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CHERENKOV GRAVITATIONAL WAVES

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It is believed that Lorentz invariance might not be an exact symmetry of nature at all energy scales, and this possibility has been primarily motivated from quantum gravity. In particular, modified dispersion relations are considered as encapsulating quantum gravity phenomenology. In the present work we propose a class of Lorentz invariance violating phenomenological dispersion relations, different for each particle species, to study the generalized vacuum Cherenkov radiation process. We start by identifying the kinematic regions where the process is allowed, and then compute the energy loss rate due to the emission of vacuum electromagnetic and gravitational Cherenkov radiation. Afterwards, we obtain the constraints for the Lorentz invariance breaking parameters of protons and gravitons via the existence or absence of vacuum gravitational Cherenkov radiation using ultra high energy cosmic ray detections.

I. INTRODUCTION

Electromagnetic Cherenkov radiation occurs when an electrically charged particle travelling through an optical medium has a velocity v larger than lights phase velocity c_γ in that medium. This phenomenon was first detected by P. A. Cherenkov in 1934 [1], and a theoretical explanation was given by I. M. Frank and I. E. Tamm three years later [2]. Cherenkov radiation (CR) can be understood microscopically in the following way [3]. When the particle travels at a velocity $v \leq c_\gamma$, the polarized atoms close to the trajectory emit out of phase radiation, leading to destructive interference. However, when the particle has a velocity $v > c_\gamma$, the wave trains emitted by atoms and molecules are in phase, and the constructive interference taking place in a Mach cone, known as the Cherenkov cone, results in the coherent radiation observed in the direction perpendicular to the cone. Equivalently, this phenomenon can be interpreted both qualitatively and quantitatively as an emission process $a \to a + \gamma$ where a charged particle a emits a photon γ: the particle outruns the electromagnetic field, causing the emission of radiation because of the accumulation of wavefronts propagating from the particle [3, 4]. The rate of energy loss in an optical medium with refractive index n is described classically by the Frank-Tamm formula [2], and receives small corrections when considering quantum mechanics [5]. This formula can also be obtained from the quantum field theory formalism [4]. For the process to be allowed the velocity of the particle must be greater than the phase velocity of light, and thus Cherenkov emission is not possible in vacuum in the framework of special relativity (SR) since $c_\gamma = c$.

Nevertheless, over the last decades there has been a growing interest in studying theoretical frameworks which suggest that Lorentz invariance (LI) may not be an exact symmetry of nature at all energy scales [6]. Such theories appear mainly in the context of quantum gravity (QG) [7, 8], where, if LI is violated, the Planck energy $E_{\rm Pl} \approx 10^{19}$ GeV is expected to be the scale where this symmetry is strongly violated. Yet there is a large energy gap between the highest energy particles detected, Ultra High Energy Cosmic Rays (UHECRs) $\approx 10^{11}$ GeV, and the Planck scale, there should be an interpolation of LI violation (LIV) to the low energy regime, where these particles could be sensitive to small departures from LI. Focusing on astrophysical phenomena, the most straightforward way to implement LIV is to consider modified dispersion relations (MDRs) for particles maintaining the 4−momentum conservation laws. In this scenario kinematics of a wide variety of processes may be affected in what are called threshold effects depending on the MDRs considered, shifting or adding new energy thresholds of existing processes, or allowing completely new reactions.

Vacuum CR may be allowed in a LI violating frame, and it has been exhaustively studied in the electromagnetic sector [9–12] to impose restrictive constraints to LI violating parameters (LIVPs) using high energy astrophysical particles. This idea has also been extended to study the vacuum gravitational CR [4, 13–17], where an arbitrary particle can lose energy due to the emission of gravitons rather than photons. With the recent observations of gravitational waves (GWs) [18, 19], combined constraints for LIVPs were obtained in the gravitational sector.

These studies, however, only modify the dispersion relation in the gravitational sector or the matter sector, but combined effects have not been investigated yet with non-trivial MDRs. The main aim of the present work is to consider MDRs both in the gravitational and the matter sector simultaneously to obtain combined restrictions in the LIVPs via the existence or absence of vacuum gravitational CR.

The remainder of this master in science (MSc) is structured as follows. In Sec. II we propose a phenomenological MDR and study the kinematics of the generalised Cherenkov radiation (GCR) process $a \to a + b$. In Sec. III we apply the results obtained in Sec. II to compute the rate of energy loss of the particle (Sec. III A) for vacuum electromagnetic and gravitational CR (Secs. III B and III C). Sec. IV is devoted to impose constraints in LIVPs using UHECRs, and, finally, in Sec. V we discuss the main results of this work and its limitations. Along this work we use natural units, *i.e.*, we set $\hbar = c = 1$.

II. KINEMATICS OF GENERALISED CHERENKOV RADIATION

A. General aspects of the MDR

At the present it is not clear how LI might be broken, but there are clues about its phenomenological manifestations. One of them is the possibility of new particle decays due to LIV. In order to study the kinematics of this processes, it has previously been proposed [10] that each particle a has, besides its own mass m_a, its own maximum velocity c_a, which is asymptotically achievable if the particle has a non-vanishing mass. The dispersion relation in this scenario is given by

$$E_a^2 = m_a^2 c_a^4 + \boldsymbol{p}_a^2 c_a^2. \tag{1}$$

On the other hand, the Cherenkov effect is usually studied in terms of the refractive index n of an optical medium, which modifies the maximum attainable velocity of light in vacuum c as $c_\gamma = cn^{-1}$. Hereafter we will set $c = 1$. Following this idea, we consider that it is convenient to parameterize the deviation of the maximum attainable velocity c_a for each particle species in terms of a particular refractive index n_a, so that $c_a = n_a^{-1}$ and Eq. (1) can be written as

$$E_a^2 = m_a^2 n_a^{-4} + \boldsymbol{p}_a^2 n_a^{-2}. \tag{2}$$

The existence of this refractive index n_a can be interpreted in the context of quantum gravity as the space-time foam acting as a medium [20].

In this work we propose a phenomenological dispersion relation where the refractive index n_a of a particle a can depend on its energy as a power law

$$n_a = 1 + \mathbb{A}^{(\alpha)} E_{\boldsymbol{p}_a}^{\alpha}. \tag{3}$$

The deviation from unity is the dominant term, hence we are not summing in α. Both the coefficient $\mathbb{A}^{(\alpha)}$, with units $[\mathbb{A}^{(\alpha)}] = [E^{-\alpha}]$, and the exponent α are allowed to be different for each particle species. From now on we shall denote $\mathbb{A} \equiv \mathbb{A}^{(\alpha)}$ except when discussing different values of α. As LI is almost an exact symmetry we will assume that $|\mathbb{A}E_{\boldsymbol{p}_a}^{\alpha}| \ll 1$. Replacing the refractive index (3) in the MDR (2) and considering terms up to first order in $|\mathbb{A}E_{\boldsymbol{p}_a}^{\alpha}|$ one gets

$$E_{\boldsymbol{p}_a}^2 = m_a^2 + \boldsymbol{p}_a^2 - 2\mathbb{A}E_{\boldsymbol{p}_a}^{\alpha}(2m_a^2 + \boldsymbol{p}_a^2). \tag{4}$$

Note that Eq. (4) is invariant under translations and rotations but not under Lorentz boosts due to the dependence of n_a on the energy of the particle. Thus, this MDR only holds in a particular preferred frame, which is usually chosen to be in rest with respect to the CMB. The relative velocity between Earth and the CMB is of order $\mathcal{O}(10^{-3})$ [21] and its effects can be neglected when imposing constraints on the LIVPs.

It should be noted that for the MDR given by Eq. (2) the group velocity and the phase velocity are different in general. The maximum attainable velocity of the particle $c_a = n_a^{-1} = 1 - \mathbb{A}p^{\alpha} + \mathcal{O}(\mathbb{A}^2)$ is referred to the phase velocity, which differs from the group velocity defined as $v_a(E) = \partial E/\partial p$. Indeed, for high energy particles with $p \gg m$ one has, using Eq. (4) and neglecting the mass in the last term,

$$v_a = 1 - \frac{m^2}{2p^2} - (\alpha+1)\mathbb{A}p^{\alpha} + \mathcal{O}(\mathbb{A}^2), \tag{5}$$

where p is the module of the 3−momentum of the particle. We then see that the maximum achievable group velocity, $v_a(m = 0) = 1 - (\alpha+1)\mathbb{A}p^{\alpha}$, is different from that given by c_a when $\alpha \neq 0$. As we will see, for the processes we want to study, the kinematics are governed by the relation between the maximum attainable phase velocities c_a of the particles, or, equivalently, the relation between the refractive indexes n_a.

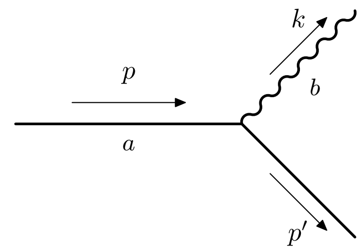

FIG. 1. Tree level GCR process where the particle a (continuous line) emits a massless particle b (wiggly line).

The MDR (4) could be used to study the shift in the threshold energy or new thresholds of existing reactions. If the mass of the particle is high enough compared to the LI violating term, then the usual dispersion relation holds and kinematics remain, essentially, unmodified. However, if the LI breaking term is comparable to the mass of the particle, significant deviations are expected in the kinematics of a wide variety of reactions. As the thresholds of the processes are determined by the mass of the particles, modifications to those thresholds are appreciable when the third term in the MDR (4) is of the same order as the mass m_a. For high energy particles $p \gg m$ the energy $p_{\rm dev}$ at which the deviation becomes important can be estimated by

$$p_{\rm dev} \sim \left|\frac{m^2}{2\mathbb{A}}\right|^{1/(\alpha+2)}. \tag{6}$$

In this MSc we are interested in studying the existence of entirely new reactions that are forbidden in SR; in particular we will consider the GCR process (see Fig. 1), *i.e.*, the two body decay process $a \to a + b$ where a particle a emits a massless particle b. Here we mention that forbidden processes in the LI scenario cannot be studied in the scope of Doubly Special Relativity (DSR): the deformed 4−momentum composition produces cancellations with the LI violating terms in such a way that this phenomenology is not possible [7].

B. Derivation of the Cherenkov angle

Perhaps the key feature of CR is the angle of emission θ_c between the 3−momentum $\boldsymbol{p}$ of the initial particle a and the 3−momentum $\boldsymbol{k}$ of the emitted particle b. If we define $E_{\boldsymbol{p}} := E_a(\boldsymbol{p})$, $E_{\boldsymbol{p}'} := E_a(\boldsymbol{p}')$ and $E_{\boldsymbol{k}} := E_b(\boldsymbol{k})$, where $\boldsymbol{p}'$ is the 3− momentum of the particle a after the emission, then the conservation of energy and 3−momentum, which are written as

$$E_a(\boldsymbol{p}) = E_b(\boldsymbol{k}) + E_a(\boldsymbol{p}'), \qquad \boldsymbol{p} = \boldsymbol{k} + \boldsymbol{p}', \tag{7}$$

allow to obtain θ_c in terms of the module of the 3-momentum $\boldsymbol{p}$ and $\boldsymbol{k}$. Note that Eq. (7) assumes the usual 4−momentum composition, hence DSR theories will not be considered.

Let us use the MDR given by Eq. (2) with refractive index (3) to obtain the Cherenkov angle θ_c or, equivalently, $\cos\theta_c$. Let m be the mass of particle a and denote $n_{\boldsymbol{p}} := n_a(\boldsymbol{p})$, $n_{\boldsymbol{p}-\boldsymbol{k}} := n_a(\boldsymbol{p}-\boldsymbol{k})$ and $n_{\boldsymbol{k}} := n_b(\boldsymbol{k})$, where we already applied the conservation of 3−momentum. From Eq. (2) we see that the energies of the particles in the GCR process are given by

$$\begin{aligned} E^2_{\boldsymbol{p}} &= m^2 n^{-4}_{\boldsymbol{p}} + p^2 n^{-2}_{\boldsymbol{p}}, \\ E^2_{\boldsymbol{p}-\boldsymbol{k}} &= m^2 n^{-4}_{\boldsymbol{p}-\boldsymbol{k}} + (p^2 + k^2 - 2pk\cos\theta_c) n^{-2}_{\boldsymbol{p}-\boldsymbol{k}}, \\ E^2_{\boldsymbol{k}} &= k^2 n^{-2}_{\boldsymbol{k}}, \end{aligned} \tag{8}$$

where we have used $(\boldsymbol{p}-\boldsymbol{k})^2 = p^2 + k^2 - 2pk\cos\theta_c$ and denoted the module of the 3−momentum of the particles a and b as p and k, respectively. On the other hand, combining the conservation of energy and 3−momentum we have $E^2_{\boldsymbol{p}-\boldsymbol{k}} = [E_{\boldsymbol{p}} - E_{\boldsymbol{k}}]^2$. Substituting here the energies given by Eq. (8) we get

$$\begin{aligned} 2pk n^{-2}_{\boldsymbol{p}-\boldsymbol{k}} \cos\theta_c =& p^2 (n^{-2}_{\boldsymbol{p}-\boldsymbol{k}} - n^{-2}_{\boldsymbol{p}}) + m^2 (n^{-4}_{\boldsymbol{p}-\boldsymbol{k}} - n^{-4}_{\boldsymbol{p}}) \\ &+ k^2 (n^{-2}_{\boldsymbol{p}-\boldsymbol{k}} - n^{-2}_{\boldsymbol{k}}) + 2 n^{-1}_{\boldsymbol{k}} k \left[n^{-2}_{\boldsymbol{p}} p^2 + n^{-4}_{\boldsymbol{p}} m^2 \right]^{1/2}. \end{aligned} \tag{9}$$

Note that in the LI scenario, which corresponds to $n_a = n_b = 1$, the RHS of Eq. (9) is always greater than the LHS unless $\cos\theta_c > 1$; hence, the process is forbidden. The limit case where the particle a has no mass is of no interest, as the emission rate, which we compute in Sec. III, vanishes for $\cos\theta_c = 1$.

Now let $\mathbb{A}$ and α be the LIVP and exponent of the refractive index of the particle a, and $\mathbb{B}$ and β those of the particle b, such that

$$n_a(\boldsymbol{p}) = 1 + \mathbb{A} E^{\alpha}_a(\boldsymbol{p}), \qquad n_b(\boldsymbol{k}) = 1 + \mathbb{B} E^{\beta}_b(\boldsymbol{k}). \tag{10}$$

Assuming that $|\mathbb{A}E^{\alpha}_{\boldsymbol{p}}|, |\mathbb{A}E^{\alpha}_{\boldsymbol{p}-\boldsymbol{k}}|, |\mathbb{B}E^{\alpha}_{\boldsymbol{k}}| \ll 1$ we can replace these refractive indexes in Eq. (9) and expand up to first order to obtain

$$\begin{aligned} 2pk\cos\theta =& 4pk\mathbb{A}E^{\alpha}_{\boldsymbol{p}-\boldsymbol{k}}\cos\theta - 2\mathbb{A}(p^2+2m^2)[E^{\alpha}_{\boldsymbol{p}-\boldsymbol{k}} - E^{\alpha}_{\boldsymbol{p}}] \\ &+ 2k(p^2+m^2)^{1/2}\left[1 - \mathbb{B}E^{\beta}_{\boldsymbol{k}}\right]\left[1 - 2\mathbb{A}\frac{(p^2+2m^2)E^{\alpha}_{\boldsymbol{p}}}{p^2+m^2}\right]^{1/2} \\ &- 2k^2[\mathbb{A}E^{\alpha}_{\boldsymbol{p}-\boldsymbol{k}} - \mathbb{B}E^{\beta}_{\boldsymbol{k}}]. \end{aligned} \tag{11}$$

This is an implicit equation in $\cos\theta$ as the energy $E_{\boldsymbol{p}-\boldsymbol{k}}$ depends on $(\boldsymbol{p}-\boldsymbol{k})^2$, which in turn depends on $\cos\theta$.

In Sec. IV we will consider UHECR observations to constrain the LI breaking parameters $\mathbb{A}$ and $\mathbb{B}$. For these particles $p \gg m$, and it can also be checked *a posteriori* that the threshold is greatly modified, that is, the momentum is much larger than the right hand side of Eq. (6). We can then consider terms up to $\mathcal{O}(m^2/p^2)$ and neglect first order products of the LI violating terms with the quotient m^2/p^2. Applying these additional approximations in Eq. (11) one gets

$$\cos\theta_c := 1 - \Theta_c \tag{12}$$

where

$$\Theta_c := \frac{p-k}{pk}\left\{(p-k)\mathbb{A}E^{\alpha}_{p-k} + \mathbb{B}kE^{\beta}_{k} - \mathbb{A}pE^{\alpha}_{p}\right\} - \frac{m^2}{2p^2}. \tag{13}$$

As Θ_c is already of first order in the LI violating terms and m^2/p^2, the 3−vectors $\boldsymbol{p}$ and $\boldsymbol{k}$ involved in the energies are considered collinear and no second order corrections are taken into account. In other words, Θ_c does not depend on $\cos\theta_c$ and, consequently, Eq. (12) completely characterizes the Cherenkov angle θ_c in terms of the initial 3−momentum of the particle a and the 3−momentum carried off by the particle b.

C. Threshold condition

Attending to Eq. (12), the Cherenkov process $a \to a + b$ is allowed as long as $\Theta_c > 0$ such that $\cos\theta_c < 1$, and hence the threshold is determined by the equation

$$\Theta_c = 0. \tag{14}$$

The study of the threshold condition is easier to carry out if the energies in Eq. (13) are substituted by the momentum, which is correct up to first order, giving

$$\Theta_c = \frac{p-k}{p} f(k) - \frac{m^2}{2p^2}, \tag{15}$$

$$f(k) := \mathbb{B}k^{\beta} - \frac{\mathbb{A}}{k}\left\{p^{\alpha+1} - (p-k)^{\alpha+1}\right\}. \tag{16}$$

In what follows we will consider $k \le p$, and this is satisfied particularly when studying the threshold condition. Indeed, if $k > p$ then the massless particle would be antiparallel to the incoming particle, and this is forbidden by the threshold theorem [22]: *if $E_{\boldsymbol{p}}$ is a strictly monotonically increasing function of p for $p > 0$ for all particles, then all thresholds for processes with two particle final states occur when the final momentum are parallel.* The monotonicity condition is satisfied as the LI violating terms are much smaller than the momentum of the particle, and $E_{\boldsymbol{p}}$ is also a rotational-invariant function of $\boldsymbol{p}$ as has been previously discussed.

Before focusing on the threshold condition, it is possible to establish whether the process is allowed or forbidden attending to the signs of the parameters $\mathbb{A}$ and $\mathbb{B}$. Two cases can be studied without loss of generality.

1. $\underline{\mathbb{A} > 0 \text{ and } \mathbb{B} < 0}$. It is easy to see from Eq. (15) that $\Theta_c < 0$ for all $k < p$, as the energy of the incoming particle is always greater than the energy of the particles in the final state. Therefore, *Cherenkov emission is forbidden.* In this scenario $n_a > n_b$ and the intuitive condition that the particle a must have a greater phase velocity than the particle b in order for the process to be allowed is fulfilled.

2. $\underline{\mathbb{A} < 0 \text{ and } \mathbb{B} > 0}$. In this case we see that all the terms in Eq. (15) are positive except the correction $m^2/(2p^2)$ due to the mass of the particle. For particles with energies many orders of magnitude greater than the one established by the RHS of Eq. (6) this term might be neglected and, thus, *Cherenkov emission is allowed* for $k \lesssim p$. Now we have that $n_a < n_b$, which fits with the idea that the process is possible since the particle a has a greater phase velocity than the particle b.

For the remaining two cases, the sign of Θ_c depends on the signs of $\mathbb{A}$ and $\mathbb{B}$ but also on the momentum p and k. It is then necessary to obtain the threshold momentum of the emitted particle b, namely k_{th}, in order to determine the values of k for which the process is allowed. We remark that the following results of this section are valid for $\alpha \geq 0$ and $\beta \geq 0$[1].

Before doing so, we can study the relation between $\mathbb{A}$ and $\mathbb{B}$ when the emitted particle b has arbitrarily low momentum, $k \to 0$, or carries off all of the momentum of the particle a, $k \to p$. In the former case, it is easy to see from Eq. (15) that Cherenkov emission is possible if:

$$\begin{cases} \mathbb{B} - (\alpha+1)\mathbb{A}p^{\alpha} \geq \dfrac{m^2}{2p^2}, & \beta = 0; \\ \mathbb{A} \leq -\dfrac{m^2}{2(\alpha+1)p^{\alpha+2}}, & \beta > 0. \end{cases} \tag{17}$$

Note that the first case corresponds to a constant refractive index n_b. The process is not allowed when $k = p$ and $m \neq 0$, but for sufficiently high energy particles such that the mass may be neglected the process is allowed as long as

$$\mathbb{B}p^{\beta} - \mathbb{A}p^{\alpha} \geq 0. \tag{18}$$

The threshold condition is recovered setting an equal sign in Eqs. (17) and (18).

[1] Recent theoretical results [23] showed that QG phenomenology in the infrared (*i.e.*, negative values of the exponents) could be modified, but this has not been yet studied in the context of particles of astrophysical origin. We thus not consider $\alpha, \beta < 0$ throughout this work.

It is also possible to prove in the massless limit that if the threshold condition $f(k_{\text{th}}) = 0$ has a solution, then k_{th} is unique. Let us first focus on the case where $\mathbb{A} > 0$ and $\mathbb{B} > 0$. We start by calculating the derivative of $f(k)$, which is given by

$$f'(k) = \beta\mathbb{B}k^{\beta-1} + \mathbb{A}k^{-2}p^{\alpha+1}g(k), \tag{19}$$

$$g(k) = 1 - \left(1 - \frac{k}{p}\right)^{\alpha}\left(1 + \frac{\alpha k}{p}\right). \tag{20}$$

It is easy to check that, for the physical momentum $k \in [0, p]$, $g'(k) > 0$ and $g(0)$, $g(p) > 0$; hence $g(k) > 0$. The remaining terms in Eq. (19) are positive, so we also have that $f'(k) > 0$; in other words, $f(k)$ is a monotonically increasing function. Therefore, if $f(0) < 0$ and $f(p) > 0$, the solution k_{th} is unique. The condition $f(0) < 0$ is satisfied when Eq. (17) does not hold, and $f(p) > 0$ when Eq. (18) is fulfilled. The threshold momentum k_{th} here establishes the minimum momentum that particle b must have in order to have Cherenkov emission with momentum $k \in [k_{\text{th}}, p]$, and it is permitted as long as $k_{\text{th}} < p$. The limit case $k_{\text{th}} = p$ occurs setting an equal sign in Eq. (18).

The prove is similar when $\mathbb{A} < 0$ and $\mathbb{B} < 0$. In this case one can check that $f(k)$ is a monotonically decreasing function, and so there exists a unique solution k_{th} when $f(0) > 0$ and $f(p) < 0$. The first condition is met when Eq. (17) is satisfied, and the second one when Eq. (18) is not fulfilled. Hence, k_{th} establishes the maximum momentum that particle b can have up to where the Cherenkov effect is allowed, and thus $k \in [0, k_{\text{th}}]$. The process is permitted when $k_{\text{th}} > 0$, and the limit case $k_{\text{th}} = 0$ is given setting an equal sign in Eq. (17).

When mass is considered in the previous analysis, one has to take into account that $\Theta_c(p) < 0$ always. The discussion when $\mathbb{A} < 0$ and $\mathbb{B} < 0$ holds: if $f(0) > 0$ there is only one solution to the thresholds condition and it is the maximum momentum that particle b can have. In the case where $\mathbb{A} > 0$ and $\mathbb{B} > 0$, when $f(0) < 0$ it is possible to have two solutions, so that $k \in [k_{\text{min}}, k_{\text{max}}]$ with $k_{\text{max}} < p$, or no solutions.

The threshold condition $\Theta_c = 0$ does not have a general solution k_{th} for arbitrary α and β since it is a polynomial equation of degree $\max\{\alpha+1, \beta+1\}$. Nevertheless, particular solutions for small values of α and β can be derived, and further simplified in the massless limit $m \to 0$.

For example, consider the case where $\alpha = 1$ and $\beta = 0$. One easily checks from Eq. (16) that the threshold momentum for $m = 0$ is given by

$$k_{\text{th}}(\alpha = 1, \beta = 0) = 2p - \frac{\mathbb{B}}{\mathbb{A}}. \tag{21}$$

Let us start discussing the case $\mathbb{A}, \mathbb{B} > 0$, where the threshold momentum is the minimum value of k for the process to be allowed. Then, $k_{\text{min}} > p$ if $\mathbb{B} < \mathbb{A}p$ and the process is forbidden, as can be seen from Eq. (18). Otherwise, the process is allowed with minimum momentum given by Eq. (21); in particular, the process is allowed

α	β	$k_{\rm th}$
0	0	No threshold
0	$\mathbb{R}^+$	$\left(\frac{\mathbb{A}}{\mathbb{B}}\right)^{1/\beta}$
1	0	$2p - \frac{\mathbb{B}}{\mathbb{A}}$
1	1	$\frac{2\mathbb{A}p}{\mathbb{A}+\mathbb{B}}$
1	2	$\frac{1}{2\mathbb{B}}\left(-\mathbb{A} \pm \sqrt{\mathbb{A}^2 + 8\mathbb{A}\mathbb{B}}\right)$
2	0	$\frac{1}{2\mathbb{A}}\left(3\mathbb{A}p \pm \sqrt{4\mathbb{A}\mathbb{B} - 3\mathbb{A}^2p^2}\right)$
2	1	$\frac{1}{2\mathbb{A}}\left(3\mathbb{A}p + \mathbb{B} \pm \sqrt{\mathbb{B}^2 + 6\mathbb{A}\mathbb{B}p - 3\mathbb{A}^2p^2}\right)$
2	2	$\frac{1}{2(\mathbb{A}-\mathbb{B})}\left(3\mathbb{A}p \pm \sqrt{12\mathbb{A}\mathbb{B} - 3\mathbb{A}^2p^2}\right)$

TABLE I. Threshold momentum $k_{\rm th}$ for the particle b that satisfy the condition $f(k) = 0$. The $\pm$ signs in the quadratic solutions depend on the signs of $\mathbb{A}$ and $\mathbb{B}$, and these are chosen so that $k_{\rm th}$ is positive for those LIVPs that fulfill either the condition (17) or (18). We emphasize again that $k_{\rm th}$ establishes the minimum momentum of particle the b when $\mathbb{A}, \mathbb{B} > 0$, and a maximum momentum when $\mathbb{A}, \mathbb{B} < 0$.

for all k when $\mathbb{B} > 2\mathbb{A}p$, see Eq. (17). The analysis is reversed when considering $\mathbb{A}, \mathbb{B} < 0$. Now Eq. (21) establishes the maximum momentum up to where the process is allowed. As long as $\mathbb{B} < 2\mathbb{A}p$ the process is permitted, and for $\mathbb{B} < \mathbb{A}p$ it is possible for all values of k; for $\mathbb{B} > 2\mathbb{A}p$ Cherenkov emission is not possible as $k_{\max} < 0$.

A similar analysis can be carried out solving Eq. (14) for different values of α and β, and study whether the process is allowed or not attending to the signs of $\mathbb{A}$ and $\mathbb{B}$ and to Eqs. (17) and (18). Table I illustrates the solutions to the threshold condition in the massless limit for all the values $\alpha, \beta \in \{0, 1, 2\}$. In the particular case $\alpha = \beta = 0$ the process is allowed only when $\mathbb{B} > \mathbb{A}$ even though there is not a threshold momentum.

III. DYNAMICS OF GENERALISED CHEREKOV RADIATION

In this section we focus on the calculation of the rate of energy loss $\mathrm{d}E/\mathrm{d}t$ of the particle a in the GCR process $a \to a + b$. We present the general procedure for the tree-level diagram shown in Fig. 1 for an arbitrary interaction vertex, and then we will study the particular cases where the emitted particle is either a photon γ (electromagnetic CR) or a graviton h (gravitational CR).

A. Decay rate and energy loss

It is well known [24] that the differential decay rate of a two body process in an arbitrary reference frame, LI scenario is given by

$$\mathrm{d}\Gamma = \frac{1}{8\pi^2 E_{\boldsymbol{p}}} \frac{\mathrm{d}^3\boldsymbol{p}'}{2E_{\boldsymbol{p}'}} \frac{\mathrm{d}^3\boldsymbol{k}}{2E_{\boldsymbol{k}}} \delta^{(4)}(p - p' - k)\overline{|\mathcal{M}(p \to p', k)|^2}. \tag{22}$$

Here $\boldsymbol{p}$ is the 3$-$momentum of the initial particle, $\boldsymbol{p}'$ and $\boldsymbol{k}$ are the 3$-$momentum of the final particles, and $\mathcal{M}(p \to p', k)$ is the Lorentz invariant matrix element which depends on the interaction vertex considered. The Dirac delta imposes the conservation of the energy and the 3$-$momentum, and the 3$-$momentum differentials come from the Lorentz invariant phase space (LIPS).

It is important to understand whether Eq. (22) is allowed to be used in a LI breaking scenario, as is the case with our MDRs. When LI is not broken, the canonical commutation relations between creation and annihilation operators of the field are $[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$ when the usual factor $(2E_{\boldsymbol{p}})^{-1/2}$ in the momentum integral of the field operators is included. This factor on the field operators is reflected in the structure of the LIPS in Eq. (22). This is no longer true when LI does not hold. Indeed, the wave function of the particles involved in the process will have a different normalization condition, including corrections due to the LI breaking term in the MDR. Hence, to ensure the canonical normalization condition, field operators will no longer be normalized by a factor $(2E_{\boldsymbol{p}})^{-1/2}$ and the LIPS in Eq. (22) should receive additional corrections. As we shall see during this section, the matrix elements $\mathcal{M}$ that we will consider are at least of the same order as the LI violating terms; thus, as long as we compute the decay rate in the preferred frame this correction appears as a normalization factor and can be neglected. Eq. (22) is then valid for the purposes of this work [12].

Let us now compute the integrals in Eq. (22). First of all, if we use the conservation of the 3$-$momentum we can immediately integrate over $\boldsymbol{p}'$ and obtain

$$\mathrm{d}\Gamma = \frac{1}{8\pi^2 E_{\boldsymbol{p}}} \frac{1}{2E_{\boldsymbol{p}-\boldsymbol{k}}} \frac{\mathrm{d}^3\boldsymbol{k}}{2E_{\boldsymbol{k}}} \delta(E_{\boldsymbol{p}} - E_{\boldsymbol{p}-\boldsymbol{k}} - E_{\boldsymbol{k}}) \cdot \cdot\overline{|\mathcal{M}(p \to p', k)|^2}, \tag{23}$$

where $\mathcal{M}$ is assumed to be evaluated at $p' = (E_{\boldsymbol{p}-\boldsymbol{k}}, \boldsymbol{p} - \boldsymbol{k})$. To integrate over $\boldsymbol{k}$ we can use spherical coordinates and fix the 3$-$momentum $\boldsymbol{p}$ along the z axis. Note that this is possible since the MDR is invariant under rotations (and thus is the decay rate). As will be seen, the matrix element $\mathcal{M}(p \to p', k)$ for the processes we consider does not depend on the azimutal angle, so

$$\Gamma = \frac{1}{16\pi E_{\boldsymbol{p}}} \int \mathrm{d}k\, k^2 \int_{-1}^{1} \mathrm{d}\cos\theta\, \frac{1}{E_{\boldsymbol{p}-\boldsymbol{k}} E_{\boldsymbol{k}}} \cdot \cdot\delta(E_{\boldsymbol{p}} - E_{\boldsymbol{p}-\boldsymbol{k}} - E_{\boldsymbol{k}})\overline{|\mathcal{M}(p \to p', k)|^2}. \tag{24}$$

On the other hand, elemental properties of the Dirac delta allow us to write

$$\delta(E_{\boldsymbol{p}} - E_{\boldsymbol{p}-\boldsymbol{k}} - E_{\boldsymbol{k}}) = 2E_{\boldsymbol{p}-\boldsymbol{k}}\delta(E^2_{\boldsymbol{p}-\boldsymbol{k}} - [E_{\boldsymbol{p}} - E_{\boldsymbol{k}}]^2). \tag{25}$$

When deriving the threshold condition in Sec. II, it can be shown from Eq. (11) that

$$E_{\boldsymbol{p}-\boldsymbol{k}}^2 - [E_{\boldsymbol{p}} - E_{\boldsymbol{k}}]^2 = 2pk\left(\cos\theta_c - 1 + \Theta_c\right), \quad (26)$$

which substituting in Eq. (25) gives

$$\delta(E_{\boldsymbol{p}} - E_{\boldsymbol{p}-\boldsymbol{k}} - E_{\boldsymbol{k}}) = \frac{E_{\boldsymbol{p}-\boldsymbol{k}}}{pk}\delta(\cos\theta_c - 1 + \Theta_c). \quad (27)$$

Finally, integrating over $\cos\theta_c$ one gets

$$\Gamma = \frac{1}{16\pi p E_{\boldsymbol{p}}}\int \mathrm{d}k\, H(\Theta_c)\overline{|\mathcal{M}(p\to p', k)|^2}. \quad (28)$$

The integration limits for k are determined by the threshold condition (14), being $k = 0$ and $k = p$ the lowest and highest integration limits possible, respectively. $H(\Theta_c)$ is the Heaviside function and establishes whether the process is kinematically allowed or forbidden.

Eq. (28) is valid for any process $a \to a + b$ with the MDR proposed with a matrix element $\mathcal{M}$ that does not depend on the azimutal angle. However, the emission rate does not allow us to impose any constraint on the LIVPs unlike the energy loss. To consider the energy carried off by the particle b we must insert the energy k of the particle in the integral of Eq. (28), obtaining the following energy loss rate of the particle a:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{16\pi p E_{\boldsymbol{p}}}\int \mathrm{d}k\, kH(\Theta_c)\overline{|\mathcal{M}(p\to p', k)|^2}. \quad (29)$$

To evaluate the matrix element $\mathcal{M}$ we will consider, for simplicity, the case where the particle a is a complex scalar field, even though the constraints obtained in Sec. IV for the LIVPs are applied to fermionic particles. It is expected that spin corrections are of order $\mathcal{O}(1)$ and thus are not significant enough to modify the order of magnitude of the constraints [4].

B. Electromagnetic Cherenkov radiation

Let us consider the assignation of 4−momentum given in Fig. 1, where the particle b is now a photon γ. Feynman rules for a charged complex scalar field applied to this process give the following matrix element [25]:

$$i\mathcal{M} = -ie_a(p^\mu + p'^\mu)\epsilon_\mu(k). \quad (30)$$

Here ϵ_μ is the polarization vector of the photon, and e_a the electric charge of the particle a. The analysis we will carry out does not distinguish between polarizations, hence we must sum over the two transverse polarization states of the photon when computing the squared matrix element. Using the conservation of 3−momentum we get

$$\overline{|\mathcal{M}|^2} = \sum_\epsilon e_a^2\left|(2p^\mu - k^\mu)\epsilon_\mu(k)\right|^2. \quad (31)$$

The physical polarizations are perpendicular to the 4−momentum of the photon, therefore $k^\mu\epsilon_\mu(k) = 0$. The product $p^\mu\epsilon_\mu(k)$ is easily calculated taking into account that the two polarizations are perpendicular to each other and the angle between $\boldsymbol{p}$ and $\boldsymbol{k}$ is θ_c. This yields

$$\overline{|\mathcal{M}|^2} = 4e_a^2p^2\sin^2\theta_c = 8e_a^2p^2\Theta_c + \mathcal{O}(\Theta_c^2), \quad (32)$$

where we used Eq. (12) and considered the LI violating terms up to first order, and denoted $p \equiv |\boldsymbol{p}|$. Substituting this matrix element in Eq. (29) the electromagnetic energy loss rate is then given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{e_a^2}{2\pi}\int \mathrm{d}k\, k\Theta_c(k)H(\Theta_c). \quad (33)$$

Note that we also approximated $p/E_{\boldsymbol{p}} \simeq 1 + \mathcal{O}(\Theta_c)$ outside the integral as its corrections in the energy loss rate would become of order $\mathcal{O}(\Theta_c^2)$.

The key feature of this emission rate is that it is of first order in Θ_c, and hence in the LI breaking parameters $\mathbb{A}$, $\mathbb{B}$ and m^2/p^2. We have checked that the results obtained here coincide with those in [4] when considering $\mathbb{A} = 0$ and $\beta = 0$. Similar MDRs have been used to study vacuum electromagnetic CR [11, 12] and it was shown that for $\alpha = \beta \neq 0$ a significant fraction of the energy of the particle a is emitted almost immediately when a has an energy above the threshold of the process. We do not present the results but we have checked that this happens when $\alpha \neq \beta$ and neither are equal to 0. In particular, when $\alpha = \beta = 0$, the energy loss rate is suppressed by the difference of the maximum attainable velocities of the particles, $\mathrm{d}E/\mathrm{d}t \sim (c_a^2 - c_\gamma^2)\alpha E^2$, where α is the fine structure constant [10]. Thus, a kinematical analysis is enough to establish stringent constraints to LIVPs and it is not necessary to consider propagation effects, *i.e.*, compute the energy loss rate. In this scenario the study of the threshold condition in Sec. II C should be carried out obtaining the threshold momentum of the particle a, namely p_{th}. Constraints can then be imposed using high energy astrophysical observations and considering the energy of the particle a detected as the threshold energy of the process [9–12].

C. Gravitational Cherenkov radiation

We now consider the case where the particle b is a graviton h. The Feynman rules for the process $a \to a + h$ with a a complex scalar field can be found in [26]:

$$i\mathcal{M} = -i\sqrt{4\pi G_{\mathrm{N}}}(m^2\eta_{\mu\nu} + C_{\mu\nu\rho\sigma}p^\rho p'^\sigma)\epsilon^{\mu\nu}, \quad (34)$$

$$C_{\mu\nu\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}. \quad (35)$$

Here G_{N} is the gravitational constant, and $\epsilon^{\mu\nu}$ is the polarization tensor for the graviton field $h_{\mu\nu}$. This tensor is traceless ($\epsilon_{\mu\nu}\eta^{\mu\nu} = 0$) and transverse ($\epsilon_{\mu\nu}k^\mu = 0$), and can be constructed in terms of the polarization vectors of massive vector bosons [26]. As in the electromagnetic case, we do not distinguish between the polarizations of the graviton and thus we sum over the two physical polarization states. Using Eq. (35) and substituting in the matrix element (34) one finds

$$\overline{|\mathcal{M}|^2} = 16\pi G_{\rm N} \sum_{\epsilon} |p_\mu p_\nu \epsilon^{\mu\nu}|^2 = 16\pi G_{\rm N} p^4 \sin^4\theta_c, \quad (36)$$

where $p \equiv |\boldsymbol{p}|$. Instead of the electric charge e_a^2 we have the gravitational constant times the 3−momentum squared $G_{\rm N}p^2$, and the emission is reduced for small opening angles, having $\sin^4\theta_c$ rather than $\sin^2\theta_c$. Substituting the matrix element (36) in the energy loss rate (29) and considering the lowest order in Θ_c in Eq. (12) we obtain

$$\frac{{\rm d}E}{{\rm d}t} = 4G_{\rm N}p^2 \int {\rm d}k\, k\Theta_c^2(k) H(\Theta_c). \quad (37)$$

We first notice that the energy loss rate (37) of the vacuum gravitational CR, due to the tensor nature of the gravitational field h, is of order $\mathcal{O}(\Theta_c^2)$, whereas the energy loss rate (33) of the electromagnetic CR is of order $\mathcal{O}(\Theta_c)$ because of the vector nature of the EM field. On the other hand, the electromagnetic coupling constant is much stronger than the gravitational coupling: the factor $e_a^2/(2\pi)$ in Eq. (33) is typically of order $\mathcal{O}(10^{-2})$, while in contrast the factor $G_{\rm N}p^2$ in (37) is of order $\mathcal{O}(10^{-14})$ for the highest energy particles observed in the universe. Therefore, as expected, the energy loss rate because of gravitational CR is much smaller than the one caused by the emission of electromagnetic CR. In the former case the decay rate of a particle of astrophysical origin can be comparable to its travel time and it must be taken into account in order to constrain the LIVPs. Unlike for the electromagnetic CR, a kinematic analysis is not enough to determine an upper bound for $\mathbb{A}$ and $\mathbb{B}$.

We shall also remark that it is possible to have both electromagnetic and gravitational CR emission at the same time. In that scenario, the electromagnetic process is much more efficient as we have discussed and it is the dominant source of energy loss. Hence, constraints coming from gravitational CR can be considered as long as the LIVP for the photon $\mathbb{B}_\gamma$ is such that the electromagnetic process is always forbidden.

The indefinite integral in Eq. (37) can be computed in the general case $m \neq 0$ and $\alpha, \beta \geq 0$ in terms of hypergeometric functions, but we do not give the general solution here. Remember that we also need to calculate the threshold momentum $k_{\rm th}$ for the integration limits, which is not possible to obtain analytically for arbitrary values of α and β. Nevertheless, we have checked that the results in [16] are recovered when taking $\mathbb{A} = 0$, and also those in [4] if we additionally impose $\beta = 0$.

IV. CONSTRAINTS ON PHENOMENOLOGY OF LIV USING GRAVITATIONAL CHERENKOV RADIATION

We are now interested in obtaining constraints for the LIVPs $\mathbb{A}$ and $\mathbb{B}$ in n_a and n_b using the energy loss rate of vacuum gravitational CR (37). High energy astrophysical observations will be used for this purpose, and we shall argue why UHECRs offer the most stringent bounds. Different values of α and β are considered and the corresponding phenomena is discussed in Figure 2.

A. Imposing constraints from energy loss rate

In Sec. III we saw that the decay rate of a particle of astrophysical origin due to gravitational CR is expected to be of the same order as the time of propagation, which implies that the energy loss rate has to be taken into account in order to impose constraints on the LIVPs of both the particle a and the theory of gravity. Computing the integral in Eq. (37) and solving the differential equation would give the maximum time travel t possible for a given momentum p in terms of $\mathbb{A}$, $\mathbb{B}$ [4, 15]. Thus, constraints on $\mathbb{A}$ and $\mathbb{B}$ can be imposed using high energy astrophysical observations if the distance travelled ct and the momentum p are known.

However, we have argued that it is not possible to obtain the maximum time travel t for arbitrary values of α and β in terms of $\mathbb{A}$ and $\mathbb{B}$, as $k_{\rm th}$ cannot be computed. In this scenario, the condition at which damping from gravitational CR becomes relevant for a particle with energy p travelling for a time t may be estimated as ${\rm d}E/{\rm d}t \geq p/t$ [16]. Then, we establish that values of $\mathbb{A}$ and $\mathbb{B}$ that satisfy ${\rm d}E/{\rm d}t \geq p/t$ are excluded by gravitational CR, and those that satisfy ${\rm d}E/{\rm d}t \leq p/t$ are allowed by observation.

To illustrate how constraints can be estimated, let us consider the simple case where $\alpha = \beta = 0$. Attending to Eqs. (17) and (18) we see that the process is allowed when $\mathbb{B} \geq \mathbb{A}$. Note that this condition is equivalent to the particle a having a greater phase velocity than the particle b, $c_a > c_b$. Integration of Eq. (37) in the massless limit is straightforward and yields

$$\frac{{\rm d}E}{{\rm d}t} = \frac{G_{\rm N}}{3} p^4 (\mathbb{A} - \mathbb{B})^2, \quad (38)$$

so damping from CR is not significant if

$$0 \leq \mathbb{B} - \mathbb{A} \leq \sqrt{\frac{3}{G_N t p^3}}. \quad (39)$$

Here we see that constraints are more stringent for high energy particles that have travelled a long distance ct; note, however, that energy plays a more relevant role since the dependence is of the form $(tp^3)^{-1/2}$. Another relevant feature is that the constraint is only imposed for the difference between the LIVPs, so one of the them remains unfixed: as long as this difference is small enough, the values of $\mathbb{A}$ and $\mathbb{B}$ can be arbitrarily large. One should be careful with this last statement as our results have been derived assuming that $|\mathbb{A}p^\alpha|$ and $|\mathbb{B}k^\beta|$ are very small compared to unity. We have checked that both of these features, the dependence on p and t and the impossibility of fixing both $\mathbb{A}$ and $\mathbb{B}$ simultaneously, are present for different values of α and β.

It is also worth to mention the case where $\mathbb{A} = 0$ and β is a positive real number. Performing a similar calculation in the massless limit $m = 0$ shows that damping from gravitational CR is not relevant if

$$0 < \mathbb{B}p^{\beta} \leq \frac{\sqrt{(\beta+1)(\beta+2)(2\beta+3)}}{\sqrt{2G_{\mathrm{N}}tp^3}}, \tag{40}$$

which corresponds to the result derived in [16]. The constraint on $\mathbb{B}p^{\beta}$ exhibits the same dependence on the energy and the time travel as Eq. (39), and we see that the value of $\mathbb{B}$ decreases for higher values of β. This has been used to derive much more stringent constraints in the gravitational sector using the absence of vacuum CR for $\beta \gtrsim 0$ rather than using direct detection of GWs, in particular the events GW150914 and GW151226 [18].

B. Observational constraints

UHECRs [27] are particles with energies above 1 EeV (10^9 GeV), whose origin is most likely extragalactic and are mainly composed of protons and heavy nuclei [28]. At such high energies these particles interact with the intergalactic photon background through the GZK (Greissen-Zatsepin-Kuz'min) effect: protons may lose energy due to photo-pion production, and heavy nuclei through photodissociation interaction. In both cases the mean free path is of order $200 - 300$ Mpc for energies around 100 EeV [7]. UHECRs offer the best scenario to test LI using CR; in fact, these are the most energetic particles observed in the universe and are sensitive to the decay time of gravitational CR thanks to their long propagation distances.

In order to obtain realistic bounds [17], let us consider that the UHECR is a proton of energy $p = 10^{11}$ GeV. The source of these particles is still unknown [27] but it is expected that are produced in active galactic nucleus; the nearest is found at a few Mpc, so we may take $ct = 10$ Mpc. Here we assume that the proton behaves as a complex scalar field and has no inner structure. Regarding the former, we already have mentioned that spin corrections are not expected to affect the order of magnitude of the constraints on $\mathbb{A}_{\mathrm{proton}} \equiv \mathbb{A}$ and $\mathbb{B}_{\mathrm{graviton}} \equiv \mathbb{B}$. Attending to the latter, a detailed analysis of the gravitational CR emitted by a proton would require to consider its partonic structure. For soft emitted gravitons this inner structure might be neglected and the pointlike approximation is valid [12]; for hard emitted gravitons it becomes important and the emission rate of quarks and gluons should be computed. This would substantially increase the number of LIVPs considered as $\mathbb{A}_{\mathrm{quark}} \neq \mathbb{A}_{\mathrm{gluon}}$ in general, complicating the analysis. This study is out of the scope of this work, but it is worth mentioning that this has been studied with more simple MDRs which consider $\alpha = \beta = 0$ [13, 14]. In short, we will assume that the proton has its own effective MDR, whose structure could be understood in terms of the composition of the MDRs of its constituents (quarks and gluons).

Computing the threshold momentum k_{th} numerically for $m = m_{\mathrm{proton}}$, performing the integral in Eq. (37) and imposing the condition $\mathrm{d}E/\mathrm{d}t \leq p/t$ we have obtained the allowed $\mathbb{A} - \mathbb{B}$ parameter space for $\alpha = 2$, $\beta = 0$ and $\alpha = 1$, $\beta = 2$ shown in Fig. 2. Values of $\mathbb{A}$ and $\mathbb{B}$ that define the dark-grey region permit Cherenkov emission without significant damping, whereas in the light-grey region the process is kinematically allowed but forbidden by the observation because of the damping effect. In both cases we see that no constraints are imposed in the region with $\mathbb{A} > 0$ and $\mathbb{B} < 0$, and also when $\mathbb{A}, \mathbb{B} > 0$ and $\mathbb{B}p^{\beta} - \mathbb{A}p^{\alpha} < 0$. The latter is due to Eq. (18), since the threshold momentum is $k_{\mathrm{min}} > p$. When both $\mathbb{A}$ and $\mathbb{B}$ are negative the region where kinematics does no permit Cherenkov emission is different in both cases. Attending to Eq. (17), the case where $\alpha = 2$ and $\beta = 0$ establishes that $\mathbb{B} - 3\mathbb{A}p^2 \geq 0$ so that the threshold momentum is $k_{\mathrm{max}} > 0$; by contrast, when $\alpha = 1$ and $\beta = 2$ the maximum momentum exists when $\mathbb{A} \leq 0$. Note that for the electromagnetic CR the allowed region would be extremely small and the threshold constraints obtained for the LIVPs would be very close to those derived by demanding the observed time travel to be longer than the lifetime of the particle a.

The excluded region appears in both cases for $|\mathbb{A}\boldsymbol{p}^{\alpha}| \sim |\mathbb{B}\boldsymbol{p}^{\beta}| \lesssim 10^{-17}$ when $p = 10^{11}$ GeV and $ct = 10$ Mpc. Other positive integer values of α and β exhibit constraints of the same order of magnitude for these energies and travelled distances. This allows us to check that LIVPs significantly modify the threshold condition of the process since $|\mathbb{A}p^{\alpha}| \gg (m/p)^2$, see Eq. (6). Hence, the approximations performed in Sec. II are consistent and the analyses in the massless limit are accurate.

As we have previously advanced, Fig. 2 shows that it is possible to have arbitrarily large values of $\mathbb{A}$ and $\mathbb{B}$ where vacuum gravitational CR is allowed by observation as long as its difference is small enough. We have checked the same happens for many other positive integer values of α and β. This fact can be understood in the following way. When computing the energy loss rate, Eq. (37), it is necessary to obtain the integration limits from the threshold condition $\Theta_c = 0$ for some given $\mathbb{A}$ and $\mathbb{B}$. Considering values of $\mathbb{A}$ and $\mathbb{B}$ where the process is allowed for some k, an increase in $\mathbb{A}$ and $\mathbb{B}$ will cause an increase in the value of Θ_c and thus in $\mathrm{d}E/\mathrm{d}t$. However, if the difference between $\mathbb{A}$ and $\mathbb{B}$ is small enough, the region of k where the Cherenkov emission is possible will be narrower; this reduces the value of $\mathrm{d}E/\mathrm{d}t$ as the integration interval is smaller, and it can be such that the condition $\mathrm{d}E/\mathrm{d}t \leq p/t$ is still satisfied. Remember that Eq. (13) for Θ_c was obtained assuming $|\mathbb{A}E_{\boldsymbol{p}}^{\alpha}| \sim |\mathbb{B}E_{\boldsymbol{k}}^{\beta}| \ll 1$, and thus our model is not predictive for large values of $\mathbb{A}, \mathbb{B}$.

In this analysis we notice that when $\beta > 0$, given a small value of $\mathbb{A} < 0$ we obtain an upper bound for $\mathbb{B}$

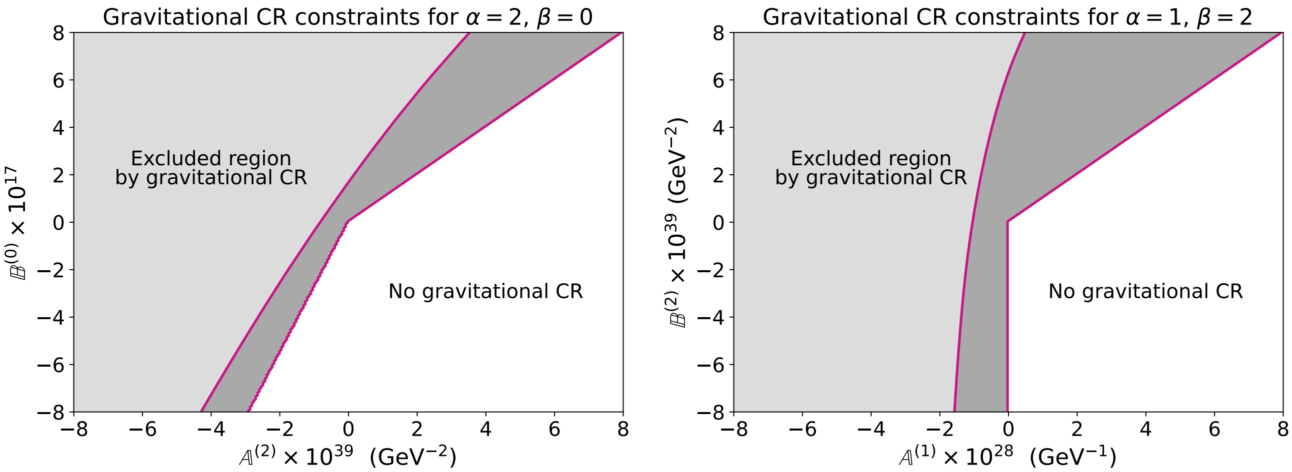

FIG. 2. Constraints on the model for $\alpha = 2$, $\beta = 0$ (left panel) and $\alpha = 1$, $\beta = 2$ (right panel) using an UHECR with $p = 10^{11}$ GeV and $ct = 10$ Mpc. The dark-gray region in both figures are the values of $\mathbb{A}_{\text{proton}} \equiv \mathbb{A}$ and $\mathbb{B}_{\text{graviton}} \equiv \mathbb{B}$ that satisfy the condition $\mathrm{d}E/\mathrm{d}t \leq p/t$. There, the damping of the UHECR due to gravitational CR is not significant, and it is possible for the particle to reach Earth while emitting gravitons. The light-gray region shows that Cherenkov emission is kinematically allowed but does not satisfy $\mathrm{d}E/\mathrm{d}t \leq p/t$. Hence, our model excludes those values of the LIVPs. The white region corresponds to $(\mathbb{A}, \mathbb{B})$ where CR is not permitted, as $\Theta_c < 0$ for all possible values of k.

but not a lower bound. This is not the case for positive values of $\mathbb{A}$ and $\mathbb{B}$, since fixing one of them constraints the other LIVP; the same happens for negative values of $\mathbb{A}$ and $\mathbb{B}$ when $\beta = 0$. Therefore, constraints derived for $\mathbb{A}$ from a different process may not fix $\mathbb{B}$ for $\beta \neq 0$. Note, however, that given a value of $\mathbb{B}$, determined by direct observation of GWs, it is always possible to obtain an upper bound for $\mathbb{A}$. Unfortunately, these measures of the parameter $\mathbb{B}$ are not stringent, so the bounds obtained for $\mathbb{A}$ are not competitive with those obtained from threshold constraints in the QED sector [11, 12].

V. CONCLUSIONS

Vacuum Cherenkov emission occurs as an emission process of a massless particle and it is possible when LI is broken [3, 4, 6, 9–17]. The easiest way to implement LIV is through MDRs, which are known as encapsulating QG phenomenology [7, 8]. In this work we proposed a new class of LI violating MDRs inspired on the classical electromagnetic Cherenkov emission in an optical medium with refractive index n, different for each particle species and with a power law dependence on the energy. Kinematics of the GCR process have been studied and the momentum configurations for which the process is allowed have been derived. This conditions were implemented to obtain the energy loss rate due to GCR, and particular emphasis has been made in the electromagnetic and gravitational vacuum CR. The latter results served to impose constraints on the LIVPs of UHECRs, assumed pointlike and spinless protons, and the graviton.

We would like to remark an important feature: the parameter space of the LI violating terms is not bounded, which implies that gravitational vacuum CR is permitted as long as the difference between them is small enough. Different processes, such as the decay of a graviton in two high energy particles, could be used to restrict the allowed parameter space, but the quantum nature of gravity has not been resolved yet and no observations of gravitons have been made. Note that this problem is not present in the QED sector. On the one hand, the characteristic time of the analog processes mentioned is much smaller than the travelled distance of astrophysical particles; hence, constraints can be imposed using only kinematics. On the other hand, electrons and photons of high energy have been detected, and thus the parameter space of the LIVPs of these particles can be restricted using the absence of vacuum electromagnetic CR and photon decay, and the change in the threshold energy of the photon annihilation process [11, 12].

Nevertheless, even though this MSc focused on the study of threshold effects, and in particular particle decays, MDRs have a much broader scope to restrict LIVPs. Constraints in the gravitational sector may be imposed using time of flight delay in multi-messenger detections. At the present only one multi-messenger detection has been reported, the GW signal GW170817 with the gamma-ray signal from a kilonova [19], but this restrictions using the time delay between GWs and photons have been applied only to impose limits on the Standard Model extension (SME) parameters in [17]. Another possibility, which is not considered in the MDR proposed here, is the vacuum birefringence phenomenon caused by the difference in the propagation velocity between the states h_+ and $h_\times$ of GWs. This has also been studied in the multi-messenger detection mentioned above, and constraints have been imposed for SME parameters [19].

In conclusion, our work establishes a systematic method to study two body particle decays in a LI violating scenario and impose constraints through high energy astrophysical observations. Extensions to more body interaction processes can be carried out following the steps exemplified in Sec. II, but these are not of much interest in the gravitational sector. We also want to emphasize the importance of using complementary observations, such as direct detection of GWs, in order to improve the bounds on the parameters introduced. This will be reinforced with future detections of UHECRs and multi-messenger events, which may allow us to understand the elusive nature of QG.

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