How a "monster" started a trend

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It came as a general shock when, in 1872, and during a presentation before the Berlin Academy, K. Weierstrass provided the classical example of a function that was continuous everywhere but differentiable nowhere (see Figure 1). The particular example was defined as

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),$$

where $0 < a < 1$, b is any odd integer and $ab > 1 + 3\pi/2$.

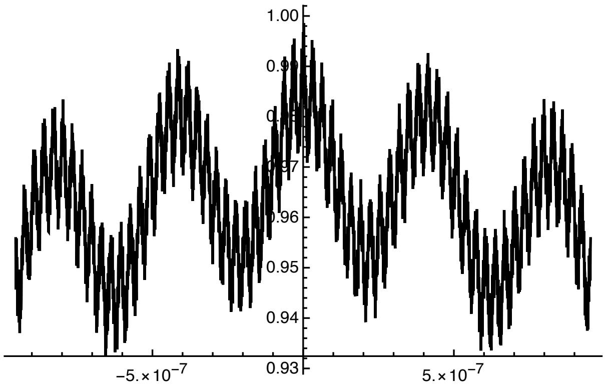

Figure 1: A sketch of Weierstrass' monster for $a = 1/2$ and $b = 13$ on the interval $[-2^{-20}, 2^{-20}]$.

This apparent *shock* was a consequence of the general thought that most mathematicians shared: A continuous function must have derivatives at a significant set of points (even A.M. Ampère attempted to give a theoretical justification for this). Although the first published example is certainly due to Weierstrass, already in 1822 the Czech mathematician B. Bolzano exhibited a continuous nowhere differentiable function. Later, it was learnt that mathematicians such as M. Ch. Cellérier (1830), B. Riemann (1861), or H. Hankel (1870) had already constructed functions of this type. After 1872 many other mathematicians such as M.G. Darboux (1874), G. Peano (1890), D. Hilbert (1891), T. Takagi (1903), W. Sierpiński (1912), G.H. Hardy (1916), or S. Banach (1931) also constructed similar functions.

In the literature, this example is widely known as Weierstrass' monster. This famous example led mathematician Vladimir I. Gurariy (1935-2005) to ask the following question: *How many examples like Weierstrass' are there for us to find?* This, apparent, innocent question has been thoroughly studied since the 1960's. In 1966 Gurariy proved that the set of Weierstrass' monsters on $[0, 1]$ contains (except for $\{0\}$) an infinite dimensional vector space. Afterwards, Fonf, Gurariy and Kadeč (1999) showed that the set of Weierstrass' monster on $[0, 1]$ is spaceable (that is, there is a closed, infinite dimensional subspace $X \subset \mathcal{C}[0, 1]$, the Banach space of continuous functions on $[0, 1]$, every non-zero element of which is nowhere differentiable on $[0, 1]$). This cascade of results led Aron, Gurariy and the author ([2]) to introduce in 2004 the terminology of *lineability* and (as already mentioned above) *spaceability.* In other words... How often (in linear terms) can we expect a "bad" property to happen?

This search for linear subspaces of elements enjoying certain "pathological" property became, in the last decade, a sort of a trend in many different areas of Mathematics and, as a consequence of this, a vast literature on this topic has recently been built, having the American Mathematical Society introducing the new classification 15A03 and 46B87 for this topic. Thus, a property that (apparently) seems very uncommon or even "rare" might end up being "everywhere" (in a linear sense!). This is an example of how a mathematical object that, at first, no one believed that it could even exist ends up being *a*lgebraically generic, that is, linearly "everywhere". Other properties that are lineable are, for instance, being differentiable and nowhere monotone ("differentiable monster") or being a real valued function attaining every real value on each real interval (also called being "everywhere surjective"). The interested reader could take a look at [1–4] in order to find a very large selection of results within this topic.

References

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[3] L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda, *Linear subsets of nonlinear sets in topological vector spaces*, Bull. Amer. Math. Soc. (N.S.) **51** (2014), no. 1, 71–130.

[4] P. H. Enflo, V. I. Gurariy, and J. B. Seoane-Sepúlveda, *Some results and open questions on spaceability in function spaces*, Trans. Amer. Math. Soc. **366** (2014), no. 2, 611–625.