School Choice with Transferable Students' Characteristics*

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Abstract

We generalize the school choice problem by defining schools priorities on (potentially transferable) students characteristics. Taking into account students preferences, school s priorities, and schools available seats, a school choice program selects an (extended) matching that is an assignment of students to schools and a final allocation of characteristics. We define the Student Exchange with Transferable Characteristics (SETC) class of mechanisms. Each SETC produces a stable extended matching that is not Pareto dominated by another stable extended matching. Moreover, any constrained efficient extended matching that Pareto improves upon a stable extended matching can be obtained via an algorithm in the SETC class. When schools' priorities are fully transferable among students, a particular algorithm in the SETC is equivalent to the application of the Top Trade Cycle Algorithm starting from the Student Optimal Stable Matching.

Keywords: School Choice, Transferable Characteristics, Priorities, Stability, Constrained Efficiency.

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1 Introduction

The school choice problem considers mechanisms for assigning students to schools. Students are considered to be strategic agents and the schools seats are objects to be distributed among students. In the late 80's many US states stated to give each student the option to choose the school she wants to attend. Today, many US cities and elsewhere around the world (UK, Sweden, Spain among others) are using school choice programs.

In a school choice program, each student submits a list of preferences of schools to a central placement authority, such as the school district. This central authority then decides as to which student will be placed in each school. This decision is based on students' preferences over schools and on schools' priority rankings that determines who will get a seat in case that a school is over demanded. In the recent years, a bast majority of school authorities have reformed their allocation algorithms and abandoned earlier mechanisms based on Immediate Acceptance of demands, to algorithms based on Gale and Shapley's Deferred Acceptance Algorithm (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005a,b; Pathak, 2016). Under Immediate Acceptance (IA), families demanded their best preferred school, and each school granted a seat to as many demanding students as seats available according to each school priority ranking. In subsequent steps, rejected students demanded a set in the remaining schools that had vacant seats in the same fashion. Deferred Acceptance (DA) (Gale and Shapley, 1962) applies the same logic, but acceptance is only temporary. Rejected students apply in subsequent phases to any school that has not yet rejected them, and may obtain a seat at expense of the students that have not been rejected in the initial phase.

In the canonical school choice problem (Abdulkadiroğlu and Sönmez (2003); Balinski and Sönmez (1999)) priorities are a primitive of the model. However, school districts use several criteria in determining a priority order for a school. For example, in Boston, the first priority for a school is given to the students who are in the same walk zone and who have a sibling attending that school, the second priority is given to those who only have a sibling attending that school, the third priority is given to those who are only in the same walk zone, and, finally, the fourth priority is given to all other students (Abdulkadiroğlu et al. (2005a,b)). Other school districts use a cardinal priorities. in Spain, all school districts use point systems. There are a number of categories. Depending on students' characteristics, each student is awarded a number of points for each category at each school. The more points a student obtains across all categories at a given school, the higher the priority the student has for a seat in that school.[1] When the number of potential students is large, these priority rankings based on categories lead to coarse partitions of the set of students and rely on randomly generated tie-breaking rules to strictly order all possible participant.

In cases of multiple tie-breaking (MTB) the tie-breaking is a source of inefficiency that can be solved by trading this characteristic. We have the example of Amsterdam where the immediate acceptance MTB (IA-MTB) rule was initially adopted for equity reasons. The MTB reduces the chances that over-demanded schools systematically reject a student that has a bad lottery draw. It spreads the pain better Arnosti (2016(@). In 2014 the association of Amsterdam school boards (OSVO) decided to move from IA-MTB with ex-post trade to a system of deferred acceptance with multiple tie breaking (DA-MTB) without ex-post trade. This decision was challenge in court. In 2015 a lawsuit by families who wished to switch assignments (in fact switch the tie break in each school). The courts rejected this claim on the grounds that DA-MTB will not be strategy-proof if a post allocation trade is allowed. Latter, the OSVO backed on their initial decision and moved to a single tie-breaking rule DA- STB.[2] In our setting OSVO could as well have made this characteristic transferable. This is because, in our setting, the exchange of school among two student that has identical characteristics but for a different school specific tie breaking does not generate a fairness violation.That would have allowed the city to maintain the DA-MTB introduced in equity grounds while correcting its possible efficiency consequences and limiting this correction to the effects of the tie-breaking rule while respecting any additional priority source.

Also, priority structures need to be carefully designed to achieve their objectives regardless on how those structures are form. We have an example of this ordinal priority

[1]In Spain students may receive points at a given school for socio-economic characteristics (family income, number of siblings), siblings attending the school, also (non negligible) legacy points if her parents or older siblings attended that school, as well as school specific tie-breaking points. See Górtazar et al. (2020); Casalmiglia et al. (2020) for detailed descriptions of Barcelona and Madrid point systems.

[2]The issue is discussed in https://www.nemokennislink.nl/publicatiesschoolstrijd-in-amsterdam/ (Schoolstrijd in Amsterdam) (Arnout Jaspers, Kennislink, July 1, 2015). This reference is taken from Ashlagi et al. (2019). Also the reader can access https://medium.com/social-choice/ why-a-dutch-court-stopped-high-school-students-from-exchanging- schools-1315303a48b6 (accessed may 21st 2020).

structure in the case of Boston where unintentionally the precedence rule at the moment to apply the 50-50 reserve system for neighborhood students undermined intended policy and led to the elimination of walk zone reserves in Boston's public school match Dur et al. (2018). In Madrid, where a point system to form cardinal priorities is in place, abolishing school choice proximity points does not seem to have been an effective public policy for reducing school segregation under tIA (Górtazar et al., 2020). We show that in our model in both in the cases of ordinal and cardinal priorities the consequences of removing a characteristic form consideration has different consequences that allow the agents to exchange this particular characteristic.

We propose a class of algorithms named Student Exchange with Transferable Characteristics (SETC). Each algorithm in this class gives a matching of students to schools and a redistribution of transferable characteristics such that the matching is stable with respect to the new allocation of characteristics and it is not Pareto dominated by another stable matching with respect to some admissible redistribution of characteristics. Moreover, each such pair of a matching and a redistribution of characteristics can be obtained by an algorithm in the SETC class (Theorem 1).

One particular algorithm in this class the Top Trade SETC algorithm (TTSETC) has the same outcome that the Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) Kesten (2010) when schools priorities are monotonous and fully transferable (Theorem 2). None of the mechanism in the class of the SETC is strategy proof. However, if we give every student the set of characteristics resulting from a mechanism in the SETC class and apply the SOSM the allocation will be identical and students will reveal their true preferences as a dominant strategy (Corollary 1). Therefore, we can consider the outcome of the an algorithm in the SETC class as the resulting of applying the SOSM after students trade their characteristics in a way that, ex-post, will not violate fairness.

1.1 Related Literature

The school choice problem was first introduced by Balinski and Sönmez (1999). In this paper the authors introduce the idea of fairness to allocate seats to students. In Abdulkadiroğlu and Sönmez (2003) the same problem is analyzed from a mechanism design perspective. the authors compare the SOSM by Gale and Shapley (1962) they show that this mechanism is produce fair or stable allocations and it is strategy-proof. They also study an adaptation of Gale's top trading cycles procedure: the TTCM. They show that the TTCM is Pareto efficient and strategy-proof.

It is not possible to reconcile this three properties in the school choice problem. In general, fair allocations are not efficient and the level of inefficiency of a fair allocation can be severe (Dur and Morrill, 2017; Kesten, 2010; Abdulkadiroğlu et al., 2009)

We can find in the literature several attempts to ameliorate the conflict between fairness and efficiency by weakening the notion of stability. In this matter, (Kesten, 2010) focus on the idea of 'consent.' In Kesten's proposal students can renounce to their priorities over schools that will not be allocate to them but, nevertheless will generate inefficiencies if the final allocations is produce by the SOSM. This is the logic behind Kesten's EADAM. The EADAM gives students incentives to consent and finds a constrained efficient matching. This idea of consent is also present in (Tang and Yu, 2014) that introduce an algorithm that is computationally simpler than the EADAM. Both in (Kesten, 2010) and (Tang and Yu, 2014) when all the students consent to waive their priorities the algorithm produce efficient allocations. Alcalde and Romero-Medina (2017) propose an alternative weakening of fairness: α-equitability. In the case of α-equitability a matching with a priority violation is unfair only if a student's objection to that priority violation can not be counter-objected by another student. Ehlers and Morrill (2019) also relax the fairness constraint. In its case, a student's priority at a school needs to be respected only if there exists a matching in which that student is assigned to the school. Ehlers and Morrill (2019) propose a stable set of legal matchings that are not dominated in fairness terms by any other legal matching, and (illegal) matchings are dominated by a legal matching. Finally, Alva and Manjunath (2019) also weaken stability presenting the concept of stable domination. A rule is stable-dominating if it selects an allocation that Pareto improves some stable allocation at every preference profile. They show that the SOSM is the only stable-dominating and strategy-proof rule.

The paper most closely related to ours is Dur et al. (2019). They propose another form of weakening stability so the final (fair) allocation is as efficient as possible. they propose the notion of partial stability. Under partial stability some specific priorities of some students at some schools are ignored. Once the set of priorities to be ignored is determined the paper explores the the idea that possible welfare gains can be captured by the improvement cycles proposed by Erdil and Ergin (2008) in the context of coarse schools' priorities. Our paper follows the same approach. We look for admissible improvement cycles by justifying the violation of initial priorities via exchange of transferable characteristics. Beyond the formal similarities conceptually the two papers have considerable differences. First, our primitives are not priorities but individual characteristics. Second, the possible welfare gains we capture are derived form the trade of characteristics and this trade will be allowed or not depending on whether the characteristic is tradeable and a potential trade does not violate fairness. That is, the resulting extended matching is justified with the final allocation of transferable characteristics and does not generate complaints of students that would like to improve the school that they are assigned to. Third, the resulting extended matching is a new allocation of school seats and student characteristics. Clearly all students that are relocated to a new school will be better off. However, the SETC trades characteristics and contrary to the stable improvement cycles algorithm in Erdil and Ergin (2008) some of the students that participate in this trade do not change schools but they change characteristics. Those students will participate in the characteristic trade but they will be weakly better off. This will allow to other students to trade their characteristics. Finally, on a technical note, our extended priorities framework does not require the introduction of additional conditions on the set of priorities that may be ignored. Our results only requires that school priorities are monotonic in students characteristics.[3]

After this brief literature review, the rest of the paper is organized as follows. In Section 2, we introduce the model and notation. In Section 3 we present our main results that we prove in Section 4. In Section 5 we relate our framework of transferable characteristics to the school choice with consent proposed by Kesten (2010). In Section 6 we conclude.

2 Notation and Definitions

We present the standard school choice problem and then introduce the extended model with partially transferable characteristics.

Let I be a finite set of students and S a finite set of schools where students have to be

[3]Assumption 1 in Dur et al. (2019). See Remark 1 in Section 2.

allocated. Each student i is equipped with a strict preference P_i over $S \cup \{\varnothing\}$,[4] where $\{\varnothing\}$ stands for the option of being unassigned. We denote by R_i the weak preference relation associated to P_i defined in the standard way and by P a generic students' preference profile. Let $\mathcal{P}$ denote the set of all students's preference profiles. Each school s has a limited number of seats available q_s.

A matching is a function $\mu : I \to S \cup \{\varnothing\}$ such that (i) for each $i \in I$, $\mu(i) \in S \cup \{\varnothing\}$ and (ii) for each $s \in S$, $\#\mu^{-1}(s) \leq q_s$. A matching μ' ***Pareto dominates*** the matching μ if for each $i \in I$, $\mu'(i)\ R_i\ \mu(i)$, and for some $j \in I$, $\mu'(j)\ P_j\ \mu(j)$.

The final component of the school choice problem is the priorities of schools. Each school ranks prospective students according to a priority order. Our contribution to the literature is to explore the structure of such priority orders. We consider that schools priorities may depend on different characteristics of students. Some of those characteristics are intrinsic to each student, but some characteristics can be exchanged among students. The relevant priorities for schools depend on the final allocation of such characteristics.

For each student i let $\omega(i) = (\omega^s(i))_{s \in S}$ be the initial vector of transferable characteristics that influence the position of student i at each school. Each student initial endowment consist of transferable characteristics specific to each school. For each school s, let $\Omega^s = \cup_{i \in I} \omega^s(i)$ be s's set of available transferable characteristics. For each school s let λ^s be a bijection from students to Ω^s. That is, λ^s is a permutation of s's transferable characteristics among the students. For each $i \in I$, and $s \in S$ there is $j \in I$ with $\lambda^s(i) = \omega^s(j)$, and for each $j, j' \in I$, $\lambda^s(j) \neq \lambda^s(j')$. For each s, let $\mathcal{L}^s$ be the set of all permutation of s's transferable characteristics among students. We call $\lambda = (\lambda_s)_{s \in S}$ an ***allocation of transferable characteristics***. Finally, for each student i and each allocation λ, $\lambda(i) \equiv (\lambda^s(i))_{s \in S}$. We denote by ω the initial endowment allocation of transferable characteristics.

When the characteristics are transferable, the assignment of such characteristics is relevant. Note that each admissible $\lambda \in \mathcal{L}$ can be obtained via exchange cycles of characteristics among the students. An ***extended matching*** is a pair (μ, λ) such that μ is a matching and $\lambda \in \mathcal{L}$. Let $\mathcal{M}$ be the set of all extended matchings.

[4]A strict preference is a complete, antisymmetric, and transitive binary relation.

In the extended framework, schools priorities do not compare only students, but pairs of students and the allocations of transferable characteristic that they present to the school choice process. Hence, each school is equipped with a complete, transitive, and antisymmetric binary relation $\succ_s$ on $I \times \Omega^s$. We use the notation $\succsim_s$ to refer to the weak priority relation associated to $\succ_s$.

Throughout this paper, we assume that transferable characteristics are monotonous in the sense that affect all the students in the same direction.

Monotonous Priorities For each i, j and s, for each $l, l' \in \Omega^s$: $(i, l) \succ_s (i, l')$ if and only if $(j, l) \succ_s (j, l')$.

Under monotonous priorities, for each s the set Ω^s is naturally ordered and, abusing notation, for each $\Lambda^s \subseteq \Omega^s$ we define

$$\max\{\Lambda^s\} \equiv \{l \in \Lambda^s, \text{ for each } i \in I, l' \in \Lambda^s, (i, l) \succsim_s (i, l')\}.$$

Remark 1. *Under monotonous priorities, for each school s, each $i_0, i_1, i_2, i_3 \in I$, and each extended priority $\succsim$, if*

$$\begin{gathered}\succ_s (i_1, \lambda^s(i_1)) \succ_s (i_2, \lambda^s(i_2)) \succ_s (i_3, \lambda^s(i_3)), \quad \textit{and} \\ (i_3, \max\{\lambda^s(i_0), \lambda^s(i_3)\}) \succsim_s (i_1, \lambda^s(i_1))\end{gathered}$$

then $(i_3, \max\{\lambda^s(i_0), \lambda^s(i_3)\}) \succsim_s (i_2, \lambda^s(i_2))$.

Finally, we present the stability notion that takes into account the fact that school priorities depend on the identity of the students and some transferable characteristics.

An extended matching (μ, λ) is **(ex-post) stable** if:

- μ is λ-***fair***: for each $i, j \in I$, $\mu(j)\ P_i\ \mu(i)$ implies $(j, \lambda^s(j)) \succ_{\mu(j)} (i, \lambda^s(i))$.
- μ is ***individually rational***: for each $i \in I$, $\mu(i)\ R_i\ \{\varnothing\}$
- μ is ***not wasteful***: if for no $i \in N$ and $s \in S$, $s\ P_i\ \mu(i)$ and $\#\mu^{-1}(s) < q_s$.

The interpretation of *(ex-post) stable* coincides with the natural notion of stability. An *(ex-post) stable* extended matching does not generate complaints of students that would like to improve the school that they are assigned to. The matching proposed is justified wth the final allocation of transferable characteristics.

It is worth to note that our notion of *(ex-post) stable* is parallel to *partial stability* in Dur et al. (2019) but we provide a rationale and structure to the admissible violations of the initial priorities. In the light of Remark 1, our extended priority structure does not call for the introduction of additional restrictions on the set of admissible violations of fairness as Assumption 1 in Dur et al. (2019).

We are interested in obtaining *(ex-post) stable* extended matching that are not Pareto dominated by other *(ex-post) stable* extended matchings. If there is no possibility of exchange of transferable characteristics, the students proposing deferred acceptance algorithm selects the student optimal stable extended matching (SOSM). The *(ex-post) stable* (μ, ω) is the ***student optimal stable extended matching (SOSM)*** if μ is not Pareto dominated by another (ex-post) stable extended matching (ν, ω).

When the students may exchange their transferable characteristics, we could find *(ex-post) stable* extended matchings (μ, λ) such μ Pareto dominates the SOSM matching. We focus on extended matchings that can be obtained by limited exchanges of transferable characteristics that lead to changes that justify the change of the students' school match. Given an extended matching (μ, λ), we say $(\bar{\mu}, \bar{\lambda})$ is a ***reshuffle of*** (μ, λ) if for each $i \in I$, for each $s \notin \{\mu(i), \bar{\mu}(i)\}$, $\lambda^s(i) = \bar{\lambda}^s(i)$.

We are now in condition to present the notion that captures the idea of obtaining efficient matchings that are required to satisfy fairness and stability when transferable characteristics can be traded.

An extended matching (μ, λ) is ***constrained efficient*** if it is *(ex-post) stable* and for no *(ex-post) stable* reshuffle (μ', λ'), μ' Pareto dominates μ.

Example 1. *Let* $I = \{i_1, i_2, i_3\}$*,* $S = \{s_1, s_2, s_3\}$*,* $q_{s_x} = 1$ *for* $x = 1, 2, 3$*. The preferences of students are:*

P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_1	s_1
s_1	s_2	s_2
s_3	s_3	s_3

Each school uses two criteria to determine their priorities. They consider whether students have a sibling already enrolled at the school and whether they live in the Walk-Zone of the school. These criteria determine four coarse priority classes in each school. Each school prioritizes students with ***Sibling+Walk-Zone****, and those students who have a* ***Sibling*** *but do not live in the* ***Walk-Zone*** *to students who live in its* ***Walk-Zone*** *with no enrolled* ***Sibling****. No student lives in s_3's walk-zone or has a sibling in s_3. Finally, the inverse natural order breaks ties inside each priority class.*

Assume that no student has any sibling and walk-zone characteristics are transferable. Student i_1 lives in school s_1 walk-zone, Student i_2 lives in school s_2 walk-zone, while student i_3 lives out of the school district. Hence, we can write the initial endowment allocation of transferable characteristics:

$$\begin{pmatrix} \omega(i_1) \\ \omega(i_2) \\ \omega_(i_3) \end{pmatrix} = \begin{pmatrix} (\omega^{s_1}(i_1), \omega^{s_2}(i_1), \omega^{s_3}(i_1)) \\ (\omega^{s_1}(i_2), \omega^{s_2}(i_2), \omega^{s_3}(i_2)) \\ (\omega^{s_1}(i_3), \omega^{s_2}(i_3), \omega^{s_3}(i_3)) \end{pmatrix} = \begin{pmatrix} (1,0,0) \\ (0,1,0) \\ (0,0,0) \end{pmatrix}.$$

Schools' priorities under the initial endowment allocation of transferable characteristics are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_1, 1)$	$(i_2, 1)$	$(i_3, 0)$
$(i_3, 0)$	$(i_3, 0)$	$(i_2, 0)$
$(i_2, 0)$	$(i_1, 0)$	$(i_1, 0)$

The SOSM for the initial endowment of transferable characteristics is (μ, ω) with $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$.

When students $\{i_1, i_2\}$ exchange their transferable characteristics, the allocation of exchangeable characteristics is

$$\begin{pmatrix} \lambda(i_1) \\ \lambda(i_2) \\ \lambda(i_3) \end{pmatrix} = \begin{pmatrix} (\lambda^{s_1}(i_1), \lambda^{s_2}(i_1), \lambda^{s_3}(i_1)) \\ (\lambda^{s_1}(i_2), \lambda^{s_2}(i_2), \lambda^{s_3}(i_2)) \\ (\lambda^{s_1}(i_3), \lambda^{s_2}(i_3), \lambda^{s_3}(i_3)) \end{pmatrix} = \begin{pmatrix} (0,1,0) \\ (1,0,0) \\ (0,0,0) \end{pmatrix},$$

and schools' extended priorities under λ are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_2, 1)$	$(i_1, 1)$	$(i_3, 0)$
$(i_3, 0)$	$(i_3, 0)$	$(i_2, 0)$
$(i_1, 0)$	$(i_2, 0)$	$(i_1, 0)$

The SOSM under λ is (μ', λ) with $\mu' = \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$. Clearly, μ Pareto dominates μ', and student i_3 has not justified envy for i_1 under the extended priorities obtained with the allocation of transferable characteristics λ.

3 Improvement Cycles for Extended Matchings

Our approach follows Erdil and Ergin (2008) and Dur et al. (2019) that propose a methods for finding fair Pareto improving trade cycles upon SOSM for coarse priorities with arbitrary tie-breakers and partially non-enforceable priorities respectively. In both papers, the logic behind improving cycles is parallel. For an initial stable matching, if there's a vacant position at some school, that position may be assigned to one student such that no other student with higher priority at that school prefers that vacant position to her position at the initial matching. In our extended framework this rationale cannot be applied immediately. Although the students may be willing to accept any position at a desirable school, depending on the student that exchanges the transferable characteristic some violation of fairness may appear. For this reason Pareto improvements involving two students may require the participation of additional students who just swap transferable characteristics without involving a change of school. Moreover, once a student leaves a position in a school, she may start a process similar to a vacancy chain (Blum et al., 1997). The first student in the priority ranking of the school may be admitted in the school without any need of the exchange in the transferable characteristics since the student that leaves the vacant position obtains a position at a preferred school.

Given an *(ex-post) stable* extended matching (μ, λ), for each school $j \in I$, let:

- $D_{(\mu,\lambda)}(j) = \{i \in I : \mu(j) \ R_i \ \mu(i)\}$ and $\tilde{D}_{(\mu,\lambda)}(j) = \{i \in I : \mu(j) \ P_i \ \mu(i)\}$.
- $X_{(\mu,\lambda)}(j) = \{i \in D_{(\mu,\lambda)}(j) : \forall k \in \tilde{D}_{(\mu,\lambda)}(j) \setminus \{i\}, (i, \max\{\lambda^s(i), \lambda^s(j)\}) \succ_s (k, \lambda^s(k))\}$.

The set $\tilde{D}_{(\mu,\lambda)}(j)$ contains all the students who prefer the match for student j rather than their own match. The set $D_{(\mu,\lambda)}(j)$ also includes all students who are matched to $\mu(j)$. The set $X_{(\mu,\lambda)}(j)$ includes all the students who would be willing to occupy j's position at $\mu(j)$ and there would not be any instance of envy if they are matched to $\mu(j)$ should j leave her position. The members of $X_{(\mu,\lambda)}(j)$ are those students in $D_{(\mu,\lambda)}(j)$ that either after they obtain $\lambda^{\mu(j)}(j)$ or maintaining $\lambda^{\mu(j)}(j)$ are ranked above the remaining members of $\tilde{D}_{(\mu,\lambda)}(j)$. Hence, if j moves to a preferred school and a member of $X_{(\mu,\lambda)}(j)$ gets j's position at $\mu(j)$, nobody could argue that the change violate her priority over $\mu(j)$.

Let $G = (V; E)$ be a directed graph with the set of vertices V, and the set of directed edges E, which is a set of ordered pairs of V .

For each extended matching (μ, λ), $G(\mu, \lambda) = (I; E(\mu, \lambda))$ is the (directed) application graph associated with (μ, λ) where the set of directed edges $E(\mu, \lambda) \subseteq I \times I$ is as follows: $ij \in E(\mu, \lambda)$ (that is, i points to j) if and only $i \in X_{(\mu,\lambda)}(j)$. A set of edges $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is a path if the vertices $i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}$ are distinct, and a cycle if the vertices $i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}$ are distinct and $i_1 = i_{n+1}$. A student i is involved in the cycle ϕ if there is a student j such that $ij \in \phi$. A cycle $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is solved when for each $ij \in \phi$, student i is assigned to $\mu(j)$ to obtain a new matching. Formally, we denote the solution of a cycle by the operation $\circ$ that is, $\eta = \phi \circ \mu$ if and only if for each $ij \in \phi$, $\eta(i) = \mu(j)$, and for each $i' \notin \{i_{1,} \ldots, i_n\}$ $\eta(i') = \mu(i')$. A cycle ϕ is an improvement cycle for $G(\mu, \lambda)$ if there is $ij \in \phi$ such that $i \in \tilde{D}_{(\mu,\lambda)}(j)$.

The following algorithm is built on an *(ex-post) stable* extended matching and is defined by solving cycles iteratively:

Student Exchange with Transferable Characteristics (SETC):

Step 0: Let (μ_0, λ_0) be an *(ex-post) stable* extended matching.

Step $k \geq 1$: Given an extended matching $(\mu_{k-1}, \lambda_{k-1})$,

(k.1) if there is no improvement cycle in $G(\mu_{k-1}, \lambda_{k-1})$, then the algorithm terminates and $(\mu_{k-1}, \lambda_{k-1})$ is the matching obtained,

(k.2) otherwise, solve one of the improvement cycles in $G(\mu_{k-1}, \lambda_{k-1})$ say ϕ_k let $\mu_k = \phi_k \circ \mu_{k-1}$, and define λ_k as follows. For each $i \in I$, let $s_k = \mu_k(i)$ and $s_0 = \mu_0(i)$.

- For each $s \notin \{s_0, s_k\}$, $\lambda_k^s(i) = \lambda_0^s(i)$.
- If there is no i' such that $ii' \in \phi_k$, then $\lambda_k^{s_k}(i) = \lambda_{k-1}^{s_k}(i)$.
- If there is i' such that $ii' \in \phi_k$ then $\lambda_k^{s_k}(i) = \max\{\lambda_{k-1}^{s_k}(i), \lambda_{k-1}^{s_k}(i')\}$.
- If there is j such that $\lambda_0^{s_0}(i) = \lambda_k^{s_0}(j)$, then $\lambda_k^{s_0}(i) = \lambda_0^{s_0}(j)$, otherwise $\lambda_k^{s_0}(i) = \lambda_0^{s_0}(i)$.

The next example shows the relevance for constructing improvement cycles of students who do not strictly benefit from the exchange of transferable characteristics.

Example 2. *Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3\}$, $q_{s_x} = 1$ for $x = 1, 3$; and $q_{s_2} = 2$. The preferences of students are:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_2	s_1	s_1	s_2
s_1	s_2	s_2	s_3
s_3	s_3	s_3	s_1

*Each school uses two criteria to determine their priorities. Schools 1 and 3 consider whether students have a sibling already enrolled at the school and whether they live in the Walk-Zone of the school. These criteria determine four coarse priority classes in each school. Each school prioritizes students with **Sibling+Walk-Zone**, and those students who have a **Sibling** but do not live in the **Walk-Zone** to students who live in its **Walk-Zone** with no enrolled **Sibling**. Finally, the inverse natural order breaks ties inside each priority class. School s_2 orders students according to the outcome of exam, using the **Walk-Zone** for breaking ties (and eventually with the inverse natural tie-breaker). Students i_1 and i_2 live in s_1's walk-zone. Student i_1 has a sibling in s_1 but their parents would like to move their children to s_2. Student i_4 has the highest test-score overall for s_2*

while students i_1, i_2 and i_3 have same test-score ranking. Finally, student i_4 lives in s_2's walk-zone.

$$\begin{pmatrix} \omega(i_1) \\ \omega(i_2) \\ \omega_(i_3) \\ \omega(i_4) \end{pmatrix} = \begin{pmatrix} (\omega^{s_1}(i_1), \omega^{s_2}(i_1), \omega^{s_3}(i_1)) \\ (\omega^{s_1}(i_2), \omega^{s_2}(i_2), \omega^{s_3}(i_2)) \\ (\omega^{s_1}(i_3), \omega^{s_2}(i_3), , \omega^{s_3}(i_3)) \\ (\omega^{s_1}(i_4), \omega^{s_2}(i_4), , \omega^{s_3}(i_4)) \end{pmatrix} = \begin{pmatrix} (1,0,0) \\ (1,0,0) \\ (0,0,0) \\ (0,1,0) \end{pmatrix}.$$

Schools' priorities under the initial endowment allocation of transferable characteristics are:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$(i_1, 1)$	$(i_4, 1)$	$(i_4, 0)$
$(i_2, 1)$	$(i_3, 0)$	$(i_3, 0)$
$(i_4, 0)$	$(i_2, 0)$	$(i_2, 0)$
$(i_3, 0)$	$(i_1, 0)$	$(i_1, 0)$

Moreover, $(i_3, 1) \succ_{s_1} (i_2, 1)$ and $(i_1, 1) \succ_{s_2} (i_2, 0)$.

The SOSM for the initial endowment of transferable characteristics is (μ, ω) with $\mu = \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_2)\}$. The assignment $\mu' = \{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_2)\}$ Pareto dominates μ but if i_1 and i_3 exchange their transferable characteristics then the resulting extended matching would result in justified envy since $\omega^{s_2}(i_3) = 0$ and $(i_2, 0) \succ_{s_2} (i_1, 0)$. However, if student i_4 participates in the exchange of characteristics, we would obtain the resuffle λ:

$$\begin{pmatrix} \lambda(i_1) \\ \lambda(i_2) \\ \lambda(i_3) \\ \lambda(i_4) \end{pmatrix} = \begin{pmatrix} (\lambda^{s_1}(i_1), \lambda^{s_2}(i_1), \lambda^{s_3}(i_1)) \\ (\lambda^{s_1}(i_2), \lambda^{s_2}(i_2), , \lambda^{s_3}(i_2)) \\ (\lambda^{s_1}(i_3), \lambda^{s_2}(i_3), \lambda^{s_3}(i_3)) \\ (\lambda^{s_1}(i_4), \lambda^{s_2}(i_4), \lambda^{s_3}(i_4)) \end{pmatrix} = \begin{pmatrix} (0,1,0) \\ (1,0,0) \\ (1,0,0) \\ (0,0,0) \end{pmatrix},$$

and the extended matching (μ', λ) is (ex-post) stable.

Starting from the initial SOSM, in Figure 1 we present the graph where each student points to the positions that students whose position would like to occupy (including indifference relations). In Figure 2 we show the strict improvements that would not generate

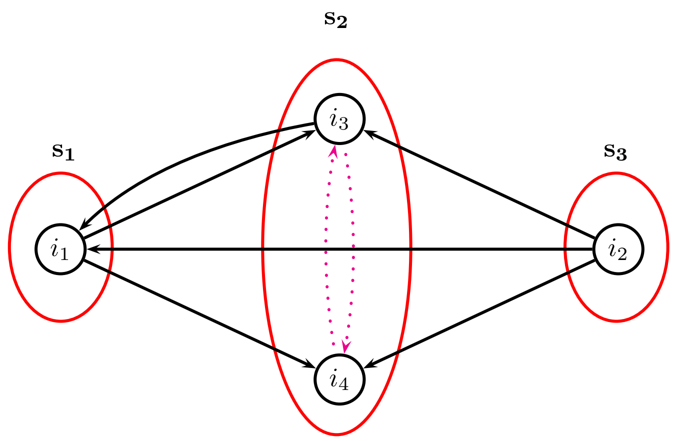

Figure 1: Example 2. Student i_x points student i_y if $i_x \in D_{(\mu,\omega)}(i_y)$. Solid lines: i_x points i_y if $i_x \in \tilde{D}_{(\mu,\omega)}(i_y)$. Dotted Lines: i_x points i_y if $i_x \in D_{(\mu,\omega)}(i_y)$, $\mu(i_x) = \mu(i_y)$.

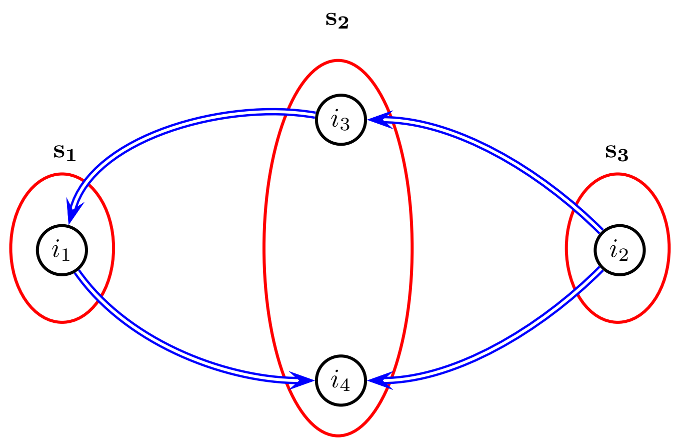

Figure 2: Example 2. Graph associated to (μ, ω). Student i_x points student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$ and $\mu(i_x) \neq \mu(i_y)$.

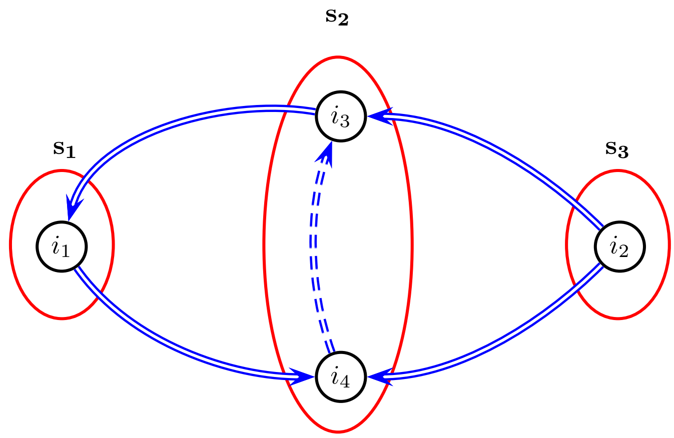

Figure 3: Example 2. $G(\mu, \omega)$. Student i_x points student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$.

justified envy and observe that no cycle can be constructed. Note that i_1 does not point i_3 because $(i_2, \omega_{i_2}^{s_2}) \succ_{s_2} (i_1, \max\{\omega_{i_1}^{s_2}, \omega_{i_3}^{s_2}\})$. Finally, in Figure 3 and we present the graph associated to (μ, ω). We observe the existence of a unique cycle $\gamma = i_1 i_4 i_3 i_1$. Solving γ generates the extended matching (μ', λ). In Figure 4, we present the graph $G(\mu', \lambda)$. The graph contains no improvement cycle and indeed the extended matching (μ', λ) is (ex-post) stable.

Remark 2. *The school priorities presented in Example 2 are consistent with point-system based priorities. Point system generate additively separable extended priorities. That is for each school s, for each pair of students i_x, i_y and each $\lambda^s, \bar{\lambda}^s \in \Omega^s$, $(i_x, \lambda^s) \succ_s (i_y, \lambda^s)$ if and only if $(i_x, \bar{\lambda}^s) \succ_s (i_y, \bar{\lambda}^s)$.*

Next, we present our main result. It turns out that starting from any *(ex-post) stable* extended matching the application of an algorithm in the SETC class always yields a constrained efficient and *(ex-post) stable* extended matching. Moreover, any constrained efficient extended stable matching can be obtained from a SETC algorithm starting at the SOSM extended matching. Hence, the SETC class identifies all the improvement cycles that yield an *(ex-post) stable* extended matchings.

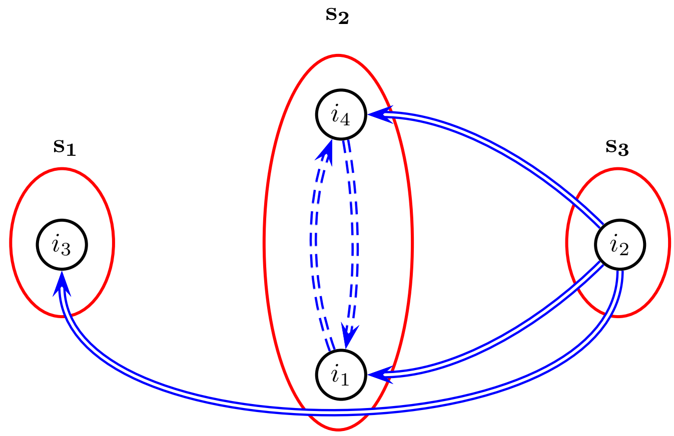

Figure 4: Example 2. $G(\mu', \lambda)$. Student i_x points student i_y if $i_x \in X_{(\mu,\omega)}(i_y)$.

Theorem 1. *For each problem, an extended matching is constrained efficient and Pareto dominates the SOSM if and only if it is obtained by an algorithm within the SETC class starting with the SOSM extended matching.*

The proof of Theorem 1 is presented in the next section. The proof follows similar arguments to the proof of Dur et al. (2019) but the extended model generates important intricacies. Transferable characteristics differ among students and only exchanges involving specific students in a school may be mutually viable. Moreover, improvement cycles may need to involve students who do not strictly improve by the exchange but facilitate the reassignment by trading their transferable characteristics.

Two immediate consequences follow from Theorem 1. Since the result of a SETC is constrained efficient and *(ex-post) stable* with respect to the final allocation of transferable characteristics, then it is the result of the SOSM for the final allocation of transferable characteristics.

Corollary 1. *For each problem and each stable matching μ_0 and each algorithm in the SETC class, if the extended matching (μ, λ) is the outcome of the SETC algorithm then (μ, λ) is the SOSM with initial endowment of transferable characteristics λ.*

We conclude this section analyzing the incentives of students to reveal their true preferences when the allocation of schools' seats is determined by an algorithm in the SETC class. For that purpose, we need to introduce further notation that relates the outcomes of different problems defined for different preference profiles.

A rule is a mapping $\Psi : \mathcal{P} \to \mathcal{M}$. The application of a SETC algorithm starting with the SOSM extended matching corresponding to each preference profile defines a rule that always selects a *(ex-post) stable* and constrained efficient extended matching. We call the class of such rules as the ***students' optimal with transferable characteristics (SOTC)*** class of rules.

Strategy-proofness A rule Ψ satisfies ***strategy-proofness*** if for each $i \in N$, each $P, P' \in \mathcal{P}$ such that for each $j \neq i$ $P_j = P'_j$ with $\Psi(P) = (\mu, \lambda)$ and $\Psi(P') = (\mu', \lambda')$, $\mu(i)\ R_i\ \mu'(i)$.

The following result is an immediate consequence of the fact that SETC rules defined By the results in Abdulkadiroğlu et al. (2009); Alva and Manjunath (2019); Kesten and Kurino (2019), since the matching selected by any SETC algorithm starting with the SOSM Pareto dominates the SOSM matching for the initial endowment of characteristics, and it results in efficient allocations, any SETC is manipulable at some profile of students preferences.

Proposition 1. *There is no rule in the SOTC class that satisfies* strategy-proofness*.*

Proof. Let A be an algorithm in the SETC, define the SOT rule Ψ that for each profile of students preferences selects the matching obtained through the application of A at that preference profile. By Theorem 1, for each preference profile the matching selected by Ψ is *(ex-post) stable* and Pareto efficient. For each $P \in \mathcal{P}$, Ψ selects an extended matching that represents a Pareto improvement upon the SOSM matching. By Abdulkadiroğlu et al. (2009), the SOSM is in the Pareto frontier of the set of rules that satisfy stability and *strategy-proofness*. Hence, Ψ violates *strategy-proofness*. $\square$

4 Proof of Theorem 1

Although Theorem 1 refers specifically to the application of SETC algorithms to the SOSM extended matching, the analysis can be carried out from any arbitrary *(ex-post)* *stable* extended matching. We study separately the proofs of necessity and sufficiency sides of the results.

4.1 Proof of "if" part

For a given problem $(R, \succsim)$ and a stable extended matching (μ_0, λ_0) consider an algorithm in the SETC class. Let K be the last step of the algorithm and (μ_k, λ_k) be the extended matching selected at $k \in \{1, \ldots, K-1\}$. A cycle is solved at each step of the algorithm, which implies that the students in the cycle are better off and no student is worse off at the new matching obtained by solving the cycle. Thus, the matching at each step *Pareto dominates* the matching in the previous step, and for each $k \geq 1$, if student j is not involved in any improvement cycle at Step k, $\tilde{D}_{(\mu_k,\lambda_k)}(j) \subseteq \tilde{D}_{(\mu_{k-1},\lambda_{k-1})}(j)$. Hence, if i points to j in $G(\mu_{k-1}, \lambda_{k-1})$ and both students are not involved in animprovement cycle at Step k then i points to j in $G(\mu_k, \lambda_k)$.

Lemma 1. *Each extended matching obtained by a SETC algorithm is stable.*

Proof. Let $(\mu_k, \lambda_k\}$ be the extended matching obtained at Step $k \in \{0, \ldots, K-1\}$. We prove the result by induction on k. The initial extended matching (μ_0, λ_0) is stable.

Fairness Assume that $(\mu_{k-1}, \lambda_{k-1})$ is fair. Take any pair of students (i, j) such that $\mu_k(j)\ P_i\ \mu_k(i)$. At each step of the algorithm, each student is either better off (she is in a solved cycle) or she is assigned to the same school as in the previous step. Let ϕ_k denote the improvement cycle solved in step k. Assume first that j is not involved in the cycle ϕ_k. Since $\mu_k(j)\ P_i\ \mu_k(i)$, $\mu_{k-1}(j)\ P_i\ \mu_{k-1}(i)$ and $i \in \tilde{D}_{(\mu_{k-1},\lambda_{k-1})}(j)$. Then, by fairness of $(\mu_{k-1}, \lambda_{k-1})$, $(j, \lambda^{\mu_{k-1}(j)}(j)) \succ_{\mu_{k-1}} (i, \lambda^{\mu_{k-1}(j)}(i))$. Since j is not involved in ϕ_k, $\lambda^{\mu_{k-1}(j)}(j) = \lambda^{\mu_k(j)}(j)$. Since $i \in \tilde{D}_{(\mu_k,\lambda_k)}(j)$, $\lambda^{\mu_{k-1}(j)}(i) = \lambda^{\mu_k(j)}(i)$. Therefore $(j, \lambda^{\mu_k(j)}(j)) \succ_{\mu_k(j)} (i, \lambda^{\mu_k(j)}(i))$. Assume now that j is involved in ϕ_k. Let $j' \in I$ be such that $j'j \in \phi_k$. Hence, $\mu_{k-1}(j')\ P_i\ \mu_{k-1}(i)$, $i \in \tilde{D}_{(\mu_k,\lambda_k)}(j')$, and $\lambda^{\mu_{k-1}(j')}(i) = \lambda^{\mu_k(j')}(i)$. Since $j'j \in \phi_k$, $(j, \max\{\lambda_{k-1}^{\mu_{k-1}(j')}(j'), \lambda_{k-1}^{\mu_{k-1}(j)} \succ_{\mu_{k-1}(j')} (i, \lambda^{\mu_{k-1}(j')}(i))$, and $(j, \lambda^{\mu_k(j)}(j)) \succ_{\mu_k(j)}$ $(i, \lambda^{\mu_k(j)}(i))$. Since i, j are arbitrary, (μ_k, λ_k) is *fair*.

Individual Rationality Since μ_0 is *individually rational*, and each student is never worse off after each step of the algorithm, the μ_K is *individually rational*.

Non-Wastefulness The initial match μ_0 is non wasteful. At each step students are assigned to better schools swapping their positions at schools, hence $\#\mu_k^{-1}(s)$ remains constant at any step of the algorithm. Assume school s has an empty slot at step k, then the school s has an empty slot at step 0. Since μ_0 is non wasteful and individually rational, for each student i with $\mu_0(i) \neq s$, $\mu_0(i)\ P_i\ s$. Since for each i, $\mu_k(i)\ R_i\ \mu_0(i)$, $\mu_k(i)\ R_i\ s$, and (μ_k, λ_k) satisfies non-wastefulness. □

Lemma 2. *For each stable extended matching* (μ, λ) *and* $j \in I$, $X_{(\mu,\lambda)}(j) \subseteq \mu(\mu^{-1}(j))\backslash\{j\}$ *if and only if* $\tilde{D}_{(\mu,\lambda)}(j) = \{\varnothing\}$.

Proof. If $\tilde{D}_{(\mu,\lambda)}(j) = \{\varnothing\}$, since $D_{(\mu,\lambda)}(j) = \mu(\mu^{-1}(j))$ and $X_{(\mu,\lambda)}(j) \subseteq D_{(\mu,\lambda)}(j)$, the result is immediate. On the other hand, if $\tilde{D}_{(\mu,\lambda)}(j) \neq \{\varnothing\}$, then by completeness and transitivity of schools' priorities there is $i \in \tilde{D}_{(\mu,\lambda)}(j)$ such that for each $i' \in \tilde{D}_{(\mu,\lambda)}(j)$, $(i, \lambda^{\mu(j)}(i)) \succsim_{\mu(j)} (i', \lambda^{\mu(j)}(i'))$. By monotonicity of priorities, $(i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(i)\}) \succ_{\mu(j)} (i, \lambda^{\mu(j)}(i))$. Therefore, $\mu(i) \neq \mu(j)$ and $i \in X_{(\mu,\lambda)}(j)$. □

Lemma 3. *Let* (μ, λ) *and* (η, λ') *be* (ex-post) stable *extended matchings such that* μ *Pareto dominates* η. *For each* $s \in S$, $\#\mu^{-1}(s) = \#\eta^{-1}(s)$.

Proof. Let $N = \{i \in I : \mu(i)\ P_i\ \eta(i)\}$. Since μ Pareto dominates η, for each $j \in I \setminus N$, $\mu(j) = \eta(j)$. Consider school s and assume that $\#(N \cap \mu^{-1}(s)) > \#(N \cap \eta^{-1}(s))$. This implies that $\#\eta^{-1}(s) < q_s$. For each $i \in N \cap \mu^{-1}(s)$, $\mu(i) = s\ P_i\ \eta(i)$, which contradicts η non-wastefulness. Hence, $\#(N \cap \mu^{-1}(s)) \leq \#(N \cap \eta^{-1}(s))$. Finally, assume to the contrary there is s such that the strict inequality holds. Summing up the inequalities across schools, the number of students in N who are assigned to some school in matching η is larger than the number of students in N that are assigned to some school in matching μ. Hence there is a student $i \in N$ such that $\eta(i) \in S$, and $\mu(i) = \{i\}$. Since η is a individually rational matching, $\eta(i)\ P_i\ \mu(i)$ which contradicts the definition of N. □

Lemma 4. *An extended matching obtained by an SETC algorithm is constrained efficient.*

Proof. Let (μ, λ) be an extended matching obtained by an SETC algorithm. By Lemma 1, (μ, λ) is *(ex-post) stable*. We show that there is no stable extended matching (ν, λ') such that ν Pareto dominates μ. Assume to the contrary, that (ν, λ') is a *(ex-post) stable* extended matching and ν dominates μ. By the definition of the SETC algorithms, there is no improvement cycle in the graph $G(\mu, \lambda)$. There are two cases:

Case 1. For each $j \in I$ $\tilde{D}_{(\mu,\lambda)} = \{\varnothing\}$. Then for each $\in I$, $X_{(\mu,\lambda)}(j) \subseteq \mu^{-1}(j) \setminus \{j\}$. Thus each student is assigned to her best school at μ and ν does not Pareto dominate μ

Case 2. There are chains in $G(\mu, \lambda)$ involving students who would like to change her assigned school, but there is no cycle. This implies that there are students who are only pointed by the students assigned to the same school.

Assume we are in Case 2. Since there is no improvement cycle, there is a set of students who are not pointed by any other student in $G(\mu, \lambda)$. Let $I_1 = \{i \mid \tilde{D}_{(\mu,\lambda)}(i) = \varnothing\}$. Let $i_1 \in I_1$ and $s_1 = \mu(i_1)$. Note that for each $j \in \mu(s_1)$, $\tilde{D}_{(\mu,\lambda)}(j) = \varnothing$ and $\mu(s_1) \subseteq I_1$ Since ν Pareto dominates μ, there does not exist any $j' \in I$, such that $\mu(j') \neq s_1$ and $\nu(j') = s_1$. Thus $\nu^{-1}(s_1) \subseteq \mu^{-1}(s_1)$. By Lemma 3, $\#\mu^{-1}(s_1) = \#\nu^{-1}(s_1)$ and we get $\mu^{-1}(s_1) = \nu^{-1}(s_1)$. Since i_1 was arbitrary, this holds for each s such that $\mu^{-1}(s) \cap I_1 \neq \varnothing$.

Next, since there is no improvement cycle in $G(\mu, \lambda)$, then there is at least a student in $I \setminus I_1$ such that she is only pointed by students in I_1. Otherwise there would be an improvement cycle or no improvement chains (Case 1). Let $I_2 = \{i \mid \tilde{D}_{(\mu,\lambda)}(i) \subseteq I_1\} \setminus I_1$ be the set of such students. Let $i_2 \in I_2$ and $s_2 = \mu(i_2)$. We first show that there is no j with $\mu(j) \neq s_2$ and $\nu(j) = s_2$. Assume to the contrary and since ν Pareto dominates μ, $s_2\ P_j\ \mu(j)$ and thus, $j \in \tilde{D}_{(\mu,\lambda)}(i_2)$. Nevertheless, by definition i_2 is only pointed by students in I_1. By the previous paragraph, for each $j \in I_1$, $\mu(j) = \nu(j)$. Hence, $\nu^{-1}(s_2) \subseteq \mu^{-1}(s_2)$. By Lemma 3, $\#\mu^{-1}(s_2) = \#\nu^{-1}(s_2)$, and therefore $\mu^{-1}(s_2) = \nu^{-1}(s_2)$.

We can continue applying the same argument iteratively, to conclude that all students in any improving chain in $G(\mu, \lambda)$ have the same assignment under μ and ν. The students who are not in a chain in $G(\mu, \lambda)$, are contained in I_1 and have the same assignment in both μ and ν. We conclude that $\mu = \nu$ and ν does not Pareto dominate μ.

□

4.2 Proof of the "only if" part

Let (μ_0, λ_0) a partially stable extended matching. We prove that each constrained efficient matching that Pareto dominates (μ_0, λ_0) can be obtained by an algorithm in the SETC class.

We use again the notion of improvement cycle we introduced in the previous subsection without making reference to the desirability graph. The following lemma is a crucial first step for the construction of improvement cycles in the desirability graph.

Lemma 5. *Let* (μ, λ) *and* $(\nu, \bar{\lambda})$ *be stable extended matchings such that* ν *Pareto dominates* μ*. Then there exists a set of disjoint improvement cycles* $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ *such that* $\nu = \gamma_k \circ \ldots \circ \gamma_1 \circ \mu$*, and there is* λ'' *obtained as in the definition of SETC such that* (ν, λ'') *is stable extended matching.*

Proof. Let $N \subseteq I$ be the set of students who strictly prefer their assignment under ν to the assignment under μ or such that $\lambda(i) \neq \lambda'(i)$. Partition the set N in three disjoint sets $N = N_1 \cup \; N_2 \cup N_3$. Define

$$
\begin{aligned}
N_1 &\equiv \{i \in N \mid \mu(i) = \nu(i) \ \&\ \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\},\\
N_2 &\equiv \{i \in M \mid \mu(i) \neq \nu(i) \ \&\ \bar{\lambda}^{\nu(i)}(i) \neq \lambda^{\nu(i)}(i)\},\\
N_3 &\equiv \{i \in N \mid \mu(i) \neq \nu(i) \ \&\ \bar{\lambda}^{\nu(i)}(i) = \lambda^{\nu(i)}(i)\}.
\end{aligned}
$$

Let $m = \#N$ and index the students in N in such that for each $j, j', j'' \in \{1, \ldots, m\}$ $i_j \in N_1$, $i_{j'} \in N_2$, $i_{j''} \in N_3$ if and only if $j < j' < j''$. Let $\tilde{G}[(\mu, \lambda), (\nu, \lambda')] = (N, E)$ be a directed graph where the edges $E \subseteq N \times N$ are constructed in the following way:

- For each $i_j \in N_1$, i_j points l if and only if $\bar{\lambda}^{\mu(i_j)}(i_j) = \lambda^{\mu(i_j)}(l)$.
- For each $i_j \in N_2$, i_j points l if and only if $\bar{\lambda}^{\nu(i_j)}(i_j) = \lambda^{\nu(i_j)}(l)$.
- For each $i_j \in N_3$, i_j points an arbitrary student in $l \in N$ such that l has not been pointed by any $i_{j'}$ with $j' < j$ and $\mu(l) = \nu(i_j)$.[5]

In the graph $\tilde{G}[(\mu, \lambda), (\nu, \bar{\lambda})]$, each student is pointed by a unique student and points to a unique student in N. Since N is finite, each student is in a cycle and no two cycles intersect. By construction each of those cycles is an improvement cycle over μ and the extended matching $(\nu, \bar{\lambda})$ is obtained solving these cycles in any order. □

Lemma 6. *Let* (μ, λ) *be an* (ex-post) stable *and* $(\nu, \bar{\lambda})$ *a* (ex-post) stable *reshuffle of* (μ, λ) *such that* ν *Pareto dominates* μ*, then there exists a sequence of cycles* $(\gamma_1, \ldots, \gamma_k)$ *such that:*

[5] Note that since $(\nu, \bar{\lambda})$ is a reshuffle of (μ, λ) such a student l exists for each $i_j \in N_3$.

- γ_1 *appears in* $G(\mu, \lambda)$.
- *for each* $k' \in \{2, \ldots, k\}$, $\gamma_{k'}$ *in* $G(\gamma_{k'-1} \circ \ldots \circ \gamma_1 \circ (\mu, \lambda))$,
- $\gamma_k \circ \gamma_{k-1} \circ \ldots \circ \gamma_1 \circ (\mu, \lambda)$.

Proof. By Lemma 5, we can construct a set of improvement cycles $\Phi = \{\phi_1, \ldots, \phi_q\}$. The result is trivial for the case where all the cycles in Φ appear in $G(\mu, \lambda)$: it follows that there are disjoint cycles in $G(\mu, \lambda)$ and solving them in any order leads to the ν and to some λ' such that (ν, λ') is an *(ex-post) stable* reshuffle of (μ, λ). To prove the alternative case, we assume that none of the cycles in ϕ appears in $G(\mu, \lambda)$. This assumption is without loss of generality because of the following observation. If a cycle $\phi \in \Phi$ appears in $G(\mu, \lambda)$, then this cycle is solved first and $\mu' = \phi \circ \mu$ is obtained. If another cycle $\phi' \in \Phi$ also appears in $G(\mu', \lambda^*)$, by the fact that all the cycles in Φ are disjoint and that if there are two students forming a link in $G(\mu, \lambda)$,and those students do not belong to ϕ, then the link also appears in $G(\mu', \lambda^*)$. Following this logic, whenever a subset of cycles Φ appear in $G(\mu, \lambda)$, these cycles are solved first, and we focus on the case where none of the improvement cycles appear in $G(\mu, \lambda)$.

To show the existence of a cycle in $G(\mu, \lambda)$ first we prove that for any $\phi \in \Phi$ and any $ij \in \phi$, there exists some $k \in I$ such that $kj \in G(\mu, \lambda)$ and $lk \in \phi'$ for some $l \in I$ and $\phi' \in \Phi$. Consider an arbitrary $\phi \in \Phi$ and $ij \in \phi$.

- if $i \in X_{(\mu,\lambda)}(j)$, then $ij \in G(\mu, \lambda)$ by construction. Moreover, i is a part of ϕ, which implies there exists $l \in I$ with $li \in \phi$.
- if $i \notin X_{(\mu,\lambda)}(j)$, there exists a student i' such that $i' \in \tilde{D}_{(\mu,\lambda)}(j)$ and $(i', \lambda^{\mu(j)}(i')) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}) \succsim_{\mu(j)} (i, \lambda^{\mu(j)}(i))$. Let k be , among those students, one such that $(k, \max\{\lambda^{\mu(j)}(k), \lambda^{\mu(j)}(j)\}) \succ_{\mu(j)} (k', \max\{\lambda^{\mu(j)}(k'), \lambda^{\mu(j)}(j)\})$ for each $k' \in D_{\mu,\lambda)}(j)$.[6] Note that $k \in X_{(\mu,\lambda)}(j)$, and therefore $kj \in G(\mu, \lambda)$. Finally, we check that k is in an improvement cycle in Φ. That is there is $\phi' \in \Phi$ such that $lk \in \phi'$ for some $l \in I$. Assume to the contrary that $\mu(k) = \nu(k)$, and $\mu(j)\ P_k\ \mu(k) = \nu(k)$. Note that $k \in X_{(\mu,\lambda)}(j)$, $i \notin X_{(\mu,\lambda)}(j)$, $\nu(i) = \mu(j)$, and $\bar{\lambda}^{\mu(j)}(i) = \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}$). Since $(k, \lambda^{\mu(j)}(k)) \succ_{\mu(j)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$, this is a contradiction, since $(\nu, \bar{\lambda})$ is *(ex-post) stable.* Thus, $\nu(k)\ P_k\ \mu(k)$, which implies that k is in an improvement cycle in Φ.

[6]By our definition of extended priorities the existence of such a student k is ensured. See Remark 1.

Thus, for any student j who is in an improvement cycle $\varphi \in \Phi$, there exists another student k such that $kj \in G(\mu,\lambda)$ and k is in an improvement cycle $\phi' \in \Phi$. Since the set of students in improvement cycles is finite, and each student is pointed at least by another student in N, and there exists a cycle γ_1 in $G(\mu,\lambda)$. Note that for each $ij \in \phi$ such that $ij \notin \gamma_1$, then $ij \notin G(\mu,\lambda)$, and $i \notin X_{(\mu,\lambda)}(j)$.

We next show that the matching $\gamma_1 \circ \mu$ Pareto dominates μ and it is weakly Pareto dominated by ν. Since $\gamma_1 \circ \mu$ solves a cycle in $G(\mu,\lambda)$ clearly $\gamma_1 \circ \mu$ Pareto dominates μ. Hence, we focus on proving that ν (weakly) Pareto dominates $\gamma_1 \circ \mu$. For any $kj \in \gamma_1$ such that $(\gamma_1 \circ \mu)(k) \neq \mu(k)$ note that $(\gamma_1 \circ \mu)(k) = \mu(j)$.

- If $kj \in \phi$ for some $\phi \in \Phi$, then $\nu(k) = \mu(j)$.
- If $kj \notin \phi$ for any $\phi \in \Phi$, we claim that $\nu(k)\ R_k\ \mu(j)$. Suppose that $\mu(j)\ P_k\ \nu(k)$, that is, $k \in \tilde{D}_{(\nu,\bar{\lambda})}(j)$. Consider the student $i \in I$ such that $ij \in \phi$ for some $\phi \in \Phi$, so $\nu(i) = \mu(j)$. By the definition of γ_1, $ij \notin G(\mu,\lambda)$. implies $\bar{\lambda}^{\mu(j)}(i) = \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\}$). Since $kj \in \gamma_1$, $kj \in G(\mu,\lambda)$ and $ij \notin G(\mu,\lambda)$, $(k, \lambda^{\mu(j)}(k)) \succ_{\mu(k)} (i, \max\{\lambda^{\mu(j)}(i), \lambda^{\mu(j)}(j)\})$, which is a contradiction becase $(\nu,\bar{\lambda})$ is *(ex-post) stable*.

Thus, under the matching $\gamma_1 \circ \mu$, each student in γ_1 is better off than under the matching μ and worse off than under the matching ν. Each remaining student is assigned to the same school to which she's assigned under μ which implies that the matching $\gamma_1 \circ \mu$ Pareto dominates μ and is weakly Pareto dominated by ν. Let λ_1 be the allocation of characteristics obtained by solving the cycle γ_1 according to the definition of the SETC algorithm. By the arguments in Lemma 1, $(\gamma_1 \circ \mu, \lambda_1)$ is *(ex-post) stable*. If the extended matching $(\gamma_1 \circ \mu)$ is equivalent to ν the proof is complete. If not we can use the same argument inductively. By Lemma 6, there is a set of distinct improvement cycles, such that the matching ν is obtained by solving these cycles over $\gamma_1 \circ \mu$ solving at each stage a cycle that appears in the graph defined by the SETC algorithm. $\square$

5 Fully Transferable Characteristics

Theorem 1 is a general result without any reference on the construction of schools priorities. In this section we discuss the implications for specific definitions of priorities.

In the case that the transferable characteristics determine completely schools priorities, when a student participates in an improvement cycle, it is equivalent to the fact of giving up completely the priorities the student have for a position at that school. With that intuition in mind, we propose a restricted domain of priorities that are completely defined by the transferable characteristics.

Fully Transferable Extended Priorities. For each $i, i', j, j'j \in I$ and $s \in S$, for each $\lambda^s, \bar{\lambda}^s \in \mathcal{L}^s$: $(i, \lambda^s) \succ_s (i', \bar{\lambda}^s)$ if and only if $(j, \lambda^s) \succ_s (j', \bar{\lambda}^s)$.

Under fully transferable priorities, the analysis of the algorithms in the SETC is simpler, since any student that desires the position of another student can obtain it with the exchange of the transferable characteristics.

Lemma 7. *Let (μ, λ) be an* (ex-post) stable *extended matching and $G(\mu, \lambda)$ the (directed) application graph associated with (μ, λ). If schools' extended priorities are fully transferable and $i \in \tilde{D}_{(\mu,\lambda)}(j)$, then $ij \in G(\mu, \lambda)$.*

Proof. Let $s = \mu(j)$. Since $\in \tilde{D}_{(\mu,\lambda)}(j)$, $s \; P_i \; \mu(i)$. Since (μ, λ) is *(ex-post) stable*, for each $j' \neq i$ such that $s \; P_{j'} \; \mu(j')$, $(j, \lambda^s(j)) \succ_s (j', \lambda^s(j'))$. Therefore, since s extended priorities are fully transferable, $(i, \max\{\lambda^s(i), \lambda^s(j)\}) \succ_s (j', \lambda^s(j'))$ and $i \in X_{(\mu,\lambda)}(j)$. □

The previous lemma implies that under fully transferable priorities improvement cycles do not need the participation of students who transfer their characteristic but remain assigned to the same school.

At this point, before we introduce a specific selection of cycles in SETC algorithms, we need additional notation.

Let (μ, λ) be an extended matching. Define the graph $\tilde{G}(\mu, \lambda) \subset G(\mu, \lambda)$ as the restriction of $G(\mu, \lambda)$ where students only point to students assigned to different schools. Hence, $E = I$ and V are such that $ij \in \tilde{G}(\mu, \lambda)$ if and only if $i \in \tilde{D}_{(\mu,\lambda)}(j)$ and $i \in X_{(\mu,\lambda)}(j)$.

Let $T_0(\mu) = I$ and recursively for each $k \geq 1$

$$B_k(\mu) = \{i \in T_{k-1}(\mu) \mid \text{ for each } j \in T_{k-1}(\mu), \; \mu(i) \; R_i \; \mu(j)\},$$

and $T_k(\mu) = T_{k-1}(\mu) \setminus B_k(\mu)$. Let k^* be the smallest integer such that $B_{k^*}(\mu) = \{\varnothing\}$, and $T(\mu) \equiv T_{k^*}$. The set $B_1(\mu)$ contains the students that are assigned at μ to their preferred school. Hence for each $j \in B_1(\mu)$ there is not j' such that $jj' \in \tilde{G}(\mu, \lambda)$.

Recursively, for each $k < k*$ and $j \in B_k(\mu)$, $jj' \in \tilde{G}(\mu, \lambda)$ implies that $j' \in B_{k'}(\mu)$ for some $k' < k$. This immediately implies that students in $I \setminus T(\mu)$ cannot participate in any Pareto improvement cycle. Moreover, if $T(\mu) = \{\varnothing\}$, then the extended matching (μ, λ) is Pareto efficient. Finally, for each extended matching (μ, λ) define the graph $G'(\mu, \lambda) \subset \tilde{G}(\mu, \lambda)$, such that each $ij \in G'(\mu, \lambda)$ if and only if $i, j \in T(\mu)$, $ij \in \tilde{G}(\mu, \lambda)$, and for each $j' \in T(\mu)$, $\mu(j)\ R_i\ \mu(j')$.

Lemma 8. *Let schools' extended priorities be fully transferable. If $T_k(\mu) \neq \{\varnothing\}$, then there is a Pareto Improvement cycle $\phi \in G'(\mu, \lambda)$ such that for each $ij \in \phi$, and each $j' \in T(\mu)$, $\mu(j)\ R_i\ \mu(j')$.*

Proof. Note that for each $i \in T(\mu)$ there is $j \in T(\mu)$ such that $i \in \tilde{D}_{(\mu,\lambda)}(j)$ and by the previous lemma, $i \in X_{(\mu,\lambda)}(j)$. Note that each $i \in T(\mu)$ points to at least some $j \in T(\mu)$ such that $\mu(i) \neq \mu(j)$. Since $T(\mu)$ is finite, there is at least one cycle in $G'(\mu, \lambda)$. □

Remark 3. *There can be several improvement cycles in $G'(\mu, \lambda)$ but either they are completely disjoint ($ij \in \phi$ implies that for no j', $ij' \in \phi'$) or if $ij \in \phi$ and $ij' \in \phi'$, $\mu(j) = \mu(j')$.*

The previous lemma shows that under fully transferable characteristics the logic of the TTC algorithm can be applied to find *(ex-post) stable* Pareto improvement cycles. This logic allows us to define a subclass of SETC algorithms that perform Pareto improvement cycles that are obtained following this TTC logic.

Top Trade SETC Algorithm:

Step 0: Let (μ_0, λ_0) be a stable extended matching.

Step $k \geq 1$: Given an extended matching $(\mu_{k-1}, \lambda_{k-1})$,

(k.1) if there is no improvement cycle in $G(\mu_{k-1}, \lambda_{k-1})$, then the algorithm terminates and $(\mu_{k-1}, \lambda_{k-1})$ is the matching obtained,

(k.2) otherwise, solve one of the improvement cycles in $\phi \in G'(\mu_{k-1}, \lambda_{k-1})$, and let $\mu_k = \phi_k \circ \mu_{k-1}$, and λ_k be defined correspondingly to the definition of SETC.

The Top Trade SETC applies the TTCM for each stable allocation. With this observation our final result relating our framework and Kesten (2010) immediately follows.

Theorem 2. *Let μ_0 be the SOSM and (μ, λ) an outcome of and Top Trade SETC algorithm under μ_0, then μ is the matching obtained with the TTCM under the initial allocation of seats μ_0.*

Proof. By Lemmata 7 and 8, under monotonous and fully transfer preferences, the Top Trade SETC algorithm is well defined. At each stage of the algorithm, there is a group of students who obtain a seat at their best preferred available school till the stage where no Pareto improvement is possible, therefore μ is the At and it coincides with the application of the TTCM for any stable matching μ. If the initial matching is the SOSM, the stable matching selected by the Top Trade SETC coincides with the application of the TTCM after the selection of the SOSM. □

We conclude by relating fully transferable priorities with students' consent and EADAM (Kesten, 2010). The motivation behind the EADAM is to explore the source of inefficiency of the SOSM due to fairness constraints and improve it on the efficiency dimension. An important observation made by Kesten (2010) is that the priority of student i at school s might not help her to get a better school under the SOSM at all. If this is the case, giving i the lowest priority at s instead of her current priority would not change her assignment and the DA would possibly select a matching that Pareto dominates the SOSM with the original priorities. Motivated by this observation, Kesten (2010) introduces the EADAM in a setting that allows students to consent for the violation of their own priorities. Under fully transferable priorities, Pareto improvements from an initially stable matching never generates a violation of fairness. This relates to the concept of consent in Kesten (2010). In that paper, if each student consents for priority violation at each school, the *Efficient Adjusted Deferred Acceptance Algorithm* results in the matching obtained with the application of the TTCM after the selection of the SOSM.

Corollary 2. *Let μ_0 be the SOSM and (μ, λ) an outcome of and Top Trade SETC algorithm under μ_0. If schools priorities are monotonous and fully transferable then μ is the outcome of the EADAM when all students consent.*

6 Conclusions

In this paper we generalize the school choice problem by defining schools priorities on (transferable) students characteristics. We define a family of algorithms– Student Exchange with Transferable Characteristics (SETC) class– that starting at a *(ex-post) stable* extended matching produce an *(ex-post) stable* extended matching that is not Pareto dominated by another *(ex-post) stable* extended matching. Moreover, any constrained efficient extended matching that Pareto improves upon a stable extended matching can be obtained via an algorithm in the SETC class. Finally, we show that a particular algorithm in the SETC class selects the outcome obtained with the iterated application of SOSM and TTTC algorithm and coincides with the Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) proposed by Kesten (2010) when all students characteristics are transferable.

Although the focus on this work has been on the application to school choice, there are further natural applications of the model. Recent research on the allocation of medical resources under triage have proved the possibilities of encompassing ethical values with the application of deferred algorithm when some resources are reserve to some groups of individuals (Pathak et al., 2020) in a situation similar to strict priorities. While this work has focused on the allocation of scarce resources when there individuals only care about getting access to one unit, our work provides techniques that allow for Pareto improvements of those allocations, when the ethical considerations may be relaxed and transfers of characteristics (as residence area or tie-breakers) are allowed.

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