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A few salient features of dissipative solitons in crystal-like lattices of active units

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ABSTRACT

A few salient soliton-like wave evolutionary features of one- and two-dimensional lattices of interacting active units are provided here. In the latter case, particular attention is given to the crystal-like triangular lattice. On the one hand, the units are coupled with nearest neighbors anharmonic forces (Morse potential). On the other hand the units are endowed with the possibility of an input-output energy balance that permits evolution to a steady state and the appearance of metastable states which on occasion can be quite long lasting. The lifetimes of such metastable states depend on the lattice parameter values and the wave front width. Eventually, all metastable states evolve to steady translational modes especially under influence of noise.

Introduction

With this contribution we wish to honor the memory of our beloved friend and great scientist Vadim S. Anishchenko. Accordingly, in view of his widespread scientific interests [1] we shall be highlighting a few salient features of the evolution of *dissipative* solitons in crystal-like lattice systems where the units behave actively. On the one hand we have that, as established in [2–6], the dissipative soliton concept demands a soliton-bearing evolutionary dynamics augmented with an adequate input-output energy balance. Thus the system is capable of sustaining the soliton propagation, as defined for conservative systems [7–9], as long as the energy balance is maintained at steady conditions. References [2–6] originated in the study of oscillatory Bénard-Maragoni instability [10,11] where predictions and experimental observations were done of Boussinesq-Korteweg-de Vries interfacial solitary wave behavior in one, two and three dimensions [5,10,11]. Let us insist that the concept of *dissipative* soliton extends the classical theory to non-conservative systems where energy (rather than being conserved) is pumped and dissipated in an appropriate balance, thus exciting and, eventually, maintaining past an instability threshold the localized structure (or a periodic nonlinear wave train). Numerous other examples of application of this concept exist in one- and two-dimensional systems [12–17]. Further it is of justice to reference [18], as referenced in [17], for its pioneering introduction of the similar dissipative soliton concept when studying the Nonlinear Schrödinger evolution equation endowed with complex coefficients and augmented with arbitrary growth/wave amplification and wave-amplitude-damping contributions albeit in balance thus allowing steady state conditions.

On the other hand, as hinted above, we shall consider the lattice units endowed with some form of “activity” which generates energy input capable of compensating the eventual dissipation along the lattice whether it would be viscosity or some other form of wave damping. Worth recalling is that the concept of *dissipative* soliton is also the natural extension of maintained *dissipative* linear waves with an underlying harmonic oscillator as a dynamical system with balanced viscous friction and energy pumping to maintain its otherwise free vibrations. To our knowledge the earliest scientist to propose this concept was Lord Rayleigh. Back in 1883, he suggested augmenting the harmonic oscillator equation with a “driving” term as an *active* friction force. Anecdotal, (disregarding time scales) he argued about the vibrations of a violin string being fed by the action of the bow feed-backed by the noisy energy stored in the soundboard which might compensate for an

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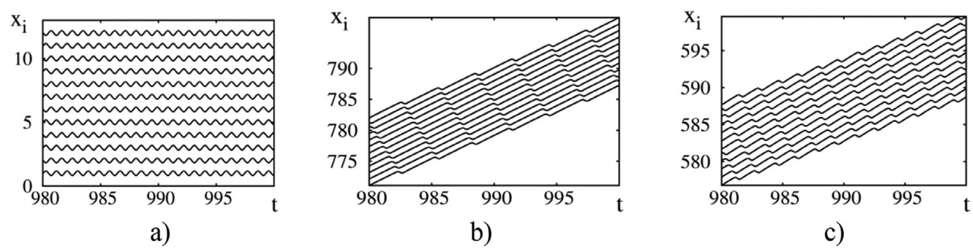

Fig. 1. According to the main text, here appear trajectories of N=12 active units in an one-dimensional lattice with excited (a) optical mode *(6,6)*, (b) one-soliton mode *(11,1)*, (c) two-soliton mode *(10,2)*.

eventual air friction. Three decades later, Van der Pol proposed a genuine nonlinear dynamics for modeling (limit cycle) sustained oscillations in electronic devices. Noteworthy also is that underlying most models used to describe axonal action potential transmission, excitability and other features in neuro-dynamics, are either the Rayleigh or the Van der Pol modeling. On occasion such an *active* force is called *negative* friction if we denote by *positive* friction, a standard, passive, friction. Such a *negative* friction pumps rather than dissipates energy [14]. There is a general setting, of particular interest to problems in bioenergetics, where we could place in adequate context Lord Rayleigh's *active* (friction) force [19,20]. Such is the case where the units are capable of extracting energy from a heat bath (in more general terms from the environment) with energy flux into the units' internal depots. The latter could be assumed to have internal dissipation that in the simplest case can be taken proportional to the instantaneous value of the energy with a constant rate of energy loss. Then the units can be assumed to be capable of transforming the stored (internal) energy into motion (kinetic energy). Yet another dissipative soliton-like wave case of interest in biology, bio-chemistry and neurobiology is discussed in [21]. The authors refer to nonlinear waves in an excitable medium where dissipation comes from a diffusion process as output adequately balancing the input provided by the excitability process.

The evolution of dissipative solitons in a crystal-like lattice system with active units has been thoroughly studied in one-dimensional (1D) and two-dimensional (2D) systems. It has been shown by various authors that there exist a number of different models of active systems with different mechanisms of energy conversion and particles interaction (see, e.g., [1]) exhibiting novel dynamical features which are not observed in conservative systems.

One-dimensional lattices

The research on dynamics of nonlinear lattices grew based upon a seminal work done at Los Alamos National laboratory (USA) by E. Fermi, J. R. Pasta and S. Ulam with the help of computer expert M. Tsingou referenced now in most nonlinear science textbooks [22]. It received high momentum with the work of Zabusky and Kruskal [7]. However, studies of the dynamics of lattices with *active* units started to progress much later. Some of the works along this direction, hence for non-conservative, non-integrable systems, were done building upon the integrable and conservative Toda lattice [8] (or its Morse potential approximation [23] augmented with *dissipative* otherwise denoted *negative* friction terms [13,24–28]. For instance, the study of a typical one-dimensional lattice of point masses starts with the dynamics of an isolated particle described by the equation

$$m\ddot{x} - \gamma_0\left(1 - \frac{\dot{x}^2}{v_0^2}\right)\dot{x} = 0 \tag{1}$$

where m is the mass of the particle, γ_0 is the negative viscous friction coefficient and v_0 is the particle steady-state velocity. Particles form an equilibrium lattice if they are connected via potential forces, in particular, corresponding to the Morse potential [23]

$$U(r) = D\left(e^{-2b(r-\sigma)} - 2e^{-b(r-\sigma)}\right) \tag{2}$$

where D is the potential well depth ($U=-D$ at $r=\sigma$), b is the potential stiffness, r is the distance between units and σ is their equilibrium distance. In view of its expression (2), like the Toda potential, is dominated by an exponential repulsive part in the limit of $r<<\sigma$. However, after its minimum, it tends asymptotically to 0 as $r\to\infty$ thus avoiding the unphysical behavior of the Toda potential. For universality in the description, dimensionless parameters can be used by rescaling quantities noting that $\omega_M = \sqrt{2Db^2/m}$ is the natural frequency of small oscillations of a particle around the minimum of the Morse potential. Then one can rescale the time as $\tau=\omega_M t$, the velocity as $v=v_0 b/\omega_M$ and $\mu=\gamma_0/m\omega_M$. To label the units, rather than using x_n one can better use $q_n=b(x_n - n\sigma)$ which accounts for the dimensionless deviation of the nth particle from its equilibrium position at $x_{n0}=n\sigma$. Overdots in equations denote time derivatives with respect to the corresponding time. Periodic boundary conditions are used for computer simulations and for illustration of results use has been done of two kinds of initial conditions. First, we apply a specified distribution of particle velocities $v_n(t=0) = v_{n0}$ at $q_n(t=0) = 0$ to excite one of the steady-state modes. Then values of velocities $v_n(t=0)$ are chosen randomly with a given statistics simulating stochastic initial conditions. Without loss of generality, it is assumed $v_0=1$ in the simulations. Then the equations are numerically integrated using a Runge–Kutta fourth-order method with appropriate control of the accuracy of calculations.

The steady-state modes (attractors) of the one-dimensional lattice, with periodic boundary conditions, look like cnoidal waves (wave trains) with a uniform spatial distribution of maxima of velocity and density of particles. When peaks of the density distribution are significantly below the distance between them, excited modes may be considered as an ordered ensemble of dissipative (discrete) solitons. Note that different modes have different average velocities and different number of maxima of the particle density along the lattice. The total number of modes in a lattice with N particles is $N+1$: two non-oscillatory modes corresponding to motion of the lattice as a whole in each direction and $N-1$ oscillatory modes. If N is even, a so-called "optical" mode with opposite signs of velocity of adjacent particles is included. Each of $N+1$ modes can be excited selectively by appropriate choice of initial conditions [13,25,26]. We denote the modes by *(k, l)* where $k+l=N$ with $k(l)$ denoting the number of particles spending more time with positive (negative) velocity. For example, for $N=12$ one may find a so-called optical mode when six particles move in opposite direction to the other six, hence denoted as *(6, 6)* mode (Fig. 1a). A mode *(11,1)* has one particle moving, say, along the negative direction ("to the left") while the others move along the positive ("to the right") direction (Fig. 1b). The average velocity of the ensemble equals zero in the first case and unity in the second case $(5/6)v_0$ (because 11 particles have velocity v_0, but one has $-v_0$). Any other

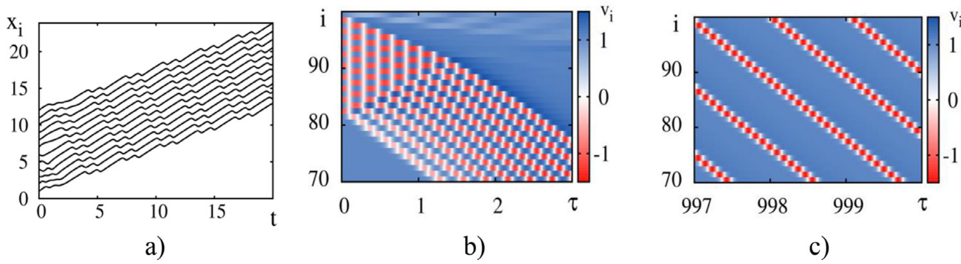

Fig. 2. Trajectories of active particles in an one-dimensional lattice with excited a) cluster of dissipative solitons (two solitons moving close to each other), b) fragment of the optical mode evolving fast to a cluster of 9 solitons, c) multi (9)-soliton mode.

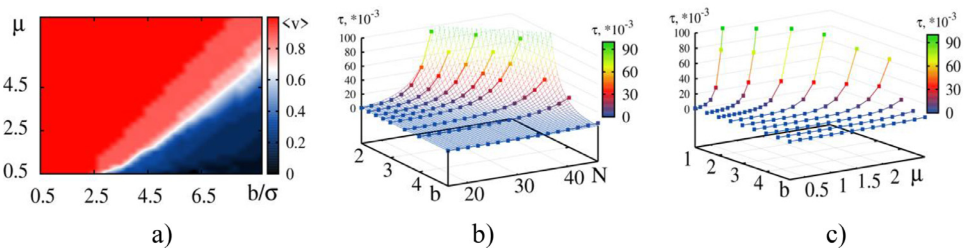

Fig. 3. Regions of realization of modes with different average velocity of units in the parametric plane ($b\sigma$,μ) a); dependence of transient time on the potential stiffness, $b\sigma$, in one-dimensional lattices with different number of particles N for $\mu = 1$, b) and for varying values of μ with $N = 20$.

ratio of particles moving to the right and to the left looks like cnoidal wave (Fig. 1c).

It can also occur that for a random initial particle velocity, a state with nonuniform distribution of particles' density maxima arises (Fig. 2a). The state is similar to the distribution of particles in the multisoliton mode of a conservative system [8]. But it does not correspond to a steady-state. It is a *metastable* state with a long-term transition from its initial state to a steady-state mode. To characterize such a state we introduce a perturbation in large enough (*12* and more particles) equilibrium lattice by an initial velocity distribution and we investigate how the mechanisms of dissipative solitons formation depends on the velocity distribution. Computer simulations show that several unevenly distributed dissipative solitons can be excited (Fig. 2a). They behave as a lattice of charged particles of the same sign and form a steady-state with uniform distribution of density peaks but the time of the transition may be quite long. This means that the distribution shown in (Fig. 2a) evolves to the one shown in (Fig. 1c). Thus, dissipative solitons are excited united in clusters in most of the cases. If we set the displacements q_n and velocities v_n of all particles to zero except the velocity of a single particle (e.g., $v_{10}=1$) at the initial time in a lattice with high stiffness ($b=3$) and low friction ($\mu=1$) one can observe the rather fast formation of a kind of localized structure similar to a fragment of the "optical mode" (Fig. 2b). Only then it evolves to a steady-state mode with an equilibrium distribution of density maxima again after a long time via a stage of a solitonic cluster (Fig. 2c).

In general, the number of solitons, *k*, arising from an initial perturbation depends on the potential stiffness *b* and on the friction coefficient μ. The number k can be identified in numerical simulations because the average velocity of the lattice depends on *k* as $<v>\approx\pm\ [(N-2k)v_0]/N$. The dependence of the average velocity on parameters is presented in Fig. 3a. It is not difficult to see that the average velocity is close to unity for a weak potential, that is for low enough values of *b* ($b\sigma\ <\ 3$). However, as *b* increases, that is, the potential becomes more "rigid" a higher number of solitons is excited. The average velocity depends on the parameter μ and grows as μ increases for $b\sigma\ >\ 3$. In such a manner, for random initial conditions, solitons are also excited in clusters, because for such conditions "pushes" can exist, which "centers" are of cluster formation. To estimate how many clusters of different length, *K*, are excited in average, a probability distribution *P(K)* of *K*-soliton clusters' excitation in lattices of different number *N* of particles has been calculated. Results show that the most frequently observed clusters in a system of $N=24$ particles, with typical values of the parameters $\mu=1$ and $b=3$, have four or five solitons. Increasing the potential stiffness *b* leads to the growth of typical cluster size. The size of clusters also depends on nonlinear friction. The most probable cluster size decreases as the coefficient μ increases. It is a consequence of the fact that large amount of friction retards the change of particle velocity direction. However, the most probable event in a lattice starting from random initial conditions is the excitation of an "optical mode" fragment. But it begins to deform almost immediately and rapidly evolves to a solitonic cluster (Fig. 2c) and then after a long transient process the cluster disintegrates into a set of isolated solitons. It appears that a fragment of the optical mode in a long enough lattice is an unstable state with a short lifetime. Therefore, it can be assumed that such states give a small contribution to macroscopic characteristics of the lattice and they can be neglected. The resulting soliton cluster is unstable and the scenario of its transformation is similar to the scenario of the optical mode fragment destruction. However, in this case the characteristic times are much longer. Thus, states with nonuniform distribution of maxima of the particle density in the lattice can be excited. Such states are metastable with long lifetimes as compared with the period of the Morse oscillations (more, say, than a hundred periods). The lifetime of the states grows exponentially with the number of particles in the lattice (Fig. 3b), and the increase of its rigidity and friction coefficient (Fig. 3c). For instance, an initially perturbed particle in the lattice with sufficiently stiff coupling can excite several dissipative solitons in its neighborhood, thus leading to forming a cluster just after a short transient period. Clusters eventually evolve to steady-state modes after a very long time.

Crystal-like triangular lattice of active units

Let us now focus attention on a crystal-like triangular lattice with only nearest-neighbor adequate anharmonic interaction between units. The latter are endowed with a reasonable form of activity otherwise behaving as point particles. For computational

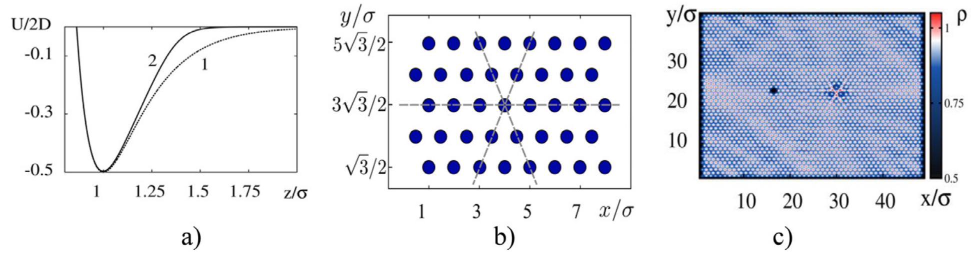

Fig. 4. Crystal-like triangular lattice. a): profiles of the standart (line 1) and modified (line 2) Morse potentials with parameter values $b\sigma = 5$, $d = 1.35\sigma$ and $\nu = 0.035\sigma$. b): schematic representation of a lattice with triangular symmetry; lines show the crystallographic axes. c): a steady state of a lattice with two defects: a vacancy at (17; 23) and interstitial particle at (30; 23).

purposes the number of units is limited to 10^3 and the lattice extension is subject to periodic boundary conditions (b. c.).

As noted above, stationary modes (attractors) in one-dimensional lattices with periodic boundary conditions are cnoidal wave trains with uniform albeit discrete spatial distribution of density maxima. When the width of density peaks is significantly narrower than the distance between them, a mode may be considered as an ordered set of dissipative discrete solitons [9]. It appears that the processes of establishing stationary modes can be very long.

For simplicity, the *activity* of the units is limited to experience *negative* friction with the following nonlinear velocity-dependent coefficient

$$\gamma\left(\vec{v}_i\right) = \tilde{\mu}\left(1 - \frac{\vec{v}_i}{\tilde{v}_0^2}\right), \tag{3}$$

where μ is the linear coefficient of negative friction, $\vec{v}_i$ is velocity of particle i and $\tilde{v}_0$ is the absolute value of stationary velocity. The velocity of each particle tends to reach the velocity $|v_i| = \tilde{v}_0$. Note that the nonlinear Rayleigh friction (3) is not sensitive to the velocity direction but only to its absolute value.

The interaction between particles is determined by the modified Morse potential [29] (Fig. 4a)

$$U(z) = D\left(e^{-2b(z-\sigma)} - 2e^{-b(z-\sigma)}\right)\cdot\frac{1}{1 + e^{\frac{z-d}{2\nu}}}, \tag{4}$$

where, as for (2), z is the distance between particles, D is the potential well depth, b is the stiffness of the potential, σ is the equilibrium distance between particles, and d and ν are the parameters of the modified potential. The parameters d and ν are chosen so that the potential and its derivative are close to zero at distances greater than the cut-off radius R (Fig. 4a). The particles interact only if the distances between them are less than R. The size of the simulation cell and the initial positions of the particles are chosen in such a way that a triangular lattice is indeed formed (Fig. 4b). The parameter values of the potential $b\sigma = 5$, $d = 1.35\sigma$ and $\nu = 0.035\sigma$ correspond to the cut-off radius $R \approx 1.6\sigma$.

The evolution of the ith particle is governed by the following dimensionless equations of motion:

$$\ddot{\vec{q}}_i - \mu\left(1 - \frac{\left|\dot{\vec{q}}_i^2\right|}{v_0^2}\right)\dot{\vec{q}}_i = \sum_{\left|\vec{q}_i^k\right|<R}\frac{q_i^k}{\left|q_i^k\right|}\frac{dU}{dq_i} \tag{5}$$

where use has been made of the dimensionless coordinate $\vec{q}_i = b\,\vec{r}_i$ (where $\vec{r}_i = \{x_i; y_i\}$ corresponds to the dimensional coordinates), $\dot{\vec{q}}_i = \frac{\omega_c}{b}\,\vec{v}_i$ is the dimensionless velocity where here the overdot means derivative with respect to the dimensionless time $\tau = \omega_c t$. Please note that later on, for convenience in the presentation of results, we will change the scale velocity to the value of $\omega_c\sigma$ which is the (linear) "sound velocity" in the corresponding one-dimensional Morse lattice with the same values of parameters. The parameter $\mu = \tilde{\mu}\frac{\omega_c}{b}$ is the dimensionless *linear negative* friction coefficient $v_0 = \tilde{v}_0\frac{b}{\omega_M}$ is the dimensionless stationary velocity value, $|\vec{q}_i^k|$ is the dimensionless distance between particles I and k and $\frac{\vec{q}_i^k}{|\vec{q}_i^k|}$ is the unit vector directed from the ith to the kth particle. Thus, the scale of time is $1/\omega_c$, and the scale of energy is $2D$. Here $\omega_c = \frac{\sqrt{2Db^2}}{m}$ is the frequency of small oscillations in the Morse potential well where m is the mass of a particle. The term $D_n\,\vec{\xi}_i$ corresponds to a white Gaussian noise source $\vec{\xi}_i = \{\xi_i^x; \xi_i^y\}$ with intensityD_n. For visualization we shall identify the lattice units with filled circles centered at the particles' coordinates with each a Gaussian radial density distribution [29].

The dynamics of the ensemble can be different according to the ratio of the characteristic values of kinetic and potential energies of lattice units associated with both their individual and collective motions. Here the dynamics of the "lattice ensemble" is considered, in which the interactions between particles prevail over their individual dynamics. Consequently, the behavior of such a lattice of active particles should reproduce certain features of the dynamics of the conservative lattice. However, the tendency of each particle to reach the stationary velocity leads to new features of lattice dynamics. Additionally, the evolution of excitations in such a lattice depends on the relation between the characteristic time scales of the system, such as the growth rate of linear perturbations, the decay time scale of nonlinear perturbations, and the Morse potential natural-oscillation period, which determines the characteristic scale of the collective dynamics.

For simplicity, we restrict consideration to the behavior of the lattice with a rather fast increase of energy in the linear stage of evolution ($\mu \approx 1.4$) and a rather short stage of nonlinear decay ($v_0 \approx 0.14\omega_c\sigma$). The main stationary state of the active lattice is the translational mode. In this state the entire lattice moves as a whole with the velocity about v_0. The displacements from equilibrium positions are absent like in (Fig. 4b). This state is an attractor of this extended system because most of the metastable states eventually evolve to the translational mode. It restores kinetic and potential energies after a localized external impact, but the direction of the center of mass motion is not stable and can change under external influences.

In addition, there exists a large set of translational modes with defects in the lattice, as presented in Fig. 4c. The existence of such states is possible because the lattice can possess a local minimum of potential energy: the particle which is knocked out of its equilibrium position can get stuck between sites of the lattice. Such defects can arise, for example, as a result of propagation of a crowdion [30]. The crowdion excitation is possible when v_0 is high enough and the stationary value of kinetic energy of the particles is comparable with the potential well depth.

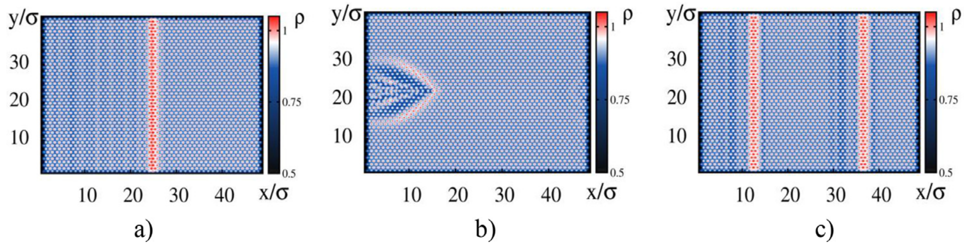

Fig. 5. Crystal-like triangular lattice. Metastable states: plane soliton a), soliton excited in a single rod b), two plane solitons c). All solitons propagate along the x-axis. Parameter values: $b\sigma = 5$, $d = 1.35\sigma$ and $\nu = 0.035\sigma$.

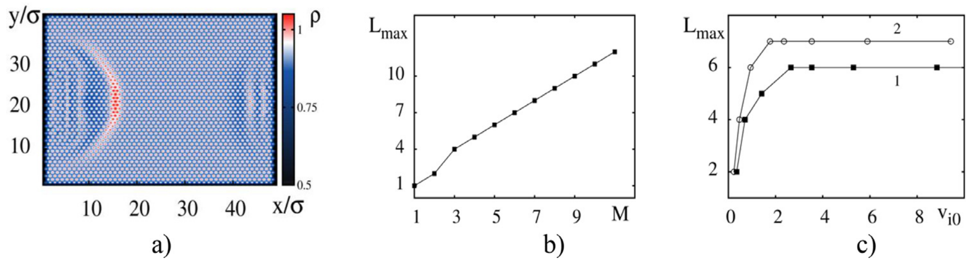

Fig. 6. Crystal-like triangular lattice. Horseshoe-like shape soliton, initially embracing $M = 15$ units (a) and dependence of maximal track lengths on the width of the plane horseshoe-like shape soliton front M in a lattice with $b\sigma = 5$, $\mu = 1.41$, $v_0 = 0.14\omega_c\sigma$ (b); dependence of maximal track lengths on the initial particle velocity for $b\sigma = 4$ (line 1) and $b\sigma = 6$ (line 2), $\mu = 1.41$ (c).

In addition to the stationary modes-attractors different metastable states can be excited by an appropriate choice of initial conditions. Let us insist that metastable states are in fact transient processes from initial conditions to the stationary modes (their lifetimes can be quite long yet they are finite). Thus metastable states may, on occasion, be considered as quasi-stationary modes, which play an important role on the macroscopic characteristics of the system. In real systems, which are under the influence of stochastic forces the disappearance of a metastable state is accompanied by the excitation of another. As a result, the system supports metastability at all times. For this reason, the determination of the features of metastable states and their lifetimes appears as a relevant question.

Results of numerical experiments show that a flat soliton-like wave (denoted below as "flat soliton") is the longest living metastable state. It is the superposition of identical one-dimensional dissipative solitons [31] in adjacent rows of particles propagating along any crystallographic axis. Solitons in each row appear as density maxima localized at least on three lattice units and moving along the row. Solitons in adjacent rows form the wavefront which is oriented perpendicular to the direction of soliton propagation.

The propagation of flat solitons along any crystallographic axis is possible (Fig. 5a). The velocity of solitons in the lattice with parameters $b\sigma = 5$, $\mu = 1.41$, $v_0 = 0.14\omega_c\sigma$ is equal to $|v_{sol}| \approx 1.3$ in units $\omega_c\sigma$ in the coordinate system moving with the simulation cell. The soliton velocity does not depend on initial conditions for given values of $b\sigma$, μ and v_0. In the lattice considered, solitons are localized on an ordered set of lattice units. In the simplest case a single soliton in a lattice with periodically boundary conditions is excited (Fig. 5b). However, in the general case the existence of a set of solitons is possible, as is illustrated in (Fig. 5c).

The soliton-like excitations considered (Fig. 6a) are metastable states. Their lifetime depends on the soliton spatial distribution (as in a one-dimensional lattice [32]) (Fig. 6b), and on the degree of perturbation done to the lattice (Fig. 6c). Perturbation can be caused by thermal fluctuations and, in computer experiments, by round-off errors in the numerical simulation.

The collapse of a soliton can be caused by modulation instability. When particles get a little perturbation in the direction perpendicular to the direction of soliton propagation, as this perturbation grows the front of the soliton deforms and then the soliton disappears. Nevertheless, the plane-wave solitons lifetimes are quite long (of the order of $10^3/\omega_c$) and solitons can pass hundreds of times through the simulation lattice space (recall we use periodic b. c.).

Let us analyze the evolution of dissipative solitons in more details. As noted above, one of the metastable states is the state with nonuniform distribution of solitons in the simulation cell. Solitons move away from each other and finally they reach a steady-state configuration because they repel and tend to a configuration with the maximal distance between them. Correspondingly, the metastable state with nonuniform distribution of solitons tends to evolve to a stationary mode with uniform distribution of solitons (Fig. 5c).

Another type of metastable states is a plane soliton-like wave with the wavefront oriented parallel to one of the crystallographic axes, which propagates perpendicularly to this axis (Fig. 5a). Such solitons have a lower lifetime because of a much greater influence of the transverse modulation instability. Moreover, there exist plane solitons with finite width (Fig. 6a). They are metastable states with linear dependence between the lifetime (or maximum track length) and the width of the front. This dependence is shown in (Fig. 6b). The initial conditions for such solitons are similar to those for the excitation of M-solitons or M-crowdions in conservative lattices [30]. By providing an initial momentum to M particles of adjacent rows along a crystallographic axis the selected units tend to form the wavefront perpendicular to the direction of propagation.

During the propagation of M-solitons the units located at the front edges transmit a pulse not only in the direction of propagation of the soliton, but also in the perpendicular direction. As a result, the front evolves to a horseshoe shape (Fig. 6a) and upon narrowing finally fades away. Correspondingly, the lattice goes to the translational mode. In the limiting case, the front width includes a single unit; this excitation may be called as one-dimensional soliton (Fig. 5b). This type of soliton-like excitation is a localized metastable state, which evolves to the translational mode as in a conservative lattice [31]. The track length of a quasi-one-dimensional soliton depends on the velocity of the initially perturbed particle v_{i0}, as shown in (Fig. 6c). There is a range of initial velocities in which the track length rapidly grows with increasing initial velocity value. Eventually, with further increase in the velocity the track length does not grow and the dependence of the track length as a function of the initial energy reaches saturation. This behavior seems to be a consequence of the decay in the velocity of the faster particles under the action of negative friction.

A chaotic (quasi-chaotic) transient process can be observed when the system starts with a random initial distribution of the particles' velocities. This state is also metastable because incoherent oscillations of particles fade away under the action of the nonlinear friction and the lattice tends to the translational mode. The lifetime of the chaotic regime depends on the number of units in the ensemble and can be hundreds or thousands of dimensionless time units.

4. Influence of noise on the behavior of dissipative solitons in a crystal-like triangular lattice

Also, it is possible to excite several types of metastable states that evolve in time to a stationary mode. Metastable states which transform to the translational mode appear as single plane solitons and states with several uniformly distributed plane solitons (lifetimes of the order of $10^3/\omega_c$), horseshoe *M*-solitons (lifetimes of the order of $10^2/\omega_c$), and quasi-one-dimensional solitons. All these metastable solitons propagate along the crystallographic axes. However, the effects arising from noise exposure («heating») of the lattice have not been previously considered in detail.

One of the metastable states of the system is a plane-wave soliton. Recently [31], a lattice with relatively soft bond $b\sigma = 3\text{–}5$ has been studied. Such potential stiffness values provide displacements of particles in order σ that correspond indeed to behavior of atoms in crystal-like structures. The lifetime of plane-wave solitons is limited because solitons in such lattices are much affected by noise. Lattices with higher rigidity provide slightly small offsets of particles (in order 0.1σ with $b\sigma = 7$). Nevertheless, increasing the potential stiffness increases the lifetime of solitons. For example, the lifetime of solitons in the lattice with $b\sigma = 7$ increases for quite long time (at least no changes in their structure were found for about $10^5 \frac{1}{\omega_c}$ and can be considered long lasting soliton-like waves). However, presumably the lifetime of a soliton can be limited due to the influence of noise, including the numerical one in simulations. This point requires a more detailed study. For this let us introduce an additive noise in Eq. (5) as follows:

$$\ddot{\vec{q}}_i - \mu\left(1 - \frac{\left|\vec{q}_i^{\,2}\right|}{v_0^2}\right)\dot{\vec{q}}_i = \sqrt{2D_n}\,\vec{\xi}_i(t) + \sum_{\left|\vec{q}_i^{\,k}\right| < R} \frac{q_i^k}{\left|q_i^k\right|}\frac{dU}{dq_i}. \tag{6}$$

Here D_n is noise intensity whereas $\vec{\xi}_i(t) = (\xi_i^x(t); \xi_i^y(t))$ is white Gaussian noise sources with two independent components for each element. The Langevin stochastic differential equations (6) are integrated here by means of a Runge-Kutta algorithm adapted for solving stochastic problems [33].

It seems that the limited lifetime of solitons in "soft" lattices ($b\sigma = 3\text{-}5$) is related to the existence of a kind of noise intensity threshold $D_{n,cr}$. Plane solitons rapidly decay when the noise intensity is greater than such a threshold. Noise with intensity lower than threshold has no significant effect on the soliton features. This threshold is quite low in lattices with $b\sigma < 6.5$ so that the numerical noise exceeds the threshold and destroys the soliton as time proceeds. Increasing $b\sigma$ makes the noise threshold increasing. At $b\sigma \approx 6.5$ the threshold becomes greater than the numerical noise and the plane-wave soliton lifetimes increasingly grows. The dependence of the critical value of the noise intensity below which the lifetime of the soliton is not limited is presented in Fig. 7. It is easy to see that resistance to noise increases rapidly when $b\sigma > 6$. The dependence of the critical value of the noise intensity on the lattice rigidity has an exponential character. Fig. 7 shows the dependence of the $\log(D_{n,cr})$ on $b\sigma$ and its linear approximation.

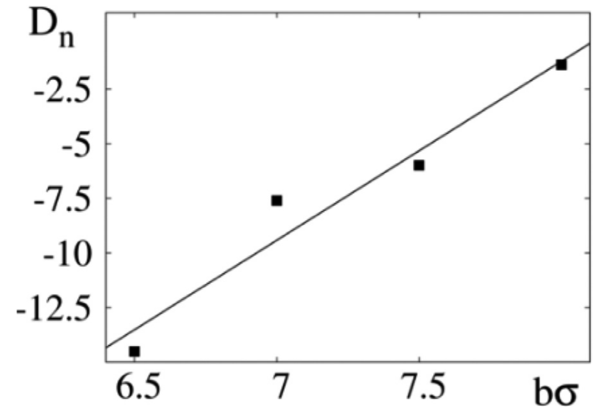

Fig. 7. Dependence of value $log(D_{n,cr})$ on lattice stiffness $b\sigma$ (dots) and its linear approximation $f(x)=8.19x-66.7$ (solid line).

As shown above in a lattice starting with stochastic initial conditions the usual tendency is towards transformation to the translational mode – a state when all particles are uniformly distributed in the triangular lattice and their velocities are unidirected (though the direction of motion of particles in translational mode can be arbitrary). In view of the above, let us start the numerical experiment with initial conditions in the form of translational mode with some noise thus "heating up" the lattice to subsequently allowing the lattice to "cool down" by turning off the noise. Achievement or non-achievement of the balance of dissipation and fluctuations is determined by the ratio between the duration and intensity of noise exposure. After "switching off" the noise a long lasting transient process starts. During this process the nonlinear friction slows down fast particles (and accelerates slow particles). As a result, the average value of particles kinetic energy slowly decreases in time. The value of parameter μ influences the duration of the transient process with higher values of μ favoring a faster transition.

Two situations are possible after a transient time lapse. In the first case, one can observe the usual translational mode with uniform spatial distribution of particles (i.e. triangular lattice with unidirected velocities of all particles as in Fig. 4b). In the second case, the particles velocities are unidirected (see low panels in Fig. 8, $p(E_k)$ shows a relative part of units having kinetic energy E_k, $\Sigma p(E_k)=1$) but the crystal lattice exhibits some heterogeneity (Fig. 8, a-d).

When the potential is hard enough ($b\sigma > 5$) the transient process can be very long. It includes two stages. First, a short stage, spatially rather chaotic oscillations decay quickly, and then different harmonic-like waves propagate along the lattice against the background of the translation mode for very long time (Fig. 9, a-d). They have different sets of kinetic energy values (Fig. 9, e-h) however their kinetic energy distribution eventually evolves to those of Fig. 8, e-d.

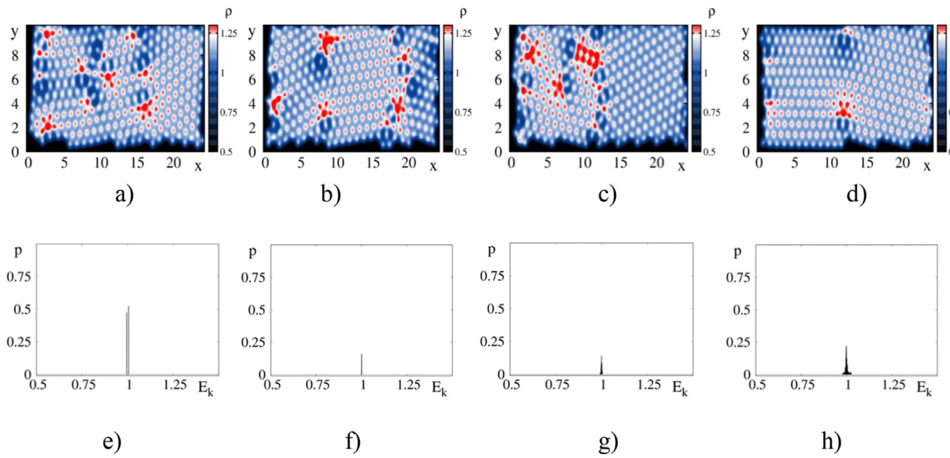

Fig. 8. Crystal-like triangular lattice. Translational mode after noise influence (from left to right - with different realizations of noise). Presented states correspond to local potential minimum; these defects are remaining in the lattice which translates in some direction "as a whole". Top panels: a)-d) spatial distribution of units. Bottom panels: e)-h) kinetic energy distribution *p(E_k)* (shows a relative part of units with kinetic energy E_k). Average translation velocity goes along the *OY* axis (see Fig. 4b). Parameter values: $b\sigma = 3,\ \mu = 7,\ D_n = 0.5,\ \tau_n = 200$.

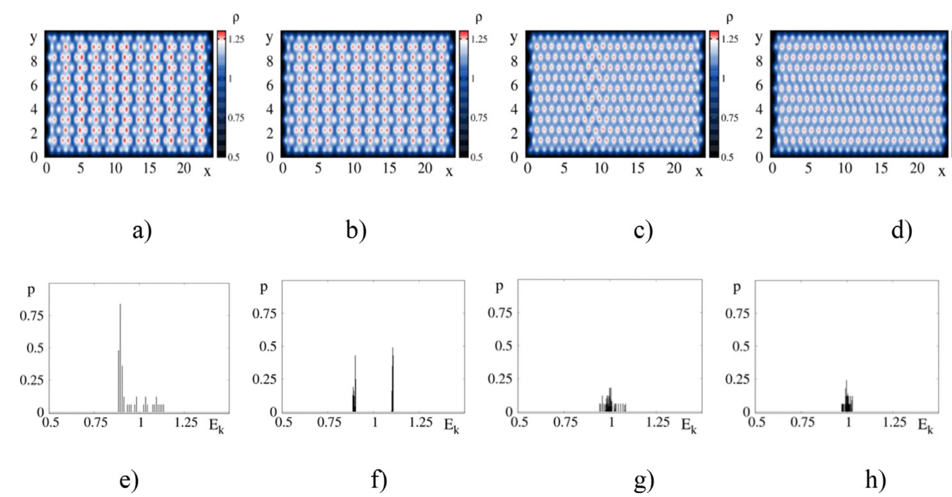

Fig. 9. Crystal-like triangular lattice. Translational mode after noise influence (from left to right – with different realizations of noise). Top panels: a)-d) spatial distribution of particles. Bottom panels: e)-h) kinetic energy distribution. Average translation velocity is along the *OY* axis. Parameter values: $b\sigma = 5,\ \mu = 7,\ D_n = 0.5,\ \tau_n = 350$.

5. Concluding remarks

A few salient soliton-like wave evolutionary features of one- and two-dimensional lattices of anharmonically (Morse potential) interacting active units have been provided here. In one-dimensional lattices, states with nonuniform distribution of maxima of the particle density can be excited. Such states are metastable, with long lifetimes as compared with the period of the Morse oscillations (more than a hundred periods), which grow exponentially with the number of units and the values of the potential stiffness and friction parameters. An initially perturbed unit in a lattice with sufficiently rigid coupling can create several dissipative solitons in its vicinity, which quickly forms a cluster. Eventually, clusters evolve to steady-state modes hence with uniform distribution of density particle peaks.

In a lattice of active units with triangular symmetry, the set of stationary modes (attractors) is represented by translational modes (lattice motion as a whole), both with a uniform distribution of particles in space and with topological defects. As for the one-dimensional case, it is possible to excite several types of metastable states that transform with time into a stationary state. Metastable states can be single plane solitons and states with several uniformly distributed plane solitons, horseshoe *M*-solitons, and quasi-one-dimensional solitons. All these metastable solitons propagate along the crystallographic axes. In the triangular lattice with high stiffness perturbed by noise influence (heating) spatially rather chaotic oscillations appear. Then they decay fast with possible onset of different harmonic-like waves propagating along the lattice against the background of the translation mode. Eventually, after very long time they evolve to the steady-state as a translational mode.

Declaration of Competing Interest

The authors declare that they have no conflicts of interest.

CRediT authorship contribution statement

N.P. Chetverikov: Methodology, Writing - original draft, Writing - review & editing. **W. Ebeling:** Writing - review & editing. **E. del Rio:** Formal analysis, Writing - review & editing. **K.S. Sergeev:** Formal analysis, Software, Visualization. **M.G. Velarde:** Conceptualization, Writing - review & editing, Supervision.

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