OPTIMAL CONSTANTS FOR CERTAIN HARDY-TYPE INTEGRAL OPERATORS

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Abstract. In recent years, the problem of obtaining optimal L^p-norm estimates for Hardy-type operators has been extensively studied. In this work, we continue this line of research by investigating the norms of polynomials in H and H^* acting on different cones of $L^p(\mathbb{R}^+)$, with $1 < p < \infty$, where H and H^* denote the classical Hardy averaging operator and its adjoint, respectively. We also determine the norms of fractional powers of H and H^*, and establish several approximation of the identity results associated with these operators. Finally, as an application of the main theorems of this study, we derive a combinatorial identity involving conjugate exponents and compute the norms of e^H and e^{H^*}.

Mathematics Subject Classification. Primary 26D15, 47B37, 46E30.
Keywords. Hardy-type inequalities, Hardy operator, adjoint of Hardy operator, sharp constants.

1. Introduction

Let $\mathcal{M}^+(\mathbb{R}^+)$ be the class of all nonnegative measurable functions on $\mathbb{R}^+ = (0, \infty)$. Let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Set

$$Hf(x) = \frac{1}{x} \int_0^x f(t)dt$$

and

$$H^* f(x) = \int_x^\infty \frac{f(t)}{t} dt,$$

provided that the integrals make sense for a function f on $\mathbb{R}^+$. These equalities define the classical Hardy operator H and its adjoint H^*.

The mappings introduced above play a central role in Analysis. For instance, H has a deep connection with one of the most important objects in Harmonic Analysis, namely the classical *Hardy–Littlewood maximal operator*, defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x, and $|Q|$ denotes the Lebesgue measure of Q. This connection arises from the fact that

$$(Mf)^* \simeq Hf^*, \tag{1}$$

where f^* denotes the decreasing rearrangement with respect to the Lebesgue measure. The relation in (1) implies that the problem of determining the boundedness of the maximal operator M on rearrangement-invariant spaces is equivalent to studying the boundedness of the much simpler operator H on the cone of nonnegative, nonincreasing functions in some Lebesgue space (see [5] for standard notations and results).

The boundedness of H on $L^p(\mathbb{R}^+)$, for $1 < p \leq \infty$, follows from what is commonly referred to as Hardy's inequality (cf. [9]):

$$\|Hf\|_{L^p(\mathbb{R}^+)} \leq p' \, \|f\|_{L^p(\mathbb{R}^+)}, \tag{2}$$

for all $f \in L^p(\mathbb{R}^+)$, where $p' = \frac{p}{p-1}$ if $1 < p < \infty$, and $p' = 1$ if $p = \infty$. Moreover, p' is the best possible constant in (2).

Regarding the boundedness of the adjoint of Hardy's operator, the authors proved in [10, p. 244] that

$$\|H^* f\|_{L^p(\mathbb{R}^+)} \leq p \, \|f\|_{L^p(\mathbb{R}^+)}, \tag{3}$$

for all $f \in L^p(\mathbb{R}^+)$, where $1 \leq p < \infty$. Furthermore, the constant p is optimal in (3).

The operator $S = H + H^*$ is also of considerable importance in Harmonic Analysis. For instance, according to Theorem 4.8 in [5], the boundedness of S on $L^p(\mathbb{R}^+)$ implies the continuity of the classical Hilbert transform $\mathcal{H}$ on $L^p(\mathbb{R})$, where

$$\mathcal{H}f(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} \, dt.$$

To see that S defines a continuous operator on $L^p(\mathbb{R}^+)$, we make use of (2) together with (3). Indeed, we have

$$\|Sf\|_p \leq (p + p') \|f\|_p, \tag{4}$$

for all $f \in L^p(\mathbb{R}^+)$ with $1 < p < \infty$. Again, the constant $p + p'$ is the best possibe in (4) (see [10]). The operator S is commonly referred to as the *Calderón operator*.

One of the main goals of this work is to extend the results of G. H. Hardy, J. Littlewood, and G. Pólya in (2), (3), and (4) by determining the optimal constant $C(p) > 0$ such that

$$\|P(H, H^*) f\|_p \leq C(p) \, \|f\|_p, \tag{5}$$

for functions $f \in L^p(\mathbb{R}^+)$ belonging to one of the following classes: general functions, nonnegative functions, or nonnegative, nonincreasing functions, where $1 < p < \infty$ and $P \in \mathbb{R}[X, Y]$ is a real polynomial with nonnegative coefficients. The sharp constant $C(p)$ in (5) is established in Theorem 3.2, which constitutes the central result of this study.

This paper is organized as follows. Section 2 introduces the notations used throughout this work. Section 3 presents the main result, namely Theorem 3.2, together with Corollary 3.4, from which we derive a combinatorial identity involving conjugate exponents in (15). In Section 4, and in particular in Theorem 4.3, we determine the norms of H^a and H^{*a} for $a > 0$. We also establish several approximation of the identity results for these operators, namely Theorem 4.5 and Theorem 4.7. Finally, in Section 5 we compute in Theorem 5.1 the norms of e^H and e^{H^*}.

2. Notations

The analysis in this work is carried out in the Lebesgue spaces $L^p(\mathbb{R}^+)$, $1 < p < \infty$, which serve as the main domain of the operators under consideration. Otherwise, we simply write $\mathcal{D}(T)$ for the domain of an operator T.

We are also interested in the following cones.

$L^p_+(\mathbb{R}^+) = \{f \in L^p(\mathbb{R}^+) : f \geq 0\}, \qquad L^p_{\text{dec}}(\mathbb{R}^+) = \{f \in L^p_+(\mathbb{R}^+) : f \text{ is nonincreasing}\}.$

For a bounded operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, we set

$$\|T\|_{L^p(\mathbb{R}^+)} := \sup_{\|f\|_p = 1} \|Tf\|_p,$$
$$\|T\|_{L^p_+(\mathbb{R}^+)} := \sup_{\substack{f \in L^p_+(\mathbb{R}^+) \\ \|f\|_p = 1}} \|Tf\|_p,$$
$$\|T\|_{L^p_{\text{dec}}(\mathbb{R}^+)} := \sup_{\substack{f \in L^p_{\text{dec}}(\mathbb{R}^+) \\ \|f\|_p = 1}} \|Tf\|_p.$$

These satisfy

$$\|T\|_{L^p_{\text{dec}}(\mathbb{R}^+)} \leq \|T\|_{L^p_+(\mathbb{R}^+)} \leq \|T\|_{L^p(\mathbb{R}^+)}.$$

Let $1 < p < \infty$ and let $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ be a bounded linear operator. For

$$Q(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} x^i y^j \in \mathbb{R}[X, Y],$$

where $m, n \in \mathbb{N}$, we define

$$Q(T, T^*) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij}\, T^i \circ T^{*j},$$

where $T^0 = T^{*0} = I$.

Note that if T is normal, i.e., $T \circ T^* = T^* \circ T$, then a straightforward calculation shows that

$$Q(T, T^*)^* = P(T, T^*),$$

where $P(x, y) = Q(y, x)$.

Throughout this work, we use the notation $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ for the Euler Gamma function, defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

We also write $p' = \frac{p}{p-1}$, for $1 < p < \infty$, to denote the conjugate exponent of p.

3. The norm of a polynomial of H and H^*

Let $n \geq 1$ be a natural number. D. W. Boyd proved (see [6, Lemma 2]) the identity

$$H^n f(x) = (H \circ H \circ \cdots \circ H) f(x) = \frac{1}{\Gamma(n)} \frac{1}{x} \int_0^x f(y)\, \ln^{n-1}\left(\frac{x}{y}\right) dy, \tag{6}$$

for all $f \in \mathcal{D}(H^n)$.

Using a similar argument, we obtain

$$H^{*n} f(x) = (H^* \circ H^* \circ \cdots \circ H^*) f(x) = \frac{1}{\Gamma(n)} \int_x^\infty \frac{f(y)}{y}\, \ln^{n-1}\left(\frac{y}{x}\right) dy, \tag{7}$$

for all $f \in \mathcal{D}(H^{*n})$.

The following result provides a formula for the composition $H^n \circ H^{*m}$ expressed as a linear combination of powers of H and H^*, where m and n are natural numbers.

Proposition 3.1. *Let $m, n \geq 1$ be natural numbers and $f \in \mathcal{M}^+(\mathbb{R}^+) \cap \mathcal{D}(H^{*m} \circ H^n)$. Then*

$$\big(H^{*m} \circ H^n\big) f(x) = \sum_{k=0}^{m-1} \binom{n+k-1}{k} H^{*m-k} f(x) + \sum_{k=0}^{n-1} \binom{m+k-1}{k} H^{n-k} f(x), \quad (8)$$

for all $x > 0$.

Proof. Let $x > 0$. Then,

$$\begin{aligned}
\big(H^{*m} \circ H^n\big) f(x) &= \frac{1}{\Gamma(m)} \int_x^\infty \frac{H^n f(y)}{y} \ln^{m-1}\left(\frac{y}{x}\right) dy \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{1}{y^2} \int_0^y f(t) \ln^{n-1}\left(\frac{y}{t}\right) dt \ln^{m-1}\left(\frac{y}{x}\right) dy \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \int_0^y \frac{f(t)}{y^2} \ln^{n-1}\left(\frac{y}{t}\right) \ln^{m-1}\left(\frac{y}{x}\right) dt dy \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_0^x \int_x^\infty \frac{f(t)}{y^2} \ln^{n-1}\left(\frac{y}{t}\right) \ln^{m-1}\left(\frac{y}{x}\right) dy dt \\
&+ \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \int_t^\infty \frac{f(t)}{y^2} \ln^{n-1}\left(\frac{y}{t}\right) \ln^{m-1}\left(\frac{y}{x}\right) dy dt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{1}{y^2} \ln^{n-1}\left(\frac{x}{t} y\right) \ln^{m-1}(y) dy dt \\
&+ \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{1}{y^2} \ln^{n-1}(y) \ln^{m-1}\left(\frac{t}{x} y\right) dy dt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \ln^{n-1}\left(\frac{x}{t} y\right) dy dt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \ln^{m-1}\left(\frac{t}{x} y\right) dy dt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \left(\ln\left(\frac{x}{t}\right) + \ln y\right)^{n-1} dy dt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \left(\ln\left(\frac{t}{x}\right) + \ln y\right)^{m-1} dy dt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \sum_{k=0}^{n-1} \binom{n-1}{k} \ln^{n-k-1}\left(\frac{x}{t}\right) \ln^k(y) dy dt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \sum_{k=0}^{m-1} \binom{m-1}{k} \ln^{m-k-1}\left(\frac{t}{x}\right) \ln^k(y) dy dt \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \ln^{n-k-1}\left(\frac{x}{t}\right) \int_1^\infty \frac{\ln^{m+k-1}(y)}{y^2} dy dt \\
&\quad + \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \ln^{m-k-1}\left(\frac{t}{x}\right) \int_1^\infty \frac{\ln^{n+k-1}(y)}{y^2} dy dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \frac{\Gamma(n)\Gamma(m+k)}{\Gamma(k+1)\Gamma(n-k)\Gamma(m)\Gamma(n)} \frac{1}{x}\int_0^x f(t)\ln^{n-k-1}\left(\frac{x}{t}\right)dt \\
&\quad + \sum_{k=0}^{m-1} \frac{\Gamma(m)\Gamma(n+k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t}\ln^{m-k-1}\left(\frac{t}{x}\right)dt \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(m+k)}{\Gamma(k+1)\Gamma(m)} H^{n-k}f(x) + \sum_{k=0}^{m-1} \frac{\Gamma(n+k)}{\Gamma(k+1)\Gamma(n)} H^{*m-k}f(x) \\
&= \sum_{k=0}^{n-1} \binom{m+k-1}{k} H^{n-k}f(x) + \sum_{k=0}^{m-1} \binom{n+k-1}{k} H^{*m-k}f(x),
\end{aligned}
$$

where we used Tonelli's Theorem in the fourth equality. We also applied (6) and (7) in the second to last equality. This concludes the proof. ☐

Note that (2), (3), and (4) says, in other words, that

$$\|H\|_{L^p(\mathbb{R}^+)} = p', \qquad \text{for } 1 < p \le \infty, \tag{9}$$

$$\|H^*\|_{L^p(\mathbb{R}^+)} = p, \qquad \text{for } 1 \le p < \infty, \tag{10}$$

$$\|H + H^*\|_{L^p(\mathbb{R}^+)} = p + p', \qquad \text{for } 1 < p < \infty. \tag{11}$$

By closely examining identities (9), (10), and (11), we observe that

$$
\begin{aligned}
\|H\|_{L^p(\mathbb{R}^+)} &= \|P(H,H^*)\|_{L^p(\mathbb{R}^+)} = P(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p', \\
\|H^*\|_{L^p(\mathbb{R}^+)} &= \|Q(H,H^*)\|_{L^p(\mathbb{R}^+)} = Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p, \\
\|H+H^*\|_{L^p(\mathbb{R}^+)} &= \|R(H,H^*)\|_{L^p(\mathbb{R}^+)} = R(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p + p',
\end{aligned}
$$

where $P(x, y) = x$, $Q(x, y) = y$, and $R(x, y) = x + y$.

It is important to note that, when working with H and H^*, two different polynomials can sometimes define the same associated operator. For instance, in [12] the following identity related to the Calderón operator is proved

$$H \circ H^* = H^* \circ H = H + H^* = S.$$

Therefore,

$$R(H, H^*) = V(H, H^*),$$

where R is as above and $V(x, y) = xy$. It is worth noting that,

$$
\begin{aligned}
p' + p &= R(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = \|R(H,H^*)\|_{L^p(\mathbb{R}^+)} = \|V(H,H^*)\|_{L^p(\mathbb{R}^+)} \\
&= V(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p'p,
\end{aligned}
$$

since p and p' are conjugate exponents.

The following theorem extends (9), (10), and (11), and constitutes one of the main results of this work.

Theorem 3.2. *Let $1 < p < \infty$ and let $Q \in \mathbb{R}[X, Y]$ be a polynomial with nonnegative coefficients. Then*

$$
\begin{aligned}
\|Q(H,H^*)\|_{L^p_{\text{dec}}(\mathbb{R}^+)} &= \|Q(H,H^*)\|_{L^p_+(\mathbb{R}^+)} = \|Q(H,H^*)\|_{L^p(\mathbb{R}^+)} \\
&= Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = Q(p', p).
\end{aligned}
$$

Proof. Let $f \in L^p(\mathbb{R}^+)$ and let $Q \in \mathbb{R}[X,Y]$ be the polynomial given by

$$Q(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}x^i y^j \in \mathbb{R}[X,Y],$$

where $c_{ij} \geq 0$ and $m, n \geq 1$ are natural numbers. By the triangle inequality and the submultiplicativity of the L^p-norm for operators, we get that

$$\begin{aligned}
\|Q(H,H^*)f\|_p \leq \|Q(H,H^*)|f|\|_p &= \left\| \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}(H^i \circ H^{*j})|f| \right\|_p \\
&\leq \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\|(H^i \circ H^{*j})|f|\|_p \\
&\leq \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\|H\|_p^i \|H^*\|_p^j \|f\|_p = Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)})\|f\|_p.
\end{aligned}$$

Thus

$$\|Q(H,H^*)\|_p \leq Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}).$$

By using (9) with (10) we obtain

$$\|Q(H,H^*)\|_p \leq Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = Q(p', p).$$

We now show that the above inequality is in fact an equality. Consider the functions $f_\epsilon, g_\epsilon : (0,\infty) \to \mathbb{R}$ given by $f_\epsilon(x) = x^{\epsilon/p - 1/p}\chi_{(0,1)}(x)$ and $g_\epsilon(x) = x^{\epsilon/p' - 1/p'}\chi_{(0,1)}(x)$, where $0 < \epsilon < 1$. Then

$$\|f_\epsilon\|_p^p = \|g_\epsilon\|_{p'}^{p'} = \frac{1}{\epsilon}.$$

On the other hand, if $0 < x < 1$ and $n \in \mathbb{N}$, an easy computation shows that

$$H^n f(x) = x^{\epsilon/p - 1/p}\frac{1}{(\epsilon/p + 1/p')^n}. \tag{12}$$

Now, as H and H^* are positive operators and $c_{ij} \geq 0$, it follows that $Q(H,H^*)$ is positive as well. So

$$\int_0^\infty |Q(H,H^*)f_\epsilon(x) \cdot g_\epsilon(x)|dx = \int_0^\infty Q(H,H^*)f_\epsilon(x) \cdot g_\epsilon(x)dx.$$

Since g_ϵ has support equal to $[0,1]$, we have that

$$\begin{aligned}
\int_0^\infty Q(H,H^*)f_\epsilon(x) \cdot g_\epsilon(x)dx &= \int_0^1 Q(H,H^*)f_\epsilon(x) \cdot g_\epsilon(x)dx \\
&= \int_0^1 \left(\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}H^i \circ H^{*j}\right) f_\epsilon(x) \cdot g_\epsilon(x)dx \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\int_0^1 \left(H^i \circ H^{*j}\right) f_\epsilon(x) \cdot g_\epsilon(x)dx.
\end{aligned}$$

Given that H and H^* commute, we obtain that

$$\int_0^1 \left(H^i \circ H^{*j}\right) f_\epsilon(x) \cdot g_\epsilon(x)dx = \int_0^1 \left(H^{*j} \circ H^i\right) f_\epsilon(x) \cdot g_\epsilon(x)dx,$$

for every $1 \leq i \leq m$ and $1 \leq j \leq n$. Observe that, as H^j is the adjoint of H^{*j}, we get that

$$\int_0^1 \left(H^{*j} \circ H^i\right) f_\epsilon(x) \cdot g_\epsilon(x) dx = \int_0^1 H^i f_\epsilon(x) \cdot H^j g_\epsilon(x) dx,$$

for every $1 \leq i \leq m$ and $1 \leq j \leq n$. Now, using (12), we have that

$$\begin{aligned}\int_0^1 H^i f_\epsilon(x) \cdot H^j g_\epsilon(x) dx &= \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \int_0^1 x^{\epsilon/p - 1/p} x^{\epsilon/p' - 1/p'} dx \\ &= \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \frac{1}{\epsilon}.\end{aligned}$$

Hence,

$$\int_0^\infty |Q(H, H^*) f_\epsilon(x) \cdot g_\epsilon(x)| dx = \frac{1}{\epsilon} \sum_{i=0}^m \sum_{j=0}^n c_{ij} \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j}.$$

Now, by using Hölder's inequality, we obtain

$$\begin{aligned}\|Q(H, H^*)\|_p &\geq \frac{\|Q(H, H^*) f_\epsilon\|_p}{\|f_\epsilon\|_p} \geq \frac{\int_0^\infty |Q(H, H^*) f_\epsilon(x) \cdot g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} \\ &= \sum_{i=0}^m \sum_{j=0}^n c_{ij} \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \to \sum_{i=0}^m \sum_{j=0}^n c_{ij} (p')^i p^j = Q(p', p),\end{aligned}$$

as $\epsilon \to 0^+$. Since the functions f_ϵ and g_ϵ are nonnegative and nonincreasing, the proof is concluded. $\square$

Remark 3.3. The condition that the coefficients of the polynomial Q be nonnegative in Theorem 3.2 is necessary, since without this hypothesis the conclusion of the theorem may fail. To illustrate this, consider the polynomial

$$Q(x, y) = y - 1,$$

for which the associated operator is $Q(H, H^*) = H^* - I$. The authors in [3, Theorem 4.2] proved that

$$\|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} = M_p^{1/p},$$

for $1 < p < 2$, where $M_p = \max_{t \in [0,1/2]} f_p(t)$ and $f_p : [0, 1/2] \to \mathbb{R}$ is defined by

$$f_p(t) = pt^{p-1} + (1 - t)^p - t^p.$$

In particular,

$$\|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} = M_p^{1/p} \geq (f_p(0))^{1/p} = 1 > p - 1 = Q(p', p),$$

for all $1 < p < 2$.

The nonnegativity of the coefficients of Q is also necessary to ensure that

$$\|Q(H, H^*)\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|Q(H, H^*)\|_{L^p_+(\mathbb{R}^+)} = \|Q(H, H^*)\|_{L^p(\mathbb{R}^+)},$$

since this equality may fail otherwise. For instance, consider again the operator $P(H, H^*)$ defined above. It was proved in [4, Theorem 1.2] that

$$\|Q(H, H^*)\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|H^* - I\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \left(\int_0^\infty |\ln(x) + 1|^p \, dx\right)^{1/p} \stackrel{\text{def}}{=} C_p^{1/p},$$

for $1 < p \le 2$. Moreover, it is shown in [4] that the function $p \mapsto C_p^{1/p}$ is strictly increasing on $\mathbb{R}^+$. Hence,

$$\begin{aligned}
\|P(H,H^*)\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)} = C_p^{1/p} < C_2^{1/2} &= \left(\int_0^\infty |\ln(x)+1|^2\,dx\right)^{1/2}\\
&= 1 = (f_p(0))^{1/p} < M_p^{1/p} = \|P(H,H^*)\|_{L^p_+(\mathbb{R}^+)}\\
&= \|P(H,H^*)\|_{L^p(\mathbb{R}^+)},
\end{aligned}$$

for all $1 < p < 2$, where in the last equality we have taken into account [3, Theorem 4.2].

Before stating our next result, we recall that by convention, $\binom{n}{0} = 1$ for all integers n; in particular, $\binom{-1}{0} = 1$. We also set the sum equal to zero whenever the upper index is less than the one at the bottom.

Corollary 3.4. *Let $m,n \in \mathbb{N}\cup\{0\}$ and $1 < p < \infty$. Then*

$$\begin{aligned}
\|H^{*m}\circ H^n\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)} &= \|H^{*m}\circ H^n\|_{L^p_+(\mathbb{R}^+)} = \|H^{*m}\circ H^n\|_{L^p(\mathbb{R}^+)}\\
&= \sum_{k=0}^{m-1}\binom{n+k-1}{k}p^{m-k} + \sum_{k=0}^{n-1}\binom{m+k-1}{k}(p')^{n-k} \qquad (13)\\
&= p^m(p')^n.
\end{aligned}$$

Proof. Let $P, Q \in \mathbb{R}[X,Y]$ be the polynomials defined by

$$P(x,y) = \sum_{k=0}^{n-1}\binom{m+k-1}{k}x^{n-k} + \sum_{k=0}^{m-1}\binom{n+k-1}{k}y^{m-k},$$

and

$$Q(x,y) = x^n y^m,$$

where $m,n \in \mathbb{N}\cup\{0\}$. Note that in the case $m=0$, we have

$$P(x,y) = x^n = Q(x,y),$$

for any $n \in \mathbb{N}\cup\{0\}$. Similarly, in the case $n=0$, we have

$$P(x,y) = y^m = Q(x,y),$$

for any $m \in \mathbb{N}\cup\{0\}$. This implies that

$$P(H,H^*) = Q(H,H^*) \quad \text{if } n=0 \text{ or } m=0.$$

In the case $m,n \ge 1$, we have seen in (8) that

$$P(H,H^*)f = Q(H,H^*)f,$$

for all $f \in \mathcal{M}^+(\mathbb{R}^+)$. Hence,

$$P(H,H^*)f = Q(H,H^*)f, \qquad (14)$$

for all $m,n \in \mathbb{N}\cup\{0\}$, and any $f \in \mathcal{M}^+(\mathbb{R}^+)$. Therefore, taking into account Theorem 3.2 and (14), we obtain (13). □

Remark 3.5. Let
$$A = \{H^{*m} \circ H^n : m, n \in \mathbb{N} \cup \{0\}\}.$$
Then $(A, \circ)$, where $\circ$ denotes composition, is an abelian semigroup with identity. In particular, Corollary 3.4 implies that the mapping
$$\begin{aligned} \varphi : A &\longrightarrow [1, \infty), \\ T &\longmapsto \|T\|_{L^p(\mathbb{R}^+)} \end{aligned}$$
is a homomorphism.

Remark 3.6. Let $1 < p < \infty$ and let $n, m \geq 1$ be natural numbers. As shown in (13), the following combinatorial identity involving p and p' holds:
$$p^m (p')^n = \sum_{k=0}^{m-1} \binom{n+k-1}{k} p^{m-k} + \sum_{k=0}^{n-1} \binom{m+k-1}{k} (p')^{n-k}. \tag{15}$$
In particular, when $m = n$, this yields
$$p^n (p')^n = \sum_{k=0}^{n-1} \binom{n+k-1}{k} \big(p^{n-k} + (p')^{n-k}\big).$$
On the other hand, since p and p' are conjugate exponents, we have
$$p^n (p')^n = (pp')^n = (p + p')^n.$$
Hence, we obtain an alternative expansion of $(p+p')^n$, which differs from the classical binomial expansion:
$$(p + p')^n = \sum_{k=0}^{n-1} \binom{n+k-1}{k} \big(p^{n-k} + (p')^{n-k}\big).$$
The coefficients $\binom{n+k-1}{k}$, for $k = 1, \cdots, n$, are known as *generalized triangular numbers* (see [2]). For a fixed n, their arrangement within Pascal's Triangle is illustrated below:

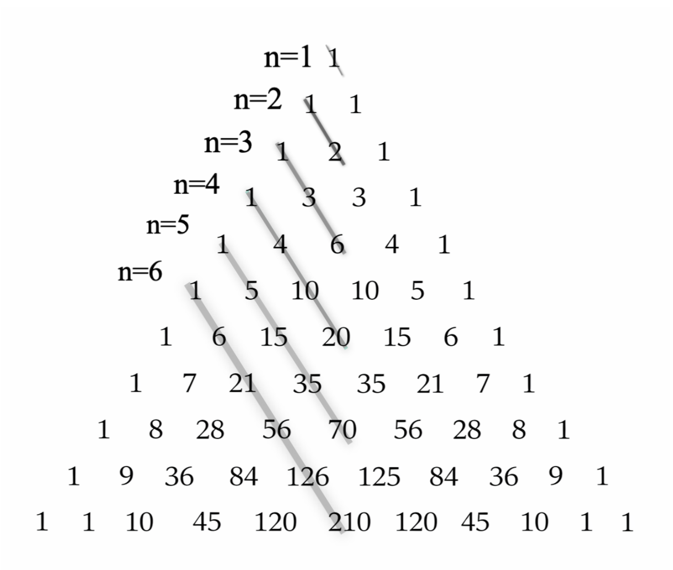

FIGURE 1. The corresponding coefficients $\binom{n+k-1}{k}$ for $n = 1, \ldots, 6$ and $k = 1, \cdots, n$.

For instance, when $n = 4$, we have

$$(p+p')^4 = \mathbf{1}\,(p^4 + (p')^4) + \mathbf{4}\,(p^3 + (p')^3) + \mathbf{10}\,(p^2 + (p')^2) + \mathbf{20}\,(p + p').$$

4. Powers of H and H^* for Real Exponents

Let $a > 0$ be a real number. For a function f on $\mathbb{R}^+$, the operators H^a and H^{*a} are given by

$$H^a f(x) = \frac{1}{\Gamma(a)}\,\frac{1}{x}\int_0^x f(t)\,\ln^{a-1}\!\left(\frac{x}{t}\right)\,dt$$

and

$$H^{*a} f(x) = \frac{1}{\Gamma(a)}\int_x^\infty \frac{f(t)}{t}\,\ln^{a-1}\!\left(\frac{t}{x}\right)\,dt,$$

whenever these integrals are well defined. Note that H^{*a} is the adjoint operator of H^a.

Observe that for $a = 1$, the operators H^1 and H^{*1} reduce to the classical Hardy operator and its adjoint, respectively. Furthermore, if a is a natural number, H^a coincides with the a-fold composition of H, as shown in (6). A similar statement holds for H^{*a}.

Our next goal is to determine the norm of the operator H^a and that of its adjoint. Before doing so, we introduce some preliminary results.

Let $1 \le p \le \infty$. We define the operator

$$T_p : L^p(\mathbb{R}^+, dt) \to L^p(\mathbb{R}, ds)$$

by

$$T_p f(s) = f(e^s)e^{s/p}. \tag{16}$$

We first note that for $1 \le p < \infty$,

$$\|T_p f\|^p_{L^p(\mathbb{R},ds)} = \int_{-\infty}^{\infty} |T_p f(s)|^p\,ds = \int_{-\infty}^{\infty} |f(e^s)|^p e^s\,ds = \int_0^\infty |f(t)|^p\,dt = \|f\|^p_{L^p(\mathbb{R}^+,dt)}.$$

In the case $p = \infty$,

$$\|Tf\|_{L^\infty(\mathbb{R},ds)} = \|f(e^s)\|_{L^\infty(\mathbb{R},ds)} = \operatorname*{ess\,sup}_{s\in\mathbb{R}} |f(e^s)| = \operatorname*{ess\,sup}_{t>0} |f(t)| = \|f\|_{L^\infty(\mathbb{R}^+,dt)}.$$

Consequently,

$$\|T_p f\|_{L^p(\mathbb{R},ds)} = \|f\|_{L^p(\mathbb{R}^+,dt)}. \tag{17}$$

Thus, T_p defines a linear isometric isomorphism between $L^p(\mathbb{R}^+, dt)$ and $L^p(\mathbb{R}, ds)$.

Let $a > 0$. Let $k_{p,a}, k^*_{p,a} : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$k_{p,a}(x) = \frac{1}{\Gamma(a)} x^{a-1} e^{-(1-1/p)x}\chi_{(0,\infty)}(x), \tag{18}$$

and

$$k^*_{p,a}(x) = k_{p,a}(-x). \tag{19}$$

The following lemma establishes several properties of these kernels.

Lemma 4.1. *Let $1 < p < \infty$ and $a > 0$. Let $k_{p,a}$ be defined as in (18). Then the following properties hold:*

(i) $\|k_{p,a}\|_{L^1(\mathbb{R},ds)} = \int_{\mathbb{R}} k_{p,a}(x)\,dx = (p')^a.$

(ii) For any $\delta > 0$,
$$\int_{|x|\geq\delta} |k_{p,a}(x)|\,dx \to 0 \quad \text{as } a \to 0^+.$$

(iii) For any $\delta > 0$,
$$\sup_{x\geq\delta} k_{p,a}(x) \to 0 \quad \text{as } a \to 0^+.$$

In particular, for all $x > 0$
$$\lim_{a\to 0^+} k_{p,a}(x) = 0.$$

(iv) If $0 < a < 1$, *then*
$$\int_0^\infty |tk'_{p,a}(t)|\,dt = (p')^a.$$

Proof. The proof of (i) is immediate, since
$$\|k_{p,a}\|_{L^1(\mathbb{R},ds)} = \int_{\mathbb{R}} k_{p,a}(x)\,dx = \frac{1}{\Gamma(a)}\int_0^\infty x^{a-1}e^{-(1-1/p)x}\,dx = \left(\frac{p}{p-1}\right)^a = (p')^a.$$

We now turn to (ii). Let $\delta > 0$. Then
$$\begin{aligned} 0 \leq \int_{|x|\geq\delta} |k_{p,a}(s)|\,ds &= \frac{1}{\Gamma(a)}\int_\delta^\infty x^{a-1}e^{-(1-\frac{1}{p})x}\,dx \\ &\leq \frac{1}{\Gamma(a)}\int_\delta^\infty \max\{1, x^{-1}\}e^{-(1-\frac{1}{p})x}\,dx \\ &= \frac{C_\delta}{\Gamma(a)}, \end{aligned}$$
for all $0 < a \leq 1$. Since $\Gamma(a) \to \infty$ as $a \to 0^+$, it follows that
$$\frac{C_\delta}{\Gamma(a)} \to 0 \quad \text{as} \quad a \to 0^+.$$
Therefore, by the squeeze theorem,
$$\int_{|x|\geq\delta} |k_{p,a}(s)|\,ds \to 0 \quad \text{as} \quad a \to 0^+.$$

We proceed to prove (iii). Without loss of generality, suppose that $a \in (0,1)$.
$$\begin{aligned} (p')^a\Gamma(a)\frac{d}{dv}k_{p,a}(v) &= (a-1)v^{a-2}e^{-(1-1/p)v} - \left(1-\frac{1}{p}\right)v^{a-1}e^{-(1-1/p)v} \\ &= v^{a-2}e^{-(1-1/p)v}\left(a-1-\left(1-\frac{1}{p}\right)v\right) < 0, \end{aligned} \tag{20}$$
for all $v > 0$. In particular, we have that $k_{p,a}$ is nonnegative, and nonincreasing. Therefore
$$\sup_{v\geq\delta} k_{p,a}(v) = k_{p,a}(\delta) = \frac{1}{(p')^a}\frac{1}{\Gamma(a)}\delta^{a-1}e^{-(1-1/p)\delta} \to 0,$$
as $a \to 0^+$, since $\Gamma(a) \to \infty$ as $a \to 0^+$.

Finally, to prove (iv), observe that when $0 < a < 1$, and using (20), it follows that

$$\begin{aligned}\int_0^\infty |tk'_{p,a}(t)|dt &= \frac{1}{\Gamma(a)}\int_0^\infty \left|t^{a-1}e^{-(1-1/p)t}\left((a-1)-t\left(1-\frac{1}{p}\right)\right|dt\\ &= \frac{(1-a)}{\Gamma(a)}\int_0^\infty t^{a-1}e^{-(1-1/p)t}dt + \left(1-\frac{1}{p}\right)\frac{1}{\Gamma(a)}\int_0^\infty t^a e^{-(1-1/p)t}dt\\ &= (1-a)(p')^a + \left(1-\frac{1}{p}\right)a(p')^{a+1} = (p')^a.\end{aligned}$$

This completes the proof. □

The following result shows that both H^a and H^{*a}, up to a change of variables, are integral operators of convolution type.

Lemma 4.2. *Let $1 < p < \infty$ and let $a > 0$. Let f be a measurable function. Then*

$$(T_p \circ H^a)f(s) = T_p f * k_{p,a}(s), \tag{21}$$

and

$$(T_p \circ H^{*a})f(s) = T_p f * k^*_{p',a}(s),$$

for all $s > 0$, where T_p is defined in (16), $k_{p,a}$ in (18), and $k^*_{p,a}$ in (19).

Proof. Let $s > 0$. On the one hand, we have

$$\begin{aligned}(T_p\circ H^a)f(s) = T_p(H^a f(s)) = e^{s/p}H^a f(e^s) &= \frac{1}{\Gamma(a)}\int_0^{e^s} e^{s/p}e^{-s}f(t)\ln^{a-1}\left(\frac{e^s}{t}\right)dt\\ &= \frac{1}{\Gamma(a)}\int_{-\infty}^{s} e^{s/p}f(e^u)(s-u)^{a-1}e^{u-s}du\\ &= \frac{1}{\Gamma(a)}\int_{-\infty}^{\infty} e^{s/p}f(e^u)(s-u)^{a-1}e^{u-s}\chi_{(0,\infty)}(s-u)du\\ &= \frac{1}{\Gamma(a)}\int_{-\infty}^{\infty} e^{u/p}f(e^u)(s-u)^{a-1}e^{-(1-1/p)(s-u)}\chi_{(0,\infty)}(s-u)du\\ &= \int_{-\infty}^{\infty} T_p f(u)k_{p,a}(s-u)du.\end{aligned}$$

Therefore

$$(T_p \circ H^a)f(s) = T_p f * k_{p,a}(s),$$

is a standard convolution on $\mathbb{R}$.

On the other hand, we have that

$$\begin{aligned}(T_p\circ H^{*a})f(s) = T_p(H^{*a} f(s)) = e^{s/p}H^{*a} f(e^s) &= \frac{1}{\Gamma(a)}\int_{e^s}^{\infty} e^{s/p}f(t)\ln^{a-1}\left(\frac{t}{e^s}\right)\frac{dt}{t}\\ &= \frac{1}{\Gamma(a)}\int_{s}^{\infty} e^{u/p}f(e^u)(u-s)^{a-1}e^{-\frac{1}{p}(u-s)}du\\ &= \frac{1}{\Gamma(a)}\int_{-\infty}^{\infty} e^{u/p}f(e^u)(u-s)^{a-1}e^{-\frac{1}{p}(u-s)}\chi_{(0,\infty)}(u-s)du\\ &= \int_{-\infty}^{\infty} T_p f(u)k_{p',a}(u-s)du\\ &= \int_{-\infty}^{\infty} T_p f(u)k^*_{p',a}(s-u)du.\end{aligned}$$

Therefore

$$(T_p \circ H^{*a})f(s) = T_p f * k^*_{p',a}(s).$$

This completes the proof. □

The following result determines the norm of H^a and its adjoint, and in particular generalizes the results in (9) and (10).

Theorem 4.3. *Let $1 \le p \le \infty$. Then*

$$\|H^a\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|H^a\|_{L^p_+(\mathbb{R}^+)} = \|H^a\|_{L^p(\mathbb{R}^+)} = (p')^a, \quad 1 < p \le \infty, \tag{22}$$

and

$$\|H^{*a}\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|H^{*a}\|_{L^p_+(\mathbb{R}^+)} = \|H^{*a}\|_{L^p(\mathbb{R}^+)} = p^a, \quad 1 \le p < \infty. \tag{23}$$

Proof. Let $a > 0$ and $f \in L^p(\mathbb{R}^+)$ with $1 < p \le \infty$. We start with the case $p = \infty$. We have trivially that

$$\begin{aligned}|H^a f(x)| &= \left|\frac{1}{\Gamma(a)}\,\frac{1}{x}\int_0^x f(t)\,\ln^{a-1}\!\left(\frac{x}{t}\right)\,dt\right| \le \frac{1}{\Gamma(a)}\,\frac{1}{x}\int_0^x |f(t)|\,\ln^{a-1}\!\left(\frac{x}{t}\right)\,dt \\ &\le \frac{1}{\Gamma(a)}\,\frac{1}{x}\int_0^x \|f\|_{L^\infty(\mathbb{R}^+,dt)}\,\ln^{a-1}\!\left(\frac{x}{t}\right)\,dt = \|f\|_{L^\infty(\mathbb{R}^+,dt)},\end{aligned}$$

for all $x > 0$. Then

$$\|H^a f\|_{L^\infty(\mathbb{R}^+,dt)} \le \|f\|_{L^\infty(\mathbb{R}^+,dt)}.$$

We now turn to the case $1 < p < \infty$. By applying (17), (21), and Young's convolution inequality (see [7, Theorem 1.2.12]), we have

$$\begin{aligned}\|H^a f\|_{L^p(\mathbb{R}^+,dt)} &= \|T_p \circ H^a f\|_{L^p(\mathbb{R},ds)} = \|T_p f * k_{p,a}\|_{L^p(\mathbb{R},ds)} \\ &\le \|T_p f\|_{L^p(\mathbb{R},ds)}\|k_{p,a}\|_{L^1(\mathbb{R},ds)} \\ &= \|f\|_{L^p(\mathbb{R}^+,dt)}(p')^a,\end{aligned}$$

where the final equality follows from (17) together with Lemma 4.1 (i). Thus,

$$\|H^a f\|_p \le (p')^a\|f\|_p,$$

for all $1 < p \le \infty$. We now show that the constant $(p')^a$ is optimal in the above inequality. To this end, consider the functions

$$f_\epsilon(x) = x^{\frac{\epsilon}{p}-\frac{1}{p}}\,\chi_{(0,1)}(x) \quad \text{and} \quad g_\epsilon(x) = x^{\frac{\epsilon}{p'}-\frac{1}{p'}}\,\chi_{(0,1)}(x),$$

where $1 \le p < \infty$ and $0 < \epsilon < 1$. If $0 < x < 1$, then

$$H^a f_\epsilon(x) = x^{\frac{\epsilon}{p}-\frac{1}{p}}\,\frac{1}{\Gamma(a)}\int_0^\infty e^{-u(\frac{\epsilon}{p}+\frac{1}{p'})}u^{a-1}du = x^{\frac{\epsilon}{p}-\frac{1}{p}}\,\frac{1}{(\epsilon/p+1/p')^a},$$

for all $1 < p < \infty$. Now, by using Hölder's inequality, we obtain

$$\|H^a\|_p \ge \frac{\|H^a f_\epsilon\|_p}{\|f_\epsilon\|_p} \ge \frac{\int_0^\infty |H^a f_\epsilon(x)\cdot g_\epsilon(x)|dx}{\|g_\epsilon\|_{p'}\|f_\epsilon\|_p} = \frac{1}{(\epsilon/p+1/p')^a} \to (p')^a,$$

as $\epsilon \to 0^+$, for all $1 < p < \infty$. In the case when $p = \infty$, we consider the function $f = \chi_{(0,1)}$. A straightforward computation shows that

$$\|H^a f_\epsilon\|_\infty = \|f_\epsilon\|_\infty = 1.$$

This ends the proof of (22).

Now we turn to (23). By a duality argument, we obtain

$$\|H^{*a}\|_{L^p(\mathbb{R}^+)} = \|H^a\|_{L^{p'}(\mathbb{R}^+)} = p^a,$$

for all $1 \leq p < \infty$. Now, consider the same functions f_ϵ and g_ϵ defined above. Then

$$\begin{aligned}
\|H^{*a}\|_p &\geq \frac{\|H^{*a}f_\epsilon\|_p}{\|f_\epsilon\|_p} \geq \frac{\int_0^\infty |H^{*a}f_\epsilon(x)\cdot g_\epsilon(x)|dx}{\|g_\epsilon\|_{p'}\|f_\epsilon\|_p} \\
&= \frac{\int_0^\infty |f_\epsilon(x)\cdot H^a g_\epsilon(x)|dx}{\|g_\epsilon\|_{p'}\|f_\epsilon\|_p} = \frac{1}{(\epsilon/p' + 1/p)^a} \to p^a,
\end{aligned}$$

as $\epsilon \to 0^+$, for all $1 < p < \infty$. In the case $p = 1$, we consider the function $f = \chi_{(0,1)}$ again. By Tonelli's theorem, we get

$$\begin{aligned}
\|H^{*a}f\|_1 &= \frac{1}{\Gamma(a)}\int_0^1\int_x^1 \frac{1}{t}\ln^{a-1}\left(\frac{t}{x}\right)dtdx = \frac{1}{\Gamma(a)}\int_0^1 \frac{1}{t}\int_0^t \ln^{a-1}\left(\frac{t}{x}\right)dxdt \\
&= \frac{1}{\Gamma(a)}\int_0^1\int_0^1 \ln^{a-1}\left(\frac{1}{x}\right)dxdt = \frac{1}{\Gamma(a)}\Gamma(a) = 1 = \|f\|_1.
\end{aligned}$$

This proves (23) and the proof is completed. $\square$

Remark 4.4. The theorem above implies in particular that

$$\|H^a\|_{L^2(\mathbb{R}^+,dt)} = \|k_{2,a}\|_{L^1(\mathbb{R},ds)} = 2^a.$$

Now, using (21) and applying the Fourier transform, we obtain

$$\widehat{(T_2\circ H^a)f}(x) = \widehat{T_2 f}(x)\cdot\widehat{k_{2,a}}(x). \tag{24}$$

Therefore, by applying Plancherel's theorem together with (24), we conclude that

$$\begin{aligned}
\|H^a f\|_{L^2(\mathbb{R}^+,dt)} &= \|(T_2\circ H^a)f\|_{L^2(\mathbb{R},ds)} = \|\widehat{(T_2\circ H^a)f}\|_{L^2(\mathbb{R},ds)} \\
&\leq \|\widehat{T_2 f}\|_{L^2(\mathbb{R},ds)}\|\widehat{k_{2,a}}\|_{L^\infty(\mathbb{R},ds)} = \|T_2 f\|_{L^2(\mathbb{R},ds)}\|\widehat{k_{2,a}}\|_{L^\infty(\mathbb{R},ds)} \\
&= \|f\|_{L^2(\mathbb{R}^+,dt)}\|\widehat{k_{2,a}}\|_{L^\infty(\mathbb{R},ds)}.
\end{aligned}$$

This means that

$$\|H^a\|_{L^2(\mathbb{R}^+,dt)} \leq \|\widehat{k_{2,a}}\|_{L^\infty(\mathbb{R},ds)}. \tag{25}$$

Now, for $x \in \mathbb{R}$, we have that

$$\widehat{k_{2,a}}(\xi) = \int_{\mathbb{R}} k_{2,a}(x)e^{-2\pi i x\xi}dx = \frac{1}{\Gamma(a)}\int_0^\infty x^{a-1}e^{-x\left(\frac{1}{2}-2\pi i\xi\right)}dx = \frac{1}{\left(\frac{1}{2}-2\pi i\xi\right)^a},$$

where, in the last equality, we used formula 6.1.1 in [1]. Therefore

$$\begin{aligned}
\|\widehat{k_{2,a}}\|_{L^\infty(\mathbb{R},ds)} &= \sup_{\xi\in\mathbb{R}}|\widehat{k_{2,a}}(\xi)| = \sup_{\xi\in\mathbb{R}}\left|\frac{1}{2}-2\pi i\xi\right|^{-a} \\
&= \sup_{\xi\in\mathbb{R}}\left(\frac{1}{4}+4\pi^2\xi^2\right)^{-a/2} = |\widehat{k_{2,a}}(0)| = 2^a.
\end{aligned}$$

Hence

$$\|H^a\|_{L^2(\mathbb{R}^+,dt)} = \|k_{2,a}\|_{L^1(\mathbb{R},ds)} = \|\widehat{k_{2,a}}\|_{L^\infty(\mathbb{R},ds)} = 2^a.$$

The following result shows that the operators H^a and its adjoint form an approximation of the identity in $L^p(\mathbb{R}^+)$; that is, $H^a f \to f$ and $H^{*a} f \to f$ in the $L^p(\mathbb{R}^+)$-norm as $a \to 0^+$.

Theorem 4.5. *Let $1 < p < \infty$ and let $f \in L^p(\mathbb{R}^+)$. Then*

$$\lim_{a\to 0^+} \|H^a f - f\|_p = 0 \tag{26}$$

and

$$\lim_{a\to 0^+} \|H^{*a} f - f\|_p = 0. \tag{27}$$

Proof. Let $1 < p < \infty$. We begin with the limit in (26). Let $a > 0$, and let $\tilde{k}_{p,a} : \mathbb{R} \to \mathbb{R}$ be the function given by

$$\tilde{k}_{p,a} := (p')^{-a} k_{p,a},$$

where $k_{p,a}$ is defined as in (18). Then $\tilde{k}_{p,a}$ meets the conditions for an approximation of the identity (see, for instance, [7, Definition 1.2.15]). Therefore, an application of [7, Theorem 1.2.19] yields

$$\lim_{a\to 0^+} \|g - g * \tilde{k}_{p,a}\|_{L^p(\mathbb{R},ds)} = 0, \tag{28}$$

for all $g \in L^p(\mathbb{R}, ds)$.

Now, by taking into account (17), Lemma 4.2, and Young's convolution inequality, we get

$$\begin{aligned}
0 \le \|(H^a - I)f\|_{L^p(\mathbb{R}^+,dt)} &= \|T_p \circ (H^a - I)f\|_{L^p(\mathbb{R},ds)} \\
&= \|T_p \circ H^a f - T_p f\|_{L^p(\mathbb{R},ds)} \\
&= \|T_p f * k_{p,a} - T_p f\|_{L^p(\mathbb{R},ds)} \\
&= \|\big((p')^a - 1\big)\, T_p f * \tilde{k}_{p,a} + (T_p f * \tilde{k}_{p,a} - T_p f)\|_{L^p(\mathbb{R},ds)} \\
&\le \big((p')^a - 1\big)\, \|T_p f * \tilde{k}_{p,a}\|_{L^p(\mathbb{R},ds)} + \|T_p f - T_p f * \tilde{k}_{p,a}\|_{L^p(\mathbb{R},ds)} \\
&\le \big((p')^a - 1\big)\, \|T_p f\|_{L^p(\mathbb{R},ds)} \|\tilde{k}_{p,a}\|_{L^1(\mathbb{R},ds)} \\
&\qquad + \|T_p f - T_p f * \tilde{k}_{p,a}\|_{L^p(\mathbb{R},ds)} \\
&= \big((p')^a - 1\big)\, \|f\|_{L^p(\mathbb{R}^+,dt)} + \|T_p f - T_p f * \tilde{k}_{p,a}\|_{L^p(\mathbb{R},ds)}.
\end{aligned}$$

Thus

$$0 \le \|(H^a - I)f\|_{L^p(\mathbb{R}^+,dt)} \le \big((p')^a - 1\big)\, \|f\|_{L^p(\mathbb{R}^+,dt)} + \|T_p f - T_p f * \tilde{k}_{p,a}\|_{L^p(\mathbb{R},ds)},$$

letting $a \to 0^+$ and using (28) together with the squeeze theorem, we conclude that

$$\|(H^a - I)f\|_{L^p(\mathbb{R}^+,dt)} \to 0, \quad \text{as } a \to 0^+.$$

This establishes (26). The proof of the limit in (27) is analogous. □

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Define

$$L_f := \left\{ x \in \mathbb{R} : \lim_{y\to x} \frac{1}{x - y} \int_y^x |f(t) - f(x)|\, dt = 0 \right\}. \tag{29}$$

By the Lebesgue Differentiation Theorem, we have

$$|\mathbb{R} \setminus L_f| = 0.$$

The following lemma will be needed.

Lemma 4.6. *Let $f \in L^1(\mathbb{R})$ and let $1 < p < \infty$. Then*

$$\lim_{a \to 0^+} f * k_{p,a}(x) = f(x),$$

for almost all $x \in \mathbb{R}$.

Proof. Let $x \in L_f$, where L_f is defined as in (29). Let $1 < p < \infty$ and let $\delta > 0$. Then

$$\begin{aligned} f * k_{p,a}(x) - f(x) &= \int_{-\infty}^{\infty} k_{p,a}(x-s)(f(s) - f(x))ds \\ &= \int_{-\infty}^{x-\delta} k_{p,a}(x-s)f(s)ds - f(x) \int_{-\infty}^{x-\delta} k_{p,a}(x-s)ds \\ &\quad + \int_{x-\delta}^{x} k_{p,a}(x-s)(f(s) - f(x))ds \\ &\quad + \int_{x}^{x+\delta} k_{p,a}(x-s)(f(s) - f(x))ds \\ &\quad + \int_{x+\delta}^{\infty} k_{p,a}(x-s)f(s)ds - f(x) \int_{x+\delta}^{\infty} k_{p,a}(x-s)ds \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Now, by the change of variable $t = x - s$, we obtain that

$$\begin{aligned} |I_1| &\leq \int_{-\infty}^{x-\delta} k_{p,a}(x-s)|f(s)|ds = \int_{\delta}^{\infty} k_{p,a}(t)|f(x-t)|dt \\ &\leq \left(\sup_{t \geq \delta} k_{p,a}(t) \right) \int_{\delta}^{\infty} |f(x-t)|dt \leq \left(\sup_{t \geq \delta} k_{p,a}(t) \right) \|f\|_{L^1(\mathbb{R},ds)} \to 0, \end{aligned}$$

as $a \to 0^+$, by Lemma 4.1 (iii).

Regarding I_2, we have

$$|I_2| = |f(x)| \int_{-\infty}^{x-\delta} k_{p,a}(x-s)ds = |f(x)| \int_{\delta}^{\infty} k_{p,a}(t)dt \to 0,$$

as $a \to 0^+$, by Lemma 4.1 (ii).

We now turn to I_3. By applying integration by parts, we get

$$\begin{aligned} I_3 &= \int_{x-\delta}^{x} k_{p,a}(x-s)(f(s) - f(x))ds = k_{p,a}(x-s) \int_{s}^{x} (f(u) - f(x))du \Big|_{s=x-\delta}^{x} \\ &\quad - \int_{x-\delta}^{x} k'_{p,a}(x-s) \int_{s}^{x} (f(u) - f(x))duds \\ &= -k_{p,a}(\delta) \int_{x-\delta}^{x} (f(u) - f(x))du \\ &\quad - \int_{x-\delta}^{x} (x-s)k'_{p,a}(x-s) \frac{1}{x-s} \int_{s}^{x} (f(u) - f(x))duds. \end{aligned}$$

Then, using Lemma 4.1 (iv), we have $\int_0^\infty |tk'_{p,a}(t)|\,dt = (p')^a < p'$ for $0 < a < 1$, and therefore

$$\begin{aligned}
|I_3| &\le k_{p,a}(\delta)\int_{x-\delta}^{x} |f(u)-f(x)|du \\
&\qquad + \max_{x-\delta\le s\le x}\left|\frac{1}{x-s}\int_s^x (f(u)-f(x))du\right| \int_{x-\delta}^{x} |(x-s)k'_{p,a}(x-s)|ds \\
&= k_{p,a}(\delta)\int_{x-\delta}^{x} |f(u)-f(x)|du \\
&\qquad + \max_{x-\delta\le s\le x}\left|\frac{1}{x-s}\int_s^x (f(u)-f(x))du\right| \int_0^{\delta} |tk'_{p,a}(t)|dt \\
&\le k_{p,a}(\delta)\int_{x-\delta}^{x} |f(u)-f(x)|du \\
&\qquad + \max_{x-\delta\le s\le x}\left|\frac{1}{x-s}\int_s^x (f(u)-f(x))du\right| \int_0^{\infty} |tk'_{p,a}(t)|dt \\
&= k_{p,a}(\delta)\int_{x-\delta}^{x} |f(u)-f(x)|du + p'\max_{x-\delta\le s\le x}\left|\frac{1}{x-s}\int_s^x (f(u)-f(x))du\right|.
\end{aligned}$$

Let $\epsilon > 0$. By Lebesgue Differentiation Theorem, there is $\delta > 0$ such that

$$p'\max_{x-\delta\le s\le x}\left|\frac{1}{x-s}\int_s^x (f(u)-f(x))du\right| < \epsilon.$$

For that same $\delta > 0$, by Lemma 4.1 (iii), there is $0 < a_0 < 1$ such that

$$k_{p,a}(\delta)\int_{x-\delta}^{x} |f(u)-f(x)|du < \epsilon,$$

for all $0 < a \le a_0$. Therefore, if $0 < a \le a_0$, then $|I_3| < 2\epsilon$.

On the other hand, making the substitution $t = x - s$, we obtain that

$$I_4 = \int_x^{x+\delta} k_{p,a}(x-s)(f(s)-f(x))ds = \int_{-\delta}^{0} k_{p,a}(t)(f(x-t)-f(x))dt = 0,$$

since $\operatorname{supp}(k_{p,a}) = [0,\infty)$. Similarly, we have that

$$I_5 = \int_{x+\delta}^{\infty} k_{p,a}(x-s)f(s)ds = \int_{-\infty}^{-\delta} k_{p,a}(t)f(x-t)dt = 0,$$

and

$$I_6 = f(x)\int_{x+\delta}^{\infty} k_{p,a}(x-s)ds = f(x)\int_{-\infty}^{-\delta} k_{p,a}(t)dt = 0.$$

This concludes the proof. $\square$

The following result shows that $H^a f \to f$ and $H^{*a} f \to f$ pointwise as $a \to 0^+$.

Theorem 4.7. *Let $f \in L^1(\mathbb{R}^+)$. Then*

$$\lim_{a\to 0^+} H^a f(x) = f(x) \tag{30}$$

and

$$\lim_{a\to 0^+} H^{*a} f(x) = f(x), \tag{31}$$

for almost every $x > 0$.

Proof. We only prove (30), as the proof of (31) is analogous. Let $x \in L_f$, where L_f is defined in (29). Note that $x > 0$. Let $1 < p < \infty$. Then

$$x^{1/p} H^a f(x) = e^{\frac{\ln x}{p}} H^a f\big(e^{\ln x}\big) = (T_p \circ H^a) f(\ln x) = (T_p f * k_{p,a})(\ln x),$$

where T_p is defined in (16), and where Lemma 4.2 has been used. Applying Lemma 4.6, we obtain

$$\lim_{a \to 0^+} (T_p f * k_{p,a})(\ln x) = T_p f(\ln x).$$

Hence

$$\lim_{a \to 0^+} x^{1/p} H^a f(x) = x^{1/p} f(x).$$

Equivalently

$$\lim_{a \to 0^+} H^a f(x) = f(x).$$

This completes the proof. □

5. Application

In this section, we investigate the norm of the exponential maps of H and H^*. Let $e^H, e^{H^*} : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ be the operators defined by

$$e^H = \sum_{n=0}^{\infty} \frac{1}{n!} H^n, \quad e^{H^*} = \sum_{n=0}^{\infty} \frac{1}{n!} H^{*n}.$$

The operators e^H and e^{H^*} are well defined and bounded on $L^p(\mathbb{R}^+)$ (see [13]).

Theorem 5.1. *Let $1 < p < \infty$. Then*

$$\|e^H\|_{L^p(\mathbb{R}^+)} = e^{\|H\|_{L^p(\mathbb{R}^+)}} = e^{p'}. \tag{32}$$

$$\|e^{H^*}\|_{L^p(\mathbb{R}^+)} = e^{\|H^*\|_{L^p(\mathbb{R}^+)}} = e^{p}. \tag{33}$$

Proof. We will prove only (32), since (33) can be established by an analogous argument. So,

$$\begin{aligned}
\|e^H\|_{L^p(\mathbb{R}^+)} &= \left\| \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} H^n \right\|_{L^p(\mathbb{R}^+)} = \lim_{N \to \infty} \left\| \sum_{n=0}^{N} \frac{1}{n!} H^n \right\|_{L^p(\mathbb{R}^+)} \\
&= \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} \|H\|^n_{L^p(\mathbb{R}^+)} = \sum_{n=0}^{\infty} \frac{1}{n!} \|H\|^n_{L^p(\mathbb{R}^+)} = e^{\|H\|_{L^p(\mathbb{R}^+)}} = e^{p'},
\end{aligned}$$

In the second equality, we have used the continuity of the operator L^p-norm, while in the third equality we apply Theorem 3.2. □

Remark 5.2. Note that, in general, the identity

$$\|e^T\|_{L^p(\mathbb{R}^+)} = e^{\|T\|_{L^p(\mathbb{R}^+)}}$$

does not hold for every bounded operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$. To illustrate this, consider the Volterra operator $V : L^2[0,1] \to L^2[0,1]$ defined by

$$Vf(x) = \int_0^x f(t)\, dt.$$

Halmos [8, Problem 188] computed the exact value of the operator norm of V. It is given by
$$\|V\|_{L^2[0,1]} = \frac{2}{\pi}.$$
The norm of the square of the Volterra operator was determined by Horgan (see [11, Table 1]), who showed that
$$\|V^2\|_{L^2[0,1]} = \frac{1}{\rho^2},$$
where ρ is the smallest positive solution of
$$(\cos\rho)(\cosh\rho) + 1 = 0.$$
Numerically,
$$\|V^2\|_{L^2[0,1]} \approx 0.28441289...$$
In particular,
$$\|V^2\|_{L^2[0,1]} < 0.285 < 0.4 < \left(\frac{2}{\pi}\right)^2 = \|V\|^2_{L^2[0,1]}.$$
Consequently, we obtain the estimate
$$\begin{aligned}\|e^V\|_{L^2[0,1]} &\leq \sum_{n=0}^{\infty} \frac{1}{n!}\|V^n\|_{L^2[0,1]} \leq 1 + \|V\|_{L^2[0,1]} + \frac{1}{2}\|V^2\|_{L^2[0,1]} + \sum_{n=3}^{\infty}\frac{1}{n!}\|V\|^n_{L^2[0,1]} \\ &< 1 + \|V\|_{L^2[0,1]} + \frac{1}{2}\|V\|^2_{L^2[0,1]} + \sum_{n=3}^{\infty}\frac{1}{n!}\|V\|^n_{L^2[0,1]} = \sum_{n=0}^{\infty}\frac{1}{n!}\|V\|^n_{L^2[0,1]} \\ &= e^{\|V\|_{L^2[0,1]}}.\end{aligned}$$
This shows that, in general,
$$\|e^T\|_{L^p(\mathbb{R}^+)} < e^{\|T\|_{L^p(\mathbb{R}^+)}}$$
may occur.

Acknowledgements

The author gratefully thanks his supervisors, Santiago Boza and Javier Soria, for their guidance and support throughout his PhD studies.

Funding

The author was partially supported by grants PID2020-113048GB-I00 and PID2024-155917NB-I00, funded by MCIN/AEI/ 10.13039/501100011033.

Data Availability Statement

No data sets were generated or analyzed during the current study.

Declarations

Code availability

Not applicable.

Conflicts of Interest

The author declares no competing interests.

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