

CHARACTERIZATION OF THE BOUNDEDNESS OF THE SEGMENT MULTIPLIER ON WEIGHTED LEBESGUE SPACES

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ABSTRACT. For $1 < p < \infty$, we characterize the boundedness of the segment multiplier on $L^p(w)$ by a large-scale Muckenhoupt condition, obtained by restricting the classical A_p condition to intervals of length at least one. The proof uses a discretization involving the discrete Hilbert transform on an associated weighted sequence space.

1. INTRODUCTION

The theory of Fourier multipliers belongs to the classical core of harmonic analysis. In the modern framework it is formulated in terms of the Fourier transform and includes, among others, the Hilbert transform and other singular integral operators. A systematic treatment of Fourier multipliers acting on L^p spaces goes back to Hörmander [7].

Let $m \in L^\infty(\mathbb{R}^n)$. The multiplier operator T with symbol m is defined, for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, by

$$\widehat{Tf}(\xi) = m(\xi) \hat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$$

denotes the Fourier transform of f . Whenever T admits a bounded extension on a given function space X , the corresponding multiplier norm of m over X is the operator norm of $T: X \rightarrow X$. Since the Fourier transform interchanges convolution and pointwise multiplication, one may also write

$$Tf = \check{m} * f,$$

where $\check{m}$ is the inverse Fourier transform of m in the distributional sense. For basic material on Fourier multipliers we refer, for example, to [5, Section 2.5].

In the one-dimensional case we single out two multipliers which will be central in what follows: the *segment multiplier* S , given by

$$Sf(x) := \int_{\mathbb{R}} \frac{\sin(x-y)}{x-y} f(y) dy, \quad x \in \mathbb{R},$$

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and the *Hilbert transform* H , defined by the principal value

$$Hf(x) := \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy, \quad x \in \mathbb{R}.$$

These two operators are among the most classical examples of Fourier multipliers. The L^p boundedness of the Hilbert transform was proved by M. Riesz [11] for $1 < p < \infty$. The segment multiplier, in turn, could be considered as the one-dimensional version of the ball multiplier, whose boundedness sparked many studies, including the fundamental observation by Fefferman [4], where he disproved any L^p boundedness besides the trivial case $p = 2$.

The symbols of S and H are

$$m_S(\xi) = \pi \chi_{[-\frac{1}{2\pi}, \frac{1}{2\pi}]}(\xi) \quad \text{and} \quad m_H(\xi) = -i\pi \operatorname{sgn}(\xi), \quad \xi \in \mathbb{R}.$$

A straightforward computation with these functions shows that both operators are related, since

$$m_S(\xi) = i \frac{m_H(\xi + \frac{1}{2\pi}) - m_H(\xi - \frac{1}{2\pi})}{2}, \quad \xi \in \mathbb{R}.$$

Hence, whenever S and H are both defined on a function space X , the triangle inequality gives

$$\|S\|_{X \rightarrow X} \leq \|H\|_{X \rightarrow X}.$$

In the scale of the unweighted Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, we know more: De Carli and Laeng [3] proved that

$$\|H\|_{L^p \rightarrow L^p} = \|S\|_{L^p \rightarrow L^p} = n_p \pi,$$

where

$$n_p := \begin{cases} \tan\left(\frac{\pi}{2p}\right), & 1 < p \leq 2, \\ \cot\left(\frac{\pi}{2p}\right), & 2 \leq p < \infty. \end{cases}$$

This indicates that the relationship between the Hilbert transform and the segment multiplier might be closer than one might expect at first glance. Beyond the classical Lebesgue spaces, a natural setting in which to study this relationship are the weighted L^p spaces, $L^p(w)$ with norm

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} = \|fw^{\frac{1}{p}}\|_p,$$

where $w \in L^1_{\text{loc}}(\mathbb{R})$ is an a.e. positive function. The boundedness of the Hilbert transform on $L^p(w)$ has been described successfully in this context by Hunt, Muckenhoupt, and Wheeden [8, Theorem 9] in terms of the A_p condition on the weight w , which also characterizes the weak-type boundedness for H and the boundedness of the maximal operator. Then, our interest now is to investigate the relationship between the boundedness of S and the A_p condition.

The main theorem of our paper, Theorem 4.3, states that the boundedness of S is characterized by a truncated version of the A_p condition (see Definition 3.1). In Proposition 3.11 we provide a weight that satisfies our condition for every $1 < p < \infty$ but is never an A_p weight so, for any such p , S is bounded on $L^p(w)$ but H is not. This shows a big qualitative difference between both operators in the weighted Lebesgue setting, compared to what was proved in [3] for $L^p(\mathbb{R})$.

The organization of this paper is as follows. In Section 2 we cover the basics of the Muckenhoupt A_p condition and some of its fundamental properties, and we also introduce its discrete version and some operators that we will need for our proofs. In Section 3 we define the truncated A_p condition, we give analogous fundamental properties and explain its relation to both the classical and the discrete A_p conditions. Finally, in Section 4 we state the main result of this paper, Theorem 4.3, which characterizes the boundedness of the segment multiplier on $L^p(w)$ in terms of the weight w .

2. PRELIMINARY DEFINITIONS AND RESULTS

If $E \subseteq \mathbb{R}$, we will denote by χ_E the characteristic function of the set E , and the Lebesgue measure of E as $|E|$. For any $1 \leq p \leq \infty$, denote its conjugate exponent by $p' = \frac{p}{p-1}$, with the usual meaning of $1' = \infty$ and $\infty' = 1$.

Whenever we write $f(s) \lesssim g(s)$, the expression denotes an inequality like $f(s) \leq Cg(s)$, where C is a constant independent of any variables. Analogously, the notation $f(s) \approx g(s)$ means the same as having both $f(s) \lesssim g(s)$ and $g(s) \lesssim f(s)$.

If I is any interval and $\lambda > 0$, we denote by λI the interval with the same center as I and λ times its length.

Definition 2.1. For $\lambda \in \mathbb{R}$, denote by T_λ the translation operator that acts as $T_\lambda f(x) = f(x - \lambda)$. For $a > 0$, denote by E_a the dilation operator that acts as $E_a f(x) = f(ax)$.

2.1. The Muckenhoupt condition. We begin by recalling the definition and main properties of the classical A_p weights, as written in [5, Section 7.1]. Results in more general measure spaces can be found in [13].

Definition 2.2. Let $1 < p < \infty$, and let w be a weight on $\mathbb{R}$; that is, a non-negative locally integrable function. We say that w satisfies the A_p condition if we have

$$\sup_I \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all bounded intervals $I \subseteq \mathbb{R}$. We denote the previous supremum by $[w]_{A_p}$, and we say that $w \in A_p$.

Remark 2.3. Given a weight w on $\mathbb{R}$ and $E \subseteq \mathbb{R}$, we will denote $w(E) = \int_E w(x) dx$.

From the definition of the A_p class, the following basic properties follow:

Proposition 2.4 ([5, Proposition 7.1.5]). *Let $1 < p < \infty$ and $w \in A_p$. Then:*

- (1) $[E_a w]_{A_p} = [w]_{A_p}$ and $[aw]_{A_p} = [w]_{A_p}$, for any $a > 0$.
- (2) $[T_\lambda w]_{A_p} = [w]_{A_p}$, for any $\lambda \in \mathbb{R}$.
- (3) $w^{-\frac{1}{p-1}} \in A_{p'}$, with constant $[w^{-\frac{1}{p-1}}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}$.
- (4) $[w]_{A_p} \geq 1$, and equality holds if and only if w is constant.
- (5) If $1 < p < q < \infty$, then $[w]_{A_q} \leq [w]_{A_p}$, so the classes A_p are increasing in p .
- (6) The A_p constant coincides with the expression

$$[w]_{A_p} = \sup_I \sup \left\{ \frac{\left(\frac{1}{|I|} \int_I |f(x)| dx \right)^p}{\frac{1}{w(I)} \int_I |f(x)|^p w(x) dx} : 0 < \int_I |f(x)|^p w(x) dx < \infty \right\}.$$

(7) *The measure $w(x)dx$ is doubling: for any $a > 1$ and any interval I , we have that $w(aI) \leq a^p[w]_{A_p} w(I)$.*

The most important property of the A_p condition is given by the fact that it characterizes the boundedness of the Hardy-Littlewood maximal operator.

Definition 2.5. For any function $f \in L^1_{\text{loc}}(\mathbb{R})$, define the Hardy-Littlewood maximal operator as

$$Mf(x) = \sup_{x \in I} \left\{ \frac{1}{|I|} \int_I |f(y)| dy \right\},$$

where the supremum is taken on all intervals $I \subseteq \mathbb{R}$ containing the point x .

Theorem 2.6 ([9, Theorem 9]). *Let w be a weight on $\mathbb{R}$ and $1 < p < \infty$. Then, M is bounded on $L^p(w)$ if and only if $w \in A_p$.*

As we stated in the introduction, the same result holds for the Hilbert transform in this range of exponents:

Theorem 2.7 ([8, Theorem 9]). *Let w be a weight on $\mathbb{R}$ and $1 < p < \infty$. Then, H is bounded on $L^p(w)$ if and only if $w \in A_p$.*

2.2. The discrete Muckenhoupt condition. On discrete weighted spaces, we can define an analogous class of A_p weights with similar properties.

Definition 2.8. Let $w = \{w_k\}_{k \in \mathbb{Z}}$ be a weight on $\mathbb{Z}$. We say that w satisfies the discrete A_p^d condition, $w \in A_p^d$, if

$$(1) \quad [w]_{A_p^d} = \sup_J \left(\frac{1}{\#J} \sum_{k \in J} w_k \right) \left(\frac{1}{\#J} \sum_{k \in J} w_k^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

where the supremum is taken over all finite intervals $J \subseteq \mathbb{Z}$, and $\#J$ denotes the cardinality of the set J . We can also write this condition as

$$[w]_{A_p^d} = \sup_{m \leq l} \left(\frac{1}{l - m + 1} \sum_{k=m}^l w_k \right) \left(\frac{1}{l - m + 1} \sum_{k=m}^l w_k^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

Many properties are shared with the continuous setting, and they can be derived by repeating the proofs or by observing that, for a discrete weight $w = \{w_k\}_k$,

$$[w]_{A_p^d} \approx [\tilde{w}]_{A_p}, \quad \text{where} \quad \tilde{w}(x) = \sum_{k \in \mathbb{Z}} w_k \chi_{[k, k+1)}(x).$$

We also refer back to the general development in [13].

Hence, for the sake of brevity, we will only state the result we will need regarding the discrete Hilbert transform.

Definition 2.9. Let $a = (a_k)_{k \in \mathbb{Z}}$ be a sequence. Its discrete Hilbert transform is the sequence $\mathcal{H}a$ with general term

$$(\mathcal{H}a)_k = \sum_{j \neq k} \frac{a_j}{k - j},$$

if the series converges.

Theorem 2.10 ([8, Theorem 10]). *Let $1 < p < \infty$ and let $w = \{w_k\}_{k \in \mathbb{Z}}$ be a discrete weight. Then, $\mathcal{H}$ is bounded on $\ell^p(w)$ if and only if $w \in A_p^d$.*

2.3. Some useful operators. We also introduce some operators that will be necessary in the proofs of the following results.

Definition 2.11. For $s > 0$, define the moving average operators:

$$(2) \quad \begin{aligned} A_s f(x) &= \frac{1}{s} \int_x^{x+s} f(t) dt, & A_{-s} f(x) &= \frac{1}{s} \int_{x-s}^x f(t) dt, \\ B_s f(x) &= \frac{1}{2s} \int_{x-s}^{x+s} f(t) dt = \frac{A_s f(x) + A_{-s} f(x)}{2}. \end{aligned}$$

Remark 2.12. All three operators in (2) are of convolution type, and their kernels are, respectively:

$$a_s(t) = \frac{1}{s} \chi_{(-s,0)}(t), \quad a_{-s}(t) = \frac{1}{s} \chi_{(0,s)}(t), \quad \text{and} \quad b_s(t) = \frac{1}{2s} \chi_{(-s,s)}(t).$$

We will prove some properties about these operators in general function spaces.

Definition 2.13. Let X be a normed space of functions. We say that its norm is monotone if $|f| \leq |g|$ a.e. implies $\|f\|_X \leq \|g\|_X$.

We say that the norm has the Fatou property if the following is satisfied: for any sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq X$ such that $|f_n| \nearrow |f|$ a.e., we have $\|f_n\|_X \nearrow \|f\|_X$.

Remark 2.14. Given any weight w and $1 < p < \infty$, the norm of $L^p(w)$ is monotone and has the Fatou property.

Proposition 2.15. *Let X be a Banach space of functions with monotone norm. If B_s is bounded on X for every $s > 0$, then for any bounded measurable set $E \subseteq \mathbb{R}$ there exists a constant $C_E > 0$ such that*

$$\|\chi_E * f\|_X \leq C_E \|f\|_X;$$

*that is, the operator given by $f \mapsto \chi_E * f$ is bounded on X .*

Proof. Let $f \in X$ and suppose that $E \subseteq I = (-R, R)$. Taking the absolute value, we find that

$$|\chi_E * f| \leq \chi_E * |f| \leq \chi_I * |f| = 2R B_R |f|.$$

Then, $\|\chi_E * f\|_X \leq 2R \|B_R\|_X \|f\|_X$, where $\|B_R\|_X$ denotes the operator norm of B_R . □

Corollary 2.16. *Let X be a Banach space of functions with monotone norm and suppose that B_s is bounded on X for each $s > 0$. Then, if $f \in L^\infty(\mathbb{R})$ with bounded support, the convolution with f is bounded on X .*

Corollary 2.17. *Let X be a Banach space of functions with monotone norm and let $s > 0$. Then, B_s is bounded on X if and only if both A_s and A_{-s} are bounded on X .*

In the proof of Theorem 4.3, we will frequently use the Poisson operator, whose boundedness can be derived from that of the averaging operators in (2). This is a result that also appears in [1] but, for completeness, we recall it here.

Definition 2.18. We denote by P the convolution operator with respect to the Poisson kernel, $\frac{1}{x^2+1}$:

$$Pf(x) = \int_{\mathbb{R}} \frac{f(y)}{1 + (x - y)^2} dy.$$

Lemma 2.19. *Let X be a Banach space of functions, whose norm $\|\cdot\|_X$ is monotone and has the Fatou property. Suppose that the operators $\{B_s\}_{s \geq 1}$ are uniformly bounded on X , $\|B_s\| \leq C$, with C independent of $s \geq 1$. Then, P is bounded on X .*

Proof. For $j \in \mathbb{N}$, let $b_j = \frac{\chi_{(-j,j)}}{2j}$ the convolution kernel of B_j , and define $g_n = \sum_{j=1}^n b_j/j^2$. Since $\{B_j\}_j$ are uniformly bounded, the operator $G_n f = g_n * f = \sum_{j=1}^n B_j f/j^2$ is bounded on X . Moreover, $\{g_n * f\}_n$ is a Cauchy sequence: if $m > n$, the difference in their corresponding terms is $G_m f - G_n f = \sum_{j=n+1}^m B_j f/j^2$, so

$$\|G_m f - G_n f\|_X \leq \sum_{j=n+1}^m \frac{1}{j^2} \|B_j f\|_X \leq C \|f\|_X \sum_{j=n+1}^m \frac{1}{j^2}.$$

Then, we can define the operator $Gf := \lim_n g_n * f$. Because of the Fatou property, we get that

$$\|Gf\|_X \leq \liminf_n \|g_n * f\|_X \leq C \sum_{j=1}^{\infty} \frac{1}{j^2} \|f\|_X,$$

so G is bounded. Using the monotone convergence theorem, we can see that G is the convolution operator with kernel $g = \sum_{j=1}^{\infty} \chi_{(-j,j)}/(2j^3)$.

Note that for $k \geq 1$ we can compare $\sum_{j=k}^{\infty} \frac{1}{j^3} \approx \frac{1}{k^2+1}$. Then, if $|x| \in (k, k+1)$, we can see that

$$g(x) = \sum_{j=k+1}^{\infty} \frac{1}{2j^3} \approx \frac{1}{k^2+1} \approx \frac{1}{x^2+1}.$$

Because both G and P have positive kernels, we have that

$$|Gf| \leq G(|f|), \quad |Pf| \leq P(|f|),$$

and also, for every non-negative function f , $Pf \approx Gf$. Hence, we obtain that

$$\|Pf\|_X \leq \|P(|f|)\|_X \approx \|G(|f|)\|_X \lesssim \|f\|_X,$$

so P is a bounded operator on X . □

3. THE TRUNCATED MUCKENHOUT CONDITION

During this work we only consider weighted $L^p(w)$ spaces over $\mathbb{R}$. We now introduce the following condition:

Definition 3.1. Let w be a weight on $\mathbb{R}$. We say that w satisfies the *truncated A_p condition* (for some level $a > 0$) if we have

$$\sup_{|I| \geq a} \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all bounded intervals $I \subseteq \mathbb{R}$ with length $|I| \geq a$. We denote the previous supremum by $[w]_{A_{p,a}}$, and we say that $w \in A_{p,a}$.

As the following proposition shows, the threshold a does not have qualitative relevance:

Proposition 3.2. *For every $a > 0$, all $A_{p,a}$ classes coincide: $A_{p,a} = A_{p,1}$. Moreover, for $w \in A_{p,1}$, we have that $[w]_{A_{p,2a}} \leq [w]_{A_{p,a}} \leq 2^p [w]_{A_{p,2a}}$.*

Proof. From the definition, it is clear that $[w]_{A_{p,2a}} \leq [w]_{A_{p,a}}$.

To prove the other inequality, consider a weight $w \in A_{p,2a}$ and note that Definition 3.1 is equivalent to having, for each interval I with $|I| \geq 2a$,

$$\left(\int_I w(x) dx \right) \left(\int_I w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq [w]_{A_{p,2a}} |I|^p.$$

Then, if $J = (c, d)$ is an interval with $a \leq |J| \leq 2a$, we have that

$$\begin{aligned} \left(\int_J w(x) dx \right) \left(\int_J w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} &\leq \left(\int_c^{c+2a} w(x) dx \right) \left(\int_c^{c+2a} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \\ &\leq [w]_{A_{p,2a}} 2^p a^p \leq 2^p [w]_{A_{p,2a}} |J|^p. \end{aligned}$$

Thus, $w \in A_{p,a}$ and $[w]_{A_{p,a}} \leq 2^p [w]_{A_{p,2a}}$. In particular, $A_{p,a} = A_{p,2a}$. Because for $a < b$, $A_{p,a} \subseteq A_{p,b}$, a quick argument shows that $A_{p,a} = A_{p,1}$ for any $a > 0$. $\square$

The truncated $A_{p,1}$ weights satisfy many of the elementary properties of the A_p class.

Proposition 3.3. *Let $1 < p < \infty$, $a > 0$, and $w \in A_{p,1}$. Then:*

- (1) $[E_b w]_{A_{p,a}} = [w]_{A_{p,ba}}$ and $[bw]_{A_{p,a}} = [w]_{A_{p,a}}$, for any $b > 0$.
- (2) $[T_\lambda w]_{A_{p,a}} = [w]_{A_{p,a}}$, for any $\lambda \in \mathbb{R}$.
- (3) $w^{-\frac{1}{p-1}} \in A_{p',1}$, with constant $[w^{-\frac{1}{p-1}}]_{A_{p',a}} = [w]_{A_{p,a}}^{\frac{1}{p-1}}$.
- (4) $[w]_{A_{p,a}} \geq 1$, and equality holds if and only if w is constant.
- (5) If $1 < p < q < \infty$, then $[w]_{A_{q,a}} \leq [w]_{A_{p,a}}$, so the classes $A_{p,1}$ are increasing in p .
- (6) The $A_{p,a}$ constant coincides with the expression

$$[w]_{A_{p,a}} = \sup_{|I| \geq a} \sup \left\{ \frac{\left(\frac{1}{|I|} \int_I |f(x)| dx \right)^p}{\frac{1}{w(I)} \int_I |f(x)|^p w(x) dx} : 0 < \int_I |f(x)|^p w(x) dx < \infty \right\}.$$

- (7) For any $b > 1$ and any interval I , we have that $w(bI) \leq b^p [w]_{A_{p,|I|}} w(I)$.

The proofs of these properties are not significantly different from Proposition 2.4, so we refer to [5, Section 7.1]. The same is true for the following characterization regarding an adequate maximal operator.

Definition 3.4. Let $a > 0$. For $f \in L^1_{\text{loc}}(\mathbb{R})$, define the truncated maximal operator M_a as

$$M_a f(x) = \sup_{x \in I, |I| \geq a} \left\{ \frac{1}{|I|} \int_I |f(y)| dy \right\}.$$

Proposition 3.5. *Let w be a weight on $\mathbb{R}$ and $1 < p < \infty$. Then, M_a is bounded on $L^p(w)$ if and only if $w \in A_{p,1}$.*

More properties can be derived in a similar manner for the truncated case. We will also prove another characterization in relation to the operators B_s introduced in (2).

Proposition 3.6. *Let $1 < p < \infty$, and let w be a weight. Then, $w \in A_{p,1}$ if and only if the family of operators $\{B_s\}_{s \geq 1}$ is uniformly bounded on $L^p(w)$.*

To prove this, we will make use of the following theorem by Heinig and Sinnamon:

Theorem 3.7 ([6, Theorem 2.2]). *Let u and v be weights on $\mathbb{R}$, and $1 < p \leq q < \infty$. Let $a, b : \mathbb{R} \rightarrow \mathbb{R}$ be increasing, differentiable functions such that $a(-\infty) = b(-\infty) = -\infty$, $a(\infty) = b(\infty) = \infty$, and $a(x) < b(x)$ for all $x \in \mathbb{R}$. For $f : \mathbb{R} \rightarrow \mathbb{C}$, define*

$$Tf(x) = \int_{a(x)}^{b(x)} f(y) dy.$$

Then, $T : L^p(u) \rightarrow L^q(v)$ if and only if

$$K = \sup \left\{ \left(\int_{a(x)}^{b(t)} u(y)^{-\frac{1}{p-1}} dy \right)^{\frac{p-1}{p}} \left(\int_t^x v(y) dy \right)^{\frac{1}{q}} : t \leq x, a(x) \leq b(t) \right\} < \infty.$$

Moreover, $K \leq \|T\| \leq 2p^{1/q}(p')^{1/p'} K$.

Remark 3.8. The theorem was originally stated for spaces over $(0, \infty)$. Doing an exponential change of variables, we obtain exactly the same statement for spaces over $\mathbb{R}$.

Proof (Proposition 3.6). We will denote

$$\langle w \rangle_s = \sup_{|I|=s} \left\{ \left(\int_I w(x) dx \right) \left(\int_I w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \right\}.$$

Using Theorem 3.7 for $a(x) = x - s$, $b(x) = x + s$ and $u = v = w$, we obtain that B_s is bounded if and only if

$$K = \sup \left\{ \left(\int_{x-s}^{t+s} w(y)^{-\frac{1}{p-1}} dy \right)^{\frac{p-1}{p}} \left(\int_t^x w(y) dy \right)^{\frac{1}{p}} : t \leq x \leq t + 2s \right\} < \infty.$$

We can bound this expression as

$$\begin{aligned} K &\leq \sup_t \left\{ \left(\int_{t-s}^{t+s} w(y)^{-\frac{1}{p-1}} dy \right)^{\frac{p-1}{p}} \left(\int_t^{t+2s} w(y) dy \right)^{\frac{1}{p}} \right\} \\ &\leq \sup_t \left\{ \left(\int_{t-s}^{t+2s} w(y)^{-\frac{1}{p-1}} dy \right)^{\frac{p-1}{p}} \left(\int_{t-s}^{t+2s} w(y) dy \right)^{\frac{1}{p}} \right\} = \langle w \rangle_{3s}^{\frac{1}{p}}. \end{aligned}$$

To give a lower bound, considering the case $x = t + s$, we have that

$$K \geq \sup_t \left\{ \left(\int_t^{t+s} w(y)^{-\frac{1}{p-1}} dy \right)^{\frac{p-1}{p}} \left(\int_t^{t+s} w(y) dy \right)^{\frac{1}{p}} \right\} = \langle w \rangle_s^{\frac{1}{p}}.$$

Since we know that $\|B_s\| \approx K/2s$, we have found that

$$(2s)^{-p} \langle w \rangle_s \leq \|B_s\|^p \lesssim (6s)^{-p} \langle w \rangle_{3s}.$$

With these inequalities, it is straightforward that

$$[w]_{A_{p,1}} = \sup_{s>1} s^{-p} \langle w \rangle_s \leq (\sup_{s>1} \|B_s\|)^p \lesssim \sup_{s>1} (3s)^{-p} \langle w \rangle_{3s} = [w]_{A_{p,3}} \approx [w]_{A_{p,1}},$$

and the result follows. $\square$

Remark 3.9. Note that we also have

$$[w]_{A_p} = \sup_{s>0} s^{-p} \langle w \rangle_s \leq \left(\sup_{s>0} \|B_s\| \right)^p \lesssim \sup_{s>0} (3s)^{-p} \langle w \rangle_{3s} = [w]_{A_p},$$

so it is true that $w \in A_p$ if and only if the operators $\{B_s\}_{s>0}$ are uniformly bounded.

Remark 3.10. A similar result was proved in [2, Theorem 2.1] and [10, Proposition 3.3]. For a family of pairwise disjoint intervals $\mathbf{I} = \{I_j\}_j$, define the following averaging operator

$$A_{\mathbf{I}}f(x) = \sum_j \left(\frac{1}{|I_j|} \int_{I_j} f(y) dy \right) \chi_{I_j}(x).$$

Then, in the particular case of $L^p(w)$ over $\mathbb{R}$, the theorem states that $w \in A_p$ if and only if the operators $A_{\mathbf{I}}$ are bounded uniformly over $\mathbf{I}$.

A natural question to ask is whether the truncated classes coincide with the classical ones. The following proposition shows that the inclusions are strict.

Proposition 3.11. *For any $1 < p < \infty$, $A_p \subsetneq A_{p,1}$.*

Proof. Let $\delta \in (0, \frac{1}{2})$, and take $M \in \mathbb{N}$ such that

$$\sum_{k=M+1}^{\infty} k2^{-k} < \delta \quad \text{and} \quad \sum_{k=M+1}^{\infty} k^{\frac{1}{p-1}} 2^{-k} < \delta.$$

For $k \geq M+1$, denote the intervals $I_k^0 = (2^{-k-1}, 2^{-k-\frac{1}{2}})$ and $I_k^1 = [2^{-k-\frac{1}{2}}, 2^{-k})$, and their union $I_k = (2^{-k-1}, 2^{-k})$. With these, define the weight w at any point $x \in \mathbb{R}$ as

$$w(x) = \begin{cases} k & x \in I_k^0, \\ \frac{1}{k} & x \in I_k^1, \\ 1 & x \notin \bigcup_{k \geq M+1} I_k. \end{cases}$$

Let us check that this weight satisfies $w \in A_{p,1} \setminus A_p$, for each $1 < p < \infty$.

To see that it is not an A_p weight, we will consider the averages over the intervals I_k : since $|I_k^0| \approx |I_k| \approx |I_k^1|$, a quick computation gives

$$\frac{1}{|I_k|} \int_{I_k} w(x) dx \approx k \quad \text{and} \quad \frac{1}{|I_k|} \int_{I_k} w(x)^{-\frac{1}{p-1}} dx \approx k^{\frac{1}{p-1}}.$$

Hence, we have that

$$\left(\frac{1}{|I_k|} \int_{I_k} w(x) dx \right) \left(\frac{1}{|I_k|} \int_{I_k} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \approx k^2 \xrightarrow[k \rightarrow \infty]{} \infty,$$

so the weight w does not belong to A_p . Note that $|I_k| \rightarrow 0$ when $k \rightarrow \infty$.

Denote $I = \bigcup_{k \geq M+1} I_k$, and observe that $|I| < \delta$. We also have

$$w(I) := \int_I w(x) dx = \sum_{k=M+1}^{\infty} \int_{I_k} w(x) dx \approx \sum_{k=M+1}^{\infty} k2^{-k} < \delta,$$

$$w^{-\frac{1}{p-1}}(I) := \int_I w(x)^{-\frac{1}{p-1}} dx = \sum_{k=M+1}^{\infty} \int_{I_k} w(x)^{-\frac{1}{p-1}} dx \approx \sum_{k=M+1}^{\infty} k^{\frac{1}{p-1}} 2^{-k} < \delta.$$

Take an interval $E \subseteq \mathbb{R}$ with $|E| > 2\delta$, which implies that $|E \setminus I| \approx |E|$. With this, we can bound the averages over E as follows:

$$\frac{1}{|E|} \int_E w(x) dx \leq \frac{1}{|E|} \int_{E \cup I} w(x) dx = \frac{1}{|E|} (w(I) + |E \setminus I|) \leq \frac{1}{|E|} (\delta + |E \setminus I|) \lesssim 1.$$

Similarly, taking the average on $w^{-\frac{1}{p-1}}$, we obtain

$$\frac{1}{|E|} \int_E w(x)^{-\frac{1}{p-1}} dx \leq \frac{1}{|E|} (w^{-\frac{1}{p-1}}(I) + |E \setminus I|) \leq \frac{1}{|E|} (\delta + |E \setminus I|) \lesssim 1.$$

Therefore, we prove that $w \in A_{p,2\delta} = A_{p,1}$, since

$$\left(\frac{1}{|E|} \int_E w(x) dx \right) \left(\frac{1}{|E|} \int_E w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \lesssim 1.$$

□

Remark 3.12. The weight in the proof of Proposition 3.11 exhibits a bad, oscillatory behavior concentrated at the origin, which makes it fail the A_p condition. However, this localized phenomenon does not impact as much the $A_{p,1}$ condition. This fact highlights the strictly large-scale nature of $A_{p,1}$ in contrast to the global one of A_p .

The truncated $A_{p,1}$ condition also has natural relations to discrete A_p^d weights by means of certain averaging operators. Next, we show some of these relations, as they will be necessary in Section 4.

Proposition 3.13. *Let w be a weight on $\mathbb{R}$, $1 < p < \infty$, and suppose that $w \in A_{p,1}$. If we define*

$$w_k = \int_{ak}^{a(k+1)} w(x) dx$$

for some $a > 0$ and for each $k \in \mathbb{Z}$, then the discrete weight $w_d = \{w_k\}_k$ satisfies the A_p^d condition.

Proof. Let $k \in \mathbb{Z}$. Note that the function $t \mapsto t^\theta$ on $\mathbb{R}^+$ is convex for any $\theta < 0$, so we can apply Jensen's inequality with $\theta = -\frac{1}{p-1}$ to get that

$$\left(\frac{1}{a} \int_{ak}^{a(k+1)} w(x) dx \right)^{-\frac{1}{p-1}} \leq \frac{1}{a} \int_{ak}^{a(k+1)} w(x)^{-\frac{1}{p-1}} dx.$$

For any interval $J = [m, l] \subseteq \mathbb{Z}$, define the real interval $\tilde{J} = [ma, (l+1)a] \subseteq \mathbb{R}$. Then, it is clear that we have

$$(3) \quad \frac{1}{\#J} \sum_{k \in J} w_k = \frac{1}{\#J} \sum_{k \in J} \int_{ak}^{a(k+1)} w(x) dx = \frac{a}{|\tilde{J}|} \int_{\tilde{J}} w(x) dx.$$

Looking at the other term in the supremum in (1), we get

$$\begin{aligned}
 \left(\frac{1}{\#J} \sum_{k \in J} w_k^{-\frac{1}{p-1}} \right)^{p-1} &= \left(\frac{1}{\#J} \sum_{k \in J} \left(\int_{ak}^{a(k+1)} w(x) dx \right)^{-\frac{1}{p-1}} \right)^{p-1} \\
 (4) \quad &\leq \left(\frac{a^{-p'}}{\#J} \sum_{k \in J} \int_{ak}^{a(k+1)} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \\
 &= \frac{1}{a} \left(\frac{1}{|\tilde{J}|} \int_{\tilde{J}} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1}.
 \end{aligned}$$

This way, we can bound the product of the left-hand sides of (3) and (4) to get:

$$\left(\frac{1}{\#J} \sum_{k \in J} w_k \right) \left(\frac{1}{\#J} \sum_{k \in J} w_k^{-\frac{1}{p-1}} \right)^{p-1} \leq \left(\frac{1}{|\tilde{J}|} \int_{\tilde{J}} w(x) dx \right) \left(\frac{1}{|\tilde{J}|} \int_{\tilde{J}} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1},$$

so taking the supremum over all intervals $J \subseteq \mathbb{Z}$, we see that w_d has the discrete A_p^d condition with constant at most $[w]_{A_{p,a}}$. $\square$

Proposition 3.14. *Let $1 < p < \infty$ and $w \in A_{p,1}$. Given $a > 0$, let $w_d = \{w_k\}_k$ be the discrete weight where $w_k = \int_{ak}^{a(k+1)} w(x) dx$ for every $k \in \mathbb{Z}$. Define the operator $V = V_a : L^p(w) \rightarrow \ell^p(w_d)$, as*

$$Vf_k = \int_{ak}^{a(k+1)} f(x) dx.$$

Then, V is well-defined, bounded and surjective.

Proof. Using Hölder's inequality, we obtain

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}} |Vf_k|^p w_k &= \sum_{k \in \mathbb{Z}} \left| \int_{ak}^{a(k+1)} f(x) w_k^{\frac{1}{p}} dx \right|^p = \sum_{k \in \mathbb{Z}} \left| \int_{ak}^{a(k+1)} f(x) w(x)^{\frac{1}{p}} \left(\frac{w_k}{w(x)} \right)^{\frac{1}{p}} dx \right|^p \\
 &\leq \sum_{k \in \mathbb{Z}} \left(\int_{ak}^{a(k+1)} |f(x)|^p w(x) dx \right) \left(\int_{ak}^{a(k+1)} \left(\frac{w_k}{w(x)} \right)^{\frac{p'}{p}} dx \right)^{\frac{p}{p'}} \\
 &= \sum_{k \in \mathbb{Z}} \left(\int_{ak}^{a(k+1)} |f(x)|^p w(x) dx \right) \left(\int_{ak}^{a(k+1)} w(x) dx \right) \left(\int_{ak}^{a(k+1)} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \\
 &\leq a^p [w]_{A_{p,a}} \sum_{k \in \mathbb{Z}} \left(\int_{ak}^{a(k+1)} |f(x)|^p w(x) dx \right) = a^p [w]_{A_{p,a}} \int_{\mathbb{R}} |f(x)|^p w(x) dx.
 \end{aligned}$$

Thus, we see that $Vf \in \ell^p(w_d)$ and that $\|V\| \leq a[w]_{A_{p,a}}^{\frac{1}{p}}$. The surjectivity of V is immediate. $\square$

It is natural to ask whether the reverse of Proposition 3.13 is true. Namely, if having the A_p^d condition on w_d implies that $w \in A_{p,1}$. The following example disproves this:

Example 3.15. Let $1 < p < \infty$, and let

$$w(x) = \begin{cases} x^{p-1}, & 0 < x < 1, \\ 1, & x \notin (0, 1). \end{cases}$$

Note that, for $x \in (0, 1)$, $w(x)^{-\frac{1}{p-1}} = \frac{1}{x}$, so $w^{-\frac{1}{p-1}} \notin L^1_{\text{loc}}$. Therefore, $w \notin A_{p,1}$.

On the other hand, a direct computation gives $w_k \approx 1$, so w_d is an A_p^d weight.

4. THE SEGMENT MULTIPLIER ON WEIGHTED LEBESGUE SPACES

Our main goal is to characterize the boundedness of the segment multiplier. For this purpose, we recall its definition as well as the truncated Hilbert transform's:

Definition 4.1. Given $\varepsilon > 0$, define the following kernel functions:

$$\text{sinc}(t) = \frac{\sin t}{t}, \quad h_\varepsilon(t) = \frac{1}{t} \chi_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)}.$$

Their convolution operators are, respectively, the segment multiplier and the truncated Hilbert transform:

$$Sf = \text{sinc} * f, \quad H_\varepsilon f = h_\varepsilon * f.$$

Remark 4.2. Let w be a weight and $1 < p < \infty$. Suppose that B_s is bounded for any $s > 0$. Then, given $\varepsilon, \delta > 0$, Corollary 2.16 implies that $H_\varepsilon - H_\delta$ is bounded on $L^p(w)$. Thus, the boundedness of H_ε and H_δ are equivalent.

We are now able to prove our main result:

Theorem 4.3. Let $1 < p < \infty$ and let w be a weight. Then, $S : L^p(w) \rightarrow L^p(w)$ is bounded if and only if $w \in A_{p,1}$.

Proof. Assume that S is a bounded operator on $L^p(w)$. For each $k \in \mathbb{Z}$ and $j \in \{1, 2\}$, denote the intervals

$$I_k = \left[\frac{\pi}{4}k, \frac{\pi}{4}(k+1) \right] \quad \text{and} \quad I_k^j = \left[\frac{\pi}{8}(2k+j-1), \frac{\pi}{8}(2k+j) \right].$$

Note that I_k^j is the j -th half of the interval I_k . Denote also $I_{k,l} = \bigcup_{m=k}^{k+l-1} I_m = \frac{\pi}{4} \cdot [k, k+l]$, for $l \geq 1$, and the intersection $I_{k,l}^j = I_{k,l} \cap \left(\bigcup_{m=0}^{\infty} I_{k+8m}^j \right)$.

Let us prove, for each $k \in \mathbb{Z}$ and $l \in \mathbb{N} \setminus 4\mathbb{N}$, that

$$(5) \quad \int_{I_{k+l,l}} w(x) dx \approx \int_{I_{k,l}} w(x) dx.$$

For $l = 1$ we have $I_{k,1} = I_k$ and $I_{k,1}^j = I_k^j$. Let $x \in I_{k+1}^j$, and note that for each $y \in I_k^j$ we have $x - y \in [\frac{\pi}{8}, \frac{3\pi}{8}]$. With this, we can obtain the following estimate

$$|S\chi_{I_k^j}(x)| = \left| \int_{I_k^j} \frac{\sin(x-y)}{x-y} dy \right| \geq \frac{\sin\left(\frac{\pi}{8}\right)}{\frac{3\pi}{8}} |I_k^j| = \frac{\sin\left(\frac{\pi}{8}\right)}{3},$$

and, hence,

$$\int_{I_{k+1}^j} w(y) dy \lesssim \int_{\mathbb{R}} |S\chi_{I_k^j}(y)|^p w(y) dy \lesssim \int_{\mathbb{R}} \chi_{I_k^j}(y)^p w(y) dy = \int_{I_k^j} w(y) dy.$$

Since $\frac{\sin x}{x}$ is an even function, we can interchange the roles of I_k^j and I_{k+1}^j to obtain the opposite inequality, so we have proved

$$\int_{I_{k+1}^j} w(x) dx \approx \int_{I_k^j} w(x) dx.$$

Adding up the equivalences for $j = 1, 2$, we obtain (5) for $l = 1$.

Let us prove (5) for $l > 1$ with $l \notin 4\mathbb{N}$. First, because it is true for $l = 1$, applying (5) sequentially for the indices $\{k, k+1, k+2, \dots, k+6\} \subseteq \mathbb{Z}$ provides

$$\int_{I_k} w(x) dx \approx \int_{I_{k+1}} w(x) dx \approx \int_{I_{k+2}} w(x) dx \approx \dots \approx \int_{I_{k+7}} w(x) dx.$$

Therefore, for any $k \in \mathbb{Z}$ and $l \in \{1, 2, \dots, 8\}$, by adding the integrals over I_m from $m = k$ to $m = k + l - 1$ we have

$$\int_{I_{k,l}} w(x) dx \approx \int_{I_k} w(x) dx,$$

so, by the definition of the sets $I_{k,l}^j$, we conclude that for any $k \in \mathbb{Z}$ and $l \geq 1$

$$\int_{I_{k,l}} w(x) dx \approx \int_{I_{k,l}^1 \cup I_{k,l}^2} w(x) dx = \int_{I_{k,l}^1} w(x) dx + \int_{I_{k,l}^2} w(x) dx.$$

Let $x \in I_{k+l,l}^j$. If $y \in I_{k,l}^j$, since $I_{n,l}^j \subseteq I_{n,l}$, we have that $x - y \leq \frac{2l\pi}{4} = l\frac{\pi}{2}$. Besides, there exist two numbers $m_x, m_y \geq 0$ such that

$$\begin{aligned} x &\in \left[\frac{\pi}{8}(2k + 2l + 16m_x + j - 1), \frac{\pi}{8}(2k + 2l + 16m_x + j) \right], \\ y &\in \left[\frac{\pi}{8}(2k + 16m_y + j - 1), \frac{\pi}{8}(2k + 16m_y + j) \right], \end{aligned}$$

so the difference satisfies $x - y \in \frac{l}{4}\pi + 2(m_x - m_y)\pi + [-\frac{\pi}{8}, \frac{\pi}{8}]$.

This way, whenever l is not a multiple of 4, it follows that $|\sin(x - y)| \geq \sin(\frac{\pi}{8})$. Then,

$$|S\chi_{I_{k,l}^j}(x)| = \left| \int_{I_{k,l}^j} \frac{\sin(x - y)}{x - y} dy \right| \geq \frac{\sin(\frac{\pi}{8})}{l\frac{\pi}{2}} |I_{k,l}^j| = \frac{\sin(\frac{\pi}{8})}{l\frac{\pi}{2}} \left(\left\lfloor \frac{l}{8} \right\rfloor + 1 \right) \frac{\pi}{8} \geq \frac{\sin(\frac{\pi}{8})}{32}.$$

With this estimate, we get that

$$\int_{I_{k+l,l}^j} w(y) dy \lesssim \int_{\mathbb{R}} |S\chi_{I_{k,l}^j}(y)|^p w(y) dy \lesssim \int_{I_{k,l}^j} w(y) dy.$$

Arguing analogously with the roles of $I_{k,l}^j$ and $I_{k+l,l}^j$ interchanged, we prove the opposite inequality. Hence, we obtain the following:

$$\int_{I_{k+l,l}} w(x) dx \approx \int_{I_{k+l,l}^1} w(x) dx + \int_{I_{k+l,l}^2} w(x) dx \approx \int_{I_{k,l}^1} w(x) dx + \int_{I_{k,l}^2} w(x) dx \approx \int_{I_{k,l}} w(x) dx.$$

Observe that the argument for the equivalence in (5) depends on the size of the sets involved, I_k , I_k^j , $I_{k,l}$ and $I_{k,l}^j$, and on their position relative to each other, but it does not depend on their absolute position. This means that we can repeat the proof with any translate of the previous sets to give, for every $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and $l \in \mathbb{N} \setminus 4\mathbb{N}$,

$$\int_{x+I_{k+l,l}} w(y) dy \approx \int_{x+I_{k,l}} w(y) dy.$$

Recall that S is a self-adjoint operator, and hence it is also bounded on $L^{p'}(w^{-\frac{1}{p-1}})$, the dual space of $L^p(w)$. Thus, we can make the same arguments and obtain the equivalence (5) for $w^{-\frac{1}{p-1}}$ in place of w .

Let us now prove the following estimate for each $k \in \mathbb{Z}$ and $l \in 2\mathbb{N} \setminus 4\mathbb{N}$:

$$\left(\int_{I_{k+l,l}} w(x) dx \right) \left(\int_{I_{k,l}} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \lesssim |I_{k,l}|^p.$$

Since we can compare each integral over $I_{n,l}$ to that over $I_{n,l}^1 \cup I_{n,l}^2$, for each $n \in \mathbb{Z}$, we need only show that

$$\left(\int_{(J_{k,l}) + \frac{\pi}{4}l} w(x) dx \right) \left(\int_{J_{k,l}} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \lesssim |I_{k,l}|^p,$$

where we denote $J_{k,l} = I_{k,l}^1 \cup I_{k,l}^2$.

Let $f = w^{-\frac{1}{p-1}} \chi_{J_{k,l}}$, and let $x \in J_{k,l} + \frac{\pi}{4}l$ and $y \in J_{k,l}$. Then, we can argue as before: first, the difference must satisfy $|x-y| \leq \frac{l\pi}{2}$. Besides this, there exist $j_x, j_y \in \{1, 2\}$ and $m_x, m_y \in \mathbb{Z}$ such that

$$\begin{aligned} x &\in \left[\frac{\pi}{8}(2k + 2l + 16m_x + j_x - 1), \frac{\pi}{8}(2k + 2l + 16m_x + j_x) \right], \\ y &\in \left[\frac{\pi}{8}(2k + 16m_y + j_y - 1), \frac{\pi}{8}(2k + 16m_y + j_y) \right], \end{aligned}$$

so the difference satisfies $x - y \in \frac{l}{4}\pi + 2(m_x - m_y)\pi + [-\frac{\pi}{4}, \frac{\pi}{4}]$.

Thus, whenever $l \in 2\mathbb{N} \setminus 4\mathbb{N}$, this interval is centered at some $n\pi + \frac{\pi}{2}$, so $|\sin(x-y)| \geq \sin(\frac{\pi}{4})$. In this case, we get

$$|Sf(x)| = \int_{J_{k,l}} \left| \frac{\sin(x-y)}{x-y} \right| w(y)^{-\frac{1}{p-1}} dy \gtrsim \frac{1}{|I_{k,l}|} \int_{J_{k,l}} w(y)^{-\frac{1}{p-1}} dy,$$

so we can finally see that

$$\begin{aligned} \left(\frac{1}{|I_{k,l}|} \int_{J_{k,l}} w(y)^{-\frac{1}{p-1}} dy \right)^p \left(\int_{J_{k+l,l}} w(y) dy \right) &\lesssim \int_{\mathbb{R}} |Sf(y)|^p \chi_{J_{k+l,l}} w(y) dy \\ &\lesssim \int_{J_{k,l}} w(y)^{-\frac{p}{p-1}} w(y) dy = \int_{J_{k,l}} w(y)^{-\frac{1}{p-1}} dy. \end{aligned}$$

Again, note that this reasoning does not depend on the absolute position of the intervals, so we can argue the same for any translates of the intervals involved.

Now, let I be an arbitrary interval with $|I| > \pi$. Let $y \in \mathbb{R}$, $k \in \mathbb{Z}$ and $l \geq 4$ such that $y + I_{k,l} \subseteq I \subseteq y + I_{k,l+1}$. Note that in the set $\{l-3, l-2, l-1, l\}$ there is one number that

also lies in $2\mathbb{N} \setminus 4\mathbb{N}$. We can see that $I_{k,l+1} \subseteq 5I_{k,l-3}$ for any $l \geq 4$. Thus, for any I of length greater than π , there exists an $l \in 2\mathbb{N} \setminus 4\mathbb{N}$ such that $y + I_{k,l} \subseteq I \subseteq 5(y + I_{k,l})$. Therefore, since $5(y + I_{k,l}) = y + I_{k-2l,5l}$ and $5l \in 2\mathbb{N} \setminus 4\mathbb{N}$, we obtain that

$$\begin{aligned} \left(\int_I w(x) dx \right) \left(\int_I w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} &\leq \left(\int_{5(y+I_{k,l})} w(x) dx \right) \left(\int_{5(y+I_{k,l})} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \\ &\lesssim |5(y + I_{k,l})|^p \lesssim |y + I_{k,l}|^p \leq |I|^p, \end{aligned}$$

so we have $w \in A_{p,1}$.

Let us prove the sufficiency of $w \in A_{p,1}$. Recall that we have the following equality

$$\frac{\sin(x-y)}{x-y} f(y) = \sin x \frac{\cos y f(y)}{x-y} - \cos x \frac{\sin y f(y)}{x-y} = \sin x \frac{f_c(y)}{x-y} - \cos x \frac{f_s(y)}{x-y},$$

with $f_c = \cos \cdot f$ and $f_s = \sin \cdot f$. With this, we can express the segment multiplier as

$$\begin{aligned} Sf(x) &= \int_{\mathbb{R}} \frac{\sin(x-y)}{x-y} f(y) dy = \int_{|x-y|>2\pi} \frac{\sin(x-y)}{x-y} f(y) dy + \int_{|x-y|\leq 2\pi} \frac{\sin(x-y)}{x-y} f(y) dy \\ &= \sin x H_{2\pi} f_c(x) - \cos x H_{2\pi} f_s(x) + \int_{|x-y|\leq 2\pi} \frac{\sin(x-y)}{x-y} f(y) dy, \end{aligned}$$

so, taking into account that $B_{2\pi}$ is a bounded operator on $L^p(w)$ (see Proposition 3.6) and that

$$\left| \int_{|x-y|\leq 2\pi} \frac{\sin(x-y)}{x-y} f(y) dy \right| \leq \int_{|x-y|\leq 2\pi} |f(y)| dy = 4\pi B_{2\pi}(|f|)(x),$$

we can bound the multiplier as

$$|Sf| \lesssim |H_{2\pi} f_c| + |H_{2\pi} f_s| + B_{2\pi}(|f|).$$

Because of Proposition 3.6, it is sufficient to check that the truncated Hilbert transform is bounded on $L^p(w)$.

Let us prove that $H_{2\pi}$ is bounded. Let $g \in L^p(w)$ and $x \in \mathbb{R}$. Put $k = \lfloor \frac{x}{\pi} \rfloor$, so that $x \in [k\pi, (k+1)\pi)$. Then, we have

$$\begin{aligned} H_{2\pi} g(x) &= \sum_{|k-j|\geq 3} \int_{j\pi}^{(j+1)\pi} \frac{g(y)}{x-y} dy + \int_{(k-2)\pi}^{x-2\pi} \frac{g(y)}{x-y} dy + \int_{x+2\pi}^{(k+3)\pi} \frac{g(y)}{x-y} dy \\ &= \frac{1}{\pi} \sum_{|k-j|\geq 3} \frac{\int_{j\pi}^{(j+1)\pi} g(y) dy}{k-j} + \sum_{|k-j|\geq 3} \int_{j\pi}^{(j+1)\pi} \left(\frac{1}{x-y} - \frac{1}{k\pi-j\pi} \right) g(y) dy \\ &\quad + \int_{(k-2)\pi}^{x-2\pi} \frac{g(y)}{x-y} dy + \int_{x+2\pi}^{(k+3)\pi} \frac{g(y)}{x-y} dy. \end{aligned}$$

The last two integrals in the right-hand side can be bounded as

$$\left| \int_{(k-2)\pi}^{x-2\pi} \frac{g(y)}{x-y} dy + \int_{x+2\pi}^{(k+3)\pi} \frac{g(y)}{x-y} dy \right| \leq \int_{x-3\pi}^{x+3\pi} \frac{|g(y)|}{2\pi} dy = 3B_{3\pi}(|g|)(x).$$

For the second term, note that if $x \in [k\pi, (k+1)\pi)$ and $y \in [j\pi, (j+1)\pi)$, then the difference satisfies $x - y \in [(k-j-1)\pi, (k-j+1)\pi]$, or, put in another way, we have $(k-j)\pi \in [x-y-\pi, x-y+\pi]$, so it follows that

$$\frac{1}{(k-j)\pi} \in \left[\frac{1}{x-y+\pi}, \frac{1}{x-y-\pi} \right].$$

With this, and with the condition $|k-j| \geq 3$, we have the following inequality:

$$\left| \frac{1}{x-y} - \frac{1}{k\pi - j\pi} \right| \lesssim \frac{1}{1 + (x-y)^2}.$$

Thus, the second term can be bounded as

$$\begin{aligned} \left| \sum_{|k-j| \geq 3} \int_{j\pi}^{(j+1)\pi} \left(\frac{1}{x-y} - \frac{1}{k\pi - j\pi} \right) g(y) dy \right| &\lesssim \sum_{|k-j| \geq 3} \int_{j\pi}^{(j+1)\pi} \frac{|g(y)|}{1 + (x-y)^2} dy \\ &\leq \sum_{j \in \mathbb{Z}} \int_{j\pi}^{(j+1)\pi} \frac{|g(y)|}{1 + (x-y)^2} dy \\ &= \int_{\mathbb{R}} \frac{|g(y)|}{1 + (x-y)^2} dy = P(|g|)(x), \end{aligned}$$

where P is the operator introduced in Definition 2.18. Considering both Proposition 3.6 and Lemma 2.19, we know that P is bounded in $L^p(w)$.

Hence, to prove the boundedness of $H_{2\pi}$, we only need to show that

$$\tilde{K}(g)(x) = \sum_{|\lfloor \frac{x}{\pi} \rfloor - j| \geq 3} \frac{\int_{j\pi}^{(j+1)\pi} g(y) dy}{\lfloor \frac{x}{\pi} \rfloor - j} = \sum_{|k-j| \geq 3} \frac{(Vg)_j}{k-j} =: (\mathcal{H}_3 Vg)_k, \quad x \in [k\pi, (k+1)\pi),$$

is a bounded operator on $L^p(w)$, where $\mathcal{H}_3$ denotes the corresponding truncated discrete Hilbert transform. It is clear that $\|\tilde{K}g\|_{L^p(w)} = \|\mathcal{H}_3 Vg\|_{\ell^p(w_d)}$, it can be immediately checked that $\mathcal{H} - \mathcal{H}_3$ is bounded on $\ell^p(w_d)$ and, because of Proposition 3.13, we know that $w_d \in A_p^d$ and $\mathcal{H}$ is bounded as well. Then,

$$\|\tilde{K}g\|_{L^p(w)} \leq \|(\mathcal{H}_3 - \mathcal{H})Vg\|_{\ell^p(w_d)} + \|\mathcal{H}Vg\|_{\ell^p(w_d)} \leq (\|\mathcal{H} - \mathcal{H}_3\| + \|\mathcal{H}\|) \|V\| \|g\|_{L^p(w)}.$$

Therefore, the boundedness for $\tilde{K}$ follows and, consequently, for $H_{2\pi}$ and S , finishing the proof. $\square$

Remark 4.4. As shown in Proposition 3.11, there are weights that satisfy the $A_{p,1}$ condition while failing to be in A_p . Because of Theorem 4.3, this means that there are weighted spaces $L^p(w)$ where the segment multiplier is bounded but the Hilbert transform is not.

Remark 4.5. A class similar in definition to $A_{p,1}$ is that of local Muckenhoupt weights; that is, weights w such that

$$[w]_{A_p^{\text{loc}}} = \sup_{|I| \leq 1} \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty.$$

Note that now the supremum covers all small intervals, $|I| \leq 1$, while it misses the large ones. This condition was first introduced by Rychkov in [12], and it satisfies analogous properties to

the ones we have proved for $A_{p,1}$. Both classes are distinct, and it is clear from their definitions that $A_{p,1} \cap A_p^{\text{loc}} = A_p$.

REFERENCES

- [1] M. F. Barea-Fernández, R. Kerman, J. Lang, and J. Soria, *Characterization of the boundedness of the segment multiplier on rearrangement-invariant spaces*, arXiv:2607.22277 (preprint).
- [2] E. I. Bereznoi, *Two-weighted estimations for the Hardy-Littlewood maximal function in ideal Banach spaces*, Proc. Amer. Math. Soc. 127 (1999), 79–87.
- [3] L. De Carli and E. Laeng, *Sharp L^p estimates for the segment multiplier*, Collect. Math. 51 (2000), 309–326.
- [4] C. Fefferman, *The multiplier problem for the ball*, Ann. of Math. (2nd Ser.) 94 (1971), 330–336.
- [5] L. Grafakos, *Classical Fourier Analysis*, Grad. Texts in Math. 249, Springer-Verlag, 3rd edition, 2008.
- [6] H. P. Heinig and G. Sinnamon, *Mapping properties of integral averaging operators*, Stud. Math. 129, No. 2 (1998), 157–177.
- [7] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–140.
- [8] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and the Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251.
- [9] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.
- [10] Z. Nieraeth, *The Muckenhoupt condition*, J. Funct. Anal. 290, no. 1 (2026).
- [11] M. Riesz, *Sur les fonctions conjuguées*, Math. Z. 27 (1927), 218–244.
- [12] V. S. Rychkov, *Littlewood–Paley theory and function spaces with A_p^{loc} weights*, Math. Nachr. 224 (2001), 145–180.
- [13] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer, Berlin, 1989.

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