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Modified gravity in neutron stars

Pablo Navarro Moreno

Directores

Aneta M. Wojnar
Felipe J. Llanes-Estrada

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Modified gravity in neutron stars

Pablo Navarro Moreno
Univ. Complutense de Madrid, dept. Física Teórica, Plaza de las Ciencias 1, 28040 Madrid, Spain.

We study neutron stars within modified theories of gravity. We consider $R-$squared gravity in which $f(R) = R + \alpha R^2$. In particular, we compute the action of the theory in the *Einstein frame* and obtain a TOV-like system of differential equations. We perform a numerical solution analyzing properties of different families of neutron stars and different values of the parameter α. Families of neutron stars are obtained integrating the system with different equations of state. We also study the maximum latent heat allowed by a first order phase transition before the star collapses into a black hole. Finally, we study the slowly rotating star approach in order to compute several physical magnitudes of interest. We also compare the results with those in General Relativity.

I. INTRODUCTION

General Relativity (GR), proposed by A. Einstein in 1915, is currently the accepted theory of gravity. It has been widely tested in the weak field regime. Many astrophysical observations, such as solar system tests, binary pulsars and gravitational-wave (GW) phenomena [1, 2], support GR predictions. However, in the strong field regime, GR might fail at some point. It is advisable to continue testing the theory against relatively simple alternatives in new regimes.
The most popular modified gravity theories nowadays are $f(R)$ theories. They are natural generalizations of GR in which the Ricci scalar R in the action is replaced by a function of it. By construction, these theories introduce dimensionfull parameters which must be constrained by observations. $f(R)$ theories are a particular case of scalar-tensor theories, which include both a tensor field and a scalar field to mediate the gravitational interaction [3, 4]. Scalar-tensor theories become important in inflationary cosmology [5].
Neutron stars (NS) are the most compact not collapsed objects we can observe and so, among the best laboratories to test our theory of gravity. In particular, their high density allows us to test the large stress-energy tensor regime. Studying NSs in modified gravity might help constrain the parameters of the theory. Observations through, for example, x-ray emissions, binary radio systems and gravitational waves put some limits in the mass and radius of these astrophysical objects. Table I shows some observational data from [6, 7].

Name	$\bar{\Omega}(\mathrm{rad}\cdot\mathrm{ms}^{-1})$	$M(M_\odot)$
J0337+1715	2.299	1.4401(15)
J0348+0432	0.161	2.01(4)
J0509+380	0.082	1.34(8)
J0453+1559	0.137	1.559(5)
J1012+5307	1.195	1.72(16)

TABLE I: Angular velocity and mass of a few well-measured pulsars [6, 7].

NS masses span the range 1-2 $M_\odot$ and their radii 10-13 km. The main problem when working with NS is that we do not know the equation of state (EoS) that describes the matter inside them with arbitrary accuracy. This uncertainty difficulties constraining the parameters of the theory. In order to describe NS within modified gravity we must use EoS independent of any astrophysical observable [8, 9].
The aim of this project is to study neutron stars within alternative theories of gravity. In particular, we are considering R^2-gravity, in which $f(R) = R + \alpha R^2$. We study both the static (computing the typical mass-radius diagrams for different families of stars and we analyze the maximum latent heat that a star can support before collapsing) and the rotating star [10, 11] (computing the moment of inertia and other observables) for different EoS and values of the parameter of the theory α. We compare the results with our previous work [12] taking the limit of GR.
This work is organized as follows. In Section II we discuss the Tolman-Oppenheimer-Volkoff (TOV) equations of hydrostatic equilibrium in a quite general formalism. In Section III we study a simple scalar-tensor theory minimally coupling a real scalar field to the Einstein-Hilbert action. In Section IV we study $f(R)$ theories, in particular $R-$squared gravity, and their equivalence to particular scalar-tensor theories in the *Einstein frame*. In Section V we present the EoS used to solve the system. In Section VI we compute the field equations and solve them numerically. We also explain how the numerical algorithm is built. In Section VII we analyze the maximum latent heat allowed by a first order phase transition before the star collapses. In Section VIII we discuss the slowly rotating star approximation and finally, in Section IX, we conclude our work.

II. STATIC STAR IN MODIFIED GRAVITY

There is a quite generic family of modified gravity theories whose field equations can be written as [13]

$$\sigma(\chi)(G_{\mu\nu} - W_{\mu\nu}) = \kappa T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ the

stress-energy one ($\kappa = 8\pi$) and $\sigma(\chi)$ is the coupling to the gravitational field due to other fields or gravitational curvature invariants χ, which acts as a gravitational weakening/strengthening parameter. $W_{\mu\nu}$ is a symmetric tensor that may include additional terms depending on the theory considered and it shifts the geometrical contribution of the theory from GR. We will work in geometrized units, where $G = c = 1$. Notice that we recover the general relativistic field equations taking $\sigma(\chi) = 1$ and $W_{\mu\nu} = 0$.
We consider, as first approximation, the energy-momentum tensor of a perfect fluid, $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$, where u_μ is the 4-velocity of an observer comoving with the fluid that satisfies $u^2 = -1$.
In these theories of modified gravity the energy-momentum tensor $T_{\mu\nu}$ might not be conserved. However, consistency with the Bianchi identity for $G_{\mu\nu}$ requires that $T^{\rm eff}_{\mu\nu} = \frac{1}{\sigma(\chi)}T_{\mu\nu} + \frac{1}{\kappa}W_{\mu\nu}$ is conserved.
A static and spherically symmetric geometry is given by the metric

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2)$$

From equations (1) and the definition of the Ricci tensor one can compute the field equations for $R_{\mu\nu}$

$$\begin{aligned} R_{tt} = -e^{2(\nu-\lambda)}\left[\nu_{,rr} + \nu_{,r}\left(\nu_{,r} - \lambda_{,r} + \frac{2}{r}\right)\right] = \\ = \frac{\kappa}{2\sigma}(\rho + 3p)e^{2\nu} + W_{tt} + \frac{e^{2\nu}W}{2}, \end{aligned} \quad (3)$$

$$\begin{aligned} R_{rr} = \nu_{,rr} + \nu_{,r}\left(\nu_{,r} - \lambda_{,r}\right) - \frac{2}{r}\lambda_{,r} = \\ = \frac{\kappa}{2\sigma}(\rho - p)e^{2\lambda} + W_{rr} - \frac{e^{2\lambda}W}{2}, \end{aligned} \quad (4)$$

$$\begin{aligned} R_{\theta\theta} = -1 + e^{-2\lambda} + re^{-2\lambda}\left(\nu_{,r} - \lambda_{,r}\right) = \\ = \frac{\kappa}{2\sigma}(\rho - p)r^2 + W_{\theta\theta} - \frac{r^2 W}{2}, \end{aligned} \quad (5)$$

where W is the trace of $W_{\mu\nu}$. Introducing the Schwarzschild-like mass parameter, which is a function of r inside the star,

$$e^{2\lambda(r)} = \left(1 - \frac{2M}{r}\right)^{-1} \quad (6)$$

and, through some manipulations of the field equations, an equation for the mass function follows, given by

$$M(r) = \int_0^r \left(4\pi\tilde{r}^2\frac{\rho(\tilde{r})}{\sigma(\tilde{r})} - \frac{W_{tt}(\tilde{r})}{2e^{2\nu(\tilde{r})}}\tilde{r}^2\right)d\tilde{r}. \quad (7)$$

With the help of some algebra on equations (3)-(5) we find the following equations which will be useful in later calculations,

$$\lambda_{,r} = \frac{1 - e^{2\lambda}}{2r} - \frac{\kappa e^{2\lambda}r}{2\sigma}Q, \quad (8)$$

$$\nu_{,r} = \frac{e^{2\lambda} - 1}{2r} - \frac{\kappa e^{2\lambda}r}{2\sigma}\Pi, \quad (9)$$

where the effective energy density and pressure are

$$Q(r) := \rho(r) + \frac{\sigma(r)}{\kappa}W_{tt}(r)e^{-2\nu(r)}, \quad (10)$$

$$\Pi(r) := p(r) + \frac{\sigma(r)}{\kappa}W_{rr}(r)e^{-2\lambda(r)}. \quad (11)$$

From Eq.(8) and (11) follows the differential equation for the effective pressure

$$\left(\frac{\Pi}{\sigma}\right)_{,r} = \frac{p_{,r}}{\sigma} - \frac{p\sigma_{,r}}{\sigma^2} + \frac{W_{rr,r}}{\kappa}e^{-2\lambda} - 2\frac{W_{rr}}{\kappa}e^{-2\lambda}\lambda_{,r}. \quad (12)$$

Enforcing the conservation of $T^{\mu\nu}_{\rm eff}$ ($\nabla_\mu T^{\mu\nu}_{\rm eff} = 0$) together with Eq.(12) we obtain the hydrostatic equilibrium equation

$$\begin{aligned} \left(\frac{\Pi}{\sigma}\right)_{,r} = -\frac{M}{r^2}\left(\frac{Q}{\sigma} + \frac{\Pi}{\sigma}\right)\left(1 + \frac{4\pi r^3\frac{\Pi}{\sigma}}{M}\right)e^{2\lambda} + \\ + \frac{2\sigma}{\kappa r}\left(\frac{W_{\theta\theta}}{r^2} + \frac{W_{rr}}{e^{2\lambda}}\right). \end{aligned} \quad (13)$$

Let us notice that Eq.(7) can be rewritten as

$$M(r) = \int_0^r \left(4\pi\tilde{r}^2\frac{Q(\tilde{r})}{\sigma(\tilde{r})}\right)d\tilde{r}. \quad (14)$$

Equations (9), (13) and (14) are the generalized Tolman-Oppenheimer-Volkoff (TOV) equations in modified gravity. Notice that taking $\sigma(\psi^i) = 1$ and $W_{\mu\nu} = 0$ in them, the TOV equations from general relativity are recovered, given by

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r), \quad (15)$$

$$-\frac{dp(r)}{dr} = (\rho(r) + p(r))\frac{m(r) + 4\pi r^3 p(r)}{r^2(1 - 2m(r)/r)}, \quad (16)$$

$$\frac{d\nu}{dr} = -\frac{1}{\rho(r) + p(r)}\frac{dp(r)}{dr}, \quad (17)$$

where we denoted by $m(r)$ the stellar mass in the general relativistic limit and where λ and ν are the components of the interior static and spherically-symmetric metric.

III. SCALAR-TENSOR THEORY

A first modified-gravity theory which can be cast as Eq.(1) is scalar-tensor gravity. In it, the gravitational interaction is described by the metric tensor g (in the same way as in general relativity) and an additional scalar field ϕ. We can include such ϕ, through a minimal coupling, in the interacting action given by [13]

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,[R - \nabla_\mu\phi\nabla^\mu\phi - 2V(\phi)] + S_M[g_{\mu\nu},\chi]. \tag{18}$$

From this action the equations of motion follow,

$$G_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - \nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}V(\phi) = \kappa T_{\mu\nu}, \tag{19}$$

$$\Box\phi - \frac{dV(\phi)}{d\phi} = 0, \tag{20}$$

where $T_{\mu\nu}$ does not include the contribution of the scalar field, present in the left-hand side of Eq.(19). Computing

$$\Box\phi = g^{\rho\sigma}\nabla_\rho\nabla_\sigma\phi = g^{\rho\sigma}\left(\partial_\rho\partial_\sigma\phi - \Gamma^\alpha_{\rho\sigma}\partial_\alpha\phi\right) \tag{21}$$

we can explicitly write Eq.(20) for ϕ as

$$\phi_{,rr} + \left[\nu_{,r} - \lambda_{,r} + \frac{2}{r}\right]\phi_{,r} - e^{2\lambda}\frac{dV(\phi)}{d\phi} = 0. \tag{22}$$

Comparison of Eq.(19) with Eq.(1) provides

$$W_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi + \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}V(\phi), \tag{23}$$

$$\sigma = 1. \tag{24}$$

Considering a spherically symmetric and static spacetime, the scalar field is a function of the radial coordinate $\phi = \phi(r)$, exclusively. Thus,

$$W_{tt} = \frac{1}{2}e^{2(\nu-\lambda)}\phi_{,r}^2 + e^{2\nu}V(\phi), \tag{25}$$

$$W_{rr} = \frac{1}{2}\phi_{,r}^2 - e^{2\lambda}V(\phi), \tag{26}$$

$$W_{\theta\theta} = -\frac{1}{2}e^{-2\lambda}r^2\phi_{,r}^2 - r^2V(\phi). \tag{27}$$

Let us notice that Eq.(19) can be written, identifying the energy-momentum tensor of the scalar field $T^{(\phi)}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla_\rho\phi\nabla^\rho\phi + 2V(\phi))$, as

$$G_{\mu\nu} = \kappa T^{\text{eff}}_{\mu\nu}, \tag{28}$$

where we defined $T^{\text{eff}}_{\mu\nu} \equiv \kappa^{-1}T^{(\phi)}_{\mu\nu} + T^{(M)}_{\mu\nu}$ as the total energy-momentum tensor. Equations (28) are the field equations of general relativity with a modified energy-momentum tensor. Therefore, we can solve the equations in modified gravity in two different ways: the general equations (9)-(14) or through TOV equations from general relativity (15)-(17) with a modified energy-momentum tensor $T^{\text{eff}}_{\mu\nu}$, whose components are

$$T^{\text{eff}}_{tt} = e^{2\nu}\left(\rho + \frac{\phi_{,r}^2}{2\kappa e^{2\lambda}} + \frac{V(\phi)}{\kappa}\right) = e^{2\nu}\tilde{\rho}, \tag{29}$$

$$T^{\text{eff}}_{rr} = e^{2\lambda}\left(p + \frac{\phi_{,r}^2}{2\kappa e^{2\lambda}} - \frac{V(\phi)}{\kappa}\right) = e^{2\lambda}\tilde{p}_1, \tag{30}$$

$$T^{\text{eff}}_{\theta\theta} = r^2\left(p - \frac{\phi_{,r}^2}{2\kappa e^{2\lambda}} - \frac{V(\phi)}{\kappa}\right) = r^2\tilde{p}_2, \tag{31}$$

$$T^{\text{eff}}_{\phi\phi} = r^2\sin^2\theta\left(p - \frac{\phi_{,r}^2}{2\kappa e^{2\lambda}} - \frac{V(\phi)}{\kappa}\right) = r^2\sin^2\theta\tilde{p}_2. \tag{32}$$

Note that having a radial scalar field strengthens the pressure in the radial direction, so the TOV equations from general relativity are valid except for a modification in (17) due to the inhomogeneity introduced by ϕ.

IV. $f(R)$ THEORIES

As motivated in Sec. I, $f(R)$ theories are of great interest and can be cast in the form of Eq.(1) too. They are built modifying the Einstein-Hilbert action of general relativity [4, 24], replacing the scalar curvature R with a function of the same that introduces additional parameters in the theory. There exist different formalisms for $f(R)$ theories, such as the metric formalism or the Palatini one [18, 29] (in this, the metric tensor g and the connection Γ are independent variables). Here we will only study the first one, in which the metric is the only dynamic field in the spacetime. The action is given by

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu},\chi), \tag{33}$$

where S_M is the action of the matter fields χ. A theory formulated over a stable vacuum must obey $\frac{d^2f}{dR^2} \geq 0$ and

$\frac{df}{dR} > 0$ [4]. Varying the action with respect to the metric tensor we obtain the field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]\, f'(R) = \kappa T_{\mu\nu}. \tag{34}$$

Particularly, if $f(R) = R$, we recover the Einstein field equations of general relativity.

$f(R)$ theories are equivalent to Brans-Dicke scalar-tensor theories with a potential of the scalar field and $\omega_{BD} = 0$ (ω_{BD} is the Brans-Dicke coupling constant) in the *Jordan frame*. To show this, we rewrite (33) as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(f(\psi) + f'(\psi)(R-\psi)\right) + S_M(g_{\mu\nu},\chi), \tag{35}$$

where $'$ now denotes derivative with respect to a new field ψ. Varying this action with respect to ψ and imposing $f''(\psi) \neq 0$ follows that $R = \psi$. Substituting this in (35) yields (33). Namely, eliminating ψ from the theory defined by (35) returns to $f(R)$. Now we define the scalar field $\varphi \equiv f'(\psi)$ with potential

$$U(\varphi) \equiv \psi(\varphi)f'(\psi(\varphi)) - f(\psi(\varphi))\ , \tag{36}$$

so that Eq. (35) becomes

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(\varphi R - U(\varphi)\right) + S_M(g_{\mu\nu},\chi), \tag{37}$$

the action of a scalar-tensor theory with non-minimal coupling.

In particular, we can work with a quadratic [5] $f(R)$ theory with functional form $f(R) = R + \alpha R^2$, where $\alpha > 0$ to obtain nominally stable solutions [4]. From the definition of φ and $U(\varphi)$ we then get

$$U(\varphi) = \frac{1}{4\alpha}(\varphi-1)^2. \tag{38}$$

The field equations and the equation of the motion of φ can be obtained varying the action (37) with respect to $g_{\mu\nu}$ and φ respectively [25], leading to

$$G_{\mu\nu} = \frac{\kappa}{\varphi}T_{\mu\nu} - \frac{1}{2\varphi}g_{\mu\nu}U(\varphi) + \frac{1}{\varphi}(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi), \tag{39}$$

$$3\Box\varphi + 2U(\varphi) - \varphi\frac{dU}{d\varphi} = \kappa T. \tag{40}$$

If one wishes to isolate R in the action of Eq.(37), removing the φ factor that multiplies it, one should study the field equations in the *Einstein frame*, which is related to the *Jordan frame* through a conformal transformation[1]. We will denote quantities in the *Einstein frame* by $*$. The conformal factor is given by the following redefinition of the scalar field

$$\phi = \frac{\sqrt{3}}{2}\log\varphi. \tag{41}$$

Introducing the conformal factor $\Omega^2(\phi) = \varphi = e^{\frac{2\phi}{\sqrt{3}}}$,

$$g^*_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}. \tag{42}$$

It is straightforward to see that $g^{*\mu\nu} = \Omega^{-2}g^{\mu\nu}$ since $g^{\mu\alpha}g_{\alpha\nu} = g^{*\mu\alpha}g^*_{\alpha\nu} = \delta^\mu_\nu$. The metric determinant transforms as

$$\sqrt{-g^*} = \sqrt{\det(\Omega^2 \mathrm{g}_{\mu\nu})} = \sqrt{(\Omega^2)^4(-g)} = \Omega^4\sqrt{-g} \tag{43}$$

and the curvature scalar as

$$R^* = \Omega^{-2}(R - 6\nabla_\mu\nabla^\mu\log\Omega - 6\nabla^\mu\log\Omega\nabla_\mu\log\Omega). \tag{44}$$

From this expression follows that

$$R = \Omega^2(R^* + 6\Box^*\log\Omega - 6\nabla^{*\mu}\log\Omega\nabla^*_\mu\log\Omega). \tag{45}$$

Recalling that $\nabla^*_\mu\log\Omega = \frac{1}{\sqrt{3}}\partial_\mu\phi$, $\nabla^{*\mu}\log\Omega = \frac{1}{\sqrt{3}}g^{*\mu\nu}\partial_\nu\phi$ and the previous results, the action in the *Einstein frame* is written as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g^*}\,\big(R^* - 2g^{*\mu\nu}\nabla^*_\mu\phi\nabla^*_\nu\phi - \\ -V(\phi)\big) + S_M(\Omega^{-2}g^*_{\mu\nu},\chi), \tag{46}$$

where the potential takes the form

$$V(\phi) = \frac{1}{4\alpha}\left(1 - e^{-\frac{2}{\sqrt{3}}\phi}\right)^2. \tag{47}$$

Before computing the field equations it is convenient to check what happens to the physical magnitudes, such as the pressure and energy density, whenever a conformal transformation is performed. Notice that ρ and p are the physical energy density and pressure, the ones which enter the equation of state. To the energy momentum tensor which they produce we apply the conformal transformation (42), $T^*_{\mu\nu} = \Omega^{-2}T_{\mu\nu}$. Rising both indices we

[1] Notice that the conformal transformation might be singular.

get $T^{*\mu\nu} = \Omega^{-6}T^{\mu\nu}$, and its trace is conformally transformed to $T^* = \Omega^{-4}T$. From the definition of the energy-momentum tensor for a perfect fluid immediately follows that $\rho^* = \Omega^{-4}\rho$ and $p^* = \Omega^{-4}p$. Lastly, the 4-velocity of the comoving observer with the fluid is transformed as $u^*_\mu = \Omega u_\mu$ and $u^{*\mu} = \Omega^{-1}u^\mu$.
Taking variations in Eq.(46) with respect to $g^*_{\mu\nu}$ and ϕ [4] we get

$$G^*_{\mu\nu} = \kappa T^{*(M)}_{\mu\nu} + 2\partial_\mu\phi\partial_\nu\phi - g^*_{\mu\nu}\partial^\rho\phi\partial_\rho\phi - \\ - \frac{1}{2}g^*_{\mu\nu}V(\phi) = \kappa T^{*(M)}_{\mu\nu} + 2T^{*(\phi)}_{\mu\nu}, \quad (48)$$

$$\Box^*\phi - \frac{1}{4}\frac{dV}{d\phi} = \frac{\kappa}{2\sqrt{3}}T^*, \quad (49)$$

where $\Box^* = \nabla^*_\mu\nabla^{*\mu}$. Notice that the equation of motion has an additional term due to the coupling with matter present in S_M.
In Sec. VI, we are going to solve the static star computing the field equations in the *Einstein frame*. Thus, it is convenient to compute the divergence of the energy-momentum tensor therein. From the Bianchi identities, using (49) and taking into account that the connection is symmetric, we arrive at

$$\nabla^*_\mu T^{*\mu}_\nu = -\frac{1}{\sqrt{3}}T^*\nabla^*_\nu\phi. \quad (50)$$

V. EQUATIONS OF STATE

In this section we present the equations of state (EoS) employed to solve the field equations. The EoS relate the thermodynamic variables that describe the state of matter under certain physical conditions. As we can see from equations (53)-(56), we need one more equation to solve the modified TOV system. Assuming that the fluid that constitutes the neutron star is a barotropic one, the equation of state takes the form $p = p(\rho)$. The EoS used are shown in FIG.(1): the most rigid (red) and the softest (green) and some typical intermediate case.
This EoS band, provided by the research group [9], is valid for cold hadronic matter. It is obtained from Chiral Perturbation Theory (ChPT) for low densities and perturbative Quantum Chromodynamics (pQCD) for high densities. Intermediate densities are obtained interpolating between both branches. Furthermore, this family of EoS is not constrained by any astrophysical observable so it can be used in extensions of general relativity, such as in scalar-tensor or $f(R)$ theories here.
All the EoS of the family satisfy the stability and causality conditions given by $\dfrac{dp}{d\rho} \geq 0$ and $\dfrac{dp}{d\rho} \leq 1$ respectively. Additionally to these conditions, the family of EoS also

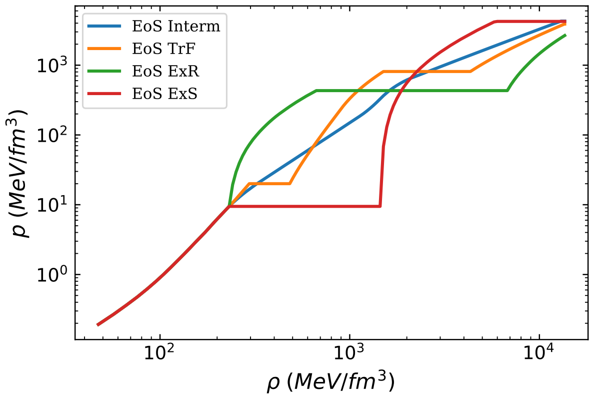

FIG. 1: Equations of state in logarithmic scale. Pressure (p) as a function of the energy density (ρ) in MeV/fm^3. The green line is the stiffest EoS in the low-density neutron star region, while the red line represents the softest allowed EoS there.

satisfies thermodynamic consistency $p = \displaystyle\int n(\mu)d\mu$ for a causal $n(\mu)$.
In general, a stiff EoS has a large slope in the pressure-energy density diagram that can even saturate causality ($c_s^2 \lesssim 1$). The most rigid EoS employed in this work is EoS ExR (Extremely Rigid) and the softest is EoS ExS (Extremely Soft). Upon increasing the stellar mass, a larger slope in the EoS will lead to an increase in the radius, a smaller slope to a decreased one. In this work we analyze these extreme cases and some intermediate ones in section VI.
Moreover, notice that some of the EoS in FIG.(1) present first order phase transitions (horizontal straight lines) given by

$$dp/d\rho = 0. \quad (51)$$

In the following results we may identify some of the effects that these phase transitions have on the properties of neutron stars.

VI. SOLUTION TO FIELD EQUATIONS

Let us solve the problem in the *Einstein frame* in which the equations take a form closest to that of the general relativistic TOV system. We consider that the static and spherically symmetric metric is given by an expresion analogous to Eq.(2),

$$ds^2_* = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (52)$$

Once solved, inverting the conformal transformation, the field equations (48) and (49) return the components of the metric to the *Jordan frame*. The components G^*_{tt}

and G^{*}_{rr} together with the divergence of $T^{*}_{\mu\nu}$ immediately yield the modified-gravity TOV equations,

$$\frac{1}{r^2}\frac{d}{dr}\left[r\left(1-e^{-2\lambda}\right)\right]=\frac{8\pi}{\Omega^4}\rho+e^{-2\lambda}\phi'^2+\frac{1}{2}V(\phi), \quad (53)$$

$$\frac{2}{r}e^{-2\lambda}\frac{d\nu}{dr}+\frac{\left(e^{-2\lambda}-1\right)}{r^2}=\frac{8\pi}{\Omega^4}p+e^{-2\lambda}\phi'^2-\frac{1}{2}V(\phi), \quad (54)$$

$$\frac{dp}{dr}=-(\rho+p)\left[\frac{d\nu}{dr}-\frac{1}{\sqrt{3}}\frac{d\phi}{dr}\right]. \quad (55)$$

Equation (49) is explicitly written as

$$\phi''+\left[\frac{d\nu}{dr}-\frac{d\lambda}{dr}+\frac{2}{r}\right]\phi'=\frac{\kappa}{2\sqrt{3}}e^{2\lambda}\Omega^{-4}(3p-\rho)+\frac{1}{4}e^{2\lambda}\frac{dV}{d\phi}. \quad (56)$$

Notice that ρ and p are the physical energy density and pressure in the *Jordan frame* respectively, related by the equation of state. Thus, the system of differential equations (53)-(56) together with a given EoS, completely determine the problem of the static star in this modified theory of gravity.
The initial conditions and boundary conditions that we will impose are the following, naturally arising in the *Einstein frame*. Pressure in the center of the star is a given value $p(0)=p_c$. Integrating the system for different central pressures will produce a family of solutions. The radius R of the star will be determined by the condition $p(R)=0$. The star's physical radius will then be

$$R_s=R\Omega^{-1}(\phi(R)). \quad (57)$$

$\lambda(0)=0$ guarantees regularity of the metric and on the other hand, $\frac{d\phi}{dr}(0)=0$ that of the scalar field. We will also impose our spacetime to be asymptotically flat so that $\lim_{r\to\infty}V(\phi)=0$ and then $\lim_{r\to\infty}\phi=0$. Moreover $\lim_{r\to\infty}\nu(r)=0$. This guarantee asymptotic flatness in both *Einstein* and *Jordan frames*.
Due to the form of the conformal factor and to the fact that the scalar field decreses exponentially at infinity, we find that the masses of the star in both the *Einstein* and the *Jordan frames* are the same. In order to obtain the mass we only need to compare the exterior metric towards infinity with that given by the Schwarzschild metric, so that

$$e^{2\lambda(r)}\longrightarrow\left(1-2\frac{M}{r}\right)^{-1}. \quad (58)$$

This results in the total mass of the star, given by

$$M=\lim_{r\to\infty}\frac{r}{2}\left(1-e^{-2\lambda(r)}\right). \quad (59)$$

The physical magnitudes of interest of the static star are its mass and radius given by equations (57) and (59). The system of differential equations (53)-(56) + EoS completely determines the problem. Since the EoS employed are computerized, we need to develop an algorithm to solve the problem numerically. We use a fourth order Runge-Kutta algorithm to address the differential equations. A complication that arises is that we have boundary conditions, we want the scalar field to vanish at infinity. Therefore, the Runge-Kutta algorithm is combined with a shooting method (that works with a bisection method) that ensures the boundary conditions. Notice that the $\nu(r)$ function does not appear explicitly in any of the equations (53)-(56) (only its derivative). Thus, we can solve the system for a given $\nu(0)=\nu_c$ and then obtain the appropriate function by subtracting the value it takes at infinity $\nu(r)\longrightarrow\nu(r)-\nu(\infty)$ (which continues to satisfy the differential equations). However, additional problems arise when numerically solving the system. Notice, as stated in [4], that the system of differential equations is stiff, with increasing stiffness as α decreases. A poor guess for the initial condition of the scalar field during the shooting method, makes $\phi(r)$, and consequently the other functions, eventually diverge. For lower values of α it is much more difficult to obtain the desired ϕ_c due to the precision of the computer. In order to optimize the running time we can obtain a reasonably small interval on which the desired ϕ_c lies. Then we can truncate the scalar field by hand at a distance in which the scalar field has decreased enough. The flowchart is shown in FIG.(2). A sample of the code developed to compute the mass-radius diagrams can be found at `https://github.com/hyliano53/Modified-Gravity-NS.git`.

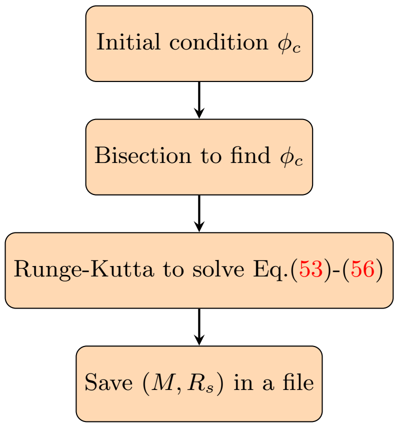

FIG. 2: Flowchart providing a visual representation of the algorithm programmed in FORTRAN.

We plot the $\lambda(r)$ function, the pressure profile and the scalar field in FIG.(3). As we can see, λ reaches its maximum in the interior of the star. On the other hand, the pressure decreases monotonically until it vanishes at the edge of the star. In the last plot we can see again that the results are consistent with the boundary conditions, since the scalar field exponentially decreases outside the star, vanishing at large r.
In FIG.(4) we plot the typical mass-radius diagram for

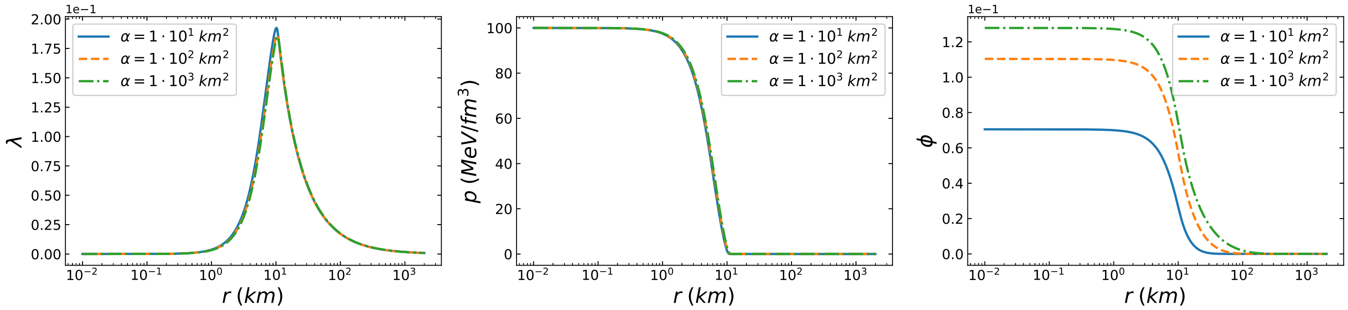

FIG. 3: NS structure obtained computing the solution of the field equations for the EoS Interm shown in FIG.1. The results correspond to a star of 100 $\mathrm{MeV/fm^3}$ central pressure. Left panel: metric function $\lambda(r)$. We can see small differences between the different values of the parameter α. Mid panel: Pressure profile $p(r)$. We can barely see any difference among the values of α. Right panel: Scalar field profile $\phi(r)$ which is clearly larger for the larger α values.

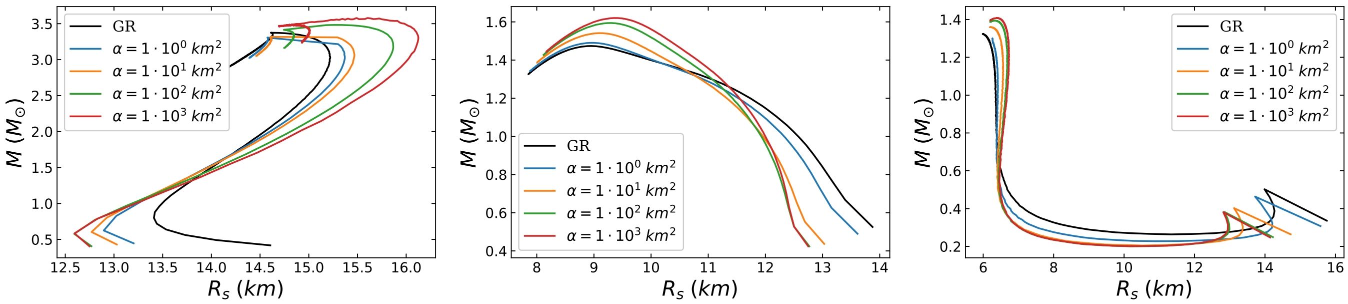

FIG. 4: Mass-radius diagrams for different EoS from FIG.(1) and several values of α. Left panel: EoS ExR. Mid panel: EoS Interm (whitout phase transition). Right panel: EoS ExS

different EoS and several values of the parameter of the theory. Notice that it is a common behaviour that the mass of the star grows with the α parameter for higher central pressures while it decreases with α for lower central pressures. Furthermore, note that the limit of GR is recovered in the limit $\alpha \longrightarrow 0$ whilst the largest difference with GR is found for the greatest value of α.

We can also find kinks in the mass-radius diagrams due to the non-differentiability introduced by the phase transitions. For instance, we can see a kink near $(14.6\,\mathrm{km}, 3.5\,\mathrm{M_\odot})$ in the EoS ExR GR diagram (and the analogous shifted kinks for $\alpha \neq 0$ curves).

Finally, let us notice that for values of the parameter α greater than 10^3 km^2 the mass-radius diagram does not significantly change and so, we can not reach the typical 2 $M_\odot$ for the EoS in the mid and right panel of FIG.(4). Their combination with typical EoS short of the stiffest can thus be excluded.

VII. LATENT HEAT

We now turn to an equation of state with a first order phase transition at p_0 from ρ_1 to $\rho_2 > \rho_1$. First, we consider a relativistic star described by the TOV equations (15)-(17). If the central pressure exceeds p_0, then we obtain a star with a core in a new phase. If lower than p_0 then we have a phase-homogeneous star.

The intensity of a first order phase transition is quantified by the latent heat, which can be defined by [17]

$$L = p_0 \frac{\rho_2 - \rho_1}{\rho_1 \rho_2}. \tag{60}$$

The analysis of Seidov in GR employing the limit of small core [15] predicts a critical value such that for larger L the star is unstable. If the phase transition is long enough then gravitational collapse occurs. The strongest phase transition allowed in GR (for a small core) is

$$\rho_2 - \rho_1 = \rho_1 \left(\frac{1}{2} + \frac{3}{2} \frac{p_0}{\rho_1} \right). \tag{61}$$

In the context of modified gravity we can integrate the TOV-like system (53)-(56) for all the equations of state and obtain an approximate limit for the maximum latent heat allowed. By integrating the different EoS in [30], we obtain the results shown in FIG.(5). Increasing the number of EoS and the variety of latent heats tried, we could obtain the maximum latent heat allowed for each α, and thus better resolve $L_{\max}(\alpha)$. Also note that the EoS with the maximum latent heat is EoS ExR.

We see in FIG.(5) that the maximum latent heat supported by the star increases with α. For $\alpha > 20$ collapse ceases being the tightest constraint on L, since microscopic physics limits L before. If nuclear physics would allow a greater band then one could eventually construct EoS with greater phase transitions and then find $L_{\max}$ for $\alpha > 20$. However, we restrict ourselves to the limit established by the band of equations of state in FIG.(1).

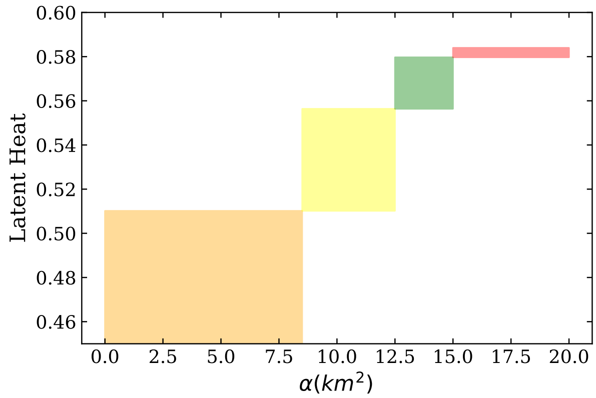

FIG. 5: Approximate limits to the maximum latent heat allowed within the EoS band of FIG.(1).

VIII. ROTATING STAR: A FIRST APPROACH

In this section we will study a slowly rotating star. We will consider that a star is slowly rotating if the changes in pressure or energy density due to rotation are small corrections. This implies that the particles in the surface of the star move with non relativistic velocities, i.e. $\bar{\Omega}R \ll 1$, where $\bar{\Omega}$ is the angular velocity of the surface of the star as seen by an observer at infinity[2]. We will call $L(r,\theta)$ the angular velocity experienced by an observer in free fall towards the star, due to the dragging of the fluid. Then, we define $\varpi \equiv \bar{\Omega} - L$ as the relative angular velocity.

Following Hartle & Thorne [19, 20], the most general stationary axisymmetric metric takes the form

$$ds_*^2 = -H^2dt^2 + Q^2dr^2 + r^2K^2(d\theta^2 + \sin^2\theta(d\phi - Ldt)^2), \tag{62}$$

where H, Q, K and L are functions of r and θ. The metric of this spacetime behaves in the same way under reversal in the direction of rotation as under a reversal in the direction of time. Due to this, an expansion of H, Q and K can only contain even powers of $\bar{\Omega}$ whilst an expansion of L can only contain odd powers of the angular velocity. In this work we are only considering terms of order $\bar{\Omega}$ so that $L(r,\theta) = \omega(r,\theta) + O(\bar{\Omega}^3)$ and the previous metric can be rewritten in the following way

$$ds_*^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2\omega dt d\phi, \tag{63}$$

where $\omega(r,\theta)$ is linear in $\bar{\Omega}$. In order to find the angular velocity we need to compute the field equation

$$R^{*t}{}_{\phi} = 8\pi T^{*t}{}_{\phi}. \tag{64}$$

The 4-velocity of the rotating fluid is given by $u^\mu = (u^t, 0, 0, \bar{\Omega}u^t)$. The normalization of u gives, up to first order in $\bar{\Omega}$, $u^t = e^{-\nu}$. The RH side of Eq.(64) is easily computed from the perfect fluid's energy-momentum tensor. It is also straightforward to compute the LH side using the identity [19]

$$(-g^*)^{-1/2}R^{*t}{}_{\phi} = \partial_\beta\left[(-g^*)^{-1/2}g^{*t\alpha}\Gamma^{*\beta}_{\phi\alpha}\right]. \tag{65}$$

After some manipulations one finds the following equation for $\varpi(r,\theta)$,

$$\frac{e^{\nu-\lambda}}{r^4}\partial_r\left[e^{-\nu-\lambda}r^4\partial_r\varpi\right] + \frac{1}{r^2\sin^3\theta}\times \times\,\partial_\theta\left[\sin^3\theta\partial_\theta\varpi\right] = 16\pi\Omega^{-4}(\rho+p)\varpi. \tag{66}$$

We can now expand in Legendre polynomials so that

$$\varpi(r,\theta) = \sum_{l=1}^{\infty}\varpi_l(r)\frac{dP_l}{d\cos\theta}, \tag{67}$$

and substituting back into Eq.(66) we arrive at the equation

$$\frac{e^{\nu-\lambda}}{r^4}\partial_r\left[e^{-\nu-\lambda}r^4\frac{d\varpi_l}{dr}\right] + \frac{2-l(l+1)}{r^2}\varpi_l = 16\pi\Omega^{-4}(\rho+p)\varpi_l. \tag{68}$$

The asymptotic exterior solution takes the form $\varpi \longrightarrow ar^{-l-2} + br^{l-1}$. Taking into account that $\varpi \longrightarrow \bar{\Omega} - \dfrac{2J}{r^3}$, with J the total angular momentum of the star, we can conclude that $l = 1$ and therefore ϖ_l vanish $\forall\ l \geq 2$. Thus, $\varpi_1 \equiv \varpi(r)$ and the equation for ϖ is

$$\frac{e^{\nu-\lambda}}{r^4}\frac{d}{dr}\left[e^{-\nu-\lambda}r^4\frac{d\varpi(r)}{dr}\right] = 16\pi\Omega^{-4}(\rho+p)\varpi(r), \tag{69}$$

with boundary conditions $\lim\limits_{r\to\infty}\varpi = \bar{\Omega}$ and $\dfrac{d\varpi(0)}{dr} = 0$. The first condition recovers the angular velocity as seen by the observer at infinity whilst the second condition guarantees the regularity at the center of the star.

An observationally accesible quantity is the moment of inertia of the star, defined by

[2] Do not confuse $\bar{\Omega}$, the angular velocity with Ω, the conformal factor.

$$I = \frac{J}{\bar{\Omega}}. \tag{70}$$

Outside the star the term $e^{-\nu-\lambda}r^4\dfrac{d\varpi}{dr}$ in Eq.(69) is constant and has to match the interior solution at $r = R$. From this fact, an integral equation for the angular momentum of the star follows,

$$e^{-\nu-\lambda}r^4\frac{d\varpi}{dr}\bigg|_0^R = 16\pi\int_0^R dr\Omega^{-4}(\rho+p)r^4e^{\lambda-\nu}\varpi = kJ. \tag{71}$$

The constant k is fixed by the Newtonian limit

$$J_{Newt} = \frac{8\pi}{3}\bar{\Omega}\int_0^R dr(\rho+p)r^4. \tag{72}$$

In the Newtonian limit $p \ll \rho$, $\phi(r) = 0$ (and the conformal factor is $\Omega = 1$), there is no dragging ($\omega = 0$) and $\nu(r) = \lambda(r)$, so that $k = 6$. Finally, we have obtained an equation for the moment of inertia

$$I = \frac{8\pi}{3}\int_0^R dr\Omega^{-4}(\rho+p)r^4e^{-\nu+\lambda}\left(\frac{\varpi(r)}{\bar{\Omega}}\right). \tag{73}$$

In FIG.(6) we show the total angular momentum of a family of stars as a function of their mass for different angular velocities. Notice the kink around 0.65 $M_\odot$ due to the first order phase transition present in EoS TrF (orange line in FIG.(1)).

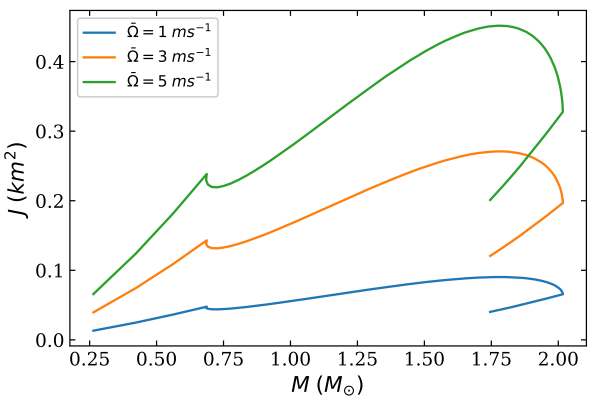

FIG. 6: Angular moment of a family of stars (EoS TrF). Different angular velocities for the same parameter $\alpha = 10$ km^2.

In FIG.(7) we display the moment of inertia of a family of stars as a function of their mass. Again, we can see a clear non-analyticity due to the phase transition present in EoS TrF. According to our previous results [12], the angular momentum of the star increases with the angular velocity and the same happens for the moment of inertia. In FIG.(8) we show the adimensional angular momentum χ, defined as $\chi = J/M^2$, accessible for example through gravitational radiation in binary mergers [1, 2]. We can also see a clear ridge due to the phase transition.

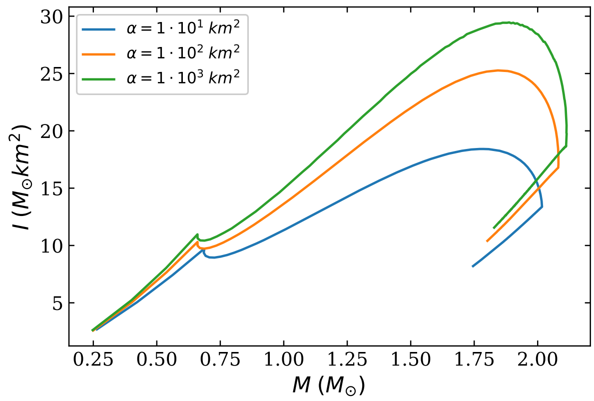

FIG. 7: Moment of inertia of a family of stars (EoS TrF). Different α values for the same angular velocity $\bar{\Omega} = 1$ ms^{-1}.

Finally, let us notice that if one computes the field equations up to second order in the angular velocity (which is beyond the scope of this work), one could obtain the mass correction and the radius correction (and then calculate the ellipticity, that gives us an idea of the deformation of the star due to rotation) [12, 19, 20].

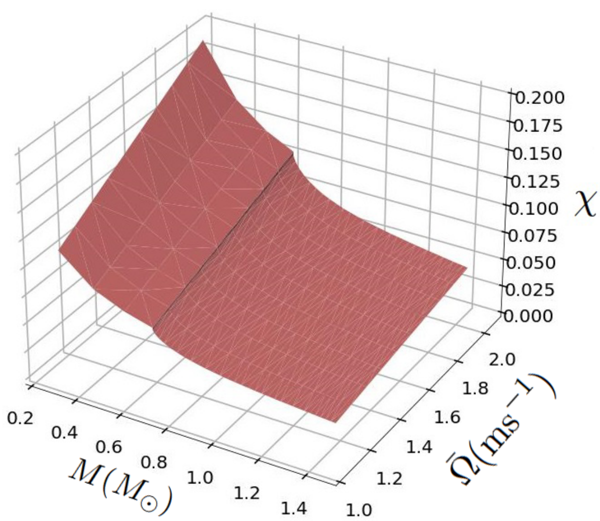

FIG. 8: Adimensional angular momentum χ as a function of the mass M and the angular velocity of the star $\bar{\Omega}$ with $\alpha = 10^3$ km^2 (EoS TrF).

IX. CONCLUSIONS

In this work we have studied neutron star properties in modified gravity. First, we have reviewed a general formalism in order to write the Tolman-Oppenheimer-Volkoff equations for a quite general family of theories. Then we have studied $f(R)$ theories and their equivalence to scalar-tensor theories, computing a complicated TOV-like system of differential equations that must be solved numerically. In particular, we studied R^2-gravity following [3, 4]. Next, we presented the set of equations of state used along this work. The EoS employed have interesting properties due to the first order phase transitions and the fact that they are not constrained by any astrophysical observable, making them useful for our study in modified gravity [8, 9]. We have programmed a numerical algorithm in order to solve the system.

We have solved the static star and computed the mass-radius diagram for several families of stars. In particular, we found non-analytical points due to the phase transitions, and observed how the diagrams change with α. This gave us a consistency check and allowed us to study how the theory drifts away from general relativity. Additionally, we have computed the maximum latent heat as a function of the parameter of the theory following the idea of Seidov in GR [15]. We note that this bounds could improve with the use of more equations of state of the band. In the future, some observational experiment could measure this observables.

Finally, we have studied the slowly rotating star, which is a good approximation for most known pulsars. We computed the angular velocity equation in modified gravity in the same way H&T did in general relativity [19, 20]. We calculated some physical observables, such as the moment of inertia, which could be measured by future experiments. We studied how the angular momentum of the family of stars changes with the angular velocity and showed that the first order phase transitions present in the EoS leaves a clear kink in the $J(M)$ and $I(M)$ diagrams. We also showed a ridge in the $\chi(M, \bar{\Omega})$ diagram due to the non-analyticity in the EoS. We concluded this work studying how the moment of inertia changes with the parameter of the theory. Again, future observations could constrain the free parameter α of the theory comparing with these numerical calculations.

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[1] R. Abbott *et al.* [LIGO Scientific and Virgo], Astrophys. J. Lett. **913** (2021) no.1, L7.
[2] B. P. Abbott *et al.* [LIGO Scientific and Virgo], Phys. Rev. X **9** (2019) no.1, 011001.
[3] K. V. Staykov, D. D. Doneva and S. S. Yazadjiev, Phys. Rev. D **93** (2016), 084010.
[4] S. S. Yazadjiev, D. D. Doneva, K. D. Kokkotas and K. V. Staykov, JCAP **06** (2014), 003 doi:10.1088/1475-7516/2014/06/003 [arXiv:1402.4469 [gr-qc]].
[5] A. A. Starobinsky, Phys. Lett. B **91** (1980), 99-102 doi:10.1016/0370-2693(80)90670-X.
[6] F. Özel and P. Freire, Ann. Rev. Astron. Astrophys. **54** (2016), 401-440, doi:10.1146/annurev-astro-081915-023322
[7] J. Antoniadis, T. M. Tauris, F. Ozel, E. Barr, D. J. Champion and P. C. C. Freire, [arXiv:1605.01665 [astro-ph.HE]].
[8] E. Lope-Oter and A. Wojnar, [arXiv:2306.00870 [gr-qc]].
[9] E. Lope-Oter and F. J. Llanes-Estrada, Eur. Phys. J. A **58** (2022) no.1, 9 doi:10.1140/epja/s10050-021-00656-9.
[10] D. D. Doneva and S. S. Yazadjiev, JCAP **11** (2016), 019 doi:10.1088/1475-7516/2016/11/019.
[11] K. V. Staykov, *et al.*, JCAP **10** (2014), 006 doi:10.1088/1475-7516/2014/10/006.
[12] P. N. Moreno, F. J. Llanes-Estrada and E. Lope-Oter, Annals Phys. **459** (2023), 169487 doi:10.1016/j.aop.2023.169487.
[13] A. Wojnar and H. Velten, Eur. Phys. J. C **76** (2016), 697 doi:10.1140/epjc/s10052-016-4549-z.
[14] D. D. Doneva, C. J. Krüger, K. V. Staykov and P. Y. Yordanov, Phys. Rev. D **108** (2023), 044054 doi:10.1103/PhysRevD.108.044054.
[15] Z. Seidov, Sov. Astron., **15** 347 (1971).
[16] R. M. Wald, Chicago Univ. Pr., 1984, doi:10.7208/chicago/9780226870373.001.0001.
[17] E. Lope-Oter and F. J. Llanes-Estrada, Phys. Rev. C **105** (2022), L052801 doi:10.1103/PhysRevC.105.L052801.
[18] G. J. Olmo, Int. J. Mod. Phys. D **20** (2011), 413-462 doi:10.1142/S0218271811018925.
[19] Hartle J B 1967 Astrophys. J. 150 1005–1029.
[20] Hartle J B and Thorne K S 1968 Astrophys. J. 153 807.
[21] J. L. Blázquez-Salcedo, L. M. González-Romero, F. S. Khoo, J. Kunz and V. Preut, Front. Astron. Space Sci. **9** (2022), 1005108 doi:10.3389/fspas.2022.1005108.
[22] K. Yagi and M. Stepniczka, Phys. Rev. D **104** (2021) no.4, 044017 doi:10.1103/PhysRevD.104.044017.
[23] E. Berti, E. Barausse, V. Cardoso, L. Gualtieri, *et al.* Class. Quant. Grav. **32** (2015), 243001 doi:10.1088/0264-9381/32/24/243001 [arXiv:1501.07274 [gr-qc]].
[24] S. Bhattacharyya, Class. Quant. Grav. **40** (2023) no.21, 215013 doi:10.1088/1361-6382/acfcff.
[25] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82** (2010), 451-497 doi:10.1103/RevModPhys.82.451.
[26] M. Aparicio Resco, Á. de la Cruz-Dombriz, F. J. Llanes Estrada and V. Zapatero Castrillo, Phys. Dark Univ. **13** (2016), 147-161 doi:10.1016/j.dark.2016.07.001.
[27] C. E. Mota, J. M. Z. Pretel and C. O. V. Flores, [arXiv:2402.13360 [gr-qc]].
[28] J. A. S. Fortunato *et al.*, [arXiv:2402.15395 [gr-qc]].
[29] G. J. Olmo, *et al.*, Phys. Rept. **876** (2020), 1-75, doi:10.1016/j.physrep.2020.07.001.
[30] E. Lope-Oter and A. Wojnar, [arXiv:2402.03914 [gr-qc]].