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Consecuencias gravitacionales de materia invariante TDiff

Gravitational consequences of TDiff invariant matter

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We consider a scalar field which couples to gravity via arbitrary functions of the metric determinant, so that the matter action is no longer invariant under Diffeomorphisms (Diffs) but only under Transverse Diffeomorphisms (TDiffs). We perform an effective description of the scalar field by treating it as a perfect fluid, consider some limiting cases, and study its energy conditions. Whereas Diff invariance guarantees the conservation of the energy-momentum tensor in General Relativity, for TDiff invariant matter the conservation is satisfied only on solutions to the Einstein equations, and imposes one additional constraint on the metric tensor. Finally, we also consider some particular examples.

I. INTRODUCTION

The theory of General Relativity (GR) was first proposed in 1915, and remains our current best description of gravity. In more recent years, we have entered the era of precision cosmology, and observations indicate that our Universe presents an accelerated expansion [1, 2]. Possible explanations for this behavior within the framework of GR include a cosmological constant, or an additional dynamical field referred to as "dark energy" [3, 4] (usually modeled by a scalar field such as quintessence or k-essence [5]). However, it is also possible that the current accelerated expansion indicates a breakdown of GR at cosmological scales and, therefore it may be necessary to modify the theory [3, 4].

Focusing on the latter option, interest has grown in theories which present a broken Diffeomorphism (Diff) invariance [6]. The most popular alternative is the so-called Unimodular Gravity (UG, see for instance [7] for a review), first proposed by Einstein in 1919. In UG, the metric determinant g is taken to be a constant, non-dynamical field, and in this manner the invariance of GR under the full group of diffeomorphisms is broken down to only invariance under the more restrictive Transverse Diffeomorphisms (TDiffs) and, in addition, Weyl rescalings. The focus in this work is placed on TDiffs, which are coordinate transformations with unit Jacobian (without assuming Weyl invariance). Infinitesimally, if we consider a coordinate transformation $x^\mu \to \hat{x}^\mu = x^\mu + \xi^\mu(x)$, then what we do is require that $\partial_\mu \xi^\mu = 0$ (hence the name "transverse"). An immediate and important consequence of TDiff invariance is that we can no longer distinguish between tensors and tensor densities, since the Jacobian must equal 1. In particular, the metric determinant is a true scalar and thus symmetry does not fix the function accompanying terms in Lagrangian functions $\mathcal{L}$ to be $\sqrt{|g|}$ [8]. An accessible introduction to TDiffs may be found in the Appendix of reference [6].

In this work we do not consider the breaking of Diff invariance in the gravitational action. Rather, the symmetry breaking from Diff to TDiff shall be taken to occur explicitly in the matter action (consequently, however, affecting the full theory) via different couplings of a scalar field ψ to gravity. Although it is possible to consider TDiff invariant visible matter as it is done in [6, 8] (where it is seen that the phenomenological viability of the models implies a particular relation between the couplings), the idea of the present work is to explore the implications of this more reduced symmetry group in a general way.

The full Diff invariance of GR implies that the energy-momentum tensor is automatically conserved on the solutions to the equations of motion of the theory (this is shown in any GR textbook, see [9] for a review). However, when Diff symmetry is broken down to TDiff, this conservation is no longer an automatic consequence of the field equation, but it is only implied by the Bianchi identities. This imposes additional constraints on the metric that shall be considered in detail.

The main aim of the present work is to perform a general study which allows us to gain some intuition on the new phenomenology that can be described within this framework. By "general study" we refer to the fact that no assumptions are made as to which is the geometry of spacetime. We do make an assumption regarding the existence of a preferred time direction, in particular that given by the derivative $\partial_\mu \psi$ of the field, and we shall see that this is equivalent to considering a perfect fluid.

The work is organized as follows: in Section II we review some definitions and techniques, and present the TDiff scalar field model under consideration (together with two limiting cases that shall be studied throughout). In Section III we describe the scalar field as a perfect fluid, and its energy conditions are considered in Section IV. Section V presents a detailed analysis of the energy-momentum tensor conservation in the potential and kinetic regimes, and its consequences on the coupling functions. Some particular cases of couplings are considered in Section VI and, finally, Section VII is devoted to the main conclusions of the work. In Appendix A we include some additional calculations, which may be skipped without losing the thread of the discussion.

II. PRELIMINARY CONCEPTS

We present in this section a short review of the definitions and techniques employed throughout the work, and introduce the model we shall study. Our conventions include units in which $\hbar = c = 1$ and the usage of metric signature $(+,-,-,-)$.

A. Presenting the model

The total action we shall consider in this work is

$$S = S_{EH} + S_m \,, \tag{1}$$

the gravitational action being the Einstein-Hilbert action

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\,\sqrt{|g|}R\,, \tag{2}$$

and the matter action taken to be of the form

$$S_m = \int d^4x\,\tilde{\mathcal{L}}_m\,. \tag{3}$$

In the above expression we find the so-called Lagrangian density $\tilde{\mathcal{L}}_m = \tilde{\mathcal{L}}_m(\Psi, \partial_\mu\Psi, g_{\mu\nu})$, which depends on the matter fields Ψ, their first derivatives $\partial_\mu\Psi$, and the metric $g_{\mu\nu}$. We remark that $\tilde{\mathcal{L}}_m$ is a TDiff scalar, since the theory is assumed to be.

Applying to (1) the stationary action principle with respect to variations in the spacetime metric yields the Einstein field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{4}$$

We find in these equations the Energy-Momentum Tensor (EMT), whose definition is the usual one in GR,

$$T^{\mu\nu} = \frac{-2}{\sqrt{|g|}}\frac{\delta S_m}{\delta g_{\mu\nu}}\,, \quad T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_m}{\delta g^{\mu\nu}}\,, \tag{5}$$

One may worry about whether these definitions extend to a situation in which we are explicitly changing the matter action with respect to that in GR, as we shall see in this work, but in fact it makes sense since we will not be altering the gravitational sector (indeed, we use S_{EH}). In other words, we are not altering "the left hand side of Einstein's equations", so we want whatever comes out on the other side to be associated to the EMT in the usual manner. As a final note regarding energy-momentum tensors, since it will be useful throughout the work, we also recall here that for a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}\,, \tag{6}$$

with ρ the energy density of the fluid, p its pressure, and u^μ represents a timelike unit vector field ($u^2 \equiv u_\mu u^\mu = 1$) which we interpret as the velocity field of the fluid.

Consider a matter field Ψ described by a tensor of n indices, whose components are ψ^A ($A = 1, \ldots, 4^n$). The equations of motion (EoM) for this field are obtained by considering (independent) variations $\delta\psi^A$ in their functional form ($\psi^A \to \psi^A + \delta\psi^A$) such that they vanish at the boundaries of the spacetime, and then imposing the stationary action principle. This leads to the Euler-Lagrange EoM, which hold for each of the components:

$$\frac{\partial\tilde{\mathcal{L}}_m}{\partial\psi^A} - \partial_\mu\left(\frac{\partial\tilde{\mathcal{L}}_m}{\partial(\partial_\mu\psi^A)}\right) = 0\,. \tag{7}$$

Before presenting the model, let us recall the following relation for vector fields:

$$\nabla_\mu v^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}v^\mu\right)\,, \tag{8}$$

which shall be of use later on. We also remark here that throughout the work the action of the covariant derivative on (TDiff) tensors maintains the usual definitions.

The model we shall study in this work is that of a scalar field $\psi(x)$ in which the kinetic and potential terms are coupled not only differently than in GR, but also differently from each other. The matter action reads [8]

$$S_m = \int d^4x\left\{\frac{f_k(g)}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - f_v(g)V(\psi)\right\}\,. \tag{9}$$

Here, $f_k(g)$ and $f_v(g)$ are arbitrary functions of the metric determinant $g = \det(g_{\mu\nu})$, and the subscripts make reference to the kinetic and potential terms, respectively (the GR limit would correspond to $f_k, f_v \propto \sqrt{|g|}$). It is argued in references [6, 8] that in order to avoid violations of the Weak Equivalence Principle both functions should coincide, i.e. $f_k = f_v \equiv f$. Although this condition makes the model phenomenologically viable for matter in the visible sector, it may not be necessarily true in the dark sector. With this possible application in mind, we shall in this work allow both functions to differ from each other. It is also interesting to note that even though the couplings are not the usual they are still minimal, meaning that there is no coupling of the field to the curvature (second derivatives of the metric). We finally remark that the matter action (9) is in general not invariant under the full group of diffeomorphisms (Diff invariance is only restored in the GR limit we mentioned above, as it is easy to verify). Nevertheless, a moment's reflection reveals that our model will be invariant under the reduced group of TDiff symmetries, since in that case the Jacobian is unity and the metric determinant is a scalar, together with functions of it.

Having clarified these interesting aspects of the matter action, we proceed now to obtaining from equations (7) and (9) the EoM for the scalar field:

$$\partial_\mu\big(f_k(g)\partial^\mu\psi\big) + f_v(g)V'(\psi) = 0\,, \tag{10}$$

where $\partial^\mu\psi = g^{\mu\nu}\partial_\nu\psi$ and $V'(\psi) = dV/d\psi$ (in general, a prime will denote differentiation with respect to its argument). Using definition (5), the associated EMT turns out to be

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \left\{ \frac{1}{2} f_k(g) \partial_\mu \psi \partial_\nu \psi + \right. \\ \left. + g \left[f_v'(g) V(\psi) - \frac{1}{2} f_k'(g) \partial_\alpha \psi \partial^\alpha \psi \right] g_{\mu\nu} \right\} , \tag{11}$$

which is equivalent to the one presented in reference [6].

We shall now present a couple of simple limiting regimes of our model which will be studied throughout the work: potential domination and kinetic domination. These are interesting not only because they greatly simplify the treatment, but also because they help us in gaining intuition about the underlying physics which we may then use. On the one hand, possible applications of a dominant kinetic term include Galileon models for dark energy [10] (which are actually purely kinetic), as well as cases in which the field is rapidly changing and the kinetic term dominates over the potential (e.g. in fast-roll scenarios at the end of inflation). On the other hand, the dominant potential behavior may be found in cases where the field is slowly varying (e.g. slow-roll inflation) or, with a dark sector application in mind, it also allows us to study dark energy models.

1. *Potential domination*

In the potential domination limit we effectively neglect the kinetic term, so that

$$S_m \simeq \int d^4x \, \left[-f_v(g) \, V(\psi) \right] \tag{12}$$

and the EoM simplify to

$$f_v V' = 0 \implies V' = 0 \,, \tag{13}$$

where we assume that $f_v \neq 0$ because otherwise we do not have potential domination. The fact that $V' = 0$ implies in turn that the field takes on the constant value $\psi = \psi_0$ which is the extremum of the potential, i.e. $V'(\psi_0) = 0$, and as a result

$$V(\psi) = V(\psi_0) \equiv V_0 = \text{const.} \tag{14}$$

Finally, we see that within this dominant potential approximation the EMT is written as

$$T_{\mu\nu} = \frac{2g}{\sqrt{|g|}} f_v' V g_{\mu\nu} \,. \tag{15}$$

2. *Kinetic domination*

In the dominant kinetic limit, we effectively neglect the potential term. Thus,

$$S_m \simeq \int d^4x \, \frac{f_k(g)}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \tag{16}$$

and the EoM become

$$\partial_\mu \left(f_k \partial^\mu \psi \right) = 0 \,. \tag{17}$$

Within this approximation, the EMT is

$$T_{\mu\nu} = \frac{1}{\sqrt{|g|}} \left[f_k \partial_\mu \psi \partial_\nu \psi - g f_k' (\partial \psi)^2 g_{\mu\nu} \right] \,. \tag{18}$$

B. Review of Energy Conditions

The so-called Energy Conditions (ECs) are a set of requirements that one imposes on the EMT upon arguing that they are "physically reasonable" [11, 12]. The most widely used are the Null, the Weak, the Strong, and the Dominant Energy Conditions (NEC, WEC, SEC, and DEC, respectively), and are defined as follows [3, 11–13]. Given any timelike and null vectors v^μ and k^μ, respectively,

$$\begin{array}{ll} \text{(i) NEC:} & T_{\mu\nu} k^\mu k^\nu \geq 0 \,, \\ \text{(ii) WEC:} & T_{\mu\nu} v^\mu v^\nu \geq 0 \,, \\ \text{(iii) SEC:} & \left(T_{\mu\nu} - \frac{1}{2} T^\alpha{}_\alpha \, g_{\mu\nu} \right) v^\mu v^\nu \geq 0 \,, \\ \text{(iv) DEC:} & \text{WEC \& } F^\mu \equiv -T^{\mu\nu} v_\nu \text{ causal.} \end{array} \tag{19}$$

For the case of a perfect fluid, they are translated into conditions on ρ and p as:

$$\begin{array}{ll} \text{(i) NEC:} & \rho + p \geq 0 \,, \\ \text{(ii) WEC:} & \rho + p \geq 0 \,, \quad \rho \geq 0 \,, \\ \text{(iii) SEC:} & \rho + p \geq 0 \,, \quad \rho + 3p \geq 0 \,, \\ \text{(iv) DEC:} & \rho \geq |p| \geq 0 \,. \end{array} \tag{20}$$

C. Review of EMT conservation

The EMT conservation equations are written as

$$\nabla_\alpha T^{\alpha\nu} = 0 \,. \tag{21}$$

When working with a perfect fluid, it is common practice to project them onto the directions longitudinal and transverse to the fluid's velocity. For the former, one must simply contract with the velocity u_ν, while for the latter one must act with the orthogonal projector $h^\mu{}_\nu = \delta^\mu_\nu - u^\mu u_\nu$. As a result, we respectively have [12]

$$\dot{\rho} + (\rho + p) \nabla_\mu u^\mu = 0 \,, \tag{22a}$$

$$(\rho + p) \dot{u}^\mu - \left(g^{\mu\nu} - u^\mu u^\nu \right) \nabla_\nu p = 0 \,, \tag{22b}$$

where we use the notation $\dot{} \equiv u^\mu \nabla_\mu$. For a scalar function ϕ, it is also the case that $\dot{\phi} = u^\mu \partial_\mu \phi = d\phi / d\tau$, with τ the parameter of the curve.

Since later on it will be of interest to study purely transverse equations, we shall here introduce a triplet of linearly independent transverse vectors $(w_1^\mu, w_2^\mu, w_3^\mu)$, which we collectively denote as $\vec{w}^\mu$. They satisfy

$$u_\mu \vec{w}^\mu = 0 \,, \tag{23}$$

and we understand that it holds for all three of them. We shall also denote the projection of the derivatives on the transverse directions as

$$\vec{\nabla} \equiv \vec{w}^{\mu} \nabla_{\mu} \equiv (w_1^{\mu} \nabla_{\mu}\,,\ w_2^{\mu} \nabla_{\mu}\,,\ w_3^{\mu} \nabla_{\mu})\,. \tag{24}$$

III. THE PERFECT FLUID APPROACH

We shall in this section show that, under the assumption of the field having a timelike derivative $\partial_\mu \psi$, our model is equivalent to considering a perfect fluid. If we wish to write the EMT (11) in the perfect fluid form (6), we must find appropriate correspondences between quantities, and a natural one to begin with seems to be

$$u^{\alpha} \equiv \frac{\partial^{\alpha} \psi}{\sqrt{(\partial \psi)^2}} \tag{25}$$

for the velocity, where $(\partial\psi)^2 = \partial_\mu \psi \partial^\mu \psi$. Note that this correspondence only makes sense if the derivative of the field is a timelike vector, and this shall be our assumption throughout the work. Of course, it is a rather strong assumption that does not hold in many interesting situations (such as a static field). Nevertheless, it is a reasonable assumption in cosmological scenarios.

In order to find the energy density, it is useful to recall that $T_{\mu\nu} u^\mu u^\nu = \rho$ for a perfect fluid, and so from our EMT (11) and definition (25) we obtain

$$\rho = \frac{2}{\sqrt{|g|}} \left\{ \frac{1}{2} f_k (\partial\psi)^2 + g \left[f_v' V - \frac{1}{2} f_k' (\partial\psi)^2 \right] \right\}\,. \tag{26}$$

Using all of the above in order to find our last unknown, it follows that

$$p = \frac{-2g}{\sqrt{|g|}} \left[f_v' V - \frac{1}{2} f_k' (\partial\psi)^2 \right]\,. \tag{27}$$

With these correspondences, we may translate our EMT (11) into that of a perfect fluid as we intended, and study its behavior. Let us note here that ρ and p are both TDiff scalars. We shall now study our two limiting cases of potential and kinetic domination in the context of the perfect fluid approach.

A. Potential domination in the perfect fluid

In the potential domination regime of our perfect fluid, we find that ρ and p are related by a characteristic equation of state (EoS):

$$p = -\rho = \frac{-2g}{\sqrt{|g|}} f_v' V\,, \tag{28}$$

that is, we have a barotropic fluid $p = p(\rho)$ with the simple EoS $p = w\rho$, where $w = -1$ for any function f_v.

B. Kinetic domination in the perfect fluid

The kinetic domination regime of our perfect fluid leads to ρ and p taking the simplified form

$$\rho = \frac{(\partial\psi)^2}{\sqrt{|g|}} (f_k - g f_k')\,, \quad p = \frac{(\partial\psi)^2}{\sqrt{|g|}} g f_k'\,. \tag{29}$$

It thus follows that we again have a barotropic fluid, whose EoS parameter may now be expressed as

$$w = \frac{p}{\rho} = \frac{g f_k'}{f_k - g f_k'}\,. \tag{30}$$

Finally, we introduce for future reference the variable

$$F \equiv \frac{g f_k'}{f_k}\,, \tag{31}$$

in terms of which the EoS parameter in the kinetic regime is written

$$w = \frac{F}{1-F}\,. \tag{32}$$

Note that the GR limit (i.e. Diff invariance) implies that $f_k \propto \sqrt{|g|} \Leftrightarrow F = 1/2 \Leftrightarrow w = 1$. In other words, in the GR limit we have stiff matter ($p = \rho$).

IV. ENERGY CONDITIONS

In this section we focus on the ECs of our model from the perfect fluid description. Before doing so, however, we remark that with the EMT written in the completely general form (11), the NEC is translated into

$$f_k \geq 0\,. \tag{33}$$

We thus find that the NEC is satisfied whenever the kinetic term is non-negative or, in other words, whenever it is not a ghost field [14]. On another note, the network of implications (DEC $\Rightarrow$ WEC $\Rightarrow$ NEC $\Leftarrow$ SEC) means that if the NEC is violated then so are all the others, and so the minimum requirement if we wish to satisfy some of the ECs is simply that $f_k \geq 0$.

Let us now see what the perfect fluid ECs (20) translate into. The WEC may be written as

$$f_k \geq 0 \quad \& \quad \frac{(\partial\psi)^2}{2} (f_k - g f_k') + g f_v' V \geq 0\,, \tag{34}$$

while the SEC becomes

$$f_k \geq 0 \quad \& \quad \frac{(\partial\psi)^2}{2} (f_k + 2g f_k') - 2g f_v' V \geq 0\,. \tag{35}$$

Regarding the DEC, the absolute value of the pressure we find in (20) implies that this condition splits into two possible cases as follows:

$$
\begin{aligned}
&p \leq 0: \quad f_v' V - \frac{(\partial\psi)^2}{2} f_k' \leq 0 \quad \& \quad f_k \geq 0\,, \\
&p > 0: \quad \begin{cases} f_v' V - \frac{(\partial\psi)^2}{2} f_k' > 0\,, \\ \frac{(\partial\psi)^2}{2}\left(f_k - 2g f_k'\right) + 2g f_v' V \geq 0\,. \end{cases}
\end{aligned} \tag{36}
$$

With the ECs written as above, we have full generality. Nevertheless, in order to simplify their treatment and gain some further insight on their physical implications, we shall now study our two limiting cases of potential and kinetic domination.

A. ECs in potential domination

We previously found that in the potential domination regime of our perfect fluid the EoS was simply $w = -1$. It is easy to see that for such an EoS the NEC is trivially satisfied (the inequality saturates), while the others translate to:

$$
\begin{aligned}
&\text{WEC:} \quad f_v' V \leq 0\,, \quad \text{SEC:} \quad f_v' V \geq 0\,, \\
&\text{DEC:} \quad \begin{cases} p \leq 0: \quad f_v' V \leq 0\,, \\ p > 0: \quad f_v' V > 0 \quad \& \quad f_v' V \leq 0 \quad \text{(contrad.)} \end{cases}
\end{aligned} \tag{37}
$$

We note first of all that if the pressure is negative or zero then the DEC is equivalent to the WEC (since the second inequality trivially saturates), whereas if it is positive then the DEC can never be satisfied. We also find that the only case where the ECs could be simultaneously satisfied is when $f_v' V = 0$. This either means that $V = V_0 = 0$ (which cannot happen if we want potential domination) or $f_v(g) = \text{const}$. It is interesting to note that in the latter case the vanishing of the EMT (15) means that a scalar field with only a potential term that couples through a constant function does not gravitate (in the sense that it does not affect the geometry of spacetime). Thus, even though it satisfies all ECs, such a matter field would be impossible to be detected through gravitational observations. On the other hand, if we want the energy density measured by a comoving observer to be positive (i.e. the WEC to be fulfilled, and hence the DEC as well), then this immediately prevents the SEC from being satisfied. This is not surprising since it is what happens with a vacuum energy, whose EMT takes the form $T_{\mu\nu} \propto g_{\mu\nu}$ as in the present case.

B. ECs in kinetic domination

As we have demonstrated for the general case, the NEC implies $f_k \geq 0$. Nevertheless, $f_k \neq 0$ in kinetic domination, and hence we must consider $f_k > 0$. The rest of the ECs translate as:

$$
\begin{aligned}
&\text{WEC:} \quad f_k \geq 0\,, \quad f_k - g f_k' \geq 0\,, \\
&\text{SEC:} \quad f_k \geq 0\,, \quad f_k + 2g f_k' \geq 0\,, \\
&\text{DEC:} \quad \begin{cases} p \leq 0: \quad f_k' \geq 0 \quad \& \quad f_k \geq 0\,, \\ p > 0: \quad f_k' < 0 \quad \& \quad f_k - 2g f_k' \geq 0\,. \end{cases}
\end{aligned} \tag{38}
$$

V. CONSERVATION OF THE EMT

In GR the conservation of the EMT on the solutions to the EoM is an immediate consequence of the full Diff invariance of the theory (and hence provides no additional information). In a TDiff theory where the symmetry is broken (but the gravitational action is still the Einstein-Hilbert one) the conservation is not trivially satisfied but it is a consequence of the Bianchi identities (which imply that $G_{\mu\nu}$ is divergenceless). Thus, the solutions to the EoM have to fulfill this additional relation. Note that in a Diff invariant theory we have a 4 gauge degrees of freedom, which allow us to fix 4 components of the metric. On the other hand, in a TDiff theory, the constraint $\partial_\mu \xi^\mu = 0$ of the allowed transformations implies that we only have 3 gauge degrees of freedom, and hence only 3 components of the metric may be fixed (see references [6, 15] for a discussion in a cosmological context). The additional metric component is actually physical, so we need an extra equation to find it: this is the constraint we obtain from the fact that the EMT conservation equation is not trivially satisfied. We shall consider in this section our limiting cases of potential and kinetic domination and obtain the corresponding constraints on the metric.

A. EMT conservation in potential domination

The conservation of the EMT in potential domination is quite straightforward as a consequence of the simple EoS $w = -1$. Indeed, in this case the longitudinal projection (22a) is simply $\dot\rho = 0$, which in turn also means that $\dot p = -\dot\rho = 0$. Taking this into account, the transverse projection (22b) simplifies down to $\partial^\mu p = 0$. Now substituting equation (28) in $\partial^\mu p = 0$, and recalling that $V = V_0 = \text{const.}$ on the solutions to the EoM, we obtain

$$
\left(\frac{1}{2} f_v' + g f_v''\right) \partial_\mu g = 0\,. \tag{39}
$$

In this way, if the coupling function f_v is left arbitrary, it follows that we must have a constant determinant:

$$
\partial_\mu g = 0 \implies g = \text{const.}\,, \tag{40}
$$

and we recover UG as solutions in this regime of more general TDiff theories. Another possibility is to allow the metric determinant to change, but then we must require

$$
f_v(g) = A\sqrt{|g|} + B\,, \tag{41}
$$

with A and B constants of integration. Thus, only particular theories (with this f_v) do not restrict the form of g due to the conservation of the EMT. Expression (41) is the most general solution, and different limiting cases may be explored. On the one hand, if we set $B = 0$, we recover GR as a particular solution ($f_v \propto \sqrt{|g|}$) as it should be expected. On the other hand, setting $A = 0$ means that the coupling is done via a constant function which, as we previously discussed, leads to a vanishing EMT and therefore the field does not gravitate. As a final comment, we note that no further information may be gained from the longitudinal projection of the EMT conservation, since substituting (28) in $\dot{\rho} = 0$ yields

$$\left(\frac{1}{2}f_v' + g f_v''\right)\frac{dg}{d\tau} = 0\,, \tag{42}$$

which is trivially satisfied using equation (39).

B. EMT conservation in kinetic domination

The study of the EMT conservation in the kinetic domination regime turns out to be rather more involved, and for this reason we have divided the analysis into smaller, more accessible parts. Firstly, we will rewrite the kinetic EoM in terms of perfect fluid quantities, and also find what their solutions must satisfy. We will also consider the longitudinal and transverse projections of the EMT conservation equation for our particular case, and separately study them in order to see what constraints the EMT conservation imposes on the theory. In passing, we will obtain a very simple expression for the energy density in a general TDiff theory.

1. *EoM and conservation equations*

We shall begin by expanding the kinetic EoM (17) as

$$0 = \partial^\mu\psi\partial_\mu f_k + f_k\partial_\mu\partial^\mu\psi\,. \tag{43}$$

If we denote the normalization constant of the velocity by $N = \sqrt{(\partial\psi)^2}$, so that $\partial^\alpha\psi = N u^\alpha$, and divide the above equation by f_k (which we assume to be different from zero in this section of kinetic domination), we obtain

$$0 = N u^\mu\partial_\mu\left(\ln f_k\right) + \partial_\mu\left(N u^\mu\right)\,. \tag{44}$$

Using equation (8), we can write the second term as

$$\partial_\mu\left(\frac{N\sqrt{|g|}u^\mu}{\sqrt{|g|}}\right) = N\left\{u^\mu\partial_\mu\left(\ln\frac{N}{\sqrt{|g|}}\right) + \nabla_\mu u^\mu\right\}\,. \tag{45}$$

In this expression we recognize the expansion scalar of the congruence, $\theta = \nabla_\mu u^\mu$. This quantity may be directly related to the fractional rate of change of the congruence's cross-sectional volume δV as [13]

$$\theta = \frac{1}{\delta V}\frac{d}{d\tau}(\delta V) = u^\mu\partial_\mu(\ln\delta V)\,. \tag{46}$$

Taking everything into account, we rewrite equation (44) (after simplifying a common factor of N) as

$$\frac{d}{d\tau}\left(\ln\frac{f_k}{\sqrt{|g|}}N\delta V\right) = 0\,, \tag{47}$$

where we have used $u^\mu\partial_\mu = \frac{d}{d\tau}$. Recalling $N = \sqrt{(\partial\psi)^2}$, it finally follows that the solutions to the EoM satisfy

$$(\partial\psi)^2 = \frac{C_\psi(x)}{\left(\delta V f_k/\sqrt{|g|}\right)^2}\,, \tag{48}$$

with $C_\psi(x)$ a function subject to the constraint

$$\dot{C}_\psi(x) = u^\mu\partial_\mu C_\psi(x) = 0\,. \tag{49}$$

It will also be useful to rewrite the EoM in terms of perfect fluid quantities. Taking into account equations (29), we may express N in terms of ρ and p as

$$N^2 = \frac{(\rho+p)\sqrt{|g|}}{f_k}\,, \tag{50}$$

and so the EoM (47) takes the form

$$\frac{d}{d\tau}\left(\ln\sqrt{\frac{f_k}{\sqrt{|g|}}(\rho+p)\delta V^2}\right) = 0\,. \tag{51}$$

From this expression we find that

$$\dot{\rho} + \dot{p} = -(\rho+p)\frac{d}{d\tau}\left(\ln\frac{\delta V^2 f_k}{\sqrt{|g|}}\right)\,. \tag{52}$$

But we also know, from equation (22a), that

$$\dot{\rho} = -(\rho+p)\frac{d}{d\tau}\left(\ln\delta V\right)\,, \tag{53}$$

and so from equations (52) and (53), it follows:

$$\dot{p} = -(\rho+p)\frac{d}{d\tau}\left(\ln\frac{\delta V f_k}{\sqrt{|g|}}\right)\,. \tag{54}$$

These are longitudinal results, in the sense that they hold along the integral curves of the tangent vector field u^μ.

Consider now equation (22b), the projection of the EMT conservation equation onto directions transverse to the fluid's velocity. In that expression, we recognize terms with $\dot{u}^\mu$, u^μ, and $\partial^\mu p$, but it turns out that these three quantities are related. It follows from the definition of the pressure in equation (29) that

$$p = N^2\frac{g f_k'}{\sqrt{|g|}} = \frac{N^2 F f_k}{\sqrt{|g|}}\,, \tag{55}$$

where F was defined in equation (31). Since we wish to take its derivative, let us begin by computing the following:

$$\begin{aligned}\partial^\mu(N^2) &= \nabla^\mu(N^2) = \nabla^\mu(\nabla_\alpha\psi\nabla^\alpha\psi) = 2\nabla_\alpha\psi\nabla^\mu\nabla^\alpha\psi = \\ &= 2N u_\alpha\nabla^\alpha(N u^\mu) = 2N(\dot{N}u^\mu + N\dot{u}^\mu) = \\ &= 2N^2\left(u^\mu\frac{d}{d\tau}(\ln N) + \dot{u}^\mu\right)\,,\end{aligned} \tag{56}$$

where we have recalled that for a torsionless connection $\nabla^\mu\nabla^\alpha\psi = \nabla^\alpha\nabla^\mu\psi$ when acting on scalar functions. Using this result when differentiating equation (55), we find

$$\partial^\mu p = 2p\left\{u^\mu \frac{d}{d\tau}(\ln N) + \dot{u}^\mu + \frac{1}{2}\partial^\mu\left(\ln\frac{Ff_k}{\sqrt{|g|}}\right)\right\}. \tag{57}$$

From this expression we may solve for $\dot{u}^\mu$,

$$\dot{u}^\mu = \frac{1}{2}\partial^\mu\left(\ln\frac{p\sqrt{|g|}}{Ff_k}\right) - u^\mu\frac{1}{2}\frac{d}{d\tau}(\ln N^2). \tag{58}$$

Now, using equations (50) and (55) we can write

$$N^2 = \frac{p\sqrt{|g|}}{Ff_k} = \frac{(\rho+p)\sqrt{|g|}}{f_k}, \tag{59}$$

and inserting the resulting $\dot{u}^\mu$ in (22b), we obtain the transverse equation

$$\begin{aligned}0 = u^\mu\dot{p} - \partial^\mu p + \frac{\rho+p}{2}\Bigg\{\partial^\mu\left(\ln\frac{(\rho+p)\sqrt{|g|}}{f_k}\right) \\ - u^\mu\frac{d}{d\tau}\left(\ln\frac{(\rho+p)\sqrt{|g|}}{f_k}\right)\Bigg\}.\end{aligned} \tag{60}$$

It is immediate to see that the equation is trivially satisfied if we contract with u_μ, as we should expect from transversality.

2. *Longitudinal results*

It is possible to relate the coupling function $f_k(g)$ with the congruence's cross-sectional volume δV by considering our longitudinal equations (53) and (54), together with the EoS in the kinetic limit (32). Since our starting point are the scalars ρ and p, throughout this study a dot on any given quantity will denote $\frac{d}{d\tau}$.

Before proceeding, however, we remark that at different points we shall be interested in dividing by ρ and by $(1+w)$, which of course can only be done when they are different from zero. On the one hand, the case $\rho = 0$ corresponds to $f_k \propto g$, as follows from equation (29), and this is the only possibility (indeed, f_k and $(\partial\psi)^2$ never vanish in kinetic domination). We shall in what follows assume that $\rho \neq 0$. On the other hand, we see from (30) that $w = -1$ can occur either when $f_k = 0$ (impossible in the kinetic limit) or when $gf_k' = \infty$. Assuming well behaved coupling functions with no divergences, we have $w \neq -1$ always.

After this discussion, we can now proceed. Using the EoS and equation (53), the derivative of the pressure is

$$\dot{p} = \dot{w}\rho + w\dot{\rho} = \dot{w}\rho - w\rho(1+w)\frac{d}{d\tau}(\ln\delta V). \tag{61}$$

Equating this expression with equation (54), dividing through by ρ, and reorganizing terms we find

$$\dot{w} = (1+w)\left\{(w-1)\frac{d}{d\tau}(\ln\delta V) - \frac{d}{d\tau}\left(\ln\frac{f_k}{\sqrt{|g|}}\right)\right\}. \tag{62}$$

Recalling what quantities are simply functions of the determinant, it is possible to write

$$\dot{w} = w'\,\dot{g} = (1+w)\frac{F'}{1-F}\,\dot{g}, \tag{63}$$

$$\frac{d}{d\tau}\left(\frac{f_k}{\sqrt{|g|}}\right) = \frac{f_k}{\sqrt{|g|}}\left(F - \frac{1}{2}\right)\frac{\dot{g}}{g}, \tag{64}$$

where we have used equation (32). Substitution gives

$$\begin{aligned}&(1+w)\frac{F'}{1-F}\,\dot{g} = \\ &= (1+w)\left\{(w-1)\frac{d}{d\tau}(\ln\delta V) - \left(F-\frac{1}{2}\right)\frac{\dot{g}}{g}\right\}.\end{aligned} \tag{65}$$

Dividing through by $(1+w)$ and rearranging, we obtain

$$\left\{\frac{F'}{1-F} + \frac{F-1/2}{g}\right\}\dot{g} = (w-1)\frac{d}{d\tau}(\ln\delta V). \tag{66}$$

After dividing this equation by $(w-1)$ ($w = 1$ would correspond to considering the GR limit) and taking into account that from equation (32) we have that

$$w - 1 = \frac{2F-1}{1-F}, \tag{67}$$

it turns out that everything simplifies neatly on the LHS:

$$\begin{aligned}&\frac{1}{w-1}\left\{\frac{F'}{1-F} + \frac{F-1/2}{g}\right\} = \\ &= \frac{1}{2}\left\{\frac{2F'}{2F-1} + \frac{1}{g} - \frac{f_k'}{f_k}\right\} = \frac{1}{2}\frac{d}{dg}\ln\left|(2F-1)\frac{g}{f_k}\right|.\end{aligned} \tag{68}$$

In this way, we have from (66) that

$$\frac{1}{2}\left(\frac{d}{dg}\ln\left|(2F-1)\frac{g}{f_k}\right|\right)\dot{g} = \frac{d}{d\tau}(\ln\delta V). \tag{69}$$

Recognizing a total derivative in the LHS and multiplying the whole equation by 2 it follows that

$$\frac{d}{d\tau}\ln\left|(2F-1)\frac{g}{f_k}\right| = \frac{d}{d\tau}\left(\ln\delta V^2\right). \tag{70}$$

Finally, then, we obtain the following longitudinal constraint:

$$(2F-1)\frac{g}{f_k} = C_g(x)\delta V^2, \tag{71}$$

with $C_g(x)$ a function which must satisfy

$$\dot{C}_g(x) = u^\mu\partial_\mu C_g(x) = 0. \tag{72}$$

Note that the longitudinal constraint (71) generalizes for an arbitrary metric the result obtained in reference [6] for Robertson-Walker, where $\delta V = a^3$ and $C_g = \text{const}$.

3. Transverse results

We now focus on the transverse part of the EMT conservation, and in order to do so we contract the transverse equation (60) with the previously introduced triplet of transverse vectors $\vec{w}_\mu$, obtaining

$$0 = \frac{\rho + p}{2} \vec{\nabla} \left(\ln \frac{(\rho + p)\sqrt{|g|}}{f_k} \right) - \vec{\nabla} p \tag{73}$$

(we remark that throughout this section $\vec{\nabla} = \vec{w}^\mu \partial_\mu$). Simple algebraic manipulations now lead to

$$\vec{\nabla}(\rho - p) = (\rho + p) \vec{\nabla} \ln \left(\frac{f_k}{\sqrt{|g|}} \right) , \tag{74}$$

which using the EoS may as well be written as

$$(1 - w)\vec{\nabla}\rho - \rho\vec{\nabla}w = \rho(1 + w)\vec{\nabla} \ln \left(\frac{f_k}{\sqrt{|g|}} \right) . \tag{75}$$

Recalling once again that w and $(f_k/\sqrt{|g|})$ are actually only functions of the determinant, we have

$$\vec{\nabla}w = w'\vec{\nabla}g = (1 + w)\frac{F'}{1 - F}\vec{\nabla}g , \tag{76}$$

$$\vec{\nabla} \left(\frac{f_k}{\sqrt{|g|}} \right) = \frac{f_k}{\sqrt{|g|}} \left(F - \frac{1}{2} \right) \frac{\vec{\nabla}g}{g} , \tag{77}$$

where we have again used equation (32). Substituting back in equation (75) and rearranging, it follows that

$$(1 - w)\vec{\nabla}\rho = \rho(1 + w) \left\{ \frac{F'}{1 - F} + \frac{F - 1/2}{g} \right\} \vec{\nabla}g . \tag{78}$$

We may now safely divide both sides of the equation by $\rho(1 - w)$, finally arriving at

$$\vec{\nabla} \ln \rho = -\frac{1 + w}{2} \vec{\nabla} \ln \left| (2F - 1)\frac{g}{f_k} \right| . \tag{79}$$

where we have used equation (68). Now, we have already obtained a (longitudinal) constraint in equation (71), and we wonder if the transverse projection of the EMT conservation equations might provide us with any additional information. To this end, we shall substitute the longitudinal constraint (71) in the RHS of equation (79). Recalling also that $1 + w = \frac{1}{1-F}$ and that on the solutions (48) to the EoM

$$\rho = \frac{(\partial\psi)^2}{\sqrt{|g|}} f_k(1 - F) = \frac{C_\psi \sqrt{|g|}(1 - F)}{f_k \delta V^2} , \tag{80}$$

it follows that

$$\vec{\nabla} \ln \left| \frac{C_\psi \sqrt{|g|}(1 - F)}{f_k \delta V^2} \right| = \frac{-1}{2(1 - F)} \vec{\nabla} \ln \left| C_g \delta V^2 \right| . \tag{81}$$

A straightforward calculation (see Appendix A) finally yields the transverse constraint

$$\vec{\nabla} \left(C_g C_\psi \right) = \vec{w}^\mu \partial_\mu \left(C_g C_\psi \right) = 0 . \tag{82}$$

This is an additional condition which relates the function C_g from the longitudinal constraint to the solutions of the EoM, which depend on C_ψ, and it does so in a very particular way. Indeed, we find in (82) that the derivative of the product $C_g C_\psi$ vanishes when projected onto the transverse directions. However, we know that the derivative also vanishes when projected along the longitudinal direction, since $\dot{C}_\psi = \dot{C}_g = 0$ and hence $\frac{d}{d\tau}\left(C_g C_\psi \right) = 0$. Consequently, we find that

$$C_g C_\psi = \text{const.} \equiv c_\rho , \tag{83}$$

i.e. the product is actually a constant (which we have denoted as c_ρ for later convenience), and the two functions are inversely proportional to each other.

4. Expression for the energy density

We will now derive a simple expression for the energy density ρ. In order to do so, we begin with its definition in equation (29), substitute the solutions to the EoM (48), recall (67), and use the longitudinal constraint (71):

$$\begin{aligned} \rho &= \frac{(\partial\psi)^2}{\sqrt{|g|}} f_k(1 - F) = \frac{C_\psi(1 - F)}{\sqrt{|g|}} \frac{|g|}{f \delta V^2} = \\ &= \frac{C_\psi}{\sqrt{|g|}} \frac{2F - 1}{w - 1} \frac{(-g)}{f \delta V^2} = \frac{C_g C_\psi}{(1 - w)\sqrt{|g|}} . \end{aligned} \tag{84}$$

Thus, recalling the consequence (83) of the transverse constraint, we finally obtain the following simple relation:

$$\rho = \frac{c_\rho}{(1 - w)\sqrt{|g|}} . \tag{85}$$

We remark that this expression is well-defined as long as $\rho \neq 0$ and we are not in the GR limit, but other than that it is completely general and valid for all geometries. In particular, it is useful in the study of cosmological perturbations in TDiff scenarios, and it simplifies the treatment that was made in reference [6]. Moreover, since $w = w(g)$, it also shows that the only dependence of both the energy density and the pressure is in the metric determinant, and this means that the possible perturbations of this fluid shall be adiabatic.

VI. SOME PARTICULAR CASES

In this section we shall consider some particular cases of interest. In the potential domination regime we found that the conservation of the EMT on the solutions to the EoM implied either a constant determinant or the particular form (41) for the coupling function f_v. However, in the kinetic domination regime we did not find such a strong constraint, and different models for f_k shall be studied, in particular a power-law and an exponential.

A. Power-law coupling

We begin then with case where the coupling function has the form of a power-law, i.e.

$$f_k(g) = C|g|^{\alpha}\,, \tag{86}$$

with C and α some constant parameters. Note that, in such a case, $F = gf_k'/f_k = \alpha$ and so it follows that the EoS parameter is also constant:

$$w = \frac{\alpha}{1-\alpha}\,. \tag{87}$$

The particular case $\alpha = 0$ means that $f_k = C$ is constant, and also that the EoS is $w = 0$ (i.e. non-relativistic matter). Away from the GR limit, we find from equation (85) that the energy density in these couplings satisfies

$$\rho \propto \frac{1}{\sqrt{|g|}}\,, \tag{88}$$

while the longitudinal constraint (71) gives

$$|g| \propto (C_g \delta V^2)^{\frac{1}{1-\alpha}} = (C_g \delta V^2)^{(1+w)}\,. \tag{89}$$

We note that if $C_g(x) = \text{const.}$ the evolution simplifies:

$$|g| \propto \delta V^{2(1+w)} \implies \rho \propto \delta V^{-(1+w)}\,, \tag{90}$$

and this may be of use in cosmological settings. Next we focus on the ECs in the dominant kinetic regime (38). In this regime we ask that $C \neq 0$, and the ECs translate to

$$\begin{aligned} \text{NEC:} \quad & C > 0\,, \\ \text{WEC:} \quad & C > 0\,, \quad \alpha \leq 1\,, \\ \text{SEC:} \quad & C > 0\,, \quad \alpha \geq -1/2\,, \\ \text{DEC:} \quad & C > 0\,, \quad \alpha \leq 1/2 \end{aligned} \tag{91}$$

They are graphically represented in Figure 1. For $C > 0$ and $\alpha \in [-1/2, 1/2]$ all of the ECs are satisfied, and in this range we could find dark matter in $\alpha = 0$ (which gives $w = 0$ as discussed). If we wish to have accelerated expansion, then the SEC must be violated and this may give rise to different dark energy models. In general, it opens up a wide range of interesting phenomenology for the dark sector to be studied. Finally, we note that this analysis for the power-law model could also apply to a more general coupling function which may be expressed as a power series.

B. Exponential coupling

We now consider that the coupling is an exponential,

$$f_k(g) = Ce^{\beta g}\,, \tag{92}$$

with C and β some constant parameters. In this case, the variable $F = \beta g$ is not a constant, and so neither is the EoS parameter w, which from equation (32) reads

$$w = \frac{\beta g}{1-\beta g}\,. \tag{93}$$

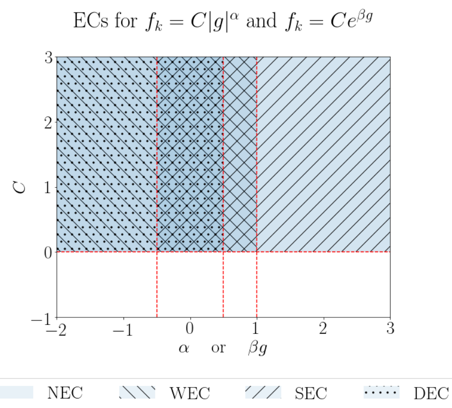

FIG. 1. Regions of validity of the ECs for the two couplings (the axes extend infinitely). The same diagram is valid for both, understanding the horizontal axis as α when considering the power-law and as βg when considering the exponential.

The study of the evolution of the energy density and the metric determinant is thus not particularly simple or illuminating (except perhaps for the case $\beta = 0$, which gives a constant f_k and non-relativistic matter, as was discussed above), so we focus on the ECs in the dominant kinetic regime (38). These take the form:

$$\begin{aligned} \text{NEC:} \quad & C > 0\,, \\ \text{WEC:} \quad & C > 0\,, \quad \beta g \leq 1\,, \\ \text{SEC:} \quad & C > 0\,, \quad \beta g \geq -1/2\,, \\ \text{DEC:} \quad & C > 0\,, \quad \beta g \leq 1/2\,, \end{aligned} \tag{94}$$

where we have again assumed that $C \neq 0$. We may graphically represent them in the same way as we did the power-law, see Figure 1. Nevertheless, the fact that the metric determinant explicitly appears in these ECs means that there will be an evolution, as opposed to the previous case. Since the metric determinant is allowed to vary, it may happen that at some points in spacetime a given EC is satisfied but at others it is not. As a particular example, suppose we choose $\beta > 0$, so that the product βg is always negative. In these cases, the WEC is always satisfied (which guarantees a positive energy density measured by the comoving observer), and so is the DEC (which ensures a causal flow of energy). However, there may be a regime in which the SEC holds which evolves to one where it does not, which in a cosmological context may give us an evolution from a non-accelerated expansion to an accelerated one [6].

VII. CONCLUSIONS

In this work, we have explored the consequences of breaking the Diff invariance of the matter sector down to TDiff in general contexts (i.e. the analysis has been purely geometrical, without assuming a spacetime metric). We have considered a scalar field model which couples to gravity via arbitrary functions of the metric determinant, and studied its limiting cases of potential and kinetic domination. We have performed a description of the model in terms of a perfect fluid, and in doing so we have assumed that the derivative of the field is a timelike vector. The ECs have also been analyzed in each of the two regimes, and we have obtained expressions involving derivatives of the coupling functions and their relations to quantities such as the metric determinant (in the case of f_k) or the potential (in the case of f_v). Let us remark again the fact that the couplings to gravity which we have considered are minimal. In particular, the kinetic term is a canonical one (meaning quadratic), and in situations in which it dominates we have a rather flexible EoS due to the variety of coupling functions f_k. In GR the only possibility is $w = 1$, which means that one must include non-standard (i.e. non-quadratic) kinetic terms in order to allow for new situations (this is for instance the case of k-inflation [16] and k-essence [5]).

An important focus of the work has been the study of the EMT conservation on the solutions to the EoM, which is not automatically satisfied in a TDiff theory, and we have found that it imposes further constraints on the metric. For the potential domination regime we can recover either a UG limit as a possibility (i.e. the metric determinant has to be constant), or obtain that the coupling function has a particular form (which contains GR as a particular limit) to allow the metric determinant to vary. The study of the kinetic domination regime and the constraints it implied resulted in a particularly simple expression for the energy density (only valid in a TDiff theory) in terms of the metric determinant, which could be very useful in perturbative analysis.

We have also studied some particular cases of coupling functions in the kinetic regime. For a power-law model, the evolution of the energy density simplifies further, and the study of the ECs for general values of the exponent provide us with regions where we could for instance find dark matter models. For exponential models, on the other hand, the study of the ECs reveals that the evolution of the metric determinant gives rise to particular models that can cross from regions where some EC is satisfied to other where can be violated. In particular, we conclude that exponential models with a negative exponent satisfy all energy conditions except for the SEC, which will be violated or satisfied in different regions depending on the value of the metric determinant. This is particularly interesting in cosmological contexts, where we could find the transition from a non-accelerated expansion to an accelerated one.

Finally, the present work considers a scalar field as a particular case study with which we may gain some intuition on the physical implications of TDiff invariant gravity. Future work on other interesting situations not considered here includes the study of a spherically symmetric scalar field, as well as the possibility of considering different types of fields (not necessarily scalar). The case in which more than one field is present in a TDiff theory may also be of interest, for instance, in considering possible interactions within the dark sector.

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Appendix A: Calculation of the transverse constraint

Our starting point is equation (81), which we rewrite here for convenience:

$$\vec{\nabla}\ln\left|C_\psi\frac{1-F}{\left(f_k/\sqrt{|g|}\right)\delta V^2}\right| = \frac{-1}{2(1-F)}\vec{\nabla}\ln\left|C_g\delta V^2\right|. \tag{A1}$$

In what follows, we shall abbreviate notation and assume that all logarithms have an absolute value sign included, i.e. $\ln x \equiv \ln|x|$ for the following calculations. Expanding the above expression and rearranging some multiplicative factors, we obtain

$$-2(1-F)\left[\vec{\nabla}\ln C_\psi + \vec{\nabla}\ln(1-F) - \vec{\nabla}\ln\left(f_k/\sqrt{|g|}\right) - \vec{\nabla}\ln\delta V^2\right] = \vec{\nabla}\ln C_g + \vec{\nabla}\ln\delta V^2\,, \tag{A2}$$

and grouping like terms in the RHS it follows that

$$-2(1-F)\left[\vec{\nabla}\ln C_\psi + \vec{\nabla}\ln(1-F) - \vec{\nabla}\ln\left(f_k/\sqrt{|g|}\right)\right] = \vec{\nabla}\ln C_g + (2F-1)\vec{\nabla}\ln\delta V^2\,. \tag{A3}$$

Let us now focus on the LHS of the above equation, which we may write as

$$-2(1-F)\vec{\nabla}\ln C_\psi - 2\vec{\nabla}(1-F) + (1-F)(2F-1)\frac{\vec{\nabla}g}{g} = -2(1-F)\vec{\nabla}\ln C_\psi + (2F-1)\left[\vec{\nabla}\ln(2F-1) + (1-F)\frac{\vec{\nabla}g}{g}\right], \tag{A4}$$

where we have made use of the fact that differentiating a constant yields zero, so that we could very well write

$$-2\vec{\nabla}(1-F) = 2\vec{\nabla}F = \vec{\nabla}(2F) = \vec{\nabla}(2F-1) = (2F-1)\frac{\vec{\nabla}(2F-1)}{(2F-1)} = (2F-1)\vec{\nabla}\ln(2F-1)\,. \tag{A5}$$

Having rewritten the LHS, we may straightforwardly rearrange the result so that we obtain

$$-2(1-F)\vec{\nabla}\ln C_\psi = \vec{\nabla}\ln C_g + (2F-1)\left[\vec{\nabla}\ln\delta V^2 - \vec{\nabla}\ln(2F-1) - (1-F)\frac{\vec{\nabla}g}{g}\right]. \tag{A6}$$

Next we focus on the square bracket on the RHS of the above expression, which using our longitudinal constraint will actually simplify greatly:

$$\vec{\nabla}\ln\delta V^2 - \vec{\nabla}\ln(2F-1) - \frac{\vec{\nabla}g}{g} + F\frac{\vec{\nabla}g}{g} = \vec{\nabla}\ln\left(\frac{\delta V^2}{g(2F-1)}\right) + g\frac{f_k'}{f_k}\frac{\vec{\nabla}g}{g} = \vec{\nabla}\ln\left(\frac{1}{C_g f_k}\right) + \vec{\nabla}\ln f_k = -\vec{\nabla}\ln C_g \tag{A7}$$

Substituting, we obtain

$$-2(1-F)\vec{\nabla}\ln C_\psi = 2(1-F)\vec{\nabla}\ln C_g\,, \tag{A8}$$

which after simplification finally yields the transverse constraint

$$\vec{\nabla}\left(C_g C_\psi\right) = 0\,. \tag{A9}$$