Optimal non-absolute domains for the Cesàro operator minus the identity

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Abstract We characterize the optimal non-absolute domain for the Cesàro operator minus the identity $(C - \mathrm{I})$, in the sequence space $\ell^p(\mathbb{N})$, $1 \le p \le \infty$, and compare the results obtained with the case of C, showing the different behavior in both cases. We also address this question for the Copson operator C^*.

Dedicated to the memory of Professor Guido Weiss,
an excellent advisor, a brilliant mathematician,
and an outstanding person.

1 Introduction

We are going to consider the classical Cesàro averaging operator acting on the sequence $x = \{x_j\}_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$:

$$Cx(n) = (Cx)_n = \frac{x_1 + \cdots + x_n}{n}$$

as well as the Copson operator:

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The first author acknowledges the support of the GARF Analysis group at the Complutense University of Madrid (https://www.ucm.es/garf/), where most of this work has been developed during a research stay.

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The second author has been partially supported by the Spanish Government grant PID2020-113048GB-I00, funded by MCIN/AEI/10.13039/501100011033, and Grupo UCM-970966.

$$C^*x(n) = (C^*x)_n = \sum_{k=n}^{\infty} \frac{x_k}{k}$$

(depending on each particular case, and in order to clarify the calculations, for the elements of a sequence $x \in \mathbb{R}^{\mathbb{N}}$, we will use the notations x_n or $x(n)$, preferably the first one).

Our main goal in this work is the characterization of the optimal domains for C minus the identity on the classical sequence spaces $\ell^p(\mathbb{N})$. Motivations for these considerations come twofold: on the one-hand, there is already a great interest in this topic for the Hardy operator

$$Sf(x) = \frac{1}{x}\int_0^x f(t)\,dt \quad x > 0,$$

(the continuous equivalent version of C), but for non-negative functions, as shown in [5, 7, 12, 13, 14]. In particular, if X is a rearrangement invariant (r.i.) Banach function space [1, Definition II.4.1], BFS for short, for which $S : X \to X$ is bounded, then the class of functions for which $S(|f|) \in X$ is known to be much larger than X and, in fact, not even a subspace of $(L^1 + L^\infty)(\mathbb{R}^+)$ [7, Theorem 2.6]. It is also an easy exercise to prove that the only non-negative function $f \in L^1(\mathbb{R}^+)$ such that $Sf \in L^1(\mathbb{R}^+)$ is the zero function (similarly for C and $\ell^1(\mathbb{N})$). Some consideration in the discrete setting of sequence spaces can be also found, for C and C^*, in [2, 3, 6]. On the other-hand, no much is known when positivity is dropped from the definition of the domain (see [10] for some preliminary results, dealing more with duality properties of what the author refers to as the non-absolute domain: the space of sequences x for which $Cx \in \ell^p(\mathbb{N})$).

Moreover, it is also well-known that subtracting the identity from an averaging operator provides some additional regularity and smoothness [9], which is the setting in which we are going to work. In particular, we want to consider the following problem: given a discrete BFS $X = X(\mathbb{N})$ (mostly, $X = \ell^p(\mathbb{N})$), study conditions for a general sequence x so that $(Cx - x) \in X$. That is, determine the conditions to describe the optimal domain:

$$\mathrm{Dom}[C - \mathrm{I}, X] = \{x = \{x_j\}_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : (Cx - x) \in X\}.$$

Observe that, this time, we do not assume a priori that the sequence $\{x_j\}_{j\in\mathbb{N}}$ is non-negative. It is worth noticing that $(Cx - x) \equiv 0$ if and only if x is a constant sequence. Hence, the optimal domain will always be invariant under the addition of constants (the kernel of the operator $C - \mathrm{I}$). Thus, if $C : X \to Y$ is bounded and

$$X + \mathbb{R} = \{x + c : x \in X, c \in \mathbb{R}\},$$

then, since X is r.i., we have that $X \subset Y$ and $X + \mathbb{R} \subset \mathrm{Dom}[C - \mathrm{I}, Y]$.

The paper is structured as follows: in Section 2 we start by showing some general results for a BFS X. We then establish a useful tool, in Proposition 2, for getting precise norm estimates. Our main result is Theorem 1, where we fully characterize $\mathrm{Dom}[C - \mathrm{I}, \ell^p(\mathbb{N})]$, $1 \leq p \leq \infty$. As a consequence of these results, we can see that while $\mathrm{Dom}[C, \ell^1(\mathbb{N})] \not\subset \ell^\infty(\mathbb{N})$, however $\mathrm{Dom}[C - \mathrm{I}, \ell^1(\mathbb{N})] \subset \ell^1_0(\mathbb{N}) + \mathbb{R}$. In Section 3 we briefly describe the behavior at the end-point $p = 1$, in terms of the weak-type space $\ell^{1,\infty}(\mathbb{N})$. Finally, in Section 4 we consider the study of optimal domains for the Copson operator C^* minus the identity, and show the analogous results in Theorem 2.

In what follows, we will use the standard notation $A \lesssim B$ to denote the existence of a positive constant $K > 0$ (independent of the main parameters defining A and B) such that $A \leq KB$ (analogously for the notation $A \gtrsim B$). If both $A \lesssim B$ and $A \gtrsim B$ hold true, we will write $A \approx B$.

2 Domain for Cesàro type operators on sequence spaces

We start with the following auxiliary result (see, e.g., [11, end of page 2]):

Lemma 1 *Let X, Y be two BFS and assume T is a positive operator (i.e., $Tf \geq 0$, if $f \geq 0$), such that $Tf \in Y$, whenever $f \in X$. Then, $T : X \to Y$, boundedly.*

We now prove a general result on a discrete BFS X, that completes the observation made in the previous section (we refer to [1, Definition I.2.3] for the definition of X', the associate space of X):

Lemma 2 *Let X be a discrete* BFS *and let X' be its associate space. Then, the following statements are equivalent:*

(i) $C : X \to X$ *and* $C^* : X \to X$ *are bounded.*
(ii) $\mathrm{Dom}[C - \mathrm{I}, X] = X + \mathbb{R}$ *and* $\mathrm{Dom}[C - \mathrm{I}, X'] = X' + \mathbb{R}$.
(iii) $\mathrm{Dom}[C^* - \mathrm{I}, X] = X$ *and* $\mathrm{Dom}[C^* - \mathrm{I}, X'] = X'$.

Proof We start by proving that *(i)* is equivalent to *(ii)*: As we have already observed, the fact that $C : X \to X$ is bounded, immediately gives that $X + \mathbb{R} \subset \mathrm{Dom}[C - \mathrm{I}, X]$. Since $C^* : X \to X$ is equivalent to $C : X' \to X'$, we also get $X' + \mathbb{R} \subset \mathrm{Dom}[C - \mathrm{I}, X']$.

For the reverse inclusions, if $x \in \mathrm{Dom}[C - \mathrm{I}, X]$, let $y = Cx - x \in X$. Then

$$x_2 - x_1 + 2y_2 = 0 \quad \text{and} \quad x_n - x_1 + \frac{n}{n-1} y_n = -\sum_{k=2}^{n-1} \frac{y_k}{k-1}, \ n \geq 3. \tag{1}$$

In fact, the equality is trivial for $n = 2$. If $n = 3$, then $-y_2 - 3y_3/2 = -(x_1 + x_2)/2 + x_2 - (x_1 + x_2 + x_3)/2 + 3x_3/2 = x_3 - x_1$. If we now assume (1) to hold for a given $n \geq 3$, then

$$
\begin{aligned}
-\sum_{k=2}^{n}\frac{y_k}{k-1} &= x_n - x_1 + \frac{n}{n-1}y_n - \frac{y_n}{n-1} = \frac{x_1+\cdots+x_n}{n} - x_1\\
&= x_{n+1} - x_1 + \frac{x_1+\cdots+x_{n+1}}{n} - \frac{n+1}{n}x_{n+1}\\
&= x_{n+1} - x_1 + \frac{n+1}{n}y_{n+1}.
\end{aligned}
$$

Now, observe that $C^* : X \to X$ implies that $\{1/k\}_{k\in\mathbb{N}} \in X'$. In fact, using [1, Theorem I.2.7] and [1, Corollary II.6.8]

$$
\begin{aligned}
\left\|\left\{\frac{1}{k}\right\}_{k\in\mathbb{N}}\right\|_{X'} &= \sup_{\|x\|_X=1}\sum_{k\in\mathbb{N}}\frac{|x_k|}{k} = \sup_{\|x\|_X=1}\|C^*(|x|)\|_{\ell^\infty(\mathbb{N})}\\
&\le \sup_{\|x\|_X=1}\|C^*(|x|)\|_X = \|C^*\|_{X\to X} < \infty.
\end{aligned}
$$

Thus, if we call $\sigma y(k) = y_{k+1}$ (the left shift operator), we have that $\sigma : X \to X$ and, for $n \ge 3$,

$$
\sum_{k=2}^{n-1}\frac{y_k}{k-1} = C^*(\sigma y)(1) - C^*(\sigma y)(n-1),
$$

since $y \in X$, using (1) we obtain

$$
x_n = x_1 - C^*(\sigma y)(1) - \frac{n}{n-1}y_n + C^*(\sigma y)(n-1), \tag{2}
$$

and taking into account that

$$
\left\{\frac{n+1}{n}y_{n+1} + C^*(\sigma y)(n)\right\}_{n\in\mathbb{N}} \in X^{\mathbb{N}},
$$

we finally deduce that $x \in X + \mathbb{R}$. Similarly, since $C^* : X' \to X'$ is equivalent to $C : X \to X$, reversing the role of X and X' in the previous argument we also obtain that $\mathrm{Dom}[C - \mathrm{I}, X'] \subset X' + \mathbb{R}$.

Conversely, to prove that *(ii)* implies *(i)*, if $\mathrm{Dom}[C - \mathrm{I}, X] = X + \mathbb{R}$, pick $x \in X$. Then, $Cx - x \in X$ and hence $Cx \in X$. Thus, using Lemma 1, we conclude that $C : X \to X$ is bounded. By a similar and dual argument, we can also get that $C^* : X \to X$.

We now show that *(i)* implies *(iii)*: The fact that $C^* : X \to X$ is bounded gives that $X \subset \mathrm{Dom}[C^* - \mathrm{I}, X]$. Since $C^* : X' \to X'$ we also get that $X' \subset \mathrm{Dom}[C^* - \mathrm{I}, X']$.

For the reversed inclusion, if $x \in \mathrm{Dom}[C^* - \mathrm{I}, X]$, let $y = C^*x - x \in X$. As in (1), one can prove by induction that

$$
x_n = \frac{1}{n-1}\sum_{k=1}^{n-1}y_k - y_n = (Cy)_{n-1} - y_n, \quad n \ge 2, \tag{3}
$$

with x_1 arbitrary. Hence $x \in X$. Similarly, since $C : X' \to X'$ we get also that $\mathrm{Dom}[C^* - \mathrm{I}, X'] \subset X'$.

To finish, we prove that *(iii)* implies *(i)*: If $\mathrm{Dom}[C^* - \mathrm{I}, X] = X$, pick $x \in X$. Then $C^*x - x \in X$ and hence $C^*x \in X$. Thus, using Lemma 2.1 we conclude that $C^* : X \to X$ is bounded. By a dual argument we also get that $C : X \to X$. ◻

Remark 1 We observe that, in Lemma 2, it is not true that $C : X \to X$ if and only if $\mathrm{Dom}[C - \mathrm{I}, X] = X + \mathbb{R}$, since $C : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$, but we will prove in Theorem 1 that $\mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})] \neq \ell^\infty(\mathbb{N})$.

Let us now consider the case of the Cesàro operator C on $\ell^1(\mathbb{N})$. Recall that $\ell^1_0(\mathbb{N})$ denotes the subspace of $\ell^1(\mathbb{N})$ sequences with vanishing sum. As we have mentioned above, it is well-known that C is not bounded on $\ell^1(\mathbb{N})$ and, moreover, $x \equiv 0$ is the only non-negative sequence satisfying that $Cx \in \ell^1(\mathbb{N})$. With more generality, we can prove the following (see [10] for some related results):

Proposition 1

(i) If x is non-negative sequence such that $Cx \in \ell^1(\mathbb{N})$, then $x \equiv 0$. That is,

$$\ell^1_+(\mathbb{N}) \cap \mathrm{Dom}[C, \ell^1(\mathbb{N})] = \{0\}.$$

(ii) There exists $x \in \ell^1(\mathbb{N}) \setminus \{0\}$ such that $Cx \in \ell^1(\mathbb{N})$. Moreover, any such sequence satisfies that $\sum_{n=1}^\infty x_n = 0$. That is,

$$\ell^1(\mathbb{N}) \cap \mathrm{Dom}[C, \ell^1(\mathbb{N})] = \ell^1_0(\mathbb{N}) \cap \mathrm{Dom}[C, \ell^1(\mathbb{N})] \neq \{0\}.$$

(iii) There exists $x \notin \ell^\infty(\mathbb{N})$ such that $Cx \in \ell^1(\mathbb{N})$ and there exists $y \in \ell^1_0(\mathbb{N})$ such that $Cy \notin \ell^1(\mathbb{N})$. That is,

$$\ell^1_0(\mathbb{N}) \not\subset \mathrm{Dom}[C, \ell^1(\mathbb{N})] \quad \textit{and} \quad \mathrm{Dom}[C, \ell^1(\mathbb{N})] \not\subset \ell^\infty(\mathbb{N}). \tag{4}$$

Proof The proof of *(i)* follows from the remark that, for every $j, n \in \mathbb{N}$, we have that $x_n \geq x_j \delta_j(n)$, where

$$\delta_j(n) = \begin{cases} 0, & \text{if } j \neq n \\ 1 & \text{if } j = n. \end{cases} \tag{5}$$

Now, for $j \in \mathbb{N}$ fixed:

$$Cx(n) \geq x_j C\delta_j(n) = \begin{cases} 0, & \text{if } n \leq j-1, \\ \dfrac{x_j}{n}, & \text{if } n \geq j. \end{cases}$$

Thus, $Cx \in \ell^1(\mathbb{N})$ if and only if $x_j = 0$, for every $j \in \mathbb{N}$.

In order to find $x \in \ell^1(\mathbb{N})$ such that $Cx \in \ell^1(\mathbb{N})$, we observe that if $y = Cx$, then,

$$x_1 = y_1, x_2 = 2y_2 - y_1, \ldots, x_k = ky_k - (k-1)y_{k-1},\ k \in \{2, 3, \ldots\}. \tag{6}$$

Therefore, the series $\sum_{n=1}^{\infty} x_n$ converges, if and only if there exists $\lim_{k\to\infty} ky_k$. Now, since $y \in \ell^1(\mathbb{N})$, then $\lim_{k\to\infty} ky_k = 0$. In fact, if $\lim_{k\to\infty} |ky_k| = l > 0$, then we could find $k_0 \in \mathbb{N}$ such that $|y_k| > \frac{l}{2k}$, for $k \geq k_0$, contradicting the fact that $y \in \ell^1(\mathbb{N})$. Therefore, we have proved that $\ell^1(\mathbb{N}) \cap \mathrm{Dom}[C, \ell^1(\mathbb{N})] = \ell_0^1(\mathbb{N}) \cap \mathrm{Dom}[C, \ell^1(\mathbb{N})]$.

To finish the proof of *(ii)*, take a positive sequence $y \in \ell^1(\mathbb{N})$ so that $\{ky_k\}_{k\in\mathbb{N}}$ decreases and $\lim_{k\to\infty} ky_k = 0$ (e.g., $y_k = 1/k^2$), and define x as in (6). In this case,

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} ky_k = 0.$$

Moreover, $x_k \leq 0$, for $k \in \{2, 3, \dots\}$ and hence

$$\sum_{n=1}^{\infty} |x_n| = |x_1| - \sum_{n=2}^{\infty} x_n = 2y_1 < \infty.$$

To prove *(iii)*, we consider

$$x_j = \begin{cases} k, & \text{if } j = 2^k, k \in \mathbb{N}, \\ -k, & \text{if } j = 2^k + 1, k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $x \notin \ell^\infty(\mathbb{N})$ and

$$Cx(j) = \begin{cases} \dfrac{k}{2^k}, & \text{if } j = 2^k, k \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

which is clearly in $\ell^1(\mathbb{N})$. Finally, with

$$y_1 = \frac{-1}{\log 2} \quad \text{and} \quad y_n = \frac{-1}{\log(n+1)} + \frac{1}{\log n} \geq 0, \quad n \geq 2$$

we have that $y \in \ell_0^1(\mathbb{N})$ and

$$\{Cy(n)\}_{n\in\mathbb{N}} = \left\{\frac{-1}{n\log(n+1)}\right\}_{n\in\mathbb{N}} \notin \ell^1(\mathbb{N}).$$

We will now fix our attention to the study of the optimal domain in all different $\ell^p(\mathbb{N})$ spaces, $1 \leq p \leq \infty$. We will see that $C - \mathrm{I}$ enjoys a different behavior at the end-points $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$. Recall that, given a weight w (a sequence with $w_n > 0$, for every $n \in \mathbb{N}$), and $1 \leq p < \infty$, we define

$$\ell^p(w, \mathbb{N}) = \left\{\{x_j\}_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|x\|_{\ell^p(w,\mathbb{N})} \stackrel{\text{def}}{=} \left(\sum_{j=1}^{\infty} |x_j|^p w_j\right)^{1/p} < \infty\right\},$$

and analogously if $p = \infty$. With this notation, $\ell^p(\mathbb{N}) = \ell^p(\mathbf{1}, \mathbb{N})$, where $\mathbf{1}$ is the constant weight $\mathbf{1}_n = 1$, for every $n \in \mathbb{N}$. Similarly,

$$\ell_0^p(w, \mathbb{N}) = \ell^p(w, \mathbb{N}) \cap \ell_0^1(\mathbb{N}),$$

with $\|x\|_{\ell_0^p(w,\mathbb{N})} = \|x\|_{\ell^p(w,\mathbb{N})} + \|x\|_{\ell^1(\mathbb{N})}$.

We need first a previous result (which follows using the Closed Graph Theorem), proving the continuity of several inclusions. Observe that, with the previous definition, $\ell_0^p(w, \mathbb{N})$ is a Banach space (i.e., the cancellation property holds true in the limit, since $\ell_0^1(\mathbb{N})$ is closed in $\ell^1(\mathbb{N})$).

Proposition 2 *Let w be a weight. Then,*

(i) $\ell_0^1(w, \mathbb{N}) \subset \mathrm{Dom}[C - \mathrm{I}, \ell^1(\mathbb{N})] \iff \|Cx - x\|_{\ell^1(\mathbb{N})} \lesssim \|x\|_{\ell^1(w,\mathbb{N})} + \|x\|_{\ell_0^1(\mathbb{N})}$.

(ii) $(\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell_0^1(\mathbb{N})) \subset \ell_0^1(w, \mathbb{N}) \iff \|x\|_{\ell_0^1(w,\mathbb{N})} \lesssim \|Cx\|_{\ell^1(\mathbb{N})} + \|x\|_{\ell_0^1(\mathbb{N})}$.

(iii) $\ell^\infty(w, \mathbb{N}) \subset \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})] \iff \|Cx - x\|_{\ell^\infty(\mathbb{N})} \lesssim \|x\|_{\ell^\infty(w,\mathbb{N})}$.

Proof Observe that, in all *(i), (ii),* and *(iii)*, the right-hand side immediately implies the embeddings on the left. Thus, it suffices to prove the implications "$\Rightarrow$".

To prove *(i)*, let us see the continuity of the embedding:

$$\|Cx - x\|_{\ell^1(\mathbb{N})} \lesssim \|x\|_{\ell^1(w,\mathbb{N})} + \|x\|_{\ell_0^1(\mathbb{N})}.$$

Using the Closed Graph Theorem, it suffices to show that

$$\left.\begin{array}{r} x^N \xrightarrow{\ell_0^1(w,\mathbb{N})} 0 \\ Cx^N - x^N \xrightarrow{\ell^1(\mathbb{N})} y \end{array}\right\} \Longrightarrow y \equiv 0. \tag{7}$$

Now, since

$$\sum_{n=1}^{\infty} |C(x^N)(n) - x^N(n) - y_n| \to 0 \text{ and } \sum_{n=1}^{\infty} |x^N(n)| w_n \to 0, \text{ as } N \to \infty,$$

then $|C(x^N)(n) - x^N(n) - y_n| \to 0$ and $|x^N(n)| \to 0$, for every $n \in \mathbb{N}$. Hence,

$$|C(x^N)(n) - y_n| \to 0, \text{ as } N \to \infty \text{ and for every } n \in \mathbb{N}.$$

Finally, for a fixed $n \in \mathbb{N}$,

$$C(x^N)(n) = \frac{1}{n} \sum_{k=1}^{n} x^N(k) \longrightarrow y_n = 0,$$

which proves (7).

To prove *(ii)*, let us see the continuity of the embedding:

$$\|x\|_{\ell_0^1(w,\mathbb{N})} \lesssim \|Cx\|_{\ell^1(\mathbb{N})} + \|x\|_{\ell_0^1(\mathbb{N})}.$$

Using Proposition 1 *(ii)* and the Closed Graph Theorem, it suffices to show that

$$\left.\begin{array}{l} x^N \xrightarrow{\ell_0^1(\mathbb{N})} 0 \\ x^N \xrightarrow{\ell_0^1(w,\mathbb{N})} y \end{array}\right\} \Longrightarrow y \equiv 0,$$

but this follows easily from the fact that $|x_n^N - y_n|w_n \to 0$ and $|x_n^N| \to 0$, as $N \to \infty$ and for very $n \in \mathbb{N}$.

Finally, to prove *(iii)*, let us see the continuity of the embedding:

$$\|Cx - x\|_{\ell^\infty(\mathbb{N})} \lesssim \|x\|_{\ell^\infty(w,\mathbb{N})}. \tag{8}$$

Using the Closed Graph Theorem, it suffices to show that

$$\left.\begin{array}{r} x^N \xrightarrow{\ell^\infty(w,\mathbb{N})} 0 \\ Cx^N - x^N \xrightarrow{\ell^\infty(\mathbb{N})} y \end{array}\right\} \Longrightarrow y \equiv 0. \tag{9}$$

Given the sequence $\{x^N\}_{N\in\mathbb{N}} \subset \ell^\infty(w,\mathbb{N})$, fix $n \in \mathbb{N}$. Then, for every $N \in \mathbb{N}$,

$$|y_n| \le |y_n - (Cx^N)(n) + x^N(n)| + |(Cx^N)(n)| + |x^N(n)|.$$

Observe that, given $\varepsilon > 0$, we can find $N_1 \in \mathbb{N}$ such that, for all $N \ge N_1$,

$$|y_n - (Cx^N)(n) + x^N(n)| \le \|y - (Cx^N - x^N)\|_{\ell^\infty(\mathbb{N})} < \frac{\varepsilon}{3}.$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that, for all $N \ge N_2$,

$$|x^N(n)| = \frac{1}{w_n}|w_n x^N(n)| \le \frac{1}{w_n}\|x^N\|_{\ell^\infty(w,\mathbb{N})} < \frac{\varepsilon}{3}.$$

Finally, there exists $N_3 \in \mathbb{N}$ such that, for all $N \ge N_3$,

$$\begin{aligned} |(Cx^N)(n)| &\le \frac{1}{\min\limits_{1\le j\le n}\{w_j\}} \frac{w_1|x^N(1)| + \cdots + w_n|x^N(n)|}{n} \\ &\le \frac{1}{\min\limits_{1\le j\le n}\{w_j\}} \|x^N\|_{\ell^\infty(w,\mathbb{N})} < \frac{\varepsilon}{3}. \end{aligned}$$

Thus, with $n \in \mathbb{N}$ fixed, given $\varepsilon > 0$, if $N_0 = \max\{N_1, N_2, N_3\}$, we have that for every $N \geq N_0$,

$$|y_n| < \varepsilon, \qquad \text{for every } \varepsilon > 0,$$

and, hence, $y_n = 0$, for every $n \in \mathbb{N}$, which proves (9). $\square$

Theorem 1 (Optimal domain for $C - \mathrm{I}$ on $\ell^p(\mathbb{N})$, $1 \leq p \leq \infty$)

(i) Case $p = 1$*:* $\mathrm{Dom}[C - \mathrm{I}, \ell^1(\mathbb{N})] = (\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N})) + \mathbb{R}$. *In addition,*

a. $\ell^1_0(w, \mathbb{N}) \subset (\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N})) \iff w_n \gtrsim \log(n+1)$. *Hence, the logarithmic space* $\ell^1_0(\log(n+1), \mathbb{N})$ *is the largest* $\ell^1_0(w, \mathbb{N})$ *space contained in* $(\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N}))$ *and, moreover,*

$$\ell^1_0(\log(n+1), \mathbb{N}) \subsetneq (\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N})).$$

b. $(\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N})) \subset \ell^1_0(w, \mathbb{N}) \iff w_n \lesssim 1$. *Hence,* $\ell^1_0(\mathbb{N})$ *is the smallest* $\ell^1_0(w, \mathbb{N})$ *space containing* $(\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N}))$ *and, moreover,*

$$(\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N})) \subsetneq \ell^1_0(\mathbb{N}).$$

(ii) Case $1 < p < \infty$*:* $\mathrm{Dom}[C - \mathrm{I}, \ell^p(\mathbb{N})] = \ell^p(\mathbb{N}) + \mathbb{R}$.

(iii) Case $p = \infty$*:*

a. $\ell^\infty(w, \mathbb{N}) \subset \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})] \iff w_n \gtrsim 1$. *Hence,* $\ell^\infty(\mathbb{N})$ *is the largest* $\ell^\infty(w, \mathbb{N})$ *space contained in* $\mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})]$ *and, moreover,*

$$\ell^\infty(\mathbb{N}) \subsetneq \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})].$$

b. $\mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})] \subset \ell^\infty(w, \mathbb{N}) \iff w_n \lesssim 1/\log(n+1)$. *Hence,* $\ell^\infty(\{1/\log(n+1)\}_n, \mathbb{N})$ *is the smallest weighted* $\ell^\infty(w, \mathbb{N})$ *space containing* $\mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})]$ *and, moreover,*

$$\mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})] \subsetneq \ell^\infty(\{1/\log(n+1)\}_n, \mathbb{N}).$$

Proof We start by proving *(i)*. Using the equality:

$$\frac{x_1 + \cdots + x_{n+1}}{n+1} - x_{n+1} = \frac{n}{n+1}\left(\frac{x_1 + \cdots + x_n}{n} - x_{n+1}\right),$$

the hypothesis $Cx - x \in \ell^1(\mathbb{N})$ is equivalent to $Cx - \sigma x \in \ell^1(\mathbb{N})$, where, as before, $\sigma x(n) = x_{n+1}$ is the left shift operator. For convenience, we will then work with this second condition.

Assume now that $Cx - \sigma x \in \ell^1(\mathbb{N})$. Let us first see that, necessarily, x is a convergent sequence (to some value $\alpha \in \mathbb{R}$): We define the sequence $d = Cx - \sigma x$. Then, inductively as in (1), we have that

$$x_2 = x_1 - d_1, x_3 = x_1 - \frac{d_1}{2} - d_2, \dots, x_{n+1} = x_1 - \frac{d_1}{2} - \dots - \frac{d_{n-1}}{n} - d_n.$$

Therefore,

$$\lim_{n\to\infty} x_n = x_1 - \sum_{j=1}^{\infty} \frac{d_j}{j+1}.$$

Setting $\alpha = x_1 - \sum_{j=1}^{\infty} d_j/(j+1)$, let us now prove that $x-\alpha = \{x_j - \alpha\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$. In fact,

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n - \alpha| &= |x_1 - \alpha| + |x_2 - \alpha| + \sum_{n=3}^{\infty} \Big| x_1 - \sum_{j=1}^{n-2} \frac{d_j}{j+1} - d_{n-1} - \Big(x_1 - \sum_{j=1}^{\infty} \frac{d_j}{j+1} \Big) \Big| \\ &= |x_1 - \alpha| + |x_2 - \alpha| + \sum_{n=3}^{\infty} \Big| \sum_{j=n-1}^{\infty} \frac{d_j}{j+1} - d_{n-1} \Big| \\ &\le |x_1 - \alpha| + |x_2 - \alpha| + \sum_{n=3}^{\infty} \sum_{j=n-1}^{\infty} \frac{|d_j|}{j+1} + \sum_{n=3}^{\infty} |d_{n-1}| \\ &\le |x_1 - \alpha| + |x_2 - \alpha| + \sum_{j=2}^{\infty} |d_j| + \sum_{n=3}^{\infty} |d_{n-1}| < \infty.\end{aligned}$$

Before proving that $x - \alpha \in \ell^1_0(\mathbb{N})$ (i.e., the cancellation property), let us show that $x - \alpha \in \mathrm{Dom}[C, \ell^1(\mathbb{N})]$:

$$Cx - x = C(x-\alpha) - (x-\alpha) \in \ell^1(\mathbb{N}) \text{ and } x - \alpha \in \ell^1(\mathbb{N}) \Rightarrow C(x-\alpha) \in \ell^1(\mathbb{N}),$$

which means $x - \alpha \in \mathrm{Dom}[C, \ell^1(\mathbb{N})]$. Now, using Proposition 1-*(ii)*, we also conclude that $x - \alpha \in \ell^1_0(\mathbb{N})$. Thus, $\mathrm{Dom}[C - \mathrm{I}, \ell^1(\mathbb{N})] \subset (\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N})) + \mathbb{R}$.

Conversely, if $x - \alpha \in (\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N}))$, then:

$$C(x-\alpha) \in \ell^1(\mathbb{N}) \text{ and } (x-\alpha) \in \ell^1(\mathbb{N}) \Rightarrow Cx - x = C(x-\alpha) - (x-\alpha) \in \ell^1(\mathbb{N}),$$

i.e., $x \in \mathrm{Dom}[C - \mathrm{I}, \ell^1(\mathbb{N})]$. Observe that, by (4), the intersection in *(i)* cannot be simplified.

Let us prove *(i)-(a)*. Using Proposition 2 *(i)*, if we assume that $\ell^1_0(w, \mathbb{N}) \subset (\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell^1_0(\mathbb{N}))$, then, with $x^N = \delta_1 - \delta_N$, $N \ge 2$ (see (5)), we have

$$\log(N+1) \approx \|Cx^N - x^N\|_{\ell^1(\mathbb{N})} \lesssim \|x^N\|_{\ell^1(w,\mathbb{N})} + \|x^N\|_{\ell^1(\mathbb{N})} = w_1 + w_N + 2 \lesssim w_N.$$

Conversely, since $\ell^1_0(\log(n+1), \mathbb{N}) \subset \ell^1_0(\mathbb{N})$, using the first equality of *(i)*, we only need to show that $\ell^1_0(\log(n+1), \mathbb{N}) \subset \mathrm{Dom}[C, \ell^1(\mathbb{N})]$: If $x \in \ell^1_0(\log(n+1), \mathbb{N})$, then

$$\sum_{n=1}^{\infty}|Cx(n)| = \sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=n+1}^{\infty}x_k\right| \le \sum_{k=1}^{\infty}|x_k|\sum_{n=1}^{k}\frac{1}{n} \approx \sum_{k=1}^{\infty}|x_k|\log(k+1) < \infty.$$

To finish this part, it suffices to observe that, if

$$x_j = \begin{cases} \dfrac{1}{j\log^2 j}, & \text{if } j \text{ is even} \\ \dfrac{-1}{(j+1)\log^2(j+1)}, & \text{if } j \text{ is odd} \end{cases}$$

then, $\{x_j\}_{j\in\mathbb{N}} \in \ell_0^1(\mathbb{N}) \setminus \ell_0^1(\log(n+1), \mathbb{N})$, and

$$Cx(j) = \begin{cases} 0, & \text{if } j \text{ is even} \\ \dfrac{-1}{j(j+1)\log^2(j+1)}, & \text{if } j \text{ is odd} \end{cases}$$

which clearly satisfies that $Cx \in \ell^1(\mathbb{N})$. Hence,

$$\ell_0^1(\log(n+1), \mathbb{N}) \subsetneq (\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell_0^1(\mathbb{N})).$$

Let us now see *(i)-(b)*. If $(\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell_0^1(\mathbb{N})) \subset \ell_0^1(w, \mathbb{N})$, using Proposition 2 *(ii)*, with the sequence $x^N = \delta_N - \delta_{N+1}$, we conclude that $Cx^N = \frac{1}{N}\delta_N$, and

$$w_N \le \|x^N\|_{\ell_0^1(w,\mathbb{N})} = w_N + w_{N+1} \lesssim \|Cx^N\|_{\ell^1(\mathbb{N})} + \|x^N\|_{\ell_0^1(\mathbb{N})} = (1 + 1/N) \lesssim 1.$$

Conversely, if $w_n \lesssim 1$, then trivially

$$(\mathrm{Dom}[C, \ell^1(\mathbb{N})] \cap \ell_0^1(\mathbb{N})) \subset \ell_0^1(\mathbb{N}) \subset \ell_0^1(w, \mathbb{N}).$$

That the embedding is strict follows from (4).

The proof of *(ii)* is a direct consequence of Lemma 2 and the fact that, for $1 < p < \infty$, both operators $C, C^* : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ are bounded (as was proved in [8, Theorems 326 and 331] and [4]).

We now consider *(iii)*, the end-point case $p = \infty$. It is clear that if $w \gtrsim 1$, then $\ell^\infty(w, \mathbb{N}) \subset \ell^\infty(\mathbb{N}) \subset \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})]$. If we now assume the embedding $\ell^\infty(w, \mathbb{N}) \subset \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})]$, then necessarily $w_n > 0$, for every $n \in \mathbb{N}$, since, otherwise, we could find $x \in \ell^\infty(w, \mathbb{N})$, with $x_n = \infty$, which clearly does not belong to $\mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})]$ (we need $x \in \ell^1_{\mathrm{loc}}(\mathbb{N})$ to define C). Let us see that, in fact, $\inf_{n\in\mathbb{N}} w_n > 0$, which is equivalent to $w \gtrsim 1$.

Take now $x = e_n = \{0, \overset{n-1}{\ldots}, 0, 1, 0, \dots\}$, $n \ge 2$. Then, using (8),

$$\begin{aligned}w(n) &\gtrsim \|Ce_n - e_n\|_{\ell^\infty(\mathbb{N})}\\ &\approx \left\| \left\{0, \overset{n-1}{\ldots}, 0, \frac{1}{n}, \frac{1}{n+1}, \ldots \right\} - \{0, \overset{n-1}{\ldots}, 0, 1, 0, \ldots\} \right\|_{\ell^\infty(\mathbb{N})} \\ &\gtrsim \left|\frac{1}{n} - 1\right| \gtrsim \frac{1}{2}.\end{aligned} \tag{10}$$

Hence, $w \gtrsim 1$ and $\ell^\infty(\mathbb{N})$ is the largest $\ell^\infty(w, \mathbb{N})$ space contained in $\mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})]$. To finish the proof of *(iii)-(a)*, we pick $x = \{\log n\}_{n\in\mathbb{N}} \notin \ell^\infty(\mathbb{N})$. On the other hand,

$$(Cx)(n) = \frac{1}{n}\log(n!) \quad \text{ and } \quad (Cx)(n) - x_n = \log\left(\frac{(n!)^{1/n}}{n}\right)$$

But,

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{n+1}{(n+1)(1+1/n)^n} = \frac{1}{e}.$$

Thus, $(Cx - x) \in \ell^\infty(\mathbb{N})$; i.e.,

$$x = \{\log n\}_{n\in\mathbb{N}} \in \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})] \setminus \ell^\infty(\mathbb{N}). \tag{11}$$

We now consider the proof of *(iii)-(b)*. Since $w_n \lesssim 1/\log(n+1)$ implies that $\ell^\infty(\{1/\log(n+1)\}_n, \mathbb{N}) \subset \ell^\infty(w, \mathbb{N})$, it suffices to see that $\mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})] \subset \ell^\infty(\{1/\log(n+1)\}_n, \mathbb{N})$. In fact, if $x \in \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})]$ and we set $y = Cx - x \in \ell^\infty(\mathbb{N})$, using (1) we obtain, for $n \geq 2$:

$$\begin{aligned}\frac{|x_n|}{\log(n+1)} &= \left| x_1 - \frac{n}{n-1} y_n + \sum_{k=2}^{n-1} \frac{y_k}{k-1} \right| \frac{1}{\log(n+1)}\\ &\lesssim \left(|x_1| + |y_n| + \sum_{k=2}^{n-1} \frac{|y_k|}{k-1} \right) \frac{1}{\log(n+1)}\\ &\lesssim |x_1| + \|y\|_{\ell^\infty(\mathbb{N})} (1 + \log n) \frac{1}{\log(n+1)} \lesssim |x_1| + \|y\|_{\ell^\infty(\mathbb{N})}.\end{aligned}$$

Thus, $x \in \ell^\infty(\{1/\log(n+1)\}_n, \mathbb{N})$. Let us see that the embedding is strict. Consider

$$x = \{x_n\}_{n\in\mathbb{N}} = \{(-1)^n \log n\}_{n\in\mathbb{N}} \in \ell^\infty\left(\frac{1}{\log(n+1)}\right).$$

Then,

$$\frac{x_1+\cdots+x_{2n}}{2n}-x_{2n}=\frac{-\log\left(\prod_{j=1}^{n}(2j-1)\right)+\log\left(\prod_{j=1}^{n}2j\right)}{2n}-\log 2n$$
$$=\frac{1}{2n}\log\left(\frac{2^n n!}{\prod_{j=1}^{n}(2j-1)}\right)-\log 2n$$
$$=\log\left(\frac{2^n n!}{(2n)^{2n}\prod_{j=1}^{n}(2j-1)}\right)^{\frac{1}{2n}}.$$

But,

$$\frac{\frac{2^{n+1}(n+1)!}{(2n+2)^{2n+2}\prod_{j=1}^{n+1}(2j-1)}}{\frac{2^n n!}{(2n)^{2n}\prod_{j=1}^{n}(2j-1)}}=2\frac{(2n)^{2n}}{(2n+2)^{2n+2}}\frac{n+1}{2n+1}$$
$$=2\frac{n+1}{2n+1}\frac{1}{(2n+2)^2}\frac{1}{(1+1/n)^{2n}}\underset{n\to\infty}{\longrightarrow}0.$$

Hence,

$$(Cx-x)(2n)\underset{n\to\infty}{\longrightarrow}-\infty \quad \text{and} \quad x\notin \mathrm{Dom}[C-\mathrm{I},\ell^\infty(\mathbb{N})]. \tag{12}$$

Finally, let us prove that $\ell^\infty(\{1/\log(n+1)\}_n,\mathbb{N})$ is the smallest $\ell^\infty(w,\mathbb{N})$ space containing $\mathrm{Dom}[C-\mathrm{I},\ell^\infty(\mathbb{N})]$. In fact, as we have already observed in (11), $x=\{\log n\}_{n\in\mathbb{N}}\in \mathrm{Dom}[C-\mathrm{I},\ell^\infty(\mathbb{N})]$. Therefore, if $x\in\ell^\infty(w,\mathbb{N})$, then

$$w_n\lesssim\frac{1}{\log(n+1)}\quad\Longrightarrow\quad \ell^\infty\left(\frac{1}{\log(n+1)}\right)\subset\ell^\infty(w,\mathbb{N}).$$

Remark 2 As a consequence of Theorem 1, we can prove that, for all values $1\le p\le\infty$, $\mathrm{Dom}[C-\mathrm{I},\ell^p(\mathbb{N})]$ is not a space satisfying the lattice property (i.e., if $|y|\le|x|$ and x belongs to the space, so does y). In fact, if $p=1$,

$$x=e_1-e_2\in\mathrm{Dom}[C-\mathrm{I},\ell^1(\mathbb{N})],\ \text{but } y=|e_1-e_2|=e_1+e_2\notin\mathrm{Dom}[C-\mathrm{I},\ell^1(\mathbb{N})].$$

For $1<p<\infty$, it suffices to consider

$$x = \left\{\frac{1}{n} + 1\right\}_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}) + \mathbb{R}, \text{ but } y = \left\{(-1)^n\left(\frac{1}{n} + 1\right)\right\}_{n\in\mathbb{N}} \notin \ell^p(\mathbb{N}) + \mathbb{R}.$$

Similarly, as proved in (11) and (12), for $p = \infty$,

$$\{\log n\}_{n\in\mathbb{N}} \in \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})], \text{ but } \{(-1)^n \log n\}_{n\in\mathbb{N}} \notin \mathrm{Dom}[C - \mathrm{I}, \ell^\infty(\mathbb{N})].$$

3 Weak-type estimates

We are now going to consider some extensions, at the end-point $p = 1$, of the classical weak-type estimates for the Cesàro operator. In particular, we will study the domain $\mathrm{Dom}[C - \mathrm{I}, \ell^{p,\infty}]$, where, for $1 \le p < \infty$ and x^* being the decreasing rearrangement of the sequence x,

$$\ell^{p,\infty}(\mathbb{N}) = \left\{x = \{x_j\}_{j\in\mathbb{N}} : \|x\|_{\ell^{p,\infty}(\mathbb{N})} \stackrel{\text{def}}{=} \sup_{n\in\mathbb{N}} n^{1/p} x_n^* < \infty\right\}.$$

Let us first observe that, if $1 < p < \infty$, since both C and C^* are bounded in $\ell^{p,\infty}(\mathbb{N})$, and using Lemma 2, then

$$\mathrm{Dom}[C - \mathrm{I}, \ell^{p,\infty}(\mathbb{N})] = \ell^{p,\infty}(\mathbb{N}) + \mathbb{R},$$

as in Theorem 1 *(ii)*. Now, if $p = 1$, we have the following results:

Proposition 3 *Let w be a weight in $\mathbb{N}$. Then,*

$$\ell^1(w, \mathbb{N}) + \mathbb{R} \subset \mathrm{Dom}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})] \iff w_n \gtrsim 1.$$

Hence, $\ell^1(\mathbb{N})$ is the largest $\ell^1(w, \mathbb{N})$ space contained in $\mathrm{Dom}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})]$ *and, moreover,*

$$\ell^1(\mathbb{N}) + \mathbb{R} \subsetneq \mathrm{Dom}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})].$$

Proof Since $w_n \gtrsim 1$ implies that $\ell^1(w, \mathbb{N}) \subset \ell^1(\mathbb{N})$, using that $\ell^1(\mathbb{N}) \subset \ell^{1,\infty}(\mathbb{N})$ and

$$\|Cx\|_{\ell^{1,\infty}(\mathbb{N})} \le \|Cx^*\|_{\ell^{1,\infty}(\mathbb{N})} = \sup_{n\in\mathbb{N}} n\left(\frac{1}{n}\sum_{k=1}^n x_k^*\right) = \|x^*\|_{\ell^1(\mathbb{N})} = \|x\|_{\ell^1(\mathbb{N})},$$

we finally get that $\ell^1(w, \mathbb{N}) + \mathbb{R} \subset \mathrm{Dom}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})]$.

Conversely, if we fix $N \ge 2$ and pick $x = \delta_N$, then as in (10),

$$\|C\delta_N - \delta_N\|_{\ell^{1,\infty}(\mathbb{N})} = \max\left\{1 - 1/N, \sup_{k\ge 2} \frac{k}{k+N}\right\} = 1 \lesssim w_N.$$

To prove that the inclusion is strict, it suffices to consider the alternating sequence $x = \{(-1)^n/n\}_{n\in\mathbb{N}} \notin \ell^1(\mathbb{N}) + \mathbb{R}$. Then

$$|(Cx - x)(n)| = \frac{1}{n}\left|\sum_{k=1}^{n} \frac{(-1)^k}{k} - (-1)^n\right| \lesssim \frac{1}{n}.$$

Thus, since $\{1/n\}_{n\in\mathbb{N}} \in \ell^{1,\infty}(\mathbb{N})$, then $(Cx - x) \in \ell^{1,\infty}(\mathbb{N})$ and we conclude the result. $\square$

We end this section with some further results dealing with $\mathrm{Dom}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})]$, and give also a complete description when restricted to the cone of decreasing sequences. We begin with some useful definitions and an interesting weak-type estimate for the Copson operator C^*:

Definition 1 Given a positive sequence (a weight) w, we define the weighted weak-type space

$$\ell^{1,\infty}(w, \mathbb{N}) = \{x = \{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|x\|_{\ell^{1,\infty}(w,\mathbb{N})} = \sup_{n\in\mathbb{N}} w_n x_n^* < \infty\}.$$

Observe that, with this definition, $\ell^{1,\infty}(\mathbb{N}) = \ell^{1,\infty}(\{n\}_{n\in\mathbb{N}}, \mathbb{N})$. Also, for reasons that will be clarified in the next proposition, we have that

$$\ell^{1,\infty}(\mathbb{N}) \subset \ell^{1,\infty}\left(\left\{\frac{n}{1+\log n}\right\}_{n\in\mathbb{N}}, \mathbb{N}\right) \subset \ell^p(\mathbb{N}), \text{ for } p > 1.$$

Definition 2 Given a sequence space X, we denote by

$$X_{\mathrm{dec}} = \{x = (x_n)_n \in X : x_n \geq 0 \text{ and } x_n \geq x_{n+1},\ n \in \mathbb{N}\}.$$

Lemma 3 *Let $p > 0$. Then, $\ell^{1,\infty}(\mathbb{N}) \subset \ell^p(w, \mathbb{N})$, with embedding constant $A > 0$, if and only if $w \in \ell^\infty(\mathbb{N})$. Moreover*

$$\max\left\{\|w\|_\infty^{1/p}, \left(\sum_{j=1}^{\infty} \frac{w_j}{j}\right)^{1/p}\right\} \leq A \leq \left(\sum_{j=1}^{\infty} \frac{w_j^*}{j}\right)^{1/p}.$$

In particular, if w is a decreasing weight, then $\ell^{1,\infty}(\mathbb{N}) \subset \ell^p(w, \mathbb{N})$, with embedding constant $A > 0$, if and only if $A = \left(\sum_{j=1}^{\infty} \frac{w_j}{j}\right)^{1/p} < \infty$.

Proof For the necessity part, taking $x = \delta_j$, we clearly get that

$$\|x\|_{\ell^p(w,\mathbb{N})} = w_j^{1/p} \leq A\|x\|_{1,\infty} = A.$$

Thus, $\|w\|_\infty^{1/p} \leq A$. Now, with $x_n = 1/n$, $\|x\|_{1,\infty} = 1$ and

$$\|x\|_{\ell^p(w,\mathbb{N})} = \left(\sum_{j=1}^{\infty} \frac{w_j}{j}\right)^{1/p} \leq A.$$

Conversely, using Hardy and Littlewood's inequality [1, Theorem II.2.2],

$$\|x\|_{\ell^p(w,\mathbb{N})} = \left(\sum_{j=1}^{\infty} |x_j|^p w_j\right)^{1/p} \le \left(\sum_{j=1}^{\infty} (x_j^*)^p w_j^*\right)^{1/p} \le \|x\|_{1,\infty}\left(\sum_{j=1}^{\infty} \frac{w_j^*}{j}\right)^{1/p}.$$

Proposition 4

(i) $C^* : \ell^{1,\infty}(\mathbb{N}) \longrightarrow \ell^{1,\infty}\left(\left\{\frac{n}{1+\log n}\right\}_{n\in\mathbb{N}}, \mathbb{N}\right)$.

(ii) $C^* : \ell^{1,\infty}(\mathbb{N}) \not\longrightarrow \ell^{1,\infty}(\mathbb{N})$.

(iii) $C^* : \ell^{1,\infty}(\mathbb{N})_{\text{dec}} \longrightarrow \ell^{1,\infty}(\mathbb{N})$.

Proof We start with *(i)*. We observe that, if $x \in \ell^{1,\infty}(\mathbb{N})$, then the series $\sum_{k=1}^{\infty} \frac{x_k}{k}$ converges (absolutely) and $C^*(x)$ is well defined. Now,

$$\left|\sum_{k=n}^{\infty} \frac{x_k}{k}\right| \le \sum_{k=n}^{\infty} \frac{|x_k|}{k}$$

and, hence,

$$\left(\sum_{k=n}^{\infty} \frac{x_k}{k}\right)^* \le \sum_{k=n}^{\infty} \frac{|x_k|}{k}.$$

Now, using Lemma 3, with $w = (\overbrace{0, ..., 0}^{n-1}, 1/n, 1/(n+1), ...)$ and $p = 1$, we have that $w^* = \{1/(k+n-1)\}_{k\in\mathbb{N}}$ and

$$\sum_{k=n}^{\infty} \frac{|x_k|}{k} = \|x\|_{\ell^1(w,\mathbb{N})} \le \sum_{k=1}^{\infty} \frac{1}{k(k+n-1)} \|x\|_{1,\infty} \lesssim \frac{1+\log n}{n} \|x\|_{1,\infty}.$$

Finally,

$$\|C^*(x)\|_{\ell^{1,\infty}\left(\left\{\frac{n}{1+\log n}\right\}_{n\in\mathbb{N}},\mathbb{N}\right)} = \sup_{n\in\mathbb{N}} \frac{n}{1+\log n}\left(\sum_{k=n}^{\infty} \frac{x_k}{k}\right)^* \lesssim \|x\|_{1,\infty}.$$

The proof of *(ii)* goes as follows. For a fixed odd number $m \in \mathbb{N}$, we consider the sequence $x^m = (1/m, 1/(m-1), ..., 1, 0, ..., 0, ...)$ and set $j = (m+1)/2$. Then, $(x^m)^* = (1, 1/2, ..., 1/m, 0, ..., 0...)$ and $\|x^m\|_{1,\infty} = 1$. On the other hand, with the same value of $j = (m+1)/2$,

$$\begin{aligned}\|C^* x^m\|_{1,\infty} \ge jC^* x^m(j) &= \frac{m+1}{2} \sum_{k=(m+1)/2}^{m} \frac{1}{k(m-k+1)} \\ &= \frac{m+1}{2}\frac{1}{m+1}\left(\sum_{k=(m+1)/2}^{m} \frac{1}{k} + \sum_{k=1}^{(m+1)/2} \frac{1}{k}\right) \gtrsim \sum_{k=1}^{(m+1)/2} \frac{1}{k} \underset{m\to\infty}{\longrightarrow} \infty.\end{aligned}$$

To finish, if we now select $y \in \ell^{1,\infty}(\mathbb{N})_{\mathrm{dec}}$, then $(C^*y)^*(n) \le C^*(|y|)(n) = C^*(y^*)(n)$ and

$$n(C^*y)^*(n) \le n \sum_{k=n}^{\infty} \frac{\|y\|_{1,\infty}}{k^2} \lesssim \|y\|_{1,\infty}.$$

Remark 3 We observe that Proposition 4 *(ii)* shows that $\ell^{1,\infty}(\mathbb{N})$ is not an interpolation quasi-Banach space between $\ell^1(\mathbb{N})$ and $\ell^p(\mathbb{N})$, for every $1 < p < \infty$.

Proposition 5

(i) $\mathrm{Dom}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})] \subset \ell^{1,\infty}\left(\left\{\frac{n}{1+\log n}\right\}_{n\in\mathbb{N}}, \mathbb{N}\right) + \mathbb{R}$.

(ii) $\mathrm{Dom}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})] \not\subset \ell^{1,\infty}(\mathbb{N}) + \mathbb{R}$.

(iii) $\mathrm{Dom}_{\mathrm{dec}}[C, \ell^{1,\infty}(\mathbb{N})] = \ell^1_{\mathrm{dec}}(\mathbb{N}) = \mathrm{Dom}_{\mathrm{dec}}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})] \cap \ell^{1,\infty}(\mathbb{N})$.

Proof Using (2), if $x \in \mathrm{Dom}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})]$ and $y = Cx - x \in \ell^{1,\infty}(\mathbb{N})$, then

$$x_n = \left(x_1 - \sum_{k=2}^{\infty} \frac{y_k}{k-1}\right) - \frac{n}{n-1} y_n + \sum_{k=n}^{\infty} \frac{y_k}{k-1},$$

and the result of *(i)* follows using Proposition 4 *(i)*.

The proof of *(ii)* follows similarly using now Proposition 4 *(ii)*.

For the first equality in *(iii)*, we already know that $\ell^1(\mathbb{N}) \subset \mathrm{Dom}[C, \ell^{1,\infty}(\mathbb{N})]$ and hence

$$\ell^1_{\mathrm{dec}}(\mathbb{N}) \subset \mathrm{Dom}_{\mathrm{dec}}[C, \ell^{1,\infty}(\mathbb{N})].$$

Conversely, if $x\downarrow$ and $Cx \in \ell^{1,\infty}(\mathbb{N})$, then

$$\sup_n n(Cx)^*_n = \sup_n \sum_{k=1}^{n} x^*_n = \|x\|_{\ell^1(\mathbb{N})} < \infty.$$

That $\ell^1_{\mathrm{dec}}(\mathbb{N}) \subset \mathrm{Dom}_{\mathrm{dec}}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})] \cap \ell^{1,\infty}(\mathbb{N})$ is trivial. Finally, if

$$x \in \mathrm{Dom}_{\mathrm{dec}}[C - \mathrm{I}, \ell^{1,\infty}(\mathbb{N})] \cap \ell^{1,\infty}(\mathbb{N}),$$

then $Cx = (C - \mathrm{I})x + x \in \ell^{1,\infty}(\mathbb{N})$ and $x \in \mathrm{Dom}_{\mathrm{dec}}[C, \ell^{1,\infty}(\mathbb{N})]$. □

4 Domain for Copson type operators on sequence spaces

In this final section, we are going to study the analogous results of Section 2, but for the Copson operator C^*. We start with some properties for C^*, similar to those proved in Proposition 1 for C.

Proposition 6

(i) There exists $x \in \ell^\infty(\mathbb{N}) \setminus \{0\}$ *such that* $C^*x \in \ell^\infty(\mathbb{N})$; *that is,*

$$\ell^\infty(\mathbb{N}) \cap \mathrm{Dom}[C^*, \ell^\infty(\mathbb{N})] \neq \{0\}.$$

(ii) There exists $x \notin \ell^\infty(\mathbb{N})$ *such that* $C^*x \in \ell^\infty(\mathbb{N})$*; that is,*

$$\mathrm{Dom}[C^*, \ell^\infty(\mathbb{N})] \not\subset \ell^\infty(\mathbb{N}).$$

Proof To prove *(i)*, take $x = (-1)^n$, $n \geq 1$. To show *(ii)*, pick $x = (-1)^n \sqrt{n}$, $n \geq 1$. Then, $x \notin \ell^\infty(\mathbb{N})$ but $C^*x \in \ell^\infty(\mathbb{N})$. □

Proposition 7 *Let* $w > 0$. *Then,*

$$C^* : \ell^\infty(w, \mathbb{N}) \longrightarrow \ell^\infty(\mathbb{N})$$

is bounded if and only if $A = \sum_{n=1}^\infty \frac{1}{nw_n} < \infty$.

Proof We prove first the sufficiency of the condition; i.e., if $A = \sum_{n=1}^\infty \frac{1}{nw_n} < \infty$ then the operator is bounded. By homogeneity it is sufficient to prove that $\|C^*x\|_\infty \leq A$, for all x such that $|x_n|w_n \leq 1$. Now,

$$\|C^*x\|_\infty = \sup_n \Big| \sum_{k=n}^\infty \frac{x_k}{k} \Big| \leq \sup_n \sum_{k=n}^\infty \frac{1}{kw_k} = A.$$

Hence C^* is bounded and $\|C^*\|_\infty \leq A$.

To prove the necessity, take $x_N = (\frac{1}{w_1}, \frac{1}{w_2}, \dots, \frac{1}{w_N}, 0, \dots, 0, \dots)$ (a sequence for every $N \in \mathbb{N}$). Since C^* is bounded, we get

$$\|C^*x_N\|_\infty = \sup_n \sum_{k=n}^N \frac{1}{kw_k} = \sum_{k=1}^N \frac{1}{kw_k} \leq \|C^*\|_\infty,$$

for any N, which implies that $A = \sum_{k=1}^\infty \frac{1}{kw_k} \leq \|C^*\|_\infty$. Hence, A is also the best constant in the inequality. □

Theorem 2 (Optimal domain for $C^* - \mathrm{I}$ **on** $\ell^p(\mathbb{N})$**,** $1 \leq p \leq \infty$**)**

(i) Case $p = 1$*:*

a. $\ell^1(w, \mathbb{N}) \subset \mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})] \iff w_n \gtrsim 1$. *Hence,* $\ell^1(\mathbb{N})$ *is the largest weighted space* $\ell^1(w, \mathbb{N})$ *contained in* $\mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})]$ *and the inclusion is strict.*

b. $\mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})] \subsetneq \ell^{1,\infty}(\mathbb{N})$.

(ii) Case $1 < p < \infty$*:* $\mathrm{Dom}[C^* - \mathrm{I}, \ell^p(\mathbb{N})] = \ell^p(\mathbb{N})$.

(iii) Case $p = \infty$*:* $\mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})] = \mathrm{Dom}[C^*, \ell^\infty(\mathbb{N})] \cap \ell^\infty(\mathbb{N})$*. Moreover,*

a. *if* $W = \left\{(w_n)_n : w_n \gtrsim 1 \text{ and } \sum_{n=1}^\infty \frac{1}{n w_n} < \infty\right\}$*, then*

$$\ell^\infty(w, \mathbb{N}) \subset \mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})] \Longleftrightarrow w \in W.$$

b. $\mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})] \subset \ell^\infty(w, \mathbb{N}) \Longleftrightarrow w_n \lesssim 1$*. Hence,* $\ell^\infty(\mathbb{N})$ *is the smallest weighted space* $\ell^\infty(w, \mathbb{N})$ *that contains* $\mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})]$ *and the inclusion is strict.*

Proof We prove first *(i)-(a)*. If we assume that $\ell^1(w, \mathbb{N}) \subset \mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})]$; i.e., $\|C^*x - x\|_1 \lesssim \|x\|_{\ell^1(w,\mathbb{N})}$, taking $x^N = \delta_N$, for $N \geq 2$, we have

$$(C^*x^N - x^N)_n = \begin{cases} \frac{1}{N}, & \text{if } 1 \leq n \leq N-1, \\ -\frac{N-1}{N}, & \text{if } n = N, \\ 0, & \text{if } n > N. \end{cases} \tag{13}$$

Hence, we have

$$1 \leq \frac{2(N-1)}{N} \lesssim w_N = \|x^N\|_{\ell^1(w,\mathbb{N})}.$$

Conversely, if $w_n \gtrsim 1$, $\ell^1(w, \mathbb{N}) \subset \ell^1(\mathbb{N}) \subset \mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})]$, since C^* is bounded from $\ell^1(\mathbb{N})$ into $\ell^1(\mathbb{N})$. That the inclusion is strict follows by considering the sequence $x_n = 1/(n+1)$, $n \geq 1$, which belongs to $\mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})]$ but not to $\ell^1(\mathbb{N})$. In fact,

$$\begin{aligned} (C^* - \mathrm{I})\,(x) &= \sum_{k=n}^\infty \frac{1}{k(k+1)} - \frac{1}{n+1} = \sum_{k=n}^\infty \left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{1}{n+1} \\ &= \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} = y_n, \end{aligned}$$

which shows that $x \in \mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})]$.

Let us now see *(i)-(b)*. Let $x \in \mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})]$. Hence, $y = C^*x - x \in \ell^1(\mathbb{N})$ and using (3) we get that

$$x_n = (Cy)_{n-1} - y_n, \quad n \geq 2.$$

Thus, using Proposition 3 we conclude that $x \in \ell^{1,\infty}$. Finally, to prove that the embedding is strict, we define

$$x_n = \frac{1}{n-1} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} - \frac{(-1)^n}{n}.$$

As above, using (3), we have that $y_n = (C^*x - x)(n) = \frac{(-1)^n}{n} \notin \ell^1(\mathbb{N})$ and, hence, $x \notin \mathrm{Dom}[C^* - \mathrm{I}, \ell^1(\mathbb{N})]$, but

$$|x_n| \leq \frac{1}{n-1}\left|\sum_{k=1}^{n-1} \frac{(-1)^k}{k}\right| + \frac{1}{n} \lesssim \frac{1}{n} \in \ell^{1,\infty}.$$

The proof of *(ii)* is a direct consequence of Lemma 2 and the fact that, for $1 < p < \infty$, both C and C^* are bounded on $\ell^p(\mathbb{N})$.

We consider now the equality $\mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})] = \mathrm{Dom}[C^*, \ell^\infty(\mathbb{N})] \cap \ell^\infty(\mathbb{N})$ in *(iii)*. Assume firstly that $y = C^*x - x \in \ell^\infty(\mathbb{N})$. As above, using (3), since $(Cy)_n \in \ell^\infty(\mathbb{N})$, when $y \in \ell^\infty(\mathbb{N})$, we necessarily get that $x \in \ell^\infty(\mathbb{N})$. Also, the equality

$$\sum_{k=n}^{\infty} \frac{x_k}{k} = x_n + y_n, \quad n \geq 2$$

implies

$$\|C^*x\|_\infty \leq \|y\|_\infty + \|x\|_\infty < \infty;$$

i.e., $x \in \mathrm{Dom}[C^*, \ell^\infty(\mathbb{N})]$.

Conversely, if $x \in \mathrm{Dom}[C^*, \ell^\infty(\mathbb{N})] \cap \ell^\infty(\mathbb{N})$, then trivially $y = C^*x - x \in \ell^\infty(\mathbb{N})$.

For the proof of *(iii)-(a)*, let $w \in W$. Then, $w_n \gtrsim 1$ (i.e., $\ell^\infty(w, \mathbb{N}) \subset \ell^\infty(\mathbb{N})$) and hence, by the previous equality, it is enough to show that $\ell^\infty(w, \mathbb{N}) \subset \mathrm{Dom}[C^*, \ell^\infty(\mathbb{N})]$, which follows from Proposition 7.

Conversely, if $\ell^\infty(w, \mathbb{N}) \subset \mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})]$, we get that $\ell^\infty(w, \mathbb{N}) \subset \mathrm{Dom}[C^*, \ell^\infty(\mathbb{N})]$ and $\ell^\infty(w, \mathbb{N}) \subset \ell^\infty(\mathbb{N})$, which as before, and from Proposition 7, imply that $w \in W$.

To prove the necessity part of *(iii)-(b)*, we choose $x^N = \delta_N$. The embedding and (13) imply $w_N \lesssim 1$.

For the sufficiency part, if $w_n \lesssim 1$, using *(iii)* we get the embeddings

$$\mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})] \subset \ell^\infty(\mathbb{N}) \subset \ell^\infty(w, \mathbb{N}).$$

That $\mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})] \subsetneq \ell^\infty(\mathbb{N})$ follows by taking $x \equiv 1$, which is in $\ell^\infty(\mathbb{N})$ but not in $\mathrm{Dom}[C^* - \mathrm{I}, \ell^\infty(\mathbb{N})]$. □

Acknowledgements We would like to thank the anonymous referees for some important comments and clarifications, which have greatly improved the final exposition of this work.

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