How a Mendeleev's study led to a mathematical problem of great interest: The Markov Inequality

Gustavo Adolfo Muñoz Fernández

In 1887, the author of the periodic table of elements, Dmitri Mendeleev, published a study [1] that highlighted the relationship between the specific gravity of an alcohol-water solution and the percentage by weight of alcohol in the solution. By plotting the specific gravity against the percentage of alcohol, he observed that the resulting graph could be well approximated using arcs of quadratic polynomials. Irregularities emerged at the points where the arcs intersected. Mendeleev was uncertain whether these anomalies were genuine or merely the outcome of measurement errors. Figure 1 shows one example of the kind of graphs obtained by Mendeleev.

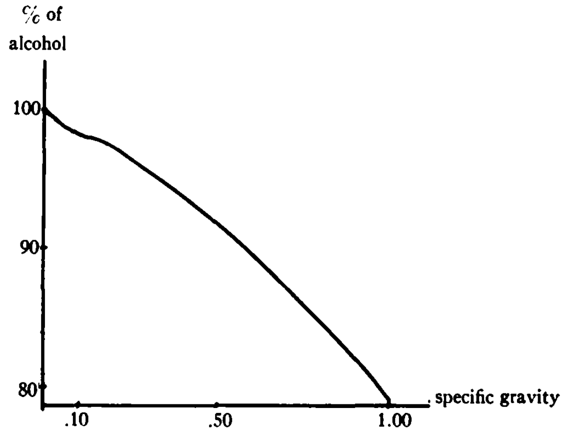

Figure 1: Extracted from [2]

To resolve this, Mendeleev posed the following problem:

Given a quadratic polynomial $P(x)$, what is the maximum value of the derivative $P'(x)$ over a given interval $[a, b]$? The motivation behind this question lies in the following reasoning: If the slope of one arc exceeds at some point the maximum slope of an adjacent arc, it is clear that the two arcs correspond to different quadratic polynomials.

Mendeleev's problem can be normalized as follows: If P is a polynomial of degree at most n and we define $\|P\| = \max\{|P(x)| : x \in [-1, 1]\}$, what is the best (in the sense of smallest) constant M_n in the inequality $\|P'\| \leq M_n\|P\|$ for any polynomial P of degree at most n?

Mendeleev himself found that $M_2 = 4$. The problem was later generalized by A. A. Markov in 1889 [3], proving that $M_n = n^2$. A further generalization was carried out by V. A. Markov, the brother of A. A. Markov, publishing in 1892 (see for instance [4]) that

$$M_n^k = \frac{n^2(n^2 - 1^2)(n^2 - 2^2) \cdots (n^2 - (k-1)^2)}{1 \cdot 3 \cdot 5 \cdots (2k-1)},$$

where M_n^k is the best constant in the inequality $\|P^{(k)}\| \leq M\|P\|$ for all polynomials P of degree at most n, being $P^{(k)}$ its k-th derivative $(1 \leq k \leq n)$. These inequalities, known today as Markov inequalities, have been studied in more general contexts, showing that they also hold, for example, in real Banach spaces [5].

Interestingly, Markov's inequalities have a completely different form when we consider complex polynomials. Indeed, if $\|Q\|_{\mathbb{D}} = \max\{|Q(z)| : |z| \in \mathbb{D}\}$ is the norm of a (complex) polynomial Q over the unit disc $\mathbb{D}$ in $\mathbb{C}$, then

$$\|P^{(k)}\|_{\mathbb{D}} \leq n(n-1)\cdots(n-k+1)\|P\|_{\mathbb{D}}$$

for every polynomial P of degree at most n and $n(n-1)\cdots(n-k+1)$ is optimal. The previous result follows straightforwardly from the well-known Bernstein Inequality for trigonometric polynomials.

References

[1] D. Mendeleev, Investigation of Aqueous Solutions Based on Specific Gravity (Russian), St. Petersburg, 1887.

[2] R. P. Boas Jr. Inequalities for the Derivatives of Polynomials, Mathematics Magazine, 42:4 (1969), 165-174.

[3] A. Markov, On a problem of D. I. Mendeleev (Russian), Zapiski Imp. Akad. Nauk, 62 (1889) 1-24.

[4] R. J. Duffin and A. C. Schaeffer, On some inequalities of S. Bernstein and W. Markoff for derivatives of polynomials, Bull. Amer. Math. Soc., 43 (1938) 289-297.

[5] L. A. Harris. A proof of Markov's theorem for polynomials on Banach spaces. J. Math. Anal. Appl. 368 (2010), no. 1, 374–381.