [1]:

$$\sigma = \int_0^{E'_{max}} \int_{4\pi} \frac{d^2\sigma(E, E', \theta, \phi)}{dE'd\Omega} dE' d\Omega \quad (51)$$

To determine the cross section, it is necessary to compute the transition probability between the initial and final states. When the coupling constant is small, as is the case for the electromagnetic and weak interaction, we can make a perturbative expansion. Working to lowest order, the matrix element of the probability amplitude is of the form [29]:

$$S_{fi} = -i \int \bar{\Psi}_f^\dagger(x) V(x) \Psi_i(x) d^4x \quad (52)$$

where ψ_i and ψ_f are the initial and final state wavefunctions, respectively, and $\bar{\Psi}_f = \Psi_f^\dagger \gamma_0$. For an electron in an electromagnetic field A^μ the interaction potential, $V(x)$, has the following expression:

$$V(x) = -e\gamma_0\gamma_\mu A^\mu(x) \quad (53)$$

This leads us to Eq. 54:

$$S_{fi} = ie \int \bar{\Psi}_f(x) \not{A} \Psi_i(x) d^4x \quad (54)$$

Using Maxwell's equations,

$$\square A^\mu(x) = J^\mu(x) \quad (55)$$

we see that the electromagnetic field $A^\mu(x)$ is associated with a nuclear current J^μ ; where $\square = \partial_\mu \partial^\mu$ is the d'Alembert operator and we have chosen Lorentz gauge: $\partial_\mu A^\mu = 0$.

To solve the inhomogeneous wave equation, Eq. 55, we use the method of Green's function. For this, we introduce Green's function in coordinate space [52]:

$$\square D_F(x - y) = \delta^{(4)}(x - y) \quad (56)$$

This function gives us the fundamental solution to the d'Alembert operator. We can obtain the function in momentum space by making the Fourier transform:

$$D_F(q^2) = -\frac{1}{q^2} \quad (57)$$

Therefore:

$$D_F(x - y) = \int \frac{d^4q}{(2\pi)^4} e^{iq(x-y)} \frac{-1}{q^2 + i\epsilon} \quad (58)$$

With Green's function fully defined, we obtain the solution to Eq. 55:

$$A^\mu(x) = \int d^4y D_F(x-y) J^\mu(y) \quad (59)$$

Therefore, we get that the Scattering Amplitude is of the form [29]:

$$S_{fi} = -ie \int d^4x \int d^4y \int \frac{d^4q}{(2\pi)^4} \bar{\Psi}_f(x) \gamma_\mu \Psi_i(x) e^{-iq(x-y)} \frac{1}{q^2 + i\epsilon} J^\mu(y) \quad (60)$$

Cross section with PWBA

To proceed with the evaluation of the transition amplitude, we use Born approximation, as explained before, meaning we consider only the exchange of one virtual photon. Further, we describe the initial and final states of the scattered electron as relativistic plane-wave functions which are solutions to the Dirac equation. Hence the plane-wave Born approximation or PWBA.

$$\Psi_i(x) = \sqrt{\frac{m_e}{\epsilon_i V}} u(k_i, s_i) e^{-ik_i \cdot x} \quad (61)$$

$$\bar{\Psi}_f(x) = \sqrt{\frac{m_e}{\epsilon_f V}} \bar{u}(k_f, s_f) e^{ik_f \cdot x} \quad (62)$$

where $u(k, s)$ is the Dirac spinor for an electron with four-momentum $k^\mu = (E, \vec{k})$ in natural units [29]. We have chosen the normalization used by Bjorken and Drell, [53].

Introducing these wavefunctions into the transition amplitude, we obtain:

$$S_{fi} = -i \sqrt{\frac{m_e}{\epsilon_i V}} \sqrt{\frac{m_e}{\epsilon_f V}} \int d^4x \int d^4y [\bar{u}(k_f, s_f) \gamma_\mu u(k_i, s_i)] e^{i(k_f - k_i) \cdot x} \cdot \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \frac{1}{q^2 + i\epsilon} J^\mu(y) \quad (63)$$

Applying the delta function $\delta^4(k_f - k_i - q)$, which enforces local momentum conservation at the leptonic vertex, we can compute the integration over d^4q . This way, we reach a simplified version of the scattering matrix:

$$\begin{aligned} S_{fi} &= \frac{iem}{\sqrt{\epsilon_i \epsilon_f} V} \bar{u} \gamma_\mu u \int d^4y J^\mu(y) \int d^4q \frac{1}{q^2 + i\epsilon} e^{iqy} \delta^{(4)}(k_f - k_i - q) = \\ &= \frac{iem}{\sqrt{\epsilon_i \epsilon_f} V} \bar{u} \gamma_\mu u \int d^4y J^\mu(y) \frac{1}{q^2 + i\epsilon} e^{iqy} \quad ; \quad \begin{cases} q = k_f - k_i \\ q^2 = q_\mu q^\mu = \omega^2 - |\vec{q}|^2 \end{cases} \end{aligned} \quad (64)$$

We can clearly see that the matrix element S_{fi} is factorized into three physical components: the leptonic current in momentum space, the Feynman propagator $\frac{1}{q^2}$ and the nuclear or hadronic transition current $J^\mu(y)$.

To model the response of the nuclear target, we must specify the hadronic current J_{nuc}^μ . We can write this current as a matrix element of a single-nucleon current operator acting between the initial and final nuclear wavefunctions:

$$J_{\text{nuc}}^\mu(0) = \bar{\Psi}_{f_{\text{nuc}}} \hat{J}_N^\mu \Psi_{i_{\text{nuc}}} \quad (65)$$

Note that this can be applied in the case of the Impulse Approximation, which will be explained later, where we assume that only one nucleon interacts with the lepton.

For the current operator, we use the most general electromagnetic current for a free relativistic nucleon:

$$\hat{J}_N^\mu = F_1(Q^2)\gamma^\mu + \frac{i\kappa}{2M_p}F_2(Q^2)\sigma^{\mu\nu}q_\nu \quad (66)$$

where F_1 and F_2 are the Dirac and Pauli form factors.

Alternatively, by applying Gordon's decomposition for on-shell Dirac spinors, Eq. (66) can be rewritten in a different form:

$$\hat{J}_N^\mu = (F_1(Q^2) + \kappa F_2(Q^2))\gamma^\mu - \frac{\kappa F_2(Q^2)}{2M_p}(P_f + P_i)^\mu \quad (67)$$

Eqs. (66) and (67) are mathematically equivalent for free, on-shell particles. However, nucleons bound inside a nucleus are off-shell. This will lead to different off-shell extrapolations depending on which operator is chosen, commonly known as the *CC1* and *CC2* prescriptions proposed by de Forest [48].

Using plane-waves for the protons:

$$\Psi_{i_{nuc}} = \sqrt{\frac{M_p}{E_I V}} u_N(P_I, S_I) e^{-iP_I \cdot y} \quad ; \quad \Psi_{f_{nuc}} = \sqrt{\frac{M_p}{E_F V}} \bar{u}_N(P_F, S_F) e^{iP_F \cdot y} \quad (68)$$

we obtain the final expression for the matrix element of the transition amplitude.

$$S_{fi} = \frac{ie^2 m_e M_p}{V^2 \sqrt{\epsilon_i \epsilon_f E_I E_F}} \bar{u} \gamma_\mu u \langle J^\mu \rangle \frac{1}{q^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(k_i - k_f + P_I - P_F); \quad (69)$$

Differential cross section

To obtain the differential cross section from the transition amplitude, we need to calculate the transition rate, which is the transition probability per unit volume and unit time [29]:

$$w_{fi} = \frac{|S_{fi}|^2}{VT} = (2\pi)^4 \delta^{(4)}(P_I + k_i - P_F - k_f) \frac{1}{V^4} \frac{m_e^2 M_p^2}{\epsilon_i \epsilon_f E_I E_F} \frac{e^4}{q^4} |\bar{u} \gamma_\mu u \langle J^\mu \rangle|^2 \quad (70)$$

The term involving the spinors can be solved by taking traces:

$$\begin{aligned} |\eta|^2 &= \frac{1}{2} \sum_{s_i, s_f} |\bar{u} \gamma_\mu u \langle J^\mu \rangle|^2 = \frac{1}{2} \frac{1}{4m_e^2} \text{Tr} [(\not{k}_f + m_e) \gamma^\mu (\not{k}_i + m_e) \gamma^\nu] \langle J_\mu \rangle \langle J_\nu \rangle^\dagger = \\ &= \frac{1}{2m_e^2} \left[k_f^\mu k_i^\nu + k_f^\nu k_i^\mu - (k_f \cdot k_i) g^{\mu\nu} + m_e^2 g^{\mu\nu} \right] \langle J_\mu \rangle \langle J_\nu \rangle^\dagger = \\ &= \frac{1}{2m_e^2} \left[k_f^\mu \langle J_\mu \rangle k_i^\nu \langle J_\nu \rangle^\dagger + k_f^\nu \langle J_\nu \rangle^\dagger k_i^\mu \langle J_\mu \rangle - |\langle J \rangle|^2 (k_f \cdot k_i - m_e^2) \right] \end{aligned} \quad (71)$$

This expression can be rewritten in terms of $q^\mu = k_i^\mu - k_f^\mu$ and $Q^\mu = k_i^\mu + k_f^\mu$:

$$|\eta|^2 = \frac{1}{2} \sum_{s_i, s_f} |\bar{u} \gamma^\mu u \langle J_\mu \rangle|^2 = \frac{1}{4m^2} \left[|(k_i^\mu + k_f^\mu) \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right] \quad (72)$$

The cross section represents the probability of finding the particles in the final state for each particle in the initial state. Therefore, to calculate the cross section, we need the transition rate, the initial flux and the number of final states of interest [29], [27]:

$$\text{Cross section} = \frac{w_{fi}}{\text{initial flux}} \times \text{number of final states} \quad (73)$$

For the initial flux, it can be written as a function of the relative velocity between the initial particle and the target.

$$\text{initial flux} = |\vec{v}_{rel}| \frac{1}{V} \frac{1}{V} \quad (74)$$

where the number of particles per unit volume is $1/V$.

Lastly, given our choice of normalization, the number of final states are of the form:

$$n^o \text{ final states} \Rightarrow \frac{V d^3 \vec{k}_f}{(2\pi)^3} \frac{V d^3 \vec{P}_F}{(2\pi)^3} \quad (75)$$

Inserting Eqs. 70, 74 and 75, we arrive at the expression for the differential cross section:

$$d\sigma = \frac{m_e^2 M_p^2}{\epsilon_i \epsilon_f E_I E_F} \frac{e^4}{q^4} (2\pi)^4 \delta^{(4)}(P_I + k_i - P_F - k_f) \frac{d^3 \vec{k}_f}{(2\pi)^3} \frac{d^3 \vec{P}_F}{(2\pi)^3} \frac{1}{|\vec{v}_{rel}|} \frac{1}{4m_e^2} \cdot \left[|(k_i^\mu + k_f^\mu) \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right] \quad (76)$$

where the relative velocity is:

$$\vec{v} = \frac{\vec{p}}{E} \Rightarrow |\vec{v}_{rel}| = \left| \frac{\vec{k}_i}{\epsilon_i} - \frac{\vec{P}_I}{E_I} \right| = \left| \frac{E_I \vec{k}_i - \epsilon_i \vec{P}_I}{\epsilon_i E_I} \right| = \frac{1}{\epsilon_i E_I} \left[(E_I \vec{k}_i - \epsilon_i \vec{P}_I)^2 \right]^{1/2} \quad (77)$$

We can continue to work with this expression to simplify it. For that, we switch to spherical coordinates and rewrite the differential in terms of the energy:

$$\epsilon_f^2 = k_f^2 + m^2 \Rightarrow 2k_f dk_f = 2\epsilon_f d\epsilon_f \quad (78)$$

$$d^3 k_f = |\vec{p}_f|^2 dk_f d\Omega_f = |\vec{k}_f| \epsilon_f d\epsilon_f d\Omega_f \quad (79)$$

$$d\sigma = \frac{M_p^2 e^4}{q_\mu^4} \delta^{(4)}(P_I + k_i - P_F - k_f) \frac{1}{4 \cdot (2\pi)^2} |\vec{k}_f| |\vec{P}_F| d\epsilon_f d\Omega_f dE_F d\Omega_F \cdot \frac{1}{|E_I \vec{k}_i - \epsilon_i \vec{P}_I|} \cdot \left[|(k_i^\mu + k_f^\mu) \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right] \quad (80)$$

Additionally, if we introduce the Mott cross section,

$$\left(\frac{d\sigma}{d\Omega} \right)_{Mott} = \frac{\alpha^2 \cos^2(\theta/2)}{4\epsilon_i^2 \sin^4(\theta/2)} \quad (81)$$

$$\sigma_M = \frac{4\alpha^2 \epsilon_f^2 \cos^2(\theta/2)}{q_\mu^4} \quad (82)$$

we arrive at a simplified expression of the differential cross section [26]:

$$d\sigma = |\vec{P}_F| E_F \sigma_M \frac{M_p^2}{4\epsilon_f E_F \cos^2(\theta/2)} \delta^{(4)}(P_I + k_i - P_F - k_f) \frac{1}{|E_I \vec{k}_i - \epsilon_i \vec{P}_I|} \cdot \left[|(k_i^\mu + k_f^\mu) \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right] d\epsilon_f d\Omega_f dE_F d\Omega_F \quad (83)$$

In Eq. 83, the transition matrix element squared can be factored into the contraction of the leptonic and hadronic tensors [27], [26]:

$$4m_e^2|\eta|^2 = L_{\mu\nu}H^{\mu\nu} = 4\epsilon_i\epsilon_f\cos^2\left(\frac{\theta}{2}\right)\omega_{\mu\nu}W^{\mu\nu} \quad (84)$$

In this way, we are able to isolate all the dependence on the nuclear currents in terms of structure functions and other factors that depend only on the electrons' kinematics.

If instead of electrons we are working with neutrinos, we need to use the neutral leptonic current, [26]:

$$\begin{aligned} L_{\mu\nu}^z &= \sum_{s_i, s_f} |\bar{u}\gamma_\mu(1 \mp \gamma_5)u|^2 = \text{Tr} [k_f\gamma_\mu(1 \mp \gamma_5)k_i\gamma_\nu(1 \mp \gamma_5)] = \\ &= 8 \left[\pm i\epsilon^{\mu\alpha\nu\beta}k_{i\alpha}k_{f\beta} + k_f^\mu k_i^\nu + k_f^\nu k_i^\mu - (k_f \cdot k_i)g^{\mu\nu} \right] \end{aligned} \quad (85)$$

In this case, we can assume $m_\nu \approx 0$

$$\sigma'_M = 4 \times 4\epsilon_i\epsilon_f\cos^2\left(\frac{\theta}{2}\right) \left[\frac{g^2}{4\pi} \right] \quad (86)$$

$$L_{\mu\nu}J^\mu J^{\nu\dagger} = 4 \times 4\epsilon_i\epsilon_f\cos^2\left(\frac{\theta}{2}\right)\omega_{\mu\nu}W^{\mu\nu} \quad (87)$$

In this case, for the nuclear current, we need to add an axial term since neutrinos interact via the weak force:

$$\hat{J}^\mu = F_1\gamma^\mu + F_2\frac{i\kappa\sigma^{\mu\nu}q_\nu}{2M} + G_1\gamma^\mu\gamma^5 = (F_1 + \kappa F_2)\gamma^\mu - \frac{\kappa F_2}{2M}(P_F + P_I)^\mu + G_1\gamma^\mu\gamma^5 \quad (88)$$

B PWIA and Elementary Off-Shell Cross Section

The elementary cross section, $\sigma_{ep} \equiv \frac{d\sigma}{d\Omega_f}$ can be defined from evaluating the interaction of an electron scattering off a free, moving nucleon. This derived quantity represents the fundamental building block of the factorization approximation [33], [34] (commonly leading to the de Forest prescriptions, such as σ_{cc1} or σ_{cc2}) [48]. We begin with the general expression for the differential cross section (Eq. 83). Let's recall that the differential cross section that we obtained was of order six:

$$\begin{aligned} \frac{d^6\sigma}{d\epsilon_f d\Omega_f dE_F d\Omega_F} &= |\vec{P}_F| E_F \sigma_M \frac{M_p^2}{4\epsilon_f E_F \cos^2(\frac{\theta}{2})} \delta^{(4)}(P_I + k_i - P_F - k_f) \\ &\cdot \frac{1}{|E_I \vec{k}_i - \epsilon_i \vec{P}_I|} \left[|k p_i^\mu + k_f^\mu \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right] \end{aligned} \quad (89)$$

We first integrate over the entire phase space of the outgoing nucleon ($d^3\vec{P}_F$), knowing that:

$$|\vec{P}_F| E_F dE_F d\Omega_F = |\vec{P}_F|^2 d|\vec{P}_F| d\Omega_F = d^3\vec{P}_F \quad (90)$$

We can also separate the four-dimensional Dirac delta into its energy and momentum components

$$\delta^{(4)}(P_I + k_i - P_F - k_f) = \delta(E_I + \epsilon_i - E_F - \epsilon_f) \delta^{(3)}(\vec{P}_I + \vec{q} - \vec{P}_F) \quad (91)$$

The three-dimensional momentum delta function forces the final nucleon momentum to perfectly conserve the momentum transfer: $\vec{P}_F = \vec{P}_I + \vec{q}$. The integral over $d^3\vec{P}_F$ evaluates to 1, leaving us with:

$$\frac{d\sigma}{d\Omega_f} = \int \sigma_M \frac{M_p^2}{4\epsilon_f E_F \cos^2(\theta/2)} \frac{\left[|(k_i^\mu + k_f^\mu) \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right]}{|E_I \vec{k}_i - \epsilon_i \vec{P}_I|} \delta(E_I + \epsilon_i - E_F - \epsilon_f) d\epsilon_f \quad (92)$$

To integrate over $d\epsilon_f$, we must use the properties of the energy delta function:

$$\int \delta(g(x)) dx = \frac{1}{|g'(x_0)|} \quad (93)$$

where, in our case, $g(x) = g(\epsilon_f)$ and x_0 is such that $g(\epsilon_f) = E_I + \epsilon_i - E_F - \epsilon_f = 0$. We must note that due to the conservation of momentum, E_F depends on ϵ_f through k_f :

$$E_F = \sqrt{M_p^2 + |\vec{P}_I + \vec{k}_i - \vec{k}_f|^2} \quad (94)$$

Therefore, the derivative with respect to ϵ_f yields

$$\frac{\partial g}{\partial \epsilon_f} = -(1 + \frac{\partial E_F}{\partial \epsilon_f}) \quad (95)$$

Using the chain rule and Eq. 94, we obtain the value of this differential:

$$2E_F \frac{\partial E_F}{\partial \epsilon_f} = \frac{\partial}{\partial \epsilon_f} (P_I + k_i - k_f)^2 = 2(P_I + k_i - k_f) \frac{\partial}{\partial \epsilon_f} (-k_f) \Rightarrow \frac{\partial E_F}{\partial \epsilon_f} = -\frac{P_F}{E_F} \frac{\partial k_f}{\partial \epsilon_f} \quad (96)$$

If we consider the electron as ultrarelativistic, then $|\vec{k}_f| \approx \epsilon_f$.

$$\frac{\partial g}{\partial \epsilon_f} = -(1 - \frac{P_F k_f}{E_F \epsilon_f}) \quad (97)$$

This gives rise to the kinematic recoil factor (f_{rec}), which accounts for the phase space distortion due to the recoil of the target:

$$f_{rec} = \left| \frac{\partial g}{\partial \epsilon_f} \right| = 1 - \frac{P_F k_f}{E_F \epsilon_f} \quad (98)$$

Integrating over $d\epsilon_f$ evaluates the expression at the exact kinematic point where energy is conserved. We finally define the elementary off-shell electron-proton cross section (σ_{ep}) as:

$$\sigma_{ep} \equiv \frac{d\sigma}{d\Omega_f} = \sigma_M \frac{M_p^2}{4\epsilon_f E_F \cos^2(\theta/2)} \frac{1}{|E_I \vec{k}_i - \epsilon_i \vec{P}_I|} \frac{1}{f_{rec}} \cdot \left[|(k_i^\mu + k_f^\mu) \langle J_\mu \rangle|^2 + q^2 |\langle J_\mu \rangle|^2 \right] \quad (99)$$

If we compare the obtained equation for σ_{ep} , Eq. 99, with the full six-fold differential cross section, Eq. 73, we can obtain the values for the kinematic factor K and the spectral function.

For the spectral function, σ_{ep} corresponds to a single free nucleon, while the term $\langle J^\mu \rangle$ from the differential cross section accounts for the total nucleus. Under PWIA, we assume that the electron interacts only with one nucleon while the rest stay as spectators. Therefore, the probability of that said nucleon to have certain momentum p_m and energy E_m will be defined by the spectral function as:

$$S(p_m, E_m) \approx \frac{|\langle J_\mu \rangle_{\text{nuclear}}|^2}{|\langle J_\mu \rangle_{\text{nucleon}}|^2} \quad (100)$$

For the kinematic part, it is important to note that the expression for σ_{ep} derived above implies an integration over the final nucleon's phase space ($d^3 \vec{P}_F = |\vec{P}_F| E_F dE_F d\Omega_F$). However, in an exclusive ($e, e'p$) coincidence experiment, the outgoing proton is explicitly detected, meaning its phase space is not integrated out. Because the final nucleon phase space is preserved, the kinematic factor K explicitly retains the $|\vec{P}_F| E_F$ term that was absorbed during the free-scattering integration, $K \propto |\vec{P}_F| E_F$.

C Longitudinal-Transversal separation

In this electron-nucleus scattering problem we have the electron scattering plane, given by x, y, z , and the proton scattering plane, which have a vector in common, the transferred momentum, as seen in Figure 2. The angle between those two planes is ϕ_F . We can make explicit all the dependence of the cross-sections in this angle, by re-expressing the spatial coordinates of the hadronic current in the in-plane (in the hadron scattering plane). It is obvious that the nuclear current expressed in the nuclear scattering plane cannot depend on the ϕ_F , and actually it cannot depend on any electron variable. Thus we analyze the components of the hadronic four-current $J^\mu = (\rho, \vec{J})$, where $\rho \equiv J^0$ represents the nuclear charge density and $\vec{J}$ is the spatial current vector. We write this hadronic current $\vec{J}$ into a reference frame aligned with the momentum transfer vector $\vec{q}$ and in the proton scattering plane. According to the kinematics defined in Figure 2, the spatial components of the current can be expressed as:

$$\vec{J} = J_q \hat{q} + J_\perp \hat{n}_\perp + J_\parallel \hat{n}_\parallel \quad (101)$$

$$\hat{n}_\perp \equiv (-\sin \phi_F, \cos \phi_F, 0) \quad (102)$$

$$\hat{n}_\parallel \equiv (\cos \phi_F, \sin \phi_F, 0) \quad (103)$$

$$\vec{P}_F \equiv |\vec{P}_F| (\sin \gamma \cos \phi_F, \sin \gamma \sin \phi_F, \cos \gamma) \quad (104)$$

This establishes the relationship between spatial components of the hadronic current in the electron scattering plane (x, y, z) and in the proton scattering plane ($\parallel, \perp, z$):

$$J_\perp = -\sin \phi_F J_x + \cos \phi_F J_y \quad (105)$$

$$J_\parallel = \cos \phi_F J_x + \sin \phi_F J_y \quad (106)$$

By imposing Lorentz invariance and parity conservation, $\omega_{\mu\nu} W^{\mu\nu}$ can be written in terms of the longitudinal (parallel to the direction of the momentum transferred), transversal and interference components [26], [27], [33], [35].

$$\omega_{\mu\nu} W^{\mu\nu} = \omega_L W_L + \omega_T W_T + \omega_{TT} W_{TT} + \omega_{LT} W_{LT} \quad (107)$$

The purely kinematic coefficients come directly from the leptonic tensor. They depend on the electron's macroscopic scattering parameters and are given exactly by:

$$\omega_L = 1 \quad (108)$$

$$\omega_T W_T = \omega_\parallel W_\parallel + \omega_\perp W_\perp \quad (109)$$

$$\omega_\parallel = \tan^2 \left(\frac{\theta}{2} \right) - \frac{q_\mu^2}{|\vec{q}|^2} \cos^2 \phi_F \quad (110)$$

$$\omega_\perp = \tan^2 \left(\frac{\theta}{2} \right) - \frac{q_\mu^2}{|\vec{q}|^2} \sin^2 \phi_F \quad (111)$$

$$\omega_{TT} = \frac{q_\mu^2}{|\vec{q}|^2} \quad (112)$$

$$\omega_{LT} = -2 \left[\tan^2 \left(\frac{\theta}{2} \right) - \frac{q_\mu^2}{|\vec{q}|^2} \right]^{\frac{1}{2}} \quad (113)$$

To ensure gauge invariance, the electromagnetic current must be conserved ($q_\mu J^\mu = 0$). This yields the condition $\omega \rho - |\vec{q}| J_q = 0$, which allows us to eliminate the longitudinal current component in

favor of the charge density ρ , or vice versa:

$$J_q = \frac{\omega}{|\vec{q}|} \rho \quad (114)$$

Applying this conservation law and using the projected coordinate system, we obtain the following nuclear structure functions:

$$W_L = |\rho|^2 + \frac{\omega^2}{|\vec{q}|^2} |J_q|^2 + \frac{\omega}{|\vec{q}|} 2\mathcal{R}(\rho J_q^\dagger) = \frac{q_\mu^4}{|\vec{q}|^4} |\rho|^2 \quad (115)$$

$$W_{\parallel} = |J_{\parallel}|^2 \quad (116)$$

$$W_{\perp} = |J_{\perp}|^2 \quad (117)$$

$$W_{TT} = \sin(2\phi_F) \mathcal{R}(J_{\perp} J_{\parallel}^\dagger) \quad (118)$$

$$W_{LT} = \mathcal{R} \left\{ \left(\rho - \frac{\omega}{|\vec{q}|} J_q \right) (-\sin\phi_F J_{\perp}^\dagger + \cos\phi_F J_{\parallel}^\dagger) \right\} = \mathcal{R} \left\{ \left(\rho - \frac{\omega}{|\vec{q}|} J_q \right) J_x^\dagger \right\} \quad (119)$$

To evaluate the structure functions at the single-nucleon vertex, we must calculate the different tensor components $\langle J^\mu J^{\nu\dagger} \rangle$. To do this, we need to take the average over the initial nucleon spins and sum over the final nucleon spins for a nucleon transitioning from initial momentum P_I to final momentum P_F .

$$W_{EM}^{\mu\nu} = \frac{1}{2} \sum_{s_I, s_F} |\bar{u}(P_F, s_F) \hat{J}_N^\mu u(P_I, s_I)|^2 \quad (120)$$

If we use the Gordon-decomposed single-nucleon current, Eq. 67 and take the traces, we obtain:

$$\begin{aligned} \langle J^\mu J^\nu \rangle_{EM} &= \frac{1}{2M^2} (F_1 + \kappa F_2)^2 [P_I^\mu P_F^\nu + P_I^\nu P_F^\mu - g^{\mu\nu} (P_I \cdot P_F - M^2)] \\ &\quad + Q^\mu Q^\nu \left[\frac{\kappa^2 F_2^2}{8M^4} (P_I \cdot P_F) + \frac{\kappa^2 F_2^2}{8M^2} - \frac{\kappa F_2}{2M^2} (F_1 + \kappa F_2) \right] \end{aligned} \quad (121)$$

By projecting this tensor into the momentum-aligned coordinate system ($\parallel, \perp, z$) defined previously, we obtain the explicit expressions for the single-nucleon electromagnetic structure functions:

$$\langle \rho^2 \rangle_{EM} = \frac{1}{4M^2} \left[\left(F_1^2 - \frac{\kappa^2 F_2^2}{4M^2} q_\mu^2 \right) (E_I + E_F)^2 - |\vec{q}|^2 (F_1 + \kappa F_2)^2 \right] \quad (122)$$

$$\langle J_q^2 \rangle_{EM} = \frac{1}{4M^2} \left[-(F_1 + \kappa F_2)^2 (E_F - E_I)^2 + \left(F_1^2 - \frac{\kappa^2 F_2^2}{4M^2} q_\mu^2 \right) \left(2|\vec{P}_F| \cos \gamma - |\vec{q}| \right)^2 \right] \quad (123)$$

$$W_{\perp, EM} = \langle J_{\perp}^2 \rangle_{EM} = \frac{1}{4M^2} (F_1 + \kappa F_2)^2 (-q_\mu^2) \quad (124)$$

$$W_{\parallel, EM} = \langle J_{\parallel}^2 \rangle_{EM} = W_{\perp, EM} + \frac{1}{M^2} |\vec{P}_F|^2 \sin^2 \gamma \left[F_1^2 - \frac{\kappa^2 F_2^2}{4M^2} q_\mu^2 \right] \quad (125)$$

$$W_{TT, EM} = \mathcal{R} \langle J_{\perp} J_{\parallel}^\dagger \rangle_{EM} = 0 \quad (126)$$

Explicitly separating the leptonic kinematic dependence (including ϕ_F) in the response functions shows that, by combining measurements at fixed energy and momentum transfer under different electron kinematics, one can in principle extract all components of the hadronic current. This demonstrates the main strength of coincidence ($e, e'p$) scattering.

In the case of the neutral current, there is an imaginary term in the leptonic tensor, Eq. 85,

so we have two additional contributions: TT' and LT'

$$\omega_{TT'} = \mp 2 \tan\left(\frac{\theta}{2}\right) \sqrt{\tan^2\left(\frac{\theta}{2}\right) - \frac{q_\mu^2}{|\vec{q}|^2}} \quad (127)$$

$$\omega_{LT'} = \pm 2 \tan\left(\frac{\theta}{2}\right) \quad (128)$$

$$W_{TT'} = \Im(J_\parallel J_\perp^\dagger) = \Im(J_x J_y^\dagger) \quad (129)$$

$$W_{LT'} = \Im\left\{\left(\rho - \frac{\omega}{|\vec{q}|} J_q\right) (\sin\phi_F J_\parallel^\dagger + \cos\phi_F J_\perp^\dagger)\right\} \quad (130)$$

Using plane waves for the initial and final nucleon wave functions, the nuclear current with the axial component from Eq. 88 and summing and averaging over the nucleon spinors we calculate $\frac{1}{2} \sum_{\sigma_I \sigma_F} |\bar{u}(\sigma_F, \vec{P}_F) J^\mu u(\sigma_I, \vec{P}_I)|^2$ as before:

$$\begin{aligned} \langle J^\mu J^\nu \rangle &= 2(F_1 + \kappa F_2)^2 [P_I^\mu P_F^\nu + P_I^\nu P_F^\mu - g^{\mu\nu} (P_I P_F - M^2)] + Q^\mu Q^\nu \left[\frac{\kappa^2 F_2^2}{2M^2} P_I P_F + \frac{\kappa^2 F_2^2}{2} - 2\kappa F_2 (F_1 + \kappa F_2) \right] \\ &\quad + 2G_1^2 [P_I^\mu P_F^\nu + P_I^\nu P_F^\mu - g^{\mu\nu} (P_I P_F + M^2)] + i2(F_1 + \kappa F_2) G_1 \cdot \epsilon^{\mu\alpha\beta} P_{I\alpha} P_{F\beta} \end{aligned} \quad (131)$$

And we obtain the explicit expressions for the structure functions:

$$\langle \rho^2 \rangle = G_1^2 [-4M^2 - |\vec{q}|^2 + (E_I + E_F)^2] + \left[F_1^2 - \frac{F_2^2 \kappa^2}{4M^2} q_\mu^2 \right] (E_I + E_F)^2 - |\vec{q}|^2 (F_1 + \kappa F_2)^2 \quad (132)$$

$$\begin{aligned} \langle J_q^2 \rangle &= G_1^2 [4M^2 + 4P_I^z P_F^z - q_\mu^2] - (F_1 + \kappa F_2)^2 (E_F - E_I)^2 + \\ &\quad + \left[F_1^2 - \frac{\kappa F_2^2}{4M^2} q_\mu^2 \right] (2|\vec{P}_F| \cos \gamma - |\vec{q}|)^2 \end{aligned} \quad (133)$$

$$\langle J_\perp^2 \rangle = G_1^2 [4M^2 - q_\mu^2] + (F_1 + \kappa F_2)^2 (-q_\mu^2) = W_\perp \quad (134)$$

$$\langle J_\parallel^2 \rangle = \langle J_\perp^2 \rangle + 4|\vec{P}_F|^2 \sin^2 \gamma \left[G_1^2 + F_1^2 - q_\mu^2 \frac{\kappa^2 F_2^2}{4M^2} \right] = W_\parallel \quad (135)$$

$$\mathcal{R}\langle J_\perp J_\parallel \rangle = 0 \Rightarrow W_{TT} = 0 \quad (136)$$

$$2\mathcal{R}(\rho J_x^\dagger) = 4|\vec{P}_F| \sin \gamma \cos \phi_F (E_I + E_F) \left[G_1^2 + F_1^2 - \frac{\kappa F_2^2}{4M^2} q_\mu^2 \right] \quad (137)$$

$$\Im\langle J_\perp J_\parallel \rangle = -4(F_1 + \kappa F_2) G_1 \left[(E_F - E_I) |\vec{P}_F| \cos \gamma - E_F |\vec{q}| \right] = W_{TT'} \quad (138)$$

$$\Im(\rho J_y^\dagger) = 4(F_1 + \kappa F_2) G_1 |\vec{P}_F| |\vec{q}| \sin \gamma \cos \phi_F \quad (139)$$

$$\begin{aligned} 2\mathcal{R}(J^0 J^3) &= 4G_1^2 (E_I P_F^z + E_F P_I^z) + 2(F_1 + \kappa F_2)^2 (E_I - E_F) |\vec{q}| + \\ &\quad + 2(E_I + E_F) (2|\vec{P}_F| \cos \gamma - |\vec{q}|) \left[F_1^2 - \frac{\kappa^2 F_2^2}{4M^2} q_\mu^2 \right] \end{aligned} \quad (140)$$

D Experimental resolution versus natural width of the one-hole states

When we study scattering reactions off nuclei where one nucleon is emitted, like it is the case for $(e, e'p)$ or $(p, 2p)$, what we obtain in the final state is a residual nucleus that is left in an excited state, denoted as a hole state. When we analyze the energy of these hole states in the detectors, we would expect discrete exact values represented as perfect peaks or lines. However, in experiments these peaks present a certain width, due to two fundamental reasons:

- The experimental resolution from experiments. Detectors and spectrometers are not perfect and have a certain minimum range they can cover. Anything smaller than this range can't be solved or distinguished by the detector and instead is seen blurry, making the discrete peak turn into a gaussian distribution of a certain width.
- The natural width of the states. This limit is imposed by Heisenberg's Uncertainty principle ($\Delta E \cdot \Delta t \geq \hbar/2$), where depending on the half time of the level, its energy band will be broader or narrower.

Focusing first on this latter point, there is a clear difference between the p-shells and the s-shells. When a proton is taken out from an outer shell (a p-shell), the hole left is not very deep and the residual nucleus is left in an excited state of low energy, below the threshold to emit another proton or neutron. Therefore, this state is very stable ($\Delta t \gg$), making its energy uncertainty almost zero. Therefore, these hole states present discrete peaks with a width that is negligible. On the other hand, when the proton emitted comes from the s-shell, the excitation energy is much larger unstable, emitting another nucleon through the strong force in a very short period of time. The result in the spectrum is a broad distribution, called a resonance.

To look at how this distribution is and its width, we can turn to experiments like the one detailed in *De-excitation γ -rays from the s-hole state in ^{15}N associated with proton decay in ^{16}O* by K. Kobayashi et al. [42]. In this article they studied the gamma rays that come from the de-excitation of the s-hole state of ^{15}N , which had been previously created through the $^{16}\text{O}(p, 2p)^{15}\text{N}$ reaction. This study was done with the aim of better understanding the de-excitation of the nucleus for neutrino neutral current interactions in water Cherenkov detectors, such as the Super-Kamiokande.

A beam of protons of 392 MeV was directed at a target of ice to obtain the desired reaction $^{16}\text{O}(p, 2p)^{15}\text{N}$, obtaining the excitation energy spectrum of ^{15}N , shown in Figure 44. In this graph we can see the peaks from the p-shell corresponding to the hole states $1/2^-$ and $3/2^-$ from the shell p, which appear at 0 MeV and 6.32 MeV, respectively, and present a small width due to the spectrometer's resolution. After 16 MeV, the resonance from the s-hole state starts. To understand what happens in the s-shell, they measured the two final protons and the gamma rays in coincidence. Firstly, they make sure that a hole state in the s-shell was created by measuring the energy of the two emitted protons and obtaining the excitation energy of the ^{15}N residual nucleus. If this energy is above the energy region of the p-shell then they know they have taken a proton from a deep shell. Then, they measure in coincidence the gamma rays that come from the de-excitation of the s-hole state by establishing a gate or time window. To distinguish the substructure of this shell, they use different energy regions of E_x and gamma energy spectrum, E_γ , in three cuts.

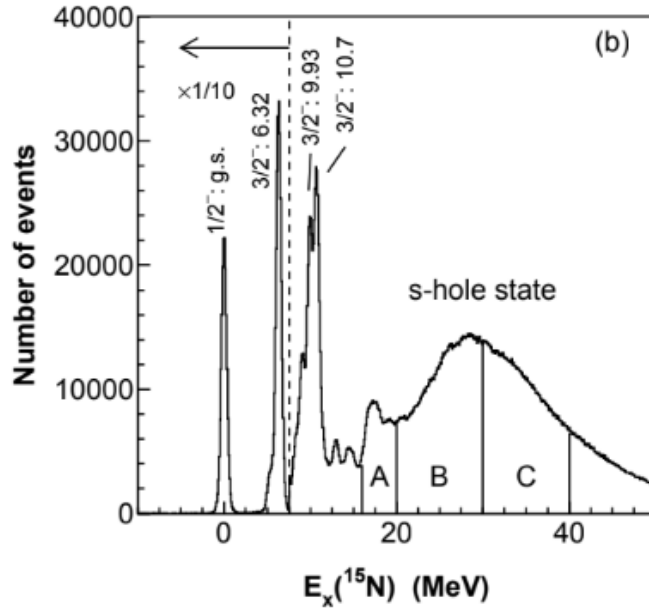


Figure 43: The excitation energy spectrum of ^{15}N induced by the $^{16}\text{O}(p, 2p)^{15}\text{N}$ reaction. The region shown by an arrow is scaled by 1/10. In the analyses of the γ -ray from the s-hole state, the coincident data with two protons is used in the regions A (Ex=16–20 MeV), B (Ex=20–30 MeV), and C (Ex=30–40 MeV). [42]

As mentioned before, the s-shell is unstable; therefore, it will decay emitting gamma rays. However, gamma rays are not emitted directly, but instead ^{15}N will break into different particles such as a proton, a neutron, deuterium and tritium. It will be the daughter nuclei ^{14}C , ^{14}N , ^{13}C and ^{12}C , the ones that will emit secondary gamma rays as shown in the following picture:

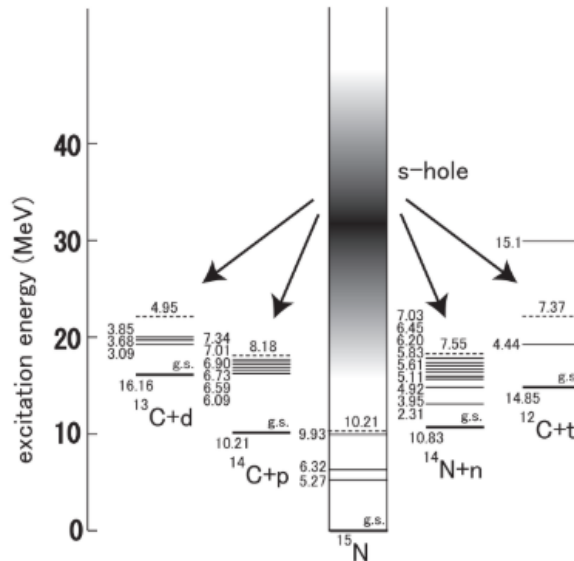


Figure 44: A decay scheme from the s-hole state in ^{15}N [42]

By measuring the gamma rays we can see the population of each state, represented in Table II from reference [42]. With this, we can determine which is the most relevant decay branch and eliminate the background noise from the reaction, obtaining a distribution that is 100% a pure s-hole state. In this way, we show why the s-shell has a broad distribution in comparison to the discrete peaks from the p-shells.

Regarding the experimental resolution, it is the one responsible for the width of the p-shell peaks seen in experimental results and graphs. This is related to the facility and the spectrometer used and it is generally dependent on three factors: the raw energy resolution of the incident beam, the momentum resolution of both spectrometers (electron and proton arms), and the effect of dispersion matching, which cancels most of the beam's energy-spread contribution to the final resolution.

The matching dispersion mechanism is a technique that consists of canceling out the variation in the initial electron energy. Since the initial beam is not perfectly monochromatic, its energy variation causes a blurring of the final resolution of the experiment. To counteract this, before the electron hits the target, two magnets are placed to deviate the motion of the electron. Because more energetic electrons are deviated differently to less energetic ones creating a relation between the transversal position of the electron and its energy. In this way, by knowing the position of impact of the electron, the optical spectrometers can be adjusted reducing the experimental resolution [19].

We can compare the different facilities that carried out $(e, e'p)$ experiments to see which one had the best resolution in energy.

NIKHEF-K was able to fully exploit the dispersion matching, reducing its 1 MeV raw beam spread to achieve record resolutions of 122 keV for $H(e, e'p)$ and 150 keV for $C(e, e'p)$ [19].

In the case of Saclay ([20], [54]) and Bates/MIT ([21]), both facilities demonstrated high-resolution potential in single-arm tests (Bates achieved a group-best 1.1×10^{-4} single-arm momentum resolution), but resolutions of several MeV due to incomplete two-arm dispersion matching during actual coincidence physics runs.

MAMI ([22], [55]) and JLab ([23], [56]) generate naturally narrow beams with minimal energy spread ($\Delta E/E \approx 10^{-4}$). Therefore, their coincidence resolutions (0.5 – 1.8 MeV) are limited primarily by straggling targets and spectrometer optics rather than beam quality.

Finally, RIKEN SCRIT is a slightly different case. It uses a compact microtron and storage ring to trap radioactive ions for elastic scattering. It operates with a coarser momentum resolution ($\sim 10^{-3}$) because its primary goal is charge-density mapping rather than fine missing-energy shell spectroscopy [24].

In the following Table there is a brief description of each experiment and its missing energy resolution:

Table 10: Facility Comparison Table: Missing Energy Resolution in $(e, e'p)$ Experiments

Facility	Accelerator Type & Energy	Record / Typical Resolution (FWHM)	Experimental Resolution Notes
NIKHEF-K (Amsterdam)	Linac (MEA) + AmPS Ring 500 – 900 MeV	~ 122 keV (record with H) 150 keV (typical C)	Achieved the best absolute coincidence resolution by fully implementing dispersion matching to cancel the beam spread [19].
Saclay / ALS (France)	Linac (ALS) 600 – 700 MeV	Several MeV (in coincidence)	Reached ~ 160 keV in single-arm commissioning, but coincidence physics runs fell short of its 100 keV design goal by nearly an order of magnitude [20].
MAMI / A1 (Mainz)	Racetrack Microtron Cascade 14 – 1508 MeV	~ 100 – 200 keV (H/D optimal) 0.6 – 1.8 MeV (complex nuclei)	Microtron phase-focusing provides an intrinsically narrow beam (30–100 keV wide); resolution is limited mainly by target effects [22] [55].
JLab / CEBAF (USA)	Superconducting RF Linac Original 6 GeV, upgraded to 12 GeV	~ 0.5 MeV (hypernuclear spectroscopy) 0.9 – 1.8 MeV (typical heavy nuclei)	Features the narrowest intrinsic beam quality ($\Delta E/E \approx 10^{-4}$); performance is limited by target straggling and angular acceptance [23].
Bates / MIT (USA)	Linac + Recirculator + Ring 100 MeV – 1 GeV	2.5 – 4.7 MeV (in coincidence)	Demonstrated group-best single-arm momentum resolution (1.1×10^{-4}), but suffered poor two-arm coincidence resolution due to incomplete matching [21].
RIKEN SCRIT (Japan)	Microtron + SR2 Storage Ring 100 – 700 MeV	Momentum resolution $\sim 10^{-3}$ (coarse for E_m)	Specialized compact setup for scattering off trapped online-produced radioactive ions; not optimized for missing-energy shell spectroscopy [24].