OPTIMAL CONSTANTS FOR CERTAIN HARDY-TYPE INTEGRAL OPERATORS

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Abstract. In recent years, the problem of establishing optimal estimates in L^p-norms for Hardy-type operators has been extensively considered. In this work, we continue this line of research by analyzing the norm of polynomials in H and H^*, where H and H^* denote the classical Hardy averaging operator and its adjoint, respectively. We also establish the norms of H and H^* raised to real powers, derive a combinatorial formula involving conjugate exponents, and determine the norms of the exponentials of H and H^*.

1. Introduction

Let $\mathcal{M}^+(\mathbb{R}^+)$ be the class of all nonnegative measurable functions on $\mathbb{R}^+ = (0,\infty)$. Let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Set

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt$$

and

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t}dt,$$

provided that the integrals make sense for a function f on $\mathbb{R}^+$. These equalities define the classical Hardy operator H and its adjoint H^*.

The mappings introduced above play a central role in Analysis. For instance, H has a deep connection with one of the most important objects in Harmonic Analysis, namely the classical *Hardy–Littlewood maximal operator*, defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x, and $|Q|$ denotes the Lebesgue measure of Q. This connection arises from the fact that

$$(Mf)^* \simeq Hf^*, \tag{1}$$

where f^* denotes the decreasing rearrangement with respect to the Lebesgue measure. The relation in (1) implies that the problem of determining the boundedness of the maximal operator M on rearrangement-invariant spaces is equivalent to studying the boundedness of the much simpler operator H on the cone of nonnegative, nonincreasing functions in some Lebesgue space (see [4] for standard notations and results).

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The boundedness of H on $L^p(\mathbb{R}^+)$, for $1 < p \leq \infty$, follows from what is commonly referred to as Hardy's inequality (cf. [7]):

$$\|Hf\|_{L^p(\mathbb{R}^+)} \leq p' \, \|f\|_{L^p(\mathbb{R}^+)}, \tag{2}$$

for all $f \in L^p(\mathbb{R}^+)$, where $p' = \frac{p}{p-1}$ if $1 < p < \infty$, and $p' = 1$ if $p = \infty$. Moreover, p' is the best possible constant in (2).

Regarding the boundedness of the adjoint of Hardy's operator, the authors proved in [8, p. 244] that

$$\|H^* f\|_{L^p(\mathbb{R}^+)} \leq p \, \|f\|_{L^p(\mathbb{R}^+)}, \tag{3}$$

for all $f \in L^p(\mathbb{R}^+)$, where $1 \leq p < \infty$. Furthermore, the constant p is optimal in (3).

The operator $S = H + H^*$ is also of considerable importance in Harmonic Analysis. For instance, according to Theorem 4.8 in [4], the boundedness of S on $L^p(\mathbb{R}^+)$ implies the continuity of the classical Hilbert transform $\mathcal{H}$ on $L^p(\mathbb{R})$, where

$$\mathcal{H}f(x) = \text{p.v.}\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} \, dt.$$

To see that S defines a continuous operator on $L^p(\mathbb{R}^+)$, we make use of (2) together with (3). Indeed, we have

$$\|Sf\|_p \leq (p + p')\|f\|_p, \tag{4}$$

for all $f \in L^p(\mathbb{R}^+)$ with $1 < p < \infty$. Again, the constant $p + p'$ is the best possibe in (4) (see [8]). The operator S is commonly referred to as the *Calderón operator*.

One of the main goals of this work is to extend the results of G. H. Hardy, J. Littlewood, and G. Pólya in (2), (3), and (4) by determining the optimal constant $C(p) > 0$ such that

$$\|P(H, H^*)f\|_p \leq C(p) \, \|f\|_p, \tag{5}$$

for functions $f \in L^p(\mathbb{R}^+)$ belonging to one of the following classes: general functions, nonnegative functions, or nonnegative, nonincreasing functions, where $1 < p < \infty$ and $P \in \mathbb{R}[X, Y]$ is a real polynomial with nonnegative coefficients. The sharp constant $C(p)$ in (5) is established in Theorem 3.2, which constitutes the central result of this study.

This paper is organized as follows. Section 2 introduces the notations used throughout this work. Section 3 presents the main result, namely Theorem 3.2 along with Corollary 3.4, from which we derive a combinatorial formula involving conjugate exponents. In Section 4, specifically in Theorem 4.1, we establish the norms of H^a and H^{*a} for real number $a > 0$. Finally, in Section 5 we compute in Theorem 5.1 the norms of e^H and e^{H^*}.

2. Notations

The analysis in this work is carried out in the Lebesgue spaces $L^p(\mathbb{R}^+)$, $1 < p < \infty$. We also consider the cones

$$L^p_+(\mathbb{R}^+) = \{f \in L^p(\mathbb{R}^+) : f \geq 0\}, \qquad L^p_{\text{dec}}(\mathbb{R}^+) = \{f \in L^p_+(\mathbb{R}^+) : f \text{ is nonincreasing}\}.$$

For a bounded operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, we set

$$\|T\|_{L^p(\mathbb{R}^+)} := \sup_{\|f\|_p=1} \|Tf\|_p,$$

$$\|T\|_{L^p_+(\mathbb{R}^+)} := \sup_{\substack{f\in L^p_+(\mathbb{R}^+)\\ \|f\|_p=1}} \|Tf\|_p,$$

$$\|T\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)} := \sup_{\substack{f\in L^p_{\mathrm{dec}}(\mathbb{R}^+)\\ \|f\|_p=1}} \|Tf\|_p.$$

These satisfy

$$\|T\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)} \le \|T\|_{L^p_+(\mathbb{R}^+)} \le \|T\|_{L^p(\mathbb{R}^+)}.$$

Let $1 < p < \infty$ and let $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ be a bounded linear operator. For

$$Q(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} x^i y^j \in \mathbb{R}[X, Y],$$

where $m, n \in \mathbb{N}$, we define

$$Q(T, T^*) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij}\, T^i \circ T^{*j},$$

where $T^0 = T^{*0} = I$.

Note that if T is normal, i.e., $T \circ T^* = T^* \circ T$, then a straightforward calculation shows that

$$Q(T, T^*)^* = P(T, T^*),$$

where $P(x, y) = Q(y, x)$.

3. The norm of a polynomial of H and H^*

Let $n \ge 1$ be a natural number, and let f be a measurable function defined on $\mathbb{R}^+$. Denote by $\Gamma : (0, \infty) \to \mathbb{R}^+$ the Euler Gamma function, defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

D. W. Boyd proved (see [5, Lemma 2]) the following equality

$$H^n f(x) = (H \circ H \circ \overset{n)}{\cdots} \circ H) f(x) = \frac{1}{\Gamma(n)} \frac{1}{x} \int_0^x f(y) \ln^{n-1}\left(\frac{x}{y}\right) dy. \tag{6}$$

Using an analogous reasoning, we find

$$H^{*n} f(x) = (H^* \circ H^* \circ \overset{n)}{\cdots} \circ H^*) f(x) = \frac{1}{\Gamma(n)} \int_x^\infty \frac{f(y)}{y} \ln^{n-1}\left(\frac{y}{x}\right) dy, \tag{7}$$

whenever the defining integral exists almost everywhere. The following result provides a formula for the composition $H^n \circ H^{*m}$ expressed as a linear combination of powers of H and H^*, where m and n are natural numbers.

Proposition 3.1. *Let $m, n \ge 1$ be natural numbers and $f \in \mathcal{M}^+(\mathbb{R}^+)$. Then*

$$\left(H^{*m} \circ H^n\right) f(x) = \sum_{k=0}^{m-1} \binom{n+k-1}{k} H^{*m-k} f(x) + \sum_{k=0}^{n-1} \binom{m+k-1}{k} H^{n-k} f(x), \tag{8}$$

for all $x > 0$.

Proof. Let $x > 0$. Then,

$$
\begin{aligned}
\big(H^{*m} \circ H^n\big) f(x) &= \frac{1}{\Gamma(m)} \int_x^\infty \frac{H^n f(y)}{y} \ln^{m-1}\left(\frac{y}{x}\right) dy \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{1}{y^2} \int_0^y f(t) \ln^{n-1}\left(\frac{y}{t}\right) dt \ln^{m-1}\left(\frac{y}{x}\right) dy \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \int_0^y \frac{f(t)}{y^2} \ln^{n-1}\left(\frac{y}{t}\right) \ln^{m-1}\left(\frac{y}{x}\right) dtdy \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \int_0^x \int_x^\infty \frac{f(t)}{y^2} \ln^{n-1}\left(\frac{y}{t}\right) \ln^{m-1}\left(\frac{y}{x}\right) dydt \\
&+ \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \int_t^\infty \frac{f(t)}{y^2} \ln^{n-1}\left(\frac{y}{t}\right) \ln^{m-1}\left(\frac{y}{x}\right) dydt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{1}{y^2} \ln^{n-1}\left(\frac{x}{t}y\right) \ln^{m-1}(y) dydt \\
&+ \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{1}{y^2} \ln^{n-1}(y) \ln^{m-1}\left(\frac{t}{x}y\right) dydt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \ln^{n-1}\left(\frac{x}{t}y\right) dydt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \ln^{m-1}\left(\frac{t}{x}y\right) dydt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \left(\ln\left(\frac{x}{t}\right) + \ln y\right)^{n-1} dydt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \left(\ln\left(\frac{t}{x}\right) + \ln y\right)^{m-1} dydt \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \int_1^\infty \frac{\ln^{m-1}(y)}{y^2} \sum_{k=0}^{n-1} \binom{n-1}{k} \ln^{n-k-1}\left(\frac{x}{t}\right) \ln^k(y) dydt \\
&\quad + \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \int_1^\infty \frac{\ln^{n-1}(y)}{y^2} \sum_{k=0}^{m-1} \binom{m-1}{k} \ln^{m-k-1}\left(\frac{t}{x}\right) \ln^k(y) dydt \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \ln^{n-k-1}\left(\frac{x}{t}\right) \int_1^\infty \frac{\ln^{m+k-1}(y)}{y^2} dydt \\
&\quad + \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{1}{\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \ln^{m-k-1}\left(\frac{t}{x}\right) \int_1^\infty \frac{\ln^{n+k-1}(y)}{y^2} dydt \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(n)\Gamma(m+k)}{\Gamma(k+1)\Gamma(n-k)\Gamma(m)\Gamma(n)} \frac{1}{x} \int_0^x f(t) \ln^{n-k-1}\left(\frac{x}{t}\right) dt \\
&\quad + \sum_{k=0}^{m-1} \frac{\Gamma(m)\Gamma(n+k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m)\Gamma(n)} \int_x^\infty \frac{f(t)}{t} \ln^{m-k-1}\left(\frac{t}{x}\right) dt \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(m+k)}{\Gamma(k+1)\Gamma(m)} H^{n-k} f(x) + \sum_{k=0}^{m-1} \frac{\Gamma(n+k)}{\Gamma(k+1)\Gamma(n)} H^{*m-k} f(x)
\end{aligned}
$$

$$= \sum_{k=0}^{n-1} \binom{m+k-1}{k} H^{n-k} f(x) + \sum_{k=0}^{m-1} \binom{n+k-1}{k} H^{*m-k} f(x),$$

where we used Tonelli's Theorem in the fourth equality. We also applied (6) and (7) in the second to last equality. This concludes the proof. ☐

Note that (2), (3), and (4) says, in other words, that

$$\|H\|_{L^p(\mathbb{R}^+)} = p', \qquad \text{for } 1 < p \leq \infty, \tag{9}$$
$$\|H^*\|_{L^p(\mathbb{R}^+)} = p, \qquad \text{for } 1 \leq p < \infty, \tag{10}$$
$$\|H + H^*\|_{L^p(\mathbb{R}^+)} = p + p', \qquad \text{for } 1 < p < \infty. \tag{11}$$

By closely examining identities (9), (10), and (11), we observe that

$$\begin{aligned}
\|H\|_{L^p(\mathbb{R}^+)} &= \|P(H,H^*)\|_{L^p(\mathbb{R}^+)} = P(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p', \\
\|H^*\|_{L^p(\mathbb{R}^+)} &= \|Q(H,H^*)\|_{L^p(\mathbb{R}^+)} = Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p, \\
\|H + H^*\|_{L^p(\mathbb{R}^+)} &= \|R(H,H^*)\|_{L^p(\mathbb{R}^+)} = R(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p + p',
\end{aligned}$$

where $P(x, y) = x$, $Q(x, y) = y$, and $R(x, y) = x + y$.

It is important to note that, when working with H and H^*, two different polynomials can sometimes define the same associated operator. For instance, in [10] the following identity related to the Calderón operator is proved

$$H \circ H^* = H^* \circ H = H + H^* = S.$$

Therefore,

$$R(H, H^*) = V(H, H^*),$$

where R is as above and $V(x, y) = xy$. It is worth noting that,

$$\begin{aligned}
p' + p = R(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) &= \|R(H,H^*)\|_{L^p(\mathbb{R}^+)} = \|V(H,H^*)\|_{L^p(\mathbb{R}^+)} \\
&= V(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = p'p,
\end{aligned}$$

since p and p' are conjugate exponents.

The following theorem extends (9), (10), and (11), and constitutes one of the main results of this work.

Theorem 3.2. *Let $1 < p < \infty$ and let $Q \in \mathbb{R}[X, Y]$ be a polynomial with nonnegative coefficients. Then*

$$\begin{aligned}
\|Q(H,H^*)\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)} = \|Q(H,H^*)\|_{L^p_+(\mathbb{R}^+)} &= \|Q(H,H^*)\|_{L^p(\mathbb{R}^+)} \\
&= Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = Q(p', p).
\end{aligned}$$

Proof. Let $f \in L^p(\mathbb{R}^+)$ and let $Q \in \mathbb{R}[X, Y]$ be the polynomial given by

$$Q(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} x^i y^j \in \mathbb{R}[X, Y],$$

where $c_{ij} \geq 0$ and $m, n \geq 1$ are natural numbers. By the triangle inequality and the submultiplicativity of the L^p-norm for operators, we get that

$$\begin{aligned}\|Q(H,H^*)f\|_p \leq \|Q(H,H^*)|f|\|_p &= \left\| \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}(H^i \circ H^{*j})|f| \right\|_p \\ &\leq \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\|(H^i \circ H^{*j})|f|\|_p \\ &\leq \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\|H\|_p^i\|H^*\|_p^j\|f\|_p = Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)})\|f\|_p.\end{aligned}$$

Thus

$$\|Q(H,H^*)\|_p \leq Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}).$$

By using (9) with (10) we obtain

$$\|Q(H,H^*)\|_p \leq Q(\|H\|_{L^p(\mathbb{R}^+)}, \|H^*\|_{L^p(\mathbb{R}^+)}) = Q(p', p).$$

We now show that the above inequality is in fact an equality. Consider the functions $f_\epsilon, g_\epsilon : (0,\infty) \to \mathbb{R}$ given by $f_\epsilon(x) = x^{\epsilon/p-1/p}\chi_{(0,1)}(x)$ and $g_\epsilon(x) = x^{\epsilon/p'-1/p'}\chi_{(0,1)}(x)$, where $0 < \epsilon < 1$. Then

$$\|f_\epsilon\|_p^p = \|g_\epsilon\|_{p'}^{p'} = \frac{1}{\epsilon}.$$

On the other hand, if $0 < x < 1$ and $n \in \mathbb{N}$, an easy computation shows that

$$H^n f(x) = x^{\epsilon/p-1/p}\frac{1}{(\epsilon/p + 1/p')^n}. \tag{12}$$

Now, as H and H^* are positive operators and $c_{ij} \geq 0$, it follows that $Q(H,H^*)$ is positive as well. So

$$\int_0^\infty |Q(H,H^*)f_\epsilon(x)\cdot g_\epsilon(x)|dx = \int_0^\infty Q(H,H^*)f_\epsilon(x)\cdot g_\epsilon(x)dx.$$

Since g_ϵ has support equal to $[0,1]$, we have that

$$\begin{aligned}\int_0^\infty Q(H,H^*)f_\epsilon(x)\cdot g_\epsilon(x)dx &= \int_0^1 Q(H,H^*)f_\epsilon(x)\cdot g_\epsilon(x)dx \\ &= \int_0^1 \left(\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}H^i \circ H^{*j}\right) f_\epsilon(x)\cdot g_\epsilon(x)dx \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\int_0^1 \left(H^i \circ H^{*j}\right) f_\epsilon(x)\cdot g_\epsilon(x)dx.\end{aligned}$$

Given that H and H^* commute, we obtain that

$$\int_0^1 \left(H^i \circ H^{*j}\right) f_\epsilon(x)\cdot g_\epsilon(x)dx = \int_0^1 \left(H^{*j} \circ H^i\right) f_\epsilon(x)\cdot g_\epsilon(x)dx,$$

for every $1 \leq i \leq m$ and $1 \leq j \leq n$. Observe that, as H^j is the adjoint of H^{*j}, we get that

$$\int_0^1 \left(H^{*j} \circ H^i\right) f_\epsilon(x)\cdot g_\epsilon(x)dx = \int_0^1 H^i f_\epsilon(x)\cdot H^j g_\epsilon(x)dx,$$

for every $1 \leq i \leq m$ and $1 \leq j \leq n$. Now, using (12), we have that

$$\begin{aligned}\int_0^1 H^i f_\epsilon(x) \cdot H^j g_\epsilon(x) dx &= \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \int_0^1 x^{\epsilon/p - 1/p} x^{\epsilon/p' - 1/p'} dx \\ &= \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \frac{1}{\epsilon}.\end{aligned}$$

Hence,

$$\int_0^\infty |Q(H, H^*) f_\epsilon(x) \cdot g_\epsilon(x)| dx = \frac{1}{\epsilon} \sum_{i=0}^m \sum_{j=0}^n c_{ij} \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j}.$$

Now, by using Hölder's inequality, we obtain

$$\begin{aligned}\|Q(H, H^*)\|_p &\geq \frac{\|Q(H, H^*) f_\epsilon\|_p}{\|f_\epsilon\|_p} \geq \frac{\int_0^\infty |Q(H, H^*) f_\epsilon(x) \cdot g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} \\ &= \sum_{i=0}^m \sum_{j=0}^n c_{ij} \frac{1}{(\epsilon/p + 1/p')^i} \frac{1}{(\epsilon/p' + 1/p)^j} \to \sum_{i=0}^m \sum_{j=0} c_{ij} (p')^i p^j = Q(p', p),\end{aligned}$$

as $\epsilon \to 0^+$. Since the functions f_ϵ and g_ϵ are nonnegative and nonincreasing, the proof is concluded. $\square$

Remark 3.3. The condition that the coefficients of the polynomial Q be nonnegative in Theorem 3.2 is necessary, since without this hypothesis the conclusion of the theorem may fail. To illustrate this, consider the polynomial

$$Q(x, y) = y - 1,$$

for which the associated operator is $Q(H, H^*) = H^* - I$. The authors in [2, Theorem 4.2] proved that

$$\|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} = M_p^{1/p},$$

for $1 < p < 2$, where $M_p = \max_{t \in [0,1/2]} f_p(t)$ and $f_p : [0, 1/2] \to \mathbb{R}$ is defined by

$$f_p(t) = pt^{p-1} + (1-t)^p - t^p.$$

In particular,

$$\|Q(H, H^*)\|_{L^p(\mathbb{R}^+)} = M_p^{1/p} \geq (f_p(0))^{1/p} = 1 > p - 1 = Q(p', p),$$

for all $1 < p < 2$.

The nonnegativity of the coefficients of Q is also necessary to ensure that

$$\|Q(H, H^*)\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|Q(H, H^*)\|_{L^p_+(\mathbb{R}^+)} = \|Q(H, H^*)\|_{L^p(\mathbb{R}^+)},$$

since this equality may fail otherwise. For instance, consider again the operator $P(H, H^*)$ defined above. It was proved in [3, Theorem 1.2] that

$$\|Q(H, H^*)\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \|H^* - I\|_{L^p_{\text{dec}}(\mathbb{R}^+)} = \left(\int_0^\infty |\ln(x) + 1|^p \, dx \right)^{1/p} \stackrel{\text{def}}{=} C_p^{1/p},$$

for $1 < p \leq 2$. Moreover, it is shown in [3] that the function $p \mapsto C_p^{1/p}$ is strictly increasing on $\mathbb{R}^+$. Hence,

$$\begin{aligned}\|P(H,H^*)\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)} &= C_p^{1/p} < C_2^{1/2} = \left(\int_0^\infty |\ln(x)+1|^2\,dx\right)^{1/2}\\ &= 1 = (f_p(0))^{1/p} < M_p^{1/p} = \|P(H,H^*)\|_{L^p_+(\mathbb{R}^+)}\\ &= \|P(H,H^*)\|_{L^p(\mathbb{R}^+)},\end{aligned}$$

for all $1 < p < 2$, where in the last equality we have taken into account [2, Theorem 4.2].

Before stating our next result, we recall that by convention, $\binom{n}{0} = 1$ for all integers n; in particular, $\binom{-1}{0} = 1$. We also set the sum equal to zero whenever the upper index is less than the one at the bottom.

Corollary 3.4. *Let $m, n \in \mathbb{N} \cup \{0\}$ and $1 < p < \infty$. Then*

$$\begin{aligned}\|H^{*m} \circ H^n\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)} &= \|H^{*m} \circ H^n\|_{L^p_+(\mathbb{R}^+)} = \|H^{*m} \circ H^n\|_{L^p(\mathbb{R}^+)}\\ &= \sum_{k=0}^{m-1}\binom{n+k-1}{k}p^{m-k} + \sum_{k=0}^{n-1}\binom{m+k-1}{k}(p')^{n-k} \qquad (13)\\ &= p^m (p')^n.\end{aligned}$$

Proof. Let $P, Q \in \mathbb{R}[X,Y]$ be the polynomials defined by

$$P(x,y) = \sum_{k=0}^{n-1}\binom{m+k-1}{k}x^{n-k} + \sum_{k=0}^{m-1}\binom{n+k-1}{k}y^{m-k},$$

and

$$Q(x,y) = x^n y^m,$$

where $m, n \in \mathbb{N} \cup \{0\}$. Note that in the case $m = 0$, we have

$$P(x,y) = x^n = Q(x,y),$$

for any $n \in \mathbb{N} \cup \{0\}$. Similarly, in the case $n = 0$, we have

$$P(x,y) = y^m = Q(x,y),$$

for any $m \in \mathbb{N} \cup \{0\}$. This implies that

$$P(H,H^*) = Q(H,H^*) \quad \text{if } n = 0 \text{ or } m = 0.$$

In the case $m, n \geq 1$, we have seen in (8) that

$$P(H,H^*)f = Q(H,H^*)f,$$

for all $f \in \mathcal{M}^+(\mathbb{R}^+)$. Hence,

$$P(H,H^*)f = Q(H,H^*)f, \qquad (14)$$

for all $m, n \in \mathbb{N} \cup \{0\}$, and any $f \in \mathcal{M}^+(\mathbb{R}^+)$. Therefore, taking into account Theorem 3.2 and (14), we obtain (13). $\square$

Remark 3.5. Let

$$A = \{H^{*m} \circ H^n : m, n \in \mathbb{N} \cup \{0\}\}.$$

Then $(A, \circ)$, where $\circ$ denotes composition, is an abelian semigroup with identity. In particular, Corollary 3.4 implies that the mapping

$$\begin{aligned} \varphi : A &\longrightarrow [1, \infty), \\ T &\longmapsto \|T\|_{L^p(\mathbb{R}^+)} \end{aligned}$$

is a homomorphism.

Remark 3.6. Let $1 < p < \infty$ and let $n \geq 1$ be a natural number. One the one hand, equality (13) implies

$$p^n (p')^n = \sum_{k=0}^{n-1} \binom{n+k-1}{k} \left(p^{n-k} + (p')^{n-k}\right).$$

On the other hand, since p and p' are conjugate exponents, we have

$$p^n (p')^n = (pp')^n = (p + p')^n.$$

Hence

$$(p + p')^n = \sum_{k=0}^{n-1} \binom{n+k-1}{k} \left(p^{n-k} + (p')^{n-k}\right).$$

The coefficients $\binom{n+k-1}{k}$, for $k = 1, \cdots, n$, are known as *generalized triangular numbers* (see [1]). For a fixed n, their arrangement within Pascal's Triangle is illustrated below:

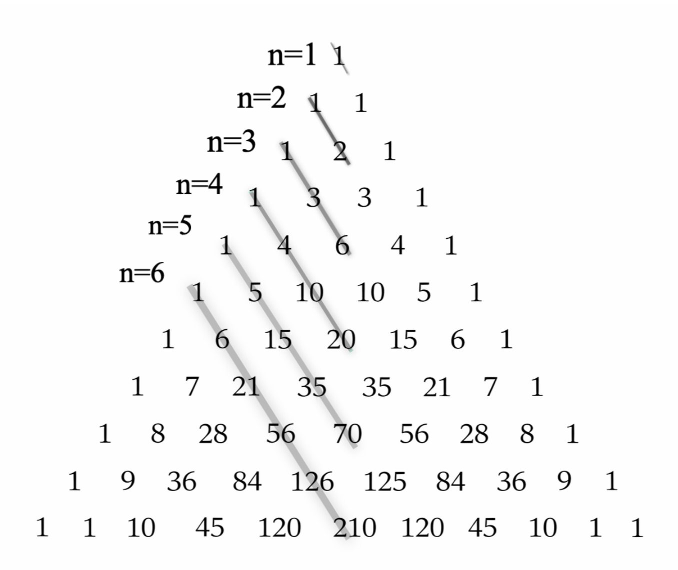

FIGURE 1. The corresponding coefficients $\binom{n+k-1}{k}$ for $n = 1, \ldots, 6$ and $k = 1, \cdots, n$.

For instance, when $n = 4$, we have

$$(p + p')^4 = \mathbf{1}\,(p^4 + (p')^4) + \mathbf{4}\,(p^3 + (p')^3) + \mathbf{10}\,(p^2 + (p')^2) + \mathbf{20}\,(p + p').$$

4. Powers of H and H^* for Real Exponents

Let $a>0$ be a real number. For a function f on $\mathbb{R}^+$, the operators H^a and H^{*a} are given by

$$H^a f(x)=\frac{1}{\Gamma(a)}\,\frac{1}{x}\int_0^x f(t)\,\ln^{a-1}\Big(\frac{x}{t}\Big)\,dt,$$

and

$$H^{*a} f(x)=\frac{1}{\Gamma(a)}\int_x^\infty \frac{f(t)}{t}\,\ln^{a-1}\Big(\frac{t}{x}\Big)\,dt,$$

whenever these integrals are well defined. Note that H^{*a} is the adjoint operator of H^a.

Observe that for $a=1$, the operators H^1 and H^{*1} reduce to the classical Hardy operator and its adjoint, respectively. Furthermore, if a is a natural number, H^a coincides with the a-fold composition of H, as shown in (6). A similar statement holds for H^{*a}.

The following result extends (9) and (10).

Theorem 4.1. *Let $1\le p\le\infty$. Then*

$$\|H^a\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)}=\|H^a\|_{L^p_+(\mathbb{R}^+)}=\|H^a\|_{L^p(\mathbb{R}^+)}=(p')^a,\quad (1<p\le\infty),\tag{15}$$

and

$$\|H^{*a}\|_{L^p_{\mathrm{dec}}(\mathbb{R}^+)}=\|H^{*a}\|_{L^p_+(\mathbb{R}^+)}=\|H^{*a}\|_{L^p(\mathbb{R}^+)}=p^a,\quad (1\le p<\infty).\tag{16}$$

Proof. Let $a>0$ and $f\in L^p(\mathbb{R}^+)$ with $1<p\le\infty$. Using the change of variables $t=xe^{-u}$, we obtain

$$H^a f(x)=\frac{1}{\Gamma(a)}\,\frac{1}{x}\int_0^x f(t)\,\ln^{a-1}\Big(\frac{x}{t}\Big)\,dt=\frac{1}{\Gamma(a)}\int_0^\infty f(xe^{-u})u^{a-1}e^{-u}du.$$

Set

$$k(u)=\frac{1}{\Gamma(a)}u^{a-1}e^{-u},\qquad k(u)\ge 0,\qquad \int_0^\infty k(u)\,du=1.$$

We start with the case $p=\infty$. We have trivially that

$$\begin{aligned}|H^a f(x)|=\left|\int_0^\infty f(xe^{-u})k(u)du\right|&\le\int_0^\infty |f(xe^{-u})|k(u)du\\ &\le\|f\|_\infty\int_0^\infty k(u)du=\|f\|_\infty.\end{aligned}$$

For $1<p<\infty$, by Minkowski's integral inequality,

$$\begin{aligned}\|H^a f\|_p&=\left(\int_0^\infty\left|\int_0^\infty f(xe^{-u})k(u)\,du\right|^p dx\right)^{1/p}\\ &\le\int_0^\infty\left(\int_0^\infty |f(xe^{-u})|^p dx\right)^{1/p} k(u)\,du.\end{aligned}$$

For fixed u, using the change of variables $y=xe^{-u}$, $dx=e^u dy$,

$$\int_0^\infty |f(xe^{-u})|^p dx=\int_0^\infty |f(y)|^p e^u dy=e^u\|f\|_p^p.$$

Hence
$$\left(\int_0^\infty |f(xe^{-u})|^p dx\right)^{1/p} = e^{u/p}\|f\|_p.$$
Therefore
$$\|H^a f\|_p \le \|f\|_p \int_0^\infty e^{u/p} k(u)\, du.$$
Now compute
$$\begin{aligned}\int_0^\infty e^{u/p} k(u)\, du &= \frac{1}{\Gamma(a)} \int_0^\infty u^{a-1} e^{-u(1-1/p)} du \\ &= \frac{1}{\Gamma(a)} \cdot \frac{\Gamma(a)}{(1-1/p)^a} = \left(\frac{p}{p-1}\right)^a = (p')^a.\end{aligned}$$
Thus,
$$\|H^a f\|_p \le (p')^a \|f\|_p,$$
for all $1 < p \le \infty$. We now show that the constant $(p')^a$ is optimal in the above inequality. To this end, consider the functions
$$f_\epsilon(x) = x^{\frac{\epsilon}{p} - \frac{1}{p}} \chi_{(0,1)}(x) \quad \text{and} \quad g_\epsilon(x) = x^{\frac{\epsilon}{p'} - \frac{1}{p'}} \chi_{(0,1)}(x),$$
where $1 \le p < \infty$ and $0 < \epsilon < 1$. If $0 < x < 1$, then
$$H^a f_\epsilon(x) = x^{\frac{\epsilon}{p} - \frac{1}{p}} \frac{1}{\Gamma(a)} \int_0^\infty e^{-u(\frac{\epsilon}{p} + \frac{1}{p'})} u^{a-1} du = x^{\frac{\epsilon}{p} - \frac{1}{p}} \frac{1}{(\epsilon/p + 1/p')^a},$$
for all $1 < p < \infty$. Now, by using Hölder's inequality, we obtain
$$\|H^a\|_p \ge \frac{\|H^a f_\epsilon\|_p}{\|f_\epsilon\|_p} \ge \frac{\int_0^\infty |H^a f_\epsilon(x) \cdot g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} = \frac{1}{(\epsilon/p + 1/p')^a} \to (p')^a,$$
as $\epsilon \to 0^+$, for all $1 < p < \infty$. In the case when $p = \infty$, we consider the function $f = \chi_{(0,1)}$. A straightforward computation shows that
$$\|H^a f_\epsilon\|_\infty = \|f_\epsilon\|_\infty = 1.$$
This ends the proof of (15).

Now we turn to (16). By the duality argument, we obtain
$$\|H^{*a}\|_{L^p(\mathbb{R}^+)} = \|H^a\|_{L^{p'}(\mathbb{R}^+)} = p^a,$$
for all $1 \le p < \infty$. To conclude the proof of (16), we consider the same functions f_ϵ and g_ϵ defined above. Then
$$\begin{aligned}\|H^{*a}\|_p &\ge \frac{\|H^{*a} f_\epsilon\|_p}{\|f_\epsilon\|_p} \ge \frac{\int_0^\infty |H^{*a} f_\epsilon(x) \cdot g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} \\ &= \frac{\int_0^\infty |f_\epsilon(x) \cdot H^a g_\epsilon(x)| dx}{\|g_\epsilon\|_{p'} \|f_\epsilon\|_p} = \frac{1}{(\epsilon/p + 1/p)^a} \to p^a,\end{aligned}$$
as $\epsilon \to 0^+$, for all $1 < p < \infty$. In the case $p = 1$, we consider the function $f = \chi_{(0,1)}$ again. By Tonelli's theorem, we get
$$\begin{aligned}\|H^{*a} f\|_1 &= \frac{1}{\Gamma(a)} \int_0^1 \int_x^1 \frac{1}{t} \ln^{a-1}\left(\frac{t}{x}\right) dt dx = \frac{1}{\Gamma(a)} \int_0^1 \frac{1}{t} \int_0^t \ln^{a-1}\left(\frac{t}{x}\right) dx dt \\ &= \frac{1}{\Gamma(a)} \int_0^1 \int_0^1 \ln^{a-1}\left(\frac{1}{x}\right) dx dt = \frac{1}{\Gamma(a)} \Gamma(a) = 1.\end{aligned}$$

This completes the proof. □

Let $C_c^1\mathbb{R}^+)$ be the set of continuously differentiable functions with compact support.

Proposition 4.2. *Let $f \in C_c^1(\mathbb{R}^+)$. Then*

$$\lim_{a\to 0^+} \frac{1}{\Gamma(a)}\frac{1}{x}\int_0^x f(t)\ln^{a-1}\left(\frac{x}{t}\right)dt = f(x), \tag{17}$$

for all $x \in \mathbb{R}^+$, and

$$\lim_{a\to 0^+} \frac{1}{\Gamma(a)}\int_x^\infty \frac{f(t)}{t}\ln^{a-1}\left(\frac{t}{x}\right)dt = f(x), \tag{18}$$

for all $x \in \mathbb{R}^+$.

Proof. We start by proving (17). Let $f \in C_c^1(\mathbb{R}^+)$ and let $x > 0$. Then

$$\frac{1}{\Gamma(a)}\int_0^x f(t)\ln^{a-1}\left(\frac{x}{t}\right)dt = \frac{1}{\Gamma(a)}\int_0^x f(t)t\ln^{a-1}\left(\frac{x}{t}\right)\frac{dt}{t}.$$

Now, using integration by parts we get

$$\begin{aligned}
&\frac{1}{\Gamma(a)}\int_0^x f(t)t\ln^{a-1}\left(\frac{x}{t}\right)\frac{dt}{t} \\
&\quad= \frac{1}{\Gamma(a+1)}f(t)t\ln^{a}\left(\frac{x}{t}\right)\Bigg|_{t=x}^{t=0} + \frac{1}{\Gamma(a+1)}\int_0^x (f(t)t)'\ln^{a}\left(\frac{x}{t}\right)dt \\
&\quad= \lim_{t\to 0^+}\frac{1}{\Gamma(a+1)}f(t)t\ln^{a}\left(\frac{x}{t}\right) + \frac{1}{\Gamma(a+1)}\int_0^x (f(t)t)'\ln^{a}\left(\frac{x}{t}\right)dt.
\end{aligned}$$

So

$$\frac{1}{\Gamma(a)}\int_0^x f(t)t\ln^{a-1}\left(\frac{x}{t}\right)\frac{dt}{t} = \frac{1}{\Gamma(a+1)}\int_0^x (f(t)t)'\ln^{a}\left(\frac{x}{t}\right)dt,$$

for all $a > 0$.

Since $f \in C_c^1(\mathbb{R}^+)$, there exists $M > 0$ such that

$$\left|(f(t)t)'\right| \le M, \qquad \text{for all } t \in \mathbb{R}^+.$$

Consequently,

$$\left|\frac{1}{\Gamma(a+1)}(f(t)t)'\ln^{a}\Big(\frac{x}{t}\Big)\right| \le M\,\frac{1}{\Gamma(a+1)}\ln^{a}\Big(\frac{x}{t}\Big),$$

for all $0 < t \le x$. Assume now that $0 < a \le 1$. Using the elementary inequality $u^a \le 1 + u$ for all $u \ge 0$, we obtain

$$\frac{1}{\Gamma(a+1)}\ln^{a}\Big(\frac{x}{t}\Big) \le \frac{1}{\Gamma(a+1)}\Big(1+\ln\Big(\frac{x}{t}\Big)\Big) \le K\Big(1+\ln\Big(\frac{x}{t}\Big)\Big)$$

for all $0 < t \le x$, where $K > 0$. The function $t \mapsto 1 + \ln\big(\frac{x}{t}\big)$ is integrable on $(0, x)$, since

$$\int_0^x \Big(1+\ln\Big(\frac{x}{t}\Big)\Big)\,dt = 2x.$$

Therefore, by the Dominated Convergence Theorem,

$$\begin{aligned}\lim_{a\to 0^+}\frac{1}{\Gamma(a+1)}\int_0^x (f(t)t)'\ln^a\left(\frac{x}{t}\right)dt &= \int_0^x (f(t)t)'\lim_{a\to 0^+}\frac{\ln^a\left(\frac{x}{t}\right)}{\Gamma(a+1)}dt\\ &= \int_0^x (f(t)t)'\,dt = f(x)x,\end{aligned}$$

from which it follows (17) for continuously differentiable functions with compact support.

Now, we turn to (18). Let $a>0$ and $x>0$. Then, by applying integration by parts, we have

$$\begin{aligned}\frac{1}{\Gamma(a)}\int_x^\infty \frac{f(t)}{t}\ln^{a-1}\left(\frac{t}{x}\right)dt &= \frac{1}{\Gamma(a+1)}f(t)\ln^a\left(\frac{t}{x}\right)\Bigg|_{t=x}^{t=\infty}\\ &\quad -\frac{1}{\Gamma(a+1)}\int_x^\infty f'(t)\ln^a\left(\frac{t}{x}\right)dt.\end{aligned}$$

Thus

$$\frac{1}{\Gamma(a)}\int_x^\infty \frac{f(t)}{t}\ln^{a-1}\left(\frac{t}{x}\right)dt = -\frac{1}{\Gamma(a+1)}\int_x^\infty f'(t)\ln^a\left(\frac{t}{x}\right)dt,$$

for all $a>0$. Now, using a similar justification as before to apply the Dominated Convergence Theorem, and invoking this result, we obtain

$$\begin{aligned}\lim_{a\to 0^+}\frac{1}{\Gamma(a+1)}\int_x^\infty \left(\frac{d}{dt}f(t)\right)\ln^a\left(\frac{t}{x}\right)dt &= \int_x^\infty \left(\frac{d}{dt}f(t)\right)\lim_{a\to 0^+}\frac{\ln^a\left(\frac{t}{x}\right)}{\Gamma(a+1)}dt\\ &= \int_x^\infty \left(\frac{d}{dt}f(t)\right)dt = -f(x),\end{aligned}$$

from which (18) follows for all $f\in C_c^1(\mathbb{R}^+)$ and $x>0$. $\square$

5. Application

In this section, we investigate the norm of the exponential maps of H and H^*. Let $e^H, e^{H^*}: L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)$ be the operators defined by

$$e^H=\sum_{n=0}^\infty \frac{1}{n!}H^n,\quad e^{H^*}=\sum_{n=0}^\infty \frac{1}{n!}H^{*n}.$$

The operators e^H and e^{H^*} are well defined and bounded on $L^p(\mathbb{R}^+)$ (see [11]).

Theorem 5.1. *Let $1<p<\infty$. Then*

$$\|e^H\|_{L^p(\mathbb{R}^+)} = e^{\|H\|_{L^p(\mathbb{R}^+)}} = e^{p'}. \tag{19}$$

$$\|e^{H^*}\|_{L^p(\mathbb{R}^+)} = e^{\|H^*\|_{L^p(\mathbb{R}^+)}} = e^{p}. \tag{20}$$

Proof. We will prove only (19), since (20) can be established by an analogous argument. So,

$$\begin{aligned}\|e^H\|_{L^p(\mathbb{R}^+)} &= \left\|\lim_{N\to\infty}\sum_{n=0}^N \frac{1}{n!}H^n\right\|_{L^p(\mathbb{R}^+)} = \lim_{N\to\infty}\left\|\sum_{n=0}^N \frac{1}{n!}H^n\right\|_{L^p(\mathbb{R}^+)}\\ &= \lim_{N\to\infty}\sum_{n=0}^N\frac{1}{n!}\|H\|^n_{L^p(\mathbb{R}^+)} = \sum_{n=0}^\infty \frac{1}{n!}\|H\|^n_{L^p(\mathbb{R}^+)} = e^{\|H\|_{L^p(\mathbb{R}^+)}} = e^{p'},\end{aligned}$$

In the second equality, we have used the continuity of the operator L^p-norm, while in the third equality we apply Theorem 3.2. $\square$

Remark 5.2. Note that, in general, the identity

$$\|e^T\|_{L^p(\mathbb{R}^+)} = e^{\|T\|_{L^p(\mathbb{R}^+)}}$$

does not hold for every bounded operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$. To illustrate this, consider the Volterra operator $V : L^2[0,1] \to L^2[0,1]$ defined by

$$Vf(x) = \int_0^x f(t)\,dt.$$

Halmos [6, Problem 188] computed the exact value of the operator norm of V. It is given by

$$\|V\|_{L^2[0,1]} = \frac{2}{\pi}.$$

The norm of the square of the Volterra operator was determined by Horgan (see [9, Table 1]), who showed that

$$\|V^2\|_{L^2[0,1]} = \frac{1}{\rho^2},$$

where ρ is the smallest positive solution of

$$(\cos\rho)(\cosh\rho) + 1 = 0.$$

Numerically,

$$\|V^2\|_{L^2[0,1]} \approx 0.28441289...$$

In particular,

$$\|V^2\|_{L^2[0,1]} < 0.285 < 0.4 < \left(\frac{2}{\pi}\right)^2 = \|V\|^2_{L^2[0,1]}.$$

Consequently, we obtain the estimate

$$\begin{aligned}
\|e^V\|_{L^2[0,1]} &\leq \sum_{n=0}^{\infty} \frac{1}{n!}\|V^n\|_{L^2[0,1]} \leq 1 + \|V\|_{L^2[0,1]} + \frac{1}{2}\|V^2\|_{L^2[0,1]} + \sum_{n=3}^{\infty} \frac{1}{n!}\|V\|^n_{L^2[0,1]} \\
&< 1 + \|V\|_{L^2[0,1]} + \frac{1}{2}\|V\|^2_{L^2[0,1]} + \sum_{n=3}^{\infty} \frac{1}{n!}\|V\|^n_{L^2[0,1]} = \sum_{n=0}^{\infty} \frac{1}{n!}\|V\|^n_{L^2[0,1]} \\
&= e^{\|V\|_{L^2[0,1]}}.
\end{aligned}$$

This shows that, in general,

$$\|e^T\|_{L^p(\mathbb{R}^+)} < e^{\|T\|_{L^p(\mathbb{R}^+)}}$$

may occur.

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