Dispersion relations and the QCD spectra

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Outline

Introduction: why the strong interactions are hard

Part I: dispersion relations for low-energy Hadron Physics

- Analiticity, unitarity and crossing symmetry

Part II: strong strong interactions

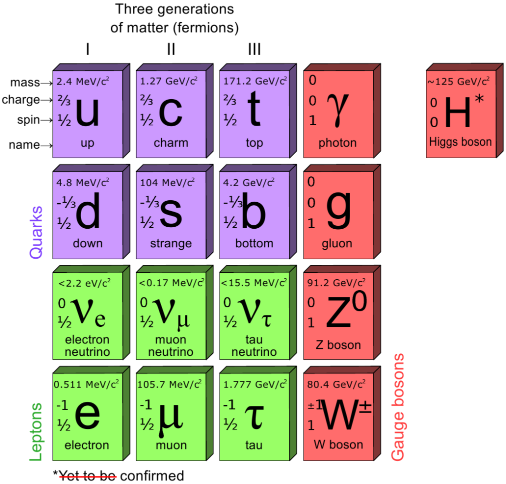
- Pion-pion scattering and the lightest scalar meson
- Pion-nucleon scattering and the nucleon mass

Part III: strong interactions and electromagnetism

- Isospin-breaking corrections to $e^+e^- \to \pi^+\pi^-$ and the muon $g-2$

Summary / Outlook

The Standard Model of particle physics



- **matter** particles:
  quarks and leptons
- **force carriers:** gauge bosons

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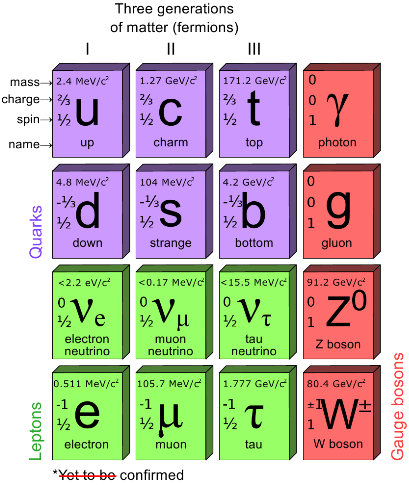

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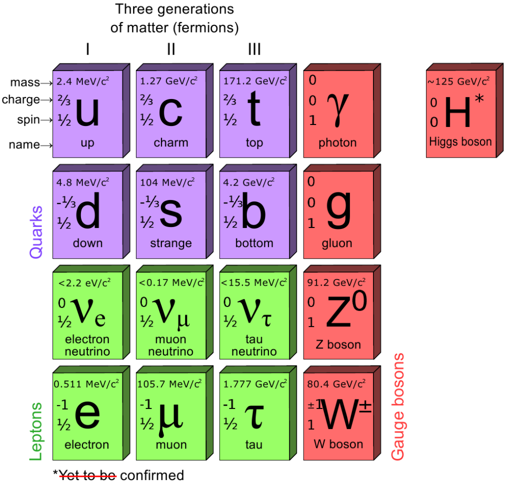

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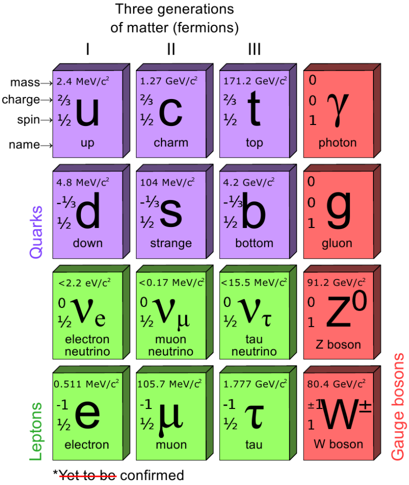

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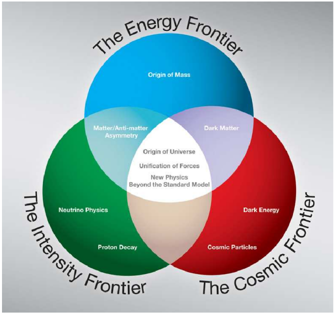
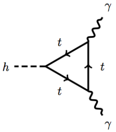
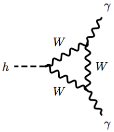


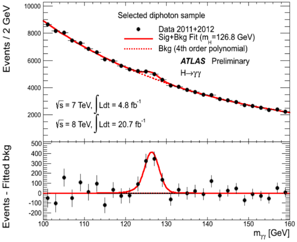
High-energy frontier

- discoveries at high energies:

  ↪ production of new particles







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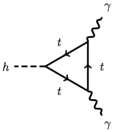
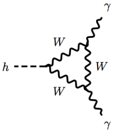
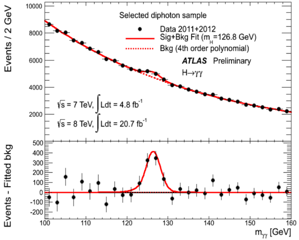
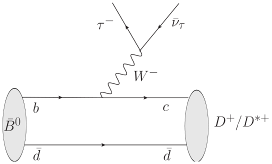






↪ production of new particles

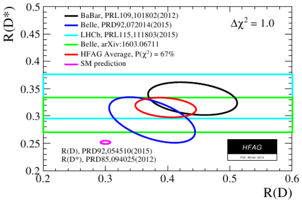
Precision frontier



- discoveries in precision experiments:

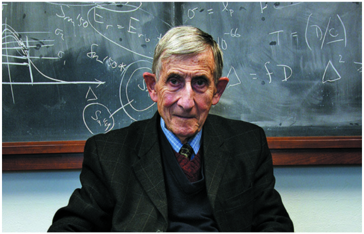
↪ effects of virtual particles

$$R_{D^{(*)}} = \frac{\mathcal{B}\left(\overline{B} \to D^{(*)}\tau\bar{\nu}_\tau\right)}{\mathcal{B}\left(\overline{B} \to D^{(*)}l\bar{\nu}_l\right)}$$



Breaking through frontiers

Freeman Dyson on 16 discoveries awarded the Nobel Prize between 1945 and 2008:



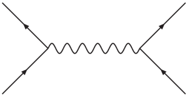
“four discoveries on the energy frontier, four on the rarity frontier, eight on the accuracy frontier. Only a quarter of the discoveries were made on the energy frontier, while half of them were made on the accuracy frontier. **For making important discoveries, high accuracy was more useful than high energy**.”

(Freeman Dyson, review of The Lightness of Being, F. Wilczek, The New York Review of Books, April 2009)

Perturbation theory and coupling constants

- expand amplitudes in powers of coupling constants α

  $\hookrightarrow$ scattering amplitude $\propto \alpha(\cdots)$

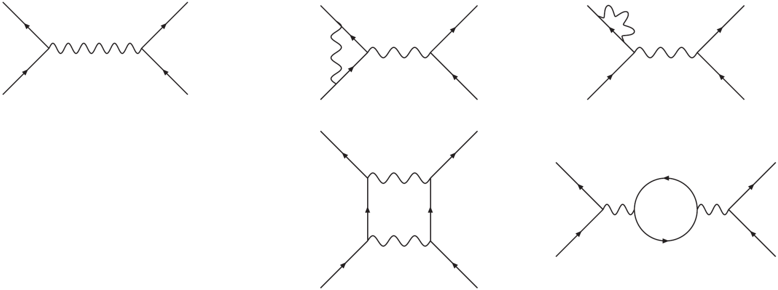


Why the strong interactions are hard?

Perturbation theory and coupling constants

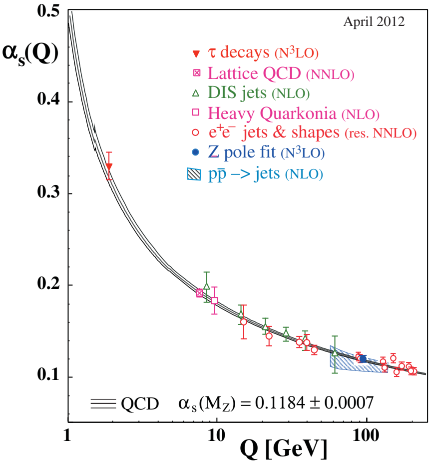
- expand amplitudes in powers of coupling constants α

  $\hookrightarrow$ scattering amplitude $\propto \alpha(\cdots) + \alpha^2(\cdots)$



- works well for electromagnetic and weak interactions: $\alpha \sim 10^{-2}$

Running coupling of QCD



[Bethke et al. 2012]

- **Asymptotic freedom** at high energies ("weak QCD")

- QCD strongly coupled at low energies

  $\hookrightarrow$ **confinement** ("strong QCD")

  no quarks + gluons, only (color-neutral) hadrons

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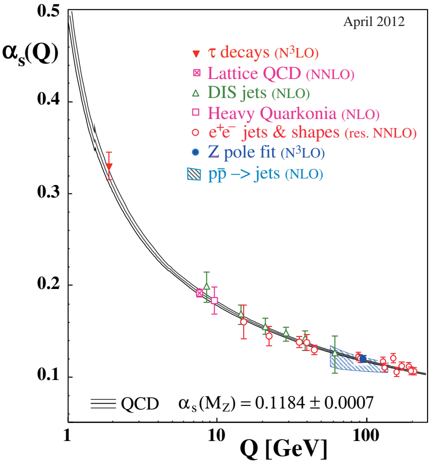

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- **Asymptotic freedom** at high energies ("weak QCD")
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  $\hookrightarrow$ **confinement** ("strong QCD")

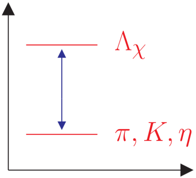
  no quarks + gluons, only (color-neutral) hadrons
- At the typical hadronic scale 1 GeV

  $\hookrightarrow$ Perturbation theory fails

  $\hookrightarrow$ **Need non-perturbative methods**

1. **Effective field theories**:

$\hookrightarrow$ symmetries, separation of scales



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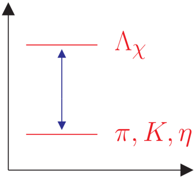
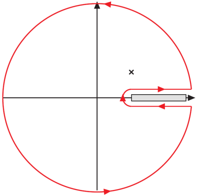


2. **Dispersion relations**: analyticity ($\simeq$ causality), unitarity ($\simeq$ probability conservation), crossing symmetry

   $\hookrightarrow$ Cauchy's theorem, analytic structure



Non-perturbative methods for low-energy hadron physics

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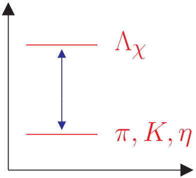
   $\hookrightarrow$ symmetries, separation of scales



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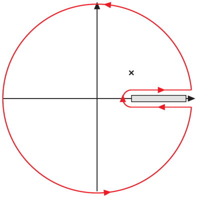
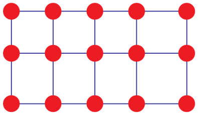


3. **Lattice**: Monte-Carlo simulation

   $\hookrightarrow$ solve discretized version of QCD numerically



Part I:

Dispersion relations:

analyticity, unitarity and crossing symmetry

Dispersion relations in a nutshell

- Dispersion relations: **analyticity**, **crossing**, **unitarity**

  ▷ analitycity constrains the energy depedence of scattering amplitude

  ▷ crossing symmetry connects different physical regions

  ▷ unitarity constrains imaginary part

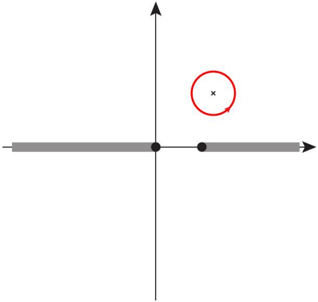
- **model independent** aproach

- **analytic continuation** for the complex plane

  ↪ resonances, unphysical regions

- Cauchy's Theorem

$$t(s) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{ds'\, t(s')}{s' - s}$$



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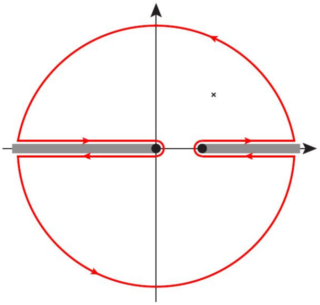
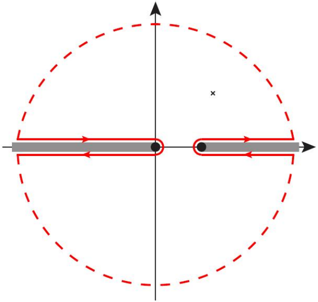


- Dispersion relation

$$t(s) = \frac{1}{\pi}\int_{\text{cuts}} \frac{ds' \,\mathrm{Im}\, t(s')}{s'-s}$$

$\hookrightarrow$ **analyticity**



From Cauchy's theorem to dispersion relations

- Dispersion relation

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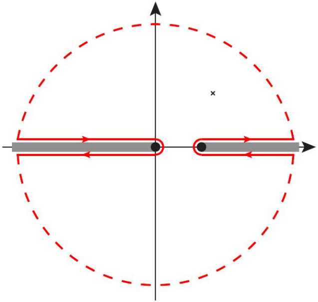
$\hookrightarrow$ **analyticity**

- Subtractions

$$t(s) = t(0) + \frac{s}{\pi}\int_{\text{cuts}} \frac{ds'\,\mathrm{Im}\,t(s')}{s'(s'-s)}$$



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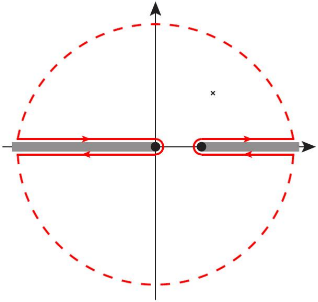
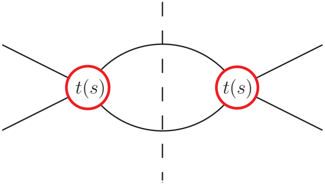
- Imaginary part from **unitarity**

$\hookrightarrow$ forward direction: optical theorem





$$\text{Im}\, t(s) = \sigma(s)|t(s)|^2, \quad t(s) = \frac{\eta(s)e^{2i\delta(s)}-1}{2i\sigma(s)}, \quad \sigma(s) = \sqrt{1-\frac{4m_\pi^2}{s}}$$

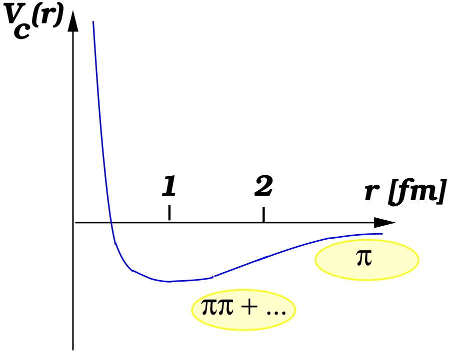
Part II: meson sector

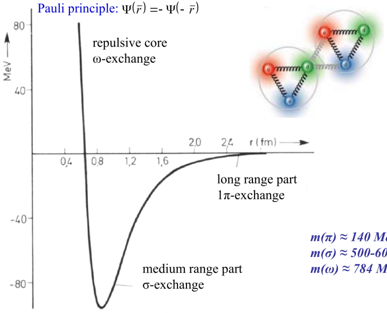
Pion-pion scattering and

the lightest scalar meson

- lightest scalar-isoscalar meson

  ↪ attractive part of nucleon-nucleon interaction





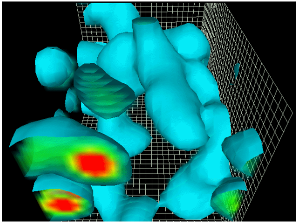
m(π) ≈ 140 MeV/c²
m(σ) ≈ 500-600 MeV/c²
m(ω) ≈ 784 MeV/c²

Yukawa Potential: $\hat{V}(r_1, r_2) = -\frac{g^2}{4\cdot\pi}\cdot\frac{e^{-\mu r}}{r}$

The lightest scalars in Hadron Physics

- vacuum quantum numbers

  $\hookrightarrow$ spontaneous chiral symmetry breaking



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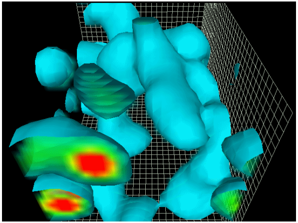
  ↪ lightest glueball expected to be a scalar



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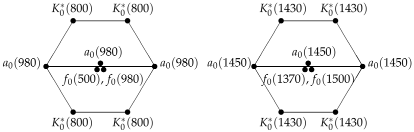

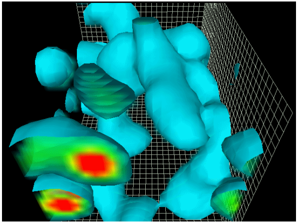

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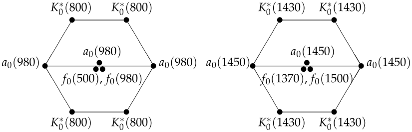
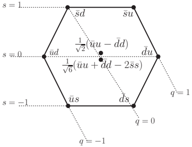
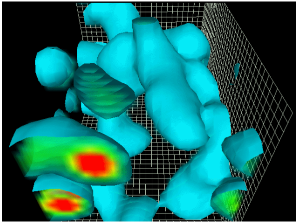


- inverted hierarchy: ordinary mesons?





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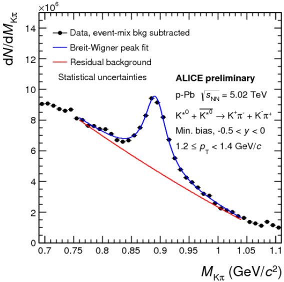

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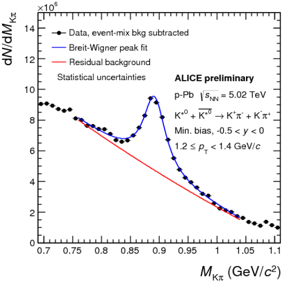
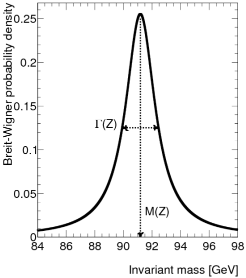
  $\hookrightarrow$ Breit-Wigner resonances





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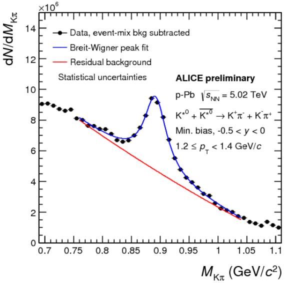
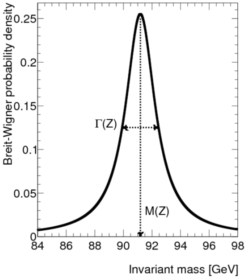
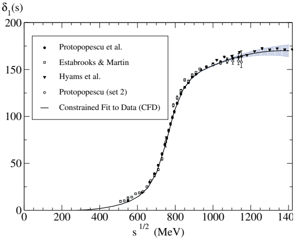
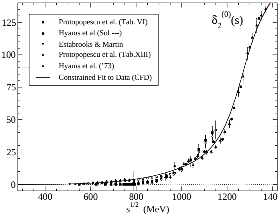
- resonance mass

  ↪ energy where $\delta = 90^\circ$



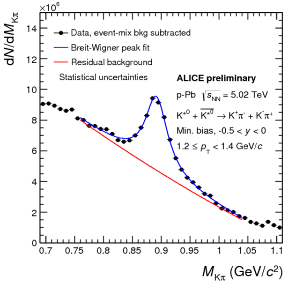


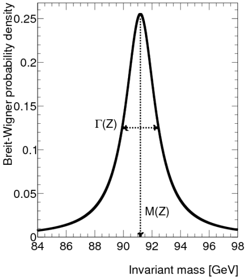


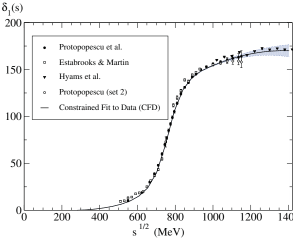


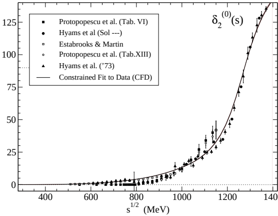
Why the scalars are tough?

- ordinary resonances are narrow
  ↪ Breit-Wigner resonances
- resonance mass
  ↪ energy where $\delta = 90^\circ$
- lightest scalars are broad









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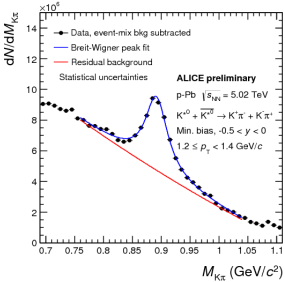
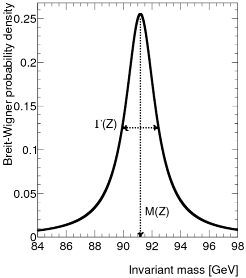
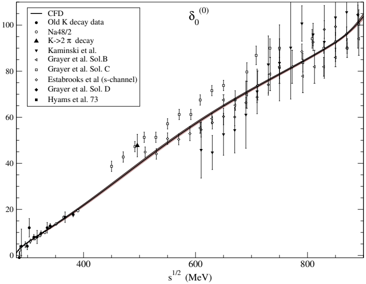
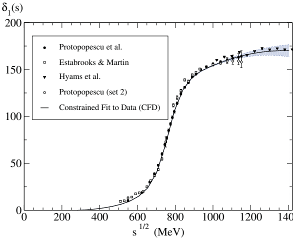
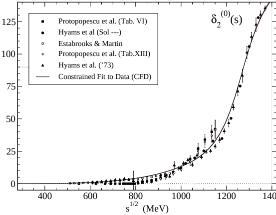




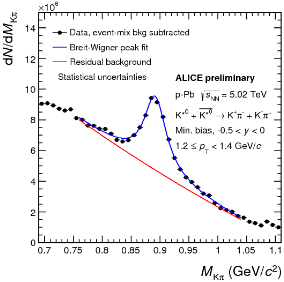


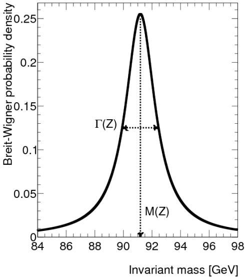


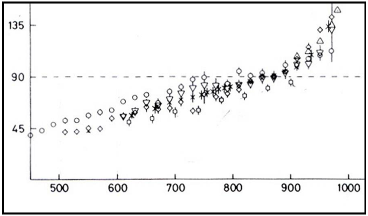


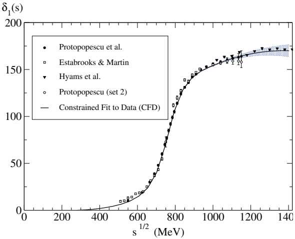
Why the scalars are tough?

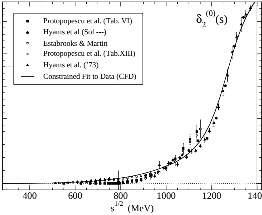
- ordinary resonances are narrow
  - $\hookrightarrow$ Breit-Wigner resonances
- resonance mass
  - $\hookrightarrow$ energy where $\delta = 90^\circ$
- lightest scalars are broad
  - $\hookrightarrow$ no Breit-Wigner shape



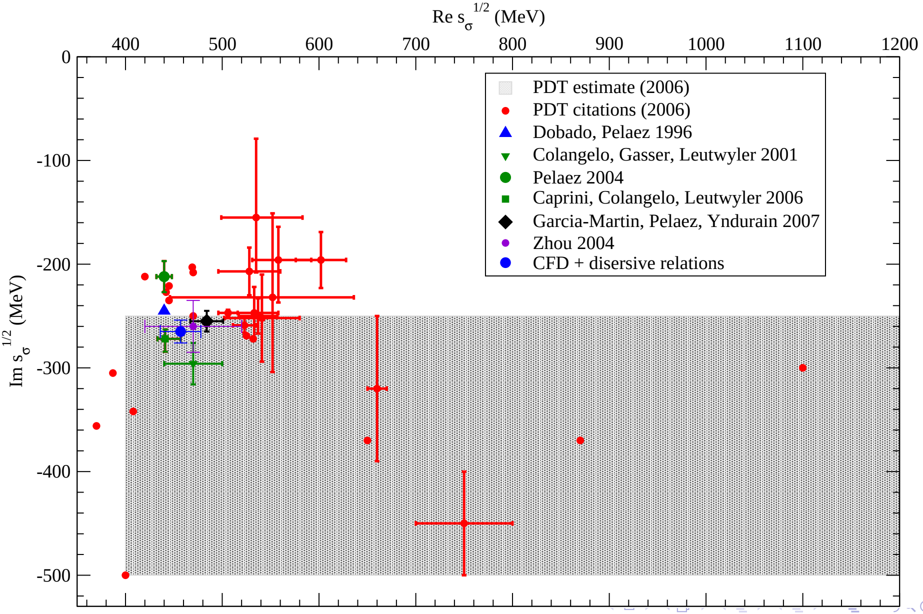








$f_0(500)$ pole position until 2012



Why $\pi\pi$ scattering?

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  $\hookrightarrow$ **good understanding** of strong interactions at **low energies**

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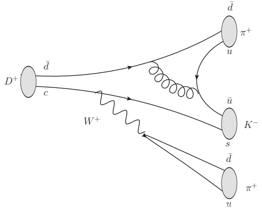

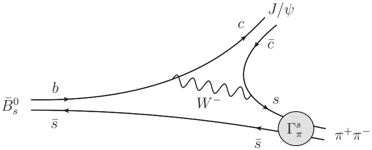

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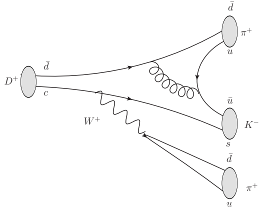
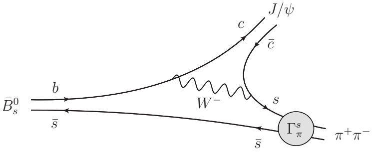
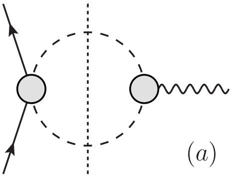


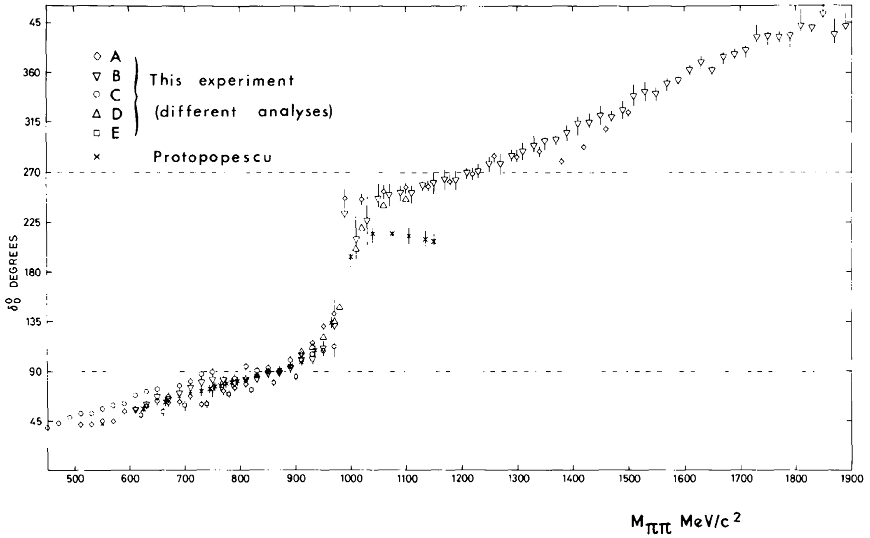


- **first inelastic correction to many observables**

  $\hookrightarrow$ nucleon structure, muon g-2, proton radius puzzle



Experimental $\pi\pi$ status



G. Grayer et al., High statistics study of the reaction $\pi^- p \to \pi^- \pi^+ n$

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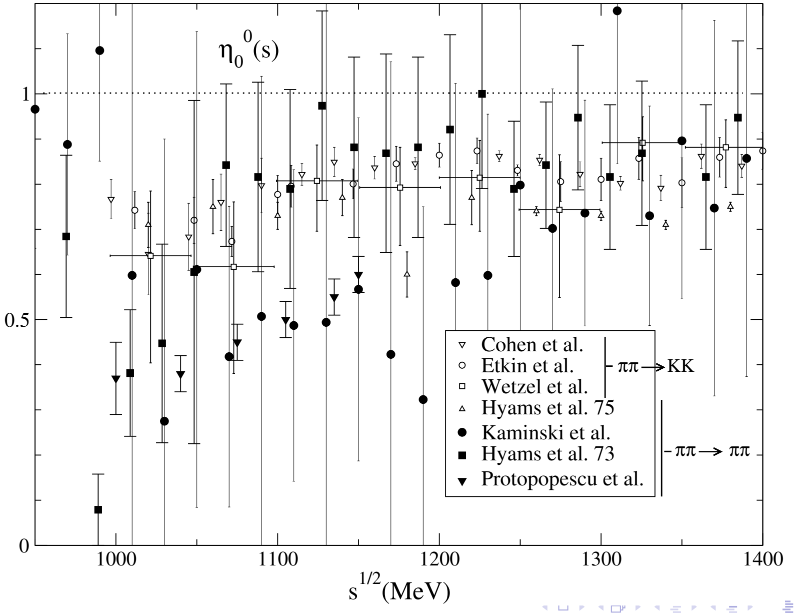
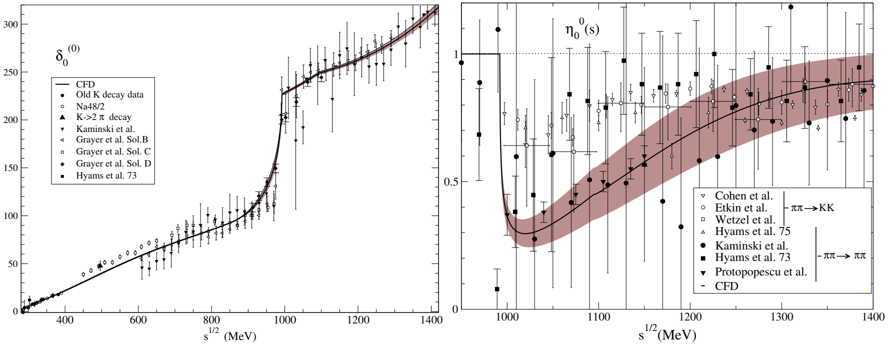
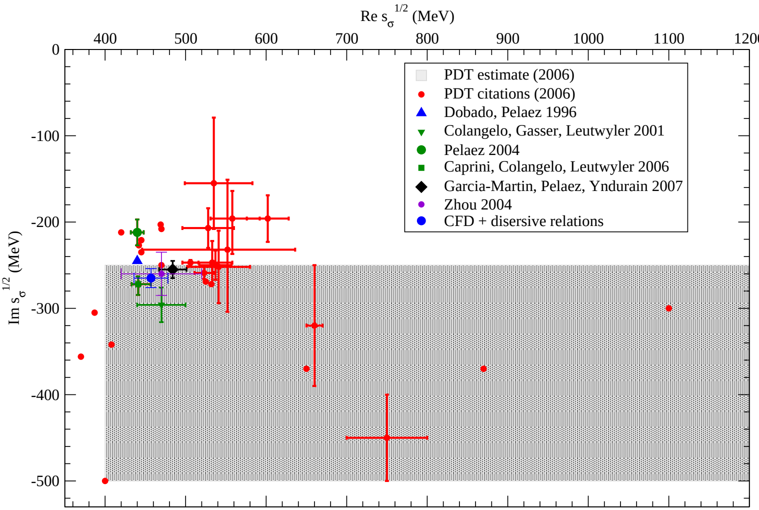


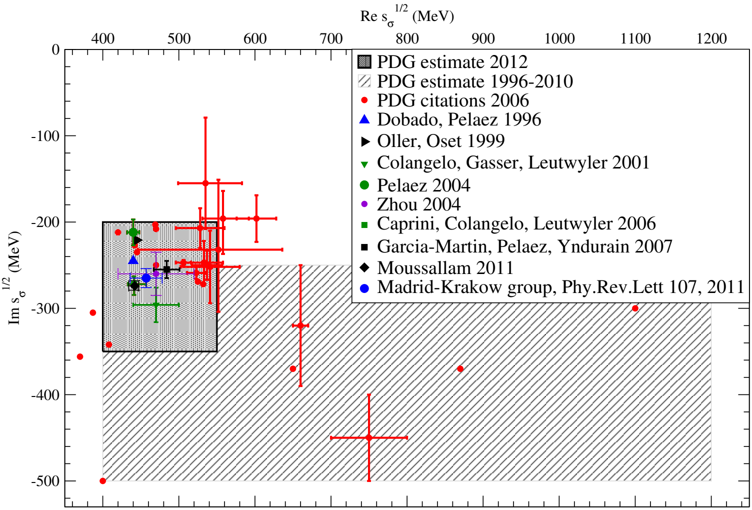
Roy-equations: $\pi\pi$ results



[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain 2011]

$f_0(500)$ pole before 2012

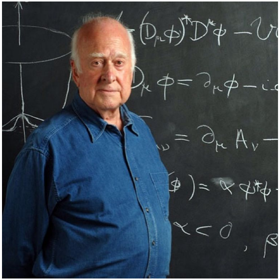



Re $s_\sigma^{1/2}$ (MeV)
400
500
600
700
800
900
1000
1100
1200
0
-100
-200
-300
-400
-500
Im $s_\sigma^{1/2}$ (MeV)
PDG estimate 2012
PDG estimate 1996-2010
PDG citations 2006
Dobado, Pelaez 1996
Oller, Oset 1999
Colangelo, Gasser, Leutwyler 2001
Pelaez 2004
Zhou 2004
Caprini, Colangelo, Leutwyler 2006
Garcia-Martin, Pelaez, Yndurain 2007
Moussallam 2011
Madrid-Krakow group, Phy.Rev.Lett 107, 2011

Part II: baryon sector

Pion-nucleon scattering and the nucleon mass

Where does the nucleon mass come from?

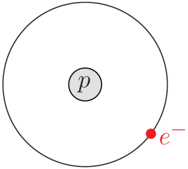


$\hookrightarrow$ haven't we discovered the Higgs boson,

giving mass to all elementary particles of the Standard Model?

Where does the nucleon mass come from?

Well, not quite:



$$m_{\rm atom} = m_{\rm proton} + m_{\rm electron} - E_{\rm binding}$$
$$= 938\,{\rm MeV} + 512\,{\rm keV} - 13.6\,{\rm eV}$$

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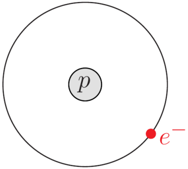

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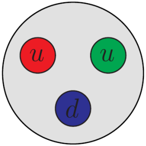

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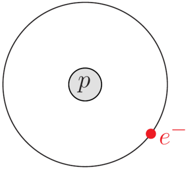

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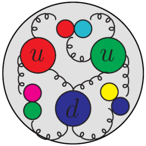

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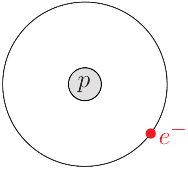

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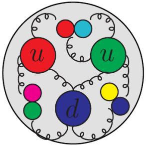

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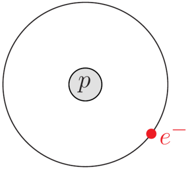

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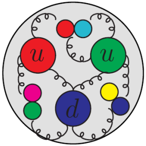

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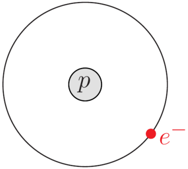

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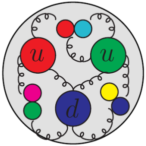

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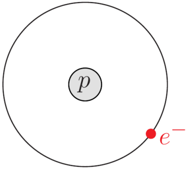

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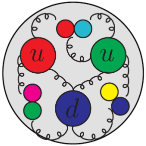

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  - ▷ from pion-nucleon scattering: "sigma term" $\sigma_{\pi N} \doteq$ part of m_N due to m_{quark}

Role of the pion-nucleon σ-term

- scalar coupling of the nucleon

$$\langle N|m_q\bar{q}q|N\rangle = f_q m_N, \quad f_q = \frac{\sigma_{\pi N}}{2m_N}$$

$\hookrightarrow$ Dark Matter detection

$\hookrightarrow$ $\mu \to e$ conversion in nuclei

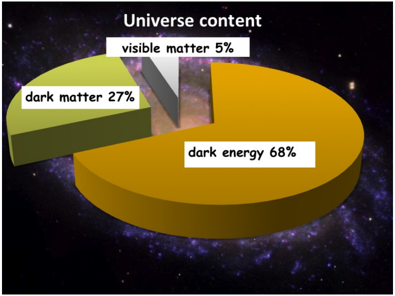
- Condensates in nuclear matter

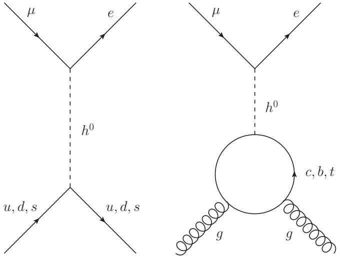
$$\frac{\langle\bar{q}q\rangle(\rho)}{\langle 0|\bar{q}q|0\rangle} = 1 - \frac{\rho\,\sigma_{\pi N}}{F_\pi^2 M_\pi^2}$$

- CP violation in πN couplings

$\hookrightarrow$ hadronic EDMs

$$g_{\eta N} \propto -\left(\frac{\sigma_s}{m_s} - \frac{\sigma_{\pi N}}{2\hat{m}}\right)\bar{\theta}$$





The σ-term and pion-nucleon scattering

- $\sigma_{\pi N}$ closely related to πN scattering at the Cheng–Dashen point

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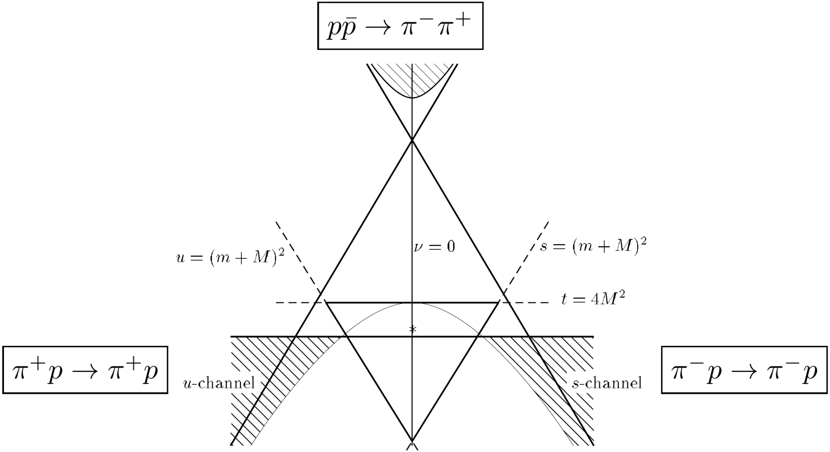

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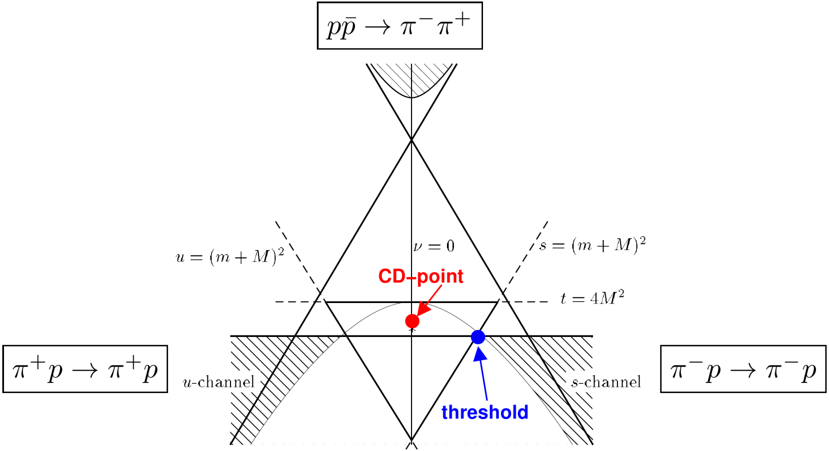

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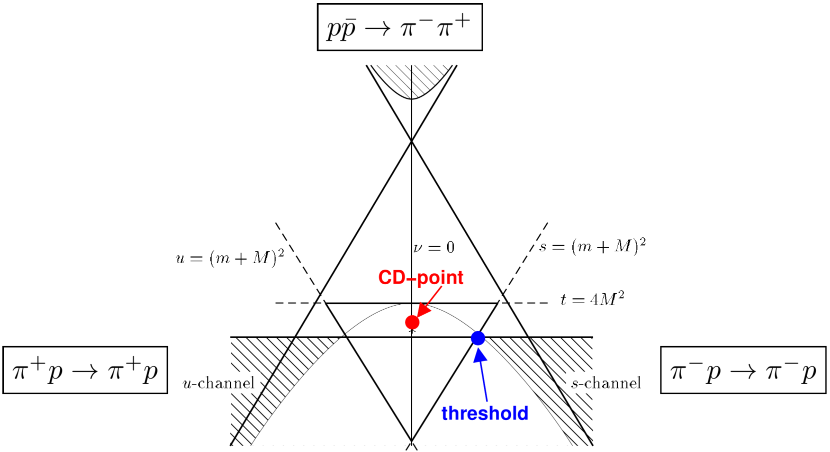

- evaluation of scattering amplitude at an **unphysical** point

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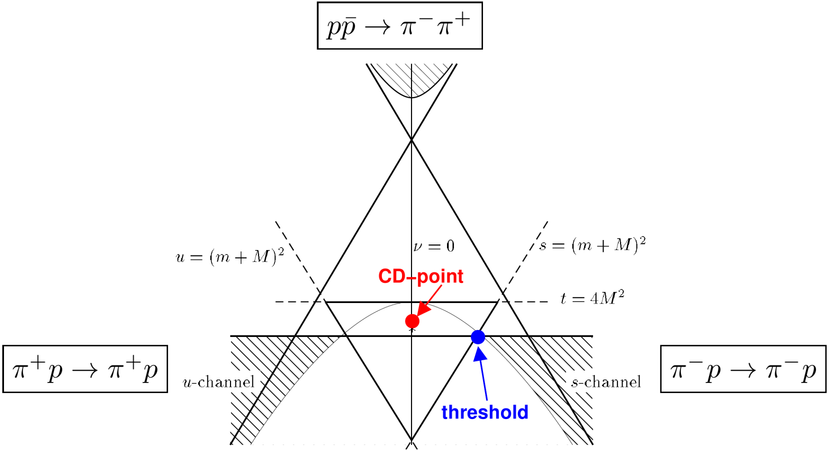
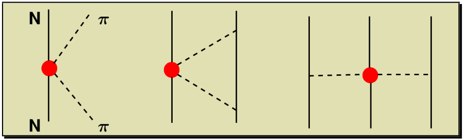


- evaluation of scattering amplitude at an **unphysical** point

  ↪ **dispersion relations**

Pion-nucleon scattering

- **low energies**: test chiral dynamics in the baryon sector
- **higher energies**: resonances, baryon spectrum
- πN scattering appears as subprocess in *NN* and 3*N* forces

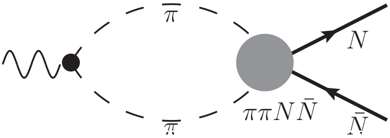


- crossed channel $\pi\pi \to N\bar{N}$: nucleon form factors

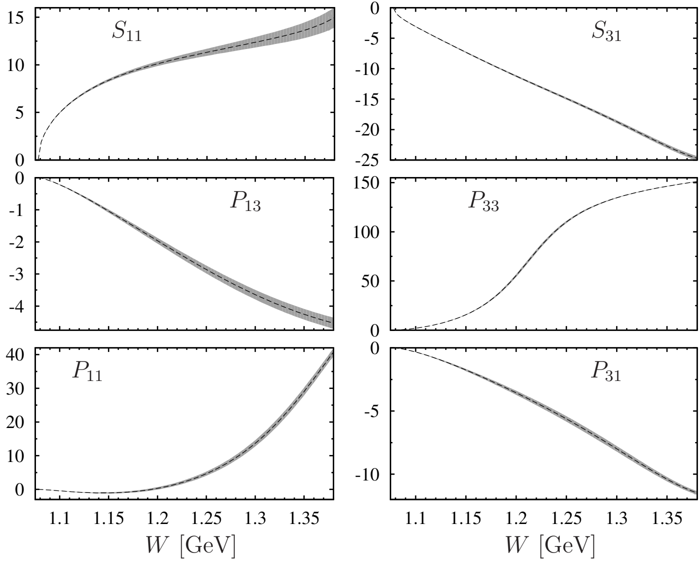
$\hookrightarrow$ probe the structure of the nucleon

Particle	J^P	overall	$N\gamma$	$N\pi$	$\Delta\pi$	$N\sigma$	$N\eta$
N	$1/2^+$	****					
$N(1440)$	$1/2^+$	****	****	****	****	***	
$N(1520)$	$3/2^-$	****	****	****	****	**	****
$N(1535)$	$1/2^-$	****	****	****	***	*	****
$N(1650)$	$1/2^-$	****	****	****	***	*	****
$N(1675)$							
$N(1680)$							
$N(1700)$							
$N(1710)$							
$N(1720)$							
$N(1860)$							

Particle	J^P	overall	$N\gamma$	$N\pi$	$\Delta\pi$
$\Delta(1232)$	$3/2^+$	****	****	****	
$\Delta(1600)$	$3/2^+$	****	****	***	****
$\Delta(1620)$	$1/2^-$	****	****	****	****
$\Delta(1700)$	$3/2^-$	****	****	****	****
$\Delta(1750)$	$1/2^+$	*	*	*	
$\Delta(1900)$	$1/2^-$	***	***	***	*
$\Delta(1905)$	$5/2^+$	****	****	****	**
$\Delta(1910)$	$1/2^+$	****	***	****	**
$\Delta(1920)$	$3/2^+$	***	***	***	***



Dispersive results: s-channel pw



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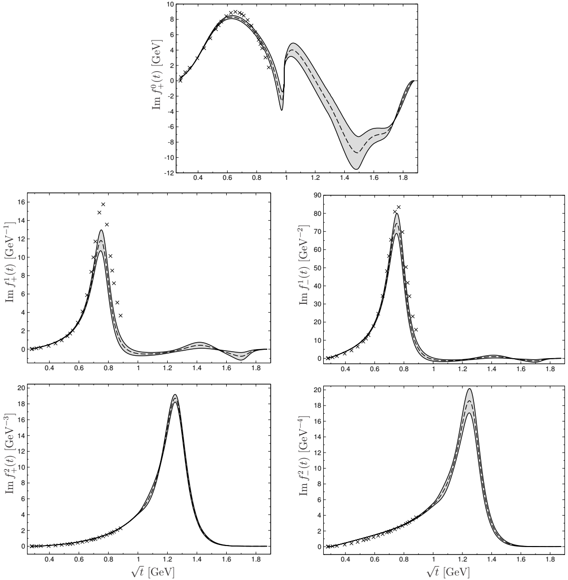

Pion–nucleon sigma term results

- Use Roy equations to extract the pion–nucleon sigma term

$$\sigma_{\pi N} = (59.1 \pm 3.5)\,\text{MeV}$$

[Hoferichter, JRE, Kubis, Meißner 2015]

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- recent lattice determination of $\sigma_{\pi N}$ at (almost) the physical point
  - ▷ BMW $\sigma_{\pi N} = 38(3)(3)$MeV [Durr et al. 2015]
  - ▷ χQCD $\sigma_{\pi N} = 44.4(3.2)(4.5)$MeV [Yang et al. 2015]
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- sigma-term puzzle

Part III:

Isospin-breaking corrections to $e^+e^- \to \pi^+\pi^-$ and the muon $g-2$

The anomalous magnetic moment of the muon

Contribution	Value $\times 10^{11}$
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)
Experiment	116 592 061(41)
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	251(59)

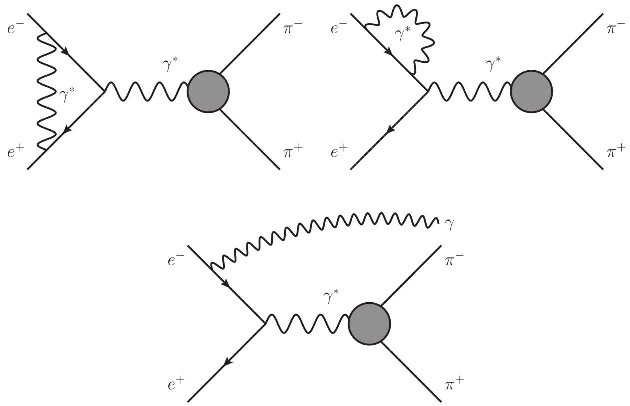
- HVP dominant source of theory uncertainty
- relative size $\sim 0.6\%$

  $\hookrightarrow$ radiative corrections in $e^+e^- \to \pi^+\pi^-$ must be under control
- RC evaluation based on models so far

  $\hookrightarrow$ a **dispersive** approach could lead to **model-independent** results

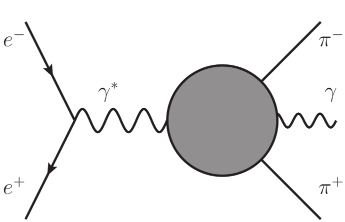
Radiative corrections to $e^+e^- \to \pi^+\pi^-$

- Initial State Radiation:



can be calculated in QED in terms of $F_\pi^V(s)$

- Final State Radiation:

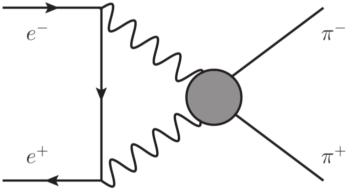


- requires hadronic matrix elements beyond $F_\pi^V(s)$
- known in ChPT to one loop

[Kubis, Meißner 2001]

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

- Interference terms



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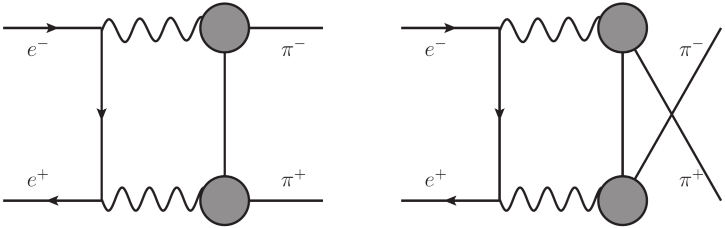
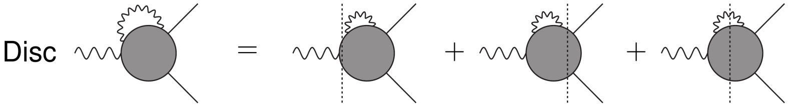


- also require hadronic matrix elements beyond $F_\pi^V(s)$ other than in the π-exchange approximation
- one-pion contributions analyzed dispersively [Colangelo, Hoferichter, Monnard, JRE. (in preparation)] do not contribute to the total cross section and will be ignored



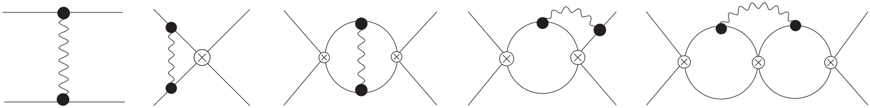
- Neglecting intermediate states beyond 2π, unitarity reads

$$\text{Im}\, F_V^{\pi,\alpha}(s) = \int d\phi_2 F_V^{\pi}(s) \times T_{\pi\pi}^{\alpha *}(s,t) + \int d\phi_3 F_V^{\pi,\gamma}(s,t) \times T_{\pi\pi}^{\gamma *}(s,t') + \int d\phi_2 F_V^{\pi,\alpha}(s) \times T_{\pi\pi}(s,t)^*$$

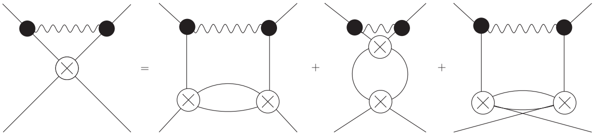
$\hookrightarrow$ need $T_{\pi\pi}^{\alpha}$ as well as $T_{\pi\pi}^{\gamma}$ and $F_\pi^{V,\gamma}$ as input

- The DR for $F_\pi^{V.\alpha}(s)$ takes the form of an integral equation

- Radiative corrections to $\pi\pi$ scattering required as input

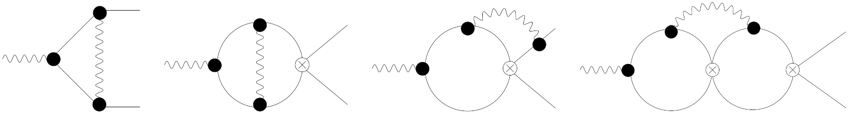


- Compute them using analyticity, unitarity
  using as input dispersive results for F_V^π, $T_{\pi\pi}$ and $A_{\pi\pi}^{\gamma\gamma}$
- Non-trivial problem. E. g. triangle topology in the t-channel

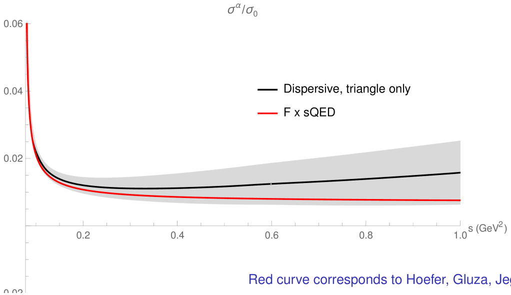


$\hookrightarrow$ s-channel cut requires a double-spectral function

Having evaluated all the following diagram



the results for $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$ look as follows: [Colangelo, Monnard, JRE (in progress)]



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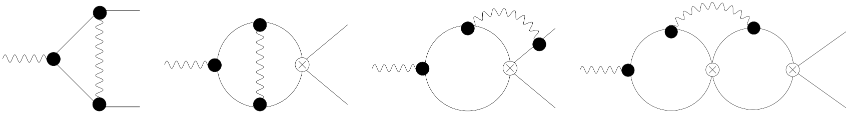

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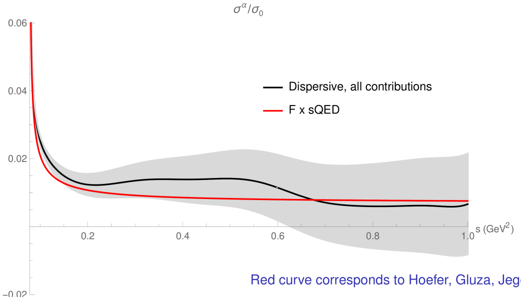

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Impact on a_μ^{HVP}

Ideally one would use the calculated RC directly in the data analysis (future?).

To get an idea of the impact we did the following:

- remove RC from the measured $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$
- fit with the dispersive representation for F_π^V
- insert back the RC

The impact on a_μ^{HVP} is evaluated by comparing to the result obtained by removing RC

$$10^{11}\Delta a_\mu^{\mathrm{HVP}} = \begin{cases} 10.2 \pm 0.5 \pm 5 & \text{FsQED} \\ 10.5 \pm 0.5 & \text{triangle} \\ 13.2 \pm 0.5 & \text{full} \end{cases}$$

3 pieces of modern strong-interaction physics using dispersive techniques:

- Dispersion relations: analyticity, unitarity, crossing symmetry
  - ▷ respect all symmetries
- Pion–pion scattering and the lightest scalar meson
  - ▷ $f_0(500)$ meson properties finally settled
- Pion–nucleon scattering and the sigma term
  - ▷ high-precision determination of the pion–nucleon sigma term
- Radiative-corrections to $e^+e^- \to \pi^+\pi^-$
  - ▷ new formalism for evaluating dispersively RC and its contribution to a_μ^{HVP}

Thank you

Spare slides

Dispersion relations: crossing symmetry

- scattering amplitude in 2 variables (e.g. CMS energy & angle)
- relativistic QM: "antiparticles = particles going backward in time"

  $\hookrightarrow$ **crossing symmetry**

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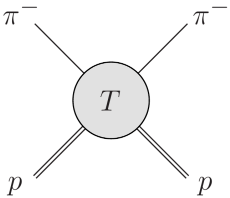

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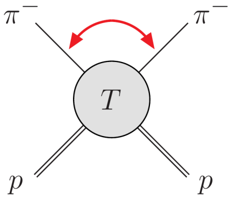

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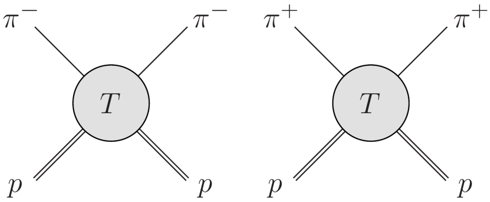

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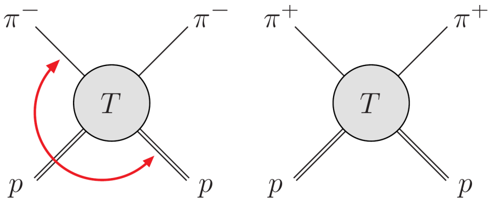

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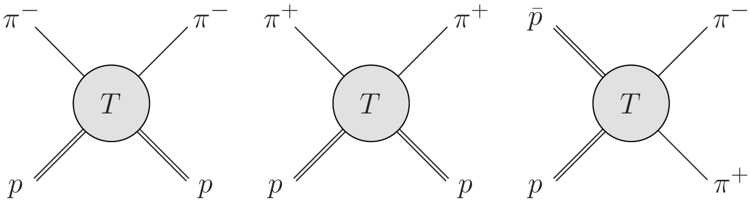

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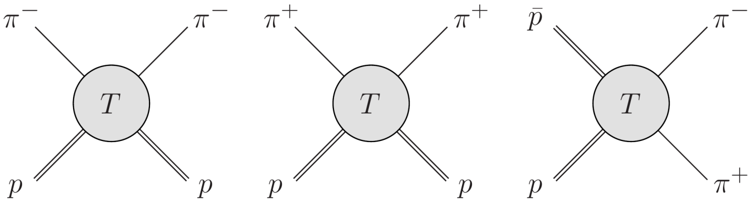

- 3 different processes described by same scattering amplitude T (for very different energies/angles)

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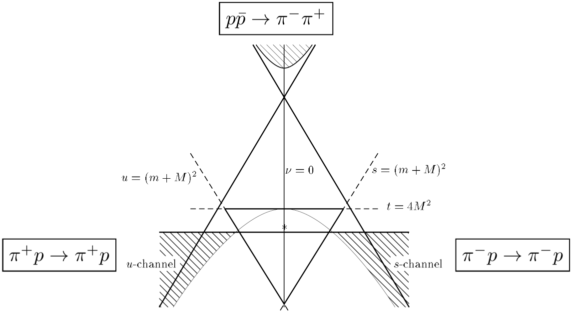

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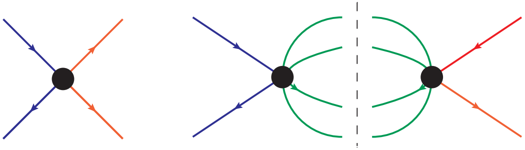

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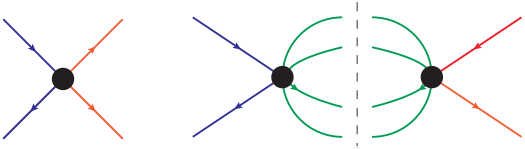

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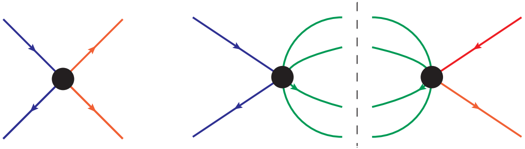

- $\mathrm{Im}\, t_{IJ} \neq 0$ above the first production threshold $\Rightarrow$ Right-Hand-Cut (RHC)
- $\mathrm{Im}\, t_{IJ} \neq 0$ production threshold in the crossed channel $\Rightarrow$ Left-Hand-Cut (LHC)

Analyticity: all **singularities** of scattering amplitudes have a **dynamical origin**

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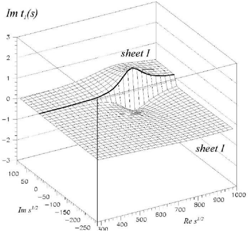

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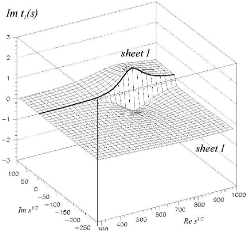

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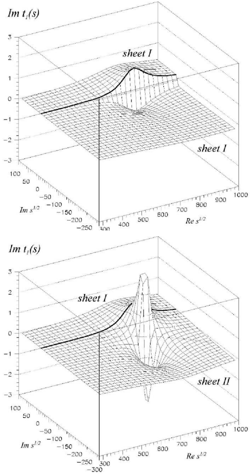

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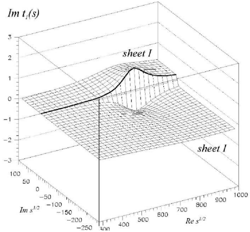
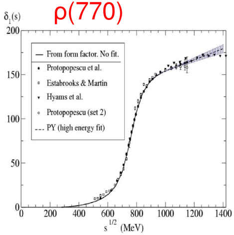


ρ(770)



From dispersion relations to Roy equations

- Start from twice-subtracted DRs

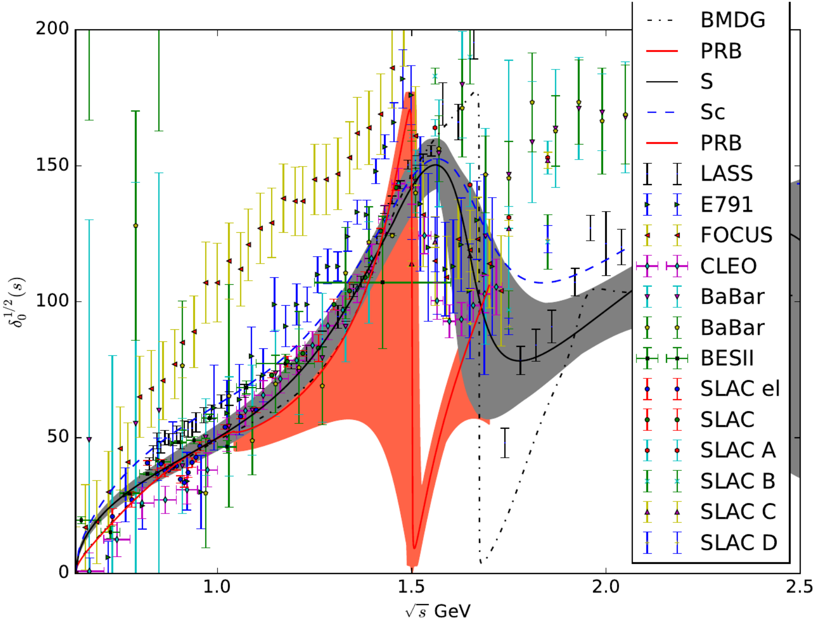
$$T^I(s,t) = c(t) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2}\left[\frac{s^2}{(s'-s)} - \frac{u^2}{(s'-u)}\right] \mathrm{Im}\, T^I(s',t)$$

- Subtraction functions c(t) are determined via crossing symmetry
- PW-projection and expansion yields the **Roy-equations** [Roy (1971)]

$$t_J^I(s) = ST_J^I(s) + \sum_{J'=0}^{\infty}(2J'+1)\sum_{I'=0,1,2}\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s',s)\,\mathrm{Im}\, t_{J'}^{I'}(s')$$

- $K_{JJ'}^{II'}(s',s) \equiv$ kernels $\Rightarrow$ analytically known

Experimental πK status



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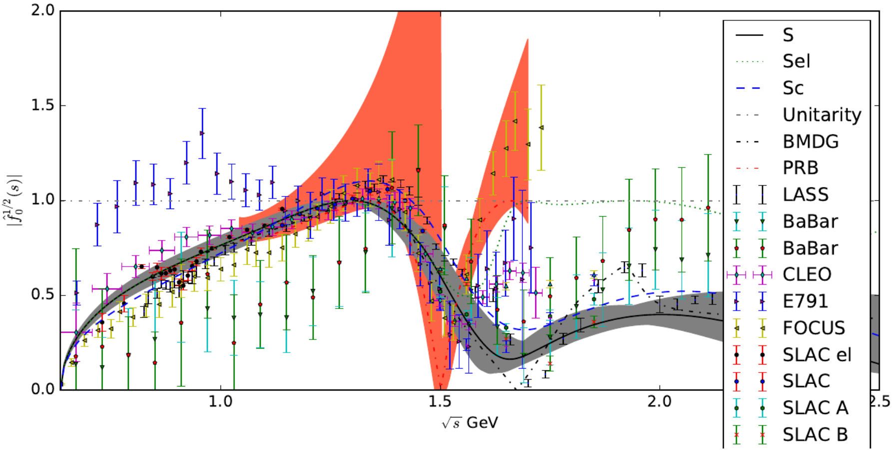
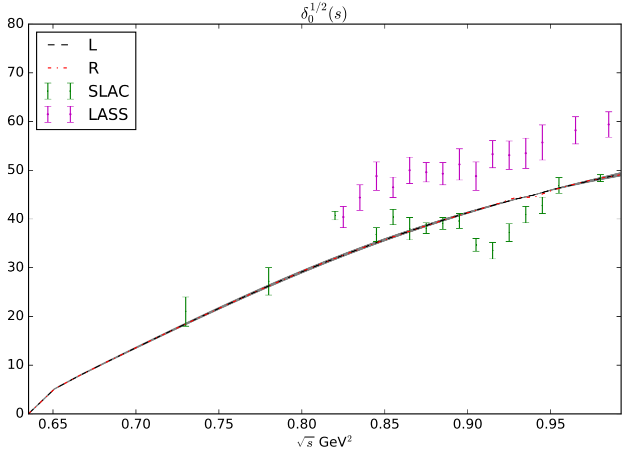
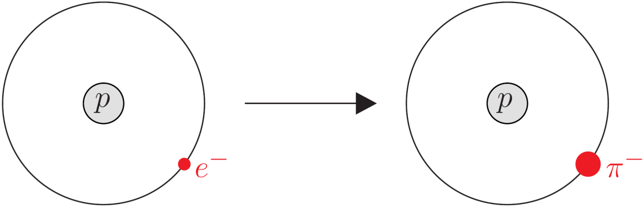


Roy-equations: πK results



[Colangelo, Maurizio, JRE 2018]

- pionic atoms: atoms with $e^- \to \pi^-$
- calculate energy levels as for hydrogen in quantum mechanics



Pionic atoms and pion–nucleon sigma term

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  - ▷ ground state energy shift ϵ_{1s}
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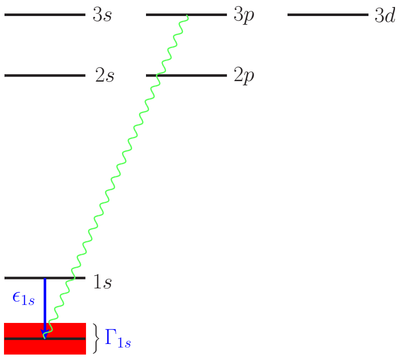
    $\pi^- p \to \pi^0 n \longrightarrow$ width Γ_{1s}



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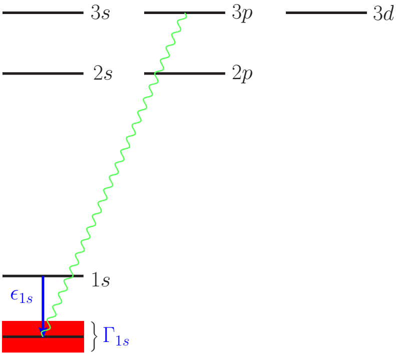
    $$\pi^- p \to \pi^0 n \longrightarrow \text{width } \Gamma_{1s}$$

- linked to πN scattering at threshold:

$$\epsilon_{1s} \propto T(\pi^- p \to \pi^- p) \propto a^+ + a^-$$

$$\Gamma_{1s} \propto T(\pi^- p \to \pi^0 n) \propto |a^-|^2$$

[Deser, Goldberger, Baumann, Thirring 1954]



Pionic atoms and pion–nucleon sigma term

- Measurements of πH and πD at PSI 1995-2010 [Gotta et al. 2008, Hennebach et al. 2014]

$$\epsilon_{1s}^{\pi H} = (-7.086 \pm 0.009)\,\text{eV}, \quad \epsilon_{1s}^{\pi D} = (2.356 \pm 0.031)\,\text{eV}, \quad \Gamma_{1s}^{\pi H} = (0.823 \pm 0.019)\,\text{eV},$$

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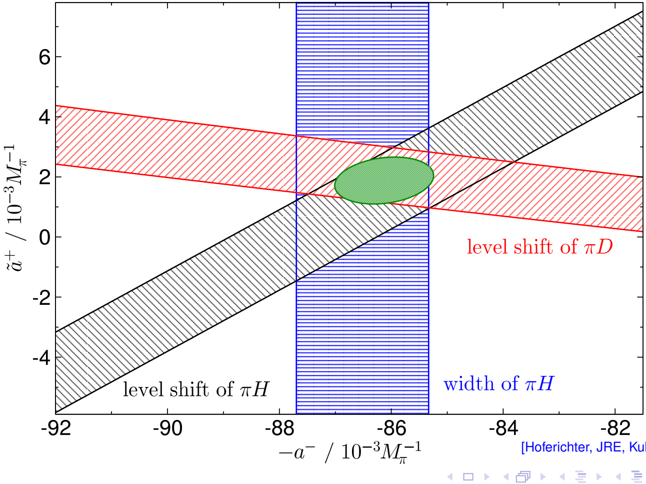


[Hoferichter, JRE, Kubis, Meißner 2015]

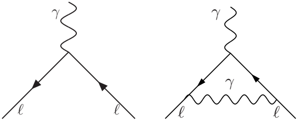
Perturbation theory and coupling constants

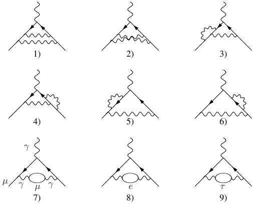
- anomalous magnetic moment of the muon: $g_\mu = 2(1 + a_\mu)$

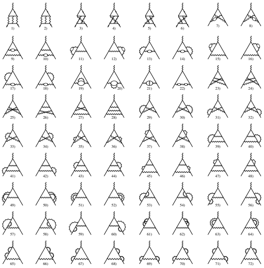
$$a_\mu^{exp} = 116592089(63) \times 10^{-11}, \quad a_\mu^{SM} = 116591813(58) \times 10^{-11}$$

- QED contribution computed at fifth order

$$a_{\mu,\,QED}^{SM} = \frac{1}{2}\frac{\alpha}{\pi} - 0.33\left(\frac{\alpha}{\pi}\right)^2 + 1.18\left(\frac{\alpha}{\pi}\right)^3 - 1.91\left(\frac{\alpha}{\pi}\right)^4 + 9.17\left(\frac{\alpha}{\pi}\right)^5$$







- Particle physics at the **precision frontier**

  $\hookrightarrow$ **good understanding** of strong interactions at **low energies**

- Low-energy regime of QCD [Fritzsch, Gell-Mann, Leutwyler 73]

$$\mathcal{L}_{QCD} = \sum_{q=u,d,s} i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q \left(\bar{q}_R q_L - \bar{q}_L q_R\right)$$

  $\hookrightarrow$ QCD invariant under **flavor rotations** of the q_L and q_R fields in the **massless limit**

- Symmetry group: $SU(N_f)_L \times SU(N_f)_R$

- $N_f = 2$, m_u and m_d **very small**

  $\hookrightarrow$ $SU(2)_L \times SU(2)_R$ should be a nearly **perfect symmetry** [Nambu 60]

- $N_f = 3$, $m_s \sim \Lambda_{QCD}$

  $\hookrightarrow$ $SU(3)_L \times SU(3)_R$ larger corrections

Spontaneous chiral symmetry breaking

- Chiral symmetry is not "visible" in the spectrum
- Chiral symmetry is realized in the Nambu-Goldstone mode

  $\hookrightarrow$ **spontaneous breaking** of **chiral symmetry**
- π, K, η are the **Goldstone Bosons** from Chiral Symmetry Spontaneous Breakdown

  $\hookrightarrow$ relevant degrees of freedom at **low energies**
- They are not exactly massless

  $\hookrightarrow$ chiral symmetry is not exact
- GB are weakly interacting massless

  $\hookrightarrow$ Calls for an effective theory $\Rightarrow$ **Chiral Perturbation Theory**

Spontaneous chiral symmetry breaking

- Chiral symmetry is not "visible" in the spectrum

  ▷ no parity doublets

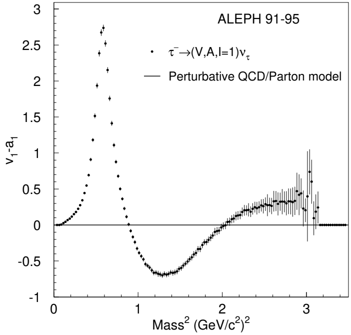
  ▷ $\langle 0|AA|0\rangle \neq \langle 0|VV|0\rangle$

- **Spontaneous breaking** of **chiral symmetry**

  $\hookrightarrow$ π, K, η are the **Goldstone Bosons** from Chiral Symmetry Spontaneous Breakdown

$$\langle 0|\partial^\mu J_\mu^{5\,a}(0)|\pi^b(p)\rangle = \delta^{ab} f_\pi m_\pi^2$$

$$\langle 0|[Q^{A,a}, \bar{q}\gamma_5 t^b q]|0\rangle = -\frac{2}{3}\delta_{ab}\langle 0|\bar{q}q|0\rangle$$



[Schael et al. (ALEPH Collaboration) 2005]

- Chiral QCD lagrangian invariant under $U(1)_L \times U(1)_R$

$$\langle 0|\partial^\mu J_\mu^{5,0}|\eta'(p)\rangle = f_\pi m_{\eta'}^2 = \langle 0|\frac{3g^2}{32\pi^2} F^{a\mu\nu} F^a_{\mu\nu}|\eta'(p)\rangle$$

$\hookrightarrow$ does not vanish in the chiral limit

Chiral Pertubation Theory

- **Most general theory** compatible with **QCD symmetries**
- **Degrees of freedom**: Goldstone Bosons of the spontaneous chiral symmetry breaking
- **Power counting**: based on the scale separation

  $\hookrightarrow$ dimensional counting in p and m_q
- **Breakdown scale**: $\Lambda_{\chi_l} = \dfrac{m_\pi^2}{(4\pi F_\pi)^2} \sim 0.014, \quad \Lambda_{\chi_s} = \dfrac{M_k^2}{(4\pi F_\pi)^2} \sim 0.18$

  $\hookrightarrow$ Non-perturbative effects: resonances, M_σ, M_ρ, $\cdots$
- **Higher energy degrees of freedom**: LECs

	$N_f = 2$		$N_f = 3$		
p^2	F, B	**2**	F_0, B_0	**2**	[Weinberg 66]
p^4	ℓ_i, h_i	**10**	L_i, H_i	**12**	[Gasser, Leutwyler 84, 85]
p^6	c_i	**56**	C_i	**94**	[Bijenens, Colangelo, Ecker 99, 00]

- $\mathcal{O}(p^6)$ LECs: some control from kinematical dependence $\longrightarrow$ dispersive techniques

Goldstone boson interactions: pion-pion and pion-kaon scattering

- What does ChPT say about $\pi\pi$ and πK scattering?

$$A_{\pi\pi}(s,t) = \frac{s - m_\pi^2}{F_\pi^2} + \cdots$$ [Weinberg 66]

$$A_{\pi K}(s,t) = \frac{m_\pi^2 + m_K^2 - s}{2F_\pi^2} + \cdots$$ [Weinberg 1966, Griffith 69]

$\hookrightarrow$ parameter free prediction at LO

- Higher order **corrections**:

	$N_f = 2$		$N_f = 3$		$\pi\pi$	πK	
p^2	F, B	**2**	F_0, B_0	**2**			[Weinberg 66]
p^4	ℓ_i, h_i	**10**	L_i, H_i	**12**	**4**	**7**	[Gasser, Leutwyler 84, 85]
p^6	c_i	**56**	C_i	**94**	**2**	**32**	[Bijenens, Colangelo, Ecker 99, 00]

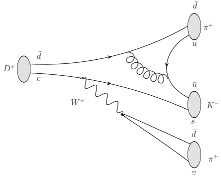
$\triangleright\mathcal{O}(p^6)$ LECs: some control from kinematic dependence $\longrightarrow$ dispersive techniques

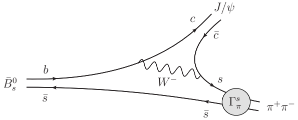
poorly known via quark-mass dependence $\longrightarrow$ lattice [Bijnens, Jemos 11, Bijnens, Ecker 14]

- What is the size of **higher orders** ?

Goldstone boson interactions: pion-pion and pion-kaon scattering

- π's and K's appear as **final state interactions** of “almost every” **hadronic process**





$\hookrightarrow$ **CP-violation** and **New Physics** searches

- What does ChPT say about $\pi\pi$ and πK scattering?

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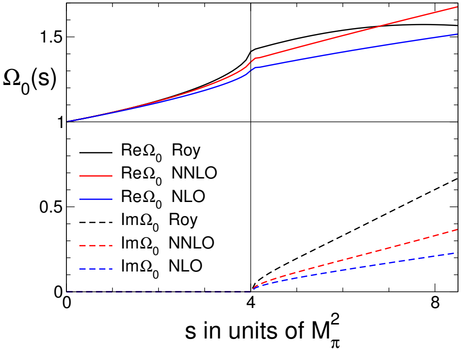
$\hookrightarrow$ parameter free prediction

- What is the size of higher **corrections**?

ChPT vs dispersion theory

- How large are rescattering effects?

  $\hookrightarrow$ look at $F^s_\pi(s) = P(s)\Omega_0(s), \quad \Omega_0(s) = \exp \frac{s}{\pi} \int_{s_\pi}^{\infty} \frac{ds'}{s'} \frac{\delta^0_0(s')}{s'-s-i\epsilon}$



[courtesy of Heiri Leutwyler 2015]

- ChPT converges rapid at subthreshold
- Slow convergence already at threshold

  $\hookrightarrow$ dispersion theory for the energy dependence, ChPT for subraction constants

$\pi\pi$ scattering lengths: ChPT predictions

- Simplest process in two flavors

  $\hookrightarrow$ test chiral dynamics in the light-quark sector

- **LO prediction** for the S-wave $\pi\pi$ scattering lenths

$$a^0 = \frac{7m_\pi^2}{32\pi F_\pi^2} = 0.16 m_\pi^{-1}, \qquad a^2 = \frac{-m_\pi^2}{16\pi F_\pi^2} = -0.045 m_\pi^{-1}.$$ [Weinberg 66]

- **NLO correction**: $a^0 = 0.20 m_\pi^{-1}$, $a^2 = -0.042 m_\pi^{-1}$. [Gasser, Leutwyler 84]

- **NNLO correction**: $a^0 = 0.217 m_\pi^{-1}$, $a^2 = -0.0413 m_\pi^{-1}$. [Bijnens et al. 96]

  $\hookrightarrow$ large corrections for a_0: LO $\xrightarrow{26\%}$ NLO $\xrightarrow{8\%}$ NNLO

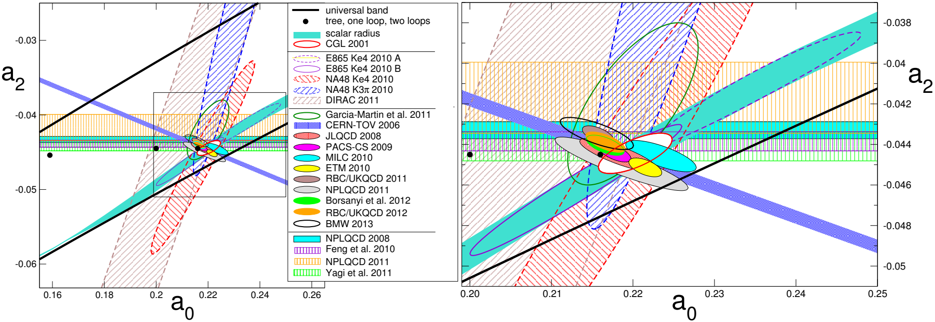
$$a^0 = \frac{7m_\pi^2}{32\pi F_\pi^2}\left(1 - \frac{9m_\pi^2}{32\pi^2 F_\pi^2}\log\left(\frac{m_\pi^2}{\mu^2}\right) + \cdots\right)$$

- large prefactor: slow convergence of the chiral series

  $\hookrightarrow$ strong curvature at threshold due to final state interactions

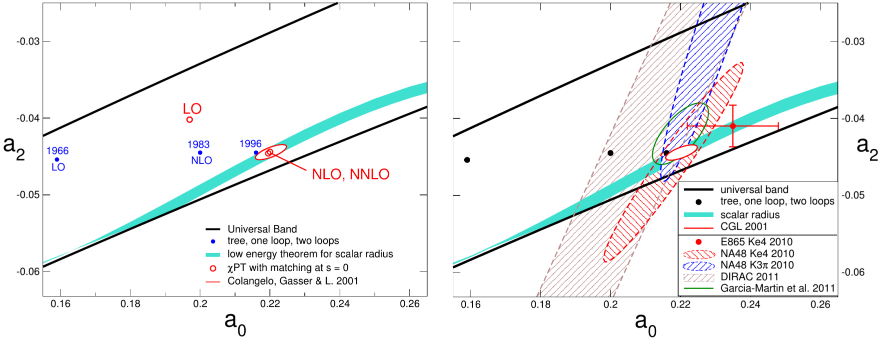
- Experiment:
  - ▷ K_{l4} and K_{l3} decays: $a^0 = (0.222 \pm 0.014)m_\pi^{-1}$, $a^2 = (-0.0432 \pm 0.0097)m_\pi^{-1}$.
    [Batley et al. (NA48/2) 10]
  - ▷ $\pi\pi$ atoms: $|a^0 - a^2| = \left(0.264^{+0.033}_{-0.020}\right) m_\pi^{-1}$. [Adeva et al. (DIRAC) 05]
- Lattice:
  - ▷ a^2 well determined since 90's [Sharpe et al. 92, Gupta et al. 93, Kuramashi et al. 93, Fukugita et al. 95, Aoki et al. 02, Du et al. 04, Chen et al. (NPLQCD) 06, Li et al. (CLQCD) 07, Beane et al. (NPLQCD) 08, Feng et al. (ETMC) 10, Dudek et al. (Had. Spec.) 11, Beane et al. (NPLQCD) 12, Dudek et al. (Had. Spec.) 12, Helmes et al. (ETMC) 15]
  - ▷ a^0 difficult due to disconnected contribution $\longrightarrow$ recent high-precision results
    [Fu 13, Bai et al. (UKQCD & RBC) 15, Briceño et al. (Had. Spec.) 17, Liu et al. (ETMC) 17]
- Dispersive theory:
  - ▷ Roy equations + ChPT: $a^0 = (0.220 \pm 0.005)m_\pi^{-1}$, $a^2 = (-0.0444 \pm 0.0010)m_\pi^{-1}$
    [Ananthanarayan et al. 01, Colangelo et al. 01]
  - ▷ Roy equations + data: $a^0 = (0.220 \pm 0.008)m_\pi^{-1}$, $a^2 = (-0.042 \pm 0.004)m_\pi^{-1}$.
    [Garcia-Martin et al. 11]

$\pi\pi$ scattering lengths



[courtesy of Heiri Leutwyler 2015]

↪ Extremely **successfull test** of QCD



-0.03
-0.04
-0.05
-0.06
a_2
1966
LO
LO
1983
NLO
1996
NLO, NNLO
Universal Band
tree, one loop, two loops
low energy theorem for scalar radius
χPT with matching at s = 0
Colangelo, Gasser & L. 2001
0.16
0.18
0.2
0.22
0.24
0.26
a_0
universal band
tree, one loop, two loops
scalar radius
CGL 2001
E865 Ke4 2010
NA48 Ke4 2010
NA48 K3π 2010
DIRAC 2011
Garcia-Martin et al. 2011

Motivation: Why πK scattering?

- **Low energies:** test chiral dynamics in the strange-quark sector

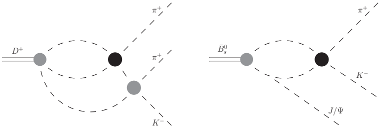
  $\triangleright$ Scattering lengths lowest energy observables

  $\hookrightarrow$ Spontaneous and explicit chiral symmetry breaking

- **Higher energies:** resonances, hadron spectrum

  $\hookrightarrow$ $\kappa(800)$ non-ordinary meson, PDG "needs confirmation"

- **Input for Heavy-meson decays**: CP-violation and New Physics searches



- **Crossed channel $\pi\pi \to \bar{K}K$**: first inelastic contribution to $\pi\pi$ scattering

  $\hookrightarrow$ $\Gamma(f_0(500) \to \bar{K}K)$ nature of the σ meson

  $\hookrightarrow$ Nucleon form factors, $g-2$ ...

Pion-kaon scattering lengths: ChPT

- Simplest scattering process involving strangeness

  $\hookrightarrow$ test chiral dynamics in the strange-quark sector

- Two independent amplitudes: $I_s = \{1/2, 3/2\}$ or $I_\pm = \{+, -\}$,
  $T^{1/2} = T^+ + 2T^-, \quad T^{3/2} = T^+ - T^-.$

- LO prediction: $a^- = \frac{m_\pi m_K}{8\pi(m_\pi + m_K)F_\pi^2} + \mathcal{O}(m_i^4), \qquad a^+ = \mathcal{O}(m_i^4)$ [Weinberg 66]

- NLO: $a^-_{LECs} = \frac{2 m_K m_\pi^3}{\pi(m_\pi + m_K)F_\pi^4} L_5 + \mathcal{O}(m_i^6)$ [Bernard, Kaiser, Meißner 91]

$$a^+_{LECs} = \frac{2 m_K^2 m_\pi^2}{\pi(m_\pi + m_K)F_\pi^4}\left(4(L_1 + L_2 - L_4) + 2L_3 - L_5 + 2(2L_6 + L_8)\right) + \mathcal{O}(m_i^6)$$

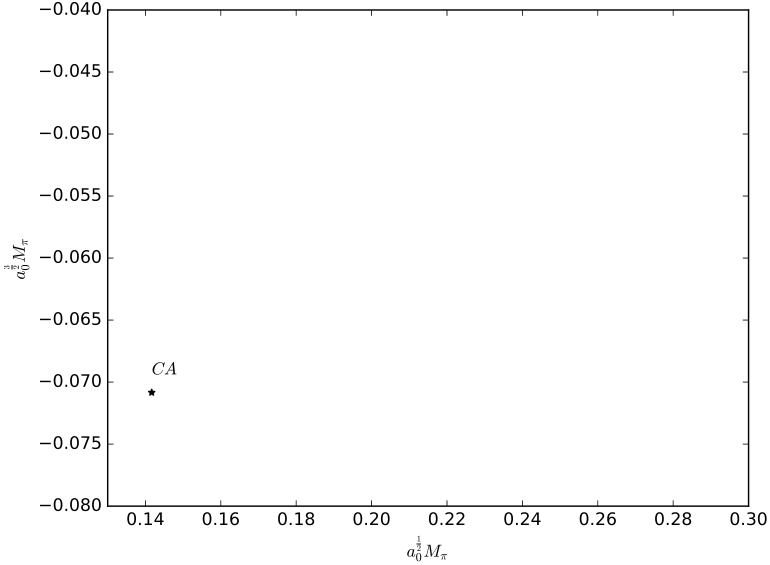
- Size of higher order corrections?

  $\hookrightarrow$ Low Energy Theorem

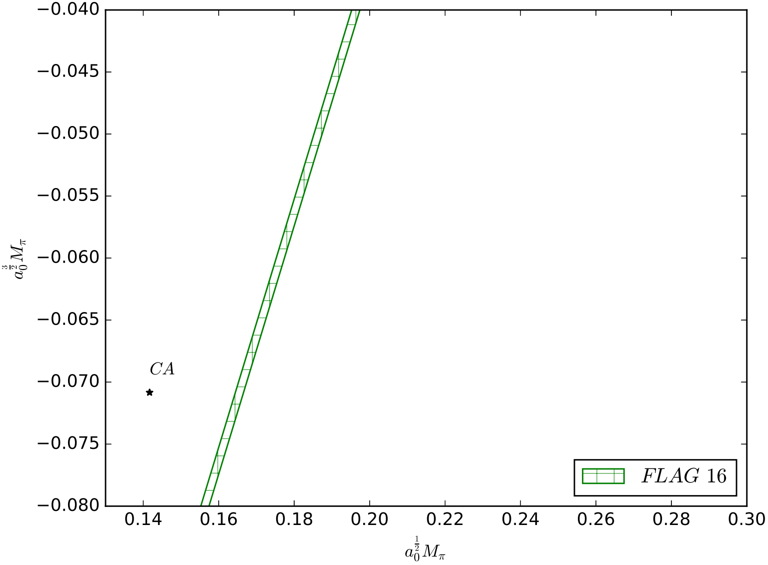
$$a^-\big|_{N^nLO} \propto a^-\big|_{LO}\left(\frac{m_\pi}{4\pi F_\pi}\right)^2\left(\frac{m_K}{4\pi F_\pi}\right)^{2n}, \quad n \geq 2$$ [Weinberg 66]

- NNLO: C_{1-32}, 10 for a^- and 23 for a^+ [Bijnens,Dhonte,Talavera 2004], [Bijnens, Ecker 2014]

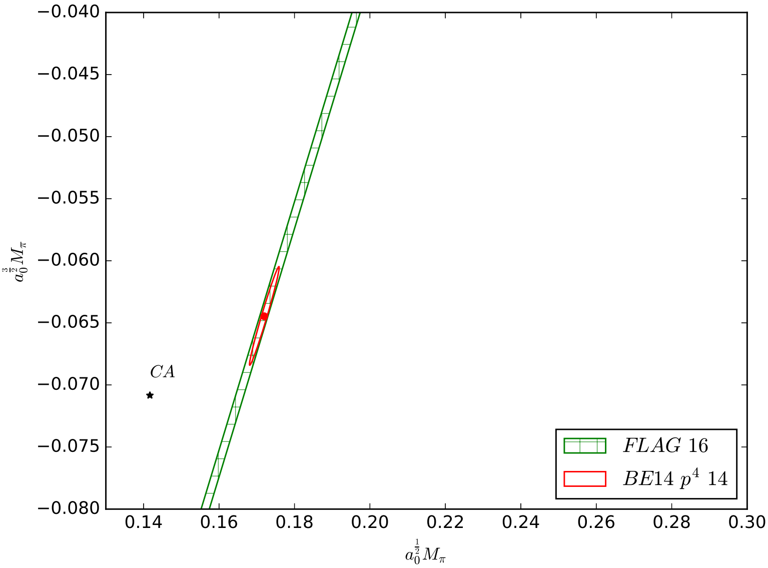
πK scattering lengths

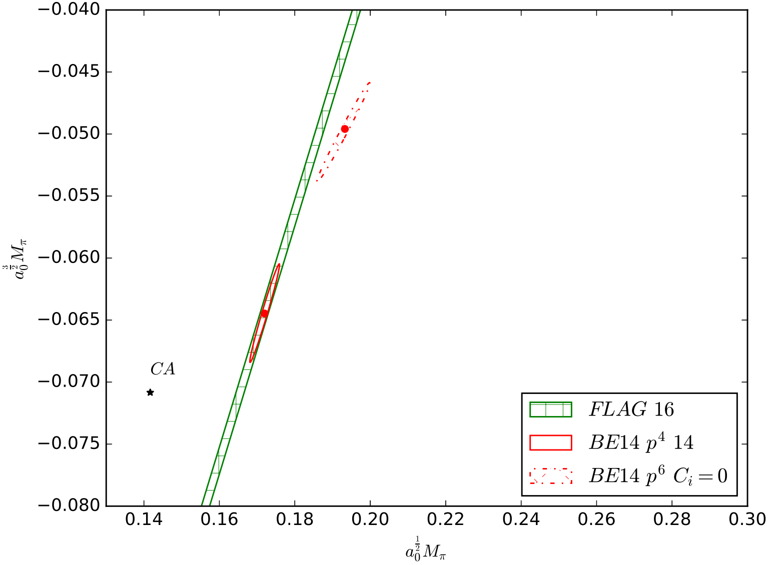


πK scattering lengths



πK scattering lengths





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−0.065
−0.070
−0.075
−0.080
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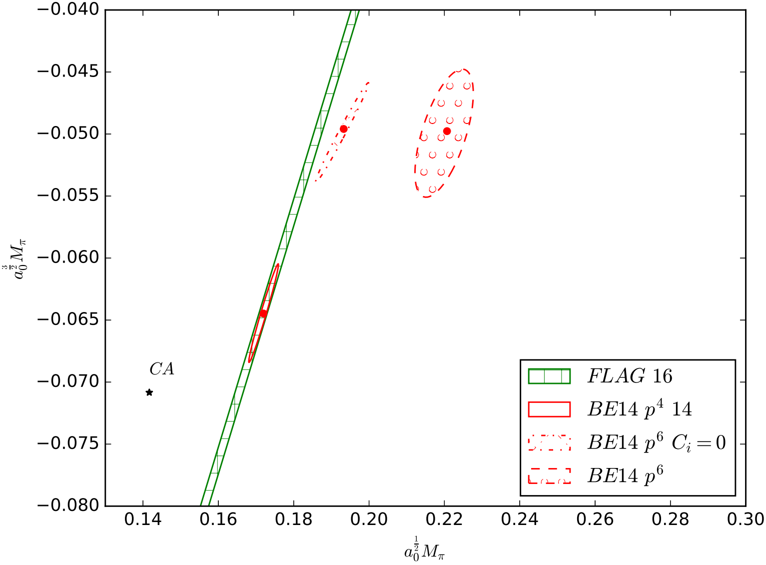
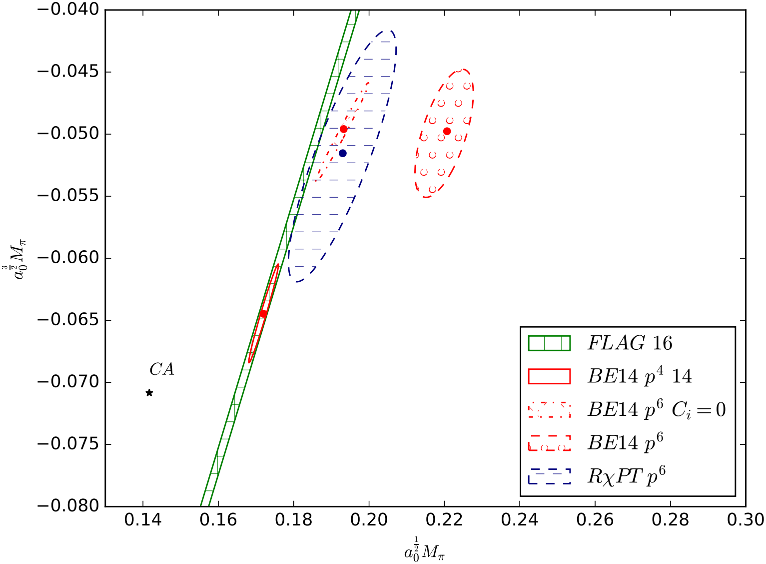
0.16
0.18
0.20
0.22
0.24
0.26
0.28
0.30
$a_0^{\frac{3}{2}} M_\pi$
$a_0^{\frac{1}{2}} M_\pi$
CA
$FLAG\ 16$
$BE14\ p^4\ 14$
$BE14\ p^6\ C_i = 0$

πK scattering lengths

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CA
$FLAG$ 16
$BE14$ p^4 14
$BE14$ p^6 $C_i = 0$
$BE14$ p^6
$R\chi PT$ p^6

- **Experimental values**: DIRAC collaboration

  ▷ lifetime of πK atoms at CERN ⇒ isovector scattering length

$$\Gamma_{1S} \propto \left| T_{\left(\pi^+ K^- \to \pi^0 K^0\right)} \right|^2 \propto |a^-|^2$$

[Deser, Goldberger, Baumann, Thirring 1954]

  ▷ Current result:

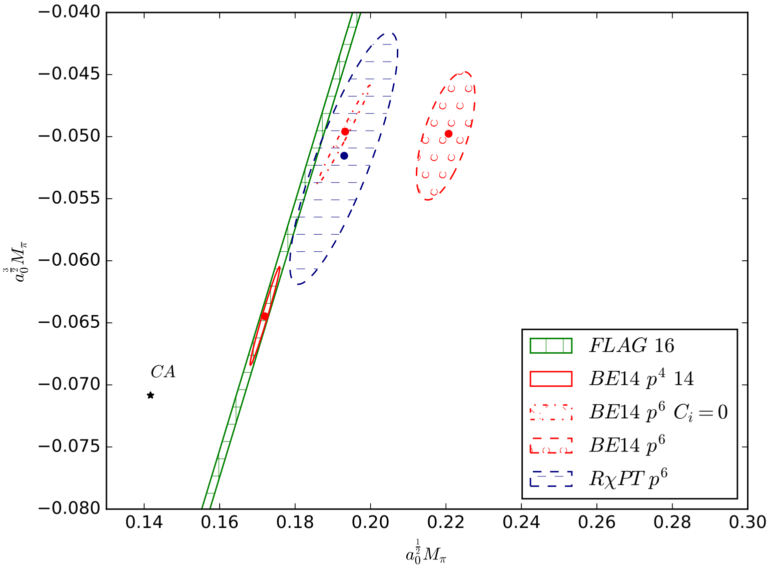
$$a^- = (-0.072^{+0.031}_{-0.020}) m_\pi^{-1}$$

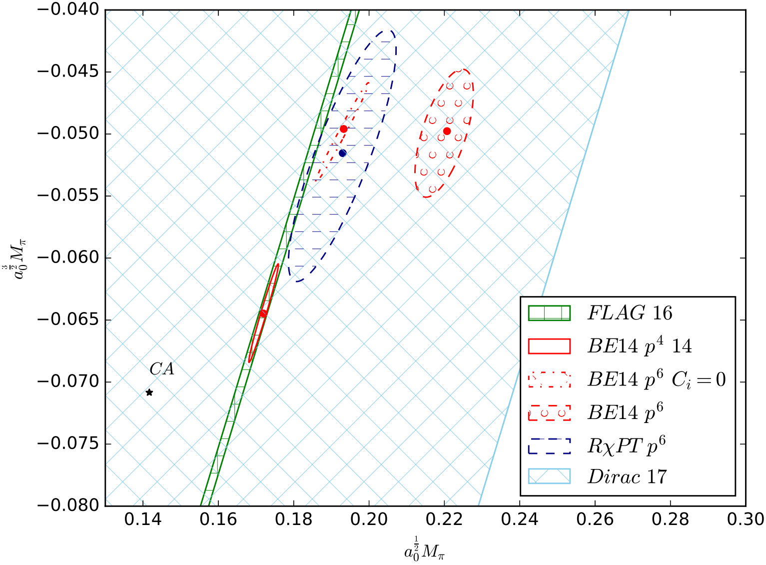
[DIRAC 2017]

  ↪ **huge uncertainties**

  ▷ Room for improvement: near future increase statistics by 10

πK scattering lengths

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$BE14\ p^6$
$R\chi PT\ p^6$
$Dirac\ 17$

Current status

- **Lattice analysis**: unquenched results only

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  ▷ Constraint from semileptonic K_{l3} decays [Flynn, Nieves 2007]

$$a^{1/2} = 0.179(17)(14)m_\pi^{-1}$$

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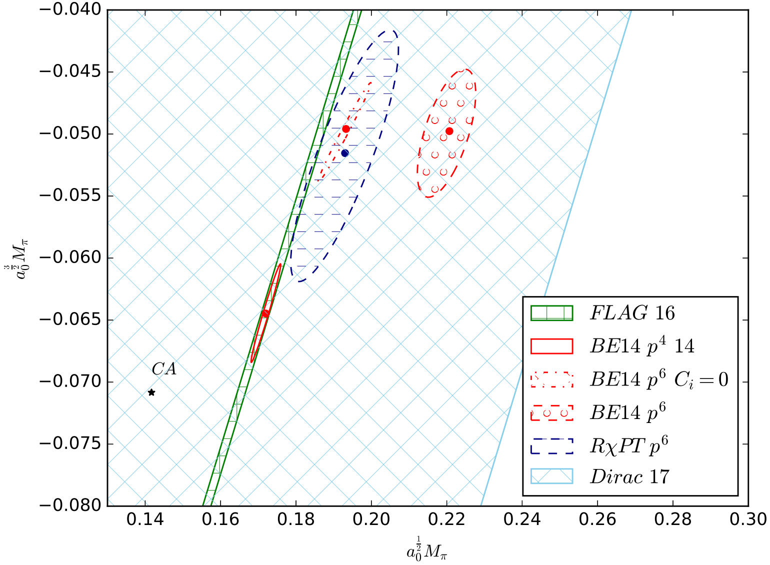

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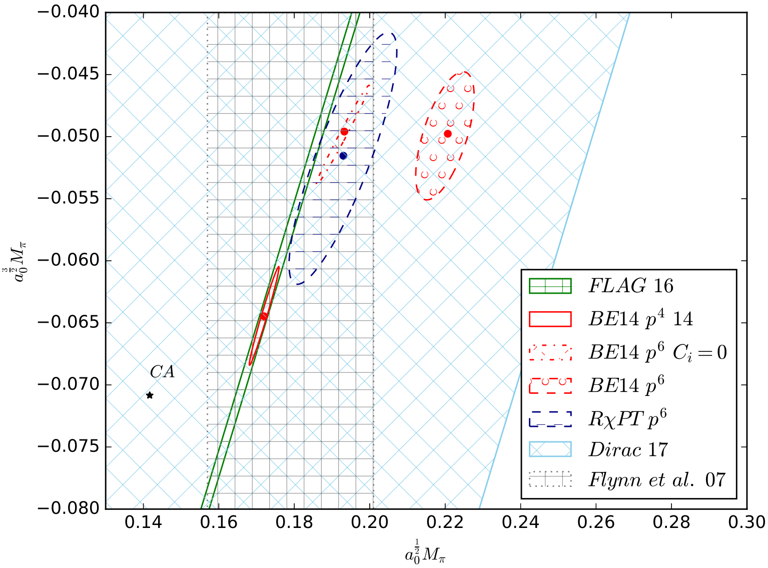

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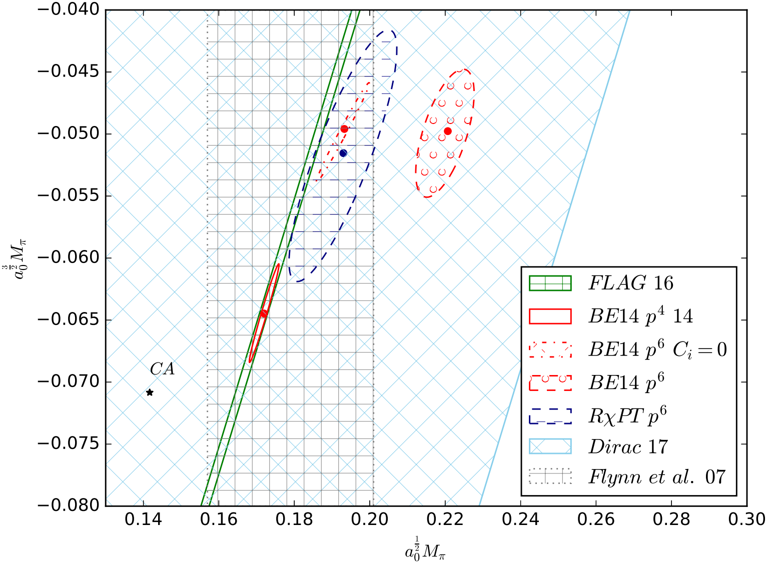

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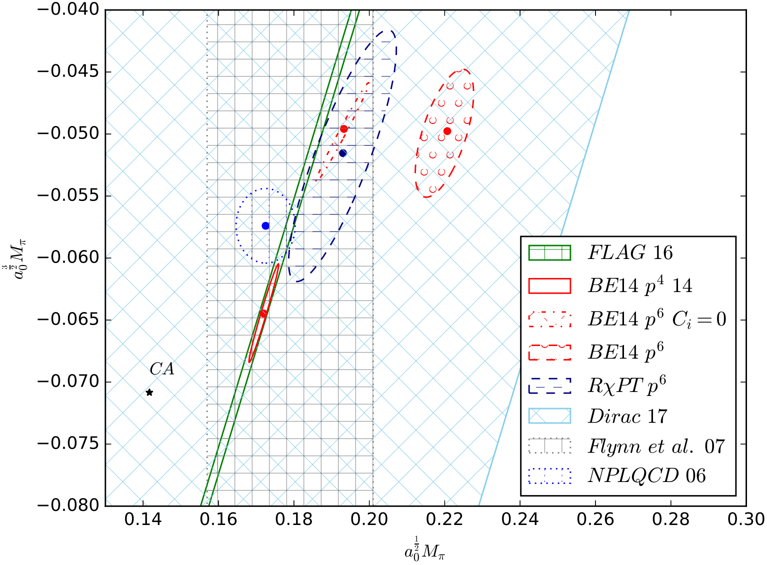

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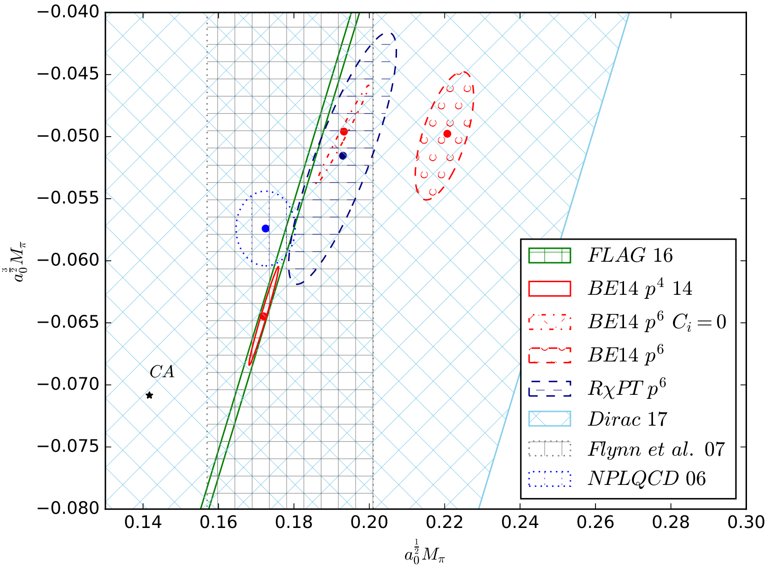

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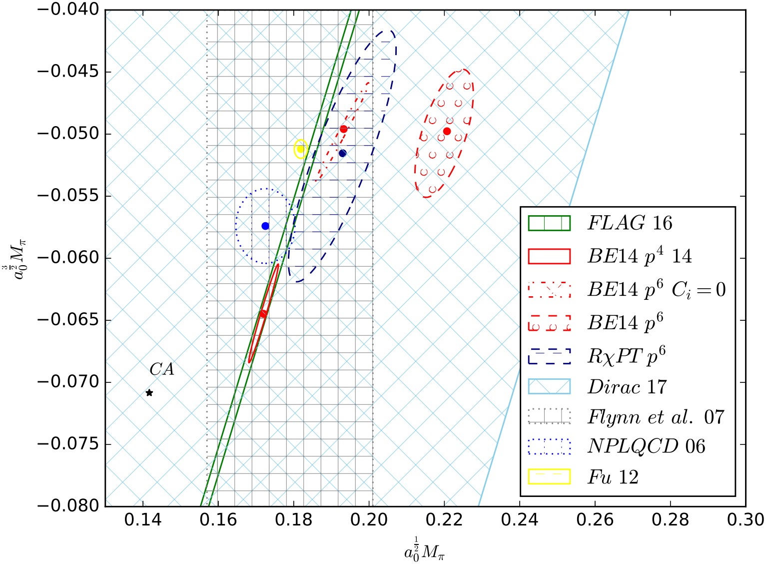

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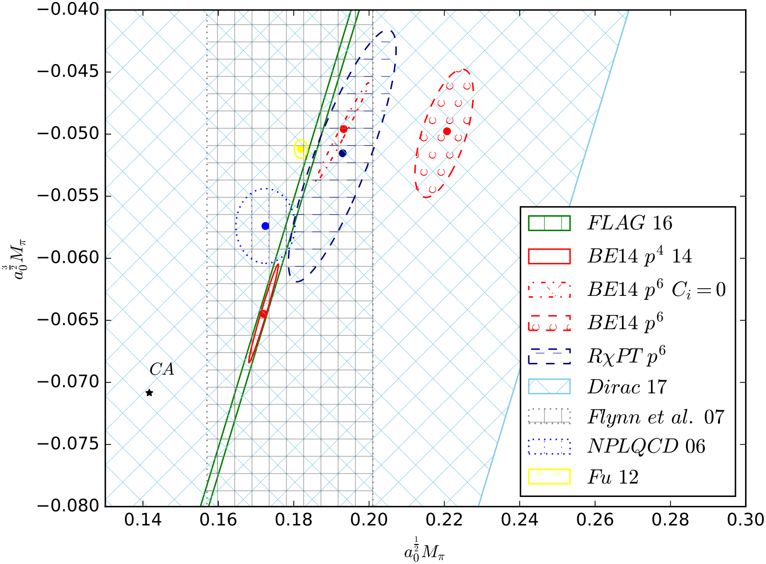
▷ PACs: dynamical $N_f = 2 + 1$ fermions, $m_\pi = 170 - 710$ MeV [PACs-Cs 2014]

$$a^{1/2} = 0.142(14)(27)m_\pi^{-1}, \quad a^{3/2} = -0.047(2)(2)m_\pi^{-1}$$

Current status



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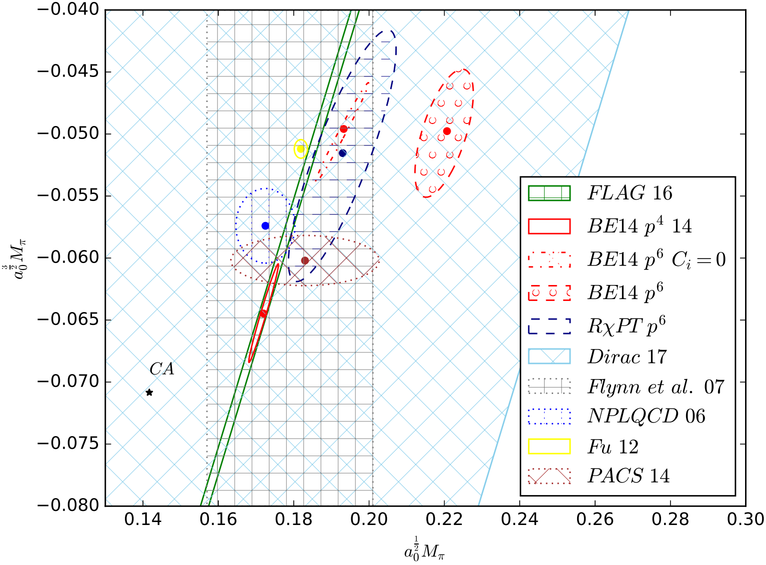

Current status

- **Dispersive determination**: Roy-Steiner equations analysis

  ▷ Most precise results up to date

$$a^{1/2} = 0.224(22)m_\pi^{-1}, \quad a^{3/2} = -0.045(8)m_\pi^{-1}$$

[Büttiker, Descontes-Genon, Moussallam 2003]

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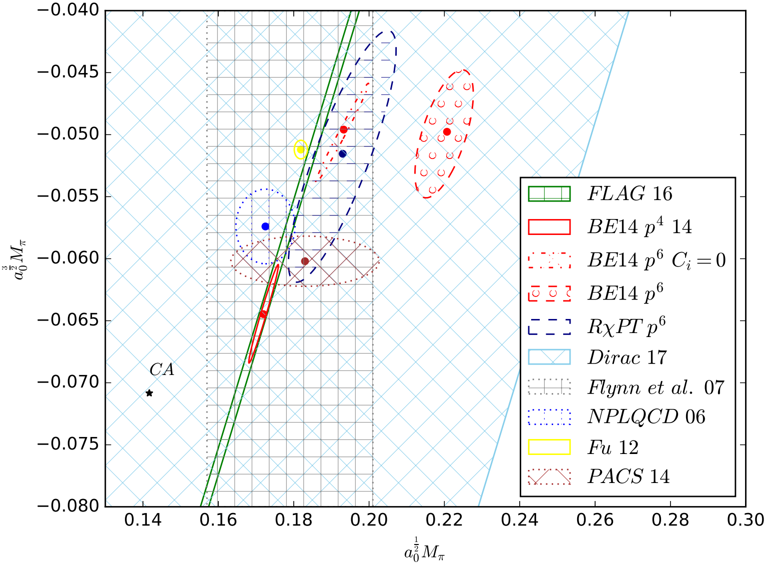

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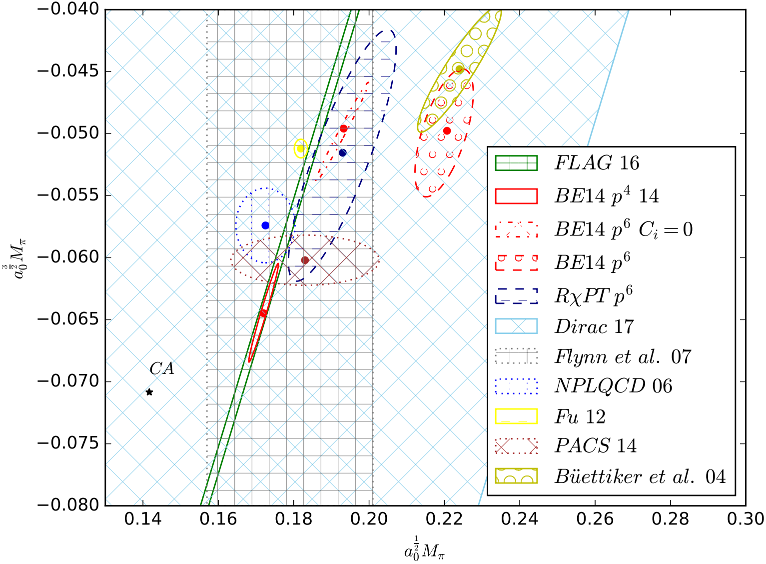

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- **This talk**: where does this discrepancy come from? [Colangelo, Maurizio, JRE, in progress]

- **Dispersive determination**: Roy-Steiner equations analysis

  ▷ Most precise results up to date

$$a^{1/2} = 0.224(22)m_\pi^{-1}, \quad a^{3/2} = -0.045(8)m_\pi^{-1}$$

[Büttiker, Descontes-Genon, Moussallam 2003]

  ↪ More than 3.5σ from $\mathcal{O}(p^4)$ ChPT results

- **This talk**: where does this discrepancy come from? [Colangelo, Maurizio, JRE, in progress]

  ↪ Overview of Roy and Roy-Steiner equation

Dispersion relations in a nutshell

- Effective field theories ⇒ systematically improvable but

  ▷ number of LECs increase rapidly

  ▷ **convergence** problems: low-lying resonances, strong rescattering effects

- Dispersion relations: **analyticity**, **crossing**, **unitarity**

  ▷ analitycity constrains the energy depedence of scattering amplitude

  ▷ crossing symmetry connects different physical regions

  ▷ unitarity constrains imaginary part

- **Roy(-Steiner) eqs.** =Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

  ↪ **model independent** aproach

  ↪ **analytic continuation** to the complex plane ⇒ resonances, unphysical regions

Motivation: Why Roy-Steiner equations?

Roy(-Steiner) eqs. = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries**: analyticity, unitarity, crossing
- **Model independent** $\Rightarrow$ the actual parametrization of the data is irrelevant once it is used in the integral.
- Framework allows for systematic improvements (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with high precision:
  - $\pi\pi$-scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - πK-scattering: [Büttiker et al. (2004)]
  - $\gamma\gamma \to \pi\pi$ scattering: [Hoferichter et al. (2011)]

Solving Roy-equations: flow information

- **Roy-equations** rigorously valid for a finite energy range

  $\Rightarrow$ introduce a matching point s_m

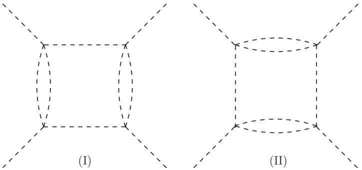
- only partial waves with $J \leq J_{\rm max}$ are solved
- Assume **isospin limit**
- **Input**
  - High-energy region: ${\rm Im} t^I_J(s)$ for $s \geq s_m$ and for all J
  - Higher partial waves: ${\rm Im} t^I_J(s)$ for $J > J_{\rm max}$ and for all s
  - Inelasticities $\eta(s)$
- **Output**
  - Self-consistent solution for $\delta_{IJ}(s)$ for $J \leq J_{\rm max}$ and $s_{\rm th} \leq s \leq s_m$
  - Subtraction constants

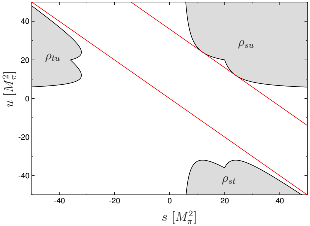
- Convergence for $T^l(s,t)$ guaranteed for $t < 4m^2$
- Where does the partial wave expansion converge?
- Assumption: Mandelstam analyticity [Mandelstam (1958,1959)]

$$T(s,t) = \frac{1}{\pi^2}\iint ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

$\hookrightarrow$ integration on the support of the double spectral densities ρ

- Boundaries of ρ

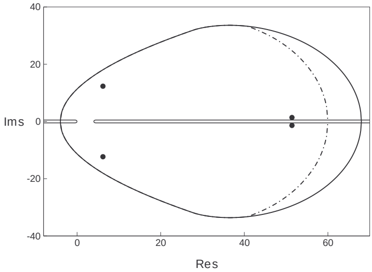




- Lehmann ellipses

  $\hookrightarrow$ largest ellipses, which do not enter any ρ [Lehmann (1958)]

- $t_{IJ}(s)$ known in the Lehmann ellipsis



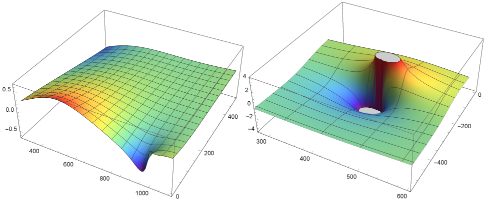
- **Resonances**

  $\hookrightarrow$ poles on unphysical Riemann sheets

- $S^{II}(s - i\epsilon, t) = S^{I}(s + i\epsilon, t)$

  $\hookrightarrow t^{II}_{IJ}(s) = t_{IJ}(s) \cdot (\mathbb{1} + 2i\,\Sigma(s) t_{IJ}(s))^{-1}$

- Elastic scattering: **II** RS is **known** exactly



- Coupled channels

$$t_{IJ}(s) = \begin{pmatrix} t^{(11)}_{IJ}(s) & t^{(12)}_{IJ}(s) \\ t^{(12)}_{IJ}(s) & t^{(22)}_{IJ}(s) \end{pmatrix}$$

$$\Sigma(s) = \begin{pmatrix} \sigma_1(s) & 0 \\ 0 & \sigma_2(s) \end{pmatrix}$$

$\hookrightarrow$ III and IV RS require crossed channels

Roy-equations: existence and uniqueness

- Solution characterized by subtraction constants and high-energy input (a, A)
- **Existence** and **uniqueness** depends on δ_i dynamically at s_m

$$m = \sum_i m_i, \qquad m_i = \begin{cases} \left\lfloor \frac{2\delta_i(s_\mathrm{m})}{\pi} \right\rfloor & \text{if } \delta_i(s_\mathrm{m}) > 0, \\ -1 & \text{if } \delta_i(s_\mathrm{m}) < 0, \end{cases}$$

$\lfloor x \rfloor \Rightarrow$ largest integer $\leq x$.

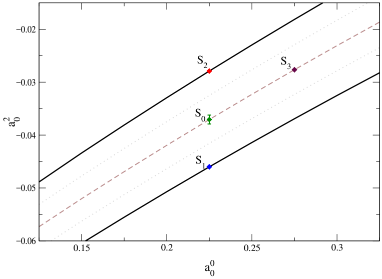
[Gasser, Wanders 1999, Wanders 2000]

- $m = 0$, a unique solution exists for any (a, A)
- $m > 0$, m-parameter family of solutions for any (a, A)
- $m < 0$, only for a specific choice of the input constrained by $|m|$ conditions
- **Physical solution** characterized by *smooth* matching

$\hookrightarrow$ non-cusp condition for each partial wave

Universal band for $\pi\pi$ scattering

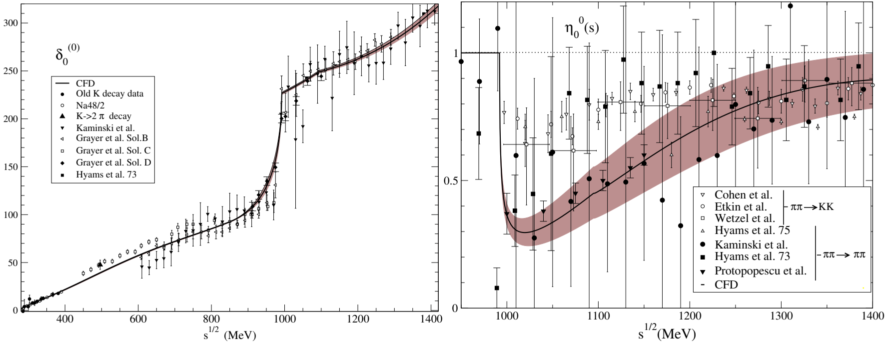
- What happens in $\pi\pi$ scattering? [Ananthanarayan et al. (2001)]
  - ▷ two-subtraction constants: a_0 and a_2
  - ▷ m=0, unique solution for any combination of scattering lengths
  - ▷ non-cusp for $\delta_0^0(s_m)$ condition removes cusps for $\delta_1^1(s_m)$ and $\delta_0^2(s_m)$
  - ↪ $\pi\pi$ scattering case ⇒ universal band in the a_0^0, a_0^2 plane [Ananthanarayan et al. (2001)]



  - ▷ Similar problem in πN scattering [Hoferichter et al. (2015)]
  - ↪ What happens in πK scattering?

Roy-equations: $\pi\pi$ results

- Solution for the $\pi\pi$ S0-wave

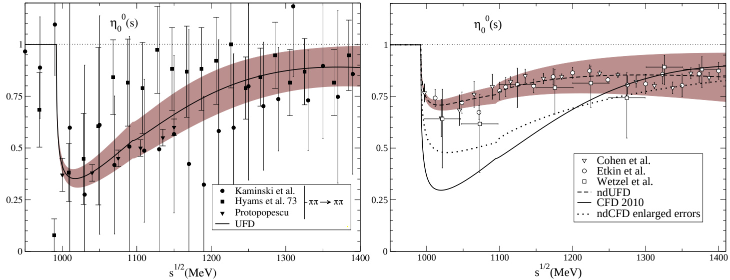


[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain 11]

- The dip vs no-dip ⇒ long-standing controversy [Pennington, Bugg, Zou, Achasov, · · ·]

  ↪ no clear preference for any of the two scenarios in previous works

- Is it possible to satisfy Roy Equations in a non-dip scenario?



- How do the dip vs non-dip solutions satisfy Roy equations

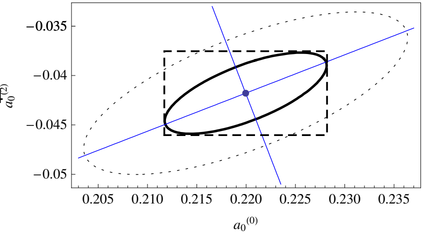
	dip	non-dip	enlarged errors
χ^2-like	1.0	**3.5**	1.7

[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain 11]

↪ the non-dip scenario is **rejected** by DR

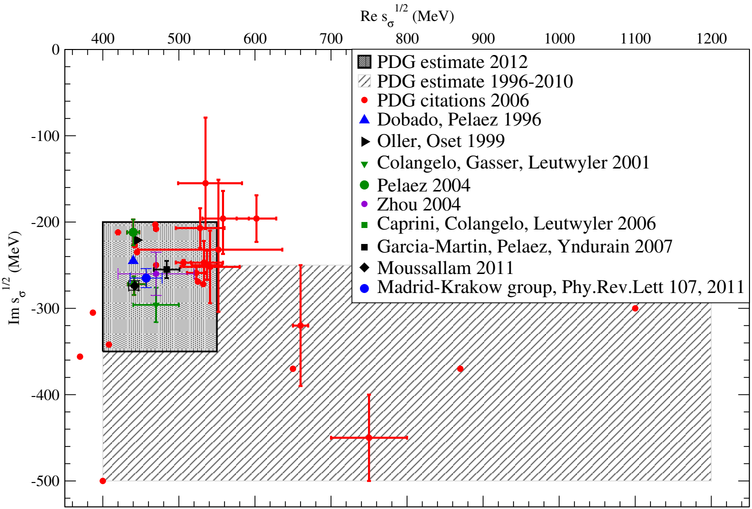
- Threshold expansion: $t_J^{(I)}(s) \simeq M_\pi p^{2J} \left(a_J^{(I)} + b_J^{(I)} p^2 + \cdots \right)$

	CFD	SR
$a_0^{(0)}$	0.221 $\pm$ 0.009	0.220$\pm$ 0.008
$a_0^{(2)}$	-0.043 $\pm$ 0.008	-0.042$\pm$ 0.004
$a_0^{(0)} - a_0^{(2)}$	0.264 $\pm$ 0.009	0.262 $\pm$ 0.009
$2a_0^{(0)} - 5a_0^{(2)}$	0.657 $\pm$ 0.043	0.650 $\pm$ 0.015
$b_0^{(0)}$	0.278 $\pm$ 0.007	0.278 $\pm$ 0.005
$b_0^{(2)}$	-0.080 $\pm$ 0.009	-0.082 $\pm$ 0.004
a_1(x10^3)	38.5 $\pm$ 1.2	38.1 $\pm$ 0.9
b_1(x10^3)	5.07 $\pm$ 0.26	5.37 $\pm$ 0.14
$a_2^{(0)}$(x10^4)	18.8 $\pm$ 0.4	17.8 $\pm$ 0.3
$a_2^{(2)}$(x10^4)	2.8 $\pm$ 1.0	1.85 $\pm$ 0.18
$b_2^{(0)}$(x10^4)	-4.2 $\pm$ 0.3	-3.5 $\pm$ 0.2
$b_2^{(2)}$(x10^4)	-2.8 $\pm$ 0.8	-3.3 $\pm$ 0.1
a_3(x10^5)	5.1 $\pm$ 1.3	5.65 $\pm$ 0.21
b_3(x10^5)	-4.6 $\pm$ 2.5	-4.06 $\pm$ 0.27

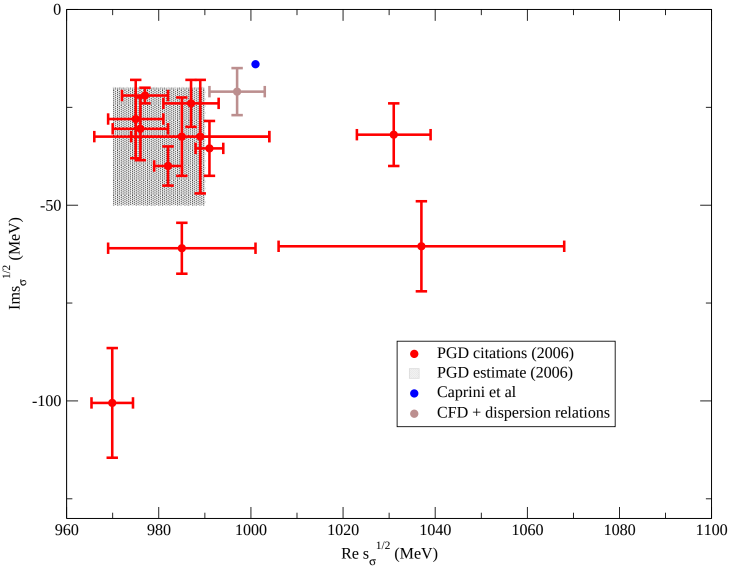


[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain 11]

$f_0(500)$ pole 2012



$f_0(980)$ pole 2010



$f_0(980)$ pole 2012