

Linear and bilinear Yano's extrapolation theory on
Lorentz spaces

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Introduction

Given an operator T , an important question in harmonic analysis is whether this operator is bounded on L^p , i.e., if there exists a constant $C > 0$ such that

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p}.$$

There are many cases in which the norm of T on L^p is finite for a certain range of p and blows up as p approaches some critical value $p = p_0$. The point p_0 at which T loses its boundedness properties is called an *endpoint*, and it becomes valuable to obtain a suitable estimate for the operator T at such endpoint. For example, the norm of the Hardy-Littlewood maximal operator M in $(0, 1)$ behaves like $(p - 1)^{-1}$, so it is unbounded at the endpoint $p_0 = 1$. Nevertheless, it is still interesting to determine for which integrable functions f , the function Mf is also locally integrable.

Yano's theorem deals with the problem of finding estimates at the endpoint $p_0 = 1$ when the norm of T on L^p for $p > 1$ behaves like $(p - 1)^{-\alpha}$ for some $\alpha > 0$.

The origins of Yano's Extrapolation Theory go back to 1928, when Titchmarsh [98] proved that if \mathcal{C} is the conjugate function on the interval $(0, 1)$

$$\mathcal{C}f(x) := p.v. \int_0^1 f(x - y) \cot y \, dy,$$

then $\mathcal{C}f \in L^1(0, 1)$ whenever $f \log(2 + |f|)^{1+\varepsilon}$ is integrable for some $\varepsilon > 0$. Almost at the same time, Zygmund [103] claimed that the above result should remain true for $\varepsilon = 0$. One year later, Titchmarsh [99] published a second paper where, using that

$$\mathcal{C} : L^p(0, 1) \longrightarrow L^p(0, 1), \quad \|\mathcal{C}\| \lesssim \frac{1}{p - 1},$$

he obtained Zygmund's conclusion. In the same year, Zygmund [104] also proved the result using a different argument.

In 1951, inspired by the ideas of Titchmarsh, Yano [102] published the following theorem.

Theorem 1 (Yano, 1951). *Let (X, μ) and (Y, ν) be finite measure spaces. If T is a sublinear operator such that, for every $1 < p \leq 2$ and for some $\alpha > 0$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu), \quad \|T\| \lesssim \frac{1}{(p-1)^\alpha}.$$

Then,

$$T : L(\log L)^\alpha(\mu) \longrightarrow L^1(\nu),$$

where $L(\log L)^\alpha(\mu) \subset L^1(\mu)$ is the space of functions such that

$$\int_X |f(x)| \left(1 + \log^+ |f(x)|\right)^\alpha d\mu(x) < \infty.$$

The relevance of Yano's theorem also relies on the fact that is optimal in the sense that we cannot expect to obtain a better endpoint estimate with these hypotheses. Indeed, in the particular case of the maximal operator M , for $1 < p \leq 2$,

$$M : L^p(0, 1) \longrightarrow L^p(0, 1), \quad \|M\| \lesssim \frac{1}{p-1},$$

and $Mf \in L^1(0, 1)$ if and only if $f \in L \log L(0, 1)$.

Since its original formulation and its extension to σ -finite measure spaces (see [14]), different versions of Yano's theorem have appeared in the literature, leading to a very complete extrapolation theory.

Some variants of Yano's theorem involve spaces in the hypotheses that differ from those in Theorem 1, such as $L^{p,1}$ or $L^{p,\infty}$, among others. Many of these variants have been motivated by questions related to the convergence of Fourier series. In fact, in 1966 Carleson [15] proved that the almost everywhere convergence of the Fourier series of functions in $L^2(0, 1)$ follows from the boundedness of the Carleson operator \mathcal{C} from $L^2(0, 1)$ into $L^{2,\infty}(0, 1)$. At that time, it was already known (see [69]) that there exist functions in $L^1(0, 1)$ whose Fourier series diverges almost everywhere. However, in 1967, Hunt [56] proved that

$$\mathcal{C} : L^p(0, 1) \longrightarrow L^p(0, 1), \quad \|\mathcal{C}\| \lesssim \frac{1}{(p-1)^2},$$

for $1 < p < \infty$ and, therefore, the almost everywhere convergence of the Fourier series of functions in $L(\log L)^2(0, 1)$ followed from Yano's theorem. To improve this space, a possible approach is to revisit Hunt's proof and observe that $L^p(0, 1)$ estimates on \mathcal{C} stem from the fact that

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(0,1)} \lesssim \frac{1}{p-1}|E|^{1/p}, \quad (1)$$

for every measurable set E . This type of estimate suggests that one might attempt to establish an analogue of Yano's theorem starting from restricted weak-type estimates of the form

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu), \quad \|T\| \lesssim \frac{1}{(p-1)^\alpha},$$

so that the norm of \mathcal{C} grows like $(p-1)^{-1}$ rather than $(p-1)^{-2}$. In 1996, Antonov [3] used (1) to establish the almost everywhere convergence of the Fourier series for functions in $L \log L \log \log L(0, 1)$. Following Antonov's ideas, Carro and Martín [24] proved an analogue of Yano's theorem with restricted weak-type estimates, concluding convergence in the same space. Other authors (see [91], [92], [17]) have also developed several versions of Yano's result using different types of spaces in the hypotheses.

Regarding the endpoint, there also exist variants of Yano's theorem where $p_0 \neq 1$. Moreover, other endpoints naturally arise in the study of classical operators in harmonic analysis. For instance, the norm of the Hilbert transform is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, blows up as p approaches ∞ , and is unbounded on $L^\infty(\mathbb{R})$. To address this type of problem, in 1959, Zygmund [105, p. 119] proved a dual version of Yano's result for cases where the endpoint is $p_0 = \infty$. Furthermore, any $1 \leq p \leq \infty$ can act as the endpoint for a particular operator. This is the case, for example, of the Bochner-Riesz operator of order α in \mathbb{R}^2 , denoted by \mathcal{B}^α , with $0 < \alpha \leq 1/2$. Carleson and Sjölin [16] showed in 1972 that

$$\mathcal{B}^\alpha : L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2) \quad \text{if and only if} \quad \frac{4}{3+2\alpha} < p < \frac{4}{1-2\alpha}.$$

Hence, we have that $p_0 = \frac{4}{3+2\alpha} > 1$ and $p_0 = \frac{4}{1-2\alpha} < \infty$. Therefore, there exist other versions of Yano's theorem that cover these situations (see [27], [1], [2]).

Finally, there are also extensions of Yano's theorem concerning on how the operator norm behaves. Indeed, there exist examples of operators for which the best-known norm does not grow like a power function as $(p-1)^{-\alpha}$. One notable case is the Carleson lacunary operator \mathcal{C}_{lac} (see (3.13) for a definition), which satisfies, for $1 < p \leq 2$,

$$\mathcal{C}_{\text{lac}} : L^p(\mathbb{R}) \longrightarrow L^{p,\infty}(\mathbb{R}), \quad \|\mathcal{C}_{\text{lac}}\| \lesssim \log_1 \left(\frac{1}{p-1} \right).$$

Generalizations of Yano's results with more general blow up behaviour of the operator norm can be found in [92], [1], [2].

We should also mention that, in the 1990s, Jawerth and Milman constructed an abstract theory of Yano's extrapolation, extending these results to the setting of compatible couples of Banach spaces using techniques from interpolation theory (see [59–63], [81]).

We point out that most of the proofs of Yano's extrapolation results in the context of Lorentz spaces follow a common structure, which we introduce here as the *ODR technique*. It consists of three steps.

Step 1. Optimization of Hypotheses. *Obtain an endpoint estimate for a subset \mathcal{O} (to be determined) of the space $L^{p_0}(\mu)$.* For example, in the classical Yano's theorem,

$$\mathcal{O} = \{f \in L^1(\mu) \cap L^\infty(\mu) : \|f\|_{L^\infty(\mu)} \leq 1\},$$

and the endpoint estimate obtained is

$$\|Tf\|_{L^1(\nu)} \lesssim \|f\|_{L^1(\mu)} \left(1 + \log^+ \frac{1}{\|f\|_{L^1(\mu)}}\right)^\alpha, \quad \forall f \in \mathcal{O}.$$

Step 2. Decomposition. *Decompose a function $f \in L^{p_0}(\mu)$ as a linear combination of functions in \mathcal{O} .* In the case of Yano's theorem, one considers

$$f = \sum_{i \in \mathbb{Z}} 2^i f_i, \quad \text{where} \quad f_i = \frac{f \chi_{\{2^{i-1} \leq |f| < 2^i\}}}{2^i}.$$

Step 3. Reconstruction. *Combine properly the information from Steps 1 and 2.* In our example, since T is sublinear, for any $f \in L^1(\mu)$, applying the estimate from Step 1 to each f_i yields

$$\|Tf\|_{L^1(\nu)} \lesssim \sum_{i \in \mathbb{Z}} 2^i \|Tf_i\|_{L^1(\nu)} \lesssim \sum_{i \in \mathbb{Z}} 2^i \|f_i\|_{L^1(\mu)} \left(\log_1 \frac{1}{\|f_i\|_{L^1(\mu)}}\right)^\alpha \lesssim \|f\|_{L(\log L)^\alpha(\mu)}.$$

As we shall see, Rubio de Francia's extrapolation theorem and the theory of Muckenhoupt weights provide a natural source of examples for the results developed in this book. It is well known that a weight is a locally integrable, non-negative function defined on \mathbb{R}^n and taking values in $(0, \infty)$. In particular, the Muckenhoupt classes A_p play a central role in harmonic analysis. As Muckenhoupt [82] proved in 1972, these weights precisely characterize the boundedness of the Hardy–Littlewood

maximal operator M : the operator M is bounded on $L^p(w)$ for some $p > 1$ if and only if $w \in A_p$. This characterization is significant because the Hardy-Littlewood maximal operator has often served as a model for the study of other operators. Indeed, in the years immediately following, it was shown that many other classical operators satisfy the same type of weighted inequalities, including the Hilbert transform or the Bochner-Riesz operator at the critical index.

If one is interested in studying this type of estimates, Rubio de Francia theorem (see [88], [89]) provides a useful method for obtaining A_p weighted estimates on L^p spaces for $1 < p < \infty$, based on information from a single point. More precisely, if for some fixed p and every $w \in A_p$, the operator T is bounded on $L^p(w)$, then for every $1 < p < \infty$ and every $w \in A_p$, T is bounded on $L^p(w)$.

Despite this, the endpoints $p_0 = 1$ and $p_0 = \infty$ can not be reached in general. Moreover, one cannot expect to deduce an estimate on $L^{p_0}(w)$ at these endpoints, even in the unweighted case where $w \equiv 1$. For example, the Hilbert transform, to which Rubio de Francia's extrapolation theorem applies, is unbounded on $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$. However, an appropriate reformulation of Rubio de Francia's theorem (see [20]) states that if for some fixed $1 < q < \infty$ and every $w \in A_q$, the operator T is bounded on $L^q(w)$ with a norm that behaves like $[w]_{A_q}^\alpha$ for some $\alpha > 0$ (where $[w]_{A_q}$ is the A_q constant of the weight w as defined in (1.1)), then for every $1 < p < q$ and every $w \in A_1$, T is bounded on $L^p(w)$ with a norm that blows up like $(p-1)^{-\alpha(q-1)}$. Therefore, by Yano's theorem, it follows that for every $w \in A_1$, the operator T is bounded from $L(\log L)^{\alpha(q-1)}$ into a specific Lorentz space. Hence, Yano's theory can be used to obtain A_1 weighted estimates at the endpoint $p_0 = 1$, and, using an analogous argument involving Zygmund's extrapolation result, one can also obtain A_∞ weighted estimates at the endpoint $p_0 = \infty$, as shown in [30].

All the results discussed above belong to the linear framework. Nevertheless, many fundamental questions in harmonic analysis concern operators acting on pairs of functions, and it is therefore natural to extend the previous theory to the bilinear setting. In this monograph we shall focus on this context, where several classical operators arise naturally and new endpoint phenomena appear. More precisely, for $1 \leq p_1, p_2 \leq \infty$, one may consider estimates of the bilinear operator B at the point $(p_1, p_2; p)$ in the form

$$B : L^{p_1} \times L^{p_2} \rightarrow L^p, \quad \text{where} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

The analysis of bilinear operators became an important topic in the 1970s with the work of Coifman and Meyer (see [36], [37]), who studied commutators of bilinear singular integrals, such as the bilinear Hilbert transform. The works of Lacey and Thiele (see [71], [72]) on the boundedness of this operator attracted significant

attention to this research area, leading to the development of numerous results in the setting of multilinear operators, including those by Kenig and Stein (see [66]), Muscalu, Tao, and Thiele [83], Grafakos and Li [50], and Grafakos and Torres (see [52], [53]).

To obtain estimates for bilinear operators, many linear techniques have been extended to this new bilinear context, such as bilinear Rubio de Francia extrapolation theory, which has been widely studied (see [51], [84], [76], [75]). Some attempts to extend Yano's theory to the bilinear context in the abstract setting appear in the work of Jawerth and Milman [63] (see also [5]).

One of the main particularities of the bilinear theory is the fact that, if $p < 1$, the space L^p is quasi-Banach, and it is not generally true that for any decomposition $f = \sum_i f_i$,

$$\left\| \sum_i f_i \right\|_{L^p} \leq \sum_i \|f_i\|_{L^p}.$$

Thus, the Step 3 of the ODR technique entails some difficulties. Observe, however, that the condition $p < 1$ is common in the bilinear setting, as shown in Figure 1, where the square represents all possible points with $1 \leq p_1, p_2 \leq \infty$, and the shaded area indicates those points for which $p < 1$.

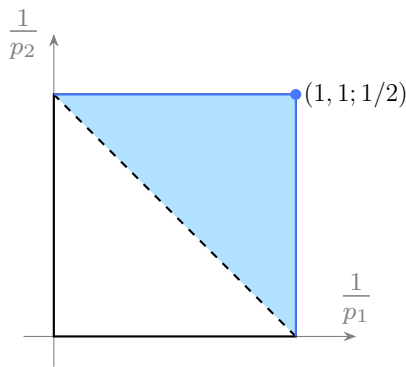


Figure 1: $(p_1, p_2; p)$ such that $1 \leq p_1, p_2 \leq \infty$ and $p < 1$.

As we will see later, another particularity of the bilinear context is that different results arise depending on how we reach the endpoint. Specifically, we may have hypotheses valid for all points near the endpoint, or only for those lying on a ray or even along a curve (as seen in [87]).

Concerning the operator norm, it may behave as a function of p alone or as a function of both p_1 and p_2 . For example, the analogue of Yano's result in the

bilinear context could involve hypotheses such as

$$B : L^{p_1} \times L^{p_2} \longrightarrow L^p, \quad \|B\| \lesssim \frac{1}{(p_1 - 1)^{\alpha_1} (p_2 - 1)^{\alpha_2}},$$

for some $\alpha_1, \alpha_2 > 0$ and for every $1 < p_1, p_2 \leq 2$, or alternatively,

$$B : L^{p_1} \times L^{p_2} \longrightarrow L^p, \quad \|B\| \lesssim \frac{1}{\left(p - \frac{1}{2}\right)^\alpha},$$

for some $\alpha > 0$ and for every $1 < p_1, p_2 \leq 2$, generating different endpoints in each case. If the norm of B follows the first form, there is not just a single endpoint $(1, 1; 1/2)$, but rather all points with $p_1 = 1$ and $1 < p_2 \leq 2$, or $p_2 = 1$ and $1 < p_1 \leq 2$, also act as endpoints.

Contributions of this monograph. The first contribution of this book is to present a comprehensive collection of results from Yano's linear theory within the context of Lorentz spaces, with proofs made as direct as possible, avoiding technicalities from the abstract theory. In particular, we have standardized and generalized the ODR technique, enabling most proofs to follow straightforwardly from it. We extend all these results to the case where the operator norm blows up as an *admissible* function, a broader class than power functions which includes all operator norm behaviours observed in the literature. We have covered the different types of spaces that could appear in the hypotheses and addressed all possible endpoints that may arise in each case.

Some of these cases were not previously addressed in the literature. In particular, the new case where our hypotheses are weak-type estimates of the form

$$T : L^p(\mu) \longrightarrow L^{p,\infty}(\nu)$$

and the endpoint is $1 < p_0 < \infty$ appears in Section 3.2. In Section 3.5, we extend the existing proof to handle the case with strong $L^{p,\infty}$ hypotheses of the form

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

when the endpoint is $p_0 > 1$ (the original proof was for $p_0 = 1$), and we provide an alternative proof for the case with endpoint $p_0 < \infty$ and we have the hypotheses for $p < p_0$. We devote Section 3.6 to address the case of having a fixed target space for all the hypothesis estimates, i.e., we consider estimates of the form

$$T : L^p(\mu) \longrightarrow \mathcal{X},$$

where \mathcal{X} is fixed for all p .

We include examples of applications of the different theorems to obtain endpoint estimates for a broad range of operators, including Fourier multipliers such as the Marcinkiewicz and Mihlin–Hörmander multipliers, and classical harmonic operators like the Hilbert transform, the spherical maximal function, the rough maximal operator, the Carleson operator, and the intrinsic square function. We derive estimates not only for the unweighted case but also for different weights.

Our second goal is the extension of Yano’s extrapolation theory to the bilinear setting, in the context of Lorentz spaces. Following the structure of the linear case, we have considered different types of hypotheses, norms that blow up like an admissible function, and various types of endpoints.

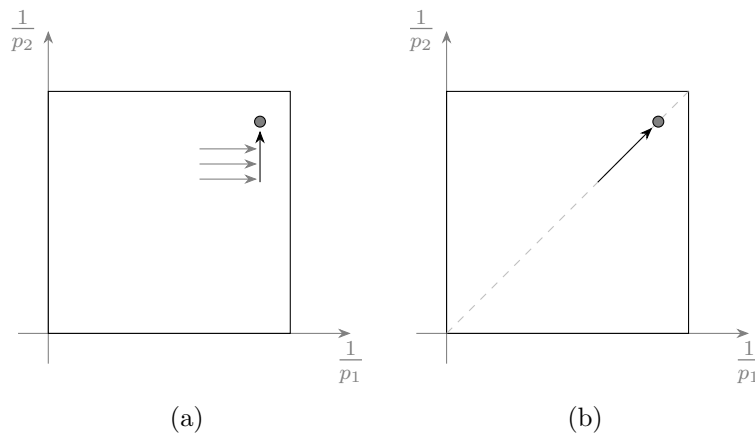


Figure 2: Techniques to approach an endpoint in the bilinear setting.

In contrast to the linear case, in this setting, we must also take into account the approach to the endpoint. We have used two techniques: linearization via an off-diagonal result (Figure 2(a)) and extrapolation by rays (Figure 2(b)). We present, as examples, two of our theorems that illustrate how we can reach the point $(1, 1; 1/2)$ in the context of finite measure spaces.

Theorem 2. *Let (X_1, μ_1) , (X_2, μ_2) , and (Y, ν) be finite measure spaces. If B is a bisublinear operator such that, for every $(p_1, p_2; p)$ with $1 < p_1, p_2 \leq 2$ and for some $\alpha_1, \alpha_2 > 0$,*

$$B : L^{p_1}(\mu_1) \times L^{p_2}(\mu_2) \longrightarrow L^p(\nu), \quad \|B\| \lesssim \frac{1}{(p_1 - 1)^{\alpha_1} (p_2 - 1)^{\alpha_2}},$$

then, for every $\varepsilon > 0$,

$$B : L(\log L)^{\alpha_1+1+\varepsilon}(\mu_1) \times L(\log L)^{\alpha_2+1+\varepsilon}(\mu_2) \longrightarrow L^{\frac{1}{2}, \infty}(\nu).$$

Theorem 3. *Let (X_1, μ_1) , (X_2, μ_2) , and (Y, ν) be finite measure spaces. If B is a bisublinear operator such that, for every $1/2 < p \leq 1$, for some $\alpha > 0$,*

$$B : L^{2p}(\mu_1) \times L^{2p}(\mu_2) \longrightarrow L^p(\nu), \quad \|B\| \lesssim \frac{1}{(p - \frac{1}{2})^\alpha},$$

then, for every $\varepsilon > 0$,

$$B : L(\log L)^{\alpha+1+\varepsilon}(\mu_1) \times L(\log L)^{\alpha+1+\varepsilon}(\mu_2) \longrightarrow L^{\frac{1}{2}, \infty}(\nu).$$

In addition, we apply bilinear Yano's extrapolation to obtain A_p weighted estimates at the endpoints not covered by the bilinear Rubio de Francia theorem (see [51]), which provides estimates of the form

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w), \quad w_1 \in A_{p_1}, w_2 \in A_{p_2},$$

where $w = w_1^{p/p_1} w_2^{p/p_2}$, at each point $(p_1, p_2; p)$ with $1 < p_1, p_2 < \infty$, based on information from a single point $(r_1, r_2; r)$ for each $w_1 \in A_{r_1}$ and $w_2 \in A_{r_2}$. By employing Yano's bilinear theory, we obtain weighted estimates at the remaining points: $(1, p_2; p)$, $(p_1, 1; p)$, $(1, \infty; 1)$, $(\infty, 1; 1)$, $(\infty, p_2; p)$, $(p_1, \infty; p)$, and $(\infty, \infty; \infty)$.

We also provide several examples of operators for which this theory yields new estimates, such as the bilinear Bochner-Riesz operator at the critical index and rough singular operators.

Outline of the book We present a brief summary of the contents of each chapter.

► **Chapter 1.** In this chapter, we establish the theoretical framework used throughout the book. We begin in Section 1.1 by setting the conventions and notations that will be assumed. Section 1.2 introduces the concepts of distribution function, decreasing rearrangement, and maximal function, along with their main properties. Our notion of admissible function is introduced in Section 1.3. Section 1.4 is devoted to defining the function spaces used in the manuscript. Finally, in Section 1.5, we define the classes of Muckenhoupt weights, and state the extrapolation theorems concerning them.

► **Chapter 2.** This chapter summarizes all the technical results related to the ODR technique that are needed for linear and bilinear Yano's extrapolation theory. In Section 2.1, we present the results concerning Step 1. In Section 2.2, we introduce the different types of decompositions we shall use, and Section 2.3 is devoted to reconstruction results.

► **Chapter 3.** The third chapter contains a review of Yano's linear extrapolation in the context of Lorentz spaces. The chapter is divided into six sections, according to the different type of spaces involved in the hypotheses: strong L^p hypotheses (Section 3.1), weak hypotheses (Section 3.2), restricted weak-type hypotheses (Section 3.3), hypotheses for characteristic functions (Section 3.4), strong $L^{p,\infty}$ hypotheses (Section 3.5), and hypotheses with fixed target space in Section 3.6. All the extrapolation proofs presented follow the ODR technique, except for those in Section 3.4 and one in Section 3.5.

► **Chapter 4.** In this chapter, we develop a multilinear extension of Yano's extrapolation theory. As in the linear case, we distinguish different types of hypotheses. The chapter is structured as follows: in Section 4.1, we shall treat the case when the target space at the endpoint estimate is Banach and Section 4.2 will be dedicated to the quasi-Banach context. Our proofs will follow the ODR technique and, in both settings, we mainly use linearization in each variable (see Theorems 4.1.3, 4.1.6 and 4.2.5). We also present results where we reach the endpoint through rays (see Theorems 4.1.8, 4.1.9, 4.2.7 and 4.2.8), and we treat the endpoint $(1, \infty; 1)$ separately (see Theorem 4.1.10 and Theorem 4.2.9).

► **Chapter 5.** The final chapter explores the relationship between bilinear Rubio de Francia and Yano's extrapolation, and uses bilinear Yano's extrapolation to get A_p weighted estimates. We then use these techniques to obtain new estimates for bilinear rough singular operators (see Theorem 5.1.1), the bilinear Bochner-Riesz operator at the critical index (see Theorem 5.2.1), and bilinear Fourier multipliers of bounded variation (see Theorem 5.3.1). Finally, in Theorems 5.4.1 and 5.4.2, we derive new estimates for bilinear Hardy operators with power weights using only bilinear Yano's theory.

Notations

C, C_1, C_2	positive universal constants
$a \lesssim b$	$a \leq Cb$ for some constant $C > 0$
$a \gtrsim b$	$a \geq Cb$ for some constant $C > 0$
$a \approx b$	$a \lesssim b$ and $b \lesssim a$
λ_f^μ	the distribution function of f in the measure space (X, μ)
f_μ^*	the decreasing rearrangement of f in the measure space (X, μ)
f_μ^{**}	the maximal function of f in the measure space (X, μ)
$f(s^+)$	$\lim_{t \rightarrow s^+} f(t)$
\inf	the infimum of a set
\sup	the supremum of a set
ess sup	the essential supremum of a set
\mathbb{N}	the set of natural numbers
\mathbb{Z}	the set of integers
\mathbb{Z}^-	the set of negative integers (without 0)
\mathbb{R}	the set of real numbers
\mathbb{R}^+	the set of strictly positive real numbers
\mathbb{R}^n	the n -dimensional Euclidean space
\mathbb{S}^{n-1}	the $(n - 1)$ -dimensional unit sphere $\{x \in \mathbb{R}^n : x = 1\}$
\mathbb{T}	the unit circle identified with the interval $[0, 1]$
$B(x, R)$	the ball with radius $R > 0$ centered at x
χ_E	the characteristic function of the set E
$ E $	the Lebesgue measure of $E \subset \mathbb{R}^n$

$\log s$	the logarithm with base e of $s > 0$
$\log^+ s$	$\max \{0, \log s\}$ for $s > 0$
$\log_n s$	$1 + \underbrace{\log^+ \dots \log^+}_n s$
$\varphi, \varphi_1, \varphi_2$	admissible functions
$I_{\varphi,p}(s)$	$s^{1/p} \varphi(\log_1 \frac{1}{s})$ for $s > 0$ and $0 < p < \infty$
$J_{\varphi,p}(s)$	$s^{1/p} \varphi(\log_1 s)$ for $s > 0$ and $0 < p < \infty$
$C^\infty(\mathbb{R}^n)$	the space of smooth functions from \mathbb{R}^n to \mathbb{R}
$\partial_j^m f$	the m -th partial derivative of $f(x_1, \dots, x_n)$ with respect to x_j
$\partial^\beta f$	$\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$
$\mathcal{S}(\mathbb{R}^n)$	$\{f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} x^\alpha \partial^\beta f(x) < \infty\}$
$L^p(\mu)$	the Lebesgue space over the measure space (X, μ)
$L^p(X)$	the Lebesgue space over the measure space (X, \cdot)
$L^1_{\text{loc}}(\mu)$	the space of functions in $L^1(\mu)$ over (μ, K) for any compact set $K \subset X$
$L^1_{\text{loc}}(X)$	the space of functions in $L^1(K)$ for any compact set $K \subset X$
$L^{p,\infty}(\mu)$	the weak Lebesgue space over the measure space (X, μ)
$L^{p,\infty}(X)$	the weak Lebesgue space over the measure space (X, \cdot)
BMO	the space of bounded mean oscillation functions
v, v_0, v_1, v_2	weights defined in \mathbb{R}^+
$V(t)$	$\int_0^t v(s) ds$
w	a weight defined in \mathbb{R}^n
$w(E)$	$\int_E w(x) dx$
$[w]_{A_p}$	the A_p constant of the weight w
$[w]_{A_p^{\mathcal{R}}}$	the $A_p^{\mathcal{R}}$ constant of the weight w
T_m	the operator associated to the Fourier multiplier m
M	the Hardy-Littlewood maximal operator
M_Ω	the rough maximal operator
\mathcal{M}	the spherical maximal operator
H	the Hilbert transform
\mathcal{C}	the Carleson operator
\mathcal{C}_{lac}	the Carleson lacunary operator
G_α	the intrinsic square function
B_H	the bilinear Hilbert transform
\mathcal{H}_v	the weighted bilinear Hardy operator
\mathcal{B}^α	the bilinear Bochner-Riesz operator of index α
B_m	the operator associated to the bilinear Fourier multiplier m

CHAPTER 1

Preliminaries

In this chapter, we establish the theoretical framework used throughout the book. We begin in Section 1.1 by setting the conventions and notations that will be assumed. Section 1.2 introduces the concepts of distribution function, decreasing rearrangement, and maximal function, along with their main properties. Our notion of *admissible function* is introduced in Section 1.3. Section 1.4 is devoted to defining the function spaces used in the text. Finally, in Section 1.5, we define the classes of Muckenhoupt weights, and state the extrapolation theorems concerning them.

1.1 Conventions

We use capital C to denote positive universal constants that do not depend on any important parameter. As usual, we write $a \lesssim b$ (or $a \gtrsim b$) if there exists a constant $C > 0$ such that $a \leq Cb$ (or $a \geq Cb$). If both $a \lesssim b$ and $a \gtrsim b$ hold, we write $a \approx b$. In particular, if f and g are functions, we write $f \lesssim g$ if $f(s) \lesssim g(s)$ for every s in the domain.

We denote by $\log s$ the logarithm of $s > 0$ with base e , and $\log^+ s = \max\{0, \log s\}$. For $n \in \mathbb{N}$, we write

$$\log_n s = 1 + \underbrace{\log^+ \dots \log^+}_n s.$$

Recall that a function f defined on a subset of \mathbb{R} is *essentially increasing* (respectively, *essentially decreasing*) if there exists an increasing function (respectively, a decreasing function) g , defined on the same subset, such that $f \approx g$.

Given a pair of quasi-normed function spaces \mathcal{X} and \mathcal{Y} , we write $T : \mathcal{X} \rightarrow \mathcal{Y}$ if the operator T is well-defined from \mathcal{X} to \mathcal{Y} and

$$\|T\|_{\mathcal{X} \rightarrow \mathcal{Y}} := \sup_{f \in \mathcal{X}} \frac{\|Tf\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}} < \infty.$$

If \mathcal{X} and \mathcal{Y} are clear from the context, we simply write $\|T\|$. Here, \mathcal{X} is referred to as the domain space, and \mathcal{Y} as the target space. If \mathcal{X}_1 , \mathcal{X}_2 , and \mathcal{Y} are quasi-normed function spaces, we write $B : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$ if the operator B is well-defined from $\mathcal{X}_1 \times \mathcal{X}_2$ to \mathcal{Y} and

$$\|B\|_{\mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}} := \sup_{\substack{f \in \mathcal{X}_1 \\ g \in \mathcal{X}_2}} \frac{\|B(f, g)\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}_1} \|g\|_{\mathcal{X}_2}} < \infty.$$

An operator T is said to be sublinear if, for every pair of functions f, g and every $a \in \mathbb{R}$, we have $|T(f + g)| \leq |Tf| + |Tg|$ and $|T(af)| = |a||Tf|$. Similarly, we say that B is bisublinear if it is sublinear in each variable.

1.2 Distribution function and decreasing rearrangement

Given a non-atomic and σ -finite measure space (X, μ) , we define the distribution function, the decreasing rearrangement, and the maximal function of a μ -measurable function f . We refer to [8, Chapter 2].

Definition 1.2.1. The *distribution function* of a μ -measurable function f is defined by

$$\lambda_f^\mu(s) := \mu(\{x \in X : |f(x)| > s\}), \quad s \geq 0.$$

Definition 1.2.2. The *decreasing rearrangement* of a μ -measurable function f is defined by

$$f_\mu^*(t) := \inf\{s \geq 0 : \lambda_f^\mu(s) \leq t\}, \quad t \geq 0.$$

Definition 1.2.3. The *maximal function* of a μ -measurable function f is defined by

$$f_\mu^{**}(t) := \frac{1}{t} \int_0^t f_\mu^*(s) ds, \quad t > 0.$$

When the measure μ is not relevant in the context, we omit it and simply write λ_f , f^* , and f^{**} , respectively. We now state some elementary properties of these functions.

Proposition 1.2.4. *Let f, g be two μ -measurable functions. Then:*

- (i) λ_f^μ, f_μ^* , and f_μ^{**} are nonnegative, decreasing functions.
- (ii) λ_f^μ and f_μ^* are right-continuous functions, and f_μ^{**} is continuous whenever $f_\mu^* \in L_{\text{loc}}^1(\mathbb{R}^+)$.
- (iii) If $|g| \leq |f|$ almost everywhere, then $\lambda_g^\mu \leq \lambda_f^\mu$, $f_g^* \leq f_f^*$, and $f_g^{**} \leq f_f^{**}$.
- (iv) For every $a \in \mathbb{R}$, $\lambda_{af}^\mu(s) = \lambda_f^\mu(s/|a|)$, $(af)_\mu^* = |a| f_\mu^*$, and $(af)_\mu^{**} = |a| f_\mu^{**}$.
- (v) We have

$$\lambda_{f+g}^\mu(s_1 + s_2) \leq \lambda_f^\mu(s_1) + \lambda_g^\mu(s_2), \quad s_1, s_2 \geq 0,$$

and

$$(f + g)_\mu^*(t_1 + t_2) \leq f_\mu^*(t_1) + g_\mu^*(t_2), \quad t_1, t_2 \geq 0.$$

- (vi) $(f + g)_\mu^{**} \leq f_\mu^{**} + g_\mu^{**}$.
- (vii) $f_\mu^* \leq f_\mu^{**}$.
- (viii) For every $0 < p < \infty$, $(|f|^p)^* = (f^*)^p$,

$$\int_X |f|^p d\mu = p \int_0^\infty s^{p-1} \lambda_f^\mu(s) ds = \int_0^\infty (f_\mu^*(t))^p dt,$$

and, for $p = \infty$,

$$\operatorname{ess\,sup}_{x \in X} |f(x)| = \inf\{s : \lambda_f^\mu(s) = 0\} = f_\mu^*(0).$$

The following formula from [29] will be useful for our purposes.

Proposition 1.2.5. *Let $\phi \in L_{\text{loc}}^1(\mathbb{R}^+)$. Then, for $0 < p < \infty$,*

$$\int_0^\infty (f_\mu^*(t))^p \phi(t) dt \approx \int_0^\infty s^{p-1} \left(\int_0^{\lambda_f^\mu(s)} \phi(t) dt \right) ds.$$

1.3 Admissible functions

We define the following set of functions as extensions of power functions.

Definition 1.3.1. An increasing, absolutely continuous, and derivable function

$$\varphi : [1, \infty) \longrightarrow [1, \infty)$$

is said to be *admissible* if:

- i) $\varphi(1) = 1$,
- ii) $\lim_{t \rightarrow \infty} \varphi(t) = \infty$,
- iii) for every $s, t \geq 1$, $\varphi(st) \leq \varphi(s)\varphi(t)$.

Throughout the book, φ will always denote an admissible function.

Remark 1.3.2. For every admissible function φ , there exists $\beta > 0$ such that, for every $s \geq 1$,

$$\varphi(s) \lesssim s^\beta.$$

To see this, let β such that $\varphi(2) = 2^\beta$. Since, for every $s \geq 1$, there exists $n \geq 0$ such that $2^n \leq s < 2^{n+1}$, we get

$$\varphi(s) \leq \varphi(2^{n+1}) \leq (\varphi(2))^{n+1} = 2^{(n+1)\beta} \leq 2^\beta s^\beta.$$

For our purposes, we can replace iii) with the weaker conditions:

- iv) for every $s, t \geq 1$, $\varphi(st) \lesssim \varphi(s)\varphi(t)$,
- v) for some $\beta > 0$ and every $s \geq 1$, $\varphi(s) \lesssim s^\beta$,

and all the theory still holds.

Example 1.3.3. If $\gamma \geq 0$ and $\beta_1, \dots, \beta_k \geq 0$,

$$\varphi(s) = s^\gamma \prod_{n=1}^k (\log_n s)^{\beta_n}$$

is an admissible function.

1.4 Function spaces

In this section, we define the function spaces that will appear in the text, explain their main properties and the relation between them. Let $0 < p < \infty$, unless indicated otherwise.

The *Lebesgue space* $L^p(\mu)$ is defined as the set of measurable functions such that

$$\|f\|_{L^p(\mu)} := \left(\int_X |f(s)|^p d\mu \right)^{1/p} < \infty.$$

For any weight (a non-negative and locally integrable function) w , if $d\mu = w(s) ds$, we write $L^p(w)$. When $w \equiv 1$, we omit it.

For every weight v defined in \mathbb{R}^+ , the *classical Lorentz space* $\Lambda_\mu^p(v)$ is defined using Lebesgue spaces as $\|f\|_{\Lambda_\mu^p(v)} = \|f_\mu^*\|_{L^p(v)}$. These spaces were first studied by Lorentz when the underlying measure space is $((0, \ell), |\cdot|)$ with $\ell < \infty$ (see [77], [78]). In particular, if for some $0 < p, q < \infty$, $v(t) = t^{\frac{p}{q}-1}$, we have that $\Lambda_\mu^p(v) = L^{q,p}(\mu)$.

The space $\Lambda_\mu^{p,\infty}(v)$ is defined by

$$\|f\|_{\Lambda_\mu^{p,\infty}(v)} := \sup_{t>0} f_\mu^*(t) V(t)^{\frac{1}{p}} < \infty,$$

where, for every $t > 0$, $V(t) := \int_0^t v(s) ds$.

The *logarithmic spaces* that we define below are special cases of $\Lambda_\mu^1(v)$. They are an extension of the *Zygmund space* $L \log L(\mu)$.

Definition 1.4.1. Let $\alpha_1, \dots, \alpha_j \geq 0$ and let φ be an admissible function. The space $L \prod_{k=1}^j (\log_k L)^{\alpha_k} \varphi(\log_j L)(\mu)$ is the set of μ -measurable functions such that

$$\|f\|_{L \prod_{k=1}^j (\log_k L)^{\alpha_k} \varphi(\log_j L)(\mu)} := \int_0^\infty f_\mu^*(t) \prod_{k=1}^j \left(\log_k \frac{1}{t} \right)^{\alpha_k} \varphi \left(\log_j \frac{1}{t} \right) dt < \infty.$$

We also introduce the *weighted gamma space* $\Gamma_\mu^{p,\infty}(v)$ that was first studied in [29].

Definition 1.4.2. Let v be a weight in \mathbb{R}^+ . The space $\Gamma_\mu^{p,\infty}(v)$ is defined by the set of μ -measurable functions such that

$$\|f\|_{\Gamma_\mu^{p,\infty}(v)} := \sup_{t>0} f_\mu^{**}(t) V(t)^{\frac{1}{p}} < \infty.$$

We observe that they are always Banach spaces.

Remark 1.4.3. From the fact that $f_\mu^* \leq f_\mu^{**}$ and Hardy's inequality, it follows that, for every $1 < p < \infty$,

$$\|f\|_{L^{p,\infty}(\mu)} \leq \|f\|_{\Gamma_\mu^{p,\infty}} \leq \frac{p}{p-1} \|f\|_{L^{p,\infty}(\mu)}.$$

On some occasions, we shall also use the following notation:

$$\|f\|_{\Lambda_\mu^{p,\infty}[V]} := \sup_{t>0} f_\mu^*(t) V(t)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{\Gamma_\mu^{p,\infty}[V]} := \sup_{t>0} f_\mu^{**}(t) V(t)^{\frac{1}{p}}.$$

Let us consider now the following extension of Zygmund exponential spaces.

Definition 1.4.4. Let $\alpha > 0$ and let w be a weight in \mathbb{R}^n . The space $E^\alpha(w)$ is the set of all μ -measurable functions such that, for some $\lambda = \lambda(f)$,

$$\sup_K \frac{\int_K \exp\left(\lambda|f(x)|\right)^{1/\alpha} w(x) dx}{\max(w(K), 1)} < \infty,$$

where the supremum extends over all measurable sets K with $w(K) := \int_K w < \infty$.

Moreover, if $V(t) = (\log_1 \frac{1}{t})^{-\alpha}$, $f \in E^\alpha(w)$ if and only if $f \in \Lambda_w^{1,\infty}[V]$ (see [30]).

Definition 1.4.5. Let w be a weight in \mathbb{R}^n , $L_\varphi^\infty(w)$ is the space of measurable functions such that

$$\|f\|_{L_\varphi^\infty(w)} = \|f\|_{L^\infty(w)} + \int_1^\infty f_w^*(s) \varphi(\log_1 s) \frac{ds}{s} < \infty.$$

If $\varphi(s) = s^\alpha$ for some $\alpha > 0$, we simply write $L_\alpha^\infty(w)$ (see [30]).

1.5 Muckenhoupt weights and extrapolation

We conclude this chapter by introducing the A_p classes of *Muckenhoupt weights*, which characterize the boundedness of the Hardy-Littlewood maximal function, defined, for $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

with the supremum taken over all cubes $Q \subset \mathbb{R}^n$ containing x . We refer to [45, Chapter 7] and [42, Chapter 7] for a complete statement of the results presented in this section. A weight w defined on \mathbb{R}^n belongs to A_p for some $1 < p < \infty$, if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} < \infty, \quad (1.1)$$

where p' is the conjugate of p . The quantity $[w]_{A_p}$ is known as the A_p constant of w . For $p = 1$, a weight $w \in A_1$ if

$$[w]_{A_1} := \text{ess sup} \frac{Mw(x)}{w(x)} < \infty.$$

The class of A_∞ weights is defined by

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p.$$

In 1972, Muckenhoupt [82] proved the following result.

Theorem 1.5.1. *For every $1 < p < \infty$ and every weight w ,*

$$M : L^p(w) \rightarrow L^p(w)$$

if and only if $w \in A_p$. Moreover,

$$\|M\|_{L^p(w) \rightarrow L^p(w)} \lesssim pp'[w]_{A_p}^{\frac{1}{p-1}}. \quad (1.2)$$

Also, for $1 \leq p < \infty$,

$$M : L^p(w) \rightarrow L^{p,\infty}(w)$$

if and only if $w \in A_p$ and

$$\|M\|_{L^p(w) \rightarrow L^{p,\infty}(w)} \lesssim [w]_{A_p}^{1/p}.$$

The behaviour of the operator norm in (1.2) was proved in [13]. Additionally, it holds (see [20]) that, for every $1 < p < \infty$ and every $w \in A_1$,

$$\|M\|_{L^p(w) \rightarrow L^p(w)} \lesssim p'[w]_{A_1}. \quad (1.3)$$

Rubio de Francia's extrapolation theorem will be fundamental for the development of the examples presented in this book, connecting weighted estimates with the endpoint extrapolation results established here. We state a modern version of the original theorem (see [88], [89], [41], [43]).

Theorem 1.5.2. *If for some $1 \leq p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0}(w) \rightarrow L^{p_0}(w), \quad \|T\| \lesssim \psi([w]_{A_{p_0}}), \quad (1.4)$$

where $\psi : (0, \infty) \rightarrow (0, \infty)$ is an increasing function, then, for $1 < p < \infty$ and every $w \in A_p$,

$$T : L^p(w) \rightarrow L^p(w),$$

with

$$\|T\| \lesssim \begin{cases} \psi \left(C_1(p')^{\frac{(p_0-1)(p-p_0)}{p-1}} [w]_{A_p}^{\frac{p_0-1}{p-1}} \right) & \text{if } 1 < p < p_0, \\ \psi \left(C_2 p^{\frac{p-p_0}{p-1}} [w]_{A_p} \right) & \text{if } p_0 < p < \infty. \end{cases}$$

Using (1.3), the following result was obtained in [20].

Corollary 1.5.3. *If T satisfies (1.4), then, for $1 < p < p_0$ and every $w \in A_1$,*

$$T : L^p(w) \longrightarrow L^p(w) \quad \|T\| \lesssim \psi \left(C(p')^{p_0-1} [w]_{A_1}^{p_0} \right).$$

We also consider the classes $A_p^{\mathcal{R}}$ of *restricted Muckenhoupt weights*, which characterize the restricted weak-type boundedness of the Hardy-Littlewood maximal operator. For $1 \leq p < \infty$, the class $A_p^{\mathcal{R}}$ is defined by the condition

$$[w]_{A_p^{\mathcal{R}}} := \sup_{E \subset Q} \left(\frac{|E|}{|Q|} \right) \left(\frac{w(Q)}{w(E)} \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and all measurable sets $E \subset Q$. It holds that $A_1^{\mathcal{R}} = A_1$ and, for every $1 < p < \infty$ and every $\epsilon > 0$, $A_p \subset A_p^{\mathcal{R}} \subset A_{p+\epsilon}$. The origin of these weights goes back to the work of Kerman and Torchinsky [67], where the following result was proved (see also [35]).

Theorem 1.5.4. *For every $1 \leq p < \infty$ and every weight w ,*

$$M : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$$

if and only if $w \in A_p^{\mathcal{R}}$. Moreover, for $1 < p < \infty$,

$$\frac{(p-1)}{p} \|M\|_{L^{p,1}(w) \rightarrow L^{p,\infty}(w)} \lesssim [w]_{A_p^{\mathcal{R}}} \leq \|M\|_{L^{p,1}(w) \rightarrow L^{p,\infty}(w)},$$

and if $p = 1$,

$$\|M\|_{L^1(w) \rightarrow L^{1,\infty}(w)} \approx [w]_{A_1}.$$

A subset of $A_p^{\mathcal{R}}$ was introduced in [22] as follows: for every $1 \leq p < \infty$,

$$\hat{A}_p := \{w : \exists f \in L^1_{\text{loc}}(\mathbb{R}^n), \exists w_1 \in A_1 : w = w_1(Mf)^{1-p}\}$$

with

$$[w]_{\hat{A}_p} = \inf [w_1]_{A_1}^{1/p}.$$

Clearly, $\hat{A}_1 = A_1$ and it also holds that $[w]_{A_p^{\mathcal{R}}} \lesssim [w]_{\hat{A}_p}$. Using this class of weights, the following restricted Rubio de Francia extrapolation theorem was proved in [22].

Theorem 1.5.5. *If for some $1 < p_0 < \infty$ and for every $w \in \hat{A}_{p_0}$,*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w), \quad \|T\| \lesssim \psi([w]_{\hat{A}_{p_0}}),$$

where $\psi : (0, \infty) \rightarrow (0, \infty)$ is an increasing function, then, for $1 \leq p < p_0$ and every $w \in \hat{A}_p$,

$$T : L^{p, \frac{p}{p_0}}(w) \longrightarrow L^{p,\infty}(w), \quad \|T\| \lesssim [w]_{\hat{A}_p}^{1-\frac{p}{p_0}} \psi \left(C [w]_{\hat{A}_p}^{\frac{p}{p_0}} \right).$$

Note that Theorem [1.5.2](#) does reach the endpoint $p = 1$, contrary to what happens in the above theorem.

CHAPTER 2

Optimization, Decomposition, Reconstruction

In 1951, inspired by the ideas of Titchmarsh, Yano [102] established the following theorem.

Theorem 2.0.1. *Let (X, μ) and (Y, ν) be finite measure spaces. If T is a sublinear operator such that, for every $1 < p \leq 2$ and for some fixed $\alpha > 0$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu), \quad \|T\| \lesssim \frac{1}{(p-1)^\alpha},$$

then

$$T : L(\log L)^\alpha(\mu) \longrightarrow L^1(\nu).$$

Yano's theorem is optimal, in the sense that we can not expect to get a better endpoint estimate with these hypotheses. Indeed, for $1 < p \leq 2$,

$$M : L^p(0, 1) \longrightarrow L^p(0, 1), \quad \|M\| \lesssim \frac{1}{(p-1)},$$

and Stein [94] proved that $Mf \in L^1(0, 1)$ if and only if $f \in L \log L(0, 1)$.

Let us present a modern proof of Yano's theorem (see, for example, [27]) that illustrates the basic structure of most proofs of the extrapolation theorems in this text.

Step 1. Optimization of Hypotheses. *To obtain an endpoint estimate for a subset \mathcal{O} (to be determined) of $L^1(\mu)$.* For example, in this case,

$$\mathcal{O} = \left\{ f \in L^1(\mu) \cap L^\infty(\mu) : \|f\|_{L^\infty(\mu)} \leq 1 \right\}.$$

Since $\mathcal{O} \subset L^p(\mu)$ for every $1 < p \leq 2$, we have that, for every $f \in \mathcal{O}$,

$$\|Tf\|_{L^1(\nu)} \leq \|Tf\|_{L^p(\nu)} (\nu(Y))^{1/p'} \approx \|Tf\|_{L^p(\nu)} \lesssim \frac{\|f\|_{L^p(\mu)}}{(p-1)^\alpha} \lesssim \frac{\|f\|_{L^1(\mu)}^{1/p}}{(p-1)^\alpha}.$$

Hence, taking the infimum over $1 < p \leq 2$ on the right hand side, we obtain

$$\|Tf\|_{L^1(\nu)} \lesssim \|f\|_{L^1(\mu)} \left(\log_1 \frac{1}{\|f\|_{L^1(\mu)}} \right)^\alpha, \quad \forall f \in \mathcal{O}. \quad (2.1)$$

Step 2. Decomposition. *To find a decomposition of a function $f \in L^1(\mu)$ as a linear combination of functions in \mathcal{O} .* In this case, we can consider

$$f = \sum_{i \in \mathbb{Z}} 2^i f_i \quad \text{where} \quad f_i = \frac{f \chi_{\{2^{i-1} \leq |f| < 2^i\}}}{2^i}.$$

Observe that $f_i \in \mathcal{O}$ and $\|f_i\|_{L^1(\mu)} \leq \lambda_f^\mu(2^{i-1})$.

Step 3. Reconstruction. *To gather the information obtained in Steps 1 and 2.* As T is sublinear, for any $f \in L^1(\mu)$, applying (2.1) to each f_i yields

$$\begin{aligned} \|Tf\|_{L^1(\nu)} &\lesssim \sum_{i \in \mathbb{Z}} 2^i \|Tf_i\|_{L^1(\nu)} \lesssim \sum_{i \in \mathbb{Z}} 2^i \|f_i\|_{L^1(\mu)} \left(\log_1 \frac{1}{\|f_i\|_{L^1(\mu)}} \right)^\alpha \\ &\lesssim \int_0^\infty \lambda_f^\mu(s) \left(\log_1 \frac{1}{\lambda_f^\mu(s)} \right)^\alpha ds \lesssim \|f\|_{L(\log L)^\alpha(\mu)}, \end{aligned}$$

where Proposition 1.2.5 is used in the last inequality.

The above steps will be called the *ODR technique* (Optimization, Decomposition, and Reconstruction). This chapter summarizes all the technical results that we will need in Chapters 3 and 4. In Section 2.1, we present the results concerning Step 1. In Section 2.2, we introduce the different types of decompositions we shall use, and Section 2.3 is devoted to reconstruction results. From now on, we omit the underlying measure spaces (X, μ) and (Y, ν) , which are σ -finite and non-atomic.

2.1 Optimization of the hypotheses

In this section, we compute the various infima that will arise. Some of them already appear in [1], [2] and [27], but we included them all for the sake of completeness.

Lemma 2.1.1. *For each fixed $1 \leq p_0 < \infty$, the following holds:*

$$\inf_{p_0 \leq p < \infty} \varphi(p) s^{-\frac{1}{p}} \lesssim \begin{cases} \varphi(p_0) \varphi(\log_1 \frac{1}{s}), & 0 < s < 1, \\ \varphi(p_0) s^{-\frac{1}{p_0}}, & s \geq 1, \end{cases}$$

where the omitted constant does not depend on p_0 .

Proof. When $0 < s < 1$, choose $p = p_0(\log_1 \frac{1}{s})$. Clearly $p_0 < p < \infty$, and

$$\varphi(p) s^{-\frac{1}{p}} = \varphi\left(p_0 \left(\log_1 \frac{1}{s}\right)\right) e^{\frac{\log \frac{1}{s}}{p_0(1+\log \frac{1}{s})}} \leq \varphi\left(p_0 \left(\log_1 \frac{1}{s}\right)\right) e \lesssim \varphi(p_0) \varphi\left(\log_1 \frac{1}{s}\right).$$

In the case $s \geq 1$, it suffices to select $p = p_0$. \square

The following functions $I_{\varphi,p}$ and $J_{\varphi,p}$ will be relevant for our purposes.

Lemma 2.1.2. *The function*

$$I_{\varphi,p}(s) := s^{\frac{1}{p}} \varphi\left(\log_1 \frac{1}{s}\right), \quad s > 0, \quad (2.2)$$

satisfies the following properties:

- a) $I_{\varphi,p}(0^+) = 0$.
- b) $I_{\varphi,p}$ is essentially increasing. In fact,

$$\int_0^s \frac{I_{\varphi,p}(t)}{t} dt \approx I_{\varphi,p}(s).$$

- c) $I_{\varphi,1}$ is subadditive.

Proof. a) By Remark 1.3.2,

$$s^{\frac{1}{p}} \leq I_{\varphi,p}(s) \leq s^{\frac{1}{p}} \left(\log_1 \frac{1}{s}\right)^\beta.$$

Taking $s \rightarrow 0$, we obtain the result.

b) We observe that

$$\int_0^s \frac{I_{\varphi,p}(t)}{t} dt \geq \varphi\left(\log_1 \frac{1}{s}\right) \int_0^s t^{\frac{1}{p}-1} dt \approx I_{\varphi,p}(s),$$

and, since $\varphi(\log_1 \cdot)$ is submultiplicative,

$$\int_0^s \frac{I_{\varphi,p}(t)}{t} dt = \int_0^1 \frac{I_{\varphi,p}(sr)}{r} dr \lesssim I_{\varphi,p}(s) \int_0^1 r^{\frac{1}{p}-1} \varphi\left(\log_1 \frac{1}{r}\right) dr \approx I_{\varphi,p}(s).$$

c) For all $s, t > 0$, since φ is increasing, we have that

$$I_{\varphi,1}(s+t) = s\varphi\left(\log_1 \frac{1}{s+t}\right) + t\varphi\left(\log_1 \frac{1}{s+t}\right) \leq I_{\varphi,1}(s) + I_{\varphi,1}(t).$$

□

Lemma 2.1.3. *The function*

$$J_{\varphi,p}(s) := s^{\frac{1}{p}} \varphi(\log_1 s), \quad s > 0, \quad (2.3)$$

satisfies the following properties:

- a) $J_{\varphi,p}(0^+) = 0$.
- b) $J_{\varphi,p}$ is essentially increasing. In fact,

$$\int_0^s \frac{J_{\varphi,p}(t)}{t} dt \approx J_{\varphi,p}(s).$$

Proof. a) By Remark 1.3.2,

$$s^{\frac{1}{p}} \leq J_{\varphi,p}(s) \leq s^{\frac{1}{p}} (\log_1 s)^\beta.$$

Letting $s \rightarrow 0$, we are done.

b) We have that

$$\int_0^s \frac{J_{\varphi,p}(t)}{t} dt \leq \varphi(\log_1 s) \int_0^s t^{\frac{1}{p}-1} dt \approx J_{\varphi,p}(s),$$

and, since

$$\varphi(\log_1 s) \leq \varphi(\log_1 rs) \varphi\left(\log_1 \frac{1}{r}\right),$$

we get

$$\int_0^s \frac{J_{\varphi,p}(t)}{t} dt = \int_0^1 \frac{J_{\varphi,p}(sr)}{r} dr \gtrsim J_{\varphi,p}(s) \int_0^1 r^{\frac{1}{p}-1} \varphi\left(\log_1 \frac{1}{r}\right)^{-1} dr \approx J_{\varphi,p}(s).$$

□

Lemma 2.1.4. *Let $1 \leq p_0 < p_1 < \infty$. For every $s > 0$,*

$$\inf_{p_0 < p \leq p_1} \varphi \left(\frac{1}{p - p_0} \right) s^{\frac{1}{p}} \lesssim I_{\varphi, p_0}(s),$$

$$\inf_{p_0 \leq p < p_1} \varphi \left(\frac{1}{p_1 - p} \right) s^{\frac{1}{p}} \lesssim J_{\varphi, p_1}(s),$$

and

$$\inf_{p_0 \leq p < \infty} \varphi(p) s^{\frac{1}{p}} \lesssim \varphi(\log_1 s).$$

Proof. By Lemma 2.1.1, setting $\frac{1}{q} = \frac{1}{p_0} - \frac{1}{p}$, we get

$$\begin{aligned} \inf_{p_0 < p \leq p_1} \varphi \left(\frac{1}{p - p_0} \right) s^{\frac{1}{p}} &\approx s^{\frac{1}{p_0}} \inf_{\left(\frac{1}{p_0} - \frac{1}{p_1}\right)^{-1} \leq q < \infty} \varphi(q) s^{-\frac{1}{q}} \\ &\lesssim s^{\frac{1}{p_0}} \varphi \left(\log_1 \frac{1}{s} \right) \chi_{(0,1)}(s) + s^{\frac{1}{p_1}} \chi_{[1,\infty)}(s) \end{aligned} \quad (2.4)$$

and, since $\varphi(1) = 1$, we conclude

$$\inf_{p_0 < p \leq p_1} \varphi \left(\frac{1}{p - p_0} \right) s^{\frac{1}{p}} \lesssim s^{\frac{1}{p_0}} \varphi \left(\log_1 \frac{1}{s} \right) = I_{\varphi, p_0}(s).$$

Similarly, if $\frac{1}{q} = \frac{1}{p} - \frac{1}{p_1}$, we have

$$\begin{aligned} \inf_{p_0 \leq p < p_1} \varphi \left(\frac{1}{p_1 - p} \right) s^{\frac{1}{p}} &\approx s^{\frac{1}{p_1}} \inf_{\left(\frac{1}{p_0} - \frac{1}{p_1}\right)^{-1} \leq q < \infty} \varphi(q) \left(\frac{1}{s} \right)^{-\frac{1}{q}} \\ &\lesssim s^{\frac{1}{p_0}} \chi_{(0,1]}(s) + s^{\frac{1}{p_1}} \varphi(\log_1 s) \chi_{(1,\infty)}(s), \end{aligned}$$

and, using that $\varphi(1) = 1$, we obtain

$$\inf_{p_0 \leq p < p_1} \varphi \left(\frac{1}{p_1 - p} \right) s^{\frac{1}{p}} \lesssim s^{\frac{1}{p_1}} \varphi(\log_1 s) = J_{\varphi, p_1}(s).$$

Finally,

$$\inf_{p_0 \leq p < \infty} \varphi(p) s^{\frac{1}{p}} \lesssim s^{\frac{1}{p_0}} \chi_{(0,1]}(s) + \varphi(\log_1 s) \chi_{(1,\infty)}(s) \leq \varphi(\log_1 s).$$

□

2.2 Decomposition results

We shall use the following decompositions.

- a) The *dyadic* decomposition over \mathbb{Z} :

$$f = \sum_{i \in \mathbb{Z}} 2^i f_i \quad \text{where} \quad f_i = \frac{f \chi_{\{2^{i-1} \leq |f| < 2^i\}}}{2^i}.$$

- b) The *k-dyadic* decomposition over \mathbb{N} :

$$f = \sum_{n \geq 0} d_n f_n \quad \text{where} \quad d_n = \underbrace{2^{2^n}}_{k\text{-times } 2} \quad \text{and} \quad f_n = \frac{f \chi_{\{d_{n-1} \leq |f| < d_n\}}}{d_n}$$

with $d_{-1} = 0$. If $k = 1$, we refer to it as the *dyadic* decomposition over \mathbb{N} .

- c) The *f*-decomposition*:

$$f = \sum_{n \geq 0} f_n \quad \text{where} \quad f_n = \begin{cases} f \chi_{\{|f| \geq f^*(1)\}} & n = 0, \\ f \chi_{\{f^*(2^n) \leq |f| < f^*(2^{n+1})\}} & n \geq 1. \end{cases} \quad (2.5)$$

- d) The *K(A₀, A₁)-decomposition*, given below.

Recall that Peetre's *K*-functional (see [8, Chapter 5] or [9, Chapter 3]) for a pair of compatible quasi-normed function spaces $\bar{A} = (A_0, A_1)$, is defined, for $t > 0$, as

$$K(f, t; \bar{A}) = \inf \left\{ \|f_0\|_{A_0} + t \|f_1\|_{A_1} : f = f_0 + f_1, f_0 \in A_0, f_1 \in A_1 \right\}.$$

The *K*-functional is an increasing concave function and, hence, is differentiable almost everywhere for $t > 0$.

Lemma 2.2.1 ([27]). *Let*

$$F(t) = K(f, t; \bar{A}),$$

and let, for every $i \in \mathbb{Z}$,

$$E_i := \{t \in (0, \infty) : F'(t) > 2^i\}.$$

Then, there exist functions $\{f_i\}_i$ such that $f = \sum_{i \in \mathbb{Z}} 2^i f_i$ and

$$K(f_i, t; \bar{A}) \lesssim \min\{t, |E_i|\}.$$

The proof is based on the *K-divisibility Theorem* (see [12, pp. 315-337]). The above decomposition of f is what we call $K(A_0, A_1)$ -decomposition.

In this work, we shall only use the following K -functionals. For $1 \leq p < \infty$,

- $K(f, t; L^p, L^\infty) \approx \left(\int_0^{t^p} f^*(s)^p ds \right)^{1/p}$,
- $K(f, t; L^{p,\infty}, L^\infty) \approx \sup_{s \leq t^p} s^{\frac{1}{p}} f^*(s)$;

and, for $1 < p < \infty$,

- $K(f, t; L^1, L^p) \approx t \left(\int_{t^{p'}}^\infty f^{**}(s)^p ds \right)^{1/p}$,
- $K(f, t; L^1, L^{p,\infty}) \approx t \sup_{s \geq t^{p'}} s^{\frac{1}{p}} f^{**}(s)$.

Lemma 2.2.2. *Fix $1 \leq p_0 < \infty$, $1 < p_1 < \infty$, a set $E \subset \mathbb{R}$ and a function f . Suppose that, for every $t > 0$,*

$$K(f, t; \bar{A}) \lesssim \min \{t, |E|\}.$$

Then,

a) if $\bar{A} = (L^{p_0}, L^\infty)$,

$$\|f\|_{L^p} \lesssim |E|^{p_0/p}, \quad \forall p_0 < p < \infty.$$

b) if $\bar{A} = (L^{p_0,\infty}, L^\infty)$,

$$\|f\|_{L^{p,\infty}} \lesssim |E|^{p_0/p}, \quad \forall p_0 < p < \infty.$$

c) if $\bar{A} = (L^1, L^{p_1})$,

$$\|f\|_{L^p} \lesssim |E|^{p_1' \left(\frac{1}{p} - \frac{1}{p_1} \right)}, \quad \forall 1 \leq p < p_1.$$

d) if $\bar{A} = (L^1, L^{p_1,\infty})$,

$$\|f\|_{L^{p,\infty}} \lesssim |E|^{p_1' \left(\frac{1}{p} - \frac{1}{p_1} \right)}, \quad \forall 1 \leq p < p_1.$$

Proof. a) We have that

$$\left(\int_0^{t^{p_0}} f^*(s)^{p_0} ds \right)^{1/p_0} \lesssim \min \{t, |E|\},$$

which is equivalent to

$$\int_0^t f^*(s)^{p_0} ds \lesssim \min \{t, |E|^{p_0}\} = \int_0^t \chi_{[0, |E|^{p_0}]}(s) ds.$$

Hence, $(f^{p_0})^{**} \lesssim (\chi_{[0, |E|^{p_0}]})^{**}$ and, [8, Corollary 4.7, p. 61] implies that, for every $p_0 < p < \infty$,

$$\|f\|_{L^p} = \|f^{p_0}\|_{L^{\frac{p}{p_0}}}^{1/p_0} \lesssim \|\chi_{[0, |E|^{p_0}]}\|_{L^{\frac{p}{p_0}}}^{1/p_0} = |E|^{p_0/p}.$$

b) In this case,

$$K(f, t; L^{p_0, \infty}, L^\infty) \approx \sup_{s \leq t^{p_0}} s^{\frac{1}{p_0}} f^*(s) \lesssim \min \{t, |E|\}.$$

Hence,

$$f^*(t) \lesssim \min \left\{ 1, \frac{|E|}{t^{\frac{1}{p_0}}} \right\},$$

and, therefore, for every $p_0 < p < \infty$,

$$\|f\|_{L^{p, \infty}} \lesssim \sup_{t > 0} t^{\frac{1}{p}} \min \left\{ 1, \frac{|E|}{t^{\frac{1}{p_0}}} \right\} = \max \left\{ \sup_{t \leq |E|^{p_0}} t^{\frac{1}{p}}, \sup_{t > |E|^{p_0}} t^{\frac{1}{p} - \frac{1}{p_0}} |E| \right\} = |E|^{p_0/p}.$$

c) Since

$$\begin{aligned} \int_0^t f^*(s) ds &\approx t^{1/p'_1} \left(\int_t^\infty s^{-p_1} ds \right)^{1/p_1} \int_0^t f^*(s) ds \\ &\leq t^{1/p'_1} \left(\int_t^\infty \frac{(\int_0^s f^*(r) dr)^{p_1}}{s^{p_1}} ds \right)^{1/p_1} \approx K(f, t^{1/p'_1}; L^1, L^{p_1}) \lesssim \min \left\{ t^{\frac{1}{p'_1}}, |E| \right\}, \end{aligned}$$

we get that $\|f\|_{L^1} \lesssim |E|$. Also,

$$\left(\int_{t^{p'_1}}^\infty f^*(s)^{p_1} ds \right)^{1/p_1} \leq \left(\int_{t^{p'_1}}^\infty f^{**}(s)^{p_1} ds \right)^{1/p_1} \lesssim \min \left\{ 1, \frac{|E|}{t} \right\} \leq 1,$$

and, therefore, $\|f\|_{L^{p_1}} \lesssim 1$. Now, for every $1 \leq p < p_1$, let $0 \leq \theta < 1$ such that $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$. Then,

$$\|f\|_{L^p} = \left(\int |f|^{(1-\theta)p} |f|^{\theta p} \right)^{\frac{1}{p}} \leq \left\| |f|^{(1-\theta)p} \right\|_{L^{\frac{1}{(1-\theta)p}}}^{\frac{1}{p}} \left\| |f|^{\theta p} \right\|_{L^{\frac{p_1}{\theta p}}}^{\frac{1}{p}} = \|f\|_{L^1}^{(1-\theta)} \|f\|_{L^{p_1}}^\theta.$$

Since $(1 - \theta) = p'_1 \left(\frac{1}{p} - \frac{1}{p_1} \right)$, we obtain

$$\|f\|_{L^p} \lesssim |E|^{p'_1 \left(\frac{1}{p} - \frac{1}{p_1} \right)}.$$

d) Using that

$$f^{**}(t) \lesssim t^{-\frac{1}{p_1}} \min \left\{ 1, \frac{|E|}{t^{\frac{1}{p_1}}} \right\},$$

we conclude that, for every $1 \leq p < p_1$,

$$\begin{aligned} \|f\|_{L^{p,\infty}} &\lesssim \sup_{t>0} t^{\frac{1}{p} - \frac{1}{p_1}} \min \left\{ 1, \frac{|E|}{t^{\frac{1}{p_1}}} \right\} \\ &= \max \left\{ \sup_{t \leq |E|^{p_1}} t^{\frac{1}{p} - \frac{1}{p_1}}, \sup_{t > |E|^{p_1}} t^{\frac{1}{p} - 1} |E| \right\} = |E|^{p'_1 \left(\frac{1}{p} - \frac{1}{p_1} \right)}. \end{aligned}$$

□

Parts a) and c) of the above result can be found in [27].

2.3 Reconstruction results

Here, we present the results related to Step 3. Let $I_{\varphi,p}$ and $J_{\varphi,p}$ be the functions described in (2.2) and (2.3).

Lemma 2.3.1. *Let $1 \leq p < \infty$. If $\{f_i\}_i$ are the functions obtained from f with the dyadic decomposition over \mathbb{Z} ,*

$$\sum_{i \in \mathbb{Z}} 2^i I_{\varphi,p} (\|f_i\|_{L^1}) \lesssim \|f\|_{\Lambda^1(v)}, \quad v(t) = t^{\frac{1}{p}-1} \varphi \left(\log_1 \frac{1}{t} \right),$$

and

$$\sum_{i \in \mathbb{Z}} 2^i J_{\varphi,p} (\|f_i\|_{L^1}) \lesssim \|f\|_{\Lambda^1(v)}, \quad v(t) = t^{\frac{1}{p}-1} \varphi (\log_1 t).$$

Proof. For the first statement, we use that $\|f_i\|_{L^1} \leq \lambda_f(2^{i-1})$ and that $I_{\varphi,p} \circ \lambda_f$ is essentially decreasing, to get

$$\sum_{i \in \mathbb{Z}} 2^i I_{\varphi,p} (\|f_i\|_{L^1}) \lesssim \sum_{i \in \mathbb{Z}} 2^i I_{\varphi,p} (\lambda_f(2^i)) \approx \int_0^\infty I_{\varphi,p} (\lambda_f(s)) ds.$$

From Lemma 2.1.2 b) and Proposition 1.2.5 it follows that

$$\int_0^\infty I_{\varphi,p}(\lambda_f(s)) ds \approx \int_0^\infty f^*(t)t^{\frac{1}{p}-1}\varphi\left(\log_1 \frac{1}{t}\right) dt.$$

The proof of the second statement follows the same steps using Lemma 2.1.3 b). \square

Lemma 2.3.2. *For any $f \in L^\infty$, let $\{f_n\}_n$ be the functions obtained from f with the f^* -decomposition (2.5), then*

$$\sum_{n \geq 0} \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right) \lesssim \|f\|_{L^\infty_\varphi}.$$

Proof. By definition, it comes clear that $\|f_n\|_{L^1} \lesssim \|f_n\|_{L^\infty} 2^{n-1}$. Therefore,

$$\begin{aligned} \sum_{n \geq 0} \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right) &\lesssim \|f\|_{L^\infty} + \sum_{n \geq 1} f^*(2^{n-1})\varphi(\log_1 2^{n-1}) \\ &\lesssim \|f\|_{L^\infty} + \sum_{n \geq 1} \int_{2^{n-2}}^{2^{n-1}} f^*(s)\varphi(\log_1 s) \frac{ds}{s} = \|f\|_{L^\infty_\varphi} + \int_{1/2}^1 f^*(s)\varphi(\log_1 s) \frac{ds}{s}. \end{aligned}$$

The result follows from the fact that

$$\int_{1/2}^1 f^*(s)\varphi(\log_1 s) \frac{ds}{s} \lesssim \|f\|_{L^\infty}.$$

\square

Finally, for the $K(A_0, A_1)$ -decomposition, we have the following lemmas, inspired by the ideas in [27].

Lemma 2.3.3. *For any function f and $\{E_i\}_i$ as in Lemma 2.2.1, we have*

$$\sum_{i \in \mathbb{Z}} 2^i I_{\varphi,1}(|E_i|) \lesssim \sup_{t>0} K(f, t; \bar{A}) + \int_0^1 \frac{K(f, s; \bar{A})}{s} \varphi'\left(\log_1 \frac{1}{s}\right) ds.$$

Proof. We assume that the right hand side of the above inequality is finite; otherwise, there is nothing to prove. Let $F(t) := K(f, t; \bar{A})$ and, therefore, since $I_{\varphi,p} \circ \lambda_{F'}$ is essentially decreasing, we have

$$\sum_{i \in \mathbb{Z}} 2^i I_{\varphi,1}(|E_i|) = \sum_{i \in \mathbb{Z}} 2^i I_{\varphi,1}(\lambda_{F'}(2^i)) \approx \int_0^\infty I_{\varphi,1}(\lambda_{F'}(s)) ds.$$

Now, in view of Lemma 2.1.2 b), we get

$$\begin{aligned} \int_0^\infty I_{\varphi,1}(\lambda_{F'}(s)) ds &\approx \int_0^\infty \int_0^{\lambda_{F'}(s)} \frac{I_{\varphi,1}(t)}{t} dt ds \\ &\approx \int_0^1 F'(s) \varphi\left(\log_1 \frac{1}{s}\right) ds + \int_1^\infty F'(s) ds = I_1 + I_2, \end{aligned}$$

where, in the second equivalence, we have used Proposition 1.2.5 and the fact that, since F is concave, $(F')^* = F'$. For I_1 , we integrate by parts to obtain

$$\begin{aligned} I_1 &= \left[F(s) \varphi\left(\log_1 \frac{1}{s}\right) \right]_0^1 + \int_0^1 \frac{F(s)}{s} \varphi'\left(\log_1 \frac{1}{s}\right) ds \\ &\lesssim F(\infty) + \int_0^1 \frac{F(s)}{s} \varphi'\left(\log_1 \frac{1}{s}\right) ds, \end{aligned}$$

where we have used that, since F is increasing,

$$\begin{aligned} 0 \leq \lim_{s \rightarrow 0} F(s) \varphi\left(\log_1 \frac{1}{s}\right) &= \lim_{s \rightarrow 0} F(s) \left(1 + \int_s^1 \varphi'\left(\log_1 \frac{1}{t}\right) \frac{dt}{t} \right) \\ &\leq F(\infty) + \int_0^1 \frac{F(t)}{t} \varphi'\left(\log_1 \frac{1}{t}\right) dt < \infty. \end{aligned}$$

Finally, since $I_2 \leq F(\infty)$, we conclude the desired result. \square

Lemma 2.3.4. *For any function f such that $\|f\|_{A_1} \leq 1$ and $\{E_i\}_i$ as in Lemma 2.2.1, we have*

$$\sum_{i \in \mathbb{Z}^-} 2^i \varphi(\log_1 |E_i|) \lesssim 1 + \int_1^\infty \frac{K(f, s; \bar{A})}{s^2} \varphi'(\log_1 s) ds.$$

Proof. Let $F(t) := K(f, t; \bar{A})$. Hence,

$$\sum_{i \in \mathbb{Z}^-} 2^i \varphi(\log_1 |E_i|) = \sum_{i \in \mathbb{Z}^-} 2^i \varphi(\log_1 \lambda_{F'}(2^i)) \lesssim \int_0^1 \varphi(\log_1 \lambda_{F'}(t)) dt.$$

Since F is an increasing concave function, it follows that, for almost every $t > 0$,

$$F'(t) \leq \frac{1}{t} \int_0^t F'(s) ds \leq \frac{F(t)}{t} \leq \|f\|_{A_1} \leq 1.$$

Thus, for every $t > 1$,

$$\varphi(\log_1 \lambda_{F'}(t)) = \varphi(1) = 1,$$

and, if $H(s) := \varphi(\log_1 s) - 1$, from Proposition 1.2.5 and the fact that $F'(t) \leq \frac{F(t)}{t}$, we obtain

$$\begin{aligned} \int_0^1 \varphi(\log_1 \lambda_{F'}(t)) dt &= 1 + \int_0^\infty \varphi(\log_1 \lambda_{F'}(t)) - 1 dt \\ &= 1 + \int_0^\infty H(\lambda_{F'}(t)) dt \approx 1 + \int_1^\infty F'(s) H'(s) ds \\ &= 1 + \int_1^\infty F'(s) \varphi'(\log_1 s) \frac{ds}{s} \lesssim 1 + \int_1^\infty \frac{F(s)}{s^2} \varphi'(\log_1 s) ds. \end{aligned}$$

□

2.3.1 The quasi-Banach setting

Now, we introduce the results required for Step 3 when the target space is quasi-Banach. To overcome the lack of triangle inequality, we shall use the following proposition from [19].

Proposition 2.3.5. *Given $0 < a < 1$ and a weight $v \in \mathbb{R}^+$, let*

$$G_V(a) := \sup_{t>0} \left(\frac{1}{t} \int_{at}^t V(s)^{-1} ds \right) V(t).$$

If $f = \sum_{n \geq 1} c_n f_n$, where c_n are non-negative numbers and $\{f_n\}_n$ are functions such that $\|f_n\|_{\Lambda^{1,\infty}[V]} \leq 1$, then, for every sequence of positive numbers $(a_n)_n$ such that $\sum_{n \geq 1} a_n \leq 1$,

$$\|f\|_{\Lambda^{1,\infty}[V]} \lesssim \sum_{n \geq 1} (1 + G_V(a_n)) c_n.$$

Lemma 2.3.6. *Let $0 < p \leq 1$. If*

$$V(t) \approx \frac{t^{1/p}}{\varphi(\log_1 t)},$$

then,

- i) for $0 < p < 1$, $G_V(a) \approx a^{-\frac{1}{p}+1}$,*
- ii) for $p = 1$, $G_V(a) = \log \frac{1}{a}$.*

Proof. We need to compute

$$G_V(a) \approx \sup_{t>0} \int_{at}^t \frac{\varphi(\log_1 s)}{s^{1/p}} ds \frac{t^{\frac{1}{p}-1}}{\varphi(\log_1 t)}.$$

i) Since $\frac{1}{p} > 1$ and φ is increasing, it follows

$$G_V(a) \lesssim \sup_{t>0} t^{\frac{1}{p}-1} \int_{at}^t s^{-1/p} ds = \sup_{t>0} t^{\frac{1}{p}-1} \left[\frac{s^{-\frac{1}{p}+1}}{-\frac{1}{p}+1} \right]_{at}^t = \frac{a^{-\frac{1}{p}+1} - 1}{\frac{1}{p} - 1} \approx a^{-\frac{1}{p}+1}.$$

Indeed, when $t \leq 1$, we can use that $\varphi(1) = 1$, to conclude

$$\int_{at}^t \frac{\varphi(\log_1 s)}{s^{1/p}} ds \frac{t^{\frac{1}{p}-1}}{\varphi(\log_1 t)} = t^{\frac{1}{p}-1} \int_{at}^t s^{-1/p} ds = \frac{a^{-\frac{1}{p}+1} - 1}{\frac{1}{p} - 1},$$

and the result follows.

ii) We can use again that φ is increasing to obtain

$$G_V(a) \lesssim \sup_{t>0} \int_{at}^t \frac{ds}{s} = \sup_{t>0} [\log s]_{at}^t = \log \frac{1}{a},$$

and $G_V(a) \approx \log \frac{1}{a}$ if $t \leq 1$. □

Lemma 2.3.7. *Let $0 < p \leq 1$. If*

$$V(t) \approx \frac{t^{1/p}}{\varphi\left(\log_1 \frac{1}{t}\right)},$$

then,

$$i) \text{ for } 0 < p < 1, G_V(a) \lesssim a^{-\frac{1}{p}+1} \varphi\left(\log_1 \frac{1}{a}\right),$$

$$ii) \text{ for } p = 1, G_V(a) \lesssim \left(\log \frac{1}{a}\right) \varphi\left(\log_1 \frac{1}{a}\right).$$

Proof. Let us compute

$$G_V(a) \approx \sup_{t>0} \int_{at}^t \frac{\varphi\left(\log_1 \frac{1}{s}\right)}{s^{1/p}} ds \frac{t^{\frac{1}{p}-1}}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

i) We have that, since $\frac{1}{p} > 1$,

$$G_V(a) \lesssim \sup_{t>0} t^{\frac{1}{p}-1} \int_{at}^t s^{-1/p} ds \varphi \left(\log_1 \frac{1}{a} \right) \approx a^{-\frac{1}{p}+1} \varphi \left(\log_1 \frac{1}{a} \right).$$

ii) In this case, we get

$$G_V(a) \lesssim \sup_{t>0} \left(\int_{at}^t \frac{ds}{s} \right) \varphi \left(\log_1 \frac{1}{a} \right) = \left(\log \frac{1}{a} \right) \varphi \left(\log_1 \frac{1}{a} \right).$$

□

Lemma 2.3.8. *If*

$$V(t) \approx \frac{t}{\varphi_1(\log_1 t) \varphi_2 \left(\log_1 \frac{1}{t} \right)},$$

then

$$G_V(a) \lesssim \left(\log \frac{1}{a} \right) \varphi_2 \left(\log_1 \frac{1}{a} \right).$$

Proof. It follows from the fact that

$$\int_{at}^t \frac{\varphi_1(\log_1 s) \varphi_2 \left(\log_1 \frac{1}{s} \right)}{s} ds \frac{1}{\varphi_1(\log_1 t) \varphi_2 \left(\log_1 \frac{1}{t} \right)} \lesssim \left(\log \frac{1}{a} \right) \varphi_2 \left(\log_1 \frac{1}{a} \right).$$

□

The following proofs are inspired by the ideas of Carro and Martín [24].

Lemma 2.3.9. *Fix $f \in L^1$ such that $\|f\|_{L^1} = 1$ and let $\{f_n\}_n$ be the functions obtained with the 2-dyadic decomposition. Then,*

i) *if $0 < p < 1$ and $\varepsilon > 0$,*

$$I_{\varphi,1}(\|f_0\|_{L^1}) + \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^{2^n} I_{\varphi,1}(\|f_n\|_{L^1}) \lesssim \|f\|_{\Lambda^1(v)},$$

$$\text{where } v(t) = \varphi \left(\log_1 \frac{1}{t} \right) \left(\log_2 \frac{1}{t} \right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)},$$

ii)

$$\sum_{n \geq 0} (\log_1 n) \varphi(\log_1 n) 2^{2^n} I_{\varphi,1}(\|f_n\|_{L^1}) \lesssim \|f\|_{\Lambda^1(v)},$$

$$\text{where } v(t) = \varphi_1 \left(\log_1 \frac{1}{t} \right) \left(\log_3 \frac{1}{t} \right) \varphi \left(\log_3 \frac{1}{t} \right).$$

Proof. i) For every $n \geq 1$,

$$\|f_n\|_{L^1} = \sum_{k=2^{n-1}}^{2^n-1} \frac{1}{2^{2^n}} \int_0^\infty f(s) \chi_{\{2^k \leq |f| < 2^{k+1}\}}(s) ds \lesssim \sum_{k=2^{n-1}}^{2^n-1} \frac{2^k}{2^{2^n}} \lambda_f(2^k).$$

Hence, since $I_{\varphi,1}$ is subadditive (Lemma 2.1.2 c)),

$$I_{\varphi,1}(\|f_n\|_{L^1}) \lesssim I_{\varphi,1} \left(\sum_{k=2^{n-1}}^{2^n-1} \frac{2^k}{2^{2^n}} \lambda_f(2^k) \right) \lesssim \sum_{k=2^{n-1}}^{2^n-1} I_{\varphi,1} \left(\frac{2^k}{2^{2^n}} \lambda_f(2^k) \right).$$

Thus, if

$$B_1 := \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^{2^n} I_{\varphi,1}(\|f_n\|_{L^1}),$$

we have that

$$\begin{aligned} B_1 &\lesssim \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^{2^n} \sum_{k=2^{n-1}}^{2^n-1} I_{\varphi,1} \left(\frac{2^k}{2^{2^n}} \lambda_f(2^k) \right) \\ &= \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \sum_{k=2^{n-1}}^{2^n-1} 2^k \lambda_f(2^k) \varphi \left(\log_1 \frac{2^{2^n}}{2^k \lambda_f(2^k)} \right). \end{aligned}$$

Using that, for every $2^{n-1} \leq k \leq 2^n$,

$$\frac{2^{2^n}}{2^k} = \frac{(2^{2^{n-1}})^2}{2^k} \leq 2^k \quad \text{and} \quad n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \lesssim \left(\log_2 2^k \right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)}, \quad (2.6)$$

we get

$$\begin{aligned} B_1 &\lesssim \sum_{n \geq 1} \sum_{k=2^{n-1}}^{2^n-1} (2^k)^2 I_{\varphi,1} \left(\frac{\lambda_f(2^k)}{2^k} \right) (\log_2 2^k)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \\ &\lesssim \int_1^\infty s I_{\varphi,1} \left(\frac{\lambda_f(s)}{s} \right) (\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} ds \\ &= \int_1^\infty \lambda_f(s) \varphi \left(\log_1 \frac{s}{\lambda_f(s)} \right) (\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} ds, \end{aligned}$$

where we have used that $I_{\varphi,1} \left(\frac{\lambda_f(s)}{s} \right)$ is essentially decreasing. Since, $s \lambda_f(s) \leq 1$ for every $s > 0$,

$$\lambda_f(s) \varphi \left(\log_1 \frac{s}{\lambda_f(s)} \right) \lesssim \lambda_f(s) \varphi \left(\log_1 \frac{1}{(\lambda_f(s))^2} \right) \approx I_{\varphi,1}(\lambda_f(s)),$$

and we obtain

$$\begin{aligned} B_1 &\lesssim \int_0^\infty I_{\varphi,1}(\lambda_f(s)) (\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} ds \\ &= \int_0^\infty \left(\int_0^{\lambda_f(s)} dI_{\varphi,1}(t) \right) (\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} ds. \end{aligned}$$

If we use Fubini, we conclude

$$\begin{aligned} B_1 &\lesssim \int_0^\infty \left(\int_0^{f^*(t)} (\log_2 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} ds \right) dI_{\varphi,1}(t) \\ &\lesssim \int_0^\infty f^*(t) (\log_2 f^*(t))^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dI_{\varphi,1}(t) \\ &\lesssim \int_0^\infty f^*(t) \left(\log_2 \frac{1}{t} \right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dI_{\varphi,1}(t) \\ &\approx \int_0^\infty f^*(t) \varphi \left(\log_1 \frac{1}{t} \right) \left(\log_2 \frac{1}{t} \right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dt, \end{aligned}$$

and the result follows because, since $\|f\|_{L^1} \leq 1$, $I_{\varphi,1}(\|f_0\|_{L^1}) \lesssim 1$.

ii) We can follow the same argument, taking into account that, instead of (2.6), we have

$$(\log_1 n) \varphi(\log_1 n) \lesssim \left(\log_3 2^k \right) \varphi \left(\log_3 2^k \right).$$

□

Lemma 2.3.10. *Let $1 \leq p_1 < \infty$. Fix $f \in L^{p_1}$ such that $\|f\|_{L^{p_1}} = 1$ and let $\{f_n\}_n$ be the functions obtained with the dyadic decomposition over \mathbb{N} . Then,*

i) *if $0 < p < 1$ and $\varepsilon > 0$,*

$$I_{\varphi,p_1}(\|f_0\|_{L^1}) + \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^n I_{\varphi,p_1}(\|f_n\|_{L^1}) \lesssim \|f\|_{\Lambda^1(v)},$$

$$\text{where } v(t) = t^{\left(\frac{1}{p_1}-1\right)} \varphi \left(\log_1 \frac{1}{t} \right) \left(\log_1 \frac{1}{t} \right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)},$$

ii)

$$\sum_{n \geq 0} (\log_1 n) 2^n I_{\varphi,p_1}(\|f_n\|_{L^1}) \lesssim \|f\|_{\Lambda^1(v)},$$

$$\text{where } v(t) = t^{\left(\frac{1}{p_1}-1\right)} \varphi \left(\log_1 \frac{1}{t} \right) \left(\log_2 \frac{1}{t} \right).$$

Proof. We only prove *i)* and *ii)* follows similarly. Since $\|f_n\|_{L^\infty} \leq 1$,

$$\|f_n\|_{L^1} \leq |\{f_n \neq 0\}| \leq |\{2^{n-1} \leq |f|\}| = \lambda_f(2^{n-1}).$$

Therefore, if

$$B_2 := \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^n I_{\varphi, p_1}(\|f_n\|_{L^1}),$$

and we use that, for any $2^{n-1} < s < 2^n$,

$$(n+1)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \lesssim (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)},$$

we get

$$\begin{aligned} B_2 &\lesssim \sum_{n \geq 0} (n+1)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} 2^n I_{\varphi, p_1}(\lambda_f(2^n)) \\ &\lesssim \sum_{n \geq 0} \int_{2^{n-1}}^{2^n} (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} I_{\varphi, p_1}(\lambda_f(2^n)) ds \\ &\lesssim \int_0^\infty (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} I_{\varphi, p_1}(\lambda_f(s)) ds. \end{aligned}$$

Finally, following the same argument as in Lemma 2.3.9, we conclude

$$\begin{aligned} B_2 &\lesssim \int_0^\infty f^*(t) (\log_1 f^*(t))^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dI_{\varphi, p_1}(t) \\ &\lesssim \int_0^\infty f^*(t) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} dI_{\varphi, p_1}(t), \end{aligned}$$

and, since $I_{\varphi, p_1}(\|f_0\|_{L^1}) \lesssim 1$, the result follows. \square

We have analogous results in the case of $J_{\varphi, p}$.

Lemma 2.3.11. *Let $1 < p_1 < \infty$. Fix $f \in L^{p_1}$ such that $\|f\|_{L^{p_1}} = 1$ and let f_n be the functions obtained with the dyadic decomposition over \mathbb{N} . Then,*

i) if $0 < p < 1$ and $\varepsilon > 0$,

$$J_{\varphi, p_1}(\|f_0\|_{L^1}) + \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_1 n) 2^n J_{\varphi_1, p_1}(\|f_n\|_{L^1}) \lesssim \|f\|_{\Lambda^1}(v),$$

where $v_1(t) = t^{\left(\frac{1}{p_1}-1\right)} \varphi_1(\log_1 t) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi\left(\log_2 \frac{1}{t}\right)$,

ii)

$$\sum_{n \geq 0} (\log_1 n) \varphi(\log_1 n) 2^n J_{\varphi_1, p_1} (\|f_n\|_{L^1}) \lesssim \|f\|_{\Lambda^1} (v),$$

$$\text{where } v(t) = t^{\left(\frac{1}{p_1}-1\right)} \varphi_1(\log_1 t) \left(\log_2 \frac{1}{t}\right) \varphi\left(\log_2 \frac{1}{t}\right).$$

Proof. We only prove i), as ii) follows in a similar manner. Let

$$B_3 := \sum_{n \geq 1} n^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_1 n) 2^n J_{\varphi_1, p_1} (\|f_n\|_{L^1}).$$

Since, for every $2^{n-1} < s < 2^n$,

$$(n+1)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_1 n) \lesssim (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_2 s),$$

we have

$$\begin{aligned} B_3 &\lesssim \sum_{n \geq 0} \int_{2^{n-1}}^{2^n} (n+1)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_1 n) J_{\varphi_1, p_1} (\lambda_f(2^n)) ds \\ &\lesssim \int_0^\infty (\log_1 s)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi(\log_2 s) J_{\varphi_1, p_1} (\lambda_f(s)) ds. \end{aligned}$$

Finally, following the same argument as in Lemma 2.3.9, we conclude

$$J_{\varphi, p_1} (\|f_0\|_{L^1}) + B_3 \lesssim \int_0^\infty f^*(t) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{p}-1\right)(1+\varepsilon)} \varphi\left(\log_2 \frac{1}{t}\right) dJ_{\varphi_1, p_1}(t),$$

and the result follows from Lemma 1.2.5 and Lemma 2.1.3. \square

Lemma 2.3.12. Fix $f \in L^\infty$. If $\{f_n\}_n$ are the functions obtained from f with the f^* -decomposition,

$$\sum_{n \geq 0} (\log_1 n) \varphi(\log_1 n) \|f_n\|_{L^\infty} \varphi_1\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right) \lesssim \|f\|_{L^\infty_{\tilde{\varphi}}}$$

where $\tilde{\varphi}(t) = \varphi_1(t)(\log_1 t)\varphi(\log_1 t)$.

Proof. It holds that $\|f_n\|_{L^1} \lesssim \|f_n\|_{L^\infty} 2^{n-1}$. Therefore,

$$\begin{aligned}
& \sum_{n \geq 0} (\log_1 n) \varphi(\log_1 n) \|f_n\|_{L^\infty} \varphi_1 \left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}} \right) \\
& \lesssim \|f\|_{L^\infty} + \sum_{n \geq 1} (\log_1 n) \varphi(\log_1 n) f^*(2^{n-1}) \varphi_1(\log_1 2^{n-1}) \\
& \lesssim \|f\|_{L^\infty} + \sum_{n \geq 1} \int_{2^{n-2}}^{2^{n-1}} f^*(s) \varphi_1(\log_1 s) (\log_2 s) \varphi(\log_2 s) \frac{ds}{s} \\
& = \|f\|_{L_{\varphi}^\infty} + \int_{1/2}^1 f^*(s) \varphi_1(\log_1 s) (\log_2 s) \varphi(\log_2 s) \frac{ds}{s} \lesssim \|f\|_{L_{\varphi}^\infty} + \|f\|_{L^\infty},
\end{aligned}$$

and we are done. □

CHAPTER 3

Review of linear Yano's extrapolation

The origins of *Yano's Extrapolation Theory* go back to 1928, when Titchmarsh [98] proved that, if \mathcal{C} is the conjugate function on the torus

$$\mathcal{C}f(x) = p.v. \int_{\mathbb{T}} f(x-y) \cot y \, dy,$$

then $\mathcal{C}f \in L^1(\mathbb{T})$ whenever $f \log(2 + |f|)^{1+\varepsilon}$ is integrable for some $\varepsilon > 0$. Almost at the same time, Zygmund announced in [103] that the above result is also true for $\varepsilon = 0$. One year later, Titchmarsh [99] published a second paper where, using that, for every $1 < p \leq p_1$,

$$\mathcal{C} : L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T}), \quad \|\mathcal{C}\| \lesssim \frac{1}{p-1},$$

he obtained Zygmund's conclusion. That same year, using a different argument, Zygmund [104] proved his announced result.

In 1951, inspired by the ideas of Titchmarsh, Yano [102] published the result that gave the theory its name (see Theorem 2.0.1). Capri and Fava [14], in 1987, extended Yano's theorem to the case where (X, μ) and (Y, ν) are σ -finite measure spaces, concluding

$$T : L(\log L)^\alpha(\mu) \longrightarrow L_{\text{loc}}^1(\nu).$$

The problem of the convergence of the Fourier series pushed forward the development of these extrapolation techniques. In 1966, Carleson [15] proved, using the boundedness of the *Carleson operator* \mathcal{C} (see (3.11)), that, for every function in

$L^2(\mathbb{T})$, its Fourier series converges almost everywhere. At that time, it was known that there exist functions in $L^1(\mathbb{T})$ whose Fourier series diverges almost everywhere (see [69]). In 1967, Hunt [56] extended Carleson's result to $L^p(\mathbb{T})$ functions with $1 < p \leq \infty$ by proving that, for every measurable set $E \subset \mathbb{T}$,

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(\mathbb{T})} \lesssim \frac{p^2}{(p-1)}|E|^{1/p}. \quad (3.1)$$

From (3.1), we get that

$$\mathcal{C} : L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T}), \quad \|\mathcal{C}\| \lesssim \frac{1}{(p-1)^2},$$

and, by Theorem 2.0.1, the almost everywhere convergence of the Fourier series of functions in the space $L(\log L)^2(\mathbb{T})$ followed. In 1969, Sjölin [91] also used (3.1) to improve the space up to $L(\log L)(\log_2 L)(\mathbb{T})$. These works inspired Soria [92] in 1989, to state the following extrapolation result for finite measure spaces. If T is a sublinear operator such that for some fixed $\alpha > 0$, for every $1 < p \leq 2$ and every measurable $E \subset \mathbb{T}$,

$$\|T\chi_E\|_{L^{1,\infty}(\mu)} \lesssim \frac{\mu(E)^{1/p}}{(p-1)^\alpha},$$

then,

$$T : B_\psi^* \rightarrow L^{1,\infty}(\mu),$$

where $\psi(t) = t(\log_1 t)^\alpha$ and B_ψ^* is a block space such that $L(\log L)(\log_2 L)(\mu) \subset B_\psi^*$.

Some years later, Carro [17], in 2000, improved Yano's original theorem for σ -finite measure spaces showing that, if, for every measurable $E \subset \mathbb{T}$,

$$\|T\chi_E\|_{L^p(\nu)} \lesssim \frac{\mu(E)^{1/p}}{(p-1)^\alpha},$$

then

$$T : L(\log L)^\alpha(\mu) \longrightarrow R(\nu),$$

where $R(\nu)$ is the space of functions such that

$$\|f\|_{R(\nu)} := \sup_{t>0} \frac{\int_{1/t}^\infty \lambda_f^\nu(s) ds}{\log_1 t} < \infty;$$

that is, assuming a weaker condition on T , she obtained the boundedness on $L(\log L)^\alpha$ and improved the range space.

Coming back to the problem of the almost everywhere convergence of Fourier series, we have to mention that in 1996, Antonov [3] proved such convergence in the space $L(\log L)(\log_3 L)(\mathbb{T})$ and Arias de Reyna [4] improved Antonov's theorem in 2002, achieving the best result of this type known so far.

In fact, Antonov's proof inspired Carro and Martín [24] to develop an extrapolation technique (see Theorem 3.3.1) that uses restricted weak type estimates of the form

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu), \quad \|T\| \lesssim \frac{1}{(p-1)^\alpha}.$$

Now, there are also operators in the literature for which the best-known operator norm does not grow up like a power function. For example, if \mathcal{C}_{lac} is the *Carleson lacunary operator* (see (3.13) for a definition), Di Plinio and Lerner [39] showed that, for $1 < p \leq 2$,

$$\mathcal{C}_{\text{lac}} : L^p(\mathbb{R}) \longrightarrow L^{p,\infty}(\mathbb{R}), \quad \|\mathcal{C}_{\text{lac}}\| \lesssim \log_1 \left(\frac{1}{p-1} \right).$$

In this case, an extension of the theorem by Carro and Martín allows to conclude an endpoint estimate. This example (among many others) motivates the extension of the extrapolation theory to cases where the blow up is not necessarily a power function (see [92], [1], [2]).

Going back to 1959, Zygmund [105, p. 119] proved a dual version of Yano's result for cases where the operator norm blows up when $p \rightarrow \infty$. Namely, if T is a sublinear operator such that, for every $1 < p < \infty$ and for some fixed $\alpha > 0$,

$$T : L^p(\mu) \longrightarrow L^p(\nu), \quad \|T\| \lesssim p^\alpha,$$

then

$$T : L^\infty(\mu) \longrightarrow E^\alpha(\nu),$$

where $E^\alpha(\nu)$ is the space of Definition 1.4.4.

Other endpoints can also be considered. Let, for example, \mathcal{B}^α be the *Bochner-Riesz operator of order α* (see (3.17) for a definition) with $0 < \alpha \leq 1/2$. It is known (see [16], [46, Chapter 5]) that

$$\mathcal{B}^\alpha : L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2) \quad \text{if and only if} \quad \frac{4}{3+2\alpha} < p < \frac{4}{1-2\alpha}.$$

In this case, if $p_0 := \frac{4}{3+2\alpha} > 1$ and $p_1 := \frac{4}{1-2\alpha} < \infty$, we have $\|\mathcal{B}^\alpha\|_{L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)} \rightarrow \infty$ when $p \rightarrow p_0$ or $p \rightarrow p_1$. It is then natural to study other extrapolation results to cover these cases (see, for example, [27], [1], [2]).

Summarizing, given any admissible function φ and any fixed endpoint $1 \leq q \leq \infty$, we are going to consider extrapolation results under hypothesis of the form

$$T : L^{p,r}(\mu) \longrightarrow L^{p,s}(\nu), \quad \|T\| \lesssim \varphi\left(\frac{1}{p-q}\right), \quad \text{for } p > q,$$

where $r \in \{1, p, \infty\}$ and $s \in \{p, \infty\}$, and similarly for the case where $p < q$. The chapter is divided into six sections, according to the different hypotheses: $r = p = s$ (Section 3.1), $r = p$ and $s = \infty$ (Section 3.2), $r = 1$ and $s = \infty$ (Section 3.3), hypotheses for characteristic functions (Section 3.4), $r = \infty = s$ (Section 3.5), and other types of estimates in Section 3.6. All the extrapolation proofs presented follow the ODR technique, except for those in Section 3.4.

This chapter will also contain applications of the extrapolation results to concrete operators, such as the maximal rough operator M_Ω , the spherical maximal operator \mathcal{M} , the Hilbert transform or the Carleson operator. As we shall see, the main source of most of these examples is the Rubio de Francia's extrapolation theorem (see Theorem 1.5.2).

Finally, we should mention and emphasize that in the 1990s, Jawerth and Milman extended extrapolation theory to the setting of compatible couples of Banach spaces using interpolation theory (see [59–62], [81], [5]). Most of the results we will present are particular cases of this more general theory; however, since we are restricted to the case of Lorentz spaces, we aim to avoid, whenever possible, technicalities from the abstract setting.

In what follows, we omit the underlying measure spaces which are σ -finite and non-atomic.

3.1 Strong type theorems

In this section, we shall assume that T is strong type (p, p) for some range of p . We treat three different cases, depending on the value of the endpoint.

3.1.1 Case endpoint $p_0 = 1$

First, we extend Yano's result to σ -finite spaces (see [27]) but in the context of admissible functions.

Theorem 3.1.1. *If for some fixed $1 < p_1 < \infty$, and for every $1 < p \leq p_1$,*

$$T : L^p \longrightarrow L^p, \quad \|T\| \lesssim \varphi\left(\frac{1}{p-1}\right), \quad (3.2)$$

then

$$T : L \varphi(\log L) \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi(\log_1 t)}.$$

Remark 3.1.2. We observe that, if (Y, ν) is finite and $\varphi(x) = x^\alpha$, $\Gamma^{1,\infty}[V] = L^1$, and we recover original Yano's theorem.

Proof. We have that if $f \in L^1 \cap L^\infty$ with $\|f\|_{L^\infty}$, for every $t > 0$ and $1 < p \leq p_1$,

$$(Tf)^{**}(t) \leq t^{-\frac{1}{p}} \|Tf\|_{L^p} \lesssim \varphi\left(\frac{1}{p-1}\right) \frac{\|f\|_{L^p}}{t^{1/p}} \lesssim \varphi\left(\frac{1}{p-1}\right) \left(\frac{\|f\|_{L^1}}{t}\right)^{1/p},$$

and, by Lemma 2.1.4 and (2.2) we obtain

$$(Tf)^{**}(t) \lesssim I_{\varphi,1}\left(\frac{\|f\|_{L^1}}{t}\right) \lesssim \frac{\varphi(\log_1 t)}{t} I_{\varphi,1}(\|f\|_{L^1}),$$

from which follows that

$$\|Tf\|_{\Gamma^{1,\infty}[V]} \lesssim I_{\varphi,1}(\|f\|_{L^1}).$$

Therefore, given any function $f \in L^1$, we can decompose it with the dyadic decomposition over \mathbb{Z} and use Lemma 2.3.1 to obtain the result. \square

Example 3.1.3. Given $\Omega \in L^1(\mathbb{S}^{n-1})$, the *rough maximal operator* (see [42, Chapter 4]) is

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} |\Omega(y')| |f(x-y)| dy, \quad x \in \mathbb{R}^n, \quad y' = \frac{y}{|y|}. \quad (3.3)$$

It holds that, for $1 < p \leq \infty$,

$$M_\Omega : L^p \longrightarrow L^p, \quad \|M_\Omega\| \lesssim p'.$$

However, it is unknown whether M_Ω is weak type (1,1). From Theorem 3.1.1, we conclude that,

$$M_\Omega : L \log L \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{\log_1 t}.$$

On the other hand, if $\Omega \in L \log L(\mathbb{S}^{n-1})$, it is known (see [34]) that M_Ω is weak type (1,1). Although the proof of this is very technical, a weaker result can be proved easily using extrapolation. Let

$$M_1(\Omega, f)(x) = \sup_{R \geq 1} \frac{1}{|B(0,R)|} \int_{B(0,R)} |\Omega(y')| |f(x-y)| dy, \quad x \in \mathbb{R}^n, \quad y' = \frac{y}{|y|}.$$

For every $p > 1$, $\Omega \in L^p(\mathbb{S}^{n-1})$ and $f \in L^1(\mathbb{R}^n)$,

$$\begin{aligned} \|M_1(\Omega, f)\|_{L^p(\mathbb{R}^n)} &\lesssim \left\| \int_{B(0,1)} |\Omega(y')| |f(\cdot - y)| dy \right\|_{L^p(\mathbb{R}^n)} \\ &+ \left\| \sup_{R \geq 1} \int_{B(0,R)/B(0,1)} \frac{|\Omega(y')|}{|y|^n} |f(\cdot - y)| dy \right\|_{L^p(\mathbb{R}^n)} = I + II. \end{aligned}$$

By Minkowski's inequality we get that $I \leq \|f\|_{L^1(\mathbb{R}^n)} \|\Omega\|_{L^p(\mathbb{S}^{n-1})}$ and that

$$II \leq \|f\|_{L^1(\mathbb{R}^n)} \|\Omega\|_{L^p(\mathbb{S}^{n-1})} \left(\int_1^\infty \frac{r^{n-1}}{r^{np}} dr \right)^{1/p} \lesssim \|f\|_{L^1(\mathbb{R}^n)} \|\Omega\|_{L^p(\mathbb{S}^{n-1})} \frac{1}{p-1}.$$

Thus, if we fix $f \in L^1(\mathbb{R}^n)$, we get that

$$M_1(\cdot, f) : L^p(\mathbb{S}^{n-1}) \longrightarrow L^p(\mathbb{R}^n), \quad \|M_1(\cdot, f)\| \lesssim \frac{\|f\|_{L^1(\mathbb{R}^n)}}{p-1},$$

and, from Theorem 3.1.1, we conclude that

$$M_1(\cdot, f) : L \log L(\mathbb{S}^{n-1}) \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{\log_1 t}, \quad \|M_1(\cdot, f)\| \lesssim \|f\|_{L^1(\mathbb{R}^n)},$$

which implies that, for $\Omega \in L \log L(\mathbb{S}^{n-1})$,

$$M_1(\Omega, \cdot) : L^1(\mathbb{R}^n) \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{\log_1 t}.$$

Example 3.1.4. The function $m \in L^\infty(\mathbb{R})$ is a *Marcinkiewicz Fourier multiplier* if is derivable and

$$\sup_{k \in \mathbb{Z}} \int_{\Delta_k} |m'(t)| dt < \infty, \quad \text{where } \Delta_k = (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1}).$$

It is known (see [79], [97]) that, for $1 < p < \infty$,

$$T_m : L^p \longrightarrow L^p, \quad \|T_m\| \lesssim (\max\{p, p'\})^{3/2}. \quad (3.4)$$

Then, by Theorem 3.1.1, we obtain the following estimate at the endpoint

$$T_m : L(\log L)^{3/2} \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{(\log_1 t)^{3/2}}.$$

We should mention that T_m locally map $L(\log L)^{1/2}$ into $L^{1,\infty}$ and for every $r < 1/2$, there exist multipliers of this type that do not locally map $L(\log L)^r$ into $L^{1,\infty}$ (see [97]).

Concerning weighted estimates, it is known (see [70], [74]) that, for $1 < p \leq 2$ and for every $w \in A_p$,

$$T_m : L^p(w) \longrightarrow L^p(w), \quad \|T_m\| \lesssim [w]_{A_p}^{\frac{1}{p-1} \left(\frac{p+2}{2}\right)}.$$

Thus, if we fix $1 < p_1 \leq 2$ and use Corollary 1.5.3, we get that, for $1 < p \leq p_1$ and every $w \in A_1$,

$$T_m : L^p(w) \longrightarrow L^p(w), \quad \|T_m\| \lesssim \frac{[w]_{A_1}^{p_1' \left(\frac{p_1+2}{2}\right)}}{(p-1) \left(\frac{p_1+2}{2}\right)}.$$

By Theorem 3.1.1, we obtain that, for every $1 < p_1 \leq 2$ and $w \in A_1$,

$$T_m : L(\log L)^{\left(\frac{p_1+2}{2}\right)}(w) \longrightarrow \Gamma_w^{1,\infty}[V], \quad V(t) = \frac{t}{(\log_1 t)^{\left(\frac{p_1+2}{2}\right)}}.$$

Taken $p_1 = 1 + \varepsilon$ with $0 < \varepsilon \leq 1$, we have that, for every $w \in A_1$,

$$T_m : L(\log L)^{\left(\frac{3}{2}+\varepsilon\right)}(w) \longrightarrow \Gamma_w^{1,\infty}[V] \quad V(t) = \frac{t}{(\log_1 t)^{\left(\frac{3}{2}+\varepsilon\right)}}.$$

Remark 3.1.5. If $V(t) = t/\varphi(\log_1 t)$, the space $\Gamma^{1,\infty}[V]$ is not comparable to $L^{1,\infty}$. Indeed, $L^{1,\infty} \not\subseteq \Gamma^{1,\infty}[V]$ because the function $f(t) = f^*(t) = t^{-1} \in L^{1,\infty}$ but

$$\|f\|_{\Gamma^{1,\infty}[V]} \geq \frac{\int_0^1 f^*(s) ds}{\varphi(\log_1 1)} = \infty.$$

On the other hand, by [26], $\Gamma^{1,\infty}[V] \not\subseteq L^{1,\infty}$, since

$$\sup_{t>0} \frac{t}{t \sup_{s \geq t} s^{-1} V(s)} \approx \sup_{t>0} \varphi(\log_1 t) = \infty.$$

Hence, if T satisfies (3.2) and is weak type (1,1),

$$T : L \varphi(\log L) \longrightarrow \Gamma^{1,\infty}[V] \cap L^{1,\infty},$$

where $\Gamma^{1,\infty}[V] \cap L^{1,\infty} \subsetneq L^{1,\infty}$.

Example 3.1.6. *Mihlin-Hörmander multipliers* are functions such that, for all multi-indices α with $|\alpha| \leq [n/2] + 1$,

$$\left| \partial_{\xi}^{\alpha} m(\xi) \right| \leq C_{\alpha} |\xi|^{-|\alpha|} \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

It is known ([80], [55], [45, Theorem 6.2.7]) that these multipliers are weak type $(1,1)$ and, for $1 < p < \infty$,

$$T_m : L^p \longrightarrow L^p, \quad \|T_m\| \lesssim \max \left\{ p, \frac{1}{p-1} \right\}.$$

Hence, in view of Remark 3.1.5,

$$T_m : L \log L \longrightarrow \Gamma^{1,\infty}[V] \cap L^{1,\infty}, \quad V(t) = \frac{t}{\log_1 t}.$$

3.1.2 Case left endpoint $p_0 > 1$

The ideas of this section are based in the results of [27]. We first introduce the function spaces we need.

Definition 3.1.7. The spaces $D_{p,\varphi}^+(\mu)$ and $R_{p,\varphi}^+(\nu)$ are defined by the norms

$$\|f\|_{D_{p,\varphi}^+(\mu)} = \|f\|_{L^p(\mu)} + \int_0^1 \frac{\left(\int_0^t [f_{\mu}^*(s)]^p ds \right)^{1/p}}{t} \varphi' \left(\log_1 \frac{1}{t} \right) dt$$

and

$$\|f\|_{R_{p,\varphi}^+(\nu)} = \sup_{t>0} \frac{\left(\int_0^t [f_{\nu}^*(s)]^p ds \right)^{1/p}}{\varphi(\log_1 t)}.$$

If $\varphi(s) = s^{\alpha}$ for some $\alpha > 0$, we simply write $D_{p,\alpha}^+(\mu)$ and $R_{p,\alpha}^+(\nu)$.

Theorem 3.1.8. *If for any fixed $1 < p_0 < p_1 < \infty$, and for every $p_0 < p \leq p_1$,*

$$T : L^p \longrightarrow L^p, \quad \|T\| \lesssim \varphi \left(\frac{1}{p-p_0} \right),$$

then

$$T : D_{p_0,\varphi}^+ \longrightarrow R_{p_0,\varphi}^+.$$

Proof. Given $f \in L^{p_0}$, we use the $K(L^{p_0}, L^\infty)$ -decomposition and Lemma 2.2.2 a) to write $f = \sum_{i \in \mathbb{Z}} 2^i f_i$ where, for every $p_0 < p \leq p_1$, we have that $\|f_i\|_{L^p} \lesssim |E_i|^{p_0/p}$. Then, for every $i \in \mathbb{Z}$,

$$\begin{aligned} \left(\int_0^t [(Tf_i)^*(s)]^{p_0} ds \right)^{1/p_0} &\leq t^{\frac{1}{p_0} - \frac{1}{p}} \|Tf_i\|_{L^p} \lesssim t^{\frac{1}{p_0} - \frac{1}{p}} \varphi \left(\frac{1}{p - p_0} \right) \|f_i\|_{L^p} \\ &\lesssim t^{1/p_0} \varphi \left(\frac{1}{p - p_0} \right) \left(\frac{|E_i|^{p_0}}{t} \right)^{1/p}. \end{aligned} \quad (3.5)$$

By Lemma 2.1.4 we get that

$$\left(\int_0^t [(Tf_i)^*(s)]^{p_0} ds \right)^{1/p_0} \lesssim t^{1/p_0} I_{\varphi, p_0} \left(\frac{|E_i|^{p_0}}{t} \right) \lesssim \varphi(\log_1 t) I_{\varphi, 1}(|E_i|). \quad (3.6)$$

Summing over $i \in \mathbb{Z}$ and taking the supremum over $t > 0$, we obtain that

$$\sup_{t > 0} \frac{\left(\int_0^t [(Tf)^*(s)]^{p_0} ds \right)^{1/p_0}}{\varphi(\log_1 t)} \lesssim \sum_{i \in \mathbb{Z}} 2^i I_{\varphi, 1}(|E_i|).$$

Finally, Lemma 2.3.3 implies that

$$\sup_{t > 0} \frac{\left(\int_0^t [(Tf)^*(s)]^{p_0} ds \right)^{1/p_0}}{\varphi(\log_1 t)} \lesssim \|f\|_{L^{p_0}} + \int_0^1 \frac{\left(\int_0^s [f^*(r)]^{p_0} dr \right)^{1/p_0}}{s} \varphi' \left(\log_1 \frac{1}{s} \right) ds.$$

□

Remark 3.1.9. Theorem 3.1.8 and its proof are also valid for $p_0 = 1$. However, in that case, we just recover Theorem 3.1.1.

Example 3.1.10. For $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$, the *spherical maximal operator* is defined as

$$\mathcal{M}f(x) = \sup_{t > 0} \left| \int_{\mathbb{S}^{n-1}} f(x - ty) d\sigma(y) \right| \quad x \in \mathbb{R}^n,$$

where σ is the normalized Lebesgue measure on the unit sphere \mathbb{S}^{n-1} . It is known (see [95], [90], [45, Theorem 6.5.1]) that, for $n \geq 3$,

$$\mathcal{M} : L^p \longrightarrow L^p, \quad \|\mathcal{M}\| \lesssim \frac{1}{p - \frac{n}{n-1}}, \quad \frac{n}{n-1} < p \leq 2. \quad (3.7)$$

As far as we are concerned, the best result at the endpoint $p = \frac{n}{n-1}$ is the one obtained in [11], where it is proved that, for $n \geq 3$,

$$\mathcal{M} : L^{\frac{n}{n-1},1} \longrightarrow L^{\frac{n}{n-1},\infty}.$$

Using Theorem 3.1.8 and (3.7), we conclude that, for $n \geq 3$,

$$\mathcal{M} : D_{\frac{n}{n-1},1}^+ \longrightarrow R_{\frac{n}{n-1},1}^+.$$

These estimates are independent because, as shown in [27], $L^{\frac{n}{n-1},1}$ is not comparable with $D_{\frac{n}{n-1},1}^+$.

To present our next example, we recall (see [45, Example 7.1.7]) that, for every $1 < p < \infty$, the function $|x|^a$ is an A_p weight if and only if $-n < a < n(p-1)$, and, for every $p_0 < p < \infty$,

$$\left[|x|^{n(p_0-1)} \right]_{A_p} \approx \frac{1}{(p-p_0)^{p-1}} = \frac{1}{(p-p_0)^{p_0-1} (p-p_0)^{p-p_0}} \approx \frac{1}{(p-p_0)^{p_0-1}}. \quad (3.8)$$

Corollary 3.1.11. *Let T be such that, for some fixed $1 < p_0 < p_1 < \infty$, for every $p_0 < p \leq p_1$ and every $w \in A_p$,*

$$T : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad \|T\| \lesssim \psi \left([w]_{A_p} \right),$$

where ψ is increasing. If $\psi(s^{p_0-1}) \approx \varphi(s)$ for some admissible function φ , then,

$$T : D_{p_0,\varphi}^+ \left(|x|^{n(p_0-1)} \right) \longrightarrow R_{p_0,\varphi}^+ \left(|x|^{n(p_0-1)} \right).$$

Proof. By (3.8), we get that, for $p_0 < p \leq p_1$,

$$T : L^p \left(|x|^{n(p_0-1)} \right) \longrightarrow L^p \left(|x|^{n(p_0-1)} \right), \quad \|T\| \lesssim \varphi \left(\frac{1}{p-p_0} \right).$$

The result follows by Theorem 3.1.8. \square

Example 3.1.12. Let H denote the Hilbert transform defined, for every $f \in \mathcal{S}(\mathbb{R})$ as follows

$$H(f)(x) := \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy, \quad x \in \mathbb{R}.$$

It is known (see [57], [86]) that, for $1 < p < \infty$,

$$H : L^p(w) \longrightarrow L^p(w) \quad \text{if and only if} \quad w \in A_p,$$

and $\|H\| \lesssim [w]_{A_p}$ for $2 < p < \infty$. As a consequence, for $p_0 > 1$, since $|x|^{n(p_0-1)} \notin A_{p_0}$, H is not bounded on $L^{p_0}(|x|^{n(p_0-1)})$, but Corollary 3.1.11 implies that

$$H : D_{p_0, p_0-1}^+ \left(|x|^{n(p_0-1)} \right) \longrightarrow R_{p_0, p_0-1}^+ \left(|x|^{n(p_0-1)} \right). \quad (3.9)$$

On the other hand (see [35]), since $|x|^{n(p_0-1)} \in A_{p_0}^{\mathcal{R}}$,

$$H : L^{p_0, 1} \left(|x|^{n(p_0-1)} \right) \longrightarrow L^{p_0, \infty} \left(|x|^{n(p_0-1)} \right). \quad (3.10)$$

We have that (3.9) and (3.10) are independent because $D_{p_0, p_0-1}^+ \left(|x|^{n(p_0-1)} \right)$ and $L^{p_0, 1} \left(|x|^{n(p_0-1)} \right)$ are not comparable.

Example 3.1.13. For $1 < p < \infty$ and $f \in L^p(\mathbb{R})$, the Carleson operator is defined as

$$\mathcal{C}f(s) := \sup_{\xi \in \mathbb{R}} \left| H \left(e^{2\pi i \xi(\cdot)} f(\cdot) \right) (s) \right|, \quad s \in \mathbb{R}. \quad (3.11)$$

It is known (see [58], [39]) that

$$\mathcal{C} : L^p(w) \longrightarrow L^p(w) \quad \|\mathcal{C}\| \lesssim [w]_{A_p}^{\max\{p', \frac{2}{p-1}\}} \log_2 [w]_{A_p}. \quad (3.12)$$

Thus, for any fixed $1 < p_0 < \infty$, we have:

(i) If $1 < p_0 < 2$,

$$\|\mathcal{C}\| \lesssim [w]_{A_p}^{\frac{2}{p_0-1}} \log_2 [w]_{A_p}, \quad p_0 < p \leq 2.$$

Hence, for $\varphi_1(s) := s^2 \log_2 s$, Corollary 3.1.11 implies

$$\mathcal{C} : D_{p_0, \varphi_1}^+ \left(|x|^{p_0-1} \right) \longrightarrow R_{p_0, \varphi_1}^+ \left(|x|^{p_0-1} \right).$$

(ii) If $2 \leq p_0 < \infty$,

$$\|\mathcal{C}\| \lesssim [w]_{A_p}^{p_0'} \log_2 [w]_{A_p}, \quad p_0 < p < \infty.$$

In this case, if $\varphi_2(s) := s^{p_0} \log_2 s$, Corollary 3.1.11 implies

$$\mathcal{C} : D_{p_0, \varphi_2}^+ \left(|x|^{p_0-1} \right) \longrightarrow R_{p_0, \varphi_2}^+ \left(|x|^{p_0-1} \right).$$

Example 3.1.14. For $f \in L^p(\mathbb{R})$ with $1 < p < \infty$, the *Carleson lacunary* operator is defined as

$$\mathcal{C}_{\text{lac}}f(s) := \sup_{a \in A} \left| H \left(e^{2\pi ia(\cdot)} f(\cdot) \right) (s) \right| \quad s \in \mathbb{R}, \quad (3.13)$$

where $A \subset \mathbb{R}$ is lacunary in the sense that there exists a constant $C > 0$ such that

$$\inf_{a \neq a' \in A} \frac{|a - a'|}{|a|} = C.$$

In [39], the authors proved that, for $1 < p < \infty$,

$$\mathcal{C}_{\text{lac}} : L^p(w) \longrightarrow L^p(w) \quad \|\mathcal{C}_{\text{lac}}\| \lesssim [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \log_1 [w]_{A_p}. \quad (3.14)$$

Hence, by Corollary 3.1.11, we get that:

(i) If $1 < p_0 < 2$, for $\varphi_3(s) := s \log_1 s$,

$$\mathcal{C}_{\text{lac}} : D_{p_0, \varphi_3}^+ \left(|x|^{p_0-1} \right) \longrightarrow R_{p_0, \varphi_3}^+ \left(|x|^{p_0-1} \right).$$

(ii) If $2 \leq p_0 < \infty$, for $\varphi_4(s) := s^{p_0-1} \log_1 s$,

$$\mathcal{C}_{\text{lac}} : D_{p_0, \varphi_4}^+ \left(|x|^{p_0-1} \right) \longrightarrow R_{p_0, \varphi_4}^+ \left(|x|^{p_0-1} \right).$$

Example 3.1.15. Consider the weight

$$w(x) := \frac{|x|^{p_0-1}}{\log_1 \frac{1}{|x|}},$$

where $1 < p_0 < \infty$. Going back to Example 3.1.12, since $w \notin A_{p_0}^{\mathcal{R}}$ (see [25]), H does not map $L^{p_0, 1}(w)$ to $L^{p_0, \infty}(w)$. However, for every fixed $p_0 < p_1 < \infty$, and for every $p_0 < p \leq p_1$,

$$[w]_{A_p} \lesssim \frac{1}{(p - p_0)^{p_0-1}},$$

which implies, following the same argument as in 3.1.11, that

$$H : D_{p_0, p_0-1}^+(w) \longrightarrow R_{p_0, p_0-1}^+(w).$$

3.1.3 Case right endpoint $p_1 < \infty$

This section follows the ideas appearing in [27]. First, we define the function spaces that we need for the theorem.

Definition 3.1.16. The spaces $D_{p,\varphi}^-(\mu)$ and $R_{p,\varphi}^-(\nu)$ are defined by the following norms

$$\|f\|_{D_{p,\varphi}^-(\mu)} = \|f\|_{L^p(\mu)} + \int_1^\infty \frac{\left(\int_s^\infty [f_\mu^{**}(r)]^p dr\right)^{1/p}}{s} \varphi'(\log_1 s) ds$$

and

$$\|f\|_{R_{p,\varphi}^-(\nu)} = \sup_{t>0} \frac{\left(\int_t^\infty [f_\nu^{**}(s)]^p ds\right)^{1/p}}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

If $\varphi(s) = s^\alpha$ for some $\alpha > 0$, we can simply write $D_{p,\alpha}^-(\mu)$ and $R_{p,\alpha}^-(\nu)$.

Theorem 3.1.17. *If for any fixed $1 < p_0 < p_1 < \infty$ and for every $p_0 \leq p < p_1$,*

$$T : L^p \longrightarrow L^p, \quad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - p}\right),$$

then,

$$T : D_{p_1,\varphi}^- \longrightarrow R_{p_1,\varphi}^-.$$

Proof. Given $f \in L^{p_1}$ with $\|f\|_{L^{p_1}} \leq 1/p_1'$, we can use the $K(L^1, L^{p_1})$ -decomposition and Lemma 2.2.2 c) to write $f = \sum_{i \in \mathbb{Z}} 2^i f_i$ where, for every $p_0 < p \leq p_1$, we have that $\|f_i\|_{L^p} \lesssim |E_i|^{p_1'(\frac{1}{p} - \frac{1}{p_1})}$.

Since $F(t) := K(f, t; L^1, L^{p_1})$ is an increasing concave function, it follows that, for almost every $t > 0$,

$$F'(t) \leq \frac{1}{t} \int_0^t F'(s) ds = \frac{F(t) - F(0)}{t} \leq \|f^{**}\|_{L^{p_1}} \leq p_1' \|f\|_{L^{p_1}} \leq 1. \quad (3.15)$$

Therefore, $E_i = \emptyset$ whenever $i \geq 0$ and, for every $i \in \mathbb{Z}^-$ and every $p_0 \leq p < p_1$, we have

$$\begin{aligned} \left(\int_t^\infty [(Tf_i)^{**}(s)]^{p_1} ds\right)^{1/p_1} &\lesssim \left(\int_t^\infty [(Tf_i)^{**}(s)]^p s^{\frac{p}{p_1}-1} ds\right)^{1/p} \\ &\lesssim t^{\frac{1}{p_1}-\frac{1}{p}} \|Tf_i\|_{L^p} \lesssim \varphi\left(\frac{1}{p_1-p}\right) \left(\frac{|E_i|^{p_1'}}{t}\right)^{\frac{1}{p}-\frac{1}{p_1}}. \end{aligned} \quad (3.16)$$

By Lemma 2.1.4 we get that

$$\left(\int_t^\infty [(Tf_i)^{**}(s)]^{p_1} ds \right)^{1/p_1} \lesssim \varphi \left(\log_1 \frac{|E_i|^{p_1}}{t} \right) \lesssim \varphi \left(\log_1 \frac{1}{t} \right) \varphi(\log_1 |E_i|).$$

Hence, summing over $i \in \mathbb{Z}$, we obtain

$$\|Tf\|_{R_{p_1, \varphi}^-} \lesssim \sum_{i \in \mathbb{Z}^-} 2^i \varphi(\log_1 |E_i|) \lesssim 1 + \int_1^\infty \frac{(\int_s^\infty [f^{**}(r)]^{p_1} dr)^{1/p_1}}{s} \varphi'(\log_1 s) ds,$$

where we have used Lemma 2.3.4 in the last inequality. The case of a general function $f \in L^{p_1}$ follows by homogeneity. \square

Example 3.1.18. The *Bochner-Riesz operator* of order $\alpha \geq 0$ in \mathbb{R}^2 is the multiplier defined by

$$\mathcal{B}^\alpha(f)(x) := \int_{\mathbb{R}^2} (1 - |\xi|^2)_+^\alpha \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad x \in \mathbb{R}^2. \quad (3.17)$$

It can be deduced from [38] (see also [16]) that, for $0 < \alpha < \frac{1}{2}$ and $4 < p < \frac{4}{1-2\alpha}$,

$$\mathcal{B}^\alpha : L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2), \quad \|\mathcal{B}^\alpha\| \lesssim \frac{1}{\left(\frac{4}{1-2\alpha} - p\right)^{5/4}}.$$

Therefore, Theorem 3.1.17 implies that

$$\mathcal{B}^\alpha : D_{\frac{4}{1-2\alpha}, \frac{5}{4}}^- \longrightarrow R_{\frac{4}{1-2\alpha}, \frac{5}{4}}^-.$$

We should mention that it is also known (see [68]) that

$$\mathcal{B}^\alpha : L_{\frac{4}{1-2\alpha}, 1} \longrightarrow L_{\frac{4}{1-2\alpha}, \infty},$$

but $D_{\frac{4}{1-2\alpha}, \frac{5}{4}}^- \not\subseteq L_{\frac{4}{1-2\alpha}, 1}$. For example, the function

$$f(t) = \frac{1}{t^{\frac{1-2\alpha}{4}} \left(\log \frac{1}{t}\right)} \chi_{[0, e^{-\frac{4}{1-2\alpha}})}(t)$$

belongs to $D_{\frac{4}{1-2\alpha}, \frac{5}{4}}^-$ but not to $L_{\frac{4}{1-2\alpha}, 1}$.

Also, since

$$\|f\|_{D_{\frac{4}{1-2\alpha}, \frac{5}{4}}^-} = |E|^{1/p} (\log_1 |E|)^{5/4} \gtrsim \|f\|_{L_{\frac{4}{1-2\alpha}, 1}} = |E|^{1/p},$$

we have that $L_{\frac{4}{1-2\alpha}, 1} \not\subseteq D_{\frac{4}{1-2\alpha}, \frac{5}{4}}^-$.

3.2 Weak type theorems

In this section, we shall assume that T is weak type (p, p) for some range of p . We treat three different cases, depending on the value of the endpoint.

3.2.1 Case endpoint $p_0 = 1$

In this case, the target space at the endpoint is quasi-Banach. The lack of a triangle inequality implies some difficulties in the reconstruction step. Thus, we need to use the *galb space* (see [100], [19]), defined as follows

$$\text{galb}(\mathcal{X}) = \left\{ (e_n)_n : \sum_n e_n f_n \in \mathcal{X}, \text{ whenever } \|f_n\|_{\mathcal{X}} \leq 1 \right\},$$

endowed with the norm

$$\|(e_n)_n\|_{\text{galb}(\mathcal{X})} := \sup_{\|f_n\|_{\mathcal{X}} \leq 1} \left\| \sum_n e_n f_n \right\|_{\mathcal{X}}.$$

It is known that

- if \mathcal{X} is a Banach space, then $\text{galb}(\mathcal{X}) = \ell^1$,
- $\text{galb}(L^{1,\infty}) = \ell \log \ell$ (see [96], [65]) where

$$\ell \log \ell = \left\{ (e_n)_n : \sum_n e_n \log_1 n < \infty \right\},$$

- if $V(t) = t/\varphi(\log_1 t)$, $\ell \log \ell \subseteq \text{galb}(\Lambda^{1,\infty}[V])$ (see [29]).

The next theorem is an extension of the work in [24] to the setting of admissible functions (see also [21]).

Theorem 3.2.1. *If for any fixed $1 < p_1 < \infty$ and, for every $1 < p \leq p_1$,*

$$T : L^p \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi\left(\frac{1}{p-1}\right),$$

then,

$$T : L \varphi(\log L) \log_3 L \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi(\log_1 t)}.$$

Proof. Let $f \geq 0$ in $L \varphi(\log L) \log_3 L$, then (see [8, p. 244]), we have that

$$A := \int f \varphi(\log_1 f) (\log_3 f) < \infty.$$

If we use the 2-dyadic decomposition, we obtain that, for every $1 < p \leq p_1$ and every $t > 0$,

$$(Tf_n)^*(t) \leq t^{-\frac{1}{p}} \|Tf_n\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p-1}\right) \left(\frac{\|f_n\|_{L^1}}{t}\right)^{1/p},$$

and, by Lemma 2.1.4 we get

$$(Tf_n)^*(t) \lesssim I_{\varphi,1}\left(\frac{\|f_n\|_{L^1}}{t}\right) \lesssim \frac{\varphi(\log_1 t)}{t} I_{\varphi,1}(\|f_n\|_{L^1}).$$

Thus, for $V(t) = t/\varphi(\log_1 t)$,

$$\|Tf_n\|_{\Lambda^{1,\infty}[V]} \lesssim I_{\varphi,1}(\|f_n\|_{L^1}),$$

and

$$\begin{aligned} \|Tf\|_{\Lambda^{1,\infty}[V]} &\leq \left\| \sum_{n \geq 0} 2^{2n} |Tf_n| \right\|_{\Lambda^{1,\infty}[V]} \lesssim \left\| \left(2^{2n} I_{\varphi,1}(\|f_n\|_{L^1}) \right)_n \right\|_{\text{galb}(\Lambda^{1,\infty}[V])} \\ &\lesssim \left\| \left(2^{2n} I_{\varphi,1}(\|f_n\|_{L^1}) \right)_n \right\|_{\ell \log \ell}. \end{aligned}$$

If we denote $c_n = \varphi(2^n) \log_1 n$ and $a_n = \int_{\{d_{n-1} \leq f < d_n\}} f \varphi(\log_1 f) \log_3 f$, we have

$$I_{\varphi,1}(\|f_n\|_{L^1}) \approx I_{\varphi,1}\left(\frac{1}{2^{2n} c_n} \int_{\{d_{n-1} \leq f < d_n\}} f \varphi(\log_1 f) (\log_3 f)\right) = I_{\varphi,1}\left(\frac{a_n}{2^{2n} c_n}\right).$$

Therefore, if we prove that, for every sequence $(a_n)_n$ such that $\|(a_n)_n\|_{\ell_1} = A$, it holds that

$$\left\| \left(2^{2n} I_{\varphi,1}\left(\frac{a_n}{2^{2n} c_n}\right) \right)_n \right\|_{\ell \log \ell} \lesssim 1,$$

we are done.

Now, let $(a_n)_n$ such that $\|(a_n)_n\|_{\ell_1} = A$,

$$\begin{aligned} &\left\| \left(2^{2n} I_{\varphi,1}\left(\frac{a_n}{2^{2n} c_n}\right) \right)_n \right\|_{\ell \log \ell} \lesssim 1 + \sum_{n \geq 1} 2^{2n} I_{\varphi,1}\left(\frac{a_n}{2^{2n} c_n}\right) \log_1 n \\ &\lesssim 1 + \sum_{n \in \text{I}} 2^{2n} I_{\varphi,1}\left(\frac{a_n}{2^{2n} c_n}\right) \log_1 n + \sum_{n \in \text{II}} 2^{2n} I_{\varphi,1}\left(\frac{a_n}{2^{2n} c_n}\right) \log_1 n, \end{aligned}$$

where

$$\text{I} = \{n \geq 1 : a_n \leq 2^{-n}\} \quad \text{and} \quad \text{II} = \{n \geq 1 : a_n > 2^{-n}\}.$$

First, since $I_{\varphi,1}$ is essentially increasing and $\varphi(t) \lesssim t^\beta$ (see Remark 1.3.2), we get

$$\begin{aligned} \sum_{n \in \mathbb{I}} 2^{2^n} I_{\varphi,1} \left(\frac{a_n}{2^{2^n} c_n} \right) \log_1 n &\lesssim \sum_{n \geq 1} 2^{2^n} I_{\varphi,1} \left(\frac{1}{2^{2^n} \varphi(2^n) 2^n \log_1 n} \right) \log_1 n \\ &= \sum_{n \geq 1} \frac{1}{\varphi(2^n) 2^n} \varphi \left(\log_1 (2^{2^n} \varphi(2^n) 2^n \log_1 n) \right) \lesssim \sum_{n \geq 1} \frac{1}{2^n} \approx 1. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{n \in \mathbb{II}} 2^{2^n} I_{\varphi,1} \left(\frac{a_n}{2^{2^n} c_n} \right) \log_1 n &\lesssim \sum_{n \geq 1} \frac{a_n}{\varphi(2^n) \log_1 n} \varphi \left(\log_1 \left(\frac{2^{2^n} \varphi(2^n) \log_1 n}{a_n} \right) \right) \log_1 n \\ &\lesssim \sum_{n \geq 1} \frac{a_n}{\varphi(2^n)} \varphi \left(\log_1 (2^{2^n} \varphi(2^n) 2^n \log_1 n) \right) \lesssim \sum_{n \geq 1} a_n = 1. \end{aligned}$$

□

Remark 3.2.2. Moreover, if for some $\beta_k > 0$, $\beta_{k+1} \cdots \beta_{k+n} \geq 0$,

$$\varphi(s) \approx \prod_{j=k}^{k+n} \left(\log_j s \right)^{\beta_j},$$

we can refine the above argument using the $(k+2)$ -dyadic decomposition. Choosing $c_n = \varphi(\log_1 d_n) \log_1 n$ and $a_n = \int_{\{d_{n-1} \leq f < d_n\}} f \varphi(\log_1 f) \log_{k+3} f$, we obtain that

$$T : L \varphi(\log L) \log_{3+k} L \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi(\log_1 t)}.$$

Example 3.2.3. It is known (see [20], [39]) that, if \mathcal{C}_{lac} is the Carleson lacunary operator (3.13), for every weight $w \in A_1$ and every $1 < p \leq 2$,

$$\mathcal{C}_{\text{lac}} : L^p(w) \longrightarrow L^{p,\infty}(w), \quad \|\mathcal{C}_{\text{lac}}\| \lesssim \log_1 \left(\frac{1}{p-1} \right).$$

Hence, by the above remark,

$$\mathcal{C}_{\text{lac}} : L(\log_2 L)(\log_4 L)(w) \longrightarrow \Lambda_w^{1,\infty}[V], \quad V(t) = \frac{t}{(\log_2 t)}.$$

3.2.2 Case left endpoint $p_0 > 1$

Theorem 3.2.4. *If for any fixed $1 < p_0 < p_1 < \infty$ and, for every $p_0 < p \leq p_1$,*

$$T : L^p \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi \left(\frac{1}{p - p_0} \right),$$

then

$$T : D_{p_0, \varphi}^+ \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t^{1/p_0}}{\varphi(\log_1 t)}.$$

Proof. It follows the same steps than the proof of Theorem 3.1.8, taking into account that, for every $1 < p_0 < p$, instead of (3.5), we have

$$(Tf_i)^{**}(t) \leq t^{-1/p} \|Tf_i\|_{\Gamma^{p,\infty}} \lesssim t^{-1/p} \|Tf_i\|_{L^{p,\infty}} \lesssim t^{-1/p} \varphi \left(\frac{1}{p - p_0} \right) \|f_i\|_{L^p}.$$

□

Example 3.2.5. Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}^+$ and $\Gamma_\alpha(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |y - x| < \alpha t\}$. The *intrinsic square function* (see [101]) is defined by

$$G_\alpha f(x) = \left(\int_{\Gamma_\alpha(x)} |A_\alpha f(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \quad x \in \mathbb{R}^n,$$

where

$$A_\alpha f(y, t) = \sup_{\phi \in \mathcal{C}^\alpha} |(\phi_t * f)(y)|,$$

with $\phi_t(y) = t^{-n} \phi(yt^{-n})$ and \mathcal{C}^α the family of functions supported in $B(0, 1)$ such that $\int \phi = 0$ and

$$|\phi(x) - \phi(x')| < |x - x'|^\alpha, \quad \forall x, x' \in \mathbb{R}^n.$$

It is known [40] that, for $2 < p \leq 3$ and every weight $w \in A_p$,

$$G_\alpha : L^p(w) \longrightarrow L^{p,\infty}(w), \quad \|G_\alpha\| \lesssim [w]_{A_p}^{\frac{1}{2}} \left(\log_1 [w]_{A_\infty} \right)^{\frac{1}{2}},$$

where $[w]_{A_\infty} := \sup_Q \frac{\int_Q M(w\chi_Q)}{w(Q)}$.

Hence, by (3.8) we get that

$$G_\alpha : L^p(|x|^n) \longrightarrow L^{p,\infty}(|x|^n), \quad \|G_\alpha\| \lesssim \frac{1}{(p-2)^{\frac{1}{2}}},$$

and, by Theorem 3.2.4, we have

$$G_\alpha : D_{2, \frac{1}{2}}^+ (|x|^n) \longrightarrow \Gamma_{(|x|^n)}^{1,\infty}[V], \quad V(t) = \frac{t^{1/2}}{(\log_1 t)^{\frac{1}{2}}}.$$

3.2.3 Case right endpoint $p_1 < \infty$

Theorem 3.2.6. *If for any fixed $1 < p_0 < p_1 < \infty$, and, for every $p_0 \leq p < p_1$,*

$$T : L^p \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi\left(\frac{1}{p_1 - p}\right),$$

then

$$T : D_{p_1, \varphi}^- \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t^{1/p_1}}{\varphi\left(\log_1 \frac{1}{t}\right)}.$$

Proof. It follows the same steps than the proof of Theorem 3.1.17. Instead of (3.16), we now have that, for every $1 < p_0 \leq p < p_1$,

$$\begin{aligned} t^{1/p_1} (Tf_i)^{**}(t) &\lesssim t^{\left(\frac{1}{p_1} - \frac{1}{p}\right)} \|Tf_i\|_{L^{p,\infty}} \lesssim t^{\left(\frac{1}{p_1} - \frac{1}{p}\right)} \varphi\left(\frac{1}{p_1 - p}\right) \|f_i\|_{L^p} \\ &\lesssim \varphi\left(\frac{1}{p_1 - p}\right) \left(\frac{|E_i|}{t^{1/p_1}}\right)^{p_1\left(\frac{1}{p} - \frac{1}{p_1}\right)}. \end{aligned}$$

The ODR technique finishes the proof. \square

3.3 Restricted weak type theorems

By restricted weak type, we mean that

$$T : L^{p,1} \longrightarrow L^{p,\infty}.$$

3.3.1 Case endpoint $p_0 = 1$

Theorem 3.3.1. *If, for any fixed $1 < p_1 < \infty$ and, for every $1 < p \leq p_1$,*

$$T : L^{p,1} \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi\left(\frac{1}{p-1}\right),$$

then,

$$T : L \varphi(\log L) \log_3 L \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi(\log_1 t)}.$$

Remark 3.3.2. Observe that the above theorem improves Theorem 3.2.1. In this case, we can follow the same proof because, for $f \in L^1 \cap L^\infty$ with $\|f\|_{L^\infty} \leq 1$, it also holds that $\|f\|_{L^{p,1}} \leq \|f\|_{L^1}^{1/p}$.

3.3.2 Case left endpoint $p_0 > 1$

We adapt the ideas in [27] to the setting of admissible functions.

Theorem 3.3.3. *If for any fixed $1 < p_0 < p_1 < \infty$ and, for every $p_0 < p \leq p_1$,*

$$T : L^{p,1} \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi\left(\frac{1}{p-p_0}\right),$$

then

$$T : \Lambda^1(v_0) \longrightarrow \Gamma^{1,\infty}[V_1]$$

where $v_0(t) = t^{\frac{1}{p_0}-1} \varphi(\log_1 \frac{1}{t})$ and $V_1(t) = \frac{t^{\frac{1}{p_0}}}{\varphi(\log_1 t)}$.

Proof. Given $f \in L^{p_0,1}$, we use the dyadic decomposition over \mathbb{Z} to get $f = \sum_{i \in \mathbb{Z}} 2^i f_i$ where $\|f_i\|_{L^\infty} \leq 1$. For every $t > 0$ and every $p_0 < p \leq p_1$,

$$(Tf_i)^{**}(t) \lesssim t^{-1/p} \|Tf_i\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p-p_0}\right) \frac{\|f_i\|_{L^{p,1}}}{t^{1/p}} \lesssim \varphi\left(\frac{1}{p-p_0}\right) \left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p}.$$

By Lemma 2.1.4, we have

$$(Tf_i)^{**}(t) \lesssim I_{\varphi,p_0}\left(\frac{\|f_i\|_{L^1}}{t}\right) \lesssim \frac{\varphi(\log_1 t)}{t^{1/p_0}} I_{\varphi,p_0}(\|f\|_{L^1}).$$

Summing over $i \in \mathbb{Z}$ and taking the supremum in $t > 0$, the result follows by Lemma 2.3.1. \square

3.3.3 Case right endpoint $p_1 < \infty$

The ideas of this case appear in [27] and we extend them to the setting of admissible functions.

Theorem 3.3.4. *If for any fixed $1 < p_0 < p_1 < \infty$ and, for every $p_0 \leq p < p_1$,*

$$T : L^{p,1} \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi\left(\frac{1}{p_1-p}\right),$$

then

$$T : \Lambda^1(v_0) \longrightarrow \Gamma^{1,\infty}[V_1],$$

where $v_0(t) = t^{\frac{1}{p_1}-1} \varphi(\log_1 t)$ and $V_1(t) = \frac{t^{\frac{1}{p_1}}}{\varphi(\log_1 \frac{1}{t})}$.

Proof. Given $f \in L^{p_1,1}$, we use the dyadic decomposition over \mathbb{Z} to get that, for every $i \in \mathbb{Z}$, for every $t > 0$ and every $p_0 \leq p < p_1$,

$$(Tf_i)^{**}(t) \lesssim t^{-1/p} \|Tf_i\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p_1 - p}\right) \frac{\|f_i\|_{L^{p_1,1}}}{t^{1/p}} \lesssim \varphi\left(\frac{1}{p_1 - p}\right) \left(\frac{\|f_i\|_{L^1}}{t}\right)^{1/p}. \quad (3.18)$$

By Lemma 2.1.4, we obtain

$$(Tf_i)^{**}(t) \lesssim J_{\varphi,p_1} \left(\frac{\|f_i\|_{L^1}}{t}\right) \lesssim \frac{\varphi(\log_1 \frac{1}{t})}{t^{1/p_1}} J_{\varphi,p_1}(\|f_i\|_{L^1}).$$

Summing over $i \in \mathbb{Z}$ and taking the supremum in $t > 0$, the result follows by Lemma 2.3.1. \square

3.3.4 Case endpoint $p_1 = \infty$

The case where the endpoint is $p_1 = \infty$ was considered first in [104] in the context of finite measure spaces, and the extension to σ -finite measure spaces was studied in [30]. Here, we extend it to the context of admissible functions.

Theorem 3.3.5. *If for a fixed $1 < p_0 < \infty$, and, for every $p_0 \leq p < \infty$,*

$$T : L^{p,1} \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi(p),$$

then

$$T : L_\varphi^\infty \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{1}{\varphi(\log_1 \frac{1}{t})}.$$

Proof. Given $f \in L^\infty$, we use the f^* -decomposition to get that $f = \sum_{n \geq 0} f_n$ with $f_n \in L^1 \cap L^\infty$ and, for every $t > 0$ and every $p_0 \leq p < \infty$,

$$(Tf_n)^{**}(t) \lesssim t^{-\frac{1}{p}} \|Tf_n\|_{L^{p,\infty}} \lesssim t^{-\frac{1}{p}} \varphi(p) \|f\|_{L^{p,1}} \lesssim \|f_n\|_{L^\infty} \varphi(p) \left(\frac{\|f_n\|_{L^1}}{t\|f_n\|_{L^\infty}}\right)^{1/p}. \quad (3.19)$$

By Lemma 2.1.4 we obtain

$$(Tf_n)^{**}(t) \lesssim \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{t\|f_n\|_{L^\infty}}\right) \lesssim \varphi\left(\log_1 \frac{1}{t}\right) \|f_n\|_{L^\infty} \varphi\left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}}\right).$$

The result follows from Lemma 2.3.2. \square

Remark 3.3.6. Recall that, if $\varphi(t) = t^\alpha$ for some $\alpha > 0$ and $V(t) = \frac{1}{\varphi(\log_1 \frac{1}{t})}$, we have that $\Gamma^{1,\infty}[V] = E^\alpha$ (see Definition 1.4.4).

Example 3.3.7. Let G_α be the intrinsic square function (see Example 3.2.5). It is known that (see [73]) that, for every $w \in A_3$,

$$G_\alpha : L^3(w) \longrightarrow L^3(w), \quad \|G_\alpha\| \lesssim [w]_{A_3}^{\frac{1}{2}}.$$

By Rubio de Francia extrapolation (see Theorem 1.5.2), we obtain that, for every $3 < p < \infty$ and every $w \in A_p$,

$$G_\alpha : L^p(w) \longrightarrow L^p(w) \quad \|G_\alpha\| \lesssim p^{\frac{1}{2}} [w]_{A_p}^{\frac{1}{2}}.$$

If we fix $w \in A_\infty$, we can take $3 < p_0 < \infty$ such that, $w \in A_{p_0}$ and, for every $p_0 \leq p < \infty$,

$$G_\alpha : L^p(w) \longrightarrow L^p(w) \quad \|G_\alpha\| \lesssim p^{\frac{1}{2}} [w]_{A_{p_0}}^{\frac{1}{2}} \approx p^{\frac{1}{2}}.$$

Consequently, Theorem 3.3.5 implies that,

$$G_\alpha : L_{\frac{1}{2}}^\infty(w) \longrightarrow E^{\frac{1}{2}}(w).$$

Example 3.3.8. Let m be a Marcinkiewicz multiplier (see Example 3.1.4). By (3.4) and Theorem 3.3.5, we can deduce that

$$T_m : L_{\frac{3}{2}}^\infty \longrightarrow E^{\frac{3}{2}}.$$

Moreover, for every $\varepsilon > 0$, if we take $p_0 := 1 + \frac{1}{2\varepsilon}$, we have (see [74]) that, for every $w \in A_{p_0}$,

$$T_m : L^{p_0}(w) \longrightarrow L^{p_0}(w), \quad \|T\| \lesssim [w]_{A_{p_0}}^{\frac{3}{2}+\varepsilon}.$$

By Rubio de Francia extrapolation, we get that, for every $p_0 \leq p < \infty$, and every $w \in A_p$,

$$T_m : L^p(w) \longrightarrow L^p(w), \quad \|T\| \lesssim (p [w]_{A_p})^{\frac{3}{2}+\varepsilon},$$

and hence, for every $w \in A_\infty$,

$$T_m : L_{\frac{3}{2}+\varepsilon}^\infty(w) \longrightarrow E^{\frac{3}{2}+\varepsilon}(w).$$

Example 3.3.9. Let \mathcal{C} be the Carleson operator (3.11). If we fix $0 < \varepsilon \leq 1$ and $p_0 := \frac{1+\varepsilon}{\varepsilon}$, (3.12) implies that, for every $w \in A_{p_0}$,

$$\mathcal{C} : L^{p_0}(w) \longrightarrow L^{p_0}(w) \quad \|\mathcal{C}\| \lesssim [w]_{A_{p_0}}^{1+\varepsilon} \log_2[w]_{A_{p_0}}.$$

Using Rubio de Francia extrapolation and Theorem 3.3.5, we get that, if $\varphi(s) = s^{1+\varepsilon} \log_2 s$, for every $w \in A_\infty$,

$$\mathcal{C} : L_\varphi^\infty(w) \longrightarrow \Gamma_w^{1,\infty}[V], \quad V(t) = \frac{1}{\varphi(\log_1 \frac{1}{t})}.$$

In the case of the Carleson lacunary \mathcal{C}_{lac} , for every $w \in A_\infty$, it follows from (3.14) that, if $\varphi(s) = s \log_1 s$,

$$\mathcal{C}_{\text{lac}} : L_\varphi^\infty(w) \longrightarrow \Gamma_w^{1,\infty}[V], \quad V(t) = \frac{1}{\varphi(\log_1 \frac{1}{t})}.$$

Example 3.3.10. Since the Hilbert transform H is bounded on $L^p(w)$ for every $w \in A_p$ with $\|H\| \lesssim p[w]_{A_p}$ when $p \geq 2$ (see [86]), from Theorem 3.3.5 it follows

$$H : L_1^\infty(w) \longrightarrow E^1(w).$$

It is also known (see [93]) that

$$H : L^\infty \longrightarrow BMO,$$

where BMO is the space of bounded mean oscillation functions. However, E^1 is not comparable with BMO . Indeed, $\log|x| \in BMO$ (see [46, Example 3.1.3]), but $\log|x| \notin E^1$ because is an increasing function. On the other hand, $h(s) = \chi_{(0,1)}(s) \log \frac{1}{s} \notin BMO$ (see [46, Example 3.1.4]) but $h(s) \in E^1$ because, for $0 < t \leq 1$,

$$h^{**}(t) = \frac{1}{t} \int_0^t \log \frac{1}{s} ds = \frac{1}{t} \left[s + s \log \frac{1}{s} \right]_0^t = \log_1 \frac{1}{t}$$

and, therefore, for $V(t) = (\log_1 \frac{1}{t})^{-1}$, $\|h\|_{\Gamma^{1,\infty}[V]} \leq 1$. Consequently,

$$H : L_1^\infty \longrightarrow BMO \cap E^1,$$

where $BMO \cap E^1 \subsetneq BMO$.

3.4 Hypotheses for characteristic functions

Here, we improve Theorem 3.1.1 by assuming weaker hypotheses. The proof follows the same argument presented in [2].

Theorem 3.4.1. *If for some fixed $1 < p_1 < \infty$, and, for every $1 < p \leq p_1$ and every measurable set E ,*

$$\|T\chi_E\|_{L^p} \lesssim \varphi\left(\frac{1}{p-1}\right) \mu(E)^{1/p},$$

then

$$T : L \varphi(\log L) \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi(\log_1 t)}.$$

Proof. Fix a measurable set $E \subseteq X$ with $\mu(E) < \infty$. For every $1 < p \leq p_1$ and $t > 0$,

$$t^{1/p}(T\chi_E)^{**}(t) \leq \|T\chi_E\|_{L^p} \lesssim \varphi\left(\frac{1}{p-1}\right) \mu(E)^{1/p}.$$

Taking the infimum over $1 < p \leq p_1$, we obtain by (2.4) that

$$(T\chi_E)^{**}(t) \lesssim \left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_1}} \chi_{(0,\mu(E)]}(t) + \left(\frac{\mu(E)}{t}\right) \varphi\left(\log_1 \frac{t}{\mu(E)}\right) \chi_{(\mu(E),\infty)}(t).$$

Now, since, for every $t < \mu(E)$,

$$\left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_1}} \approx 1 + \frac{1}{t^{\frac{1}{p_1}}} \int_t^{\mu(E)} \frac{ds}{s^{1-\frac{1}{p_1}}} \lesssim \frac{1}{t} \int_0^t \varphi\left(\log_1 \frac{t}{s}\right) ds + \frac{1}{t^{\frac{1}{p_1}}} \int_t^{\mu(E)} \frac{ds}{s^{1-\frac{1}{p_1}}},$$

and for every $t > \mu(E)$,

$$\left(\frac{\mu(E)}{t}\right) \varphi\left(\log_1 \frac{t}{\mu(E)}\right) \lesssim \frac{1}{t} \int_0^{\mu(E)} \varphi\left(\log_1 \frac{t}{s}\right) ds,$$

we get that, for $t > 0$,

$$\begin{aligned} (T\chi_E)^{**}(t) &\lesssim \frac{1}{t} \int_0^t \varphi\left(\log_1 \frac{t}{s}\right) \chi_{(0,\mu(E)]}(s) ds + \frac{1}{t^{\frac{1}{p_1}}} \int_t^\infty \chi_{(0,\mu(E)]}(s) \frac{ds}{s^{1-\frac{1}{p_1}}} \\ &= R_{1,p_1,\varphi}((\chi_E)^*)(t), \end{aligned} \tag{3.20}$$

where, for every measurable function f ,

$$R_{1,p_1,\varphi}(f^*)(t) := \frac{1}{t} \int_0^t \varphi\left(\log_1 \frac{t}{s}\right) f^*(s) ds + \frac{1}{t^{\frac{1}{p_1}}} \int_t^\infty f^*(s) \frac{ds}{s^{1-\frac{1}{p_1}}}.$$

To extend this inequality to simple functions, we take $f = \sum_{n=1}^j a_n \chi_{F_n}$, where the sets $F_1 \subseteq F_2 \subseteq \dots \subseteq F_j$ have finite measure. Thus, $f^* = \sum_{n=1}^j a_n \chi_{[0, \mu(F_n))}$, and, by (3.20), we get

$$(Tf)^{**}(t) \lesssim \sum_{n=1}^j a_j (T\chi_{F_n})^{**}(t) \lesssim \sum_{n=1}^j a_n R_{1,p_1,\varphi}((\chi_{F_n})^*)(t) = R_{1,p_1,\varphi}(f^*)(t).$$

To extend the above inequality to arbitrary measurable functions we use a density argument. In fact, we observe that

$$R_{1,p_1,\varphi}(f^*)(t) = \|f\|_{\Lambda^1(v_t)}$$

where

$$v_t(s) := \begin{cases} \frac{1}{t} \varphi \left(\log_1 \frac{t}{s} \right), & 0 < s < t, \\ \frac{1}{t} \left(\frac{t}{s} \right)^{1-\frac{1}{p_1}}, & t \leq s. \end{cases}$$

Therefore, given a sequence of simple functions such that $|f_n| \uparrow |f|$ at almost every point, we have that, for every $n, m \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{t} \sup_{\mu(E)=t} \int_E \|Tf_n| - |Tf_m|\| &\leq \frac{1}{t} \sup_{\mu(E)=t} \int_E |T(f_n - f_m)| = T(f_n - f_m)^{**}(t) \\ &\lesssim R_{1,p_1,\varphi}((f_n - f_m)^*)(t) = \|f_n - f_m\|_{\Lambda^1(v_t)}, \end{aligned}$$

where we have used [8, Prop. 3.3, p. 53] in the second one.

Since $v_t \notin L^1$, [28, Theorem 2.3.4] implies that $\Lambda^1(v_t)$ has absolutely continuous norm and, in view of Proposition 2.3.3. of the same work,

$$\lim_{n,m \rightarrow \infty} \|f_n - f_m\|_{\Lambda^1(v_t)} = 0.$$

Consequently, for every set E with finite measure, Tf_n is a Cauchy sequence in $L^1(E)$, and therefore converges to a function $T_E f$. Since our measure space is σ -finite, let us take a sequence $(E_n)_n$ of sets with finite measure such that $\cup_{n \geq 1} E_n = X$. Define $Tf(x) := T_{E_n} f(x)$ for every $x \in E_n$. Let us see that Tf is well defined. Indeed, for any pair of sets E, F such that $\mu(E \cap F) > 0$, if we fix $\varepsilon > 0$, there exists $n > 0$ such that

$$\|T_E f - T_F f\|_{L^1(E \cap F)} \leq \|T_E f - T_{E_n} f\|_{L^1(E)} + \|T_{E_n} f - T_F f\|_{L^1(F)} \leq \varepsilon,$$

and, therefore, $T_E f = T_F f$ almost everywhere in $E \cap F$. Moreover, this also implies that the definition of Tf does not depend on the sequence of sets $(E_n)_n$ chosen.

Hence,

$$\begin{aligned} (Tf)^{**}(t) &= \sup_{\mu(E)=t} \frac{1}{t} \int_E |Tf| = \sup_{\mu(E)=t} \lim_{n \rightarrow \infty} \frac{1}{t} \int_E |Tf_n| \leq \lim_{n \rightarrow \infty} \frac{1}{t} \int_0^t (Tf_n)^* \\ &\lesssim \lim_{n \rightarrow \infty} R_{1,p_1,\varphi}(f_n^*)(t) \lesssim R_{1,p_1,\varphi}(f^*)(t). \end{aligned}$$

Finally, since we have (3.20) for all measurable functions and every $t > 0$, using the properties of φ it follows that

$$(Tf)^{**}(t) \lesssim R_{1,p_1,\varphi}(f^*)(t) \lesssim \frac{\varphi(\log_1 t)}{t} \int_0^\infty \varphi\left(\log_1 \frac{1}{s}\right) f^*(s) ds,$$

and taking the supremum over $t > 0$ we are done. \square

For our next result, we need an extra condition on the operator T ; namely that T is (ε, δ) -atomic approximable (see [18]).

Theorem 3.4.2. *If for some fixed $1 < p_1 < \infty$, and, for every $1 < p \leq p_1$ and every measurable set E ,*

$$\|T\chi_E\|_{L^{p,\infty}} \lesssim \varphi\left(\frac{1}{p-1}\right) \mu(E)^{1/p},$$

and T is (ε, δ) -atomic approximable, then

$$T : L \varphi(\log L) \log_3 L \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi(\log_1 t)}.$$

Proof. From [18] follows that, for every $f \in L^1 \cap L^\infty$ such that $\|f\|_{L^\infty} \leq 1$,

$$(Tf)^*(t) \lesssim \varphi\left(\frac{1}{p-1}\right) \left(\frac{\|f\|_{L^1}}{t}\right)^{1/p}.$$

Thus, we can use the proof of Theorem 3.3.1 to conclude the result. \square

Example 3.4.3. The Carleson operator \mathcal{C} is (ε, δ) -atomic approximable (see [18]). From (3.1), we obtain

$$\mathcal{C} : L \log L \log_3 L \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t}{(\log_1 t)},$$

recovering Antonov's result.

3.5 Extrapolation on $L^{p,\infty}$

Let us introduce now some results where the hypotheses are of the form

$$T : L^{p,\infty} \longrightarrow L^{p,\infty}.$$

3.5.1 Case endpoint $p_0 \geq 1$

The particular case $p_0 = 1$ can be found in [31]. We extend the result to $p_0 > 1$ and to the setting of admissible functions. We first define the function spaces that will be used.

Definition 3.5.1. The spaces $[L \varphi(\log L) \log_3 L]$ and $D_{p,\infty,\varphi}^+$ are defined by the norms

$$\|f\|_{[L \varphi(\log L) \log_3 L]} = \|f\|_{L^{1,\infty}} + \int_0^1 \sup_{t \leq s} t f^*(t) \phi'(s) ds < \infty,$$

where $\phi(s) = -\varphi\left(\log_1 \frac{1}{s}\right) \log_3 \frac{1}{s}$, and

$$\|f\|_{D_{p,\infty,\varphi}^+} = \|f\|_{L^{p,\infty}} + \int_0^1 \frac{\sup_{t \leq s} t^{\frac{1}{p}} f^*(t)}{s} \varphi'\left(\log_1 \frac{1}{s}\right) ds.$$

We observe that, if $\varphi'(x) \approx \frac{\varphi(x)}{x}$,

$$\|f\|_{[L \varphi(\log L) \log_3 L]} = \|f\|_{L^{1,\infty}} + \int_0^1 \frac{\sup_{t \leq s} t f^*(t)}{s} \varphi'\left(\log_1 \frac{1}{s}\right) \log_3 \frac{1}{s} ds < \infty.$$

Theorem 3.5.2. *If for some fixed $1 \leq p_0 < p_1 < \infty$ and, for every $p_0 < p \leq p_1$,*

$$T : L^{p,\infty} \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi\left(\frac{1}{p - p_0}\right),$$

then:

i) If $p_0 = 1$,

$$T : [L \varphi(\log L) \log_3 L] \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi(\log_1 t)}.$$

ii) If $p_0 > 1$,

$$T : D_{p_0,\infty,\varphi}^+ \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t^{\frac{1}{p_0}}}{\varphi(\log_1 t)}.$$

Proof. Fix $g_0 \in L^{p_0, \infty}$ and define the function $H(s) = \sup_{t \leq s} t(g_0^*(t))^{p_0}$. Clearly, $(g_0^*(s))^{p_0} \leq H(s)/s$, which implies that there exists $h_0 \in L^\infty$ with $\|h_0\|_{L^\infty} \leq 1$ such that $g_0^*(s) = h_0(s) \left(\frac{H(s)}{s}\right)^{1/p_0}$.

Since $g_0 \in L^{p_0, \infty}$, $\lim_{t \rightarrow \infty} g_0^*(t) = 0$ and, by [8, Corollary 7.6, p.83], there exists a measure preserving transformation σ_0 such that $g_0(x) = g_0^*(\sigma_0(x))$ and hence,

$$g_0(x) = h_0(\sigma_0(x)) \left(\frac{H(\sigma_0(x))}{\sigma_0(x)}\right)^{1/p_0} := h_0 \sigma_0(x) H_{\sigma_0}(x). \quad (3.21)$$

We have that $(H_{\sigma_0})^*(s) = (H(s)/s)^{1/p_0}$ where H is quasi-concave (because $H(s)$ is increasing and $H(s)/s$ decreasing) and, thus, there exists a decreasing function \tilde{g} such that

$$(H_{\sigma_0})^*(s) \approx \left(\frac{\int_0^s (\tilde{g}(t))^{p_0} dt}{s}\right)^{1/p_0}.$$

Indeed, since $\|H_{\sigma_0}\|_{L^{p_0, \infty}} = \|g_0\|_{L^{p_0, \infty}}$, we have that $H_{\sigma_0} \in L^{p_0, \infty}$ and $\lim_{t \rightarrow \infty} \tilde{g}(t) = 0$. Thus, there exists f_0 such that $f_0^* = \tilde{g}$. Moreover, there also exists h_1 such that $h_1, h_1^{-1} \in L^\infty$ and a measure preserving transformation σ_1 such that

$$H_{\sigma_0}(x) = h_1(\sigma_1(x)) \left(\frac{\int_0^{\sigma_1(x)} (f_0^*(t))^{p_0} dt}{\sigma_1(x)}\right)^{1/p_0}.$$

For every $f \in L_{\text{loc}}^{p_0}$, we can define the operator

$$Sf(x) := h_1(\sigma_1(x)) \left(\frac{\int_0^{\sigma_1(x)} (f^*(t))^{p_0} dt}{\sigma_1(x)}\right)^{1/p_0}.$$

We observe that

$$Sf_0 = H_{\sigma_0}, \quad (3.22)$$

and $(Sf)^*(t) \approx \left(\frac{1}{t} \int_0^t (f^*(s))^{p_0} ds\right)^{1/p_0}$. Therefore, for $p > p_0$,

$$\|Sf\|_{L^{p, \infty}} \approx \sup_{t > 0} t^{\frac{1}{p} - \frac{1}{p_0}} \left(\int_0^t (f^*(s))^{p_0} ds\right)^{\frac{1}{p_0}} \leq \|f\|_{L^p}.$$

On the other hand, the operator $T'g := T((h_0 \circ \sigma_0) \cdot g)$, satisfies the hypothesis of the theorem and hence,

$$T' \circ S : L^p \longrightarrow L^{p, \infty}, \quad \|T' \circ S\| \lesssim \varphi\left(\frac{1}{p - p_0}\right).$$

By (3.21) and (3.22), we have

$$Tg_0(x) = T((h_0 \circ \sigma_0) \cdot H_{\sigma_0})(x) = T'(H_{\sigma_0})(x) = T'(Sf_0)(x).$$

If $p_0 > 1$, we can apply Theorem 3.2.4 to $T' \circ S$ and get

$$\begin{aligned} \|Tg_0\|_{\Gamma^{1,\infty}[V]} &= \|(T' \circ S)(f_0)\|_{\Gamma^{1,\infty}[V]} \\ &\lesssim \|f_0\|_{L^{p_0}} + \int_0^1 \frac{(\int_0^t (f_0^*(s))^{p_0} ds)^{\frac{1}{p_0}}}{t} \varphi' \left(\log_1 \frac{1}{t} \right) dt \\ &\approx \|Sf_0\|_{L^{p_0,\infty}} + \int_0^1 (Sf_0)^*(t) t^{\frac{1}{p_0}-1} \varphi' \left(\log_1 \frac{1}{t} \right) dt \\ &= \|H_{\sigma_0}\|_{L^{p_0,\infty}} + \int_0^1 (H_{\sigma_0})^*(t) t^{\frac{1}{p_0}-1} \varphi' \left(\log_1 \frac{1}{t} \right) dt \\ &\approx \|g_0\|_{L^{p_0,\infty}} + \int_0^1 \frac{\sup_{s \leq t} s^{\frac{1}{p_0}} g_0^*(s)}{t} \varphi' \left(\log_1 \frac{1}{t} \right) dt. \end{aligned}$$

If $p_0 = 1$ and $\phi(s) = -\varphi\left(\log_1 \frac{1}{s}\right) \log_3 \frac{1}{s}$, by Theorem 3.3.1 and Fubini, we get

$$\begin{aligned} \|Tg_0\|_{\Lambda^{1,\infty}[V]} &= \|(T' \circ S)(f_0)\|_{\Lambda^{1,\infty}[V]} \lesssim \|f\|_{L^{\varphi(\log L) \log_3 L}} = \int_0^\infty f_0^*(t) (-\phi(t)) dt \\ &= \|f_0\|_{L^1} + \int_0^1 f_0^*(t) (-\phi(t)) dt \lesssim \|f_0\|_{L^1} + \int_0^1 f_0^*(t) \int_t^1 \phi'(s) ds dt \\ &= \|f_0\|_{L^1} + \int_0^1 \left(\frac{1}{s} \int_0^s f_0^*(t) dt \right) s \phi'(s) ds \approx \|Sf_0\|_{L^{1,\infty}} + \int_0^1 (Sf_0)^*(s) s \phi'(s) ds, \\ &= \|H_{\sigma_0}\|_{L^{1,\infty}} + \int_0^1 (H_{\sigma_0})^*(s) s \phi'(s) ds \approx \|g_0\|_{L^{1,\infty}} + \int_0^1 \sup_{s \leq t} s^{\frac{1}{p_0}} g_0^*(s) \phi'(s) ds. \end{aligned}$$

□

3.5.2 Case left endpoint $p_1 < \infty$

We observe that, Theorem 3.5.2 can also be proved with the ODR technique using the $K(L^{p_0,\infty}; L^\infty)$ -decomposition.

Let us present the case where the endpoint is $p_1 < \infty$, using this method.

Definition 3.5.3. The space $D_{p,\infty,\varphi}^-$ is defined by the norm

$$\|f\|_{D_{p,\infty,\varphi}^-} = \|f\|_{L^{p,\infty}} + \int_1^\infty \frac{\sup_{t \geq s} t^{\frac{1}{p}} f^{**}(t)}{s} \varphi'(\log_1 s) ds.$$

Theorem 3.5.4. *If for some fixed $1 < p_0 < p_1 < \infty$ and, for every $p_0 \leq p < p_1$,*

$$T : L^{p,\infty} \longrightarrow L^{p,\infty}, \quad \|T\| \lesssim \varphi \left(\frac{1}{p_1 - p} \right),$$

then

$$T : D_{p_1,\infty,\varphi}^- \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t^{\frac{1}{p_1}}}{\varphi \left(\log_1 \frac{1}{t} \right)}.$$

Proof. Let $f \in L^{p_1,\infty}$ with $\|f\|_{L^{p_1,\infty}} \leq 1/p_1'$. Hence, we can decompose f using the $K(L^1, L^{p_1,\infty})$ -decomposition. The rest of the proof follows the same steps than the proof of Theorem 3.2.6, using part *d*) of Lemma 2.2.2 and Lemma 2.3.4. \square

3.6 Fixed target space

Theorem 3.6.1. *If for any fixed $1 < p_1 < \infty$, any fixed quasi-Banach space \mathcal{X} , and for every $1 < p \leq p_1$,*

$$T : L^p \longrightarrow \mathcal{X}, \quad \|T\| \lesssim \varphi \left(\frac{1}{p-1} \right),$$

then

$$\|Tf\|_{\mathcal{X}} \lesssim \inf_{\sum_n e_n f_n = f} \left\| \left(e_n I_{\varphi,1} (\|f_n\|_{L^1}) \right)_n \right\|_{\text{galb}(\mathcal{X})},$$

where the infimum is taken over all possible decompositions of f with $\|f_n\|_{L^\infty} \leq 1$.

Proof. If $f \in L^1$ such that $f = \sum_n e_n f_n$ and $\|f_n\|_{L^\infty} \leq 1$, since $f_n \in L^1 \cap L^\infty$, we have by Lemma 2.1.4 that

$$\|Tf_n\|_{\mathcal{X}} \lesssim \inf_{1 < p \leq p_1} \varphi \left(\frac{1}{p-1} \right) \|f_n\|_{L^1}^{\frac{1}{p}} \lesssim I_{\varphi,1} (\|f_n\|_{L^1}),$$

and, therefore,

$$\|Tf\|_{\mathcal{X}} \leq \left\| \sum_n e_n Tf_n \right\|_{\mathcal{X}} = \left\| \sum_n e_n \|Tf_n\|_{\mathcal{X}} \frac{Tf_n}{\|Tf_n\|_{\mathcal{X}}} \right\|_{\mathcal{X}} \leq \left\| (e_n \|Tf_n\|_{\mathcal{X}})_n \right\|_{\text{galb}(\mathcal{X})}.$$

\square

Example 3.6.2. Let M_Ω be the rough maximal operator on \mathbb{R}^2 (see (3.3)). It is known (see [34]) that, if $\Omega \in L \log L(\mathbb{S}^1)$, then M_Ω is weak type (1,1). However, a weaker result can be easily deduced from [33] using extrapolation. Indeed, in that work, it is proved that, for every $q > 1$ and $\Omega \in L^q(\mathbb{S}^1)$,

$$M_\Omega : L^1(\mathbb{R}^2) \longrightarrow L^{1,\infty}(\mathbb{R}^2), \quad \|M_\Omega\| \lesssim \frac{1}{(q-1)^{\frac{2}{1+\frac{1}{q}}}} \|\Omega\|_{L^q(\mathbb{S}^1)}.$$

Thus, for every $\varepsilon > 0$ and every $1 < q \leq \frac{1+\varepsilon}{1-\varepsilon}$,

$$\|M_\Omega\| \lesssim \frac{1}{(q-1)^{1+\varepsilon}} \|\Omega\|_{L^q(\mathbb{S}^1)}.$$

If we fix $f \in L^1(\mathbb{R}^2)$ and we denote $M_f(\Omega) := M_\Omega(f)$, then

$$M_f : L^q(\mathbb{S}^1) \longrightarrow L^{1,\infty}(\mathbb{R}^2), \quad \|M_f\| \lesssim \frac{\|f\|_{L^1(\mathbb{R}^2)}}{(q-1)^{1+\varepsilon}}.$$

Using Theorem 3.6.1 and following the computations in the proof of Theorem 3.2.1, we obtain

$$M_f : L(\log L)^{1+\varepsilon} \log_3 L(\mathbb{S}^1) \longrightarrow L^{1,\infty}(\mathbb{R}^2),$$

which implies that, for every $\Omega \in L(\log L)^{1+\varepsilon} \log_3 L(\mathbb{S}^1)$, M_Ω is weak type (1, 1).

CHAPTER 4

Bilinear Yano's extrapolation

The study of multilinear operators began in the 1970s with the work of Coifman and Meyer (see [36], [37]), who studied commutators of bilinear singular integrals, such as the bilinear Hilbert transform, defined as follows

$$B_H(f, g)(x) := \lim_{\varepsilon \rightarrow 0} \int_{|t| > \varepsilon} f(x-t)g(x+t) \frac{dt}{t}, \quad x \in \mathbb{R}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Years earlier Calderón had posed the question of whether B_H satisfies an estimate of the form

$$B_H : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \longrightarrow L^p(\mathbb{R}), \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \quad (4.1)$$

In particular, he conjectured that the case $p_1 = p_2 = 2$ holds. The first results concerning this problem were provided by Lacey and Thiele (see [71], [72]), who proved that (4.1) is true when $1 < p_1, p_2 \leq \infty$ and $2/3 < p < \infty$. Their work attracted significant attention to this research area, leading to the development of many results in the setting of multilinear operators, such as those by Kenig and Stein (see [66]), Muscalu, Tao, and Thiele [83], Grafakos and Li [50], and Grafakos and Torres (see [52], [53]).

In this chapter, we develop a multilinear version of Yano's extrapolation theory. We present the results only in the bilinear context to simplify the notation. The extension to the multilinear case is straight forward.

For $1 \leq p_1, p_2 \leq \infty$ and $0 < p \leq \infty$, we denote by $(p_1, p_2; p)$ the triple that satisfies Hölder's condition

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Thus, if we fix an endpoint $(q_1, q_2; q)$, we consider operators

$$B : L^{p_1, r_1} \times L^{p_2, r_2} \longrightarrow L^{p, r}$$

such that $\|B\| \rightarrow \infty$ as $p_j \rightarrow q_j$ for $j = 1, 2$, and we want to study the behaviour of B at the endpoint spaces. As done in the previous chapter, we distinguish different cases (strong, weak, etc.) depending on the values of r_1, r_2 and r where

$$r_1 \in \{1, p_1\}, \quad r_2 \in \{1, p_2\} \quad \text{and} \quad r \in \{p, \infty\}.$$

In the following, let $(p_1, p_2; p)$ denote the points where we already have an estimate, and let $1 < p_1^+, p_2^+, p_1^-, p_2^- < \infty$ be fixed numbers. If we aim to reach the point $(q_1, q_2; q)$, we could have, for example, the hypotheses for all $(p_1, p_2; p)$ such that $q_1 < p_1 \leq p_1^+$ and $p_2^- \leq p_2 < q_2$ (see Figure 4.1).

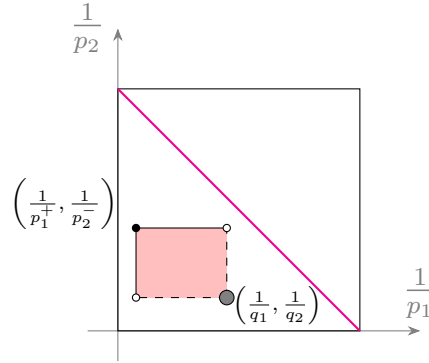


Figure 4.1: $(p_1, p_2; p)$ such that $q_1 < p_1 \leq p_1^+$ and $p_2^- \leq p_2 < q_2$.

The chapter is structured as follows: in Section 4.1, we shall treat the case when $L^{q, r}$ is Banach and Section 4.2 will be dedicated to the quasi-Banach context. Our proofs will follow the ODR technique and, in both settings, we mainly use linearization in each variable. For example, if

$$B : L^{p_1, r_1} \times L^{q_2, r_2} \longrightarrow L^{p, r}, \quad \|B\| \lesssim \varphi \left(\frac{1}{p_1 - q_1} \right),$$

and we fix $g \in L^{q_2, r_2}$, we shall consider the linear operator $T_1 := B(\cdot, g)$. Clearly, we have that

$$T_1 : L^{p_1, r_1} \longrightarrow L^{p, r}, \quad \|T_1\| \lesssim \varphi \left(\frac{1}{p_1 - q_1} \right) \|g\|_{L^{q_2, r_2}}.$$

Consequently, except in the case $q_2 = \infty$, we need to develop an off-diagonal extrapolation result.

Finally, we should mention that some partial results concerning the bilinear setting have been stated in the context of abstract extrapolation theory (see [63], [5]).

4.1 Banach setting

In this section we consider the case when $L^{q,r} = L^q$ with $q \geq 1$ or $L^{q,r} = L^{q,\infty}$ with $q > 1$.

4.1.1 Off-Diagonal extrapolation

Proposition 4.1.1. *Suppose that $1 \leq q_1, q < \infty$. If, for every $q_1 < p_1 \leq p_1^+$ and p such that*

$$\frac{1}{p} - \frac{1}{p_1} = \frac{1}{q} - \frac{1}{q_1}, \quad (4.2)$$

$$T : L^{p_1, r_1} \longrightarrow L^{p, r}, \quad \|T\| \lesssim \varphi \left(\frac{1}{p_1 - q_1} \right),$$

then

$$T : D^+ \longrightarrow R^+,$$

where D^+ and R^+ are determined by the values described in Table 4.1 (see Definition 3.1.7).

r_1	D^+	r	R^+
1	$\Lambda(v_1), v_1(t) = t^{\frac{1}{q_1} - 1} \varphi(\log_1 \frac{1}{t})$	p	$R_{q,\varphi}^+$
p_1	$D_{q_1,\varphi}^+$	∞ ($q \neq 1$)	$\Gamma^{1,\infty}[V], V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)}$

Table 4.1: Proposition 4.1.1.

Proof. $\boxed{r_1 = p_1}$ If $r = p$, given $f \in L^{q_1}$, we use the $K(L^{q_1}, L^\infty)$ -decomposition and Lemma 2.2.2 a) to get $f = \sum_{i \in \mathbb{Z}} 2^i f_i$ with $\|f_i\|_{L^{p_1}} \lesssim |E_i|^{q_1/p_1}$ for every $q_1 < p_1 \leq p_1^+$.

Since $p > q$,

$$\begin{aligned} \left(\int_0^t [(Tf_i)^*(s)]^q ds \right)^{1/q} &\leq t^{\frac{1}{q} - \frac{1}{p}} \|Tf_i\|_{L^p} \lesssim t^{\frac{1}{q_1} - \frac{1}{p_1}} \varphi \left(\frac{1}{p_1 - q_1} \right) \|f_i\|_{L^{p_1}} \\ &\lesssim t^{1/q_1} \varphi \left(\frac{1}{p_1 - q_1} \right) \left(\frac{|E_i|^{q_1}}{t} \right)^{1/p_1}, \end{aligned}$$

where we have used (4.2) in the second inequality. Therefore, Lemma 2.1.4 implies

$$\left(\int_0^t [(Tf_i)^*(s)]^q ds \right)^{1/q} \lesssim t^{1/q_1} I_{\varphi, q_1} \left(\frac{|E_i|^{q_1}}{t} \right) \lesssim \varphi(\log_1 t) I_{\varphi, 1}(|E_i|).$$

The rest of the proof follows the same argument as Theorem 3.1.8 with $p_0 = q$ in (3.6).

If $r = \infty$, we can follow the same steps than the ones in Theorem 3.2.4 because

$$t^{\frac{1}{q}}(Tf_i)^{**}(t) \lesssim t^{\frac{1}{q} - \frac{1}{p}} \|Tf_i\|_{L^{p, \infty}} \lesssim t^{\frac{1}{q_1} - \frac{1}{p_1}} \varphi \left(\frac{1}{p_1 - q_1} \right) \|f_i\|_{L^{p_1}}. \quad (4.3)$$

$\boxed{r_1 = 1}$ If $r = p$, we decompose $f \in L^{q_1, 1}$ with the dyadic decomposition over \mathbb{Z} . Thus, for every $i \in \mathbb{Z}$, $p_1^- \leq p_1 < q_1$,

$$\left(\int_0^t [(Tf_i)^*(s)]^q ds \right)^{1/q} \leq t^{\frac{1}{q} - \frac{1}{p}} \|Tf_i\|_{L^p} \lesssim t^{1/q_1} \varphi \left(\frac{1}{p_1 - q_1} \right) \left(\frac{\|f_i\|_{L^1}}{t} \right)^{1/p_1}.$$

By Lemma 2.1.4 we get

$$\left(\int_0^t [(Tf_i)^*(s)]^q ds \right)^{1/q} \lesssim t^{1/q_1} I_{\varphi, q_1} \left(\frac{\|f_i\|_{L^1}}{t} \right) \lesssim \varphi(\log_1 t) I_{\varphi, q_1}(\|f_i\|_{L^1}).$$

Finally, by summing over $i \in \mathbb{Z}$, using Lemma 2.3.1 and taking the supremum over $t > 0$, we obtain

$$\sup_{t>0} \frac{\left(\int_0^t [(Tf_i)^*(s)]^q ds \right)^{1/q}}{\varphi(\log_1 t)} \lesssim \sum_{i \in \mathbb{Z}} 2^i I_{\varphi, q_1}(\|f_i\|_{L^1}) \lesssim \|f_i\|_{\Lambda^1(v_1)}.$$

If $r = \infty$, we can follow the same argument since, for every f_i obtained using the dyadic decomposition over \mathbb{Z} , $q_1 < p_1 \leq p_1^+$ and $t > 0$, we have (4.3). \square

Using a similar argument, we can also obtain the following proposition.

Proposition 4.1.2. *Suppose that $1 < q_1, q < \infty$. If for every $q_1 < p_1 \leq p_1^+$ and p such that (4.2) is satisfied we have that*

$$T : L^{p_1, r_1} \longrightarrow \Gamma^{1, \infty}[V_p], \quad V_p(t) = \frac{t^{1/p}}{\varphi(\log_1 t)}, \quad \|T\| \lesssim \varphi_1 \left(\frac{1}{p_1 - q_1} \right),$$

then

$$T : D^+ \longrightarrow \Gamma^{1, \infty}[V], \quad V(t) = \frac{t^{1/q}}{(\varphi_1 \varphi)(\log_1 t)},$$

where D^+ is determined by the values described in Table 4.2 (see Definition 3.1.7).

r_1	D^+
1	$\Lambda(v_1), v_1(t) = t^{\frac{1}{q_1}-1} \varphi(\log_1 \frac{1}{t})$
p_1	$D_{q_1, \varphi}^+$

Table 4.2: Proposition 4.1.2.

Proposition 4.1.1 and Proposition 4.1.2 yield the following theorem for the case where the operator blows up in both variables.

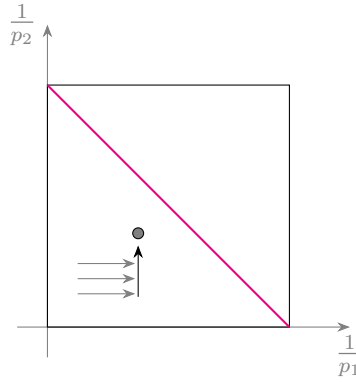


Figure 4.2: Banach setting. Theorem 4.1.3.

Theorem 4.1.3 (Figure 4.2). *Suppose that $1 \leq q_1, q_2, q < \infty$ are fixed. If, for every $q_1 < p_1 \leq p_1^+$ and $q_2 < p_2 \leq p_2^+$,*

$$B : L^{p_1, r_1} \times L^{p_2, r_2} \longrightarrow L^{p, r}, \quad \|B\| \lesssim \varphi_1 \left(\frac{1}{p_1 - q_1} \right) \varphi_2 \left(\frac{1}{p_2 - q_2} \right),$$

then

$$B : D_1^+ \times D_2^+ \longrightarrow R^+,$$

where D_1^+, D_2^+ and R^+ are determined by the values described in Table 4.3 in which $\varphi = \varphi_1 \varphi_2$ (see Definition 3.1.7).

Proof. We prove first the case when $r = p$. Let us fix p_2 such that $q_2 < p_2 \leq p_2^+$, and $g \in L^{p_2, r_2}$. Thus, the operator $T_1 := B(\cdot, g)$ satisfies that, for every $(p_1, p_2; p)$ with $q_1 < p_1 \leq p_1^+$,

$$T_1 : L^{p_1, r_1} \longrightarrow L^p, \quad \|T_1\| \lesssim \varphi_1 \left(\frac{1}{p_1 - q_1} \right) \varphi_2 \left(\frac{1}{p_2 - q_2} \right) \|g\|_{L^{p_2, r_2}}.$$

r_1	D_1^+	r_2	D_2^+
1	$\Lambda(v_1), v_1(t) = t^{\frac{1}{q_1}-1} \varphi_1(\log_1 \frac{1}{t})$	1	$\Lambda(v_2), v_2(t) = t^{\frac{1}{q_2}-1} \varphi_2(\log_1 \frac{1}{t})$
p_1	D_{q_1, φ_1}^+	p_2	D_{q_2, φ_2}^+

r	R^+
p	$R_{q, \varphi}^+$
∞ ($q \neq 1$)	$\Gamma^{1, \infty}[V], V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)}$

Table 4.3: Theorem 4.1.3.

Therefore, we can apply Proposition 4.1.1 to T_1 and obtain that, for every $(q_1, p_2; p)$,

$$T_1 : D_1^+ \longrightarrow R_{p, \varphi_1}^+, \quad \|T_1\| \lesssim \varphi_2 \left(\frac{1}{p_2 - q_2} \right) \|g\|_{L^{p_2, r_2}}.$$

Since we can do this reasoning for every $q_2 < p_2 \leq p_2^+$, we have that,

$$B : D_1^+ \times L^{p_2, r_2} \longrightarrow R_{p, \varphi_1}^+, \quad \|B\| \lesssim \varphi_2 \left(\frac{1}{p_2 - q_2} \right).$$

In particular, if we fix $f \in D_1^+$, and we consider any measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$, we have that, for every $q_2 < p_2 \leq p_2^+$ where

$$\frac{1}{p} - \frac{1}{p_2} = \frac{1}{q} - \frac{1}{q_2} = \frac{1}{q_1},$$

the operator $T_2^E := B(f, \cdot) \chi_E$ satisfies

$$\begin{aligned} \frac{\|T_2^E g\|_{L^p}}{\varphi_1(\log_1 |E|)} &= \frac{\left(\int_E [B(f, g)(s)]^p ds \right)^{1/p}}{\varphi_1(\log_1 |E|)} \leq \frac{\left(\int_0^{|E|} [(B(f, g))^*(s)]^p ds \right)^{1/p}}{\varphi_1(\log_1 |E|)} \\ &\leq \|B(f, g)\|_{R_{p, \varphi_1}^+} \lesssim \|f\|_{D_1^+} \varphi_2 \left(\frac{1}{p_2 - q_2} \right) \|g\|_{L^{p_2, r_2}}. \end{aligned}$$

Thus, for every $q_2 < p_2 \leq p_2^+$,

$$T_2^E : L^{p_2, r_2} \longrightarrow L^p, \quad \|T_2^E\| \lesssim \varphi_1(\log_1 |E|) \|f\|_{D_1^+} \varphi_2 \left(\frac{1}{p_2 - q_2} \right).$$

Proposition 4.1.1 implies that

$$T_2^E : D_2^+ \longrightarrow R_{q,\varphi_2}^+, \quad \|T_2^E\| \lesssim \varphi_1(\log_1 |E|) \|f\|_{D_1^+}.$$

Finally, since this argument is valid for every measurable set E , if we fix $g \in D_2^+$ and $t > 0$, and we consider all the sets E such that $|E| = t$, it follows from [8, Proposition 3.3, p.53] that

$$\begin{aligned} & \frac{\left(\int_0^t \left[(B(f,g))^*(s) \right]^q ds \right)^{1/q}}{\varphi_1(\log_1 t) \varphi_2(\log_1 t)} = \frac{\sup_{|E|=t} \left(\int_E \left[(B(f,g))(s) \right]^q ds \right)^{1/q}}{\varphi_1(\log_1 t) \varphi_2(\log_1 t)} \\ & = \frac{\sup_{|E|=t} \left(\int_E \left[(T_2^E g)(s) \right]^q ds \right)^{1/q}}{\varphi_1(\log_1 t) \varphi_2(\log_1 t)} \leq \frac{\sup_{|E|=t} \left(\int_0^t \left[(T_2^E g)^*(s) \right]^q ds \right)^{1/q}}{\varphi_1(\log_1 t) \varphi_2(\log_1 t)} \\ & \leq \frac{\sup_{|E|=t} \|T_2^E g\|_{R_{q,\varphi_2}^+}}{\varphi_1(\log_1 t)} \lesssim \|f\|_{D_1^+} \|g\|_{D_2^+}. \end{aligned}$$

Taking the supremum over $t > 0$ we conclude the desired result.

For the case where $r = \infty$, we also fix the second variable and apply Proposition 4.1.1 to obtain that, for every $(q_1, p_2; p)$ with $q_2 < p_2 \leq p_2^+$,

$$B : D_1^+ \times L^{p_2, r_2} \longrightarrow \Gamma^{1, \infty}[V_p], \quad V_p(t) = \frac{t^{1/p}}{\varphi_1(\log_1 t)}, \quad \|B\| \lesssim \varphi_2\left(\frac{1}{p_2 - q_2}\right).$$

Hence, if we fix $f \in D_1^+$ and consider the operator $T_2 := B(f, \cdot)$, we have that, for every $q_2 < p_2 \leq p_2^+$,

$$T_2 : L^{p_2, r_2} \longrightarrow \Gamma^{1, \infty}[V_p], \quad \|T_2\| \lesssim \|f\|_{D_1^+} \varphi_2\left(\frac{1}{p_2 - q_2}\right),$$

and the result follows by Proposition 4.1.2. \square

We can also consider the case where, the operator B satisfies that, for every $(p_1, q_2; p)$ with $p_1^- \leq p_1 < q_1$,

$$B : L^{p_1, r_1} \times L^{q_2, r_2} \longrightarrow L^{p, r}, \quad \|B\| \lesssim \varphi\left(\frac{1}{q_1 - p_1}\right).$$

In this case we need to develop the following off-diagonal extrapolation result.

Proposition 4.1.4. *Suppose that $1 < q_1 \leq \infty$ and $1 \leq q \leq \infty$. If for every $p_1^- \leq p_1 < q_1$ and p such that (4.2) is satisfied we have that*

$$T : L^{p_1, r_1} \longrightarrow L^{p, r}, \quad \|T\| \lesssim \begin{cases} \varphi\left(\frac{1}{q_1 - p_1}\right), & 1 < q_1 < \infty, \\ \varphi(p_1), & q_1 = \infty, \end{cases}$$

then

$$T : D^- \longrightarrow R^-,$$

where D^- and R^- are determined by the values described in Table 4.4 (see Definition 3.1.16).

r_1	$D^-, q_1 \neq \infty$	$D^-, q_1 = \infty$
1	$\Lambda^1(v_1), v_1(t) = t^{\frac{1}{q_1}-1} \varphi(\log_1 t)$	L_φ^∞
p_1	$D_{q_1, \varphi}^-$	

r	R^-
p ($q \neq \infty$)	$R_{q, \varphi}^-$
∞ ($q \neq 1$)	$\Gamma^{1, \infty}[V], V(t) = \frac{t^{1/q}}{\varphi(\log_1 \frac{1}{t})}$

Table 4.4: Proposition 4.1.4.

Proof. $\boxed{q_1 \neq \infty, r_1 = p_1}$ If $r = p$, fix $f \in L^{q_1}$ with $\|f\|_{L^{q_1}} \leq 1/q_1'$ and use the $K(L^1, L^{q_1})$ -decomposition and Lemma 2.2.2 c) to obtain $f = \sum_{i \in \mathbb{Z}} 2^i f_i$ with f_i with $\|f_i\|_{L^{p_1}} \lesssim |E_i|^{q_1'(\frac{1}{p_1} - \frac{1}{q_1})}$.

The rest of the proof follows the same steps than Theorem 3.1.17 with $p_1 = q_1$, using that, instead (3.16), since $p < q$, we have

$$\left(\int_t^\infty [(Tf_i)^{**}(s)]^q ds \right)^{1/q} \lesssim \left(\int_t^\infty [(Tf_i)^{**}(s)]^p s^{\frac{p}{q}-1} ds \right)^{1/p} \leq t^{\frac{1}{q} - \frac{1}{p}} \|Tf_i\|_{L^p}, \quad (4.4)$$

which yields

$$\begin{aligned} \left(\int_t^\infty [(Tf_i)^{**}(s)]^q ds \right)^{1/q} &\lesssim t^{\frac{1}{q_1} - \frac{1}{p_1}} \varphi \left(\frac{1}{q_1 - p_1} \right) \|f_i\|_{L^{p_1}} \\ &\lesssim \varphi \left(\frac{1}{q_1 - p_1} \right) \left(\frac{|E_i|^{q'_1}}{t} \right)^{\frac{1}{q_1} - \frac{1}{p_1}}. \end{aligned}$$

If $r = \infty$, as seen in Theorem 3.2.6, we follow the same argument using that

$$t^{\frac{1}{q}} (Tf_i)^{**}(t) \lesssim t^{\frac{1}{q} - \frac{1}{p}} \|Tf_i\|_{L^{p,\infty}} \lesssim \varphi \left(\frac{1}{q_1 - p_1} \right) \left(\frac{|E_i|^{q'_1}}{t} \right)^{\frac{1}{q_1} - \frac{1}{p_1}}$$

instead of (4.4).

$\boxed{q_1 \neq \infty, r_1 = 1}$ We proceed as in Theorem 3.3.4, taking into account that, instead of (3.18), we have by (4.4) that

$$\left(\int_t^\infty [(Tf_i)^{**}(s)]^q ds \right)^{1/q} \lesssim t^{\frac{1}{q} - \frac{1}{p}} \|Tf_i\|_{L^p} \lesssim t^{-\frac{1}{q_1}} \varphi \left(\frac{1}{q_1 - p_1} \right) \left(\frac{\|f_i\|_{L^1}}{t} \right)^{1/p_1}$$

when $r = p$, and

$$t^{\frac{1}{q}} Tf_i^{**}(t) \lesssim t^{\frac{1}{q} - \frac{1}{p}} \|Tf_i\|_{L^{p,\infty}} \lesssim t^{-\frac{1}{q_1}} \varphi \left(\frac{1}{q_1 - p_1} \right) \left(\frac{\|f_i\|_{L^1}}{t} \right)^{1/p_1}$$

when $r = \infty$.

$\boxed{q_1 = \infty}$ We proceed as in Theorem 3.3.5, taking into account that, instead of (3.19), we get by (4.4) that

$$\left(\int_t^\infty [(Tf_n)^{**}(s)]^q ds \right)^{1/q} \lesssim t^{-\frac{1}{p_1}} \varphi(p_1) \|f_n\|_{L^{p_1,1}} \lesssim \|f_n\|_{L^\infty} \varphi(p_1) \left(\frac{\|f_n\|_{L^1}}{t \|f_n\|_{L^\infty}} \right)^{1/p_1}$$

when $r = p$, and

$$t^{\frac{1}{q}} Tf_n^{**}(t) \lesssim t^{\frac{1}{q} - \frac{1}{p}} \|Tf_n\|_{L^{p,\infty}} \lesssim t^{-\frac{1}{p_1}} \varphi(p_1) \|f_n\|_{L^{p_1,1}} \lesssim \|f_n\|_{L^\infty} \varphi(p_1) \left(\frac{\|f_n\|_{L^1}}{t \|f_n\|_{L^\infty}} \right)^{1/p_1}$$

when $r = \infty$. □

From a similar argument, we can also obtain the following proposition.

Proposition 4.1.5. *Suppose that $1 < q_1 \leq \infty$ and $1 \leq q \leq \infty$. If for every $p_1^- \leq p_1 < q_1$ and p such that (4.2) is satisfied we have that*

$$T : L^{p_1, r_1} \longrightarrow R^-(p), \quad \|T\| \lesssim \begin{cases} \varphi\left(\frac{1}{q_1 - p_1}\right), & 1 < q_1 < \infty, \\ \varphi(p_1), & q_1 = \infty, \end{cases}$$

where $R^-(p) = R_{p, \varphi_1}^-$ or $R^-(p) = \Gamma^{1, \infty}[V_p]$ with $V_p(t) = \frac{t^{1/p}}{\varphi_1(\log_1 \frac{1}{t})}$, then

$$T : D^- \longrightarrow R^-,$$

where D^- and R^- are determined by the values described in Table 4.5 (see Definition 3.1.16).

r_1	$D^-, q_1 \neq \infty$	$D^-, q_1 = \infty$
1	$\Lambda^1(v_1), v_1(t) = t^{\frac{1}{q_1} - 1} \varphi(\log_1 t)$	L_φ^∞
p_1	$D_{q_1, \varphi}^-$	
$R^-(p)$		R^-
R_{p, φ_1}^- ($q \neq \infty$)		$R_{q, \varphi_1 \varphi}^-$
$\Gamma^{1, \infty}[V_p], V_p(t) = \frac{t^{1/p}}{\varphi_1(\log_1 \frac{1}{t})}$ ($q \neq 1$)		$\Gamma^{1, \infty}[V], V(t) = \frac{t^{1/q}}{(\varphi_1 \varphi)(\log_1 \frac{1}{t})}$

Table 4.5: Proposition 4.1.5.

From Proposition 4.1.4 and Proposition 4.1.5 we obtain the following theorem.

Theorem 4.1.6 (Figure 4.3). *Suppose that $1 < q_1, q_2, q \leq \infty$. If, for every $p_1^- \leq p_1 < q_1$ and $p_2^- \leq p_2 < q_2$,*

$$B : L^{p_1, r_1} \times L^{p_2, r_2} \longrightarrow L^{p, \infty}$$

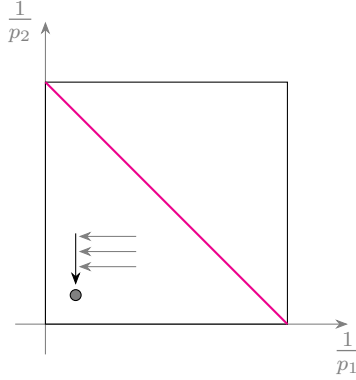


Figure 4.3: Banach setting. Theorem 4.1.6.

with

$$\|B\| \lesssim \begin{cases} \varphi_1\left(\frac{1}{q_1 - p_1}\right) \varphi_2\left(\frac{1}{q_2 - p_2}\right), & 1 < q_1, q_2 < \infty, \\ \varphi_1\left(\frac{1}{q_1 - p_1}\right) \varphi_2(p_2), & 1 < q_1 < \infty, q_2 = \infty, \\ \varphi_1(p_1) \varphi_2\left(\frac{1}{q_2 - p_2}\right), & q_1 = \infty, 1 < q_2 < \infty, \\ \varphi_1(p_1) \varphi_2(p_2), & q_1 = q_2 = \infty, \end{cases}$$

then

$$B : D_1^- \times D_2^- \longrightarrow R^-,$$

where D_1^- , D_2^- and R^- are determined by the values described in Table 4.6 in which $\varphi = \varphi_1 \varphi_2$ (see Definition 3.1.16).

Proof. We only prove the case $1 < q_1, q_2 < \infty$. The remaining cases follows using the same argument.

Fix p_2 such that $p_2^- \leq p_2 < q_2$, and $g \in L^{p_2, r_2}$. Therefore, the operator $T_1 := B(\cdot, g)$ satisfies that, for every $(p_1, p_2; p)$ such that $p_1^- \leq p_1 < q_1$,

$$T_1 : L^{p_1, r_1} \longrightarrow L^{p, r}, \quad \|T_1\| \lesssim \varphi_1\left(\frac{1}{q_1 - p_1}\right) \varphi_2\left(\frac{1}{q_2 - p_2}\right) \|g\|_{L^{p_2, r_2}}.$$

Therefore, we can apply Proposition 4.1.4 to T_1 and get that, for $(q_1, p_2; p)$,

$$T_1 : D_1^- \longrightarrow R^-(p), \quad \|T_1\| \lesssim \varphi_2\left(\frac{1}{q_2 - p_2}\right) \|g\|_{L^{p_2, r_2}},$$

r_1	$D_1^-, q_1 \neq \infty$	$D_1^-, q_1 = \infty$
1	$\Lambda^1(v_1), v_1(t) = t^{\frac{1}{q_1}-1} \varphi_1(\log_1 t)$	$L_{\varphi_1}^\infty$
p_1	D_{q_1, φ_1}^-	
r_2	$D_2^-, q_2 \neq \infty$	$D_2^-, q_2 = \infty$
1	$\Lambda^1(v_2), v_2(t) = t^{\frac{1}{q_2}-1} \varphi_2(\log_1 t)$	$L_{\varphi_2}^\infty$
p_2	D_{q_2, φ_2}^-	
r	R^-	
p ($q \neq \infty$)	$R_{q, \varphi}^-$	
∞	$\Gamma^{1, \infty}[V], V(t) = \frac{t^{1/q}}{\varphi(\log_1 \frac{1}{t})}$	

Table 4.6: Theorem 4.1.6.

where $R^-(p) = R_{p, \varphi_1}^-$ when $r = p$ and $R^-(p) = \Gamma^{1, \infty}[V_p]$ with $V_p(t) = \frac{t^{1/p}}{\varphi_1(\log_1 \frac{1}{t})}$, when $r = \infty$.

Since this argument is valid for every $q_2 < p_2 \leq p_2^+$, we have that, for every $(q_1, p_2; p)$,

$$B : D_1^- \times L^{p_2, r_2} \longrightarrow R^-(p), \quad \|B\| \lesssim \varphi_2 \left(\frac{1}{q_2 - p_2} \right).$$

If we now fix $f \in D_1^-$, the operator $T_2 := B(f, \cdot)$, satisfies that

$$T_2 : L^{p_2, r_2} \longrightarrow R^-(p), \quad \|T_2\| \lesssim \varphi_2 \left(\frac{1}{q_2 - p_2} \right) \|f\|_{D_1^-}.$$

At this point, it is enough to use Proposition 4.1.5 to conclude the result. \square

From Proposition 4.1.1 and Proposition 4.1.4, we can conclude the following theorem.

Theorem 4.1.7. *Suppose that $1 < q_1, q \leq \infty$. If, for every $q_1 < p_1 \leq p_1^+$ and $p_2^- \leq p_2 < q_2$,*

$$B : L^{p_1, r_1} \times L^{p_2, r_2} \longrightarrow L^{p, \infty}$$

with

$$\|B\| \lesssim \begin{cases} \varphi_1\left(\frac{1}{p_1 - q_1}\right) \varphi_2\left(\frac{1}{q_2 - p_2}\right), & 1 < q_2 < \infty, \\ \varphi_1\left(\frac{1}{p_1 - q_1}\right) \varphi_2(p_2), & q_2 = \infty, \end{cases}$$

then

$$B : D_1^+ \times D_2^- \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t^{1/q}}{\varphi_1(\log_1 t) \varphi_2(\log_1 \frac{1}{t})},$$

where D_1^+ , D_2^- are determined by the values described in Table 4.7 (see Definitions 3.1.7 and 3.1.16).

r_1	$D_1^+, q_1 \neq 1$	
1	$\Lambda^1(v_1), v_1(t) = t^{\frac{1}{q_1}-1} \varphi_1(\log_1 \frac{1}{t})$	
p_1	D_{q_1, φ_1}^+	
r_2	$D_2^-, q_2 \neq \infty$	$D_2^-, q_2 = \infty$
1	$\Lambda^1(v_2), v_2(t) = t^{\frac{1}{q_2}-1} \varphi_2(\log_1 t)$	$L_{\varphi_2}^\infty$
p_2	D_{q_2, φ_2}^-	

Table 4.7: Theorem 4.1.7.

4.1.2 Rays extrapolation

Let us consider now the case where the operator norm blows up in both variables simultaneously along a ray, that is, every $(p_1, p_2; p)$ satisfies

$$\frac{p}{q} = \frac{p_1}{q_1} = \frac{p_2}{q_2}. \quad (4.5)$$

Theorem 4.1.8 (Figure 4.4(a)). *Suppose that $1 \leq q_1, q_2, q < \infty$. If for every $q_1 < p_1 \leq p_1^+$ and $q_2 < p_2 \leq p_2^+$ such that (4.5) is satisfied we have that*

$$B : L^{p_1, r_1} \times L^{p_2, r_2} \longrightarrow L^{p, r}, \quad \|B\| \lesssim \varphi\left(\frac{1}{p - q}\right),$$

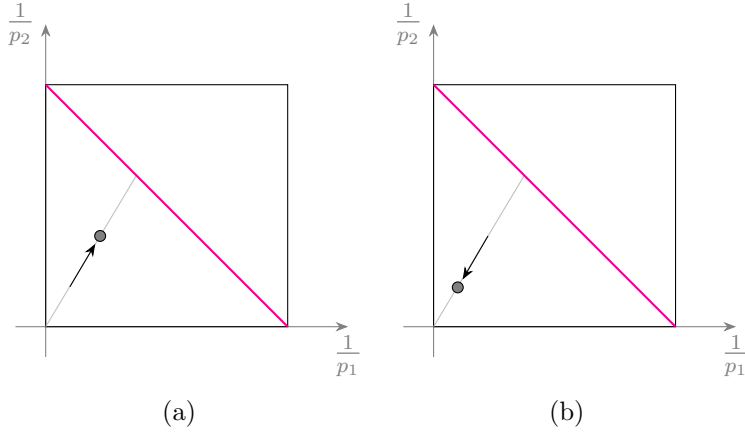


Figure 4.4: Banach setting: rays. Theorems 4.1.8 and 4.1.9.

then

$$B : D_1^+ \times D_2^+ \longrightarrow R^+,$$

where D_1^+ , D_2^+ and R^+ are determined by the values described in Table 4.8 (see Definition 3.1.7).

r_1	D_1^+	r_2	D_2^+
1	$\Lambda(v_1), v_1(t) = t^{\frac{1}{q_1}-1} \varphi(\log_1 \frac{1}{t})$	1	$\Lambda(v_1), v_1(t) = t^{\frac{1}{q_2}-1} \varphi(\log_1 \frac{1}{t})$
p_1	$D_{q_1, \varphi}^+$	p_2	$D_{q_2, \varphi}^+$

r	R^+
p	$R_{q, \varphi}^+$
∞ ($q \neq 1$)	$\Gamma^{1, \infty}[V], V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)}$

Table 4.8: Theorem 4.1.8.

Proof. $\boxed{r_1 = p_1, r_2 = p_2}$ When $r = p$, given $f \in L^{q_1}$ and $g \in L^{q_2}$, we decompose f with the $K(L^{q_1}, L^\infty)$ -decomposition and g with the $K(L^{q_2}, L^\infty)$ -decomposition to obtain $f = \sum_{i \in \mathbb{Z}} 2^i f_i$ and $g = \sum_{k \in \mathbb{Z}} 2^k g_k$ where $\|f_i\|_{L^{p_1}} \lesssim |E_i|^{q_1/p_1}$ and also $\|g_k\|_{L^{p_2}} \lesssim |F_k|^{q_2/p_2}$.

Thus,

$$\begin{aligned} \left(\int_0^t \left[(B(f_i, g_k))^*(s) \right]^q ds \right)^{1/q} &\lesssim t^{\frac{1}{q} - \frac{1}{p}} \varphi \left(\frac{1}{p-q} \right) \|f_i\|_{L^{p_1}} \|g_k\|_{L^{p_2}} \\ &\lesssim t^{\frac{1}{q}} \varphi \left(\frac{1}{p-q} \right) \left(\frac{|E_i|^q |F_k|^q}{t} \right)^{1/p}. \end{aligned}$$

By Lemma 2.1.4, we get

$$\begin{aligned} \left(\int_0^t \left[(B(f_i, g_k))^*(s) \right]^q ds \right)^{1/q} &\lesssim t^{\frac{1}{q}} I_{\varphi, q} \left(\frac{|E_i|^q |F_k|^q}{t} \right) \\ &\lesssim \varphi(\log_1 t) I_{\varphi, 1}(|E_i|) I_{\varphi, 1}(|F_k|), \end{aligned}$$

and, by Lemma 2.3.3, the result follows.

If $r = \infty$, we can follow the same argument using now that

$$(B(f_i, g_k))^{**}(t) \lesssim t^{\frac{1}{q} - \frac{1}{p}} \varphi \left(\frac{1}{p-q} \right) \|f_i\|_{L^{p_1}} \|g_k\|_{L^{p_2}} \lesssim t^{\frac{1}{q}} \varphi \left(\frac{1}{p-q} \right) \left(\frac{|E_i|^q |F_k|^q}{t} \right)^{\frac{1}{p}}.$$

$\boxed{r_1 = r_2 = 1}$ If $r = p$, given $f \in L^{q_1, 1}$ and $g \in L^{q_2, 1}$, we decompose them with the dyadic decomposition over \mathbb{Z} and we get

$$\begin{aligned} \left(\int_0^t \left[(B(f_i, g_k))^*(s) \right]^q ds \right)^{1/q} &\lesssim t^{\frac{1}{q} - \frac{1}{p}} \varphi \left(\frac{1}{p-q} \right) \|f_i\|_{L^1}^{1/p_1} \|g_k\|_{L^1}^{1/p_2} \\ &= t^{\frac{1}{q}} \varphi \left(\frac{1}{p-q} \right) \left(\frac{\|f_i\|_{L^1}^{q/q_1} \|g_k\|_{L^1}^{q/q_2}}{t} \right)^{1/p}. \end{aligned}$$

By Lemma 2.1.4 and Lemma 2.3.3 the result follows.

If $r = \infty$, we have that

$$t^{\frac{1}{q}} (B(f_i, g_k))^{**}(t) \lesssim t^{\frac{1}{q} - \frac{1}{p}} \varphi \left(\frac{1}{p-q} \right) \|f_i\|_{L^1}^{1/p_1} \|g_k\|_{L^1}^{1/p_2}.$$

$\boxed{r_1 = p_1, r_2 = 1}$ If $r = p$, given $f \in L^{q_1}$ and $g \in L^{q_2, 1}$, we decompose f using the $K(L^{q_1}, L^\infty)$ -decomposition and g with the dyadic decomposition over \mathbb{Z} and we have that,

$$\begin{aligned} \left(\int_0^t \left[(B(f_i, g_k))^*(s) \right]^q ds \right)^{1/q} &\lesssim t^{\frac{1}{q} - \frac{1}{p}} \varphi \left(\frac{1}{p-q} \right) |E_i|^{q_1/p_1} \|g_k\|_{L^1}^{1/p_2} \\ &= t^{\frac{1}{q}} \varphi \left(\frac{1}{p-q} \right) \left(\frac{|E_i|^q \|g_k\|_{L^1}^{q/q_2}}{t} \right)^{1/p}. \end{aligned}$$

By Lemma 2.1.4, Lemma 2.3.3 and Lemma 2.3.1, the result follows.

If $r = \infty$, we shall use that

$$t^{\frac{1}{q}} (B(f_i, g_k))^{**} (t) \lesssim t^{\frac{1}{q} - \frac{1}{p}} \varphi \left(\frac{1}{p - q} \right) |E_i|^{q_1/p_1} \|g_k\|_{L^1}^{1/p_2}.$$

□

If we have the hypotheses for $p < q$, we have the following theorem.

Theorem 4.1.9 (Figure 4.4 (b)). *Suppose that $1 < q_1, q_2 \leq \infty$ and $1 \leq q \leq \infty$. If for every $p_1^- \leq p_1 < q_1$ and $p_2^- \leq p_2 < q_2$ such that (4.5) is satisfied we have that*

$$B : L^{p_1, r_1} \times L^{p_2, r_2} \longrightarrow L^{p, r}, \quad \|B\| \lesssim \begin{cases} \varphi \left(\frac{1}{q - p} \right), & 1 < q < \infty, \\ \varphi(p), & q = \infty, \end{cases}$$

then

$$B : D_1^- \times D_2^- \longrightarrow R^-,$$

where D_1^- , D_2^- and R^- are determined by the values described in Table 4.9 (see Definition 3.1.16).

Proof. $\boxed{q_1 \neq \infty, q_2 \neq \infty, r_1 = p_1, r_2 = p_2}$ If $r = p$, given $f \in L^{q_1}$ such that $\|f\|_{L^{q_1}} \leq 1/q_1'$ and $g \in L^{q_2}$ such that $\|g\|_{L^{q_2}} \leq 1/q_2'$, we decompose f with the $K(L^1; L^{q_1})$ -decomposition and g using the $K(L^1; L^{q_2})$ -decomposition. Hence, we have that $f = \sum_{i \in \mathbb{Z}} 2^i f_i$ and $g = \sum_{k \in \mathbb{Z}} 2^k g_k$ where,

$$\|f_i\|_{L^{p_1}} \lesssim |E_i|^{q_1' \left(\frac{1}{p_1} - \frac{1}{q_1} \right)} \quad \text{and} \quad \|g_k\|_{L^{p_2}} \lesssim |F_k|^{q_2' \left(\frac{1}{p_2} - \frac{1}{q_2} \right)}.$$

From (3.15) follows that $E_i = F_k = \emptyset$ whenever $i, k \geq 0$. Since $\frac{p}{q} = \frac{p_1}{q_1}$, we have

$$q_1' \left(\frac{1}{p_1} - \frac{1}{q_1} \right) = q_1' \frac{q}{q_1} \left(\frac{1}{p} - \frac{1}{q} \right) = \frac{q}{q_1 - 1} \left(\frac{1}{p} - \frac{1}{q} \right)$$

and the same happens with q_2', q_2, p_2 . Therefore,

$$\left(\int_t^\infty \left[(B(f_i, g_k))^{**} (s) \right]^q ds \right)^{1/q} \lesssim \varphi \left(\frac{1}{q - p} \right) \left(\frac{|E_i|^{\frac{q}{q_1 - 1}} |F_k|^{\frac{q}{q_2 - 1}}}{t} \right)^{\frac{1}{p} - \frac{1}{q}}$$

r_1	$D_1^-, q_1 \neq \infty$	$D_1^-, q_1 = \infty$
1	$\Lambda^1(v_1), v_1(t) = t^{\frac{1}{q_1}-1} \varphi(\log_1 t)$	L_φ^∞
p_1	$D_{q_1, \varphi}^-$	
r_2	$D_2^-, q_2 \neq \infty$	$D_2^-, q_2 = \infty$
1	$\Lambda^1(v_2), v_2(t) = t^{\frac{1}{q_2}-1} \varphi(\log_1 t)$	L_φ^∞
p_2	$D_{q_2, \varphi}^-$	
r	R^-	
p ($q \neq \infty$)	$R_{q, \varphi}^-$	
∞ ($q \neq 1$)	$\Gamma^{1, \infty}[V], V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)}$	

Table 4.9: Theorem 4.1.9.

and by Lemma 2.1.4, we get

$$\left(\int_0^t \left[(B(f_i, g_k))^{**}(s) \right]^q ds \right)^{1/q} \lesssim \varphi \left(\log_1 \frac{1}{t} \right) \varphi(\log_1 |E_i|) \varphi(\log_1 |F_k|).$$

Finally, by Lemma 2.3.4, the result follows.

If $r = \infty$, we use

$$t^{\frac{1}{q}} (B(f_i, g_k))^{**}(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}} \varphi \left(\frac{1}{q-p} \right) \|f_i\|_{L^{p_1}} \|g_k\|_{L^{p_2}}.$$

$\boxed{q_1 \neq \infty, q_2 \neq \infty, r_1 = r_2 = 1}$ If $r = p$, given $f \in L^{q_1, 1}$ and $g \in L^{q_2, 1}$, we can decompose f, g using the dyadic decomposition over \mathbb{Z} and we have that

$$\begin{aligned} \left(\int_0^t \left[(B(f_i, g_k))^{**}(s) \right]^q ds \right)^{1/q} &\lesssim t^{\frac{1}{q}-\frac{1}{p}} \varphi \left(\frac{1}{q-p} \right) \|f_i\|_{L^1}^{1/p_1} \|g_k\|_{L^1}^{1/p_2} \\ &= t^{\frac{1}{q}} \varphi \left(\frac{1}{q-p} \right) \left(\frac{\|f_i\|_{L^1}^{q/q_1} \|g_k\|_{L^1}^{q/q_2}}{t} \right)^{1/p}. \end{aligned}$$

By Lemma 2.1.4 and Lemma 2.3.1 the result follows.

If $r = \infty$, we can follow the same argument using now that,

$$t^{\frac{1}{q}} (B(f_i, g_k))^{**} (t) \lesssim t^{\frac{1}{q} - \frac{1}{p}} \varphi \left(\frac{1}{q-p} \right) \|f_i\|_{L^1}^{1/p_1} \|g_k\|_{L^1}^{1/p_2}.$$

$\boxed{q_1 \neq \infty, q_2 \neq \infty, r_1 = p_1, r_2 = 1}$ If $r = p$, given $f \in L^{q_1}$ and $g \in L^{q_2, 1}$, we decompose f using the $K(L^{q_1}, L^\infty)$ -decomposition and g with the dyadic decomposition over \mathbb{Z} and we have that

$$\left(\int_0^t \left[(B(f_i, g_k))^{**} (s) \right]^q ds \right)^{1/q} \lesssim t^{\frac{1}{q}} |E_i|^{1-q_1} \varphi \left(\frac{1}{q-p} \right) \left(\frac{|E_i|^{\frac{q}{q_1-1}} \|g_k\|_{L^1}^{q/q_2}}{t} \right)^{1/p}.$$

By Lemma 2.1.4, we get

$$\begin{aligned} \left(\int_0^t \left[(B(f_i, g_k))^{**} (s) \right]^q ds \right)^{1/q} &\lesssim t^{\frac{1}{q}} |E_i|^{1-q_1} J_{\varphi, q} \left(|E_i|^{\frac{q}{q_1-1}} \|g_k\|_{L^1}^{q/q_2} t \right) \\ &\lesssim \varphi \left(\log_1 \frac{1}{t} \right) \varphi \left(\log_1 |E_i| \right) J_{\varphi, q_2} \left(\|g_k\|_{L^1} \right). \end{aligned}$$

The result follows from Lemma 2.3.4 and Lemma 2.3.1.

If $r = \infty$, we shall use that

$$t^{\frac{1}{q}} (B(f_i, g_k))^{**} (t) \lesssim t^{\frac{1}{q}} |E_i|^{1-q_1} \varphi \left(\frac{1}{q-p} \right) \left(\frac{|E_i|^{\frac{q}{q_1-1}} \|g_k\|_{L^1}^{q/q_2}}{t} \right)^{1/p}.$$

$\boxed{q_1 = q_2 = \infty}$ Given $f, g \in L^\infty$, we decompose them using the f^* -decomposition and g^* -decomposition, and, we have that

$$(B(f_n, g_m))^{**} (t) \lesssim \|f_n\|_{L^\infty} \|g_m\|_{L^\infty} \inf_{p^- \leq p < \infty} \varphi(p) \left[\frac{1}{t} \left(\frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}} \right)^{\frac{p^-}{p_1}} \left(\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}} \right)^{\frac{p^-}{p_2}} \right]^{\frac{1}{p}}.$$

By Lemma 2.1.4, we get

$$\begin{aligned} (B(f_n, g_m))^{**} (t) &\lesssim \|f_n\|_{L^\infty} \|g_m\|_{L^\infty} \varphi \left(\log_1 \left[\frac{1}{t} \left(\frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}} \right)^{\frac{p^-}{p_1}} \left(\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}} \right)^{\frac{p_1^-}{p_2}} \right] \right) \\ &\lesssim \varphi \left(\log_1 \frac{1}{t} \right) \|f_n\|_{L^\infty} \varphi \left(\log_1 \frac{\|f_n\|_{L^1}}{\|f_n\|_{L^\infty}} \right) \|g_m\|_{L^\infty} \varphi \left(\log_1 \frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}} \right). \end{aligned}$$

The result follows summing over $n, m \in \mathbb{N}$ using Lemma 2.3.2. \square

4.1.3 The endpoint $(1, \infty; 1)$

Now, we introduce the case where we reach the endpoint $(1, \infty; 1)$ by only using hypotheses where the target space is in the Banach setting.

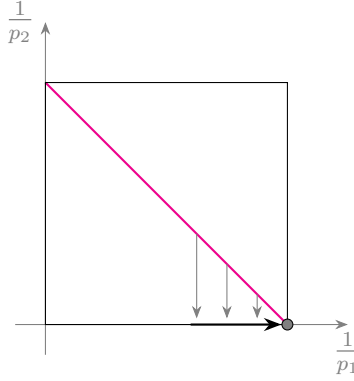


Figure 4.5: Banach setting. Theorem 4.1.10.

Theorem 4.1.10 (Figure 4.5). *If, for every $1 < p_1 \leq p_1^+$ and $(p_1^+)' \leq p_2 < \infty$,*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^p, \quad \|B\| \lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \varphi_2(p_2),$$

then

$$B : \Lambda^1(v_1) \times L_{\varphi_2}^\infty \longrightarrow \Gamma^{1,\infty}[V], \quad V(t) = \frac{t}{(\varphi_1 \varphi_2)(\log_1 t) \varphi_2(\log_1 \frac{1}{t})},$$

where $v_1(t) = (\varphi_1 \varphi_2)(\log_1 \frac{1}{t})$.

Proof. First, given $f \in L^1$ and $g \in L^\infty$, we decompose f with the dyadic decomposition over \mathbb{Z} and g using the g^* -decomposition. Hence, if we fix $i \in \mathbb{Z}$ and p_1 , we get

$$\begin{aligned} B(f_i, g_m)^{**}(t) &\lesssim t^{-\frac{1}{p}} \|B(f_i, g_m)\|_{L^p} \lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \left(\frac{\|f_i\|_{L^{p_1,1}}}{t^{1/p_1}} \right) \varphi_2(p_2) \left(\frac{\|g_m\|_{L^{p_2,1}}}{t^{1/p_2}} \right) \\ &\lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \left(\frac{\|f_i\|_{L^1}}{t} \right)^{1/p_1} \varphi_2(p_2) \|g_m\|_{L^\infty} \left(\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty} t} \right)^{1/p_2}. \end{aligned}$$

Thus, we can take the infimum over $p'_1 \leq p_2 < \infty$ using Lemma 2.1.1 to obtain

$$\begin{aligned} B(f_i, g_m)^{**}(t) &\lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \left(\frac{\|f_i\|_{L^1}}{t} \right)^{1/p_1} \|g_m\|_{L^\infty} \inf_{p'_1 \leq p_2 < \infty} \varphi_2(p_2) \left(\frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty} t} \right)^{1/p_2} \\ &\approx (\varphi_1 \varphi_2) \left(\frac{1}{p_1 - 1} \right) \left(\frac{\|f_i\|_{L^1}}{t} \right)^{1/p_1} \|g_m\|_{L^\infty} \varphi_2 \left(\log_1 \frac{1}{t} \right) \varphi_2 \left(\log_1 \frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}} \right). \end{aligned}$$

Now, we can take the infimum over $1 < p_1 \leq p_1^+$ using Lemma 2.1.4 to get

$$B(f_i, g_m)^{**}(t) \lesssim I_{\varphi_1 \varphi_2, 1} \left(\frac{\|f_i\|_{L^1}}{t} \right) \|g_m\|_{L^\infty} \varphi_2 \left(\log_1 \frac{1}{t} \right) \varphi_2 \left(\log_1 \frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}} \right),$$

which implies

$$\frac{tB(f_i, g_m)^{**}(t)}{(\varphi_1 \varphi_2) (\log_1 t) \varphi_2 (\log_1 \frac{1}{t})} \lesssim I_{\varphi_1 \varphi_2, 1} (\|f_i\|_{L^1}) \|g_m\|_{L^\infty} \varphi_2 \left(\log_1 \frac{\|g_m\|_{L^1}}{\|g_m\|_{L^\infty}} \right).$$

Summing over i, m , using Lemma 2.3.1 and Lemma 2.3.2, the result follows. \square

4.2 Quasi-Banach setting

In this section, we treat the cases where $L^{q,r}$ is a quasi-Banach space. We only consider restricted-weak type hypotheses.

4.2.1 Off-Diagonal extrapolation

As we did in the Banach setting, we shall use a linearization argument. Hence, we develop the corresponding off-diagonal extrapolation result.

Proposition 4.2.1. *Suppose that $1 \leq q_1 < \infty$ and $0 < q \leq 1$. If, for every $q_1 < p_1 \leq p_1^+$ and p such that*

$$\frac{1}{p} - \frac{1}{p_1} = \frac{1}{q} - \frac{1}{q_1}, \quad (4.6)$$

$$T : L^{p_1, 1} \longrightarrow L^{p, \infty}, \quad \|T\| \lesssim \varphi \left(\frac{1}{p_1 - q_1} \right),$$

then

$$T : D^+ \longrightarrow \Lambda^{1, \infty}[V], \quad V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)}$$

where D^+ is determined by the values described in Table 4.10 in which $\varepsilon > 0$ is arbitrary.

		D^+
$0 < q < 1$	$q_1 = 1$	$\Lambda^1(v_1), v_1(t) = \varphi\left(\log \frac{1}{t}\right) \left(\log_2 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
	$1 < q_1 < \infty$	$\Lambda^1(v_1), v_1(t) = t^{\left(\frac{1}{q_1}-1\right)} \varphi\left(\log_1 \frac{1}{t}\right) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
$q = 1$	$q_1 = 1$	$\Lambda^1(v), v_1(t) = \varphi\left(\log \frac{1}{t}\right) \left(\log_3 \frac{1}{t}\right)$
	$1 < q_1 < \infty$	$\Lambda^1(v_1), v_1(t) = t^{\left(\frac{1}{q_1}-1\right)} \varphi\left(\log_1 \frac{1}{t}\right) \left(\log_2 \frac{1}{t}\right)$

Table 4.10: Proposition 4.2.1

Remark 4.2.2. Whenever it is needed we will denote $D^+(q) = D^+$ and we observe that, if $r \geq q$, we have that $D^+(q) \subset D^+(r)$.

Proof. $q_1 = 1, 0 < q < 1$ Given $f \in L^1$ such that $\|f\|_{L^1} = 1$, we use the 2-dyadic decomposition. Then,

$$t^{1/q} (Tf_n)^*(t) \lesssim t^{\frac{1}{q}-\frac{1}{p}} \|Tf_n\|_{L^{p,\infty}} \lesssim t \varphi\left(\frac{1}{p_1 - q_1}\right) \left(\frac{\|f_n\|_{L^1}}{t}\right)^{1/p_1}.$$

By Lemma 2.1.4, we obtain

$$t^{1/q} (Tf_n)^*(t) \lesssim t I_{\varphi,1} \left(\frac{\|f_n\|_{L^1}}{t}\right) \lesssim \varphi(\log_1 t) I_{\varphi,1}(\|f_n\|_{L^1}).$$

Thus, if $V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)}$, we get that

$$\|Tf_n\|_{\Lambda^{q,\infty}[V]} \lesssim I_{\varphi,1}(\|f_n\|_{L^1}).$$

Since

$$|Tf| \leq \sum_{n \geq 0} 2^{2n} \|Tf_n\|_{\Lambda^{q,\infty}[V]} \frac{|Tf_n|}{\|Tf_n\|_{\Lambda^{q,\infty}[V]}},$$

Proposition 2.3.5 and Lemma 2.3.6 i) imply that, for every sequence $(a_n)_n$ of positive numbers such that $\sum_n a_n \leq 1$,

$$\|Tf\|_{\Lambda^{q,\infty}[V]} \lesssim \sum_{n \geq 0} a_n^{-\frac{1}{q}+1} 2^{2n} \|Tf_n\|_{\Lambda^{q,\infty}[V]} \lesssim \sum_{n \geq 0} a_n^{-\frac{1}{q}+1} 2^{2n} I_{\varphi,1}(\|f_n\|_{L^1}).$$

If we fix $\varepsilon > 0$ and choose $a_n \approx n^{-(1+\varepsilon)}$, the result follows by Lemma 2.3.9 i).

$q_1 = q = 1$ We follow the same steps using but Lemma 2.3.6 ii) to obtain

$$\|Tf\|_{\Lambda^{q,\infty}[V]} \lesssim \sum_{n \geq 0} \left(\log \frac{1}{a_n} \right) 2^{2^n} \|Tf_n\|_{\Lambda^{q,\infty}[V]} \lesssim \sum_{n \geq 0} \left(\log \frac{1}{a_n} \right) 2^{2^n} I_{\varphi,1} (\|f_n\|_{L^1}).$$

If we fix $\varepsilon > 0$ and we now choose $a_n \approx n^{-2}$, the result follows by Lemma 2.3.9 ii).

$1 < q_1 < \infty, 0 < q < 1$ We can use the dyadic decomposition over \mathbb{N} to obtain

$$\|Tf_n\|_{\Lambda^{q,\infty}[V]} \lesssim I_{\varphi,q_1} (\|f_n\|_{L^1}).$$

The result follows using Lemma 2.3.6 i), choosing $a_n \approx n^{-(1+\varepsilon)}$ and applying Lemma 2.3.10 i).

$1 < q_1 < \infty, q = 1$ In this case, we apply Lemma 2.3.6 ii) with $a_n \approx n^{-2}$, and then Lemma 2.3.10 ii). \square

Remark 4.2.3. We observe that, if $q_1 = q = 1$, we recover Theorem 3.3.1.

Similarly, we have the following proposition.

Proposition 4.2.4. *Suppose that $1 \leq q_1 < \infty$ and $0 < q \leq 1$. If for every $q_1 < p_1 \leq p_1^+$ and p such that (4.6) is satisfied we have that*

$$T : L^{p_1,1} \longrightarrow \Lambda^{1,\infty}[V_p], \quad V_p(t) = \frac{t^{1/p}}{\varphi_1(\log_1 t)}, \quad \|T\| \lesssim \varphi \left(\frac{1}{p_1 - q_1} \right),$$

then

$$T : D^+ \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t^{1/q}}{(\varphi_1 \varphi)(\log_1 t)},$$

where D^+ is determined by the values described in Table 4.10 in which $\varepsilon > 0$ is arbitrary.

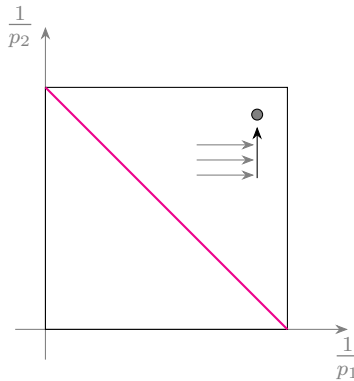


Figure 4.6: Quasi-Banach setting. Theorem 4.2.5.

Theorem 4.2.5 (Figure 4.6). *Suppose that $1 \leq q_1, q_2 < \infty$ and $0 < q \leq 1$. If, for every $q_1 < p_1 \leq p_1^+$ and $q_2 < p_2 \leq p_2^+$,*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \quad \|B\| \lesssim \varphi_1 \left(\frac{1}{p_1 - q_1} \right) \varphi_2 \left(\frac{1}{p_2 - q_2} \right),$$

then

$$B : D_1^+ \times D_2^+ \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t^{1/q}}{(\varphi_1 \varphi_2)(\log_1 t)},$$

where D_1^+, D_2^+ are determined by the values described in Table 4.11 in which $\varepsilon > 0$ is arbitrary.

		D_1^+
$0 < q < 1$	$q_1 = 1$	$\Lambda^1(v_1), v_1(t) = \varphi_1 \left(\log_1 \frac{1}{t} \right) \left(\log_2 \frac{1}{t} \right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
	$1 < q_1 < \infty$	$\Lambda^1(v_1), v_1(t) = t^{\left(\frac{1}{q_1}-1\right)} \varphi_1 \left(\log_1 \frac{1}{t} \right) \left(\log_1 \frac{1}{t} \right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
$q = 1$	$1 < q_1 < \infty$	$\Lambda^1(v_1), v_1(t) = t^{\left(\frac{1}{q_1}-1\right)} \varphi_1 \left(\log_1 \frac{1}{t} \right) \left(\log_2 \frac{1}{t} \right)$
		D_2^+
$0 < q < 1$	$q_2 = 1$	$\Lambda^1(v_2), v_2(t) = \varphi_2 \left(\log_1 \frac{1}{t} \right) \left(\log_2 \frac{1}{t} \right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
	$1 < q_2 < \infty$	$\Lambda^1(v_2), v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \varphi_2 \left(\log_1 \frac{1}{t} \right) \left(\log_1 \frac{1}{t} \right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)}$
$q = 1$	$1 < q_2 < \infty$	$\Lambda^1(v_2), v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \varphi_2 \left(\log_1 \frac{1}{t} \right) \left(\log_2 \frac{1}{t} \right)$

Table 4.11: Theorem 4.2.5

Proof. Let us fix p_2 and $g \in L^{p_2,1}$. Thus, the operator $T_1 := B(\cdot, g)$ satisfies that

$$T_1 : L^{p_1,1} \longrightarrow L^{p,\infty}, \quad \|T_1\| \lesssim \varphi_1 \left(\frac{1}{p_1 - q_1} \right) \varphi_2 \left(\frac{1}{p_2 - q_2} \right) \|g\|_{L^{p_2,1}}.$$

Therefore, if we can apply Proposition 4.2.1 to T_1 to obtain (see Remark 4.2.2)

$$T_1 : D^+(r) \longrightarrow \Lambda^{1,\infty}[V_r], \quad V_r = \frac{t^{1/r}}{\varphi_1(\log_1 t)}, \quad \|T_1\| \lesssim \varphi_2 \left(\frac{1}{p_2 - q_2} \right) \|g\|_{L^{p_2,1}},$$

where $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{p_2}$. Since $r \geq q$, $D_1^+ \subseteq D^+(r)$, and we get that

$$T_1 : D_1^+ \longrightarrow \Lambda^{1,\infty}[V_r], \quad \|T_1\| \lesssim \varphi_2 \left(\frac{1}{p_2 - q_2} \right) \|g\|_{L^{p_2,1}}.$$

Thus, for every $(q_1, p_2; p)$ such that $q_2 < p_2 \leq p_2^+$,

$$B : D_1^+ \times L^{p_2,1} \longrightarrow \Lambda^{1,\infty}[V_p], \quad \|B\| \lesssim \varphi_2 \left(\frac{1}{p_2 - q_2} \right).$$

Hence, if we now fix $f \in D_1^+$, the operator $T_2 := B(f, \cdot)$ satisfies that

$$T_2 : L^{p_2,1} \longrightarrow \Lambda^{1,\infty}[V_p], \quad \|T_2\| \lesssim \|f\|_{D_1^+} \varphi_2 \left(\frac{1}{p_2 - q_2} \right),$$

and, by Proposition 4.2.4, the result follows. \square

We also study the corresponding extrapolation result where we have the hypotheses for $p < q$.

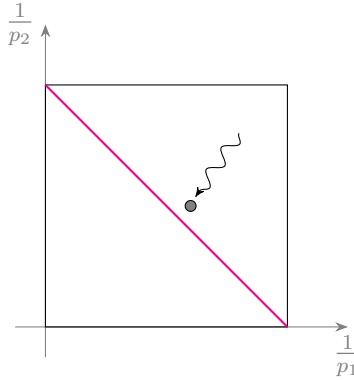


Figure 4.7: Quasi-Banach setting. Theorem 4.2.6.

Theorem 4.2.6 (Figure 4.7). *Suppose that $1 < q_1, q_2 < \infty$ and $0 < q \leq 1$. If, for every $p_1^- \leq p_1 < q_1$ and $p_2^- \leq p_2 < q_2$,*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \quad \|B\| \lesssim \varphi_1 \left(\frac{1}{q_1 - p_1} \right) \varphi_2 \left(\frac{1}{q_2 - p_2} \right),$$

then

$$B : D_1^- \times D_2^- \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t^{1/q}}{(\varphi_1 \varphi_2) \left(\log_1 \frac{1}{t} \right)}$$

where D_1^-, D_2^- are determined by the values described in Table 4.12 in which $\varepsilon > 0$ is arbitrary.

D_1^-	
$0 < q < 1$	$\Lambda^1(v_1), v_1(t) = t^{\left(\frac{1}{q_1}-1\right)} \varphi_1(\log_1 t) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)} (\varphi_1 \varphi_2) \left(\log_2 \frac{1}{t}\right)$
$q = 1$	$\Lambda^1(v_1), v_1(t) = t^{\left(\frac{1}{q_1}-1\right)} \varphi_1(\log_1 t) \left(\log_2 \frac{1}{t}\right) (\varphi_1 \varphi_2) \left(\log_2 \frac{1}{t}\right)$
D_2^-	
$0 < q < 1$	$\Lambda^1(v_2), v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \varphi_2(\log_1 t) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)} (\varphi_1 \varphi_2) \left(\log_2 \frac{1}{t}\right)$
$q = 1$	$\Lambda^1(v_2), v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \varphi_2(\log_1 t) \left(\log_2 \frac{1}{t}\right) (\varphi_1 \varphi_2) \left(\log_2 \frac{1}{t}\right)$

Table 4.12: Theorem 4.2.6

Proof. We prove the case where $0 < q < 1$. Given $f \in L^{q_1,1}$ and $g \in L^{q_2,1}$ such that $\|f\|_{L^{q_1,1}} = \|g\|_{L^{q_2,1}} = 1$, we decompose them with the dyadic decomposition over \mathbb{N} . Thus,

$$(B(f_n, g_m))^*(t) \lesssim \varphi_1\left(\frac{1}{q_1 - p_1}\right) \left(\frac{\|f_n\|_{L^1}}{t}\right)^{1/p_1} \varphi_2\left(\frac{1}{q_2 - p_2}\right) \left(\frac{\|g_m\|_{L^1}}{t}\right)^{1/p_2}.$$

Taking the infimum over $p_1^- \leq p_1 < q_1$ and over $p_2^- \leq p_2 < q_2$ independently with Lemma 2.1.4, we get

$$\begin{aligned} (B(f_n, g_m))^*(t) &\lesssim J_{\varphi_1, q_1} \left(\frac{\|f_n\|_{L^1}}{t}\right) J_{\varphi_2, q_2} \left(\frac{\|g_m\|_{L^1}}{t}\right) \\ &\lesssim \frac{(\varphi_1 \varphi_2) \left(\log_1 \frac{1}{t}\right)}{t^{\frac{1}{q}}} J_{\varphi_1, q_1} (\|f_n\|_{L^1}) J_{\varphi_2, q_2} (\|g_m\|_{L^1}). \end{aligned}$$

Hence, if $V(t) = \frac{t^{\frac{1}{q}}}{(\varphi_1 \varphi_2) \left(\log_1 \frac{1}{t}\right)}$, we obtain that

$$\|B(f, g)\|_{\Lambda^{q, \infty}[V]} \lesssim J_{\varphi_1, q_1} (\|f\|_{L^1}) J_{\varphi_2, q_2} (\|g\|_{L^1}). \quad (4.7)$$

On the other hand, since

$$|B(f, g)| \leq \sum_{n \geq 0} 2^n \|B(f_n, g)\|_{\Lambda^{q, \infty}[V]} \frac{|B(f_n, g)|}{\|B(f_n, g)\|_{\Lambda^{q, \infty}[V]}}.$$

Proposition 2.3.5 and Lemma 2.3.7 i) imply that, for every sequence $(a_n)_n$ of positive numbers such that $\sum_n a_n \leq 1$,

$$\|B(f, g)\|_{\Lambda^{q, \infty}[V]} \lesssim \sum_{n \geq 0} a_n^{-\frac{1}{q}+1} 2^n \|B(f_n, g)\|_{\Lambda^{q, \infty}[V]}.$$

We can repeat this argument with the other variable to get

$$\|B(f, g)\|_{\Lambda^{q, \infty}[V]} \lesssim \sum_{n \geq 0} a_n^{-\frac{1}{q}+1} 2^n \sum_{k \geq 0} a_k^{-\frac{1}{q}+1} 2^k \|B(f_n, g_k)\|_{\Lambda^{q, \infty}[V]}.$$

By applying (4.7) to each term we get that

$$\|B(f, g)\|_{\Lambda^{q, \infty}[V]} \lesssim \sum_{n \geq 0} a_n^{-\frac{1}{q}+1} 2^n J_{\varphi_1, q_1}(\|f_n\|_{L^1}) \sum_{k \geq 0} a_k^{-\frac{1}{q}+1} 2^k J_{\varphi_2, q_2}(\|g_k\|_{L^1}).$$

Thus, we fix $\varepsilon > 0$ and we choose $a_n \approx n^{-(1+\varepsilon)}$ and $a_k \approx k^{-(1+\varepsilon)}$ and, by Lemma 2.3.11 i), the proof is complete. \square

4.2.2 Rays extrapolation

We consider now the case where the operator blows up in both variables simultaneously along a ray. We recall that this means that every $(p_1, p_2; p)$ satisfies

$$\frac{p}{q} = \frac{p_1}{q_1} = \frac{p_2}{q_2}. \quad (4.8)$$

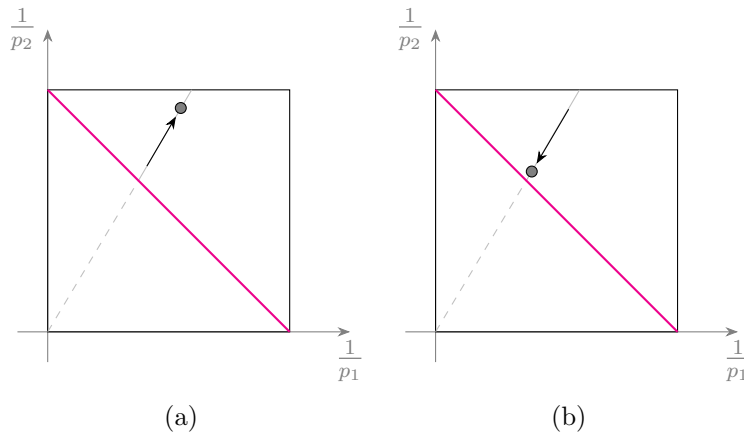


Figure 4.8: Quasi-Banach setting: rays. Theorems 4.2.7 and 4.2.8.

Theorem 4.2.7 (Figure 4.8 (a)). *Suppose that $1 \leq q_1, q_2 < \infty$ and $0 < q \leq 1$. If for every $q_1 < p_1 \leq p_1^+$ and $q_2 < p_2 \leq p_2^+$ such that (4.8) is satisfied we have that*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \quad \|B\| \lesssim \varphi \left(\frac{1}{p-q} \right),$$

then

$$B : D_1^+ \times D_2^+ \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)},$$

where D_1^+ and D_2^+ are determined by the values described in Table 4.13 in which $\varepsilon > 0$ is arbitrary.

		D_1^+
$0 < q < 1$	$q_1 = 1$	$\Lambda^1(v_1), v_1(t) = \varphi(\log_1 \frac{1}{t}) (\log_2 \frac{1}{t})^{(\frac{1}{q}-1)(1+\varepsilon)}$
	$1 < q_1 < \infty$	$\Lambda^1(v_1), v_1(t) = t^{(\frac{1}{q_1}-1)} \varphi(\log_1 \frac{1}{t}) (\log_1 \frac{1}{t})^{(\frac{1}{q}-1)(1+\varepsilon)}$
$q = 1$	$1 < q_1 < \infty$	$\Lambda^1(v_1), v_1(t) = t^{(\frac{1}{q_1}-1)} \varphi(\log_1 \frac{1}{t}) (\log_2 \frac{1}{t})$
		D_2^+
$0 < q < 1$	$q_2 = 1$	$\Lambda^1(v_2), v_2(t) = \varphi(\log_1 \frac{1}{t}) (\log_2 L)^{(\frac{1}{q}-1)(1+\varepsilon)}$
	$1 < q_2 < \infty$	$\Lambda^1(v_2), v_2(t) = t^{(\frac{1}{q_2}-1)} \varphi(\log_1 \frac{1}{t}) (\log_1 \frac{1}{t})^{(\frac{1}{q}-1)(1+\varepsilon)}$
$q = 1$	$1 < q_1 < \infty$	$\Lambda^1(v_2), v_2(t) = t^{(\frac{1}{q_2}-1)} \varphi(\log_1 \frac{1}{t}) (\log_2 \frac{1}{t})$

Table 4.13: Theorem 4.2.7

Proof. Given $f \in L^{q_1,1}$ and $g \in L^{q_2,1}$ such that $\|f\|_{L^{q_1,1}} = \|g\|_{L^{q_2,1}} = 1$, we use the dyadic decomposition over \mathbb{N} . Thus,

$$(B(f_n, g_m))^*(t) \lesssim t^{-\frac{1}{p}} \|B(f_n, g_m)\|_{L^{p,\infty}} \lesssim \varphi \left(\frac{1}{p-q} \right) \left(\frac{\|f_n\|_{L^1}^{\frac{q}{q_1}} \|g_m\|_{L^1}^{\frac{q}{q_2}}}{t} \right)^{1/p}.$$

By Lemma 2.1.4, we obtain

$$(B(f_n, g_m))^*(t) \lesssim I_{\varphi,q} \left(\frac{\|f_n\|_{L^1}^{\frac{q}{q_1}} \|g_m\|_{L^1}^{\frac{q}{q_2}}}{t} \right) \lesssim \frac{\varphi(\log_1 t)}{t^{\frac{1}{q}}} I_{\varphi,q_1}(\|f_n\|_{L^1}) I_{\varphi,q_2}(\|g_m\|_{L^1}).$$

Now, if $V(t) = \frac{t^{1/q}}{\varphi(\log_1 t)}$, we conclude

$$\|B(f_n, g_m)\|_{\Lambda^{q, \infty}[V]} \lesssim I_{\varphi, q_1}(\|f_n\|_{L^1}) I_{\varphi, q_2}(\|g_m\|_{L^1}).$$

The rest of the proof follows the same ideas than Theorem 4.2.6 but using Lemma 2.3.6 and Lemma 2.3.9 when $q_1 = 1$ and Lemma 2.3.10 when $1 < q_1 < \infty$. \square

Theorem 4.2.8 (Figure 4.8 (b)). *Suppose that $1 < q_1, q_2 < \infty$ and $0 < q \leq 1$. If for every $p_1^- \leq p_1 < q_1$ and $p_2^- \leq p_2 < q_2$ such that (4.8) is satisfied we have that*

$$B : L^{p_1, 1} \times L^{p_2, 1} \longrightarrow L^{p, \infty}, \quad \|B\| \lesssim \varphi\left(\frac{1}{q-p}\right),$$

then

$$B : D_1^- \times D_2^- \longrightarrow \Lambda^{1, \infty}[V], \quad V(t) = \frac{t^{1/q}}{\varphi(\log_1 \frac{1}{t})},$$

where D_1^- and D_2^- are determined by the values described in Table 4.14 in which $\varepsilon > 0$ is arbitrary.

D_1^-	
$0 < q < 1$	$\Lambda^1(v_1), v_1(t) = t^{\left(\frac{1}{q_1}-1\right)} \varphi(\log_1 t) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)} \varphi\left(\log_2 \frac{1}{t}\right)$
$q = 1$	$\Lambda^1(v_1), v_1(t) = t^{\left(\frac{1}{q_1}-1\right)} \varphi(\log_1 t) \left(\log_2 \frac{1}{t}\right) \varphi\left(\log_2 \frac{1}{t}\right)$
D_2^-	
$0 < q < 1$	$\Lambda^1(v_2), v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \varphi(\log_1 t) \left(\log_1 \frac{1}{t}\right)^{\left(\frac{1}{q}-1\right)(1+\varepsilon)} \varphi\left(\log_2 \frac{1}{t}\right)$
$q = 1$	$\Lambda^1(v_1), v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \varphi(\log_1 t) \left(\log_2 \frac{1}{t}\right) \varphi\left(\log_2 \frac{1}{t}\right)$

Table 4.14: Theorem 4.2.8

Proof. We follow the ideas of Theorem 4.2.7 but using Lemma 2.3.7 and Lemma 2.3.11. \square

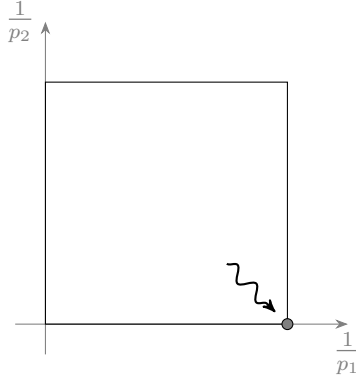


Figure 4.9: Quasi-Banach setting. Theorem 4.2.9.

4.2.3 The endpoint $(1, \infty; 1)$

Theorem 4.2.9 (Figure 4.9). *If, for every $1 < p_1 \leq p_1^+$ and $p_2^- \leq p_2 < \infty$*

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \quad \|B\| \lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \varphi_2(p_2),$$

then

$$B : \Lambda^1(v_1) \times L_{\widetilde{\varphi}_2}^\infty \longrightarrow \Lambda^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi_1(\log_1 t) \varphi_2(\log_1 \frac{1}{t})},$$

where $v_1(t) = \varphi_1(\log \frac{1}{t})(\log_3 \frac{1}{t})\varphi_2(\log_3 \frac{1}{t})$ and $\widetilde{\varphi}_2(t) := \varphi_2(t)(\log_1 t)\varphi_2(\log_1 t)$.

Proof. First, given $f \in L^1$ and $g \in L^\infty$, we decompose f using the dyadic decomposition over \mathbb{N} and g using the g^* -decomposition. Hence,

$$B(f_n, g_m)^*(t) \lesssim \varphi_1 \left(\frac{1}{p_1 - 1} \right) \left(\frac{\|f_n\|_{L^1}}{t} \right)^{1/p_1} \varphi_2(p_2) \left(\frac{\|g_m\|_{L^1}}{t} \right)^{1/p_2}.$$

Taking the infimum in each variable, by Lemma 2.1.4, we obtain

$$\sup_{t>0} \frac{t B(f_n, g_m)^*(t)}{\varphi_1(\log_1 t) \varphi_2 \left(\log_1 \frac{1}{t} \right)} \lesssim I_{\varphi,1} (\|f_n\|_{L^1}) \left[\|g_m\|_{L^\infty} \varphi_2 \left(\log_1 \frac{\|g_k\|_{L^1}}{\|g_m\|_{L^\infty}} \right) \right],$$

and now, we can proceed as in Theorem 4.2.6, using Lemmas 2.3.8, 2.3.9 and 2.3.12. \square

CHAPTER 5

Endpoint estimates for particular operators

In this chapter, we apply bilinear Yano's theory to obtain endpoint estimates for certain classes of operators, such as rough singular operators and the bilinear Bochner-Riesz multiplier at the critical index.

As in the linear case, Rubio de Francia extrapolation theory will be a fundamental tool to find interesting examples. In particular, we shall use the following result (see [51], [43]).

Theorem 5.0.1. *If, for some $(r_1, r_2; r)$ with $1 \leq r_1, r_2 < \infty$ and, for every $w_1 \in A_{r_1}$ and $w_2 \in A_{r_2}$,*

$$B : L^{r_1}(w_1) \times L^{r_2}(w_2) \longrightarrow L^r(w), \quad \|B\| \lesssim \psi \left([w_1]_{A_{r_1}}, [w_2]_{A_{r_2}} \right), \quad (5.1)$$

where ψ is an increasing function in each variable and $w = w_1^{r/r_1} w_2^{r/r_2}$, then, for $1 < p_1, p_2 < \infty$ and, for every $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$ and $w = w_1^{p/p_1} w_2^{p/p_2}$,

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \rightarrow L^p(w),$$

with

$$\|B\| \lesssim \psi \left(C_1 p_1 [w_1]_{A_{p_1}}^{\max\{1, \frac{r_1-1}{p_1-1}\}}, C_2 p_2 [w_2]_{A_{p_2}}^{\max\{1, \frac{r_2-1}{p_2-1}\}} \right). \quad (5.2)$$

We shall denote by $(r_1, r_2; r)$ the first point where we have a weighted estimate, and by $(p_1, p_2; p)$ with $1 < p_1, p_2 < \infty$ all the points obtained by Rubio de Francia extrapolation theorem. Thus, we will study the boundedness of the operator at the

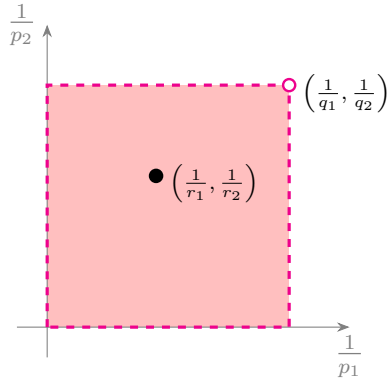


Figure 5.1: Bilinear Rubio de Francia Theorem 5.0.1.

endpoints $(q_1, q_2; q)$ which lie in the boundary of the shaded square (see Figure 5.1). Following carefully the behaviour of the constants in the proof of Theorem 5.0.1 (see [43]), one can easily obtain the following result.

Corollary 5.0.2. *If B satisfies (5.1), then, for $1 < p_1, p_2 < \infty$ and, for every $w_1 \in A_{p_1}$ and $w_2 \in A_{p_2}$, and $w = w_1^{p/p_1} w_2^{p/p_2}$,*

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w),$$

with the following operator norm:

i) If $1 < p_1 < r_1$, $1 < p_2 < r_2$ and $w_1, w_2 \in A_1$,

$$\|B\| \lesssim \psi \left(C_1 \left(\frac{1}{p_1 - 1} \right)^{r_1 - p_1} [w_1]_{A_1}^{1+r_1-p_1}, C_2 \left(\frac{1}{p_2 - 1} \right)^{r_2 - p_2} [w_2]_{A_1}^{1+r_2-p_2} \right).$$

ii) If $1 < p_1 < r_1$, $1 < p_2 < \infty$ and $w_1 \in A_1$, $w_2 \in A_{p_2}$,

$$\|B\| \lesssim \psi \left(C_1 \left(\frac{1}{p_1 - 1} \right)^{r_1 - p_1} [w_1]_{A_1}^{1+r_1-p_1}, C_2 p_2 [w_2]_{A_{p_2}}^{\max\{1, \frac{r_2-1}{p_2-1}\}} \right).$$

We observe that, in order to apply our Yano's extrapolation results after using Rubio de Francia theory, we need $w = w_1^{p/p_1} w_2^{p/p_2}$ to remain fixed for all $(p_1, p_2; p)$. This can be achieved by either setting $w_1 = w_2 = w$ (which will provide one-weight estimates at the endpoints) or by assuming that $\frac{p_1}{r_1} = \frac{p_2}{r_2} = \frac{p}{r}$ (which will provide two-weight estimates at the endpoints).

Let us start first with the one-weight case (see Figure 5.2).

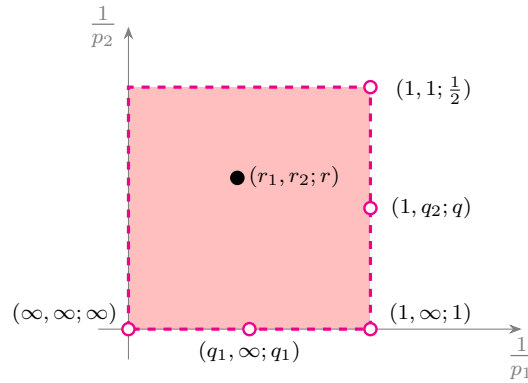


Figure 5.2: Corollary 5.0.3.

Theorem 5.0.3. *If, for some $(r_1, r_2; r)$ with $1 < r_1, r_2 < \infty$ and, for $w_1 \in A_{r_1}$, $w_2 \in A_{r_2}$, and $w = w_1^{r/r_1} w_2^{r/r_2}$,*

$$B : L^{r_1}(w_1) \times L^{r_2}(w_2) \longrightarrow L^r(w), \quad \|B\| \lesssim \varphi_1 \left([w_1]_{A_{r_1}} \right) \varphi_2 \left([w_2]_{A_{r_2}} \right),$$

then, B satisfies the estimates described in Table 5.1 in which $\varepsilon > 0$ is arbitrary.

Proof. $\boxed{(1, 1; \frac{1}{2})}$ If we set $w \in A_1$, by Corollary 5.0.2 follows that, for $1 < p_1 < r_1$ and $1 < p_2 < r_2$,

$$B : L^{p_1}(w) \times L^{p_2}(w) \longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1 \left(\left(\frac{1}{p_1 - 1} \right)^{r_1 - 1} \right) \varphi_2 \left(\left(\frac{1}{p_2 - 1} \right)^{r_2 - 1} \right) \varphi_1 \left([w]_{A_1}^{r_1} \right) \varphi_2 \left([w]_{A_1}^{r_2} \right).$$

Hence, we can use Theorem 4.2.5 to obtain the estimate.

$\boxed{(1, q_2; q)}$ Fix $1 < q_2 < \infty$ and $w \in A_1$. Then, for every $1 < p_1 < r_1$,

$$B : L^{p_1}(w) \times L^{q_2}(w) \longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1 \left(\left(\frac{1}{p_1 - 1} \right)^{r_1 - 1} \right) \varphi_1 \left([w]_{A_1}^{r_1} \right) \varphi_2 \left([w]_{A_{q_2}}^{\max\{1, \frac{r_2 - 1}{q_2 - 1}\}} \right).$$

Thus, we can fix $g \in L^{q_2}(w)$ and use Proposition 4.2.1 to get the result.

Endpoint	Weights	One-weight estimate	$\ B\ \lesssim$
$(1, 1; \frac{1}{2})$	$w \in A_1$	$B : \Lambda_w^1(v_1) \times \Lambda_w^1(v_2) \rightarrow \Lambda_w^{1,\infty}[V],$ $v_j(t) = \varphi_j \left((\log_1 \frac{1}{t})^{r_j-1} (\log_2 \frac{1}{t})^{1+\varepsilon} \right),$ $V(t) = \frac{t^2}{\varphi_1 \left((\log_1 t)^{r_1-1} \right) \varphi_2 \left((\log_1 t)^{r_2-1} \right)}$	$\varphi_1([w]_{A_1}^{r_1}) \varphi_2([w]_{A_1}^{r_2})$
$(1, q_2; q)$	$w \in A_1$	$B : \Lambda_w^1(v_1) \times L^{q_2}(w) \rightarrow \Lambda_w^{1,\infty}[V],$ $v_1(t) = \varphi_1 \left((\log_1 \frac{1}{t})^{r_1-1} (\log_2 \frac{1}{t})^{\frac{1+\varepsilon}{q_2}} \right),$ $V(t) = \frac{t^{1/q}}{\varphi_1 \left((\log_1 t)^{r_1-1} \right)}$	$\varphi_1([w]_{A_1}^{r_1}) \varphi_2 \left([w]_{A_{q_2}}^{\max\{1, \frac{r_2-1}{q_2-1}\}} \right)$
$(1, \infty; 1)$	$w \in A_1$	$B : \Lambda_w^1(v_1) \times L_{\varphi_2}^\infty(w) \rightarrow \Gamma_w^{1,\infty}[V],$ $v_1(t) = \varphi_1 \left((\log_1 \frac{1}{t})^{r_1-1} \right) \varphi_2 \left(\log_1 \frac{1}{t} \right),$ $V(t) = \frac{t}{\varphi_1 \left((\log_1 t)^{r_1-1} \right) \varphi_2 \left(\log_1 t \right) \varphi_2 \left(\log_1 \frac{1}{t} \right)}$	$\varphi_1([w]_{A_1}^{r_1}) \varphi_2([w]_{A_{r_2}})$
$(q_1, \infty; q_1)$	$w \in A_{q_1}$	$B : L^{q_1}(w) \times L_{\varphi_2}^\infty(w) \rightarrow \Gamma_w^{1,\infty}[V],$ $V(t) = \frac{t^{1/q_1}}{\varphi_2 \left(\log_1 \frac{1}{t} \right)}$	$\varphi_1 \left([w]_{A_{q_1}}^{\max\{1, \frac{r_1-1}{q_1-1}\}} \right) \varphi_2([w]_{A_{q_1}})$
$(\infty, \infty; \infty)$	$w \in A_\infty$	$B : L_{\varphi_1}^\infty(w) \times L_{\varphi_2}^\infty(w) \rightarrow \Gamma_w^{1,\infty}[V],$ $V(t) = \frac{1}{\varphi_1 \left(\log_1 \frac{1}{t} \right) \varphi_2 \left(\log_1 \frac{1}{t} \right)}$	

Table 5.1: Corollary 5.0.3

$(1, \infty; 1)$ Let $w \in A_1$. By Corollary 5.0.2, we get that, for every $1 < p_1 < r_1$ and $r_2 < p_2 < \infty$,

$$B : L^{p_1}(w) \times L^{p_2}(w) \longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1 \left(\left(\frac{1}{p_1 - 1} \right)^{r_1-1} \right) \varphi_2(p_2) \varphi_1 \left([w]_{A_1}^{r_1} \right) \varphi_2 \left([w]_{A_{p_2}} \right).$$

The endpoint estimate follows by Theorem 4.1.10.

$(q_1, \infty; q_1)$ Fix $w \in A_{q_1}$. For every $\max\{r_2, q_1\} < p_2 < \infty$, by (5.2), we get

$$B : L^{q_1}(w) \times L^{p_2}(w) \longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1 \left([w]_{A_{q_1}}^{\max\{1, \frac{r_1-1}{q_1-1}\}} \right) \varphi_2(p_2) \varphi_2 \left([w]_{A_{q_1}} \right).$$

Thus, we can fix $f \in L^{q_1}(w)$ and use Proposition 4.1.4 to get the result.

$(\infty, \infty; \infty)$ If $w \in A_\infty$, there exists $\max\{r_1, r_2\} \leq p_0 < \infty$ such that $w \in A_{p_0}$. Thus, it follows from Corollary 5.0.2 that, for every $p_0 < p_1, p_2 < \infty$,

$$B : L^{p_1}(w) \times L^{p_2}(w) \longrightarrow L^p(w),$$

with

$$\|B\| \lesssim \varphi_1(p_1) \varphi_2(p_2) \varphi_1 \left([w]_{A_{p_0}} \right) \varphi_2 \left([w]_{A_{p_0}} \right),$$

and by Theorem 4.1.6 the result follows. \square

Remark 5.0.4. At the endpoint $(1, \infty; 1)$, we can also use Theorem 4.2.9 to conclude that, for every $w \in A_1$,

$$B : \Lambda_w^1(v_1) \times L_{\tilde{\varphi}_2}^\infty(w) \rightarrow \Lambda_w^{1,\infty}[V], \quad V(t) = \frac{t}{\varphi_1((\log_1 t)^{r_1-1}) \varphi_2(\log_1 \frac{1}{t})}$$

where

$$v_1(t) = \varphi_1 \left((\log_1 \frac{1}{t})^{r_1-1} \right) \left(\log_3 \frac{1}{t} \right) \varphi_2 \left(\log_3 \frac{1}{t} \right)$$

and $\tilde{\varphi}_2(t) = \varphi_2(t) (\log_1 t) \varphi_2(\log_1 t)$.

In the points $(q_1, q_2; q)$ on the boundary of the shaded square that can be reached through a ray, we obtain two-weights estimates.

Theorem 5.0.5. *If, for some $(r_1, r_2; r)$ with $1 < r_1, r_2 < \infty$, for every $w_1 \in A_{r_1}$, $w_2 \in A_{r_2}$ and $w = w_1^{r/r_1} w_2^{r/r_2}$,*

$$B : L^{r_1}(w_1) \times L^{r_2}(w_2) \longrightarrow L^r(w), \quad \|B\| \lesssim \varphi_1 \left([w_1]_{A_{r_1}} \right) \varphi_2 \left([w_2]_{A_{r_2}} \right),$$

then, B satisfies the estimates described in Table 5.2, in which $\varepsilon > 0$ is arbitrary.

Proof. $(1, 1; \frac{1}{2})$ If we set $w_1, w_2 \in A_1$, it follows from Corollary 5.0.2 that, for every $1 < p_1 < r_1$ and $1 < p_2 < r_2$,

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w_1^{p/p_1} w_2^{p/p_2}),$$

Endpoint	Weights	Two-weight estimate	$\ B\ \lesssim$
$(1, 1; \frac{1}{2})$	$w_1, w_2 \in A_1$	$B : \Lambda_{w_1}^1(v_0) \times \Lambda_{w_2}^1(v_0) \rightarrow \Lambda_{(w_1^{1/2} w_2^{1/2})}^{1, \infty}[V],$ $v_0(t) = \varphi_1 \left((\log_1 \frac{1}{t})^{r_1-1} \right) \varphi_2 \left((\log_1 \frac{1}{t})^{r_2-1} \right) (\log_2 \frac{1}{t})^{1+\varepsilon},$ $V(t) = \frac{t^2}{\varphi_1 \left((\log_1 t)^{r_1-1} \right) \varphi_2 \left((\log_1 t)^{r_2-1} \right)}$	$\varphi_1([w_1]_{A_1}^{r_1}) \varphi_2([w_2]_{A_1}^{r_2})$
$(1, q_2; q)$	$w_1 \in A_1$ $w_2 \in A_{q_2}$	$B : \Lambda_{w_1}^1(v_1) \times \Lambda_{w_2}^1(v_2) \rightarrow \Lambda_{(w_1^q w_2^{q/q_2})}^{1, \infty}[V],$ $v_1(t) = \varphi_1 \left((\log_1 \frac{1}{t})^{r_1-1} \right) (\log_2 \frac{1}{t})^{\frac{1+\varepsilon}{q_2}}$ $v_2(t) = t^{\left(\frac{1}{q_2}-1\right)} \varphi_1 \left((\log_1 \frac{1}{t})^{r_1-1} \right) (\log_1 \frac{1}{t})^{\frac{1+\varepsilon}{q_2}}$ $V(t) = \frac{t^{1/q}}{\varphi_1 \left((\log_1 t)^{r_1-1} \right)},$	$\varphi_1([w_1]_{A_1}^{r_1}) \varphi_2 \left([w_2]_{A_{q_2}}^{\max\{1, \frac{r_2-1}{q_2}\}} \right)$
$(\infty, \infty; \infty)$	$w_1, w_2 \in A_\infty$ $\alpha, \beta > 0$ $\alpha + \beta = 1$	$B : L_\varphi^\infty(w_1) \times L_\varphi^\infty(w_2) \rightarrow \Gamma_{(w_1^\alpha w_2^\beta)}^{1, \infty}[V],$ $V(t) = \frac{1}{\varphi \left(\log_1 \frac{1}{t} \right)},$ $\varphi(s) = \varphi_1(s) \varphi_2(s)$	

Table 5.2: Corollary 5.0.5

with

$$\|B\| \lesssim \varphi_1 \left(\left(\frac{1}{p_1 - 1} \right)^{r_1-1} \right) \varphi_2 \left(\left(\frac{1}{p_2 - 1} \right)^{r_2-1} \right) \varphi_1 \left([w_1]_{A_1}^{r_1} \right) \varphi_2 \left([w_2]_{A_1}^{r_2} \right).$$

In particular, for every $1 < p_1 < \min\{r_1, r_2\}$

$$B : L^{p_1}(w_1) \times L^{p_1}(w_2) \longrightarrow L^p(w_1^{1/2} w_2^{1/2}),$$

with

$$\|B\| \lesssim \varphi_1 \left(\left(\frac{1}{p - \frac{1}{2}} \right)^{r_1-1} \right) \varphi_2 \left(\left(\frac{1}{p - \frac{1}{2}} \right)^{r_2-1} \right) \varphi_1 \left([w_1]_{A_1}^{r_1} \right) \varphi_2 \left([w_2]_{A_1}^{r_2} \right).$$

Hence, from Theorem 4.2.7 the result follows.

$\boxed{(1, q_2; q)}$ Let us fix $1 < q_2 < \infty$, $w_1 \in A_1$ and $w_2 \in A_{q_2}$. It follows from Corollary 5.0.2 that, for every $1 < p_1 < r_1$ and $1 < p_2 < \infty$ such that $p_1 = \frac{p_2}{q_2} = \frac{p}{q} > 1$,

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w_1^q w_2^{q/q_2}),$$

with

$$\|B\| \lesssim \varphi_1 \left(\left(\frac{1}{p-q} \right)^{r_1-1} \right) \varphi_1 \left([w]_{A_1}^{r_1} \right) \varphi_2 \left([w]_{A_{q_2}}^{\max\{1, \frac{r_2-1}{q_2-1}\}} \right).$$

Thus, we can use Theorem 4.2.7 to obtain the endpoint estimate.

$(\infty, \infty; \infty)$ We observe that this point can be reached through different rays (see Figure 5.3). Let us fix $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and let us move along the ray

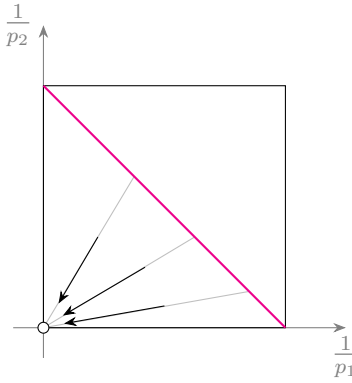


Figure 5.3: Banach setting: rays. Theorems 4.1.8 and 4.1.9.

where $\frac{p}{p_1} = \alpha$ and $\frac{p}{p_2} = \beta$. If we set $w_1, w_2 \in A_\infty$, there exists $\max\{r_1, r_2\} \leq p_0 < \infty$ such that $w_1, w_2 \in A_{p_0}$. Thus, by (5.2) we get that, for $p_0 < p_1, p_2 < \infty$ such that $\alpha p_1 = \beta p_2 = p$,

$$B : L^{p_1}(w_1) \times L^{p_2}(w_2) \longrightarrow L^p(w_1^\alpha w_2^\beta),$$

with

$$\|B\| \lesssim (\varphi_1 \varphi_2)(p) \varphi_1 \left([w_1]_{A_{p_0}} \right) \varphi_2 \left([w_2]_{A_{p_0}} \right),$$

and the endpoint estimate follows by Theorem 4.1.9. \square

5.1 Bilinear rough singular operators

Given a function Ω defined on \mathbb{S}^{2n-1} such that $\int_{\mathbb{S}^{2n-1}} \Omega = 0$, the *bilinear rough singular operator* associated to Ω (see [36]) is defined by

$$T_\Omega(f, g)(x) = p.v. \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\Omega((x, y)')}{|y, z|^{2n}} f(x-y)g(x-z)dydz, \quad x \in \mathbb{R}^n,$$

where $(y, z)' := \frac{(y, z)}{|y, z|}$. In [47] and [48] (see also [49]), the boundedness of the bilinear Hilbert transform has been used to obtain estimates for T_Ω . In fact, in [48] it is proved that, for every $(p_1, p_2; p)$ such that $1 < p_1, p_2 < \infty$,

$$T_\Omega : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$$

as long as $\Omega \in L^\infty(\mathbb{S}^{2n-1})$. On the other hand, in [7] sparse domination is used to conclude that, if $\Omega \in L^\infty(\mathbb{S}^{2n-1})$,

$$\|T_\Omega\|_{L^3(w_1) \times L^3(w_2) \rightarrow L^{3/2}(w)} \lesssim C(\|\Omega\|_{L^\infty}) [w_1]_{A_3} [w_2]_{A_3} \quad \forall w_1, w_2 \in A_3. \quad (5.3)$$

Also, from [32] it can be deduced that,

$$\|T_\Omega\|_{L^2(w_1) \times L^2(w_2) \rightarrow L^1(w)} \lesssim \|\Omega\|_{L^\infty} [w_1]_{A_2}^{3/2} [w_2]_{A_2}^{3/2}, \quad w_1, w_2 \in A_2. \quad (5.4)$$

Some results are known at the endpoints. Indeed, by [75] and (5.3), it follows the weighted boundedness of T_Ω for every $(q_1, \infty; q_1)$ with $1 < q_1 < \infty$. In the unweighted case, it is also known (see [54]) that

$$T_\Omega : L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \longrightarrow L^{1, \infty}(\mathbb{R}^n),$$

and

$$T_\Omega : L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \longrightarrow BMO.$$

As far as we know, the cases $(1, 1; 1/2)$ and $(1, q_2; q)$ with $1 < q_2 < \infty$ are open. We shall use estimates (5.3) and (5.4) to obtain new weighted estimates at these endpoints.

Theorem 5.1.1. *If $\Omega \in L^\infty(\mathbb{S}^{2n-1})$ and $\int_{\mathbb{S}^{2n-1}} \Omega = 0$, the operator T_Ω satisfies the endpoint estimates described in Table 5.3 and Table 5.4, in which $\varepsilon > 0$ is arbitrary.*

Proof. We apply Theorem 5.0.3 and Theorem 5.0.5, using estimate (5.4) in the cases where the endpoint is $(1, 1; \frac{1}{2})$ or $(1, q_2; q)$, and (5.3) in when the endpoint is $(1, \infty; 1)$ or $(\infty, \infty; \infty)$. \square

5.2 Bilinear Bochner-Riesz operators

The bilinear Bochner-Riesz operator of index $\alpha \geq 0$ is defined, for $f, g \in \mathcal{S}(\mathbb{R}^n)$, by

$$\mathcal{B}^\alpha(f, g)(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 - |\xi|^2 - |\eta|^2)_+^\alpha \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta, \quad x \in \mathbb{R}^n.$$

Endpoint	Weights	One-weight estimate	$\ T_\Omega\ \lesssim$
$(1, 1; \frac{1}{2})$	$w \in A_1$	$T_\Omega : \Lambda_w^1(v_1) \times \Lambda_w^1(v_2) \rightarrow \Lambda_w^{1,\infty}[V],$ $v_j(t) = (\log_1 \frac{1}{t})^{3/2} (\log_2 \frac{1}{t})^{1+\varepsilon},$ $V(t) = \frac{t^2}{(\log_1 t)^3}$	$[w]_{A_1}^6$
$(1, q_2; q)$	$w \in A_1$	$T_\Omega : \Lambda_w^1(v_1) \times L^{q_2}(w) \rightarrow \Lambda_w^{1,\infty}[V],$ $v_1(t) = (\log_1 \frac{1}{t})^{3/2} (\log_2 \frac{1}{t})^{\frac{1+\varepsilon}{q_2}},$ $V(t) = \frac{t^{1/q}}{(\log_1 t)^{3/2}}$	$[w]_{A_1}^3 \left([w]_{A_{q_2}}^{\max\{1, \frac{1}{q_2-1}\}} \right)^{3/2}$
$(1, \infty; 1)$	$w \in A_1$	$T_\Omega : \Lambda_w^1(v_1) \times L_1^\infty(w) \rightarrow \Gamma_w^{1,\infty}[V],$ $v_1(t) = (\log_1 \frac{1}{t})^3,$ $V(t) = \frac{t}{(\log_1 t)^3 (\log_1 \frac{1}{t})}$	$[w]_{A_1}^3 [w]_{A_3}$
$(\infty, \infty; \infty)$	$w \in A_\infty$	$T_\Omega : L_1^\infty(w) \times L_1^\infty(w) \rightarrow E^2(w)$	

Table 5.3: Corollary 5.1.1

As in the linear case, it is useful to work with the kernel associated to this operator. Indeed,

$$\mathcal{B}^\alpha(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_\alpha(y_1, y_2) f(x - y_1) g(x - y_2) dy_1 dy_2, \quad x \in \mathbb{R}^n,$$

where, for every $y_1, y_2 \in \mathbb{R}^n$,

$$K_\alpha(y, z) = \frac{\Gamma(\alpha + 1)}{\pi^\alpha} \frac{J_{\alpha+n} \left(2\pi |(y_1, y_2)| \right)}{|(y_1, y_2)|^{\alpha+n}}$$

and $J_{\alpha+n}$ denotes the standard Bessel function of order $\alpha + n$. It is known (see [10]) that for $\alpha > n - \frac{1}{2}$ and $1 \leq p_1, p_2 \leq \infty$,

$$\mathcal{B}^\alpha : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

Endpoint	Weights	Two-weight estimate	$\ T_\Omega\ \lesssim$
$(1, 1; \frac{1}{2})$	$w_1, w_2 \in A_1$	$T_\Omega : \Lambda_{w_1}^1(v_0) \times \Lambda_{w_2}^1(v_0) \rightarrow \Lambda_{(w_1^{1/2} w_2^{1/2})}^{1, \infty}[V],$ $v_0(t) = (\log_1 \frac{1}{t})^3 (\log_2 \frac{1}{t})^{1+\varepsilon},$ $V(t) = \frac{t^2}{(\log_1 t)^3}$	$[w_1]_{A_1}^3 [w_2]_{A_1}^3$
$(1, q_2; q)$	$w_1 \in A_1$ $w_2 \in A_{q_2}$	$T_\Omega : \Lambda_{w_1}^1(v_1) \times \Lambda_{w_2}^1(v_2) \rightarrow \Lambda_{(w_1^q w_2^{q/q_2})}^{1, \infty}[V],$ $v_1(t) = (\log_1 \frac{1}{t})^{3/2} (\log_2 \frac{1}{t})^{\frac{1+\varepsilon}{q_2}},$ $v_2(t) = t^{(\frac{1}{q_2}-1)} (\log_1 \frac{1}{t})^{\frac{3}{2} + \frac{1+\varepsilon}{q_2}},$ $V(t) = \frac{t^{1/q}}{(\log_1 t)^{3/2}},$	$[w_1]_{A_1}^3 \left([w_2]_{A_{q_2}}^{\max\{1, \frac{1}{q_2-1}\}} \right)^{3/2}$
$(\infty, \infty; \infty)$	$w_1, w_2 \in A_\infty$ $\alpha, \beta > 0$ $\alpha + \beta = 1$	$T_\Omega : L_2^\infty(w_1) \times L_2^\infty(w_2) \rightarrow E^2(w_1^\alpha w_2^\beta)$	

Table 5.4: Corollary 5.1.1

and $\alpha = n - \frac{1}{2}$ is the so-called critical index.

In the weighted context, it has been proved in [64] that

$$\mathcal{B}^{n-\frac{1}{2}} : L^2(w_1) \times L^2(w_2) \rightarrow L^1(w) \quad \forall w_1, w_2 \in A_2. \quad (5.5)$$

Hence, we need to find φ_1, φ_2 such that, for every $w_1, w_2 \in A_2$, (5.5) is satisfied with

$$\left\| \mathcal{B}^{n-\frac{1}{2}} \right\| \leq \varphi_1([w_1]_{A_2}) \varphi_2([w_2]_{A_2}).$$

We observe (see [13], [85]) that there exists a constant $C_n := 1/2^{n+5}$ such that, for every $w_1, w_2 \in A_2$, if $\delta_1 = [w_1]_{A_2}^{-1}$ and $\delta_2 = [w_2]_{A_2}^{-1}$, then,

$$[w_1^{1+C\delta_1}]_{A_2} \lesssim [w_1]_{A_2}^{1+C\delta_1} \quad \text{and} \quad [w_2^{1+C\delta_2}]_{A_2} \lesssim [w_2]_{A_2}^{1+C\delta_2}.$$

Using this fact, a detailed study of the proof in [64] shows that, for every $n \geq 2$ and $w_1, w_2 \in A_2$, we have

$$\mathcal{B}^{n-\frac{1}{2}} : L^2(w_1) \times L^2(w_2) \longrightarrow L^1(w), \quad \left\| \mathcal{B}^{n-\frac{1}{2}} \right\| \lesssim \max\{[w_1]_{A_2}^2, [w_2]_{A_2}^2\} [w_1]_{A_2} [w_2]_{A_2}.$$

In particular,

$$\mathcal{B}^{n-\frac{1}{2}} : L^2(w_1) \times L^2(w_2) \longrightarrow L^1(w), \quad \|\mathcal{B}^{n-\frac{1}{2}}\| \lesssim [w_1]_{A_2}^3 [w_2]_{A_2}^3.$$

Thus, by Theorem 5.0.3 and Theorem 5.0.5, we obtain the following new estimates.

Theorem 5.2.1. *The operator $\mathcal{B}^{n-\frac{1}{2}}$ satisfies the endpoint estimates described in Table 5.5 and Table 5.6, in which $\varepsilon > 0$ is arbitrary.*

Endpoint	Weights	One-weight estimate	$\ \mathcal{B}^{n-\frac{1}{2}}\ \lesssim$
$(1, 1; \frac{1}{2})$	$w \in A_1$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda_w^1(v_1) \times \Lambda_w^1(v_2) \rightarrow \Lambda_w^{1,\infty}[V],$ $v_j(t) = (\log_1 \frac{1}{t})^3 (\log_2 \frac{1}{t})^{1+\varepsilon},$ $V(t) = \frac{t^2}{(\log_1 t)^6}$	$[w]_{A_1}^{12}$
$(1, q_2; q)$	$w \in A_1$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda_w^1(v_1) \times L^{q_2}(w) \rightarrow \Lambda_w^{1,\infty}[V],$ $v_1(t) = (\log_1 \frac{1}{t})^3 (\log_2 \frac{1}{t})^{\frac{1+\varepsilon}{q_2}},$ $V(t) = \frac{t^{1/q}}{(\log_1 t)^3}$	$[w]_{A_1}^6 [w]_{A_{q_2}}^{\max\{1, \frac{1}{q_2-1}\}}$
$(1, \infty; 1)$	$w \in A_1$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda_w^1(v_1) \times L_1^\infty(w) \rightarrow \Gamma_w^{1,\infty}[V],$ $v_1(t) = (\log_1 \frac{1}{t})^6,$ $V(t) = \frac{t}{(\log_1 t)^6 (\log_1 \frac{1}{t})^3}$	$[w]_{A_1}^9$
$(\infty, \infty; \infty)$	$w \in A_\infty$	$\mathcal{B}^{n-\frac{1}{2}} : L_3^\infty(w) \times L_3^\infty(w) \rightarrow E^6(w)$	

Table 5.5: Corollary 5.2.1

The results obtained in this section have been published in [23].

Endpoint	Weights	Two-weight estimate	$\ \mathcal{B}^{n-\frac{1}{2}}\ \lesssim$
$(1, 1; \frac{1}{2})$	$w_1, w_2 \in A_1$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda_{w_1}^1(v_0) \times \Lambda_{w_2}^1(v_0) \rightarrow \Lambda_{(w_1^{1/2}w_2^{1/2})}^{1,\infty}[V],$ $v_0(t) = (\log_1 \frac{1}{t})^6 (\log_2 \frac{1}{t})^{1+\varepsilon},$ $V(t) = \frac{t^2}{(\log_1 t)^6}$	$[w_1]_{A_1}^6 [w_2]_{A_1}^6$
$(1, q_2; q)$	$w_1 \in A_1$ $w_2 \in A_{q_2}$	$\mathcal{B}^{n-\frac{1}{2}} : \Lambda_{w_1}^1(v_1) \times \Lambda_{w_2}^1(v_2) \rightarrow \Lambda_{(w_1^q w_2^{q/q_2})}^{1,\infty}[V],$ $v_1(t) = (\log_1 \frac{1}{t})^3 (\log_2 \frac{1}{t})^{\frac{1+\varepsilon}{q_2}}$ $v_2(t) = t^{(\frac{1}{q_2}-1)} (\log_1 \frac{1}{t})^{3+\frac{1+\varepsilon}{q_2}}$ $V(t) = \frac{t^{1/q}}{(\log_1 t)^3},$	$[w_1]_{A_1}^6 [w_2]_{A_{q_2}}^{\max\{1, \frac{1}{q_2-1}\}}$
$(\infty, \infty; \infty)$	$w_1, w_2 \in A_\infty$ $\alpha, \beta > 0$ $\alpha + \beta = 1$	$\mathcal{B}^{n-\frac{1}{2}} : L_6^\infty(w_1) \times L_6^\infty(w_2) \rightarrow E^6(w_1^\alpha w_2^\beta)$	

Table 5.6: Corollary 5.2.1

5.3 Bilinear Fourier multipliers of bounded variation

Let us consider now the following extension of the bounded variation functions on \mathbb{R} . Set

$$m(\xi, \eta) := \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} d\mu(t, s), \quad \xi, \eta \in \mathbb{R}, \quad (5.6)$$

where $d\mu$ is a finite measure on \mathbb{R}^2 and without loss of generality we may assume that $\mu(\mathbb{R}^2) = 1$. We observe this is the case if $m(\xi, \eta) = m_1(\xi)m_2(\eta)$ with m_j , $j = 1, 2$, two normalized ($dm_j(\mathbb{R}) = 1$, $m_j(-\infty) = 0$) bounded variation functions. For functions f and g in C_c^∞ , the following operator associated to the multiplier m was introduced in [87],

$$B_m(f, g)(x) := \int_{\mathbb{R}^2} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix(\xi+\eta)} d\xi d\eta, \quad x \in \mathbb{R}.$$

In particular, it is easy to see that

$$B_m(f, g)(x, y) = \int_{\mathbb{R}^2} H_t f(x) H_s g(y) d\mu(t, s), \quad \forall x, y \in \mathbb{R},$$

where

$$H_t f(x) := \int_t^\infty \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \quad x \in \mathbb{R}.$$

Some endpoint estimates for this operator are known. Indeed, using techniques different from those presented in this work, we established in [6] the following one-weight estimate at the endpoint $(1, 1; \frac{1}{2})$:

$$B_m : L^1(w) \times L^1(w) \longrightarrow L^{1/2, \infty}(w), \quad \forall w \in A_1,$$

and, by [87], a two-weight estimate at the endpoint $(1, q_2; q)$ with $1 < q_2 < \infty$ can also be deduced, namely

$$B_m : L^1(w_1) \times L^{q_2}(w_2) \longrightarrow L^{q, \infty}(w), \quad \forall w_1 \in A_1, w_2 \in A_{q_2}.$$

For the remaining endpoints, we can use Theorem 5.0.3 and Theorem 5.0.5 to obtain new estimates.

Theorem 5.3.1. *The operator B_m , associated to the multiplier m defined in (5.6), satisfies the endpoint estimates described in Table 5.7 and Table 5.8, in which $\varepsilon > 0$ is arbitrary.*

Endpoint	Weights	One-weight estimate	$\ B_m\ \lesssim$
$(1, \infty; 1)$	$w \in A_1$	$B_m : \Lambda_w^1(v_1) \times L_{\varphi_2}^\infty(w) \rightarrow \Gamma_w^{1, \infty}[V],$ $v_1(t) = \left(\log_1 \frac{1}{t}\right)^2,$ $V(t) = \frac{t}{(\log_1 t)^2 (\log_1 \frac{1}{t})}$	$[w]_{A_1}^2 [w]_{A_2}$
$(q_1, \infty; q_1)$	$w \in A_{q_1}$	$B_m : L^{q_1}(w) \times L_1^\infty(w) \rightarrow \Gamma_w^{1, \infty}[V],$ $V(t) = \frac{t^{1/q_1}}{(\log_1 \frac{1}{t})}$	$[w]_{A_{q_1}}^{1 + \max\{1, \frac{1}{q_1 - 1}\}}$
$(\infty, \infty; \infty)$	$w \in A_\infty$	$B_m : L_1^\infty(w) \times L_1^\infty(w) \rightarrow E^2(w)$	

Table 5.7: Theorem 5.3.1

Proof. It is immediate to see that, for every $w \in A_2$,

$$\|H_t\|_{L^2(w) \rightarrow L^2(w)} \leq C_1 \left(1 + \|H\|_{L^2(w) \rightarrow L^2(w)}\right) \leq C_2 [w]_{A_2},$$

Endpoint	Weights	Two-weight estimate	$\ B_m\ \lesssim$
$(1, 1; \frac{1}{2})$	$w_1, w_2 \in A_1$	$B_m : \Lambda_{w_1}^1(v_0) \times \Lambda_{w_2}^1(v_0) \rightarrow \Lambda_{(w_1^{1/2}w_2^{1/2})}^{1,\infty}[V],$ $v_0(t) = (\log_1 \frac{1}{t})^2 (\log_2 \frac{1}{t})^{1+\varepsilon},$ $V(t) = \frac{t^2}{(\log_1 t)^2}$	$[w_1]_{A_1}^2 [w_2]_{A_1}^2$
$(\infty, \infty; \infty)$	$w_1, w_2 \in A_\infty$ $\alpha, \beta > 0$ $\alpha + \beta = 1$	$B_m : L_2^\infty(w_1) \times L_2^\infty(w_2) \rightarrow E^2(w_1^\alpha w_2^\beta),$	

Table 5.8: Theorem 5.3.1

with C_2 independent of t . Therefore, for every $w_1, w_2 \in A_2$ and $w = w_1^{1/2}w_2^{1/2}$,

$$\begin{aligned} \|B_m(f, g)\|_{L^1(w)} &\leq \int_{\mathbb{R}^2} \|H_t f H_s g\|_{L^1(w)} |d\mu|(t, s) \\ &\leq \int_{\mathbb{R}^2} \|H_t f\|_{L^2(w_1)} \|H_s g\|_{L^2(w_2)} |d\mu|(t, s) \leq C_2^2 [w_1]_{A_2} [w_2]_{A_2} \|f\|_{L^2(w_1)} \|g\|_{L^2(w_2)}. \end{aligned}$$

Finally, we can apply Theorem 5.0.3 and Theorem 5.0.5 to this estimate. \square

5.4 The weighted bilinear Hardy operator

In [44], it has been considered the following weighted bilinear Hardy operator. For any weight

$$v : [0, 1] \times [0, 1] \longrightarrow [0, \infty),$$

$$\mathcal{H}_v(f, g)(x) := \int_{0 < t_1, t_2 < 1} f(t_1 x) g(t_2 x) v(t_1, t_2) dt_1 dt_2, \quad x \in \mathbb{R}^n.$$

When $p > 1$, it has been proved (see [44]) that, if $1 < p_1, p_2, p < \infty$, then

$$\mathcal{H}_v : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

with

$$\|\mathcal{H}_v\| = \int_{0 < t_1, t_2 < 1} t_1^{-\frac{n}{p_1}} t_2^{-\frac{n}{p_2}} v(t_1, t_2) dt_1 dt_2.$$

Hence, it follows that, if for some $\alpha_1, \alpha_2 > -1$, we set $v(t_1, t_2) = t_1^{\alpha_1} t_2^{\alpha_2}$, then, for every $\frac{n}{1+\alpha_1} < p_1 < \infty$ and $\frac{n}{1+\alpha_2} < p_2 < \infty$ such that $p > 1$, we have

$$\mathcal{H}_v : L^{p_1} \times L^{p_2} \longrightarrow L^p, \quad \|\mathcal{H}_v\| \approx \frac{p_1 p_2}{\left(p_1 - \frac{n}{1+\alpha_1}\right) \left(p_2 - \frac{n}{1+\alpha_2}\right)}.$$

Thus, from Proposition 4.1.1 and Theorem 4.1.3, the following endpoint estimates can be deduced.

Theorem 5.4.1. *If, for some $\alpha_1, \alpha_2 > -1$, we set $v(t_1, t_2) = t_1^{\alpha_1} t_2^{\alpha_2}$, then, for every $(\frac{n}{1+\alpha_1}, q_2; q)$ such that $\frac{n}{1+\alpha_2} < q_2 < \infty$ and $q > 1$, we have that*

$$\mathcal{H}_v : D_{\frac{n}{1+\alpha_1}, 1}^+ \times L^{q_2}(\mathbb{R}^n) \longrightarrow R_{q, 1}^+.$$

In particular, if $n < q_2 < \infty$, the unweighted bilinear Hardy operator satisfies that

$$\mathcal{H} : D_{n, 1}^+ \times L^{q_2}(\mathbb{R}^n) \longrightarrow R_{q, 1}^+.$$

Theorem 5.4.2. *If for some $\alpha_1, \alpha_2 > -1$ such that $\frac{n}{2 + \alpha_1 + \alpha_2} > 1$, we set $v(t_1, t_2) = t_1^{\alpha_1} t_2^{\alpha_2}$, then*

$$\mathcal{H}_v : D_{\frac{n}{1+\alpha_1}, 1}^+ \times D_{\frac{n}{1+\alpha_2}, 1}^+ \longrightarrow R_{\frac{n}{2+\alpha_1+\alpha_2}, 2}^+.$$

In particular, if $n > 2$, the unweighted bilinear Hardy operator satisfies

$$\mathcal{H} : D_{n, 1}^+ \times D_{n, 1}^+ \longrightarrow R_{\frac{n}{2}, 2}^+.$$

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